FORMERLY ACTA MATHEMATICA ACADEMIAE SCIENTIARUM HUNGARICAE

Acta Mathematica Hungarica

VOLUME 42, NUMBERS 1–2, 1983

EDITOR-IN-CHIEF

K. TANDORI

DEPUTY EDITOR-IN-CHIEF

J. SZABADOS

EDITORIAL BOARD

Á. CSÁSZÁR, P. ERDŐS, L. FEJES TÓTH, A. HAJNAL, I. KÁTAI, L. LEINDLER, L. LOVÁSZ, A. PRÉKOPA, A. RAPCSÁK, P. RÉVÉSZ, E. SZEMERÉDI, B. SZ.-NAGY

Akadémiai Kiadó, Budapest

ACTA MATH. HUNG. HU ISSN 0001—5954

ACTA MATHEMATICA
HUNGARICA

Acta Mathematica publishes papers on mathematics in English, German, French and Russian.

Acta Mathematica is published in two volumes of four issues a year by

AKADÉMIAI KIADÓ
Publishing House of the Hungarian Academy of Sciences
H-1054 Budapest, Alkotmány u. 21.

Manuscripts and editorial correspondence should be addressed to

Acta Mathematica, H-1395 Budapest, P.O.B. 428

Orders may be placed with

KULTURA Foreign Trading Company
H-1389 Budapest P.O. Box 149

or with its representatives abroad.

Acta Mathematica is indexed in Current Contents

ACTA MATHEMATICA HUNGARICA

EDITOR-IN-CHIEF
K. TANDORI

DEPUTY EDITOR-IN-CHIEF
J. SZABADOS

EDITORIAL BOARD
Á. CSÁSZÁR, P. ERDŐS, L. FEJES TÓTH, A. HAJNAL,
I. KÁTAI, L. LEINDLER, L. LOVÁSZ, A. PRÉKOPA, A. RAPCSÁK,
P. RÉVÉSZ, E. SZEMERÉDI, B. SZ.-NAGY

VOLUME 42

AKADÉMIAI KIADÓ, BUDAPEST
1983

CONTENTS

VOLUME 42

Alimov, Š. A. and *Joó, I.*, On the eigenfunction expansions associated with the Schrödinger operator having spherically symmetric potential 121
Alon, N., A note on the decomposition of graphs into isomorphic matchings 221
Bácsó, S., Eine Charakterisierung der Finslersche Räume von konstanter Krümmung 233
Вейдар, К. И. и *Салавова, К.*, О решетках N-радикалов, строгих слева радикалов, наследственных слева радикалов 81
Берман, Д. Л., Решение одной экстремальной задачи теории операторов 153
Цао Цзя-Дин, Об операторах типа Канторовича и интегральных сплайнах Шенберга .. 189
Collier, M. G. and *Kelingos, J. A.*, Periodic Beurling distributions 261
Csörgő, S., Tandori, K. and *Totik, V.*, On the strong law of large numbers for pairwise independent random variables 319
Ferenczi, M., Measures on cylindric algebras 3
Füredi, Z., An intersection problem with 6 extremes 177
Garay, B. M., Metrization and Liapunov functions. I 337
Grecu, E., Détermination des géodésiques de certains espaces riemanniens singuliers 97
Gut, A., Complete convergence and convergence rates for randomly indexed partial sums with an application to some first passage times 225
Győrvári, J., Lakunäre Interpolation mit Spilne-Funktionen 25
Győry, K., Bounds for the solutions of norm form, discriminant form and index form equations in finitely generated integral domains 45
Huhn, A. P., A reduced free product of distributive lattices. I 349
Jain, R. K., Semigroups with primary ideals of prime power 165
Joó, I. and *Alimov, Š. A.*, On the eigenfunction expansions associated with the Schrödinger operator having spherically symmetric potential 121
Karamzadeh, O. A. S., On the Krull intersection theorem 139
Kátai, I., Multiplicative functions with regularity properties. I 295
Kelingos, J. A. and *Collier, M. G.*, Periodic Beurling distributions 261
Komornik, V., On the distribution of the eigenvalues of an orthonormal system, consisting of eigenfunctions of higher order of a linear differential operator 171
Kroó, A., On the unicity of best L_1-approximation by polynomials of several variables 309
Kubacki, K. S. and *Szynal, D.*, Weak convergence of martingales with random indices to infinitely divisible laws 143
Leawitt, W. G., Upper radicals of regular classes 213
Maknys, M., On the distance between consecutive prime ideal numbers in sectors 131
Matolcsy, K., Refined extensions of syntopogenous structures and quasi-uniformities 111
Móricz, F., Lebesgue functions and multiple function series. III. 205
Mukhopadhyay, S. N. and *Pal, B. K.*, The Cesaro—Denjoy —Bochner scale of integration ... 243
Oswald, P., On some convergence properties of Haar—Fourier series in the classes $\varphi(L)$..... 279
Pal, B. K. and *Mukhopadhyay, S. N.*, The Cesaro—Denjoy—Bochner scale of integration..... 243
Petruska, G., Derivatives take every value on the set of approximate continuity points 355
Petz, D., On spectral and central states of Banach algebras 19

Салавова, К. и *Бейдар, К. И.*, О решетках N-радикалов, строгих слева радикалов, наследственных слева радикалов 81
Sebestyén, Z., A moment problem with respect to C^*-algebras 331
Szynal, D. and *Kubacki, K. S.*, Weak convergence of martingales with random indices to infinitely divisible laws 143
Tandori, K., Csörgő, S. and *Totik, V.*, On the strong law of large numbers for pairwise independent random variables 319
Tanovič-Miller, N., On strong convergence of trigonometric and Fourier series 35
Totik, V., Csörgő, S. and *Tandori, K.*, On the strong law of large numbers for pairwise independent random variables 319
Yahya, S. M., On an isomorphism 257
Yoneda, K., Fourier transforms and convolutions of generalized measures defined on the dyadic field 237

83-819 — Szegedi Nyomda — F. v.: Dobó József igazgató

CONTENTS

Ferenczi, M., Measures on cylindric algebras 3
Petz, D., On spectral and central states of Banach algebras 19
Győrvári, J., Lakunäre Interpolation mit Spline-Funktionen 25
Tanović-Miller, N., On strong convergence of trigonometric and Fourier series 35
Győry, K., Bounds for the solutions of norm form, discriminant form and index form equations in finitely generated integral domains 45
Бейдар, К. И. и Салавова, К., О решетках N-радикалов, строгих слева радикалов, наследственных слева радикалов 81
Grecu, E., Détermination des géodésiques de certains espaces riemanniens singuliers 97
Parhi, N., On non-oscillatory solutions of second order differential inequalities 101
Matolcsy, K., Refined extensions of syntopogenous structures and quasi-uniformities 111
Alimov, Š. A. and *Joó, I.*, On the eigenfunction expansions associated with the Schrödinger operator having spherically symmetrical potential 121
Maknys, M., On the distance between consecutive prime ideal numbers in sectors 131
Karamzadeh, O. A. S., On the Krull intersection theorem 139
Kubacki, K. S. and *Szynal, D.*, Weak convergence of martingales with random indices to infinitely divisible laws 143
Берман, Д. Л., Решение одной экстремальной задачи теории операторов 153
Jain, R. K., Semigroups with primary ideals of prime power 165
Komornik, V., On the distribution of the eigenvalues of an orthonormal system, consisting of eigenfunctions of higher order of a linear differential operator 171

Acta Math. Hung.
42 (1—2) (1983), 3—17.

MEASURES ON CYLINDRIC ALGEBRAS

M. FERENCZI (Budapest)

Introduction

The historical origins of the topic of present paper are related to such concepts as assignment of probabilities to logical formulas and inference by probability, to mention only the most important. Such concepts have been investigated by certain philosophers but their mathematical application was slow and not sophisticated. The understanding of the close connection between the symbolic logic and the modern algebra, and the success of the axiomatic theory of probability have promoted the application of efficient methods. For example, these methods have made possible the analysis of the connection between the logical formulas and the probabilities as the examination of probabilities defined on certain algebraic structures.

The axiomatic theory of probability considers the probability on Boolean algebras which latters are isomorphic to those algebraic structures which correspond to the formal language of the propositional logic, i.e. to the propositional Lindenbaum-algebras. As is known, the Boolean algebras are the algebraizations of the propositional logic. After the emergence of the theories describing the algebraizations of the higher-order logics one after the other, (e.g. for the predicate logic, the theory of the cylindric algebras, the theory of the polyadic algebras etc. see [6], [7], [3]) it has become possible to analyse the inference by probability even for higher-order logics as examination of probabilities defined on suitable algebraic structures. Special Boolean algebras, certain subalgebras of the forementioned algebraizations (e.g. the algebra corresponding to closed formulas, see [4] and [14]) have already been applied for certain similar purposes. It is clear that to generalize a logic in a probabilistic direction, it is interesting to take into account (see [4]) the whole structures of the foregoing logic and of the algebraization, e.g., not only elements corresponding to the closed formulas.

The aim of the present paper is the exposition of the forementioned subject, the introduction of the relevant concepts and the analysis of the most important properties of these concepts. Above all we shall concentrate on the analysis of *measures* (i.e. probabilities), but emphasising the measure — theoretic, algebraic and logical aspects rather than the probabilistic ones. In a forthcoming paper we shall return to these latter aspects as well, underline the *applications* and study the case of the cylindric *set algebras* separately (see [12], [7], too).

We would like to make the following further notes on the specific subject of the present paper:

As is well known, the infinite-dimensional locally finite-dimensional cylindric algebras (or the predicate Lindenbaum algebras isomorphic to them) are neither σ-complete, nor m-complete (where m is the dimension of the algebra), but are complete for certain joins, meets, m-joins and m-meets; namely, for those which can be

assigned to quantifiers. Moreover, the forementioned algebras have a property similar to the σ-chain condition, i.e. the forementioned (joins) meets (which can be assigned to quantifiers) are equal to (joints) meets of their *arbitrary denumerable* original components. So, we can introduce such a concept of measure from which we require only σ-continuity instead of m-continuity and even the σ-continuity is only restricted to the foregoing distinguished quantifier operations; for these operations the algebra is complete, anyway. This requirement is consistent with the foregoing properties of the algebra. Further, we require the usual finite additivity from the measures on the Boolean part of the algebra. These measures will be called KU-measures (quantifier measures) and their stronger and weaker variants will be defined.

The following fact also underlines the significance of the concept of the KU-measure: The denumerable and locally finite-dimensional Lindenbaum algebras are separable (see [8]) and atomless. Further, Horn and Tarski have proved the non-existence of σ-additive measures on such Boolean algebras. It is also known, however, that a finitely additive measure exists on every Boolean algebra. We shall show the existence of KU-measures on cylindric algebras of the forementioned type, moreover, it will be shown that the class of these measures is rather rich. Thus the KU-measures are between the σ-additive and the finitely additive measures.

The special KU-measures i.e. the KU-measures with *product-property* and the *symmetrical* KU-measures play similarly important roles among the KU-measures as the measures with product-property and the symmetrical measures do among the measures, respectively. The product-property is similar to the product-measure property. The symmetry-property has the following meaning e.g. for cylindric set algebra: the measures of the sets are invariant under one to one replacements of all the coordinates.

Section 1 contains the prerequisites, the notations and the most important basic concepts. In Section 2 we deal with the extension of measures either from a subalgebra of a fixed algebra or permitting also the expansion of the underlying algebra. Meanwhile we give a short survey on the most important relations among the structures used here. Further, we shall examine the problems of extensions in case of important classes of measures and that of the construction of a measure. Section 3 deals with the transformations of KU-measures by cylindric homomorphisms.

1. Concepts

The present paper uses a usual logical language having identity symbol. The standard system of deduction is assumed. For the sake of simplicity in the basic language we permit neither individual constants nor function symbols. We do not restrict the cardinality of individual variables and the predicate symbols in general. In the opposite case special notice will be given. We shall also refer to the *Lindenbaum algebra* (for short: L. algebra) *corresponding to* all formulas of a logical language and *to* a given theory (this latter being given by a consistent set of sentences). We consider the L. algebras *as special cylindric algebras.*

The Boolean reductum of a cylindric algebra $\mathfrak{A}$ will be denoted by $\mathfrak{A}^0$. If $\mathfrak{A}$ is a L. algebra we refer to $\mathfrak{A}^0$ as Lindenbaum Boolean algebra (for short: LB. algebra). We shall use several special subalgebras of the LB. algebra: subalgebra of the closed formulas or subalgebra of quantifier-free formulas, to give only some examples. If

we do not assume a theory, i.e. if we merely consider the L. algebra corresponding to the language, then we refer to this algebra as *basic L. algebra.* It is also known that for an arbitrary LB. algebra, the subalgebra of the closed formulas is isomorphic to a field of sets of models.

We denote in the L. algebra the element corresponding to a formula φ by $|\varphi|$. If we substitute a free variable by the variable y in the formula $\varphi(x)$, we shall write $\varphi(y)$ instead of $\varphi(x|y)$ if misunderstanding is not possible.

The connection between logic and cylindric algebra theory was elaborated in [0], [6], [12]. The knowledge of the most important basic concepts of the theory of cylindric algebras is assumed (see [6], [7]). Therefore we do not explain the meaning of the following notions: cylindric set algebra, cylindric homomorphism, cylindric ideal, the dimension set of an element a (its notation is Δa), the substitution operator (its notation is S_ξ^χ). If there is no misunderstanding, S_ξ^χ will be replaced by S_ζ. The homomorphic image of an element a is denoted by ha if h is the homomorphism. Throughout the paper α is an arbitrary but infinite ordinal. The cylindric algebras we shall consider will always be implicitly assumed to be of dimension α. By cylindric algebra we always mean only a *locally finite dimensional* cylindric algebra of dimension α. In the text we shall not mention α explicitly as no misunderstanding will be possible. The greek letters χ, ζ, ι always denote elments of α. Whenever we mention the "set of individual variables" we actually mean α. E.g. in § 2.1 the sentence "let denote the set of individual variables" actually means to say "let $X=\alpha$".

It is known that the infinite-dimensional cylindric algebras are atomless (under a slight restriction on the diagonal elements; see [6], p. 235). We assume the knowledge of the isomorphism relations between the L. algebras and the infinite-dimensional locally finite-dimensional cylindric algebras.

A Boolean algebra is called *separable* if its basic set has a denumerable subset W, such that for every element a, $a\neq 0$ in the algebra there exists a $w\in W$ for which $a>w$ (see [8]). (Evidently, if the language has countable logical variables, then the corresponding L. algebra is separable. Thus, denumerable-dimensional, locally finite-dimensional cylindric algebras with countable support are separable as well.)

Further prerequisites are: the concept of the Stone space of a Boolean algebra and the topological properties of the Stone space.

By *isomorphism* we mean the *usual* notion of *isomorphism for finite Boolean operations* (meets, joins) unless other adjectives are applied.

Throughout this paper by an infinite sequence we mean a countable sequence that is a w sequence. Specifically $\zeta_1, \zeta_2, \ldots . \zeta_n$, denotes the sequence $\langle \zeta_i : i\in n\rangle$ of length w.

By *measure* we mean a *finitely additive*, non-negative real function, normalized to 1 i.e. finitely additive probability on a Boolean algebra unless other adjectives are applied. Requiring σ-additivity we shall use the adjective "σ-additive".

Now we introduce few new concepts:

By a *Q-measure* defined on a cylindric algebra we mean a measure μ defined on $\mathscr{A}^0$, having the following property (for short: *Q-property*): For any sequence $\zeta_1, \zeta_2, \ldots, \zeta_n, \ldots$ consisting of different ordinals and for any element a the following relation holds:

$$\mu(\exists \chi a) = \sup_n \mu\left(\bigvee_{i=1}^{n} S_{\zeta_i}^\chi a\right). \tag{1}$$

By a *q-measure* defined on a cylindric algebra $\mathfrak{A}$ we mean a measure μ having the following property (for short: *q-property*): For any element a *there exists* a countable sequence $\zeta_1, \zeta_2, \ldots . \zeta_n, \ldots$ consisting of different ordinals, such that the relation (1) holds.

(It can be proved that a measure μ on a cylindric algebra is Q measure if only if for any element a, such that $\exists\chi a = 1$, and for any sequences $\zeta_1, \zeta_2, \ldots, \zeta_n, \ldots$ consisting of different ordinals, the relation

$$\sup_n \mu\left(\bigvee_{i=1}^{n} S^{\chi}_{\zeta_i} a\right) = 1 \tag{1*}$$

holds. Statement like this one can be made for q measures, as well.)

Throughout this paper from now on we shall write $\bigvee_i$ instead of $\bigvee_{i\in\omega}$ and $\sup_n$ instead of $\sup_{n\in\omega}$.

The abovementioned definitions are motivated by the following relation which holds in any locally finite dimensional cylindric algebra for any infinite sequence of ordinals $\zeta_1, \zeta_2, \ldots, \zeta_n, \ldots$ and for any element a $\exists\chi a = \bigvee_i S^{\chi}_{\zeta_i} a$.

It has the following special form in an L. algebra for any formula φ $|\exists x\varphi| = \bigvee_i |\varphi(y_i)|$.

Consequently, a measure μ defined on an L. algebra is a Q-measure if for any sequence of variables $y_1, y_2, \ldots, y_n, \ldots$ consisting of different variables and for any formula φ the following relation is valid:

$$\mu(|\exists x\varphi|) = \sup_n \mu\left(\bigvee_1^n |\varphi(y_i)|\right).$$

Therefore we can say that like σ- or m-measures reflect the σ- or m-operations of the Boolean algebras in a certain sense, the Q-measures introduced above reflect the cylindric operations of the cylindric algebras. Though a q-measure may not be a Q-measure, it will be seen that the two concepts are equivalent for important classes of measures under slight restrictions.

Now we define the product- and the symmetry-properties of the measure for cylindric algebras. A measure defined on a cylindric algebra is said to have the *product-property* if for any pair of elements a and b, $\Delta a \wedge \Delta b = \varnothing$ implies $\mu(a \wedge b) = \mu(a)\mu(b)$. For example, this property means the following for a basic L. algebra: If the formulas φ and ψ have no common, real free variable, then $\mu(|\varphi \wedge \psi|) = \mu(|\varphi|)\mu(|\psi|)$. In a certain sense the product-property corresponds to the concept of independence in the theory of probability.

A measure μ defined on a cylindric algebra is *symmetrical* if for any element a and for any mapping τ defined on the dimension set Δa, such that the τ-images are ordinals and τ assings different ordinals to different ordinals, the following relation holds:

$$\mu(a) = \mu(S_\tau a) \tag{2}$$

where S_τ denotes that substitution operation which substitutes for the elements in Δa their τ-images (see [6] p. 236).

The uniqueness of the definition of S_τ is implied by the definition of τ. The meaning of the symmetry-property is that the measure is invariant under the one to one replacements of all the ordinals.

2. Constructions of measures

2.1. Extension of measures

In this section we shall consider L. algebras only, but our statements can easily be transmitted to cylindric algebras, since locally finite cylindric algebras are all isomorphic to the L. algebras.

Let $\bar{\mathscr{A}}$ be an L. algebra of a theory Σ and of a language $\mathscr{L}$ having an arbitrary individual variable set X with $\|\mathscr{L}\| \leqq \|X\|$. Let $\bar{X}$ be the set of the new individual constants $\bar{x}$ which corresponds (one to one) to the set X of the individual variables. Let $\mathscr{L}'$ denote the language expanded by the set $\bar{X}$.

Let $\bar{\mathscr{A}}$ denote the LB. algebra *corresponding to* the language $\mathscr{L}'$ *to* the closed formulas and *to* the theory Σ. It is easy to show that $\mathscr{A}^0$ is isomorphic to $\bar{\mathscr{A}}$. Let $\bar{\mathscr{N}}$ denote the subalgebra of $\mathscr{A}$ corresponding to the quantifier-free formulas at this isomorphism.

Let us denote by $\mathscr{M}'$ the collection of those models of the foregoing expanded language $\mathscr{L}'$ for which *every element* of the universe is the interpretation of a suitable $\bar{x}$.

Let us consider the image of $\bar{\mathscr{A}}$ in the model space, models of type $\mathscr{L}'$, and the *trace* of the model space on $\mathscr{M}'$, i.e., the field of the intersections of foregoing model sets with $\mathscr{M}'$. This Boolean algebra will be denoted by $\bar{\mathscr{A}}_{\mathscr{M}'}$. It is isomorphic to $\bar{\mathscr{A}}$ (to the discussion of which we return later). Now we have

THEOREM 2.1. *Let μ be a measure on the Boolean subreductum $\mathscr{N}$ of the L. algebra $\mathscr{A}$ corresponding to a countable language and to the theory Σ where $\mathscr{N}$ is the subreductum corresponding to the quantifier-free formulas. μ can be extended to $\mathscr{A}$ as a q-measure if μ is σ-additive on the image $\bar{\mathscr{N}}_{\mathscr{M}'}$ of $\mathscr{N}$, where $\bar{\mathscr{N}}$ corresponds to $\mathscr{N}$ in $\bar{\mathscr{A}}$ and $\bar{\mathscr{N}}_{\mathscr{M}'}$ stands for the image of $\bar{\mathscr{N}}$ in $\bar{\mathscr{A}}_{\mathscr{M}'}$, at the foremention isomorphism between $\bar{\mathscr{A}}$ and $\bar{\mathscr{A}}_{\mathscr{M}'}$.*

We shall give two proofs for the theorem: the first proof will be of formal logical character and the second one of algebraic character. We shall return to the relation of the two proofs below.

PROOF I. Let a closed formula of the form $\exists x \bar{\varphi}$ be in the language $\mathscr{L}'$. Then

$$\mathscr{M}'(\exists x \bar{\varphi}) = \bigcup_1^\infty \mathscr{M}'(\bar{\varphi}(\bar{x}_i))$$

where $\mathscr{M}'(\bar{\varphi}(\bar{x}_i))$ denotes the set of models of $\mathscr{M}'$ satisfying the sentence $\bar{\varphi}(\bar{x}_i)$. (Here we made use of the fact that the language has only countable many constants.)

Relation (3) implies that $\bar{\mathscr{A}}_{\mathscr{M}'}$ is contained in the σ-algebra generated by $\bar{\mathscr{N}}_{\mathscr{M}'}$.

Since μ is σ-additive on $\overline{\mathcal{N}}_{\mathcal{M}'}$, it can be extended to the generated σ-algebra and also to $\overline{\mathcal{A}}_{\mathcal{M}'}$ as σ-additive measure; i.e.,

$$\mu(\mathcal{M}'(\exists x\bar{\varphi})) = \sup_n \mu\left(\mathcal{M}'\left(\bigvee_1^n \bar{\varphi}(\bar{x}_i)\right)\right).$$

Since $\overline{\mathcal{A}}_{\mathcal{M}'} \simeq \overline{\mathcal{A}}$ and $\overline{\mathcal{A}} \simeq \mathcal{A}^0$ we obtain a measure on $\mathcal{A}$ as well.

We have to prowe $\mu(|\exists x\varphi|) = \sup_n \mu\left(\bigvee_1^n \varphi(x_i)\right)$ where φ is such that every symbol $\bar{x}$ in $\bar{\varphi}$ is replaced by the corresponding variable x.

Since at the foregoing isomorphism $\mathcal{M}'\left(\bigvee_1^n \bar{\varphi}(\bar{x}_i)\right)$ corresponds to $\bigvee_1^n |\varphi(x_i)|$, $|\exists x\bar{\varphi}|$ corresponds to $|\exists x\varphi|$ which completes the proof. Q.E.D.

COROLLARY. *If $\mathcal{A}$ is a basic L. algebra, then every measure defined on $\overline{\mathcal{N}}_{\mathcal{M}'}$ is σ-additive (since $\overline{\mathcal{N}}_{\mathcal{M}'}$ is a perfect field of sets, see* [15] *p.* 18*). Consequently, every measure μ defined on $\mathcal{N}$ can be extended to $\mathcal{A}$ as a q-measure.*

This latter special case is proved in [4], too, but in a different context. The proposition, that the intersection of arbitrary nonempty denumerable many sets from $\overline{\mathcal{N}}_{\mathcal{M}'}$ equal to the intersection of finitely many components of the original intersection, results from the Compactness Theorem of the Logic, as well.

REMARK. If we dropped the restriction according to which the language is countable, then even this general case could be reduced to the case of Theorem 2.1, applying the generalization of the construction of directed union of the model theory.

Before turning to the second proof we recall a few concepts which are of importance also in themselves.

$\mathcal{A}$ is isomorphic to the setalgebra of the clopen sets in the Stone space of its i.e. to a set algebra whose universe is the set of the ultrafilters of $\mathcal{A}$. Because of the compactness the isomorphism (to be denoted by f) transforms any measure defined on $\mathcal{A}$ into a σ-additive measure on this setalgebra (see [15] p. 151).

An ultrafilter U of $\mathcal{A}$ is called *perfect* if for any formula of the form $\exists x\varphi$ with $|\exists x\varphi| \in U$ there exists a variable y_j of the language, such that $|\varphi(y_j)| \in U$ (see [1′] p. 196). Let us denote the set of perfect ultrafilters by $\mathcal{P}$.

Next we consider the subalgebra $\mathcal{N}$ (of quantifier-free formulas) of $\mathcal{A}^0$ and a measure μ defined on it. This algebra is transformed by the isomorphism f into an algebra $\mathcal{N}'$ in the Stone space. The 'trace' of $\mathcal{N}'$ on $\mathcal{P}$, the sets of the form $\{N \cap \mathcal{P};\ N \in \mathcal{N}'\}$, is a set algebra on $\mathcal{P}$ which will be denoted by $\mathcal{N}_p$.

A measure on $\mathcal{N}$ is said to be *σ_p-measure* if whenever there exists an isomorphism between $\mathcal{N}'$ and $\mathcal{N}_p$ the measure transformed by the isomorphism is σ-additive on $\mathcal{N}_p$.

Now we shall prove the following variant of Theorem 2.1. *If μ is a σ_p-measure on $\mathcal{N}$, then it can be extended to $\mathcal{A}$ as a q-measure.* (The equivalence of the two variants will be discussed below.)

PROOF II. Consider the joins $\bigvee_1^\infty |\varphi(x_i)|$ in $\mathcal{A}$ where φ may be any formula of the language. Since the set of these joins is countable, for any non-zero element $a \in \mathcal{A}$ there exists a perfect ultrafilter U containing a (see [1] p. 159).

In turn, this implies that $\mathscr{A}^0$ is isomorphic to $\mathscr{A}_p$ which is the trace of the image of $\mathscr{A}^0$ on the set of perfect ultrafilters in the Stone space. This isomorphism h has the following property:

$$h\left(\bigvee_1^\infty |\varphi(x_i)|\right) = \bigcup_1^\infty h(|\varphi(x_i)|) \tag{4}$$

for every formula of the form $\exists x\varphi$ (see [15] p. 84).

Thus the image of $\mathscr{N}$ is $\mathscr{N}_p$ at the isomorphism h. Denote by $\mathscr{H}'$ the σ-setalgebra generated by $\mathscr{N}_p$. $\mathscr{A}_p$ is a subalgebra of $\mathscr{H}'$. On the one hand, the former is a subset of the basic set of the latter, since each element of $\mathscr{A}$ can be obtained by finite applications of denumerable joins and meets of the elements corresponding to the quantifier-free formulas. (Consequently, the h-images can similarly be obtained from the images of the quantifier-free formulas, because of the property (4) of the isomorphism h.) On the other hand the finite Boolean operations of $\mathscr{A}_p$ and of $\mathscr{H}'$ coincide, since they are the subalgebras of the same power set algebra.

The measure μ transformed by the isomorphism can be extended from $\mathscr{N}_p$ to $\mathscr{H}'$, because of the σ_p-property. Thus a measure is obtained on $\mathscr{A}_p$ as well and thus, a measure exists then on $\mathscr{A}$ too, because of the isomorphism. It is to be proved that this latter measure is a q-measure. But this follows from

$$\mu(|\exists x\varphi|) = \mu\left(\bigvee_1^\infty |\varphi(x_i)|\right) = \mu\left(h\left(\bigvee_1^\infty |\varphi(x_i)|\right)\right) = \mu\left(\bigcup_1^\infty h(|\varphi(x_i)|)\right) =$$

$$= \sup_n \mu\left(\bigcup_1^n h(|\varphi(x_i)|)\right) = \sup_n \mu\left(\bigvee_1^n |\varphi(x_i)|\right)$$

where we did not make distinction between the notations of the original measure and of the measure transformed by the isomorphism. Here we made use of the continuity of μ on $\mathscr{H}'$.

Now we present a proposition on the extension to a *Q-measure*:

THEOREM 2.2. *Let $\mathscr{A}$ be an arbitrary L. algebra and let μ be a measure on the Boolean subreductum $\mathscr{N}$ of the quantifier-free formulas. Consider the σ-setalgebra of the Borel sets in the Stone space of $\mathscr{A}^0$ and let $\mathscr{N}'$ denote the image of $\mathscr{N}$ at the canonical isomorphism; then consider the quotient algebra C^* of the Borel sets modulo of the sets of the first category and let $\overline{\mathscr{N}}'$ be the algebra corresponding to $\mathscr{N}'$ in C^* at the canonical homomorphism. Since $\mathscr{N}''$ is isomorphic to $\mathscr{N}$, we can consider μ on $\mathscr{N}''$ as well, and assume that μ is σ-additive on $\mathscr{N}''$. Then μ can be extended to $\mathscr{A}$ as a Q-measure.*

PROOF. As is known, C^* is a complete Boolean algebra. Considering the natural mapping from $\mathscr{A}'$ — where $\mathscr{A}'$ is the image $\mathscr{A}'$ of $\mathscr{A}^0$ in the Stone space — into C^* mentioned in the theorem we have an isomorphism (not only a homomorphism) since in a compact Hausdorff space there is no non-empty open set of first category. Moreover, this isomorphism is a *complete isomorphism* (see [13] p. 118).

At this isomorphism let $\mathscr{A}''$ be the image of $\mathscr{A}'$, $\overline{\mathscr{N}}'$ be that of $\mathscr{N}'$, both are subalgebras of C^*. Since $\mathscr{N}''$ is a subalgebra of a complete Boolean algebra, thus we

can speak of a minimal generated σ-algebra with respect to C^*, wich contains $\mathscr{A}''$ because of the following property of the L. algebras: $|\exists x\varphi| = \bigvee_1^\infty |\varphi(y_j)|$ holds for any infinite sequence of variables $y_1, y_2, \ldots, y_n, \ldots$.

We denoted by μ, too, the measure transformed from $\mathscr{N}$ onto $\mathscr{N}''$. Similarly to the case of the setalgebras, by introducing the outer measure and the measurable elements, the measure can be extended to the minimal σ-algebra containing $\mathscr{N}''$ because μ is σ-additive with respect to the Boolean σ-operations on $\mathscr{N}''$. Thus a measure μ is obtained on $\mathscr{A}$ as well.

This extension has the Q-property, since for an arbitrary sequence $y_1, y_2, \ldots, y_n, \ldots$ the continuity of the measure on $\mathscr{A}'$ implies $\mu(|\exists x\varphi|) = \sup_n \mu\left(\bigvee_1^n |\varphi(y_i)|\right)$. Q.E.D.

Now we turn to the proof of such an extension theorem which requires also the expansion of the algebra — differing from our earlier investigation.

THEOREM 2.3. *Let $\mathscr{A}$ be an L. algebra corresponding to an arbitrary language and to a given theory and let μ be a measure on $\mathscr{A}^0$. Expand the language by a set of individual variables having the cardinality of the language. Then on the L. algebra corresponding to the resulting new language and to the given theory there exists a measure which is the extension of μ and which is already a q-measure.*

PROOF. We prove the theorem for the case of a countable language, but it can be generalized by transfinite induction to the non-countable case, too.

Let $z_1, z_2, \ldots, z_n, \ldots$ be the newly introduced variables. Consider all the formulas of the form $\exists x\varphi$ in the language. Let $\exists x\varphi_1, \exists x\varphi_2, \exists x\varphi_3, \ldots$ be their enumeration. Now let us assign such a finite subset $F\varphi_1$ of the new variables to the formula $\exists x\varphi_1$ that no variable occur in $F\varphi_1$ which is free in $\exists x\varphi_1$. Next assign a finite subset $F\varphi_2$ to $\exists x\varphi_2$ in such a way that no variable occur in $F\varphi_2$ which is free either in $\exists x\varphi_2$, or in $\exists x\varphi_1$, or which occurs in $F\varphi_1$. In general, a finite set $F\varphi_j$ of the newly introduced variables is assigned to $\exists x\varphi_j$ in such a way that no variable occurs, in $F\varphi_j$ which variable occurs in $F\varphi_i$ $(i=1, 2, \ldots, j-1)$ or is free in $\exists x\varphi_i$ $(i=1, 2, \ldots, j)$. This can be done.

Now consider the formulas of the form $\exists x\varphi_i \to \bigvee_{\{k: z_k \in F\varphi_i\}} \varphi(z_k)$ $(i=1, 2, \ldots)$. (In short: $\exists x\varphi_i \to \bigvee_{F\varphi_i} \varphi(z_k)$.) We show that there exists a measure μ on the expanded LB. algebra $\mathscr{B}^0$, which measure is the extension of the original one and $\mu(|\neg(\exists x\varphi_i \to \bigvee_{F\varphi_i} \varphi(z_k))|) = 0$ holds for every i. The latter property implies $\mu(|\bigvee_{F\varphi_j} \varphi(z_k)|) \geqq \mu(|\exists x\varphi|)$. Since the converse inequality is valid in every L. algebra, in the q-measure condition (1) the supremum is attained as a maximum on the sets $F\varphi_i$.

For our purpose a homomorphism from $\mathscr{B}^0$ is constructed. At this homomorphism the elements $(\neg(\exists x\varphi_i \to \bigvee_{F\varphi_i} \varphi(z_k))|$ (the elements, to be denoted by b_i) are transformed into 0. For this it is sufficient to consider the ideal $\mathscr{I}$ in $\mathscr{B}^0$ generated by the elements b_i and consider the canonical homomorphism into $\mathscr{B}^0/\mathscr{I}$.

Recalling the definition of the measure the following can be noticed: At the foregoing canonical homomorphism the image of $\mathscr{A}^0$ is isomorphic to $\mathscr{A}^0/\mathscr{I} \cap \mathscr{A}^0$, since $\mathscr{I} \cap \mathscr{A}^0 = 0$ as we shall show.

Let $|\alpha| \in \mathscr{I} \cap \mathscr{A}^0$ be arbitrary. Then the notion of the ideal generated by arbitrary elements implies the existence of finitely many elements $b_1, \ldots, b_n$ for which $|\alpha| \leqq \bigvee_1^n b_i$. Hence

$$\Sigma \vdash \alpha \to \bigvee_1^n \neg \left(\exists x \varphi_i \to \bigvee_{F\varphi_i} \varphi(z_k) \right) \tag{6}$$

is obtained where Σ is the prefixed theory. Because of the construction of the sets $F\varphi_i$ the variables of $F\varphi_n$ can occur freely only in the subformula $\bigvee_{F\varphi_n} \varphi(z_k)$ of (6). They cannot occur freely in α, either, since $|\alpha| \in \mathscr{A}^0$.

For arbitrary formulas β and γ, if x does not occur freely in β, then $\Sigma \vdash \beta \to \gamma$ implies $\Sigma \vdash \beta \to \forall x \gamma$. This implication yields the following result for our case:

$$\Sigma \vdash \alpha \to \forall z_{k_1} \ldots \forall z_{k_n} \left(\bigvee_1^n \neg \left(\exists x \varphi_i \to \bigvee_{F\varphi_i} \varphi(z_k) \right) \right)$$

where $F_i = \{z_{k_1} \ldots z_{k_n}\}$ i.e.,

$$\Sigma \vdash \alpha \to \bigvee_1^{n-1} \neg \left(\exists x \varphi_i \to \bigvee_{F\varphi_i} \varphi(z_k) \right) V \left(\neg \exists x \varphi_n V \forall z_{k_1} \ldots \forall z_{k_n} \bigvee_{F\varphi_n} \varphi(z_k) \right).$$

It is easy to see that the nth member of this latter disjunction can only be a confutation. Executing the detachment, and continuing the process by induction we get $|\alpha| \leqq 0$, i.e. $|\alpha| = 0$.

Since $\mathscr{I} \cap \mathscr{A}^0 = 0$, $\mathscr{A}^0$ is isomorphic to $\mathscr{A}^0/\mathscr{I} \cap \mathscr{A}^0$, i.e., to a subalgebra of $\mathscr{B}^0/\mathscr{I}$. The isomorphism transforms μ into this subalgebra. According to a known theorem, a measure can always be extended from a subalgebra to the whole algebra, now to $\mathscr{B}/\mathscr{I}$, as a finitely additive measure.

The latter measure, as is known, is transformed from $\mathscr{B}^0/\mathscr{I}$ to $\mathscr{B}^0$ by the canonical homomorphism. We shall prove that the transformed measure will be the measure mentioned in the theorem. Indeed, it is evidently an extension of the original measure, the measure of elements b_i will be zero, since the image of these elements is 0 for the canonical homomorphism. Q.E.D.

The proof of the following theorem follows directly from the preceding proof. We shall assume that the cardinality of the language is equal to that of the individual variables.

THEOREM 2.4. *If a measure is given on the Boolean subalgebra of the closed formulas of an L. algebra, then it can be extended to the whole L. algebra as a q-mesaure.*

This statement can be generalized to the extension from that subalgebra which corresponds to the formulas containing only the previously fixed variables $x_1, \ldots, x_n$ as free variables. Further, Theorem 2.5 can be sharpened to obtain a Q-measure at the extension.

2.2. On the relations among foregoing algebras and on special measures

As mentioned earlier now we survey the connection between the L. algebras of *all* the formulas and the LB. algebras of the *closed* formulas — corresponding to

the language expanded by the set $\overline{X}$ of constants corresponding to the individual variables in a one-to-one way (language $\mathscr{L}'$). We also look over the connection between the measures on the foregoing LB. algebras of the closed formulas satisfying the condition

$$\mu(|\exists x\varphi|) = \sup_n \mu\left(\bigvee_1^n |\varphi(\bar{x}_i)| : x_1, x_2, \ldots, x_n \in X\right) \tag{5}$$

and the KU-measures on the L. algebras of all formulas. The two proofs of Theorem 2.1 already referred implicitly to these connections, therefore we shall refer to the notations and concepts of Theorem 2.1 but already not confining ourselves to the case of the countable language.

We have the following one-to-one correspondences: The correspondence between the perfect ultrafilters in $\mathscr{A}$ and the models in $\mathscr{M}'$.

Further consider an equivalence relation on $\overline{X}$ and the collection of those models of type $\mathscr{L}$ which are the models of the given theory and their universe being formed by the forementioned equivalence classes, where the equivalence classes are chosen in all the possible ways. The correspondence between the collection of these models and the models in $\mathscr{M}'$ is a one to one correspondence too (see [1] p. 197).

A simple logical argument shows that an isomorphism correspondence exists between the LB. algebras $\mathscr{A}^0$ and $\overline{\mathscr{A}'}$ as mentioned sbove. It can be proved by the Reader.

One can attach to this isomorphism the *trace* $\mathscr{A}_p$ of the image of $\mathscr{A}^0$ in the Stone space *on* the set $\mathscr{P}$ of the perfect ultrafilters, i.e. a set algebra with universe $\mathscr{P}$; as well as the *trace* $\overline{\mathscr{A}_{\mathscr{M}'}}$ of the model set of the theory Σ *on* $\mathscr{M}'$.

Taking into account the isomorphism between $\mathscr{A}_p$ and $\mathscr{A}_{\mathscr{M}'}$ we mention that *in* $\mathscr{A}_p$ the sets of the perfect ultrafilters containing $|\varphi|$ *correspond to* a formula φ and *in* $\overline{\mathscr{A}_{\mathscr{M}'}}$ the models belonging to $\mathscr{M}'$ satisfying φ *correspond to* it. A perfect ultrafilter generates an equivalence relation on $\overline{X}$, thus it generates a model of the assumed theory whose universe consists of the equivalence classes. The collection of these models, in turn, can be considered to correspond to $\overline{\mathscr{A}_{\mathscr{M}'}}$. Hence $\mathscr{A}_p \simeq \overline{\mathscr{A}_{\mathscr{M}'}}$.

$\mathscr{A}_{\mathscr{M}'}$ is isomorphic to $\overline{\mathscr{A}}$, since if a closed formula φ has a model being the model of Σ, then it has a model in $\mathscr{M}'$, too. Indeed, it occurs only finite number of constants in φ, therefore we can choose the interpretation of the remaining constants arbitrarily. If the model is too large to saturate it with the elements of $\overline{X}$, then it can be elementarily restricted, according to the Löwenheim–Skolem theorem.

A notice with respect to the isomorphism between $\mathscr{A}^0$ and $\overline{\mathscr{A}_{\mathscr{M}'}}$ is the following: $|\exists x\varphi| = \bigvee_{y \in X} |\varphi(y)|$ holds in $\mathscr{A}$, and

$$\mathscr{M}'(\exists x\bar{\varphi}) = \bigcup_{\bar{y} \in \bar{x}} \mathscr{M}'(\bar{\varphi}(\bar{y}))$$

holds in $\overline{\mathscr{A}_{\mathscr{M}'}}$, where X is the set of all the variables and $\bar{\varphi}$ is obtained from φ by replacing any free variable y in φ by the corresponding constant symbol $\bar{y}$. As is known $|\exists x\varphi| = \bigvee_{y \in K} |\varphi(y)|$ holds in $\mathscr{A}$ even if K is any expansion of X by individual variables but $\mathscr{M}'(\exists x\bar{\varphi}) = \bigcup_{\bar{y} \in K} \mathscr{M}'(\varphi(\bar{y}))$ is false in general if we are still restricted to such models that every element of their universe is an interpretation of a constant from $\overline{X}$.

Now considering the connection between the measures satisfying the condition (5) on $\overline{\mathscr{A}}$ and the q-measures on $\mathscr{A}$ we set out from the forementioned isomorphisms. One can see that a former measure yields a latter measure on $\mathscr{A}$ but it may not be a Q-measure. Conversely, both q- and Q-measures on $\mathscr{A}$ yield a measure on $\overline{\mathscr{A}}$ satisfying the condition (5).

Now we state the following proposition on the symmetrical measures:

If on the Boolean subalgebra of the quantifier-free formulas of an L. algebra a symmetrical measure is given which can be extended to $\mathscr{A}$ as a Q-measure, then the extension will also be a symmetrical Q-measure.

The proposition can be proved by the Reader by induction on the number of blocks of quantifiers in prenex normal form.

The proposition remains valid for q-measures as well.

The importance of symmetry-property is emphasized by the following proposition. First we introduce a property denoted by δ.

Property δ: Let us assume that a q-measure satisfies the stronger restriction that for any element a there exists a sequence $\zeta_1, \zeta_2, \ldots, \zeta_n, \ldots$ consisting of different ordinals such that $\zeta_i \notin \Delta a$ and (1) holds for this sequence.

Next we prove

THEOREM 2.5. *A symmetrical q-measure defined on a cylindric algebra is a Q-measure if and only if Property δ holds.*

PROOF. If a measure μ is a Q-measure, then it has Property δ. To prove the converse it is sufficient to show that for an arbitrary element a, for an arbitrary sequence of variables $\iota_1, \iota_2, \ldots, \iota_n, \ldots$, and for the sequence $\zeta_1, \zeta_2, \ldots, \zeta_n, \ldots$ the existence of the latter being guaranteed by our property, for any integer N there exists an integer M such that

$$\mu\left(\bigvee_1^M S^{\chi}_{\iota_k} a\right) \leqq \mu\left(\bigvee_1^N S^{\chi}_{\zeta_i} a\right)$$

$\left(\mu\left(\bigvee_1^n S^{\chi}_{\iota_k} a\right) \leqq \mu(\exists \chi a) \text{ is valid for every } n.\right)$

Let M_0 be such that $i \geqq M_0$ implies $\iota_i \in \Delta a$. Then

$$\mu\left(\bigvee_{M_0}^{M_0+N-1} S^{\chi}_{\iota_k} a\right) = \mu\left(\bigvee_1^N S^{\chi}_{\zeta_i} a\right)$$

where $\iota_k \in \Delta a$.

To have this relation consider the element $\bigvee_1^n S^{\chi}_{\zeta_i} a$ and that mapping τ which assigns ι_{i+M_0-1} to ζ_i.

On the other hand the following inequality holds:

$$\mu\left(\bigvee_1^{M_0-1} S^{\chi}_{\iota_k} a \vee \bigvee_{M_0}^{M_0+N-1} S^{\chi}_{\iota_k} a\right) \geqq \mu\left(\bigvee_{M_0}^{M_0+N-1} S^{\chi}_{\iota_k} a\right).$$

This completes the proof.

REMARK. As is already mentioned, if $\mathcal{N}$ is a basic L. algebra then the symmetrical measure on $\mathcal{N}$ can always be extended to the whole algebra as a symmetrical measure. This observation and Theorem 2.3 provides a possibility for a simple construction of Q-measures.

As is known, in a Boolean algebra the elements with measure zero constitute a Boolean ideal. We have the following analogous proposition for cylindric algebra and Q-measures:

THEOREM 2.6. *If μ is a symmetrical Q-measure on a cylindric algebra then the elements with measure zero constitute a cylindric ideal.*

Theorem like this one can be proved for q-measures too.

The proofs are left to the reader.

Finally we mention the following property of those q-measures μ which have both the symmetry- and the product properties: if a is an element of zero dimension, $\mu(a)$ is either 0 or 1.

Indeed, if an element of zero dimension has a form $\exists\chi a$, then by the q-property there exists a sequence of variables $\zeta_1, \zeta_2, \ldots, \zeta_n, \ldots$ such that $\mu(\forall\chi a)=\inf_n \mu\left(\bigwedge_1^n S^{\chi}_{\zeta_i} a\right)$ is valid. In turn, the product-property implies the following relation: $\mu\left(\bigwedge_1^n S^{\chi}_{\zeta_i} a\right)=$ $=\prod_1^n \mu(S^{\chi}_{\zeta_i} a)$. However, $\mu(S_{\zeta_i} a)=\mu(S_{\zeta_j} a)$ holds for any pair (i,j) because of the symmetry. Therefore $\mu\left(\bigwedge_1^n S^{\chi}_{\zeta_i} a\right)=\mu^n(S^{\chi}_{\zeta_1} a)$ holds, implying our proposition.

3. Transformation of measures

As is mentioned, in this section we shall deal with the transformation of KU-measures of several kind by cylindric homomorphisms.

THEOREM 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary infinite-dimensional locally finite-dimensional cylindric algebras, and let h: $\mathcal{A}\to\mathcal{B}$ be a cylindric homomorphism.*

(1) *Let μ be a Q-measure on $\mathcal{B}$. If for a and b $h(a)=b$, then define a real function μ' on $\mathcal{A}$ by $\mu'(a):=\mu(b)$. Then μ' is a Q-measure on $\mathcal{A}$.*

(2) *Conversely, let μ be a Q-measure on $\mathcal{A}$. If for a and b $h(a)=b$, then define the function μ' on $\mathcal{B}$ by $\mu'(b):=\mu(a)$. If μ vanishes on the kernel of h, then μ' is a Q-measure on $\mathcal{B}$.*

PROOF. (1) It is known that μ' will be a measure on $\mathcal{A}^0$ (see [5]). We have only to prove the following: for an arbitrary sequence $\zeta_1, \zeta_2, \ldots, \zeta_n, \ldots$ $\mu'(\exists\chi a)=$ $=\sup_n \mu'\left(\bigvee_1^n S_{\zeta_i} a\right)$ holds. We know

$$\mu'(\exists\chi a)=\mu(h\exists\chi a) \quad \text{and} \quad \mu'\left(\bigvee_1^n S_{\zeta_i} a\right)=\mu\left(h\bigvee_1^n S_{\zeta_i} a\right).$$

Therefore it is sufficient to prove the corresponding sup-property between the right-hand sides of our last two equations. But the property of the cylindric homomorphism implies $h\exists\chi a=\exists\chi ha$; further, $h\bigvee_1^n S_{\zeta_i}a=\bigvee_1^n hS_{\zeta_i}a$ holds as well. Further, $hS_{\zeta_i}a=S_{\zeta_i}ha$ is valid, since $S_{\zeta_i}^{\chi}=\exists\chi(a\wedge d_{\chi\zeta_i})$ holds for the substitution operator $S_{\zeta_i}^{\chi}$. Take into account

$$h\exists\chi(a\wedge d_{\chi\zeta_i})=\exists\chi h(a\wedge d_{\chi\zeta_i})=\exists\chi(ha\wedge d'_{\chi\zeta_i})=S_{\zeta_i}^{\chi}ha$$

where $d_{\chi\zeta_i}$ and $d'_{\chi\zeta_i}$ are the corresponding diagonal elements. For the validity of the last relations, see [6]. Therefore $\bigvee_1^n hS_{\zeta_i}a=\bigvee_1^n S_{\zeta_i}ha$ is valid. Finally, since μ is a Q-measure, $\mu(\exists\chi a)=\sup_n \mu\left(\bigvee_1^n S_{\zeta_i}ha\right)$ is true.

(2) First we note that the condition is evidently necessary for μ be a measure on $\mathscr{B}^0$, since this latter implies $\mu(0)=0$.

If μ vanishes on the kernel of h, then the definition $\mu'(b):=\mu(a)$ is correct. Indeed, $h(a_1)=h(a_2)=b$ implies $\mu(a_1)=\mu(a_2)$ as can be proved by a simple calculation. It is also easy to obtain that μ' is a measure on $\mathscr{B}^0$. Now it is to be proved that μ' is a Q-measure, i.e. for an arbitrary sequence $\zeta_1,\zeta_2,\ldots,\zeta_n,\ldots$ $\mu'(\exists\chi b)=\sup_n \mu'\left(\bigvee_1^n S_{\zeta_i}b\right)$ holds.

Let a and b be such that $h(a)=b$. On the one hand now $\exists\chi b=\exists\chi ha=h\exists\chi a$ is true, i.e., by definition, $\mu'(\exists\chi b)=\mu(\exists\chi a)$. On the other hand

$$\bigvee_1^n S_{\zeta_i}b=\bigvee_1^n S_{\zeta_i}ha=\bigvee_1^n hS_{\zeta_i}a=h\bigvee_1^n S_{\zeta_i}a$$

holds, implying

$$\mu'\left(\bigvee_1^n S_{\zeta_i}b\right)=\mu\left(\bigvee_1^n S_{\zeta_i}a\right).$$

But μ is a Q-measure, yielding $\mu(\exists\chi a)=\sup_n \mu\left(\bigvee_1^n S_{\zeta_i}a\right)$ Q.E.D.

THEOREM 3.2. *Let $\mathscr{A}$ and $\mathscr{B}$ be arbitrary, infinite-dimensional, locally finite-dimensional cylindric algebras, and let $h\colon \mathscr{A}\to\mathscr{B}$ be a cylindric homomorphism. Let μ be a Q-measure on $\mathscr{B}$ having the product-property. Then the measure μ' defined in* (1) *of Theorem* 3.1 *has the product-property as well.*

PROOF. Suppose that a_1 and $a_2\in\mathscr{A}$ are arbitrary elements satisfying $\Delta a_1\cap\Delta a_2=\varnothing$. It is to be proved that $\mu'(a_1\wedge a_2)=\mu'(a_1)\mu'(a_2)$ holds.

Since in a cylindric algebra $\Delta ha\subseteq\Delta a$ holds for every element in a, $\Delta ha_1\cap\Delta ha_2=\varnothing$ is valid, too.

Taking into account $\mu'(a_1\wedge a_2)=\mu(ha_1\wedge ha_2)$, $\mu'(a_1)=\mu(ha_1)$, $\mu'(a_2)=\mu(ha_2)$ and the product-property of μ, we have $\mu'(ha_1\wedge ha_2)=\mu'(ha_1)\mu'(ha_2)$. Q.E.D.

REMARK. If a measure μ having the product-property were given even on $\mathscr{A}$ and it were transformed to $\mathscr{B}$ as a Q-measure by h, it would not be sure that the

transformed measure would have the product-property, because $\Delta ha_1 \cap \Delta ha_2 = \varnothing$ does *not* imply $\Delta a_1 \cap \Delta a_2 = \varnothing$.

THEOREM 3.3. *If the measure μ in* (1) *and* (2) *of Theorem* 3.1 *is assumed to be symmetrical too, then μ' will be symmetrical as well.*

PROOF. (1) Let μ be a symmetrical Q-measure on $\mathscr{B}$. Let $a \in \mathscr{A}$ be arbitrary and let τ be a mapping on Δa, such that it assigns different individual variables to different individual variables, that is a mapping occurring in the definition of the symmetry-property of the measure. Because of the stated property of τ one can speak of the simultaneous substitution for the elements in Δa the corresponding τ-images. Let S_τ be this simultaneous substitution operator.

It is to be proved that $\mu'(a) = \mu'(S_\tau a)$. We have the following relations: $\mu'(a) = \mu(ha)$, $\mu'(S_\tau a) = \mu'(hS_\tau a)$ and $hS_\tau a = S_\tau ha$ (a relation similar to the last one has already been applied for the ordinary substitution operator and for cylindric homomorphism).

Since $\Delta ha \subseteqq \Delta a$, τ satisfies the condition of the one-to-one correspondence restricted to Δha. Relying on the symmetry of the measure μ, we obtain $\mu(ha) = \mu(S_\tau ha)$.

(2) Now we suppose that a Q-measure μ is given on $\mathscr{A}$, inducing a measure on $\mathscr{B}$ like in Theorem 3.1.

Further, let μ be symmetrical on $\mathscr{A}$. Let $b \in \mathscr{B}$ be arbitrary. Let τ be a mapping from Δb into the class of ordinals inducing a one-to-one correspondence. Assume $h(a) = b$ for a, then $\Delta b \subseteqq \Delta a$. Extend then τ to Δa in such a way that it induces also a one-to-one correspondence on Δa to get a mapping on Δa occurring in the definition of the symmetrical measure. This extension is evidently possible. Denote the newly obtained mapping by τ'.

Evidently, $S_{\tau'} b = S_\tau b$ holds, therefore it is sufficient to prove the equality $\mu'(b) = \mu'(S_\tau b)$. Take into account $\mu'(b) = \mu(a)$ and $\mu'(S_\tau b) = \mu(S_\tau a)$ (resulting from $S_\tau ha = hS_\tau a$). Since μ is symmetrical, for τ $\mu(S_\tau a) = \mu(a)$ holds. Q.E.D.

COROLLARY. *If μ is a symmetrical Q-measure on a cylindric algebra $\mathscr{A}$, then the elements of measure zero form a cylindric ideal. Furthermore, by Theorem* 3.3 *the canonical homomorphism induces a symmetrical Q-measure on the quotient algebra of $\mathscr{A}$ modulo the foregoing ideal.*

We have to show that $\mu(a) = 0$ implies $\mu(\exists \chi a) = 0$. Choose a sequence of variables $\zeta_1, \zeta_2, \ldots, \zeta_n, \ldots$ not containing any element of Δa. Since μ is a Q-measure, $\mu(\exists \chi a) = \sup_n \left(\bigvee_1^n S_{\zeta_i} a \right)$ holds. Since $\mu(a) = 0$ and $\zeta_i \notin \Delta a$, the symmetry-property implies $\mu(a) = \mu(S_{\zeta_i} a) = 0$ for each i. In turn $\mu\left(\bigvee_1^n S_{\zeta_i} a \right) = 0$ follows for each n, since the sum of finitely many elements of zero measure is also of zero measure. Q.E.D.

As is known any cylindric algebra is isomorphic to a quotient algebra of a basis L. algebra $\mathscr{B}$ modulo of a suitable cylindric ideal $\mathscr{I}$.

Moreover if a measure μ is given on $\mathscr{B}^0$ the canonical homomorphism generate a measure on $\mathscr{B}^0/\mathscr{I}$ iff μ vanishes on $\mathscr{I}$.

Consider the property Q and the dual of condition (1*) for $\mathscr{B}/\mathscr{I}$. This dual condition induces the following properties on $\mathscr{B}$: If $|\forall x\varphi| \in \mathscr{I}$, then

$$\inf_n \mu\left(\bigwedge_1^n |\varphi(yi)|\right) = 0$$

hctds.

Let $\mathscr{B}$ be a basis L. algebra, let μ be a measure on $\mathscr{B}$, and $\mathscr{I}$ a cylindric ideal on it. So the following theorem is true.

THEOREM 3.4. *The canonical homomorphism generate a Q-measure on $\mathscr{B}/\mathscr{I}$ if and only if μ vanishes on $\mathscr{I}$ and whenever $|\forall x\varphi| \in \mathscr{I}$ then for any variable sequences $y_1, y_2, \ldots, y_n, \ldots$ consisting of different variables, the relation*

$$\inf_n \mu\left(\bigwedge_1^n |\varphi(y_i)|\right) = 0$$

holds.

A proposition like this one can be stated for q measures, as well.

References

[0] H. Andréka, T. Gergely and I. Németi, On universal algebraic construction of logics, *Studia Logica*, **36** (1977), 9—47.

[1] J. L. Bell and M. Machover, *A Course in Mathematical Logic*, North-Holland (Amsterdam, 1977).

[2] R. Chuaqui, *Foundations of statistical methods using a semantical definition of probability*, Mathematical Logic in Latin America, North-Holland (Amsterdam, 1978).

[3] J. E. Fenstad, *Representations of probabilities defined on first order languages. Sets, Models and Recursion Theory*. North-Holland (Amsterdam, 1967).

[4] H. Gaifman, Concerning measures in first order calculi, *Israel J. Math.*, **2** (1964), 1—18.

[5] P. R. Halmos, *Lectures on Boolean algebras*, Van Nostrand (Princeton, 1963).

[6] L. Henkin, J. D. Monk and A. Tarski, *Cylindric Algebras I*, North-Holland (Amsterdam, 1971).

[7] L. Henkin, J. O. Monk, A. Tarski, H. Andréka, I. Németi, *Cylindric Set Algebras*, Lecture Notes in Math., **883** (Springer, 1981).

[8] A. Horn and A. Tarski, Measures in Boolean algebras, *Trans. Amer. Math. Soc.*, **64** (1948), 467—497.

[9] J. Loš, Remarks on foundations of probability, *Proc. Intern. Congress of Mathematicians*, Stockholm, 1962.

[10] J. F. Lynch, Almost sure theories, *Annals of Math. Logic*, **18** (1980).

[11] C. Morgenstern, The measure quantifier, *The Journal of Symb. Logic* V., **44**, 1979.

[12] I. Németi, Connections between cylindric algebras and initial algebra semantics of CF languages, Math. Logic in Computer Science, *Colloq. Math. Soc. J. Bolyai*, **26**, North-Holland, (1981), pp. 561—606.

[13] J. Schwartz, Cylindric algebras with filter quantifiers, *Z. Math. Logik u. Grundl. Math*, **26**, (1980).

[14] D. Scott and P. Krauss, *Assigning probabilities to logical formulas, Aspects of inductive logic*, North-Holland (Amsterdam, 1966).

[15] R. Sikorski, *Boolean Algebras*, Springer (Berlin, 1964).

[16] C. A. Rogers, *Analytic Sets*, Academic Press (New York, 1980).

DEPARTMENT OF MATHEMATICS
POLYTECHNICAL UNIVERSITY OF BUDAPEST
BUDAPEST, STOCZEK U. 2.
H-1111, HUNGARY

(Received September 18, 1981; revised September 27, 1982)

Acta Math. Hung.
42 (1—2) (1983), 19—24.

ON SPECTRAL AND CENTRAL STATES OF BANACH ALGEBRAS

D. PETZ (Budapest)

Preliminaries and introduction

By a Banach algebra we always mean a complex complete normed algebra with unit. Let $\mathscr{A}$ be a Banach algebra. $f\in\mathscr{A}^*$ is called a state if $\|f\|=f(1)=1$.

PROPOSITION 1 ([1], p. 111). *Let $\mathscr{A}$ be a Banach algebra and $f\in\mathscr{A}^*$. Then the following conditions are equivalent*:

(i) $f(a)\in\operatorname{co}\operatorname{Sp}(a)$ $(a\in\mathscr{A})$

(ii) $f(1)=1$, $|f(a)|\leqq\varrho(a)$ $(a\in\mathscr{A})$

(iii) *f is a state with respect to any algebra norm p equivalent to the given one and satisfying $p(1)=1$.*

We say that f is a spectral state if it satisfies conditions (i)—(iii).

Each non-zero multiplicative functional is a spectral state. In fact, by the Kahane—Gleason—Żelazko theorem $f\in\mathscr{A}^*$ is multiplicative if and only if $f(a)\in\operatorname{Sp}(a)$ for every $a\in\mathscr{A}$. The following weakened version of this theorem has been proved recently by Kowalski and Slodkowski [6].

PROPOSITION 2. *If $\mathscr{A}$ is a Banach algebra and f: $\mathscr{A}\to\mathbf{C}$ satisfies*

(i) $f(0)=0$,

(ii) $f(a)-f(b)\in\operatorname{Sp}(a-b)$,

then f is multiplicative and linear.

We shall use the method of Kowalski and Slodkowski and obtain a version of Proposition 1 without the assumption on linearity.

The most common spectral state is the normalized trace on the algebra $M_n(\mathbf{C})$ of matrices. Wimmer [12] proved that it has the property

$$(*)\qquad \overline{\lim_{n\to\infty}}\,|\tau(a^n)|^{\frac{1}{n}}=\varrho(a).$$

Zemánek [14] proposed the investigation of the property (*). We shall show that the canonical trace in a continuous finite factor does not have this property.

A state f of the algebra $\mathscr{A}$ is said to be central if $f(ab-ba)=0$ for every $a, b\in\mathscr{A}$. The following example is borrowed from [1].

EXAMPLE 1. Define the product on $\mathbf{C}^2$ by the formula

$$(\alpha_1,\alpha_2)(\beta_1,\beta_2)=(\alpha_1\beta_2+\alpha_2\beta_1,\alpha_2\beta_2)$$

and let $\|(\alpha_1, \alpha_2)\| = |\alpha_1| + |\alpha_2|$. So $(0, 1)$ is a unit and $f(\alpha_1, \alpha_2) = \alpha_1 + \alpha_2$ defines a central state. f is not spectral, since $(1, 0)$ is nilpotent and $f(1, 0) = 1$.

Hence there is a central state which is not spectral but any spectral state is central.

PROPOSITION 3 ([1], p. 114). *Let* $f \in \mathcal{A}^*$ *and suppose*

$$|f(a)| \leqq \varrho(a) \quad (a \in \mathcal{A}).$$

Then $f(ab - ba) = 0$ *for every* $a, b \in \mathcal{A}$.

However, in C^*-algebras spectral and central states coincide. This follows from the formula

$$\varrho(a) = \inf_s \|s^{-1}as\|$$

where s runs on the invertible elements (see [8]). A characterization of the central states in C^*-algebras has been given by Gardner [3] with the inquality

$$|f(a)| \leqq f(|a|) \quad ((|a| = (a^*a)^{1/2}).$$

Results

Several C^*-algebras (for example, the uniformly hyperfinite ones) have exactly one central state [10].

LEMMA 1. *Let* $\mathcal{A}$ *be a Banach algebra with a unique spectral state and* $\tau: \mathcal{A} \to \mathbf{C}$ *a mapping such that*

(a1) τ *is* **R***-linear,*

(a2) $\tau(\lambda \cdot 1) = \lambda \quad (\lambda \in \mathbf{C})$,

(a3) $|\tau(a)| \leqq \varrho(a)$.

Then τ *is the spectral state of* $\mathcal{A}$.

PROOF. Take

$$\varphi_1(x) = \frac{\tau(x) - i\tau(ix)}{2} + \frac{\overline{\tau(x) + i\tau(ix)}}{2} \quad (x \in \mathcal{A}),$$

$$\varphi_2(x) = \frac{\tau(x) - i\tau(ix)}{2} - \frac{\overline{\tau(x) + i\tau(ix)}}{2} \quad (x \in \mathcal{A}).$$

Then $\varphi_k(ix) = i\varphi_k(x)$ and φ_k is (complex) linear $(k = 1, 2)$. Condition (a2) gives $\varphi_1(1) = \varphi_2(1) = 1$. Using the **R**-linearity we get

$$e^{ir}\tau(e^{-ir}x) = \frac{\tau(x) - i\tau(ix)}{2} + e^{i2r}\frac{\tau(x) + i\tau(ix)}{2}$$

for $r \in \mathbf{R}$. Since

$$|e^{ir}\tau(e^{-ir}x)| = |\tau(e^{-ir}x)| \leqq \varrho(e^{-ir}x) = \varrho(x),$$

it follows that $|\varphi_k(x)| \leqq \varrho(x)$ $(k=1, 2)$. Hence φ_1 and φ_2 are spectral states of $\mathscr{A}$ and from unicity $\varphi_1=\varphi_2$. This implies $\tau=\varphi_1$ and τ is the spectral state of $\mathscr{A}$.

We shall need some results on differentiation of Lipschitz functions.

PROPOSITION 4 [7]. *If $f: X \to \mathbf{C}$ is a Lipschitz mapping on a separable Banach space X then f has a real Gateux differential except for some zero set.*

THEOREM 1. *Let $\tau: \mathscr{A} \to \mathbf{C}$ be a mapping on a Banach algebra $\mathscr{A}$. Then the conditions*

(b1) *τ is spectral state,*
(b2) *τ satisfies conditions* (a1)—(a3),
(b3) $\tau(0)=0$ *and* $\tau(a)-\tau(b) \in \operatorname{co} \operatorname{Sp}(a-b)$

are related by (b1)$\to$(b2)$\to$(b3). *Moreover, if $\mathscr{A}$ has at most one spectral state then they are equivalent.*

PROOF. The implication (b1)$\to$(b2) is evident. In order to prove (b2)$\to$(b3) we note that

$$\operatorname{co} \operatorname{Sp}(a) = \cap \{V_p(a): p \in N\}$$

where N is the set of all algebra norms p with $p(1)=1$ and equivalent to the given norm (cf. [1], p. 23). Taking $p \in N$ and using the Hahn—Banach theorem, we can define an $f \in \mathscr{A}^*$ with the properties $f(a)=\tau(a)$, $f(1)=1$ and $\|f\|_p=1$ since $\tau(\lambda)=\lambda$ and $|\tau(a)+\lambda| \leqq \varrho(a+\lambda) \leqq \|a+\lambda\|$. Hence $\tau(a) \in V_p(a)$ for any $p \in N$.

Now we assume that $\mathscr{A}$ has at most one spectral state and prove (b3)$\to$(b2). We show that τ is $\mathbf{R}$-linear, i.e. $\tau(r_1 a+r_2 b)=r_1\tau(a)+r_2\tau(b)$ $(r_1, r_2 \in \mathbf{R},\ a, b \in \mathscr{A})$. Following the line of [6] and dropping down to the subalgebra generated by a and b we may assume that $\mathscr{A}$ is separable. It follows from (b3) that τ is a Lipschitz mapping and Proposition 4 tells us that τ has a real Gateux differential almost everywhere. Assume that τ has a differential $(D\tau)_x: \mathscr{A} \to \mathbf{C}$ at a point $x \in \mathscr{A}$. Then

$$(D\tau)_x(a) = \lim_{r \to 0} \frac{\tau(x+ra)-\tau(x)}{r} \in \operatorname{co} \operatorname{Sp}(a).$$

Hence $(D\tau)_x$ satisfies conditions (a1)—(a3) and by Lemma 1 $(D\tau)_x$ is the unique spectral state τ_0. So $\tau-\tau_0=$constant and $\tau=\tau_0$. This completes the proof.

DEFINITION. Let $\mathscr{A}$ be a Banach algebra and $a_1, a_2, \ldots, a_n \in \mathscr{A}$. The set

$$V_s(a_1, a_2, \ldots, a_n) = \{(f(a_1), f(a_2), \ldots, f(a_n)) \in \mathbf{C}^n : f \text{ is a state } f([a_i, a_j]) = 0\ (i, j \leqq n)\}$$

will be called the strong joint numerical range of $a_1, a_2, \ldots, a_n$. (Here $[a, b]$ stands for $ab-ba$.)

If $\mathscr{A}$ has a central state then $V_s(a_1, a_2, \ldots, a_n)$ is non-empty for any $a_1, a_2, \ldots, a_n \in \mathscr{A}$.

THEOREM 2. *Let $\mathscr{A}$ be a Banach algebra. $\mathscr{A}$ has a central state if and only if $V_s(a_1, a_2, \ldots, a_n)$ is nonempty for every finite subset $\{a_1, a_2, \ldots, a_n\}$ of $\mathscr{A}$.*

PROOF. If $V_s(a_1, a_2, \ldots, a_n) \neq \varnothing$ then

$$H(a_1, a_2, \ldots, a_n) = \{f \text{ is a state: } f([a_i, a_j]) = 0\ (i, j \leqq n)\}$$

is a non-void convex compact set in the state space. $f\in\bigcap H(a_1, a_2, \ldots, a_n)$ is a central state if $\{a_1, a_2, \ldots, a_n\}$ runs over the finite subsets of $\mathscr{A}$.

We remark that $V_s(a, b)$ is non-void if $[a, b]$ is not invertible. By an old result of Wintner [13] $1\notin\mathbf{C}[a, b]$. If $[a, b]$ is not invertible then $\|\lambda[a, b]+1\|\geqq 1$ and $\|\lambda+\mu[a, b]\|\geqq|\lambda|$ $(\lambda, \mu\in\mathbf{C})$. Taking $f(\lambda+\mu[a, b])=\lambda$ and using the Hahn—Banach theorem, we obtain a state f with $f([a, b])=0$.

THEOREM 3. *Let $\mathscr{A}$ be a von Neumann algebra. Then $\mathscr{A}$ is finite if and only if the strong joint numerical range of any four elements is non-empty.*

PROOF. If $\mathscr{A}$ is finite then it has a trace (i.e. a central state) and so $V_s(a_1, a_2, \ldots, a_n)\neq\varnothing$ for any $\{a_1, a_2, \ldots, a_n\}\subset\mathscr{A}$.

On the other hand, if $\mathscr{A}$ is properly infinite, then there is a projection p such that $1\sim p\sim 1-p$ (cf. [11], p. 48). It follows that there are $u, v\in\mathscr{A}$ satisfying

$$p = u^*u, \quad 1-p = v^*v, \quad uu^* = 1 \quad \text{and} \quad vv^* = 1.$$

So $1=[u, u^*]+[v, v^*]$ and we have obtained that $V_s(u, u^*, v, v^*)=\varnothing$.

H. K. Wimmer showed that the normalized trace τ on a full matrix algebra $M_n(\mathbf{C})$ satisfies

$$(*) \qquad \varlimsup_{n\to\infty} |\tau(a^n)|^{\frac{1}{n}} = \varrho(a)$$

for every $a\in M_n(\mathbf{C})$ and P. D. Johnson proved that τ is the only state having property (*) (see [12] and [5]).

THEOREM 4. *Let $\mathscr{A}$ be a finite von Neumann algebra with trivial centre (i.e. a factor). Then the canonical trace satisfies (*) if and only if $\mathscr{A}$ is finite dimensional.*

PROOF. If $\mathscr{A}$ is an infinite dimensional finite factor then it contains the hiperfinite type II_1 factor $\mathscr{R}$. Hence it is sufficient to find a unitary $u\in\mathscr{R}$ such that $\tau(u^n)=0$ for every $n\in\mathbf{N}$. (Here τ stands for the canonical trace.) We may realise $\mathscr{R}$ as a cross-product of $L^\infty(\mathbf{T}, \lambda)$ and $\mathbf{Z}$. Let $\mathbf{T}$ be the unit circle and λ the Lebesgue measure. So $L^\infty(\mathbf{T}, \lambda)$ is a commutative von Neumann algebra and $\mathbf{Z}$ can act on $L^\infty(\mathbf{T}, \lambda)$ by irrational rotation. By Lemma 5.2.3 in [9] the cross-product is the hiperfinite factor. Any non-zero element of $\mathbf{Z}$ produces a unitary in the cross-product algebra.

J. Zemánek [14] proposed to investigate states with property (*) on Banach algebras. The previous theorem tells us that a spectral state may not satisfy condition (*). The existence of a state with (*) is not obvious even on commutative Banach algebras.

THEOREM 5. *If $\mathscr{A}$ is a commutative separable Banach algebra then it possesses a state with property* (*).

PROOF. Under the conditions the maximal ideal space is metrizable and hence there is a countable family $\{\varphi_i: i\in\mathbf{N}\}$ of multiplicative states such that $\{\varphi_i(a): i\in\mathbf{N}\}$ is dense in $\mathrm{Sp}(a)$ for every $a\in\mathscr{A}$.

Let (d_n) be a sequence of positive numbers such that $\sum d_n=1$ and

$$\sum_{n=k+1}^{\infty} d_n < \exp(-k\cdot 8^k).$$

We define a state by the formula

$$\tau(a) = \sum_{n=1}^{\infty} d_n \varphi_n(a) \quad (a \in \mathscr{A}).$$

In order to prove (*), suppose that $\varepsilon > 0$ and $N_0 \in \mathbf{N}$ are given. We may assume that $\varrho(a) = 1$ and $\varepsilon < \frac{1}{2}$. Now $\varphi_n(a) = \varrho_n \exp(i\alpha_n)$ where $0 \leqq \varrho_n \leqq 1$ and $0 \leqq \alpha_n < 2\pi$. Choose $N_1 \in \mathbf{N}$ such that

$$\sum_{\substack{n<N_1 \\ \varrho_n > \sqrt{1-\varepsilon}}} d_n > \frac{1}{2} E(\varepsilon) \quad \text{where} \quad E(\varepsilon) = \sum_{\varrho_n > \sqrt{1-\varepsilon}} d_n.$$

Take an $N_2 > N_0, N_1$ such that

$$CE(\varepsilon)(1-\varepsilon)^{\frac{k}{2}} - \exp(-k) > (1-\varepsilon)^k$$

for $k > N_2$. (Here C is a constant.) By Dirichlet's theorem ([4]) there is an $l \in \mathbf{N}$ such that

$$0 \leqq lN_2\alpha_n < \frac{\pi}{4} \mod 2\pi \quad \text{for} \quad n \leqq N_2$$

and $l \leqq 8^{N_2}$. Taking $k = lN_2$ we have the following estimates:

$$|\tau(a^k)| = |\Sigma d_n \varphi_n(a)^k| \geqq \left| \sum_{n=1}^{N_2} d_n \varphi_n(a)^k \right| -$$

$$- \left| \sum_{n=N_2+1}^{\infty} d_n \varphi_n(a)^k \right| \geqq \left| \sum_{\substack{n<N_2 \\ \varrho_n > \sqrt{1-\varepsilon}}} d_n \varphi_n(a)^k \right| - \sum_{n=N_2+1}^{\infty} d_n \geqq$$

$$\geqq CE(\varepsilon)(1-\varepsilon)^{\frac{k}{2}} - \exp(-N_2 8^{N_2}) \geqq CE(\varepsilon)(1-\varepsilon)^{\frac{k}{2}} - \exp(-k) \geqq (1-\varepsilon)^k.$$

The proof is complete.

Finally we remark that a general Banach algebra is poor in spectral states. More precisely, if we assume that

$$\varrho(a) = \sup \{|\tau(a)| : \tau \text{ is a spectral state}\}$$

for every $a \in \mathscr{A}$ then it follows the subadditivity of the spectral radius and by [15], $\mathscr{A}/\mathrm{Rad}\,\mathscr{A}$ is commutative.

Acknowledgement

The author is grateful to M. Laczkovich, S. Popa, Z. Slodkowski and J. Zemánek for valuable discussions which improved essentially the first draft of the manuscript.

Added in proof (June 25, 1982). Z. Slodkowski kindly informed me about the following example. Let $\mathscr{A}$ be the disc algebra and let $\varphi \in \mathscr{A}^*$ be defined by $\varphi(f) = f(0) + \frac{1}{4}\overline{f'(0)}$. This φ is **R**-linear and $\varphi(f) \in \mathrm{co}\,\mathrm{Sp}\,f$. The example shows that (b3) does not imply (b1) in Theorem 1.

References

[1] F. F. Bonsall and J. Duncan, *Numerical ranges of Operators and of Elements of Normed Algebras*, London Math. Soc. Lecture Note Series 2, Cambridge Univ. Press (Cambridge, 1971).
[2] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Ergebnisse der Math. und ihrer Grenzgebiete 80, Springer-Verlag (Berlin—Heidelberg—New York, 1973).
[3] L. T. Gardner, An inequality characterizes the trace, *Canad. J. Math.*, **31** (1979), 1322—1328.
[4] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Calderon Press (Oxford, 1960).
[5] P. D. Johnson, Spectral radius and seminorms in finite-dimensional algebras, *Colloq. Math.*, **39** (1978), 331—341.
[6] S. Kowalski and Z. Slodkowski, A characterization of multiplicative linear functionals in Banach algebras, *Studia Math.*, **67** (1980), 215—223.
[7] P. Mankiewicz, On the differentiability of Lipschitz mappings in Frechet spaces, *Studia Math.*, **45** (1973), 15—29.
[8] G. J. Murphy and T. T. West, Spectral radius formulae, *Proc. Edinburgh Math. Soc.*, **22** (1979), 271—275.
[9] F. J. Murray and J. von Neumann, On rings of operators, IV, *Ann. of Math.*, **44** (1943), 716—808.
[10] G. K. Pedersen, *C*-algebras and their automorphism groups*, L.M.S. Monographs, Academic Press (London—New York—San Francisco, 1979).
[11] D. M. Topping, *Lectures on von Neumann algebras*, Van Nostrand Reinhold Math. Stu. 36 (Van Nostrand Reinhold Comp., 1971).
[12] H. K. Wimmer, Spectral radius and radius of convergence, *Amer. Math. Monthly*, **81** (1974), 625—627.
[13] A. Wintner, The unboundedness of quantum-mechanical matrices, *Phys. Rev.*, **71** (1947), 738—739.
[14] J. Zemánek, Letter to the author.
[15] J. Zemánek, Spectral radius characterizations of commutativity in Banach algebras, *Studia Math.*, **61** (1977), 257—268.

(Received October 20, 1981)

MATHEMATICAL INSTITUTE,
HUNGARIAN ACADEMY OF SCIENCES,
REÁLTANODA U. 13—15.,
1053 BUDAPEST,
HUNGARY

Acta Math. Hung.
42 (1—2) (1983), 25—33.

LAKUNÄRE INTERPOLATION MIT SPLINE-FUNKTIONEN

DIE FÄLLE (0, 2, 3) UND (0, 2, 4)

J. GYŐRVÁRI (Veszprém)

1. Einleitung

R. S. Mishra und K. K. Mathur konstruierten in [1] die Funktionen $R_n(x) \in R_{n,6}^{(2)}$ und $S_n(x) \in R_{n,6}^{(2)}$, für die

$$R_n(x) \in C^2[0; 1]$$

$$R_n(x_{2i}) = f(x_{2i}) \quad (i = 0, 1, 2, \ldots, m)$$

$$R_n^{(q)}(t_{2i}) = f^{(q)}(t_{2i}) \quad (i = 0, 1, 2, \ldots, m-1; \quad q = 0, 2, 3)$$

$$R_n'(x_0) = f'(x_0); \quad R_n'(x_{2m}) = f'(x_{2m})$$

ferner

$$S_n(x) \in C^2[0; 1]$$

$$S_n(x_{2i}) = f(x_{2i}) \quad (i = 0, 1, 2, \ldots, m)$$

$$S_n^{(q)}(t_{2i}) = f^{(q)}(t_{2i}) \quad (i = 1, 2, \ldots m-1) \quad (q = 0, 2, 4)$$

$$S_n'(x_0) = f'(x_0), \quad S_n'(x_{2n}) = f'(x_{2m})$$

wo

$$x_i = \frac{i}{2m} \quad (i = 0, 2, 4, \ldots, 2m);$$

$$t_{2i} = x_{2i} + \frac{2}{3}h \quad (i = 0, 1, 2, \ldots, m-1); \quad h = \frac{1}{2m}$$

bedeuten und

$$f(x_0), \ f(x_2), \ldots f(x_{2m}), \ f^{(q)}(t_0), \ f^{(q)}(t_2), \ldots,$$

$f^{(q)}(t_{2m-2})$ $(q = 0, 2, 3, 4)$, $f'(x_0)$, $f'(x_{2m})$ gegeben sind.

Von ihnen wurden folgende Sätze bewiesen: Es sei $f \in C^5[0, 1]$. Dann gelten

$$\|R_n^{(q)}(x) - f^{(q)}(x)\|_\infty \leqq 293 m^{q-5} \omega_5\left(\frac{1}{m}\right) + 4m^{q-5}\|f^{(5)}\|_\infty \quad (q = 0, 1, 2, 3, 4)$$

$$\|S_n^{(q)}(x) - f^{(q)}(x)\|_\infty \leqq 2245 m^{q-5} \omega_5\left(\frac{1}{m}\right) + 4m^{q-5}\|f^{(5)}\|_\infty \quad (q = 0, 1, 2, 3, 4)$$

In dieser Arbeit konstruieren wir die lakunären Spline-Funktionen $S_\Delta(x) \in C[0, 1]$ und $G_\Delta(x) \in C[0; 1]$, welche in jedem Interwall aus Polinomen fünften oder sechsten

Grades bestehen und folgende Bedingungen erfüllen:

$$S_k(x_k) = f(x_k) = y_k \quad (k = 0, 1, 2, \dots, m-1) \tag{1.1.a}$$

$$S_k^{(q)}(x_k) = f^{(q)}(x_k) = y_k^{(q)} \quad (q = 2, 3, \quad k = 0, 1, 2, \dots, m-1) \tag{1.1.b}$$

$$S_{m-1}(x_m) = f(x_m) = y_m \tag{1.1.c}$$

$$S_{m-1}^{(q)}(x_m) = f^{(q)}(x_m) = y_m^{(q)} \quad (q = 2, 3) \tag{1.1.d}$$

$$S_k(x_{k+1}) = S_{k+1}(x_{k+1}) = f(x_{k+1}) = y_{k+1} \quad (k = 0, 1, 2, \dots, m-2) \tag{1.2.a}$$

$$S_k^{(q)}(x_{k+1}) = S_{k+1}^{(q)}(x_{k+1}) = f^{(q)}(x_{k+1}) = y_{k+1}^{(q)} \quad (q = 2, 3, \quad k = 0, 1, 2, \dots, m-2) \tag{1.2.b}$$

$$S_0'(x_0) = f'(x_0) = y_0', \quad S_{m-1}'(x_m) = f'(x_m) = y_m'. \tag{1.3}$$

$$G_k(x_k) = f(x_k) = y_k \quad (k = 0, 1, 2, \dots, m-1) \tag{2.1.a}$$

$$G_k^{(q)}(x_k) = f^{(q)}(x_k) = y_k^{(q)} \quad (q = 2, 4, \quad k = 0, 1, 2, \dots, m-1) \tag{2.1.b}$$

$$G_{m-1}(x_m) = f(x_m) = y_m \tag{2.1.c}$$

$$G_{m-1}^{(q)}(x_m) = f^{(q)}(x_m) = y_m^{(q)} \quad (q = 2, 4) \tag{2.1.d}$$

$$G_k(x_{k+1}) = G_{k+1}(x_{k+1}) = f(x_{k+1}) = y_{k+1} \quad (k = 0, 1, 2, \dots, m-2) \tag{2.2.a}$$

$$G_k^{(q)}(x_{k+1}) = G_{k+1}^{(q)}(x_{k+1}) = f^{(q)}(x_{k+1}) = y_{k+1}^{(q)} \quad (q = 2, 4, \quad k = 0, 1, 2, \dots, m-2) \tag{2.2.b}$$

$$G_0'(x_0) = f'(x_0) = y_0', \quad G_{m-1}'(x_m) = f'(x_m) = y_m', \tag{2.3}$$

wobei $x_k = \frac{k}{m}$ $(k = 0, 1, 2, \dots, m)$, $h = \frac{1}{m}$ bedeuten und $f^{(q)}(x_0), f^{(q)}(x_1), \dots$ $\dots, f^{(q)}(x_m)$ $(q = 0, 2, 3, 4)$, $f'(x_0), f'(x_m)$ vorgegebene Werte sind.

Mit den Spline-Funktionen approximieren wir die Funktion $f \in C^5[0;1]$ und ihre Ableitungen bis zum 5. Grad (in [1] steht $q = 0, 1, 2, 3, 4$) und geben bessere Abschätzungen als die in [1] angegebenen.

2. Existenz und Unizität

a) (0, 2, 3)-*Interpolation*

SATZ 2.1. *Es sei*

$$S_\Delta(x) := \begin{cases} S_0(x), & \text{wenn} \quad x_0 \leqq x \leqq x_1 \\ S_k(x), & \text{wenn} \quad x_k \leqq x \leqq x_{k+1} \\ S_{m-1}(x), & \text{wenn} \quad x_{m-1} \leqq x \leqq x_m \end{cases} \quad (k = 1, 2, \dots, m-2)$$

wo

$$S_0(x) = y_0 + y_0'(x-x_0) + \frac{y_0''}{2}(x-x_0)^2 + \frac{y_0'''}{6}(x-x_0)^3 + a_{0,4}(x-x_0)^4 +$$

$$+ a_{0,5}(x-x_0)^5 + a_{0,6}(x-x_0)^6,$$

$$S_k(x) = y_k + a_{k,1}(x-x_k) + \frac{y_k''}{2}(x-x_k)^2 + \frac{y_k'''}{6}(x-x_k)^3 + a_{k,4}(x-x_k)^4 + a_{k,5}(x-x_k)^5,$$

$$S_{m-1}(x) = y_{m-1} + a_{m-1,1}(x-x_{m-1}) + \frac{y_{m-1}''}{2}(x-x_{m-1})^2 + \frac{y_{m-1}'''}{6}(x-x_{m-1})^3 +$$

$$+ a_{m-1,4}(x-x_{m-1})^4 + a_{m-1,5}(x-x_{m-1})^5 + a_{m-1,6}(x-x_{m-1})^6.$$

Dann sind die Koeffizienten des Polynoms durch die Bedingungen 1.1, 1.2 *und* 1.3 *eindeutig bestimmt.*

BEWEIS. Aus den Bedingungen 1.2.a und 1.2.b bekommen wir für $k=0$:

$$a_{0,4} + a_{0,5}h + a_{0,6}h^2 = \frac{1}{h^4}\left\{y_1 - y_0 - y_0'h - \frac{y_0''}{2}h^2 - \frac{y_0'''}{6}h^3\right\},$$

$$12a_{0,4} + 20a_{0,5}h + 30a_{0,6}h^2 = \frac{1}{h^2}\{y_1'' - y_0'' - y_0'''h\},$$

$$24a_{0,4} + 60a_{0,5}h + 120a_{0,6}h^2 = \frac{1}{h}\{y_1''' - y_0'''\},$$

und dieses Gleichungssystem ergibt

$$a_{0,4} = \frac{5}{h^4}\left\{y_1 - y_0 - y_0'h - \frac{y_0''}{2}h^2 - \frac{y_0'''}{6}h^3\right\} - \frac{1}{2h^2}\{y_1'' - y_0'' - y_0'''h\} + \frac{1}{12h}\{y_1''' - y_0'''\},$$

$$a_{0,5} = -\frac{6}{h^5}\left\{y_1 - y_0 - y_0'h - \frac{y_0''}{2}h^2 - \frac{y_0'''}{6}h^3\right\} + \frac{4}{5h^3}\{y_1'' - y_0'' - y_0'''h\} - \frac{3}{20h^2}\{y_1''' - y_0'''\},$$

$$a_{0,6} = \frac{2}{h^6}\left\{y_1 - y_0 - y_0'h - \frac{y_0''}{2}h^2 - \frac{y_0'''}{6}h^3\right\} - \frac{3}{10h^4}\{y_1'' - y_0'' - y_0'''h\} + \frac{1}{15h^3}\{y_1''' - y_0'''\}.$$

Für $k=1, 2, \ldots, m-2$ erhält man aus denselben Bedingungen das Gleichungssystem

$$a_{k,1} + a_{k,4}h^3 + a_{k,5}h^4 = \frac{1}{h}\left\{y_{k+1} - y_k - \frac{y_k''}{2}h^2 - \frac{y_k'''}{6}h^3\right\},$$

$$12a_{k,4} + 20a_{k,5}h = \frac{1}{h^2}\{y_{k+1}'' - y_k'' - y_k'''h\},$$

$$24a_{k,4} + 60a_{k,5}h = \frac{1}{h}\{y_{k+1}''' - y_k'''\},$$

und hieraus die Werte

$$a_{k,1}=\frac{1}{h}\left\{y_{k+1}-y_k-\frac{y_k''}{2}h^2-\frac{y_k'''}{6}h^3\right\}-\frac{3}{20}h\{y_{k+1}''-y_k''-y_k'''h\}+\frac{1}{30}h^2\{y_{k+1}'''-y_k'''\},$$

$$a_{k,4}=\frac{1}{4h^2}\{y_{k+1}''-y_k''-y_k'''h\}-\frac{1}{12h}\{y_{k+1}'''-y_k'''\},$$

$$a_{k,5}=-\frac{1}{10h^3}\{y_{k+1}''-y_k''-y_k'''h\}+\frac{1}{20h^2}\{y_{k+1}'''-y_k'''\}.$$

Die Bedingungen 1.1.c, 1.3 und 1.1.d führen zum Gleichungssystem

$$a_{m-1,1}+a_{m-1,4}h^3+a_{m-1,5}h^4+a_{m-1,6}h^5=\frac{1}{h}\left\{y_m-y_{m-1}-\frac{y_{m-1}''}{2}h^2-\frac{y_{m-1}'''}{6}h^3\right\},$$

$$a_{m-1,1}+4a_{m-1,4}h^3+5a_{m-1,5}h^4+6a_{m-1,6}h^5=y_m'-y_{m-1}''h-\frac{y_{m-1}'''}{2}h^2,$$

$$12a_{m-1,4}+20a_{m-1,5}h+30a_{m-1,6}h^2=\frac{1}{h^2}\{y_m''-y_{m-1}''-y_{m-1}'''h\},$$

$$24a_{m-1,4}+60a_{m-1,5}h+120a_{m-1,6}h^2=\frac{1}{h}\{y_m'''-y_{m-1}'''\},$$

wodurch endlich

$$a_{m-1,1}=\frac{2}{h}\left\{y_m-y_{m-1}-\frac{y_{m-1}''}{2}h^2-\frac{y_{m-1}'''}{6}h^3\right\}-\left\{y_m'-y_{m-1}''h-\frac{y_{m-1}'''}{2}h^2\right\}+$$

$$+\frac{1}{5}h\{y_m''-y_{m-1}''-y_{m-1}'''h\}-\frac{1}{60}h^2\{y_m'''-y_{m-1}'''\},$$

$$a_{m-1,4}=-\frac{5}{h^4}\left\{y_m-y_{m-1}-\frac{y_{m-1}''}{2}h^2-\frac{y_{m-1}'''}{6}h^3\right\}+\frac{5}{h^3}\left\{y_m'-y_{m-1}''h-\frac{y_{m-1}'''}{2}h^2\right\}-$$

$$-\frac{3}{2h^2}\{y_m''-y_{m-1}''-y_{m-1}'''h\}+\frac{1}{6h}\{y_m'''-y_{m-1}'''\},$$

$$a_{m-1,5}=\frac{6}{h^5}\left\{y_m-y_{m-1}-\frac{y_{m-1}''}{2}h^2-\frac{y_{m-1}'''}{6}h^3\right\}-$$

$$-\frac{6}{h^4}\left\{y_m'-y_{m-1}''h-\frac{y_{m-1}'''}{2}h^2\right\}+\frac{2}{h^3}\{y_m''-y_{m-1}''-y_{m-1}'''h\}-\frac{1}{4h^2}\{y_m'''-y_{m-1}'''\},$$

$$a_{m-1,6}=-\frac{2}{h^6}\left\{y_m-y_{m-1}-\frac{y_{m-1}''}{2}h^2-\frac{y_{m-1}'''}{6}h^3\right\}+\frac{2}{h^5}\left\{y_m'-y_{m-1}''h-\frac{y_{m-1}'''}{2}h^2\right\}-$$

$$-\frac{7}{10h^4}\{y_m''-y_{m-1}''-y_{m-1}'''h\}+\frac{1}{10h^3}\{y_m'''-y_{m-1}'''\}$$

bestimmt werden.

(b) (0, 2, 4)-*Interpolation*

SATZ 2.2. *Es sei*

$$G_\Delta(x) := \begin{cases} G_0(x) & \text{wenn} \quad x_0 \leqq x \leqq x_1 \\ G_k(x) & \text{wenn} \quad x_k \leqq x \leqq x_{k+1} \\ G_{m-1}(x) & \text{wenn} \quad x_{m-1} \leqq x \leqq x_m, \end{cases} \qquad (k = 1, 2, \ldots, m-2)$$

wo

$$G_0(x) := y_0 + y_0'(x-x_0) + \frac{y_0''}{2}(x-x_0)^2 + b_{0,3}(x-x_0)^3 + \frac{y_0^{(4)}}{24}(x-x_0)^4 +$$

$$+ b_{0,5}(x-x_0)^5 + b_{0,6}(x-x_0)^6,$$

$$G_k(x) = y_k + b_{k,1}(x-x_k) + \frac{y_k''}{2}(x-x_k)^2 + b_{k,3}(x-x_k)^3 + \frac{y_k^{(4)}}{24}(x-x_k)^4 + b_{k,5}(x-x_k)^5,$$

$$G_{m-1}(x) = y_{m-1} + b_{m-1,1}(x-x_{m-1}) + \frac{y_{m-1}''}{2}(x-x_{m-1})^2 +$$

$$+ b_{m-1,3}(x-x_{m-1})^3 + \frac{y_{m-1}^{(4)}}{24}(x-x_{m-1})^4 + b_{m-1,5}(x-x_{m-1})^5 + b_{m-1,6}(x-x_{m-1})^6.$$

Dann sind die Koeffizienten des Polynoms durch die Bedingungen 2.1, 2.2 *und* 2.3 *eindeutig bestimmt.*

BEWEIS. Auf demselben Wege, wie bei der Interpolation (0, 2, 3), bekommen wir:

$$b_{0,3} = \frac{5}{3h^3}\left\{y_1 - y_0 - y_0'h - \frac{y_0''}{2}h^2 - \frac{y_0^{(4)}}{24}h^4\right\} -$$

$$- \frac{1}{9h}\left\{y_1'' - y_0'' - \frac{y_0^{(4)}}{2}h^2\right\} - \frac{1}{216}h\{y_1^{(4)} - y_0^{(4)}\},$$

$$b_{0,5} = -\frac{1}{h^5}\left\{y_1 - y_0 - y_0'h - \frac{y_0''}{2}h^2 - \frac{y_0^{(4)}}{24}h^4\right\} +$$

$$+ \frac{1}{6h^3}\left\{y_1'' - y_0'' - \frac{y_0^{(4)}}{2}h^2\right\} - \frac{1}{90h}\{y_1^{(4)} - y_0^{(4)}\},$$

$$b_{0,6} = \frac{1}{3h^6}\left\{y_1 - y_0 - y_0'h - \frac{y_0''}{2}h^2 - \frac{y_0^{(4)}}{24}h^4\right\} -$$

$$- \frac{1}{18h^4}\left\{y_1'' - y_0'' - \frac{y_0^{(4)}}{2}h^2\right\} + \frac{7}{1080h^2}\{y_1^{(4)} - y_0^{(4)}\},$$

$$b_{k,1}=\frac{1}{h}\left\{y_{k+1}-y_k-\frac{y_k''}{2}h^2-\frac{y_k^{(4)}}{24}h^4\right\}-\frac{h}{6}\left\{y_{k+1}''-y_k''-\frac{y_k^{(4)}}{2}h^2\right\}+\frac{7}{360}h^3\{y_{k+1}^{(4)}-y_k^{(4)}\},$$

$$b_{k,3}=\frac{1}{6h}\left\{y_{k+1}''-y_k''-\frac{y_k^{(4)}}{2}h^2\right\}-\frac{1}{36}h\{y_{k+1}^{(4)}-y_k^{(4)}\},$$

$$b_{k,5}=\frac{1}{120h}\{y_{k+1}^{(4)}-y_k^{(4)}\},$$

$$b_{m-1,1}=\frac{2}{h}\left\{y_m-y_{m-1}-\frac{y_{m-1}''}{2}h^2-\frac{y_{m-1}^{(4)}}{24}h^4\right\}-\left\{y_m'-y_{m-1}''h-\frac{y_{m-1}^{(4)}}{6}h^3\right\}+$$

$$+\frac{h}{6}\left\{y_m''-y_{m-1}''-\frac{y_{m-1}^{(4)}}{2}h^2\right\}-\frac{h^3}{360}\{y_m^{(4)}-y_{m-1}^{(4)}\},$$

$$b_{m-1,3}=-\frac{5}{3h^3}\left\{y_m-y_{m-1}-\frac{y_{m-1}''}{2}h^2-\frac{y_{m-1}^{(4)}}{24}h^4\right\}+$$

$$+\frac{5}{3h^2}\left\{y_m'-y_{m-1}''h-\frac{y_{m-1}^{(4)}}{6}h^3\right\}-\frac{7}{18h}\left\{y_m''-y_{m-1}''-\frac{y_{m-1}^{(4)}}{2}h^2\right\}+\frac{h}{108}\{y_m^{(4)}-y_{m-1}^{(4)}\},$$

$$b_{m-1,5}=\frac{1}{h^5}\left\{y_m-y_{m-1}-\frac{y_{m-1}''}{2}h^2-\frac{y_{m-1}^{(4)}}{24}h^4\right\}-$$

$$-\frac{1}{h^4}\left\{y_m'-y_{m-1}''h-\frac{y_{m-1}^{(4)}}{6}h^3\right\}+\frac{1}{3h^3}\left\{y_m''-y_{m-1}''-\frac{y_{m-1}^{(4)}}{2}h^2\right\}-\frac{1}{72h}\{y_m^{(4)}-y_{m-1}^{(4)}\},$$

$$b_{m-1,6}=-\frac{1}{3h^6}\left\{y_m-y_{m-1}-\frac{y_{m-1}''}{2}h^2-\frac{y_{m-1}^{(4)}}{24}h^4\right\}+$$

$$+\frac{1}{3h^5}\left\{y_m'-y_{m-1}''h-\frac{y_{m-1}^{(4)}}{6}h^3\right\}-\frac{1}{9h^4}\left\{y_m''-y_{m-1}''-\frac{y_{m-1}^{(4)}}{2}h^2\right\}+\frac{1}{135h^2}\{y_m^{(4)}-y_{m-1}^{(4)}\}.$$

3. Probleme der Konvergenz

Wir werden die folgenden Sätze beweisen:

SATZ 3.1. *Es sei* $f\in C^5[0,1]$, *und* $S_\Delta(x)$ *sei die vorher konstruierte Spline-Funktion. Dann gelten*

$$|f^{(q)}(x)-S_\Delta^{(q)}(x)|\leqq K\omega_5(h)h^{5-q}\quad(q=0,1,2,3,4,5)$$

wo

$$K=52,\quad \textit{wenn}\quad x\in[x_0,x_1]$$
$$K=3,\quad \textit{wenn}\quad x\in[x_k,x_{k+1}]\quad(k=1,2,\ldots,m-2)$$
$$K=142,\quad \textit{wenn}\quad x\in[x_{m-1},x_m]$$

und $\omega_5(h)=\omega(h,f^{(5)})=\sup\limits_{|x-x_1|\leqq h}|f^{(5)}(x)-f^{(5)}(x_1)|.$

SATZ 3.2. *Es sei* $f\in C^5[0, 1]$, *und* $G_\Delta(x)$ *sei die vorher konstruierte Spline-Funktion. Dann gelten*

$$|f^{(q)}(x)-G_\Delta^{(q)}(x)| \leqq K_1\omega_5(h)h^{5-q} \quad (q = 0, 1, 2, 3, 4, 5)$$

wo

$$K_1 = 10, \quad \textit{wenn} \quad x\in[x_0, x_1],$$

$$K_1 = 1, \quad \textit{wenn} \quad x\in[x_k, x_{k+1}] \quad (k = 1, 2, \ldots, m-2)$$

$$K_1 = 21{,}4, \quad \textit{wenn} \quad x\in[x_{m-1}, x_m]$$

$$\textit{und} \quad \omega_5(h) = \omega(h, f^{(5)}) = \sup_{|x-x_1|\leqq h} |f^{(5)}(x)-f^{(5)}(x_1)|.$$

BEWEIS DES SATZES 3.1. Es sei $x_k\leqq x\leqq x_{k+1}$ $(k=1, 2, \ldots, m-2)$. Dann bekommen wir

$$|f^{(5)}(x)-S_k^{(5)}(x)| = |f^{(5)}(x)-120a_{k,5}| =$$

$$= \left|f^{(5)}(x)+\frac{12}{h^3}\{y''_{k+1}-y''_k-y'''_k h\}-\frac{6}{h^2}\{y'''_{k+1}-y'''_k\}\right|.$$

Wir können die nachstehenden Taylor-Formeln aufschreiben, die sich aus der Bedingung $f\in C^5[0; 1]$ ergeben:

$$f''(x) = f''(x_k)+f'''(x_k)(x-x_k)+\frac{f^{(4)}(x_k)}{2}(x-x_k)^2+$$

$$+\frac{f^{(5)}(\xi_k)}{6}(x-x_k)^3 \quad (x_k < \xi_k < x),$$

$$f'''(x) = f'''(x_k)+f^{(4)}(x_k)(x-x_k)+\frac{f^{(5)}(\eta_k)}{2}(x-x_k)^2 \quad (x_k < \eta_k < x).$$

Bei $x=x_{k+1}$ ergibt das

$$y''_{k+1}-y''_k-y'''_k h = \frac{y_k^{(4)}}{2}h^2+\frac{f^{(5)}(\xi_k)}{6}h^3,$$

$$y'''_{k+1}-y'''_k = y_k^{(4)}h+\frac{f^{(5)}(\eta_k)}{2}h^2,$$

und durch Anwendung dieser Gleichungen folgt

$$|f^{(5)}(x)-S_k^{(5)}(x)| = |f^{(5)}(x)+2f^{(5)}(\xi_k)-3f^{(5)}(\eta_k)| \leqq 3\omega_5(h).$$

Es sei $g(x) := f^{(3)}(x)-S_k^{(3)}(x)$. Laut der Bedingung 1.1b haben wir $g(x_k)=g(x_{k+1})=0$ und so existiert nach dem Satz von Rolle ein ε_k, mit

$$g'(\varepsilon_k) = f^{(4)}(\varepsilon_k)-S_k^{(4)}(\varepsilon_k) = 0 \quad \text{und} \quad x_k < \varepsilon_k < x_{k+1}.$$

Somit ist:

$$|f^{(4)}(x)-S_k^{(4)}(x)| = \left|\int_{\varepsilon_k}^{x}\{f^{(5)}(t)-S_k^{(5)}(t)\}\,dt\right| \leqq$$

$$\leqq \left|\int_{\varepsilon_k}^{x}|f^{(5)}(t)-S_k^{(5)}(t)|\,dt\right| \leqq \left|\int_{\varepsilon_k}^{x}3\omega_5(h)\,dt\right| \leqq 3\omega_5(h)h,$$

und wegen $f^{(3)}(x_k)-S_k^{(3)}(x_k)=0$ folgt daraus

$$|f^{(3)}(x)-S_k^{(3)}(x)| \leqq \left|\int_{x_k}^{x}|f^{(4)}(t)-S_k^{(4)}(t)|\,dt\right| \leqq 3\omega_5(h)h^2.$$

Ähnlich ergibt sich:

$$|f''(x)-S_k''(x)| \leqq \left|\int_{x_k}^{x}|f^{(3)}(t)-S_k'''(t)|\,dt\right| \leqq 3\omega_5(h)h^3.$$

Es sei jetzt $h(x) := f(x)-S_k(x)$.

Laut der Bedingung 1.1a sind $h(x_k)=h(x_{k+1})=0$, und hier ergibt der Satz von Rolle die Existenz eines δ_k, mit

$$h'(\delta_k) = f'(\delta_k)-S_k'(\delta_k) = 0 \quad \text{und} \quad x_k < \delta_k < x_{k+1}.$$

So erhalten wir

$$|f'(x)-S_k'(x)| \leqq \left|\int_{\delta_k}^{x}|f''(t)-S_k''(t)|\,dt\right| \leqq 3\omega_5(h)h^4$$

und schließlich

$$|f(x)-S_k(x)| \leqq \left|\int_{x_k}^{x}|f'(t)-S_k'(t)|\,dt\right| \leqq 3\omega_5(h)h^5.$$

Für $x_0 \leqq x \leqq x_1$ gelten

$$|f^{(5)}(x)-S_0^{(5)}(x)| \leqq |f^{(5)}(x)-120a_{0,5}|+720|a_{0,6}|h =$$

$$= f^{(5)}(x)+120\frac{6}{h^5}\left\{\frac{y_0^{(4)}}{24}h^4+\frac{f^{(5)}(\vartheta_0)}{120}h^5\right\}-\frac{120\cdot 4}{5h^3}\left\{\frac{y_0^{(4)}}{2}h^2+\frac{f^{(5)}(\xi_0)}{6}h^3\right\}+$$

$$+\frac{120\cdot 3}{20h^2}\left\{y_0^{(4)}h+\frac{f^{(5)}(\eta_0)}{2}h^2\right\}\Bigg|+720h\left|\frac{2}{h^6}\left\{\frac{y_0^{(4)}}{24}h^4+\frac{f^{(5)}(\vartheta_0)}{120}h^5\right\}-\right.$$

$$\left.-\frac{3}{10h^4}\left\{\frac{y_0^{(4)}}{2}h^2+\frac{f^{(5)}(\xi_0)}{6}h^3\right\}+\frac{1}{15h^3}\left\{y_0^{(4)}h+\frac{f^{(5)}(\eta_0)}{2}h^2\right\}\right| \leqq$$

$$\leqq |f^{(5)}(x)+6f^{(5)}(\vartheta_0)+9f^{(5)}(\eta_0)-16f^{(5)}(\xi_0)|+6|2f^{(5)}(\vartheta_0)+4f^{(5)}(\eta_0)-6f^{(5)}(\xi_0)| \leqq$$

$$\leqq 16\omega_5(h)+36\omega_5(h) = 52\omega_5(h),$$

und genau so erhält man auch die übrigen Abschätzungen.

Im Falle $x_{m-1} \leqq x \leqq x_m$ haben wir endlich

$$|f^{(5)}(x) - S^{(5)}_{m-1}(x)| \leqq |f^{(5)}(x) - 120a_{m-1,5}| + 720|a_{m-1,6}|h =$$

$$= |f^{(5)}(x) - \frac{120 \cdot 6}{h^5}\left\{y_m - y_{m-1} - y'_{m-1}h - \frac{y''_{m-1}}{2}h^2 - \frac{y'''_{m-1}}{6}h^3\right\} +$$

$$+ \frac{120 \cdot 6}{h^4}\left\{y'_m - y'_{m-1} - y''_{m-1}h - \frac{y'''_{m-1}}{2}h^2\right\} - \frac{120 \cdot 2}{h^3}\{y''_m - y''_{m-1} - y'''_{m-1}h\} +$$

$$+ \frac{120}{4h^2}\{y'''_m - y'''_{m-1}\}| + 720h\Big| - \frac{2}{h^6}\left\{y_m - y_{m-1} - y'_{m-1}h - \frac{y''_{m-1}}{2}h^2 - \frac{y'''_{m-1}}{6}h^3\right\} +$$

$$+ \frac{2}{h^5}\left\{y'_m - y'_{m-1} - y''_{m-1}h - \frac{y'''_{m-1}}{2}h^2\right\} - \frac{7}{10h^4}\{y''_m - y''_{m-1} - y'''_{m-1}h\} +$$

$$+ \frac{1}{10h^3}\{y'''_m - y'''_{m-1}\}| \leqq \left|f^{(5)}(x) - \frac{720}{h^5}\left\{\frac{y^{(4)}_{m-1}}{24}h^4 + \frac{f^{(5)}(\vartheta_{m-1})}{120}h^5\right\} +\right.$$

$$+ \frac{720}{h^4}\left\{\frac{y^{(4)}_{m-1}}{6}h^3 + \frac{f^{(5)}(\nu_{m-1})}{24}h^4\right\} - \frac{240}{h^3}\left\{\frac{y^{(4)}_{m-1}}{2}h^2 + \frac{f^{(5)}(\xi_{m-1})}{6}h^3\right\} +$$

$$\left. + \frac{30}{h^2}\left\{y^{(4)}_{m-1}h + \frac{f^{(5)}(\eta_{m-1})}{2}h^2\right\}\right| + 720h\Big| - \frac{2}{h^6}\left\{\frac{y^{(4)}_{m-1}}{24}h^4 + \frac{f^{(5)}(\vartheta_{m-1})}{120}h^5\right\} +$$

$$+ \frac{2}{h^5}\left\{\frac{y^{(4)}_{m-1}}{6}h^3 + \frac{f^{(5)}(\nu_{m-1})}{24}h_4\right\} - \frac{7}{10h^4}\left\{\frac{y^{(4)}_{m-1}}{2}h^2 + \frac{f^{(5)}(\xi_{m-1})}{6}h^5\right\} +$$

$$\left. + \frac{1}{10h^3}\left\{y^{(4)}_{m-1}h + \frac{f^{(5)}(\eta_{m-1})}{2}h^2\right\}\right| = |f^{(5)}(x) - 6f^{(5)}(\vartheta_{m-1}) + 30f^{(5)}(\nu_{m-1}) -$$

$$- 40f^{(5)}(\xi_{m-1}) + 15f^{(5)}(\eta_{m-1})| + |-12f^{(5)}(\vartheta_{m-1}) + 60f^{(5)}(\nu_{m-1}) - 84f^{(5)}(\xi_{m-1}) +$$

$$+ 36f^{(5)}(\eta_{m-1})| \leqq 46\omega_5(h) + 96\omega_5(h) = 142\omega_5(h),$$

und die übrigen Abschätzungen können wiederum auf dieselbe weise gewonnen werden.

Der Beweis des Satzes 3.2 kann nach dem Muster des Satzes 3.1 geführt werden.

Hier möchte ich Herrn Prof. Dr. J. Balázs meinen Dank für das Thema und für seine wertvolle Hilfe aussprechen.

Literaturverzeichnis

[1] R. S. Mishra and K. K. Mathur, Lacunary interpolation by splines, (0; 0, 2, 3) and (0; 0, 2, 4) cases, *Acta Math. Acad. Sci. Hungar.*, **36** (1980), 251—260.

(Eingegangen am 23. Oktober 1981.)

UNIVERSITÄT FÜR CHEMISCHE INDUSTRIE VESZPRÉM
LEHRSTUHL FÜR MATHEMATIK
H-8201 VESZPRÉM, PF. 158.

Acta Math. Hung.
42 (1—2) (1983), 35—43.

ON STRONG CONVERGENCE OF TRIGONOMETRIC AND FOURIER SERIES

N. TANOVIĆ-MILLER (Sarajevo)[1]

The purpose of this paper is to introduce a study of strong convergence of Fourier series and present first results in this area.

Although the concept of strong C_1 summability originated in connection with Fourier series and a notion of strong convergence developed with the first extension of this concept to Cesàro transformations C_α of order $\alpha>0$, the latter notion was never considered in Fourier analysis. The fact that strong convergence lies between absolute and ordinary convergence — the two types of convergence in regard to which Fourier series have been thoroughly investigated — leads to many interesting questions about the strong convergence of Fourier series. These questions open a whole new area that can be almost as rich and intriguing as the absolute and ordinary convergence of Fourier series are.

Deriving necessary conditions in order that a trigonometric series is strongly convergent on a set of positive measure or on a set of second category we obtain analogues of the classical results on the absolute convergence of trigonometric series, due to Denjoy and Lusin. From this and the well known theorems on strong C_1 summability of Fourier series, we deduce necessary and sufficient conditions in order that a Fourier series is strongly convergent — almost everywhere or uniformly. Next, we prove several results concerning the strong convergence of Fourier series of continuous functions. Among these, there is a simple characterization of functions of bounded variation whose Fourier series are uniformly strongly convergent and a generalization of Paley's theorem on the ordinary convergence of Fourier series with nonnegative coefficients. It can be seen already from these first results that the strong convergence of Fouriers series bears some of the properties of both absolute and ordinary convergence.

1. Notation, definitions and preliminaries

A real or complex valued sequence (s_k) is strongly C_1 summable to a limit t, with index $\lambda>0$, and we write $s_k\to t\,[C_1]_\lambda$, if

$$\frac{1}{n+1}\sum_{k=0}^{n}|s_k-t|^\lambda = o(1). \tag{1.1}$$

[1] The work on this paper was partially supported by the Research Council of S. R. Bosna and Hercegovina.

If A and B are two convergence (summability) methods we shall write $A \Rightarrow B$ if $s_k \to t$ B whenever $s_k \to t$ A. If $s_k \to t$ in the ordinary sense we shall also write $s_k \to t$ I, where I denotes the identity transformation on the set of all real or complex valued sequences.

The following properties of the strong C_1 summability can be easily established applying Hölder's inequality (for a more general statement see Theorem 1 in [7]):

(i) $[C_1]_\lambda \Rightarrow [C_1]_\mu$ for $\lambda > \mu > 0$,
(ii) $[C_1]_\lambda \Rightarrow C_1$ for $\lambda \geqq 1$,
(iii) $I \Rightarrow [C_1]_\lambda$ for any index $\lambda > 0$.

Thus for $\lambda \geqq 1$, strong C_1 summability of index λ is a natural generalization of C_1 summability. This was first considered by Hardy and Littlewood in 1913, in connection with summability of Fourier series (see [1]). Much later, in 1952, Hyslop [3] extended this concept to Cesàro transformations C_α of order α, $\alpha > 0$. Since then, strong summability methods associated with other matrix transformations were considered by various authors (see [6] and [7] and the references cited there). Introducing a concept of strong C_α summability of order $\alpha > 0$. Hyslop in [3] has also defined strong C_0 summability of index $\lambda > 0$, that is a strong summability method associated with the identity transformation $I = C_0$, called the strong convergence of index λ.

A real or complex valued sequence (s_k) is strongly convergent to a limit t with index $\lambda > 0$, and we write $s_k \to t$ $[\mathrm{I}]_\lambda$, if

$$\frac{1}{n+1} \sum_{k=0}^{n} |(k+1)(s_k - t) - k(s_{k-1} - t)|^\lambda = o(1). \tag{1.2}$$

The following simple statements can be easily verified (see also Corollary 2 of Theorem 1 in [7] and Corollary 2 of Theorem 5 in [6]):

(iv) $[I]_\lambda \Rightarrow [I]_\mu$ for $\lambda > \mu > 0$,
(v) $[I]_\lambda \Rightarrow I$ for $\lambda \geqq 1$,
(vi) For $\lambda \geqq 1$, $s_k \to t$ $[I]_\lambda$ if and only if $s_k \to t$ and

$$\frac{1}{n+1} \sum_{k=0}^{n} k^\lambda |s_k - s_{k-1}|^\lambda = o(1). \tag{1.3}$$

Statement (vi) shows that for $\lambda \geqq 1$, the above definition of strong convergence of index λ, is equivalent to the definition introduced by Hyslop in [3]. This definition is, however, more practical than Hyslop's in relation to a general concept of strong summability, discussed in [7]. From (iii), (v) and Minkowski's inequality we obtain:

(vii) For $\lambda \geqq 1$, $s_k \to t$ $[I]_\lambda$ if and only if (1.3) and $s_k \to t$ $[C_1]_\lambda$.

A real or complex valued sequence (s_k) is absolutely convergent to a limit t, with index $\lambda > 0$, and we write $s_k \to t$ $|I|_\lambda$, if $s_k \to t$ and

$$\sum_{k=1}^{\infty} k^{\lambda-1} |s_k - s_{k-1}|^\lambda < \infty. \tag{1.4}$$

If $\lambda=1$ then (1.4) implies that $s_k\to t$ for some t. In the case $\lambda>1$, (1.4) does not imply that $s_k\to t$; e.g. $s_k=\ln\ln k\to\infty$, but

$$\sum_{k=1}^{\infty} k^{\lambda-1}|s_k-s_{k-1}|^{\lambda}=O\left(\sum_{k=1}^{\infty}\frac{1}{k(\ln k)^{\lambda}}\right)<\infty.$$

The following simple statement is also contained in Corollary 2 of Theorems 1 and 2 in [7]:

(viii) $|I|_{\lambda}\Rightarrow[I]_{\lambda}\Rightarrow I$ for $\lambda\geqq 1$.

The absolute and strong convergence of index $\lambda=1$ will be called simply the absolute and strong convergence and denoted by $|I|$ and $[I]$, respectively.

It is useful to remark that if the above convergence methods are applied to sequences of real or complex valued functions, then the corresponding statements (i) through (viii) hold for the appropriate pointwise or uniform convergence on a subset of the real line, whichever the case may be.

For $p\geqq 1$ let L^p denote the set of all real or complex valued 2π-periodic functions f such that $\|f\|_p=\left(\frac{1}{2\pi}\int|f|^p\right)^{1/p}$ is finite, the integral being taken over any interval of length 2π. Let C, BV and AC denote the sets of all real or complex valued 2π-periodic functions that are continuous, of bounded variation and absolutely continuous, respectively.

For $f\in L^1$ let sf denote the Fourier series of f; s_nf, $n=0, 1, 2, \ldots$, the n-th partial sum of sf; $f_c\hat{}\,(n)$ and $f_s\hat{}\,(n)$, $n=0, 1, 2, \ldots$, the cosine and sine Fourier coefficients of f, respectively.

For $f\in L^p$, $p\geqq 1$ and a point x we shall write

$$\Phi_{x,p}(t)=\int_0^t\left|\frac{1}{2}[f(x+u)+f(x-u)]-f(x)\right|^p du. \tag{1.5}$$

We recall that, by a theorem due to Lebesgue, for almost all x, $\Phi_{x,p}(t)=o(t)$ as $t\to 0^+$.

In Section 2 we will refer to the following extensions of the Fejér—Lebesgue theorem:

THEOREM A. (i) *If* $f\in L^1$ *and* $\lambda>0$ *then* $s_nf\to f$ $[C_1]_{\lambda}$ *almost everywhere.*

(ii) *If* $f\in L^p$, $p>1$ *and* $\lambda>0$ *then* $s_nf(x)\to f(x)$ $[C_1]_{\lambda}$ *at every point* x *at which* $\Phi_{x,p}(t)=o(t)$.

(iii) *If* $f\in C$ *and* $\lambda>0$ *then* $s_nf\to f$ $[C_1]_{\lambda}$ *uniformly.*

Statements (ii) and (iii) are due to Hardy and Littlewood (see p. 490 in [1] or p. 182, Vol II, in [8]). Statement (i) was proved by Marcinkiewicz and Zygmund (see p. 493 in [1] or p. 184, Vol II, in [8]). The latter is a much more difficult result, for it is no longer necessarily true that $s_nf(x)\to f(x)$ $[C_1]_{\lambda}$ at every point x at which $\Phi_{x,p}(t)=o(t)$.

2. General trigonometric and Fourier series

The following statement is an analogue of the well known Denjoy—Lusin theorem (see p. 232, Vol. I, in [8]).

THEOREM 1. *Let* $\lambda \geqq 1$. *If a trigonometric series*

$$a_0/2+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx) \tag{2.1}$$

is $[I]_\lambda$ *convergent for* x *belonging to a set* A *of positive measure, then*

$$\frac{1}{n+1}\sum_{k=1}^{n}|ka_k|^\lambda = o(1) \text{ and } \frac{1}{n+1}\sum_{k=1}^{n}|kb_k|^\lambda = o(1) \tag{2.2}$$

that is $ka_k \to 0$ $[C_1]_\lambda$ *and* $kb_k \to 0$ $[C_1]_\lambda$.

PROOF. If (2.1) is $[I]_\lambda$ convergent on a set A, $m(A)>0$, where m denotes the Lebesgue measure, then by (vii) it follows that

$$\frac{1}{n+1}\sum_{k=1}^{n}k^\lambda|a_k\cos kx+b_k\sin kx|^\lambda = o(1) \tag{2.3}$$

for x belonging to A. By Egoroff's theorem there is a set $B\subset A$, $m(B)>0$ on which (2.3) holds uniformly. Therefore, writing $\varrho_k\cos(kx+x_k)=a_k\cos kx+b_k\sin kx$, where $\varrho_k=(a_k^2+b_k^2)^{1/2}$ we have

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=1}^{n}k^\lambda\varrho_k^\lambda\int_B|\cos(kx+x_k)|^\lambda dx = 0. \tag{2.4}$$

Since $1\leqq\lambda\leqq 2j$ for some integer j, it follows that

$$\int_B|\cos(kx+x_k)|^\lambda dx \geqq \int_B|\cos(kx+x_k)|^{2j}dx =$$

$$= \frac{1}{2^j}\int_B(1+\cos 2(kx+x_k))^j dx \geqq \frac{1}{2^j}m(B)+\frac{j}{2^j}\int_B\cos 2(kx+x_k)\,dx.$$

But $\int_B\cos 2(kx+x_k)\,dx\to 0$ by the theorem of Riemann—Lebesgue, and therefore for sufficiently large k

$$\int_B|\cos(kx+x_k)|^\lambda dx > \frac{1}{2^{j+1}}m(B).$$

Consequently from (2.4) it follows that

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=1}^{n}k^\lambda\varrho_k^\lambda = 0$$

which clearly implies (2.2).

Our next result is the category counterpart of Theorem 1 and an analogue of the corresponding theorem for absolutely convergent trigonometric series due to Lusin (see p. 752 in [1] or p. 233, Vol I, in [8]). The proof is, however, slightly more involved.

THEOREM 2. *Let* $\lambda \geqq 1$. *If a trigonometric series* (2.1) *is* $[I]_\lambda$ *convergent on a set of second category, then* (2.2) *holds.*

PROOF. Suppose that (2.1) is $[I]_\lambda$ convergent on a set A of second category. Let

$$\sigma_n(x) = \frac{1}{n+1} \sum_{k=1}^{n} k^\lambda |a_k \cos kx + b_k \sin kx|^\lambda$$

and let $E = \{x\colon \sigma_n(x) \to 0\}$. Then statement (vii) implies $A \subset E$. Clearly σ_n is continuous for each n and therefore $\{x\colon \sigma_n(x) \leqq 1/i\}$ is closed for each pair of integers i and n. Let $E_{im} = \{x\colon \sigma_n(x) \leqq 1/i \text{ for } n > m\}$, then E_{im} is closed as the intersection of closed sets. Moreover, clearly

$$E = \bigcap_{i=1}^{\infty} \bigcup_{m=1}^{\infty} E_{im}.$$

Now for each i, $\bigcup_{m=1}^{\infty} E_{im}$ is of second category, for otherwise E and therefore A would be subsets of a set of first category which is impossible by the assumption on A. Consequently, for each i there is an integer m_i such that E_{im_i} contains an interval I_i. Hence, for each i there is an integer m_i and an interval I_i such that

$$\frac{1}{n+1} \sum_{k=1}^{n} k^\lambda \varrho_k^\lambda \int_{I_i} |\cos(kx + x_k)|^\lambda dx = \int_{I_i} \sigma_n(x)\, dx \leqq \frac{1}{i} d(I_i) \tag{2.5}$$

for all $n \geqq m_i$, where $d(I_i)$ is the length of I_i and where (ϱ_k) and (x_k) are sequences as in the proof of Theorem 1. Moreover, just as in the proof of Theorem 1, $1 \leqq \lambda \leqq 2j$ for some integer j implies that

$$\int_{I_i} |\cos(kx + x_k)|^\lambda dx \geqq \int_{I_i} \cos^{2j}(kx + x_k)\, dx \geqq$$

$$\geqq \frac{1}{2^j} d(I_i) + \frac{j}{2^j} \int_{I_i} \cos 2(kx + x_k)\, dx$$

so that there is an integer k_i such that

$$\int_{I_i} |\cos(kx + x_k)|^\lambda dx > \frac{1}{2^{j+1}} d(I_i) \quad \text{for all} \quad k \geqq k_i.$$

Consequently from (2.5) and the last inequality we have

$$\frac{1}{n+1} \sum_{k=k_i}^{n} k^\lambda \varrho_k^\lambda < \frac{2^{j+1}}{i} \quad \text{for all} \quad n \geqq \max(m_i, k_i).$$

Choosing $n_i \geqq \max(m_i, k_i)$ and such that

$$\frac{1}{n_i+1}\sum_{k=1}^{k_i-1} k^\lambda \varrho_k^\lambda < \frac{1}{i}$$

we conclude that for each i there is an integer n_i depending on i, such that

$$\frac{1}{n+1}\sum_{k=1}^{n} k^\lambda \varrho_k^\lambda < (1+2^{j+1})\frac{1}{i} \quad \text{for all} \quad n \geqq n_i.$$

Consequently (2.2) holds.

THEOREM 3. *Let* $\lambda \geqq 1$.

(i) *If* $f \in L^1$ *then* $s_n f \to f$ $[I]_\lambda$ *almost everywhere if and only if its Fourier coefficients satisfy* (2.2), *that is*

$$\frac{1}{n+1}\sum_{k=1}^{n} |k f_c\hat{}\,(k)|^\lambda = o(1) \quad \text{and} \quad \frac{1}{n+1}\sum_{k=1}^{n} |k f_s\hat{}\,(k)|^\lambda = o(1). \tag{2.6}$$

(ii) *If* $f \in L^p$, $p>1$, *then* $s_n f(x) \to f(x)$ $[I]_\lambda$ *at every point* x *at which* $\Phi_{x,p}(t) = o(t)$ *if and only if its Fourier coefficients satisfy* (2.6).

(iii) *If* $f \in C$ *then* $s_n f \to f$ $[I]_\lambda$ *uniformly if and only if its Fourier coefficients satisfy* (2.6).

PROOF. The necessity part of each statement is clearly implied by Theorem 1. Conversely, if the Fourier coefficients satisfy (2.6) then (2.3) holds uniformly for all x, with $a_k = f_c\hat{}\,(k)$ and $b_k = f_s\hat{}\,(k)$. The respective converse implications then follow from (vii) and statements (i), (ii) and (iii), of Theorem A.

REMARK. Fourier series with coefficients satisfying (2.6) for $\lambda = 1$ were considered before, in regard to ordinary convergence (see for example Theorem 1 on p. 177 in [1]). This was only natural in view of the Fejér—Lebesgue theorem and the fact that $s_n f - \sigma_n f$ are majorized by $\frac{1}{n+1}\sum_{k=1}^{n} k[|f_c\hat{}\,(k)| + |f_s\hat{}\,(k)|]$, where $\sigma_n f$ is the n-th Cesàro partial sum of sf.

The following corollary clearly includes the above mentioned result and it is a simple consequence of statement (v) and Theorem 3.

COROLLARY 1. *Let* $\lambda \geqq 1$. *If* $f \in L^1$ *and its Fourier coefficients satisfy* (2.6) *then* $s_n f \to f$ *almost everywhere. If* $f \in C$ *and its Fourier coefficients satisfy* (2.6) *then* $s_n f \to f$ *uniformly.*

Our next application of Theorem 3 is a generalization of Fatou's theorem (see p. 178 in [1]):

COROLLARY 2. *If for a trigonometric series* (2.1) *the coefficients satisfy*

$$a_n = o(1/n) \quad \text{and} \quad b_n = o(1/n) \tag{2.7}$$

then (2.1) *is* $[I]_\lambda$ *convergent almost everywhere, for any* $\lambda \geqq 1$. *If furthermore* (2.1) *is the Fourier series of a continuous function then it is uniformly* $[I]_\lambda$ *convergent, for any* $\lambda \geqq 1$.

PROOF. If (2.7) holds then clearly $\sum (a_n^2+b_n^2)<\infty$ so that (2.1) is a Fourier series. Moreover (2.7) implies (2.6) for any $\lambda \geqq 1$ and the conclusions follow from Theorem 3.

3. Fourier series of continuous functions

Let $\mathscr{A}$ be the set of all $f \in C$ whose Fourier series converges absolutely. Then clearly $\mathscr{A}$ is the set of all sums of everywhere absolutely convergent trigonometric series or equivalently the set of all sums of trigonometric series with $\sum |a_k|<\infty$ and $\sum |b_k|<\infty$. Let $\mathscr{U}$ be the set of all $f \in C$ whose Fourier series converges uniformly. Then $\mathscr{U}$ is the set of all sums of uniformly convergent trigonometric series. Let $\mathscr{S}$ be the set of all $f \in C$ whose Fourier series is uniformly strongly convergent.

THEOREM 4. (i) $\mathscr{S}$ *is the set of all sums of uniformly strongly convergent trigonometric series or equivalently, the set of all* $f \in C$ *whose Fourier coefficients satisfy* (2.6) *for* $\lambda=1$.

(ii) $\mathscr{A} \subset \mathscr{S} \subset \mathscr{U}$ *properly.*

PROOF. (i) If f is the sum of a uniformly strongly convergent trigonometric series (2.1), then by (v), (2.1) is uniformly convergent, $f \in C$ and (2.1) is the Fourier series of f so that $f \in \mathscr{S}$. Conversely, if $f \in \mathscr{S}$ then again clearly f is the sum of its uniformly strongly convergent Fourier series. The second equivalent statement is just an application of (iii) in Theorem 3.

(ii) The inclusion is clear by (viii). It remains to be shown that these inclusions are both proper. To see that $\mathscr{A} \neq \mathscr{S}$, consider the series $\sum_{k=1}^{\infty} \frac{\varepsilon_k}{k} \sin kx$ where $\varepsilon_k \downarrow 0$, which is clearly uniformly convergent. Taking (ε_k) so that $\sum_{k=1}^{\infty} \frac{\varepsilon_k}{k}=\infty$ and $f(x)= \sum_{k=1}^{\infty} \frac{\varepsilon_k}{k} \sin kx$ it follows that $f \notin \mathscr{A}$, while by Theorem 3 clearly $f \in \mathscr{S}$. Thus $\mathscr{A} \neq \mathscr{S}$.

We now show that $\mathscr{S} \neq \mathscr{U}$. Let

$$\pm a_0/2+\sum_{k=1}^{\infty} \pm (a_k \cos kx+b_k \sin kx) \tag{3.1}$$

be a series obtained from (2.1) by changing the signs of the terms of the latter series in an arbitrary way. Taking $a_k=b_k=1/k$ for $k=1, 2, \ldots$, it is clear that (2.2) does not hold for any choice of the signs in (3.1), so that by Theorem 1, (3.1) does not converge strongly on any set of positive measure. However, since for any $\varepsilon>0$

$$\sum_{k=1}^{\infty} (a_k^2+b_k^2)(\ln k)^{1+\varepsilon} = 2 \sum_{k=1}^{\infty} \frac{1}{k^2} (\ln k)^{1+\varepsilon} < \infty,$$

by a result due to Zygmund and Paley, almost all series (3.1) converge uniformly (see pp. 214 and 219, Vol I, in [8]), and are therefore the Fourier series of continuous functions. Consequently $\mathscr{S} \neq \mathscr{U}$.

REMARK. By Theorem 2, the last example shows that there are continuous functions whose Fourier series converge uniformly, but are not strongly convergent on any set of second category, even of measure zero.

The following statement gives a new meaning to a classical result of Wiener about functions of bounded variation.

THEOREM 5. *If* $f\in BV$ *then* $s_nf\to f$ $[I]$ *uniformly if and only if* $f\in C$.

PROOF. If $f\in BV$ and $s_nf\to f$ $[I]$ uniformly then clearly $s_nf\to f$ uniformly and $f\in C$. Conversely, if $f\in BV$ and $f\in C$ then by Wiener's theorem (p. 205 in [1]),

$$\frac{1}{n+1}\sum_{k=0}^{n}(k+1)\,[\hat{f_c}^2(k)+\hat{f_s}^2(k)]^{1/2}=o(1).$$

Consequently (2.6) holds for $\lambda=1$ and $s_nf\to f$ $[I]$ uniformly by statement (iii) of Theorem 3.

COROLLARY 1. $BV\cap C\subset\mathscr{S}$.

REMARK 1. Corollary 1 should be compared with a well known theorem of Zygmund: if $f\in BV\cap\lambda_\alpha$ for some $\alpha>0$, then $s_nf\to f$ $|I|$ that is $f\in\mathscr{A}$, where λ_α denotes the Lipschitz class of order α and where the second condition is not superfluous (see p. 241, Vol I, in [8]). The simple example

$$f(x)=\sum_{k=2}^{\infty}\frac{1}{k\log k}\sin kx \tag{3.2}$$

given in [8], shows that $BV\cap C\not\subset\mathscr{A}$.

COROLLARY 2. $AC\subset\mathscr{S}$.

PROOF. This is a trivial consequence of Corollary 1, but it can be established directly from statement (iii) of Theorem 3. Namely, if $f\in AC$ then $|D\hat{f_s}(k)|=|k\hat{f_c}(k)|$ and $|D\hat{f_s}(k)|=|k\hat{f_s}(k)|$ so that (2.6) holds for $\lambda=1$, by Riemann—Lebesgue theorem.

REMARK 2. Corollary 2 should be compared with the corresponding theorem of Zygmund: if $f\in AC$ and $Df\in L^p$ for some $p>1$, then $s_nf\to f$ $|I|$, that is $f\in\mathscr{A}$. Again the second condition is not superfluous, for the above example (3.2) shows that $AC\not\subset\mathscr{A}$.

THEOREM 6. (i) *If* $f\in C$ *is even and* $\hat{f_c}(k)\geqq 0$, $k=0,1,\ldots$, *then* $s_nf\to f$ $[I]$ *uniformly, that is* $f\in\mathscr{S}$.

(ii) *If* $f\in C$ *is odd and* $\hat{f_s}(k)\geqq 0$, $k=1,2,\ldots$, *then* $s_n\to f$ $[I]$ *uniformly, that is* $f\in\mathscr{S}$.

PROOF. (i) This is a simple consequence of the fact that under the above assumptions $s_nf\to f$ $[I]$ (cf. Theorem 1, p. 277 in [1]), and statement (viii).

(ii) By statement (iii) of Theorem 3 it suffices to show that

$$\frac{1}{n+1}\sum_{k=1}^{n}k\hat{f_s}(k)=o(1).$$

We use the same argument as in (3) 7.2.2 [2]. Namely, if $\sigma_n f$ denotes the n-th Cesàro partial sum of sf, then clearly by Fejér—Lebesgue theorem, $\sigma_n f \to f$ uniformly, so that $f(0)=0$ and $\sigma_n(\pi/n)\to 0$. But clearly

$$\sigma_{2n}(\pi/2n) \geqq \sum_{k=1}^{n}\left(1-\frac{k}{2n+1}\right)\hat{f_s}(k)\frac{2}{\pi}\frac{k\pi}{2n} > \frac{1}{2}\frac{1}{n+1}\sum_{k=1}^{n} k\hat{f_s}(k)$$

and the conclusion follows since $\sigma_{2n}(\pi/2n)=o(1)$.

REMARK 1. As the example given by (3.2) shows, the conclusion in statement (ii) can not be substituted by $s_n f \to f$ $|I|$.

Our next result extends a well known theorem of Paley (see p. 277 in [1]). It is another example of those types of statements about ordinary convergence that can be carried over to strong, but not to absolute convergence.

THEOREM 7. *If $f\in C$ and the Fourier coefficients of f are nonnegative, then $s_n f \to f$ $[I]$ uniformly, that is $f\in\mathcal{S}$.*

PROOF. This is a trivial consequence of Theorem 6. Let g and h be given by

$$g(x)=[f(x)+f(-x)]/2 \quad \text{and} \quad h(x)=[f(x)-f(-x)]/2.$$

Then $f=g+h$ and g and h satisfy the assumptions of (i) and (ii) of Theorem 6, respectively. Thus $s_n g \to g$ $[I]$ uniformly and $s_n h \to h$ $[I]$ uniformly, so that $s_n(g+h)\to$ $\to g+h$ $[I]$ uniformly, that is $f\in\mathcal{S}$.

In conclusion we remark that while Theorems 1 through 3 show an obvious analogy between strong and absolute convergence of trigonometric and Fourier series, Theorems 5 through 7 point to a similarity with ordinary convergence in a sence that some of the properties of Fourier series valid for ordinary convergence extend to strong convergence.

References

[1] N. K. Bary, *Trigonometričeskie Rjadi* (Moscow, 1961) (in Russian).
[2] R. E. Edwards, *Fourier Series Modern Introduction*, Vols. 1 & 2, Holt, Rinehart and Winston Inc. (New York, 1967).
[3] J. M. Hyslop, Note on the strong summability of series, *Proc. Glasgow Math. Assoc.*, **1** (1952), 16—20.
[4] J. P. Kahane, *Séries de Fourier Absolument Convergentes*, Springer-Verlag (Berlin—Heidelberg—New York, 1970).
[5] A Peyerimhoff, *Lectures on Summability*, Springer-Verlag (Berlin—Heidelberg, 1969).
[6] N. Tanović-Miller, On strong summability, *Glasnik Mat.*, **14** (1979), 87—97.
[7] N. Tanović-Miller, On strong summability II, *Glasnik Mat.*, **15** (1980), 283—302.
[8] A. Zygmund, *Trigonometric Series*, Vols. I & II, Cambridge University Press (New York, 1959).

(Received November 4, 1981)

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SARAJEVO
SARAJEVO
YUGOSLAVIA

Acta Math. Hung.
42 (1—2) (1983), 45—80.

BOUNDS FOR THE SOLUTIONS OF NORM FORM, DISCRIMINANT FORM AND INDEX FORM EQUATIONS IN FINITELY GENERATED INTEGRAL DOMAINS

K. GYŐRY (Debrecen)

1. Introduction

In the present paper we give an effective procedure for solving Thue equations and norm form, discriminant form and index form equations in integral domains of finite type over **Z**. We shall also establish a relative formulation of our result. Our theorems provide a common generalization of several well-known effective results, obtained over number fields and function fields.

Let K be an algebraic number field, and R the ring of integers of K (or, more generally, a subring of K finitely generated over the ring **Z** of rational integers). Let $F(X_1, \ldots, X_m)$ be a form of degree greater than m with coefficients in K, and suppose that F is *decomposable* (i.e. that it factorizes into linear factors over some finite extension of K). Let β be a non-zero element of K, and consider the solutions $x_1, \ldots, x_m$ in R of the equation

$$F(x_1, \ldots, x_m) = \beta. \tag{1}$$

The equations of this type are of basic importance in the theory of diophantine equations and its applications. The most important cases are when F is a binary form, a norm form, a discriminant form or an index form.* There is an extensive literature of Thue equations (when $m=2$ in (1)) and of norm form, discriminant form and index form equations. (For references see e.g. [28], [37], [59], [9], [36], [5], [48], [53], [4], [62], [55], [19] and §§ 2.3 to 2.5 of the present paper.) Most of the results concerning equations of this type are finiteness theorems, stating the finiteness of the number of solutions. Many of them are effective, i.e. provide effective bounds for the height of all the solutions. The first general effective estimate for the solutions of Thue equations were established in Baker's famous papers [2], [3]. Baker's theorem was later improved and generalized respectively among others by Coates [6], [7], Feldman [11], Sprindžuk [60], [61], [63], Stark [64], Kotov [29], Kotov and Sprindžuk [32], Győry and Papp [27] and Győry [18]. In case of discriminant form, index form and norm form equations in an arbitrary number of unknowns, Győry [15], [18], [19], [20], [21], [23], Győry and Papp [25], [26], [27], Trelina [68] and Kotov [30], [31] obtained general and effective finiteness theorems.

Osgood [42], [43] and Schmidt [50], [51], [52] gave effective bounds for the height of the solutions of Thue equations considered over function fields of one variable.

---

* *Added in proof (July 29, 1983).* A further important case is when F is a form of discriminant type, i.e. $F(X_1, \ldots, X_m) = \prod_{1 \leq i < j \leq m} (X_i - X_j)^2$. Some results of [24] (concerning polynomials of given discriminant) can be deduced (with weaker estimates) from our Theorem 1 by applying it to equations of discriminant type.

It follows from a general theorem of Lang [33], [34] that if in (1) $m=2$ and K and R are of finite type over $\mathbf{Q}$ and $\mathbf{Z}$ respectively, then (1) (under the usual assumptions concerning Thue equations) has only finitely many solutions in R. In [22] we obtained general but ineffective finiteness theorems on norm form, discriminant form and index form equations in this more general situation. We showed that under certain conditions made on the linear factors of F, (1) has only finitely many solutions in any given integral domain R finitely generated over $\mathbf{Z}$.

In this paper we make effective, in a certain sense, the results of [22] and the above-quoted consequence of Lang's theorem [33], [34]. We prove that if K, R, β and the coefficients of the linear factors of F are effectively given in the sense defined in § 2.1, then we can effectively give a suitable extension ring R' of R such that all solutions of (1) in R' can be effectively determined. We shall obtain effective results in the relative case as well (when K is of finite type over a field k of characteristic 0 and R is a finitely generated algebra over k).

The theorems of the present paper are formulated and proved in a quantitative form. Apart from the form of the bounds, the above-mentioned effective finiteness theorems (obtained in the number field case) are special cases or consequences of our results. Further, our Theorems 1 and 2 generalize (with other estimates) some earlier effective results, concerning Thue equations defined over function fields.

Our main result is Theorem 1. The other results will be deduced from this theorem. In the relative case, Theorem 1 will be proved by combining a strong effective theorem of Schmidt [51] (concerning Thue equations defined over function fields) with our method used in [16] and [21] (see also [17], [19] and [22]). Then, in the absolute case, our result obtained in the relative case together with sufficiently many suitable specializations of R' into number fields will enable us to reduce the proof of Theorem 1 to its proof in the number field case. Finally, the application of our results [23] established in case of number fields will complete the proof of Theorem 1. We remark that the results of [23] have been proved by using Baker's powerful method and our graph method (see [23], [21] and [17]).

2. Results

§ 2.1. Notations and preliminary remarks on the effectiveness of the results

Let k be the ring $\mathbf{Z}$ of rational integers (absolute case) or a field of characteristic 0 (relative case). Let K_0 be a field finitely generated over the quotient field[1] k' of k, R an integral domain finitely generated over k, and K a finitely generated extension field of k', containing K_0 and R. Then we can write

$$K_0 = k'(z_1, \ldots, z_{s_0}, y_0) \tag{2.1}$$

where $z_1, \ldots, z_{s_0}$ is a transcendence basis of K_0 over k' and y_0 is integral over $k[z_1, \ldots, z_{s_0}]$. Such a set of generators $z_1, \ldots, z_{s_0}, y_0$ will be called a generating set

[1] In other words, $k'=\mathbf{Q}$ or $k'=k$ according as $k=\mathbf{Z}$ or k is a field.

over k'. Further, we have

$$R = k[y_1, \ldots, y_t] \tag{2.2}$$

and

$$K = k'(z_1, \ldots, z_s, y), \tag{2.3}$$

where $z_1, \ldots, z_s$ $(s \geqq s_0)$ is a transcendence basis of K over k', y is integral over the polynomial ring $k[z_1, \ldots, z_s]$ (i.e. $z_1, \ldots, z_s, y$ is a generating set of K over k') and $y_1, \ldots, y_t$ are algebraic over $k'(z_1, \ldots, z_s)$. We may suppose without loss of generality that K is a normal extension of $k'(z_1, \ldots, z_s)$. Let $f(X) = X^\delta + f_1 X^{\delta-1} + \ldots + f_\delta$ be the minimal polynomial of y over $k'(z_1, \ldots, z_s)$. Since $k[z_1, \ldots, z_s]$ is integrally closed, all f_i lie in $k[z_1, \ldots, z_s]$.

To avoid confusion, we adopt the notation of Schmidt [52] and denote henceforth the total degree in $z_1, \ldots, z_s$ of an element P of $k[z_1, \ldots, z_s]$ by capitalized Deg P.

Any element α of K has a unique representation (up to the unit factors of k) in the form

$$\alpha = \frac{P_0 + P_1 y + \ldots + P_{\delta-1} y^{\delta-1}}{Q}, \quad \delta = [K\colon k'(z_1, \ldots, z_s)], \tag{2.4}$$

where $P_0, \ldots, P_{\delta-1}$ and Q are relatively prime polynomials from $k[z_1, \ldots, z_s]$. We define the *Degree* of α as

$$\operatorname{Deg} \alpha = \max \{\operatorname{Deg} P_0, \ldots, \operatorname{Deg} P_{\delta-1}, \operatorname{Deg} Q\}.$$

It is easy to see by induction that

$$\operatorname{Deg}(\alpha_1 + \ldots + \alpha_r) \leqq \operatorname{Deg} \alpha_1 + \ldots + \operatorname{Deg} \alpha_r \tag{2.5}$$

and

$$\operatorname{Deg}(\alpha_1 \ldots \alpha_r) \leqq \operatorname{Deg} \alpha_1 + \ldots + \operatorname{Deg} \alpha_r + (r-1)(\delta-1) \max_{1 \leqq i \leqq \delta} \operatorname{Deg} f_i \tag{2.6}$$

for any $\alpha_1, \ldots, \alpha_r \in K$. Further, in the absolute case the *length*[2] of α will be defined as

$$L(\alpha) = \max \{L(P_0), \ldots, L(P_{\delta-1}), L(Q)\}$$

where $L(P)$ denotes the length of the polynomial $P \in \mathbf{Z}[z_1, \ldots, z_s]$ (i.e. the sum of the absolute values of the coefficients of P). It is clear that Deg α and $L(\alpha)$ are uniquely determined by the generating set $z_1, \ldots, z_s, y$ of K.

In §§ 2.2, 2.3 and 2.4 we shall derive bounds for the Degree of all the solutions in R of equations of the type (1). Further, in the absolute case bounds will be established for the length of the solutions as well. Since in case $k = \mathbf{Z}$ there are only finitely many elements in K with bounded Degree and bounded length, our estimates (in the absolute case) will imply the finiteness of the number of solutions.

Our bounds are given explicitly in terms of each parameter. These bounds depend of course on the choice of the generating set $z_1, \ldots, z_s, y$ of K and on $y_1, \ldots, y_t$. However, if the generators of a finitely generated ring or field are given in some way, it is in general a hard problem[3] to determine the transcendence degree and a

[2] We note that in the special case when K is an algebraic number field, this concept is different from the usual concept of length (cf. [6?]).

[3] We can mention e.g. the field $\mathbf{Q}(e+\pi)$ whose transcendence degree over $\mathbf{Q}$ is not known.

generating set. Hence our estimates and results will be effective only if K_0, R and K are *effectively given* over k in the following sense: K_0, R and K are given in the form (2.1), (2.2) and (2.3) respectively, and

in the absolute case the degree δ as well as the coefficients $f_1, \ldots, f_\delta \in \mathbf{Z}[z_1, \ldots, z_s]$ of $f(X)$ are given,[4] and $y_0, y_1, \ldots, y_t$ are given in the form (2.4);

in the relative case bounds are given for δ, $\max_{1 \leqq i \leqq \delta} \operatorname{Deg} f_i$, $\operatorname{Deg} y_0$ and the Degrees of the denominators of $y_1, \ldots, y_t$ in their representations of the form (2.4).

When K_0, R and K are effectively given in the above sense, in the absolute case an element α of K will be said to be *effectively given* (resp. *effectively determinable*) in the form (2.4) if α is given in the form (2.4) (resp. if there is an algorithm to give α in the form (2.4). Further, in the relative case we shall say that an $\alpha \in K$ is *effectively bounded* if we have an effective upper bound for $\operatorname{Deg} \alpha$.

We note that if α and $\beta \in K$ are effectively given (resp. effectively bounded), so are $\alpha \pm \beta$, $\alpha \cdot \beta$ and α/β $(\beta \neq 0)$.

We make now some remarks on the absolute case. Suppose that K_0, R and K are effectively given.

REMARK 1. If $\alpha \in K$ is effectively given, its characteristic polynomial relative to $K/\mathbf{Q}(z_1, \ldots, z_s)$ can be effectively determined by the theorem on symmetric functions. Thus the defining polynomial[5] of α over $\mathbf{Z}[z_1, \ldots, z_s]$ can also be effectively determined by using Kronecker's factorization algorithm (cf. [45]). Conversely, if we are given a polynomial $P(X)$ with relatively prime coefficients from $\mathbf{Z}[z_1, \ldots, z_s]$, then we can effectively decide whether $P(X)$ has a zero in K and all $\alpha \in K$ with $P(\alpha)=0$ can be effectively given. To verify this we may suppose that $\deg P$ divides δ. Consider the polynomial $F(X)=(a_0^{-1}P(X))^{\delta/\deg P}$ where a_0 denotes the leading coefficient of P. We look for α in the form

$$\alpha = a_0' + a_1' y + \ldots + a_{\delta-1}' y^{\delta-1} \tag{2.7}$$

with $a_0', \ldots, a_{\delta-1}' \in \mathbf{Q}(z_1, \ldots, z_s)$. Clearly, $P(\alpha)=0$ with some $a_0', \ldots, a_{\delta-1}' \in \mathbf{Q}(z_1, \ldots, z_s)$ if and only if $F(\alpha)=0$, when $F(X)$ is the characteristic polynomial of α. We can determine (see e.g. [5], p. 403) the elements of a dual basis $\omega_1, \ldots, \omega_\delta$ of the basis $1, y, \ldots, y^{\delta-1}$ in the form (2.7) (and so we can determine the characteristic polynomial of ω_j for all j). Then we have

$$a_j' = \operatorname{Tr}(\alpha \omega_j), \quad j = 1, \ldots, \delta.$$

Let $y^{(1)}, \ldots, y^{(\delta)}$ denote the conjugates of y over $\mathbf{Q}(z_1, \ldots, z_s)$, and, for any $\gamma \in K$, let $\gamma^{(1)}, \ldots, \gamma^{(\delta)}$ be the corresponding field conjugates of γ. The characteristic polynomial of $\alpha\omega_j$ divides the polynomial $g_j(X)=\prod_{i,l} (X-\alpha^{(i)}\omega_j^{(l)})$ in the polynomial ring $\mathbf{Q}(z_1, \ldots, z_s)[X]$, where in g_j the product is taken over all field conjugates of α and ω_j. Hence a_j' can take only finitely many values from $\mathbf{Q}(z_1, \ldots, z_s)$ and all these

[4] In the absolute case an element P of $\mathbf{Z}[z_1, \ldots, z_s]$ is considered to be given if the degree and coefficients of P, as a polynomial in $z_1, \ldots, z_s$, are actually known. Further, an $\alpha \in K$ is said to be given in the form (2.4) if $P_0, \ldots, P_{\delta-1}$ and Q are given in the representation of the form (2.4) of α.

[5] We recall that if α is of degree ν over $k'(z_1, \ldots, z_s)$ then its defining polynomial $P(X)$ over $k[z_1, \ldots, z_s]$ is a polynomial of degree ν with $P(\alpha)=0$ and with coprime coefficients in $k[z_1, \ldots, z_s]$. This polynomial P is unique up to the non-zero factors belonging to k.

can be determined. Finally, considering the polynomial

$$h(X) = \prod_{j=1}^{\delta} \left(X-(a'_0+a'_1 y^{(j)}+\ldots+a'_{\delta-1}(y^{(j)})^{\delta-1})\right)$$

for all possible values of $a'_0, \ldots, a'_{\delta-1}$ and comparing its coefficients with those of $F(X)$, we can decide if $P(\alpha)=0$ with some $\alpha\in K$, and we can determine $a'_0, \ldots, a'_{\delta-1}$ in the representation of the form (2.7) of all $\alpha\in K$ for which $P(\alpha)=0$.

REMARK 2. If the elements $\alpha_1, \ldots, \alpha_m$ of K are algebraic over K_0 and are effectively given then we can effectively decide if they are linearly independent over K_0 or not. Further, all conjugates of $\alpha_1, \ldots, \alpha_m$ over K_0 can be effectively given in view of Remark 1. In particular, we can effectively determine the degree of $K_0(\alpha_1, \ldots, \alpha_m)$ over K_0.

In our above arguments (applied in the absolute case) we tacitly used the fact that if K is effectively given over $\mathbf{Q}$ then it is computable. (For a general theory of computable algebras and fields we refer to the work [44] of Rabin. Further basic constructions can be found e.g. in Seidenberg [54].) We have a similar situation in the relative case if in particular k is computable (when the cardinality of k is obviously countably infinite). Namely, when K is given in the form (2.3) then it is also computable. Further, if in addition k has a splitting algorithm (i.e. if there exists an effective procedure for determining the product decomposition into prime factors of any polynomial $P\in k[X]$; cf. [44]) then the above definitions, remarks and arguments concerning the absolute case can be used in this situation as well. A necessary and sufficient condition for the existence of a splitting algorithm in a computable field has been given in [44].

In general the elements of R cannot be represented in a manageable form. To avoid this difficulty, we shall consider the solutions of (1) in the extension ring

$$R' = k\left[z_1, \ldots, z_s, \frac{1}{a}, y\right]$$

of R where $a\in k[z_1, \ldots, z_s]$ denotes the product of the denominators of $y_1, \ldots, y_t$ in their representations of the form (2.4). The ring R' consists of those elements of K whose denominator in (2.4) is divisible only by irreducible factors of a. It is clear that if R and K are effectively given then a can be effectively given (resp. effectively bounded). Further, if $k=\mathbf{Z}$ or if k is a computable field with a splitting algorithm, then all irreducible factors of a can be effectively determined. In this case we can effectively decide, for any effectively given $\alpha\in K$, whether $\alpha\in R'$ or not.

We mention three important special cases when K_0, R and K are effectively given.

EXAMPLE 1. If we are given the algebraic number fields $K_0\subseteq K$ and a finite set S of valuations[6] of K (with value group $\mathbf{Z}$), then K_0, K and the ring R of S-integers of K can be regarded effectively given in the above sense. (For an effective theory of algebraic number fields we refer the reader to the books [5], [73] and [67].)

[6] We adopt the terminology of Zariski and Samuel [72] and Borevich and Shafarevich [5]. We recall that an element α of K is said to be S-integral if $v(\alpha)\geqq 0$ for all $v\notin S$.

EXAMPLE 2. Let k be the ring of rational integers (resp. a field of characteristic 0), $R=k[Z_1, \ldots, Z_s]$ the polynomial ring over k in $Z_1, \ldots, Z_s$, K_0 its quotient field and $K=K_0(y)$ with an integral element y over R. If the minimal polynomial of y over K_0 (resp. an upper bound for the maximum degree of the coefficients of this polynomial) is known, then K_0, R and K are effectively given over k.

EXAMPLE 3. Let $k=\mathbf{Z}$, K_0 an algebraic number field and y an algebraic integer for which $K_0 \subseteq \mathbf{Q}(y)$. Let R be as in Example 2 and $K=\mathbf{Q}(Z_1, \ldots, Z_s, y)$. Then K_0, R and K can be regarded effectively given.

It is obvious that in Examples 2 and 3 the a defined above is equal to 1 and $R'=R[y]$. If in Example 1 K is given by the minimal polynomial $f(X)$ of a primitive integral element of K, then a can be effectively determined in terms of f and the rational primes associated with the valuations belonging to S.

Finally, we shall say that the discussed equations (1), (2.10), (2.14), (2.15) and (2.17) are *effectively given* if K_0, R, K, β and the coefficients of the linear factors of the decomposable form under consideration are effectively given (resp. effectively bounded). If this is the case, our estimates obtained for the solutions are effective. Further, in the absolute case they enable one, at least in principle, to determine all the solutions in R'. As to the solutions in R, they can be found among the solutions in R' only if the elements of R have a well-utilizable representation. If for example R has a subring of the form $R_0=\mathbf{Z}[Z_1, \ldots, Z_s, 1/b]$ with some given $b \in \mathbf{Z}[Z_1, \ldots, Z_s]$ over which R is itegral and R, as and R_0-module, admits a finite basis with effectively given elements, then all solutions in R can be effectively found. We note that in particular the polynomial rings over $\mathbf{Z}$, the rings of integers of algebraic number fields and, more generally, all S-orders of a number field possess the above property.

§ 2.2. Statement of Theorem 1

Let k, k', R, K and R' be as in § 2.1. Let $F(\mathbf{X})=F(X_1, \ldots, X_m) \in K[X_1, \ldots, X_m]$ be a form of degree $n \geqq 3$ in $m \geqq 2$ variables, and suppose that F is decomposable into linear factors over K. Let

$$F(\mathbf{X}) = l_1(\mathbf{X}) \ldots l_n(\mathbf{X})$$

be a fixed factorization of F into linear factors over K. Assume that the coefficients of $l_1, \ldots, l_n$ have Degrees $\leqq A$ and, when $k=\mathbf{Z}$, have lengths $\leqq \mathscr{A}$. Suppose that the system $\mathscr{L}$ of linear forms $l_1, \ldots, l_n$ has rank m over K (which implies $m \leqq n$) and that $\mathscr{L}$ can be divided into pairwise disjoint subsystems $\mathscr{L}_1, \ldots, \mathscr{L}_h$ such that each $\mathscr{L}_j$ $(1 \leqq j \leqq h)$ is connected (i.e. $\mathscr{L}_j$ consists of at least two linear forms and for any distinct i, i' with $l_i, l_{i'} \in \mathscr{L}_j$ there is a sequence $l_i=l_{i_1}, \ldots, l_{i_v}=l_{i'}$ in $\mathscr{L}_j$ such that, for each u with $1 \leqq u \leqq v-1$, $l_{i_u}, l_{i_{u+1}}$ have a linear combination with non-zero coefficients in K which belongs to $\mathscr{L}_j$; for this concept see e.g. [21] and [22].[7]) Further, suppose that there exists a q $(1 \leqq q \leqq m)$ with the property that, for each j with

[7] It is easy to see that if $m=2$ then every system $\mathscr{L}$ containing at least three pairwise non-proportional linear forms satisfies these conditions with $h=1$.

$1 \leqq j \leqq h$, X_q can be expressed as a linear combination with coefficients in K of the linear forms belonging to $\mathscr{L}_j$.[8]

Let $f(X)=X^\delta+f_1X^{\delta-1}+\ldots+f_\delta$ and a be defined as in § 2.1. Suppose $\max_i \operatorname{Deg} f_i \leqq D$ and, in the absolute case, $\max_i L(f_i) \leqq \mathscr{F}$. When $k=\mathbf{Z}$, r ($\geqq 0$) will denote the number of distinct rational prime factors of a, and P the greatest of these prime factors (with the convention that P and $\log P=1$ if $r=0$). Further, $\mathscr{P}$ will signify the maximum length of the non-constant irreducible factors of a in the polynomial ring $\mathbf{Z}[z_1, \ldots, z_s]$ (with $\mathscr{P}=1$ if $a \in \mathbf{Z}$).

Let β be a non-zero element of K with $\operatorname{Deg} \beta \leqq B$ and, in the case $k=\mathbf{Z}$, $L(\beta) \leqq \mathscr{B}$.

THEOREM 1. *Under the above assumptions, all solutions* $(x_1, \ldots, x_m) \in R'^m$ *of the equation* (1) *with* $x_q \neq 0$ *satisfy*

$$\max_{1 \leqq i \leqq m} \operatorname{Deg} x_i \leqq 7200s(m+1)!\,(n(\delta+1)!)^2(A+B+D+\operatorname{Deg} a+1) = sC_1 \tag{2.8}$$

and, when $k=\mathbf{Z}$,

$$\max_{1 \leqq i \leqq m} L(x_i) \leqq \exp\Big\{((s\delta+1)C_1)^{1+s(\delta+1)!(r\delta!+7)((s\delta+1)C_1)^s} \cdot (mn)^3 \cdot \tag{2.9}$$

$$\cdot\big((25(r+2)(\delta!)^2)^{10}\mathscr{F}^{2\delta}\big)^{\delta!(r\delta!+7)} P^{\delta!}(\log P)^{r\delta!+9}\big(1+n\log(\mathscr{A}\mathscr{B}\mathscr{P})\big)\Big\}$$

with the C_1 *occurring in* (2.8).

As we remarked in § 2.1, if R, K, β and the coefficients of $l_1, \ldots, l_n$ are effectively given (resp. effectively bounded), then our estimates (2.8) and (2.9) are effective. Furthermore, from our theorem follows that in the case $k=\mathbf{Z}$ the number of solutions is finite and, substituting every $\mathbf{x}=(x_1, \ldots, x_m) \in R'^m$ ($x_q \neq 0$) satisfying (2.8) and (2.9) into (1), all solutions can be effectively found. In particular, all $x_1, \ldots, x_m \in \mathbf{Z}[z_1, \ldots, z_s]$ ($x_q \neq 0$) satisfying (1) can be determined.

We note that in the relative case we cannot assert the finiteness of the number of solutions, because (1) may have infinitely many solutions in k. This is the case e.g. when k is algebraically closed and β and the coefficients of the l_i lie in k.

It is easy to see (cf. the remark following Theorem 4) that (1) may have infinitely many solutions with $x_q=0$.

When $k=\mathbf{Z}$, the finiteness of the number of solutions of (1) with $x_q \neq 0$ (under the hypotheses[9] of Theorem 1) was proved in [22], Theorem 3. Our Theorem 1 in the absolute case is an effective version of Theorem 3 of [22].

In the special case when K is an algebraic number field and R the ring of S-integers of K (with a finite set S of valuations of K), the main result of [23] gives an explicit bound for the height of all the solutions (with $x_q \neq 0$) of (1) in R (cf. Lemma 6 of the present paper). Apart from the form of the estimates, our Theorem 1 is a generalization of this result of [23] and Theorem 1 of [25].

---

[8] When $h=1$, the other hypotheses imply this condition for each q. So, in case $h=1$ the restriction $x_q \neq 0$ can be omitted from Theorem 1.

[9] The hypotheses of our Theorem 1 concerning the linear factors of F are slightly different from those of Theorem 3 of [22] and Theorem 1 of [21], but it is easily seen that these assumptions in the three theorems in question are equivalent.

§ 2.3. Bounds for the solutions of Thue equations

Our Theorem 1 has several applications. In this section we present some of its consequences concerning Thue equations.

Let k, R, K and R' be as in §§ 2.1 and 2.2. Consider the binary form

$$F(X_1, X_2) = (\alpha_{11}X_1+\alpha_{12}X_2) \dots (\alpha_{n1}X_1+\alpha_{n2}X_2)$$

where the coefficients of the linear factors are elements of K with Degrees$\leqq A$ and, in the case $k=\mathbf{Z}$, with lengths$\leqq\mathscr{A}$. Suppose that among the linear factors of F there are at least three pairwise non-proportional. Let β be a non-zero element of K with Degree$\leqq B$ and, in the absolute case, with $L(\beta)\leqq\mathscr{B}$. Consider the solutions x_1, x_2 of the Thue equation

$$F(x_1, x_2) = \beta \tag{2.10}$$

in R'.

As a simple consequence of Theorem 1 we get the following

THEOREM 2. *All solutions x_1, x_2 of* (2.10) *in R' satisfy* (2.8) *and, in case $k=\mathbf{Z}$,* (2.9) *with the choice $m=2$.*

In the absolute case Theorem 2 implies that (2.10) has only finitely many solutions in R' and these can be effectively determined, provided that (2.10) is effectively given. In particular, all solutions $x_1, x_2\in\mathbf{Z}[z_1, \dots, z_s]$ of (2.10) can be found. We note that the finiteness of the number of solutions of (2.10) follows from a general but ineffective theorem of Lang [33], [34], i.e. from a generalized version of Siegel's theorem [57] concerning integral points of curves of genus $\geqq 1$. Furthermore, Lang also gave a relative formulation of his result.

In the number field case (when K is an algebraic number field) the first general effective estimate for the integral solutions of (2.10) was obtained by Baker [2], [3]. Later, effective bounds have been established for the S-integral solutions of (2.10) (where S is a finite set of valuations of K). (Essentially Coates [6], [7] and Kotov [29]; see also Sprindžuk [60], Kotov and Sprindžuk [32], Győry [23] and the references given there.) Our Corollary 2.1 below implies these results with another estimate.

Let K be an algebraic number field of degree δ over $\mathbf{Q}$, and y a primitive integral element of K with height[10] $H(y)\leqq H$. Suppose that the discriminant of y has r prime factors whose maximum is Q. Let $F(X_1, X_2)$ be as above, and suppose that α_{ij} and β are elements of K with heights not exceeding $\mathscr{H}$. Let S be a finite set of valuations of K with cardinality t, and P the maximum of the rational primes corresponding to the valuations induced on $\mathbf{Q}$ by the elements of S (with the convention that P and $\log P=1$ if $t=0$).

COROLLARY 2.1. *Under the above hypotheses, all solutions of* (2.10.) *in S-integral x_1, x_2 of K satisfy*

$$\max\{H(x_1), H(x_2)\} < \exp\{((25(r+t+2)(\delta!)^2)^{10}H^{2\delta})^{\delta!((r+t)\delta!+8)} \cdot n^6(P+Q)^{\delta!}(\log(P+Q))^{(r+t)\delta!+9}(1+\log\mathscr{H})\}. \tag{2.11}$$

[10] As usual, $H(\alpha)$ denotes the maximum absolute value of the (relatively prime integer) coefficients in the defining polynomial over $\mathbf{Z}$ of an algebraic number α.

Next, let k be a field of characteristic 0, and consider the polynomial ring $k[Z]$ over k in the variable Z. Osgood [42], [43], Schmidt [50], [51] and Stepanov [66] obtained effective bounds for the degrees of the solutions of (2.10) in the case when β and the coefficients of F lie in the rational function field $K=k(Z)$ and $x_1, x_2 \in k[Z]$. Schmidt [51] extended these results to the case of function fields K/k in one variable, and gave an explicit bound for the heights of all the solutions of (2.10) in $x_1, x_2 \in K$.* For certain related results see also Schmidt [52] and Stepanov [66]. The bounds obtained in the works [50], [51], [52] and [66] are surprisingly good.

When $s=1$, our Theorem 2 in the relative case is due to Schmidt ([51], Theorem 1, (ii)). In [51] Schmidt uses height in place of Degree, but the argument of the proof of our estimate (2.8) together with Lemma 7 shows that we can change over from Degree to height, and conversely.

Consider now (2.10) over the polynomial ring $k[Z_1, \ldots, Z_s]$ and adopt the above notation Deg P for the total degree of every polynomial $P \in k[Z_1, \ldots, Z_s]$. From Theorem 2 we shall deduce the following

COROLLARY 2.2. *Let $F(X_1, X_2)$ be a form of degree $n \geqq 3$ with coefficients in $k[Z_1, \ldots, Z_s]$. Suppose that F has distinct linear factors and that its splitting field over $k(Z_1, \ldots, Z_s)$ has degree δ.*[11] *Further, let $0 \neq \beta \in k[Z_1, \ldots, Z_s]$ with Degree B, and let H be the maximum Degree of the coefficients of F. Then all solutions x_1, x_2 of* (2.10) *in $k[Z_1, \ldots, Z_s]$ satisfy*

$$\max\{\mathrm{Deg}\, x_1, \mathrm{Deg}\, x_2\} < 5 \cdot 10^4 s\big(n(\delta+1)!\big)^2\big(s^2(n(\delta+1)!)^3 H + B + 1\big) = C_2.$$

We remark that in the case $s=1$ this corollary was proved in Schmidt [50], Theorem 2, with a better estimate.

Finally, in the absolute case we get the following result over the polynomial ring $\mathbf{Z}[Z_1, \ldots, Z_s]$.

COROLLARY 2.3. *Let $F(X_1, X_2)$ be a form of degree $n \geqq 3$ with coefficients in $\mathbf{Z}[Z_1, \ldots, Z_s]$. Suppose that $F(1, 0) \neq 0$ and that the linear factors of F are distinct. Let $0 \neq \beta \in \mathbf{Z}[Z_1, \ldots, Z_s]$ with* Deg $\beta \leqq B$ *and $L(\beta) \leqq \mathscr{B}$, and let H and $\mathscr{H}$ be the maximum of the Degrees and the lengths respectively of the coefficients of F. Then all solutions x_1, x_2 in $\mathbf{Z}[Z_1, \ldots, Z_s]$ of* (2.10) *satisfy* (2.12) *and*

$$\max\{L(x_1), L(x_2)\} < \exp\left\{(2n)^{10^{8n^3}}\big(4(6C_2+1)^{s(3C_2)^s}\mathscr{H}\big)^{10^{n^4}}(1+\log \mathscr{B})\right\}$$

with the above C_2 and with the choice $\delta = n!$ in C_2.

This implies that (under the assumptions of Corollary 2.3) (2.10) has only finitely many solutions in $\mathbf{Z}[Z_1, \ldots, Z_s]$ and all these can be effectively determined. Apart from the form of the bounds, Baker's well-known theorem [2] on Thue equations corresponds to the case $s=0$ of our Corollary 2.3.

---

* *Added in proof (July 29, 1983).* Recently R. C. Mason (*J. London Math. Soc.*, **24** (1981), 414—426) gave an algorithm for determining all the solutions of (2.10) in x_1, x_2 ($\in K$) integral over $k[Z]$.

[11] Clearly $1 \leqq \delta \leqq n!$.

§ 2.4. Bounds for the solutions of norm form equations

We give now some applications of Theorem 1 to norm form equations.

Let k, k', K_0, R, K and R' be as in §2.1, with the parameters specified in § 2.2. Further, let $\alpha_1=1, \alpha_2, \ldots, \alpha_m$ be algebraic and linearly independent[12] elements of K over K_0 with $m\geqq 2$, and assume that $M=K_0(\alpha_1, \ldots, \alpha_m)$ is of degree $n(\geqq 3)$ over K_0. There are n K_0-isomorphisms of M into K; denote the images of α_j under these isomorphisms by $\alpha_j^{(1)}, \ldots, \alpha_j^{(n)}$. Let $l(\mathbf{X})$ denote the linear form $\alpha_1 X_1+\ldots+\alpha_m X_m$, and put $l^{(i)}(\mathbf{X})=\alpha_1^{(i)}X_1+\ldots+\alpha_m^{(i)}X_m$. Then

$$N(\alpha_1 X_1+\ldots+\alpha_m X_m) = \prod_{i=1}^{n} l^{(i)}(\mathbf{X})$$

is a decomposable form of degree n with coefficients in K_0. Such a form is called a *norm form* over K_0. Consider the solutions $\mathbf{x}=(x_1, \ldots, x_m)\in R^m$ of the *norm form equation*[13]

$$N(\alpha_1 x_1+\ldots+\alpha_m x_m) = \beta \tag{2.14}$$

where β is a non-zero element of K_0.

In case of number fields (i.e. when $k=\mathbf{Z}$ and K_0 and K are number fields), there is an extensive literature of norm form equations (see [59], [5], [48], [4], [19], [53] and the references mentioned there). When $K_0=\mathbf{Q}$, Schmidt [47] obtained a general criterion for (2.14) to have only finitely many solutions in $R=\mathbf{Z}$. This result was generalized by Schlickewei [46] to the case of an arbitrary subring R of $\mathbf{Q}$ of finite type over $\mathbf{Z}$. As it is known, these deep results are ineffective.

In the case when K_0 is an arbitrary number field, Győry and Papp [25], [27], Győry [16], [18], [19], [20], [21], [23] and Kotov [30], [31] obtained general and effective finiteness theorems concerning (2.14). In [22], we studied the equation (2.14) in the above more general situation. We showed [22] that under certain assumptions concerning $\alpha_1, \ldots, \alpha_m$, (2.14) has only finitely many solutions in R. As a consequence of our Theorem 1, we make now these results of [22] effective in the sense described in § 2.1.

Returning to the general situation discussed in §§ 2.1 and 2.2 and keeping the notations established there, we suppose that $\alpha_1, \ldots, \alpha_m$ as well as their conjugates over K_0 have Degrees$\leqq A$ and, when $k=\mathbf{Z}$, have lengths$\leqq \mathscr{A}$ (with respect to the generating set $z_1, \ldots, z_s, y$ of K; cf. § 2.1). Further, suppose Deg $\beta\leqq B$ and, in the case $k=\mathbf{Z}$, $L(\beta)\leqq \mathscr{B}$.

THEOREM 3. *Suppose that in* (2.14) α_{i+1} *is of degree at least* 3 *over* $K_0(\alpha_1, \ldots, \alpha_i)$ *for* $i=1, \ldots, m-1$. *Then all solutions* $x_1, \ldots, x_m$ *of* (2.14) *in* R' *satisfy* (2.8). *Further, when* $k=\mathbf{Z}$, (2.9) *also holds for every solution.*

---

[12] As it is known, the case when in (2.14) $\alpha_1, \ldots, \alpha_m$ are linearly dependent over K_0 can be easily reduced to this one.

[13] To avoid confusion we note that by a solution of (2.14) we mean an $\mathbf{x}=(x_1, \ldots, x_m)\in R^m$ for which $l^{(1)}(\mathbf{x}) \ldots l^{(n)}(\mathbf{x})=\beta$.

If we restrict ourselves to those solutions of (2.14) for which $x_m \neq 0$, the assumptions of Theorem 3 can be weakened. We recall that in (2.14), by assumption, $\alpha_1, \ldots, \alpha_m$ are linearly independent over K_0.

THEOREM 4. *Suppose that in* (2.14) α_m *is of degree* $\geqq 3$ *over* $K_0(\alpha_1, \ldots, \alpha_{m-1})$. *Then all solutions* $x_1, \ldots, x_m$ *of* (2.14) *in* R' *with* $x_m \neq 0$ *satisfy* (2.8). *Further, in case* $k=\mathbf{Z}$ (2.9) *also holds for all these solutions.*

We remark that in the absolute case (2.14) is effectively given if K_0, R, K, $\alpha_1, \ldots, \alpha_m$ and β are effectively given (cf. Remark 2 in § 2.1). If this is the case, then the equation (2.14) (under the assumptions of Theorem 3 or 4) has only finitely many solutions in R' (with $x_m \neq 0$ in case of Theorem 4) and all these can be effectively determined. In particular, all the solutions can be found in $\mathbf{Z}[z_1, \ldots, z_s]$.

It is easily seen that the assertion of Theorems 3 and 4 does not remain valid in general if we lower the bound 3 concerning the degrees of the α_j. Further, under the assumptions of Theorem 4, (2.14) may have infinitely many solutions $x_1, \ldots, x_m$ with $x_m = 0$.

We note that the above-mentioned effective results of [25], [16], [18], [19], [20], [21], [23], [30] and [31] can be deduced (with other estimates) from our Theorems 3 and 4 (following the argument used in the deduction of Corollary 2.1 from Theorem 2).

Consider now the equation (2.14) in the special case when k is a field of characteristic 0, $R=k[Z_1, \ldots, Z_s]$ is a polynomial ring over k in the variables $Z_1, \ldots, Z_s$ and $K_0=k(Z_1, \ldots, Z_s)$. Let $\overline{K}_0$ be a fixed algebraic closure of K_0. When $\alpha \in \overline{K}_0$, we denote by $d(\alpha)$ the maximum Degree of the coefficients of the defining polynomial of α over R. The following corollary is a consequence of Theorem 3.

COROLLARY 3.1. *Suppose that in* (2.14) $\alpha_1, \ldots, \alpha_m$ *satisfy the conditions of Theorem* 3, *and that* $d(\beta) \leqq B'$, $d(\alpha_j) \leqq A'$, $j=1, \ldots, m$. *Then all solutions* $x_1, \ldots, x_m$ *of* (2.14) *in* $k[Z_1, \ldots, Z_s]$ *have Degrees not exceeding*

$$s^3(m+1)!(2n!)^{5n!}(A'+B'+1).$$

It is clear that Theorem 4 has a similar consequence.

We note that in the case $k=\mathbf{Z}$ we could deduce similar results from Theorems 3 and 4, providing explicit bounds for both the Degrees and the lengths of the solutions $x_1, \ldots, x_m$ of (2.14) in the polynomial ring $\mathbf{Z}[Z_1, \ldots, Z_s]$.

§ 2.5. Bounds for the solutions of discriminant form and index form equations

From Theorem 1 we shall now derive explicit bounds for the solutions of discriminant form and index form equations.

Let k, k', K_0, R, K and R' be as in § 2.1 with the parameters specified in § 2.2. Further, let $1, \alpha_1, \ldots, \alpha_m$ be algebraic and linearly independent[14] elements of K

[14] The case when $1, \alpha_1, \ldots, \alpha_m$ are linearly dependent in the equation (2.15) can be easily reduced to this one.

over K_0 with $m \geqq 2$. Assume that $M = K_0(\alpha_1, \ldots, \alpha_m)$ is of degree $n (\geqq 3)$ over K_0. Using the notation of the preceding section, consider the linear forms $l^{(i)}(\mathbf{X}) = X_0 + \alpha_1^{(i)} X_1 + \ldots + \alpha_m^{(i)} X_m$, $i = 1, \ldots, n$. Then

$$D(\alpha_1 X_1 + \ldots + \alpha_m X_m) = \prod_{1 \leqq i < j \leqq n} (l^{(j)}(\mathbf{X}) - l^{(i)}(\mathbf{X}))^2$$

is a decomposable form of degree $n(n-1)$ in $X_1, \ldots, X_m$ with coefficients in K_0. Such a form is said to be a *discriminant form* over K_0. We are interested in the solutions $x_1, \ldots, x_m \in R$ of the *discriminant form equation*

$$D(\alpha_1 x_1 + \ldots + \alpha_m x_m) = \beta \tag{2.15}$$

where β is a non-zero element of K_0.

Let S and T be subrings of M containing 1 such that $S \subset T$. Suppose that K_0 is the quotient field of S, and that T, as an S-module, has a basis of the form $\omega_1 = 1, \omega_2, \ldots, \omega_n$. This holds for example in the following important special cases:

(i) $K_0 = \mathbf{Q}$, $S = \mathbf{Z}$, T is the ring of integers of M;

(ii) $K_0 =$ the rational function field $k(z)$, $S = k[z]$, T is the ring of integral functions of M (with respect to z);

(iii) S is integrally closed and $T = S[\alpha]$ with an integral element α over S.

Further examples and references can be found e.g. in [72] and [41].

It is easily verified that

$$D(\omega_2 X_2 + \ldots + \omega_n X_n) = [F(X_2, \ldots, X_n)]^2 D(1, \omega_2, \ldots, \omega_n) \tag{2.16}$$

where $D(1, \omega_2, \ldots, \omega_n)$ denotes the discriminant of the basis $\{1, \omega_2, \ldots, \omega_n\}$ over K_0, and $F(X_2, \ldots, X_n)$ is a decomposable form of degree $n(n-1)/2$ with coefficients in S. The form $F(X_2, \ldots, X_n)$ is called the *index form* of the basis $\{1, \omega_2, \ldots, \omega_n\}$ of T over S. Consider the solutions $x_2, \ldots, x_n$ in R of the *index form equation*

$$F(x_2, \ldots, x_n) = \beta. \tag{2.17}$$

Many problems can be reduced to discriminant form and index form equations (cf. [19]). In case of number fields the equations (2.15) and (2.17) have been studied by several authors. In the case $m \leqq 3$, $n \leqq 4$, $K_0 = \mathbf{Q}$, $S = \mathbf{Z}$ and $R = \mathbf{Z}$, among others Nagell [38], [39], [40], Delone [8], Delone and Faddeev [9], M. N. Gras [12], [13], Archinard [1] and Knight (cf. [49]) obtained results on the number of solutions of (2.15) and (2.17). For further references see [19]. In [15] we proved that if $K_0 = \mathbf{Q}$, $S = \mathbf{Z}$ and $R = \mathbf{Z}$, then both (2.15) and (2.17) have only finitely many solutions in $\mathbf{Z}$ and all these can be effectively determined. Certain effective generalizations have been established by Trelina [68], Győry and Papp [26] and Győry [18]. Recently, we further generalized [21], [23] these results to the case of arbitrary number fields K_0 and K.

In [22] we showed that if K_0 is a field of finite type over $\mathbf{Q}$, then both (2.15) and (2.17) have only finitely many solutions in any given integral domain R of finite type over $\mathbf{Z}$. As a consequence of Theorem 1, we make now these results of [22] effective in the sense specified in § 2.1.

Under the above assumptions, suppose that in (2.15) (resp. in (2.17)) $\alpha_1, \ldots, \alpha_m$ (resp. $\omega_1, \ldots, \omega_n$) and their conjugates over K_0 have Degrees $\leqq A$, and, in the absolute case, have lengths $\leqq \mathscr{A}$ (with respect to the generating set $z_1, \ldots, z_s, y$ of K). Let Deg $\beta \leqq B$ and, in case $k=\mathbf{Z}$, $L(\beta) \leqq \mathscr{B}$.

THEOREM 5. *All solutions* $x_1, \ldots, x_m$ *of* (2.15) *in* R' *satisfy* (2.8) *and, in case* $k=\mathbf{Z}$, (2.9) *with* $n(n-1)$, $2A$ *and* $2\mathscr{A}^2 \exp\{3sA\}$ *in place of* n, A *and* $\mathscr{A}$, *respectively.*

The following theorem is an easy consequence of Theorem 5.

THEOREM 6. *All solutions* $x_2, \ldots, x_n$ *of* (2.17) *in* R' *satisfy* (2.8) *and, when* $k=\mathbf{Z}$, (2.9) *with* $n-1$, $n(n-1)$, $2A$, $2(n+1)!(A+B+\delta D)$, $2\mathscr{A}^2 \exp\{3sA\}$ *and* $L(D(\omega_1, \ldots, \omega_n)\beta^2)$ *in place of* m, n, A, B, $\mathscr{A}$ *and* $\mathscr{B}$, *respectively.*

We remark that $L(D(\omega_1, \ldots, \omega_n)\beta^2)$ can be estimated from above in terms of the given parameters (using e.g. Lemma 4.2.5 of [69]), but the bound obtained in this way would be rather complicated.

If $k=\mathbf{Z}$ and if K_0, R, K, β and $\alpha_1, \ldots, \alpha_m$ (resp. $\omega_1, \ldots, \omega_n$) are effectively given, then (2.15) (resp. (2.17)) is effectively given (cf. § 2.1). Then, under the hypotheses of Theorem 5 (resp. of Theorem 6) the equation (2.15) (resp. (2.17)) has only finitely many solutions in R' and these can be effectively determined. In particular, all the solutions in $\mathbf{Z}[z_1, \ldots, z_s]$ can be found.

We note that if in particular K_0 and K are number fields, from Theorems 5 and 6 it is easy to deduce (with other estimates) the above-mentioned effective results obtained in the number field case.

In the special case when k is a field of characteristic 0, $R=k[Z_1, \ldots, Z_s]$ is a polynomial ring over k in $Z_1, \ldots, Z_s$ and $K_0=k(Z_1, \ldots, Z_s)$, Theorems 5 and 6 have a consequence similar to Corollary 3.1. We now present a consequence of Theorem 6. Let M be an algebraic extension of degree $n \geqq 3$ of $K_0=k(Z_1, \ldots, Z_s)$, and let T be an integral extension ring of R in M. Suppose that T, as an R-module, has a basis $\{1, \omega_2, \ldots, \omega_n\}$ with $d(\omega_i) \leqq A'$ $(i=1, \ldots, n)$ and that $0 \neq \beta \in R$ has Degree $\leqq B'$.

COROLLARY 6.1. *With the above notations and assumptions, all solutions* $x_2, \ldots, x_n$ *of* (2.17) *in* $k[Z_1, \ldots, Z_s]$ *have Degrees not exceeding*

$$2 \cdot 10^4 s^3 ((n+1)!)^{6n!}(A'+B'+1).$$

In the absolute case similar corollaries can be obtained from Theorems 5 and 6, with explicit bounds for the Degrees and the lengths of the solutions.

Our Theorems 5 and 6 have a number of applications, among others to integral elements of given discriminants and to simple ring extensions over finitely generated integral domains. Several results of [22] and [24] can be deduced (with other bounds) from Theorems 5 and 6 of the present paper. However, the more direct deductions of [24] yields better estimates.

3. Proofs

§ 3.1. Preliminary remarks concerning function fields of one variable

In §§ 3.1 and 3.2, we shall use the notations of Schmidt's paper [51].

Let k be an algebraically closed field of characteristic 0, and let K be a function field of transcendence degree 1 over k with genus g. For $x\neq 0$ in K, the (additive) height $\mathbf{H}_K(x)$ of x over K is defined by

$$\mathbf{H}_K(x) = -\sum_v \min\{0, v(x)\},$$

where v runs through the valuations of K/k with value group $\mathbf{Z}$. It is clear that $\mathbf{H}_K(x)=0$ if and only if $x\in k$. Using the additive version $\sum_v v(x)=0$ of the well-known product formula, it is easy to prove that the height has the properties

$$\mathbf{H}_K\left(\frac{1}{x}\right) = \mathbf{H}_K(x), \quad \mathbf{H}_K(x^m) = m\mathbf{H}_K(x),$$

$$\mathbf{H}_K(x+y) \leqq \mathbf{H}_K(x)+\mathbf{H}_K(y), \quad \mathbf{H}_K(xy) \leqq \mathbf{H}_K(x)+\mathbf{H}_K(y)$$

for all non-zero x, y in K and for every positive integer m.

Suppose now that we are given an element $z\in K$ with $z\notin k$. Then z is transcendental over k, and K is an algebraic extension of the rational function field $k(z)$. All valuations of $k(z)/k$ with value group $\mathbf{Z}$ are known (see e.g. [10], § 4). The valuation $u_\infty=-\deg$ of $k(z)/k$ with $u_\infty(z)=-1$ is called the "infinite valuation". The "finite valuations" of $k(z)/k$ are determined by the monic linear polynomials of $k[z]$. Namely, if the finite valuation u is associated with the monic linear polynomial $p(z)\in k[z]$, then $x=(p(z))^{u(x)}g(z)/h(z)$ for each non-zero $x\in k(z)$ where $g, h\in k[z]$ are relatively prime to $p(z)$.

Every valuation v of K/k is the extension of some valuation u of $k(z)/k$. In this case we write $v|u$. Extensions of u_∞ to K are called "infinite valuations". The other valuations of K/k are said to be "finite".

If $v|u$, denote the ramification index of v over u by e_v. Since by assumption the value group of v is $\mathbf{Z}$, hence $v(x)=e_v u(x)$ for $x\in k(z)$. Let Δ denote the degree of K over $k(z)$. Then it is well-known that $\sum_{v|u} e_v=\Delta$ and so

$$\sum_{v|u} v(x) = \Delta u(x) \tag{3.1}$$

if $x\in k(z)$. In this case we may form the height $\mathbf{H}_{k(z)}(x)$ defined over $k(z)$, and by virtue of (3.1) we obtain

$$\mathbf{H}_K(x) = \Delta\mathbf{H}_{k(z)}(x). \tag{3.2}$$

Further, if x is a non-zero element of $k(z)$ and $x=g(z)/h(z)$ with coprime polynomials $g, h\in k[z]$, then it is easily verified that

$$\mathbf{H}_{k(z)}(x) = \max(\deg g, \deg h). \tag{3.3}$$

Finally, let $f(X)=f_0X^d+\ldots+f_d=f_0(X-\alpha_1)\ldots(X-\alpha_d)$ be a polynomial, and suppose that $f_0, \alpha_1, \ldots, \alpha_d \in K$. Then we have (see (2.3) in [51])

$$\mathbf{H}_K(\alpha_i) \leqq -\sum_v \min\{v(f_0), \ldots, v(f_d)\}, \quad i = 1, \ldots, d. \tag{3.4}$$

Further, we shall need the following

LEMMA 1 (W. M. Schmidt). *Let $f(X)$ be as above. Suppose that K is the splitting field of f over $k(z)$, and that the coefficients of f are polynomials in z (they lie in $k[z]$). Then K/k has genus*

$$g \leqq (\Delta-1)d \max_{0\leqq i\leqq d} \deg f_i.$$

PROOF. This is Lemma H in [51].

§ 3.2. Schmidt's theorem on Thue equations and its application to the equation (3.7)

Let k and K be defined as in § 3.1, and suppose that K/k has genus g. Let $F(X, Y)$ be a binary form of degree n with coefficients in K, and let H be the maximum of the heights (over K) of the coefficients of F. Consider the Thue equation

$$F(x, y) = 1 \tag{3.5}$$

in $x, y \in K$.

LEMMA 2 (W. M. Schmidt [51]). *If $n \geqq 5$, then every solution $(x, y) \in K^2$ of* (3.5) *satisfies*

$$\max\{\mathbf{H}_K(x), \mathbf{H}_K(y)\} \leqq 89H+211g. \tag{3.6}$$

This important theorem is stated in [51] in a slightly different form, but it is easily seen that (3.6) is a simple consequence of Part (i) of Theorem 1 of Schmidt [51].

Let S be a finite set of valuations of K/k (with value group $\mathbf{Z}$). An element α of K is called S-integral if $v(\alpha) \geqq 0$ for every $v \notin S$. All S-integral elements of K form a ring which will be called the ring of S-integers of K. Let $|S|$ denote the cardinality of S.

Let $\lambda_1, \lambda_2, \lambda_3$ and β be non-zero S-integral elements in K. Consider the equation

$$\lambda_1\beta_1+\lambda_2\beta_2+\lambda_3\beta_3 = 0 \tag{3.7}$$

in S-integers $\beta_1, \beta_2, \beta_3$ of K. In proving Theorem 1 in the relative case, we shall use the following consequence of Schmidt's theorem quoted above.

LEMMA 3. *If $\beta_1, \beta_2, \beta_3$ is a solution of* (3.7) *in S-integers of K such that β_i divides β in the ring of S-integers for $i=1, 2, 3$, then*

$$\beta_i = \varepsilon\gamma_i, \quad i = 1, 2, 3, \tag{3.8}$$

with some ε and $\gamma_i \in K$ satisfying

$$\mathbf{H}_K(\gamma_i) \leqq 1781\left(\mathbf{H}_K(\beta)+|S|+\max_{1\leqq i\leqq 3} \mathbf{H}_K(\lambda_i)\right)+2836g. \tag{3.9}$$

An important special case of Lemma 3 is when $\beta \in k$, i.e. when $\beta_1, \beta_2, \beta_3$ are S-units of K (in other words $v(\beta_i)=0$ for every $v \notin S$). In this case (3.7) implies (3.8) and (3.9) with $\mathbf{H}_K(\beta)=0$.

PROOF OF LEMMA 3. Let $\beta_1, \beta_2, \beta_3$ be an arbitrary but fixed solution of (3.7) having the property that β_i divides β in the ring of S-integers ($i=1, 2, 3$). We shall first prove that there are ε_i, δ_i ($i=1, 2, 3$) in K with

$$\beta_i = \varepsilon_i^5 \delta_i \tag{3.10}$$

and

$$\mathbf{H}_K(\delta_i) \leqq \mathbf{H}_K(\beta) + |S| + g \tag{3.11}$$

for $i=1, 2, 3$. To prove this, we use the argument of Lemma L of Schmidt [51]. Take some valuation $v_0 \notin S$. We shall find ε_i with the property

$$v(\varepsilon_i) \geqq \frac{1}{5} v(\beta_i) \quad \text{for} \quad v \neq v_0, \tag{3.12}$$

$$v_0(\varepsilon_i) \geqq \frac{1}{5} v_0(\beta_i) - \mathbf{H}_K(\beta) - |S| - g. \tag{3.13}$$

Since by assumption β and β_i are S-integral and β_i divides β in the ring of S-integers, hence we have

$$0 \leqq \sum_{v \notin S} v(\beta_i) \leqq \sum_{v \notin S} v(\beta) \leqq \mathbf{H}_K(\beta), \quad i = 1, 2, 3.$$

Consequently, the number of v with $v(\beta_i) \neq 0$ is at most $|S| + \mathbf{H}_K(\beta)$. Replacing the right-hand sides of (3.12) and (3.13) by the next largest integers, in view of the sum formula we get a sum $\leqq -g$. So, by Riemann's theorem (see e.g. [10]) there exists indeed a non-zero ε_i in K satisfying (3.12) and (3.13). Let now δ_i be defined by (3.10). Then we have

$$\mathbf{H}_K(\delta_i) = \mathbf{H}_K(1/\delta_i) = \mathbf{H}_K(\varepsilon_i^5/\beta_i) = -\sum_v \min\{0, 5v(\varepsilon_i) - v(\beta_i)\} =$$

$$= -\min\{0, 5v_0(\varepsilon_i) - v_0(\beta_i)\} \leqq \mathbf{H}_K(\beta) + |S| + g.$$

From (3.7) we obtain

$$\left(-\frac{\lambda_1 \delta_1}{\lambda_3 \delta_3}\right)(\varepsilon_1/\varepsilon_3)^5 + \left(-\frac{\lambda_2 \delta_2}{\lambda_3 \delta_3}\right)(\varepsilon_2/\varepsilon_3)^5 = 1.$$

This is a Thue equation over K in the variables $\varepsilon_1/\varepsilon_3$, $\varepsilon_2/\varepsilon_3$. By virtue of (3.11) the sum of the height of the coefficients of this equation does not exceed

$$H = 4\left(\mathbf{H}_K(\beta) + |S| + g + \max_{1 \leqq i \leqq 3} \mathbf{H}_K(\lambda_i)\right).$$

Therefore, by Lemma 2 we have

$$\max\{\mathbf{H}_K(\varepsilon_1/\varepsilon_3), \mathbf{H}_K(\varepsilon_2/\varepsilon_3)\} \leqq 89H + 211g.$$

Writing $\varepsilon_i' = \varepsilon_i/\varepsilon_3$ and $\gamma_i = \varepsilon_i'^5 \delta_i$, we get $\beta_i = \varepsilon_3^5 \gamma_i$, $i=1, 2, 3$, where

$$\mathbf{H}_K(\gamma_i) \leqq 1781\left(\mathbf{H}_K(\beta) + |S| + \max_{1 \leqq i \leqq 3} \mathbf{H}_K(\lambda_i)\right) + 2837g$$

and this proves (3.9).

§ 3.3. Proof of the estimate (2.8) of Theorem 1

Since the absolute case can easily be reduced to the relative one, it suffices to prove (2.8) in the case when k is a field of characteristic 0. Further, the case $s=0$ being trivial, we suppose $s>0$.

Consider the representation of the form (2.4) of β and of the coefficients of the linear factors l_j. Let a_1 be the product of a and the denominators of these elements (in their representations of the form (2.4)). It is clear that

$$\text{Deg } a_1 \leqq \text{Deg } a + mnA + B = c_1. \tag{3.14}$$

Writing

$$R'' = k\left[z_1, \ldots, z_s, \frac{1}{a_1}, y\right],$$

we have $R'' \supseteq R'$. Further, β and the coefficients of the l_j lie in R''.

Let us fix an index i with $1 \leqq i \leqq s$. Let $k_i = k(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_s)$, and $\bar{k}_i$ its algebraic closure. In what follows, we write z for z_i and K_i for $\bar{k}_i(z)$. Then $k(z_1, \ldots, z_s) \subseteq K_i$. If $\omega \in k[z_1, \ldots, z_s]$, denote by ω' the element of $k_i[z]$ obtained from ω by substituting z for z_i.

We recall that the minimal polynomial of y over $k(z_1, \ldots, z_s)$ is $f(X) = X^\delta + f_1 X^{\delta-1} + \ldots + f_\delta$ with coefficients in $k[z_1, \ldots, z_s]$. Let $y^{(1)} = y, \ldots, y^{(\delta)}$ be the conjugates of y over $k(z_1, \ldots, z_s)$. Further let M_i be the splitting field of the polynomial $X^\delta + f_1' X^{\delta-1} + \ldots + f_\delta'$ over $\bar{k}_i(z)$, i.e. let $M_i = K_i(y^{(1)}, \ldots, y^{(\delta)})$. The subring

$$R_i'' = \bar{k}_i\left[z, \frac{1}{a_1'}, y^{(1)}, \ldots, y^{(\delta)}\right]$$

of M_i contains R''. Put $\Delta_i = [M_i : K_i]$. Clearly $\Delta_i \leqq \delta!$. Denote by g_{M_i} the genus of $M_i/\bar{k}_i$.

Let S denote the subset of valuations of $M_i/\bar{k}_i$ (with value group $\mathbf{Z}$) consisting of the infinite valuations and of those finite valuations v for which $v(a_1') \neq 0$. Then for the cardinality $|S|$ of S we have

$$|S| \leqq \Delta_i + \Delta_i \deg_z a_1' \leqq \Delta_i(1 + \text{Deg } a_1) \leqq \Delta_i(1 + c_1) = c_2. \tag{3.15}$$

Since $y^{(1)}, \ldots, y^{(\delta)}$ are integral over $k[z_1, \ldots, z_s]$, they are integral also over $\bar{k}_i[z]$. From this follows that $v(y^{(j)}) \geqq 0$ for each j and for every finite valuation v of $M_i/\bar{k}_i$. This shows that every element of R_i'' is S-integral.

Any element α of K has a representation of the form

$$\alpha = \frac{P_0 + P_1 y + \ldots + P_{\delta-1} y^{\delta-1}}{Q}$$

with relatively prime polynomials $P_0, \ldots, P_{\delta-1}, Q$ from $k[z_1, \ldots, z_s]$. Let $\alpha^{(j)}$ denote the conjugate of α over $k(z_1, \ldots, z_s)$ corresponding to $y^{(j)}$. Then

$$\alpha^{(j)} = \frac{P_0' + P_1' y^{(j)} + \ldots + P_{\delta-1}' (y^{(j)})^{\delta-1}}{Q'}$$

lies in M_i for each j with $1 \leqq j \leqq \delta$. This shows that the Degree of α with respect to the generating set $z_1, \ldots, z_s, y$ of K over k coincides with that of $\alpha^{(j)}$ with respect to the generating set $z_1, \ldots, z_s, y^{(j)}$ of K over k. By (3.4) we have

$$\mathbf{H}_{M_i}(y^{(j)}) \leqq -\sum_v \min\{v(f_1'), \ldots, v(f_\delta')\}$$

where the sum is taken over all valuations v of M_i with value group $\mathbf{Z}$. So, using (3.1) we get

$$\mathbf{H}_{M_i}(y^{(j)}) \leqq \delta \Delta_i \max_{1 \leqq l \leqq \delta} \deg_z f_l' \leqq \delta \Delta_i \max_{1 \leqq l \leqq \delta} \operatorname{Deg} f_l \leqq \delta \Delta_i \mathrm{D} = c_3. \tag{3.16}$$

Further, by (3.2) and (3.3) we obtain

$$\mathbf{H}_{M_i}(\alpha^{(j)}) \leqq \sum_{l=0}^{\delta-1} \mathbf{H}_{M_i}\left(\frac{P_l'}{Q'}(y^{(j)})^l\right) \leqq \sum_{l=0}^{\delta-1} \mathbf{H}_{M_i}\left(\frac{P_l'}{Q'}\right) + \tag{3.17}$$

$$+ \sum_{l=0}^{\delta-1} \mathbf{H}_{M_i}((y^{(j)})^l) \leqq \delta \Delta_i \operatorname{Deg} \alpha + \frac{(\delta-1)\delta}{2} c_3$$

for $j=1, \ldots, \delta$.

Let $\mathbf{x}=(x_1, \ldots, x_m)$ be an arbitrary but fixed solution of (1) in R'^m, and let $l_j(\mathbf{x})=\beta_j$. Since the coefficients of l_j lie in R'', hence $\beta_j \in R''$ and so β_j is S-integral in M_i for each j. Clearly β is also S-integral. Further, from (1) we get

$$\beta_1 \ldots \beta_n = \beta. \tag{3.18}$$

We shall give an explicit bound for $\max_{1 \leqq r \leqq m} \mathbf{H}_{M_i}(x_r)$. In deriving this bound, we shall use Lemmas 1 and 3 and certain arguments of [16] and [21]. In fact, our graph method (cf. [17], [21], [22]) will be utilized without using graph terminology.

For a fixed integer j, $1 \leqq j \leqq h$, let $\mathscr{I}_j$ denote the set of indices t satisfying $l_t \in \mathscr{L}_j$. We may suppose without loss of generality that $j \in \mathscr{I}_j$. Let $t \in \mathscr{I}_j \setminus \{j\}$. By assumption $\mathscr{L}_j$ is connected, i.e. there is a sequence $l_j=l_{t_1}, \ldots, l_{t_f}=l_t$ in $\mathscr{L}_j$ such that, for each e with $1 \leqq e \leqq f-1$, we have

$$\lambda'_{t_e} l_{t_e} + \lambda''_{t_{e+1}} l_{t_{e+1}} = \lambda_{t_e, e+1} l_{t_e, e+1} \tag{3.19}$$

with some $l_{t_e, e+1} \in \mathscr{L}_j$ and with $\lambda'_{t_e}, \lambda''_{t_{e+1}}, \lambda_{t_e, e+1} \in K \setminus \{0\}$. Using (2.5) and (2.6) it is easily verified that λ'_{t_e}, $\lambda''_{t_{e+1}}$ and $\lambda_{t_e, e+1}$ can be chosen so that their Degrees do not exceed $4A+2(\delta-1)D$. It is clear that the sequence $l_j=l_{t_1}, \ldots, l_{t_f}$ depends on l_t. However, we can choose these sequences so that l_{t_2} is the same linear form for each $l_t \in \mathscr{L}_j$ with $t \neq j$, and that the number of the necessary λ's is at most $3n$. Then $f \leqq n+1$.

From (3.19) we get

$$\lambda'_{t_e} \beta_{t_e} + \lambda''_{t_{e+1}} \beta_{t_{e+1}} = \lambda_{t_e, e+1} \beta_{t_e, e+1}, \quad e = 1, \ldots, f-1.$$

Further, in view of (3.18) β_{t_e}, $\beta_{t_{e+1}}$ and $\beta_{t_e, e+1}$ divides β in the ring of S-integers. We can now apply Lemma 3 and we obtain

$$\beta_{t_e} = \varepsilon_{t_e} \gamma'_{t_e}, \quad \beta_{t_{e+1}} = \varepsilon_{t_e} \gamma''_{t_{e+1}}, \quad e = 1, \ldots, f-1, \tag{3.20}$$

where $\varepsilon_{t_e}, \gamma'_{t_e}, \gamma''_{t_{e+1}} \in M_i$ with

$$\max\{\mathbf{H}_{M_i}(\gamma'_{t_e}), \mathbf{H}_{M_i}(\gamma''_{t_{e+1}})\} \leqq 1781\big(\mathbf{H}_{M_i}(\beta)+|S|+ \tag{3.21}$$

$$+\max\{\mathbf{H}_{M_i}(\lambda'_{t_e}), \mathbf{H}_{M_i}(\lambda''_{t_{e+1}}), \mathbf{H}_{M_i}(\lambda_{t_e, e+1})\}\big)+2836 g_{M_i}.$$

Using Lemma 1, (3.15) and (3.17), we get

$$1781\big(\delta\Delta_i(B+4A)+c_2\big)+\big(1781\delta(\delta+1)-726\big)c_3 = c_4$$

as an upper bound for the right-hand side of (3.21). It follows from (3.20) that

$$\beta_t = \beta_{t_f} = \varepsilon_j \delta_t \tag{3.22}$$

where $\varepsilon_j = \varepsilon_{t_1}$ has the same value for each $t \neq j$ with $l_t \in \mathscr{L}_j$, $\delta_j = \gamma'_{t_1}$ and

$$\delta_t = \frac{\gamma''_{t_2} \cdots \gamma''_{t_f}}{\gamma'_{t_2} \cdots \gamma'_{t_{f-1}}}$$

for $t \neq j$ with $l_t \in \mathscr{L}_j$. By (3.21) we have

$$\mathbf{H}_{M_i}(\delta_t) \leqq (2n-1)c_4 = c_5 \tag{3.23}$$

for all t with $l_t \in \mathscr{L}_j$.

Since $l_t(\mathbf{x}) = \beta_t$, by (3.22) we have

$$l_t(\mathbf{x}) = \varepsilon_j \delta_t \text{ for all } t \text{ with } l_t \in \mathscr{L}_j. \tag{3.24}$$

But by assumption the variable X_q can be expressed in the form

$$X_q = \sum_{t, l_t \in \mathscr{L}_j} \tau_{tj} l_t$$

with $\tau_{tj} \in K$. It is easily seen that the coefficients τ_{tj} can be chosen so that $\tau_{tj} = \tau'_{tj}/\tau_j$ and

$$\max_{t, l_t \in \mathscr{L}_j} \{\operatorname{Deg} \tau'_{tj}, \operatorname{Deg} \tau_j\} \leqq m!\big(mA+(m-1)(\delta-1)D\big), \tag{3.25}$$

whence, by (3.17) we get

$$\mathbf{H}_{M_i}(\tau_{tj}) \leqq 2m \cdot m!\big(\delta\Delta_i A+(\delta-1)\delta c_3\big).$$

Thus, with the notation

$$\mu_{jq} = \sum_{t, l_t \in \mathscr{L}_j} \tau_{tj} \delta_t$$

we have (for the component x_q of the solution $\mathbf{x}$ considered above)

$$x_q = \varepsilon_j \mu_{jq} \tag{3.26}$$

and

$$\mathbf{H}_{M_i}(\mu_{jq}) \leqq 2m^2 \cdot m!\big(\delta\Delta_i A+(\delta-1)\delta c_3\big)+mc_5 = c_6, \quad j = 1, \ldots, h. \tag{3.27}$$

(3.26) implies $\varepsilon_j = \varepsilon_1 \mu_{1q}/\mu_{jq}$ for $j=1, \ldots, h$. So, it follows from (3.22) $\beta_t = \varepsilon_1 \varrho_t$, $t=1, \ldots, n$, where $\varrho_t = \delta_t \mu_{1q}/\mu_{jq}$ and, by (3.23) and (3.27), we have

$$\mathbf{H}_{M_i}(\varrho_t) \leqq c_5+2c_6 = c_7 \tag{3.28}$$

for $t=1, \ldots, n$. From (3.18) we get now $\varepsilon_1^n(\varrho_1 \ldots \varrho_n)=\beta$, whence we have

$$\mathbf{H}_{M_i}(\varepsilon_1) \leqq \frac{1}{n}\left(\delta \Delta_i (A+B)+(\delta-1)\delta c_3\right)+c_7$$

by (3.17) and (3.28). Thus

$$\mathbf{H}_{M_i}(\beta_t) \leqq \frac{1}{n}\left(\delta \Delta_i (A+B)+(\delta-1)\delta c_3\right)+2c_7 = c_8.$$

Regard

$$l_t(\mathbf{x}) = \beta_t, \quad t = 1, \ldots, n, \tag{3.29}$$

as a system of equations in $\mathbf{x}$. By hypothesis the rank of $\mathscr{L}$ is m, hence we can form a non-vanishing determinant $\mathscr{D}$ of order m from the coefficients of $l_1, \ldots, l_n$. Further, by Cramer's rule we get

$$\mathbf{H}_{M_i}(x_r) \leqq (2m-1)m!\left(\delta \Delta_i A + \frac{(\delta-1)\delta}{2} c_3\right)+mc_8 = c_9$$

for $r=1, \ldots, m$.

We recall that $y^{(1)}, \ldots, y^{(\delta)}$ denote the conjugates of y over $k(z_1, \ldots, z_s)$. Let $R'^{(j)}$, $R''^{(j)}$, $\beta^{(j)}$, $x_r^{(j)}$, $l_t^{(j)}$ and $F^{(j)}$ be the conjugates of R', R'', β, x_r, l_t and F (over $k(z_1, \ldots, z_s)$) corresponding to $y^{(j)}$, respectively. Then it is easily verified that the equation

$$F^{(j)}(x_1, \ldots, x_m) = \beta^{(j)} \tag{3.30}$$

satisfies all conditions of our Theorem 1 with R' replaced by $R'^{(j)}$. Clearly $M_i \supseteq K$ and $R_i'' \supset R''^{(j)}$. We can apply the above argument to the solutions $(x_1^{(j)}, \ldots, x_m^{(j)}) \in R'^{(j)}$ of the equation (3.30) and we get in the same way as above that

$$\mathbf{H}_{M_i}(x_r^{(j)}) \leqq c_9 \tag{3.31}$$

for all r and j with $1 \leqq r \leqq m$ and $1 \leqq j \leqq \delta$.

From (3.31) we shall now derive an explicit bound for $\max\limits_{1 \leqq r \leqq m} \operatorname{Deg} x_r$. By the assumption $x_r \in R'$ we can write

$$x_r^{(j)} = \frac{p_{0r}+p_{1r}y^{(j)}+\ldots+p_{\delta-1,r}(y^{(j)})^{\delta-1}}{q_r}, \quad 1 \leqq r \leqq m; \quad 1 \leqq j \leqq \delta,$$

where $p_{0r}, p_{1r}, \ldots, p_{\delta-1,r}, q_r \in k[z_1, \ldots, z_s]$ are relatively prime, and q_r is divisible only by the irreducible factors of the polynomial $a \in k[z_1, \ldots, z_s]$. Denote by $p'_{0r}, \ldots, p'_{\delta-1,r}, q'_r$ the elements of $\bar{k}_i[z]$ obtained from $p_{0r}, \ldots, p_{\delta-1,r}, q_r$ by substituting z for z_i. Then we have

$$x_r^{(j)} = \frac{p'_{0r}+p'_{1r}y^{(j)}+\ldots+p'_{\delta-1,r}(y^{(j)})^{\delta-1}}{q'_r}. \tag{3.32}$$

Since $y^{(1)}, \ldots, y^{(\delta)}$ are distinct, using (3.16) and (3.31) we obtain from (3.32)

$$\mathbf{H}_{M_i}\left(\frac{p'_{lr}}{q'_r}\right) \leqq (\delta-1)\delta \cdot \delta! c_3 + \delta c_9 = c_{10} \tag{3.33}$$

for each l and r.

Let v be an arbitrary finite valuation of $M_i/\bar{k}_i$ (with value group $\mathbf{Z}$) for which $v(q_r')\neq 0$. The polynomials $p_{0r}', \ldots, p_{\delta-1,r}', q_r'$ being relatively prime over $k[z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_s]$, there is an l with $0\leqq l\leqq \delta-1$ such that $v(p_{lr}')=0$. Hence, for fixed l and r, (3.33) implies

$$0 \leqq \sum v(q_r') = \sum v\left(\frac{q_r'}{p_{l,r}'}\right) \leqq \mathbf{H}_{M_i}\left(\frac{q_r'}{p_{l,r}'}\right) = \mathbf{H}_{M_i}\left(\frac{p_{l,r}'}{q_r'}\right) \leqq c_{10}$$

where the sum is taken over all finite valuations v for which $v(q_r')\neq 0$ and $v(p_{l,r}')=0$. Taking these inequalities for $l=0, \ldots, \delta-1$ and adding them we get

$$-\sum_{\substack{v \\ v\,\text{infinite}}} v(q_r') = \sum_{\substack{v \\ v\,\text{finite}}} v(q_r') \leqq \delta c_{10}. \tag{3.34}$$

Denoting by u_∞ the infinite valuation of $\bar{k}_i(z)/\bar{k}_i$, (3.1) and (3.34) give $-\Delta_i u_\infty(q_r')\leqq \leqq \delta c_{10}$, whence

$$\deg_z q_r' \leqq \delta c_{10}/\Delta_i. \tag{3.35}$$

Finally, using (3.33) we obtain

$$|\deg_z p_{l,r}' - \deg_z q_r'| = \left|u_\infty\left(\frac{p_{l,r}'}{q_r'}\right)\right| \leqq \left|v\left(\frac{p_{l,r}'}{q_r'}\right)\right| \leqq \mathbf{H}_{M_i}\left(\frac{p_{l,r}'}{q_r'}\right) \leqq c_{10}$$

for each $v|u_\infty$ and for each l with $0\leqq l\leqq \delta-1$. This together with (3.35) gives

$$\deg_z p_{l,r}' \leqq (1+\delta/\Delta_i)c_{10}.$$

Since this is true for $z=z_1, \ldots, z_s$ (i.e. for $i=1, \ldots, s$), we have

$$\max_{1\leqq r\leqq m} \operatorname{Deg} x_r \leqq s(1+\delta/\Delta_i)c_{10}$$

whence (2.8) easily follows.

§ 3.4. Specializations in the absolute case

The specializations of the following type will play an important role in the proof of the estimate (2.9) of Theorem 1. They will enable us to reduce the general case to the number field case.

In what follows, we keep the notations of § 2.1 established in the case $k=\mathbf{Z}$. Suppose $s>0$. Let $P\in\mathbf{Z}[z_1, \ldots, z_s]$ be a polynomial of degree less than d in each z_j where d is a positive integer. Then the polynomial in one variable

$$\mathscr{S}_P(t) = P(t, t^d, \ldots, t^{d^{s-1}})$$

has the same non-zero coefficients as P and $\deg \mathscr{S}_P(t)\leqq d^s-1$. If $Q\in\mathbf{Z}[z_1, \ldots, z_s]$ is another polynomial such that Q and $P\cdot Q$ are of degree $<d$ in each z_j, then $\mathscr{S}_{P+Q}=\mathscr{S}_P+\mathscr{S}_Q$ and $\mathscr{S}_{P\cdot Q}=\mathscr{S}_P\cdot\mathscr{S}_Q$.

Let y be defined as in § 2.1 with the minimal polynomial $f(X)=X^\delta+f_1X^{\delta-1}+\ldots$ $\ldots+f_\delta$ over $\mathbf{Q}(z_1, \ldots, z_s)$. The coefficients $f_1, \ldots, f_\delta$ and the discriminant D_f

of f are elements of $\mathbf{Z}[z_1, \ldots, z_s]$ and $D_f \neq 0$. We have

$$\operatorname{Deg} D_f \leqq (2\delta-2) \max_{1 \leqq l \leqq \delta} \operatorname{Deg} f_l = c_{11}.$$

Consider the ring

$$R'' = \mathbf{Z}\left[z_1, \ldots, z_s, \frac{1}{a_1}, y\right]$$

where $a_1 \in \mathbf{Z}[z_1, \ldots, z_s]$. If $d > \max\{c_{11}, \operatorname{Deg} a_1\}$, then $\mathscr{S}_{D_f}(t)$ and $\mathscr{S}_{a_1}(t)$ are non-vanishing polynomials of degree $\leqq d^s - 1$ in t. We note that $\mathscr{S}_{D_f}(t)$ is the discriminant of the polynomial $X^\delta + \mathscr{S}_{f_1}(t) X^{\delta-1} + \ldots + \mathscr{S}_{f_\delta}(t)$. Let t_0 be a rational integer such that $\mathscr{S}_{D_f}(t_0) \neq 0$ and $\mathscr{S}_{a_1}(t_0) \neq 0$, and let $y_1, \ldots, y_\delta$ be the roots of the polynomial $X^\delta + \mathscr{S}_{f_1}(t_0) X^{\delta-1} + \ldots + \mathscr{S}_{f_\delta}(t_0) \in \mathbf{Z}[X]$ in $\mathbf{C}$. Because of $\mathscr{S}_{D_f}(t_0) \neq 0$ the numbers $y_1, \ldots, y_\delta$ are distinct. Let T_{t_0} denote the algebraic number field generated by $y_1, \ldots, y_\delta$ over $\mathbf{Q}$. The substitution $z_1 \to t_0, \ldots, z_s \to t_0^{d^{s-1}}, y \to y_j$ defines a homomorphic mapping (or, in other words a specialization; cf. [72]) of the ring R'' into the number field T_{t_0}. Namely, if $\alpha \in R''$ and

$$\alpha = \frac{P_0 + P_1 y + \ldots + P_{\delta-1} y^{\delta-1}}{Q}$$

is its representation of the form (2.4) (with relatively prime polynomials $P_0, \ldots, P_{\delta-1}, Q \in \mathbf{Z}[z_1, \ldots, z_s]$), then in view of $\mathscr{S}_{a_1}(t_0) \neq 0$ we have $\mathscr{S}_Q(t_0) \neq 0$ and

$$\frac{\mathscr{S}_{P_0}(t_0) + \mathscr{S}_{P_1}(t_0) y_j + \ldots + \mathscr{S}_{P_{\delta-1}}(t_0) y_j^{\delta-1}}{\mathscr{S}_Q(t_0)} \tag{3.36}$$

is the image of α under this mapping. We shall denote this image by $\mathscr{S}_\alpha(t_0, j)$ or, if there is no danger of confusion, simply by α_j^*. It is clear that if in particular $\alpha \in \mathbf{Z}[z_1, \ldots, z_s]$ then $\mathscr{S}_\alpha(t_0, j)$ (resp. α_j^*) coincides with the $\mathscr{S}_\alpha(t_0)$ defined above.

Let $|\overline{x}|$ denote the maximum absolute value of the conjugates of an algebraic number x in $\mathbf{C}$. We remark that if x is an integer then

$$|\overline{x}| \leqq H(x) + 1.$$

Further, if x' is another algebraic integer and $\mathbf{Q}(x, x')$ is of degree n over $\mathbf{Q}$, then we have (see e.g. [19])

$$H\left(\frac{x}{x'}\right) \leqq \left(|\overline{x}| + |\overline{x'}|\right)^n.$$

LEMMA 4. *Let $\mathscr{N}$ be a finite subset of $\mathbf{Z}$ with cardinality $|\mathscr{N}|$, and let $\alpha_1, \ldots, \alpha_k$ be non-zero elements of R''. Suppose*

$$d > (k+2)(\delta-1) \max_{1 \leqq l \leqq \delta} \operatorname{Deg} f_l + \delta(\operatorname{Deg} \alpha_1 + \ldots + \operatorname{Deg} \alpha_k) + \operatorname{Deg} a_1 = c_{12}.$$

There are at least $|\mathscr{N}| - (d^s - 1)$ elements t_0 in $\mathscr{N}$ such that the numbers $\mathscr{S}_{D_f}(t_0), \mathscr{S}_{a_1}(t_0), \mathscr{S}_{\alpha_1}(t_0, j), \ldots, \mathscr{S}_{\alpha_k}(t_0, j)$ are all different from zero, $j = 1, \ldots, \delta$, and

$$\max_{1 \leqq j \leqq \delta} H\left(\mathscr{S}_{\alpha_i}(t_0, j)\right) < \left(2\delta |t_0|^{d^s} L(\alpha_i) (\delta! \max_{1 \leqq l \leqq \delta} L(f_l))^{\delta-1}\right)^\delta \tag{3.37}$$

for all i.

PROOF. Let

$$\alpha_i = \frac{P_{0i}+P_{1i}y+\ldots+P_{\delta-1,i}y^{\delta-1}}{Q_i}, \quad i=1, \ldots, k, \tag{3.38}$$

be the representation of α_i in the form (2.4), and consider the polynomial $g_i(X)=P_{\delta-1,i}X^{\delta-1}+\ldots+P_{0i}$ in X. Its coefficients lie in $\mathbf{Z}[z_1, \ldots, z_s]$. Let R_i denote the resultant of $f(X)$ and $g_i(X)$. Since by assumption $f(X)$ is irreducible over $\mathbf{Q}(z_1, \ldots, z_s)$ and $\deg_X g_i < \deg_X f$, hence $R_i \neq 0$. Further, by a well-known theorem (see e.g. [45]) there exist polynomials $F_i(X)$, $G_i(X)$ of degree $<\delta$ with coefficients in $\mathbf{Z}[z_1, \ldots, z_s]$ such that

$$f(X)F_i(X)+g_i(X)G_i(X) = R_i, \tag{3.39}$$

whence $R_i \in \mathbf{Z}[z_1, \ldots, z_s]$. By using the determinant form of the resultant we see that

$$\operatorname{Deg} R_i \leqq (\delta-1) \max_{1\leqq l\leqq\delta} \operatorname{Deg} f_l + \delta \max_{0\leqq l\leqq\delta-1} \operatorname{Deg} P_{li}.$$

Put $H=D_f a_1 R_1 \ldots R_k$. Then

$$\operatorname{Deg} H < c_{11}+\operatorname{Deg} a_1+k(\delta-1) \max_{1\leqq l\leqq\delta} \operatorname{Deg} f_l+\delta(\operatorname{Deg}\alpha_1+\ldots+\operatorname{Deg}\alpha_k) = c_{12}.$$

By assumption $d>c_{12}$, hence $\mathcal{S}_H(t)$ is a non-vanishing polynomial of degree $\leqq d^s-1$ in t. Since

$$\mathcal{S}_H = \mathcal{S}_{D_f}\mathcal{S}_{a_1}\mathcal{S}_{R_1}\ldots\mathcal{S}_{R_k},$$

there are at least $|\mathcal{N}|-(d^s-1)$ elements t_0 in $\mathcal{N}$ such that $\mathcal{S}_{D_f}(t_0)$, $\mathcal{S}_{a_1}(t_0)$, $\mathcal{S}_{R_1}(t_0)$, ..., $\mathcal{S}_{R_k}(t_0)$ are all different from zero. Denoting by $y_1, \ldots, y_\delta$ the roots of $f_{t_0}(X)=X^\delta+\mathcal{S}_{f_1}(t_0)X^{\delta-1}+\ldots+\mathcal{S}_{f_\delta}(t_0)$ for such a t_0, from (3.38) and (3.39) we see that the numbers

$$\mathcal{S}_{\alpha_i}(t_0, j) = \frac{\mathcal{S}_{P_{0i}}(t_0)+\mathcal{S}_{P_{1i}}(t_0)y_j+\ldots+\mathcal{S}_{P_{\delta-1,i}}(t_0)y_j^{\delta-1}}{\mathcal{S}_{Q_i}(t_0)} \tag{3.40}$$

are all different from zero $(i=1, \ldots, k;\ j=1, \ldots, \delta)$.

The maximum absolute value of the coefficients of the polynomial $f_{t_0}(X)$ is at most $|t_0|^{ds} \max_{1\leqq l\leqq\delta} L(f_l)$. So, in view of Hilfssatz III of [56] we obtain

$$H(y_j) \leqq \delta!|t_0|^{ds} \max_{1\leqq l\leqq\delta} L(f_l) = c_{13}, \tag{3.41}$$

whence $\overline{|y_j|} \leqq c_{13}+1$. This together with (3.40) implies

$$\overline{|\mathcal{S}_{Q_i}(t_0)\mathcal{S}_{\alpha_i}(t_0, j)|} \leqq \delta|t_0|^{ds}L(\alpha_i)(c_{13}+1)^{\delta-1}.$$

Since

$$|\mathcal{S}_{Q_i}(t_0)| \leqq |t_0|^{ds}L(\alpha_i),$$

(3.37) easily follows.

Let again $d>\max\{c_{11}, \operatorname{Deg} a_1\}$. Let t_0 be a rational integer with $\mathcal{S}_{D_f}(t_0)\neq 0$ and $\mathcal{S}_{a_1}(t_0)\neq 0$, $y_1, \ldots, y_\delta$ the roots of $f_{t_0}(X)=X^\delta+\mathcal{S}_{f_1}(t_0)X^{\delta-1}+\ldots+\mathcal{S}_{f_\delta}(t_0)$, and $T_{t_0}=\mathbf{Q}(y_1, \ldots, y_\delta)$. Let n_T, R_T and h_T denote the degree, regulator and class number of T_{t_0} (over $\mathbf{Q}$), respectively. It is obvious that

$$n_T \leqq \delta! \tag{3.42}$$

LEMMA 5. *We have*

$$h_T R_T < \left(\delta^2 |t_0|^{d^s} \max_{1 \leqq l \leqq \delta} L(f_l)\right)^{\delta \cdot \delta!}. \tag{3.43}$$

PROOF. Let D_T denote the absolute value of the discriminant of T_{t_0} over $\mathbf{Q}$. By a theorem of Siegel [58] we have

$$h_T R_T < 16 n_T^2 D_T^{1/2} (\log D_T)^{n_T - 1}. \tag{3.44}$$

We shall now estimate D_T. Let T_i be the number field generated by y_i over $\mathbf{Q}$, with degree n_i and discriminant D_{T_i}. By a lemma of Stark [65] we have

$$D_T \Big| \prod_{i=1}^{\delta} D_{T_i}^{n_T/n_i}. \tag{3.45}$$

Let $p_1(X), \ldots, p_r(X)$ be the distinct irreducible factors of $f_{t_0}(X)$ in $\mathbf{Z}[X]$. If $y_{i_1}, \ldots, y_{i_v}$ denote the roots of p_i, then $n_{i_1} = \ldots = n_{i_v} = v$. Further, the discriminants of the number fields $T_{i_1}, \ldots, T_{i_v}$ coincide and divide the discriminant $D(p_i)$ of p_i. Consequently, from (3.45) we get

$$D_T \Big| \prod_{j=1}^{r} (D(p_j))^{n_T}. \tag{3.46}$$

Since $\prod_{j=1}^{r} D(p_j)$ divides the discriminant $D(f_{t_0})$ of f_{t_0}, it suffices to estimate $D(f_{t_0})$. But we showed in the proof of Lemma 4 that the coefficients of the polynomial $f_{t_0}(X)$ in absolute value do not exceed $|t_0|^{d^s} \max_{1 \leqq l \leqq \delta} L(f_l)$. So

$$|D(f_{t_0})| \leqq \left(2\delta^3 |t_0|^{2d^s} \left(\max_{1 \leqq l \leqq \delta} L(f_l)^2\right)^{\delta - 1} = c_{14}. \right. \tag{3.47}$$

This together with (3.46) gives

$$D_T \leqq c_{14}^{n_T}, \tag{3.48}$$

and (3.43) follows from (3.44), (3.42) and (3.48).

§ 3.5. Proof of Theorem 1 in the number field case

In the special case when $k = \mathbf{Z}$ and $s = 0$, the assertion of Theorem 1 can be deduced from the main result of [23]. To do it, we first state this result of [23] in such a form which will be more convenient for our purpose.

Let T be an algebraic number field of degree n_T. Let R_T and h_T be the regulator and class number of T, respectively. Let

$$F(X_1, \ldots, X_m) = F(\mathbf{X}) = l_1(\mathbf{X}) \ldots l_n(\mathbf{X})$$

be a decomposable form of degree $n \geqq 3$ in $m \geqq 2$ variables with linear factors

$$l_i(\mathbf{X}) = \alpha_{i1} X_1 + \ldots + \alpha_{im} X_m \in T[X_1, \ldots, X_m].$$

Let $H(\alpha_{ij}) \leqq H$. Assume that among $l_1, \ldots, l_n$ there are m linearly independent forms, and that $l_1, \ldots, l_n$ can be divided into pairwise disjoint subsystems $\mathscr{L}_1, \ldots, \mathscr{L}_h$ such that each $\mathscr{L}_j$ $(1 \leqq j \leqq h)$ is connected (for this concept see [21], [23] or § 2.2). Further,

suppose that there exists a q $(1 \leqq q \leqq m)$ such that, for each fixed j $(1 \leqq j \leqq h)$, X_q can be expressed as a linear combination of the forms belonging to $\mathscr{L}_j$.

Let S be a finite set of valuations of T with the parameters P, t specified in § 2.3, and $\mathcal{O}_s$ the ring of S-integers of T. Let β be a non-zero element of T with $H(\beta) \leqq H'$.

LEMMA 6. *With the above notations and assumptions, all solutions* $(x_1, \ldots, x_m) \in \mathcal{O}_s^m$ *of* (1) *with* $x_q \neq 0$ *satisfy*

$$\max_{1 \leqq i \leqq m} H(x_i) < \exp\left\{c_{15}(mn)^3 (R_T h_T)^{t+7} P^{n_T} (\log P)^{t+9} (1+\log(HH'))\right\} \tag{3.49}$$

where $c_{15} = (25(n_T+t+2)n_T)^{2n_T(t+11)+11t+23}$.

Lemma 6 is a special case of the main result of [23] which has been proved by combining Baker's method with our graph method (cf. [23] and [21]).

In order to be able to deduce (2.9) (in the number field case) from our Lemma 6, we remark that if $k = \mathbf{Z}$, $s = 0$ and

$$\alpha = \frac{p_0 + p_1 y + \ldots + p_{\delta-1} y^{\delta-1}}{q}$$

is the representation of the form (2.4) of an element α of R' then

$$H(\alpha) \leqq \left(2\delta \max_{1 \leqq j \leqq \delta} |f_j|\right)^{\delta(\delta-1)} (L(\alpha))^{\delta} \tag{3.50}$$

and

$$L(\alpha) \leqq \left(\left(2\delta \max_{1 \leqq j \leqq \delta} |f_j|\right)^{\delta(\delta-1)} H(\alpha)\right)^{2r+1} \tag{3.51}$$

where r denotes the number of distinct prime factors of a.

Indeed, assuming that K is imbedded into $\mathbf{C}$, we have

$$\overline{|y|} \leqq H(y)+1 = \max_{1 \leqq j \leqq \delta} |f_j| + 1.$$

This implies

$$\overline{|q\alpha|} \leqq \delta \left(\max_{1 \leqq j \leqq \delta} |f_j| + 1\right)^{\delta-1} L(\alpha),$$

whence

$$H(\alpha) \leqq (\overline{|q\alpha|} + |q|)^{\delta} \leqq \left(2\delta \max_{1 \leqq j \leqq \delta} |f_j|\right)^{\delta(\delta-1)} (L(\alpha))^{\delta}.$$

Conversely, taking the conjugates of α over $\mathbf{Q}$ we can easily see that

$$H\left(\frac{p_i}{q}\right) \leqq \left(2\delta \max_{1 \leqq j \leqq \delta} |f_j|\right)^{2\delta(\delta-1)} (H(\alpha))^2 = c_{16}^2.$$

Since by assumption $p_0, \ldots, p_{\delta-1}, q$ are relatively prime, we have $q \leqq c_{16}^{2r}$. Finally, taking again the conjugates of $q\alpha$ we obtain (3.51).

PROOF OF (2.9) IN THE CASE $s = 0$. Suppose that K is imbedded into $\mathbf{C}$, and let $y_1 = y, \ldots, y_\delta$ be the roots of the polynomial $f(X) = X^\delta + f_1 X^{\delta-1} + \ldots + f_\delta$ in $\mathbf{C}$. Let $T = \mathbf{Q}(y_1, \ldots, y_\delta)$, and let n_T, h_T, R_T and D_T be the degree, class number, regulator and absolute value of the discriminant of T, respectively. Then $T = K$.

$F(\mathbf{X})$ satisfies all conditions of Lemma 6. In view of (3.50)

$$H = (2\delta\mathscr{F})^{\delta(\delta-1)}\mathscr{A}^{\delta} \quad \text{and} \quad H' = (2\delta\mathscr{F})^{\delta(\delta-1)}\mathscr{B}^{\delta}$$

are upper bounds for the heights of the coefficients of the l_i and for the height of β, respectively.

Let S denote the set of all valuations of T, induced by the prime factors of a. The cardinality t of S is at most $r\delta!$.

If $x_1, \ldots, x_m$ is an arbitrary solution of (1) in R' with $x_q \neq 0$ then $x_1, \ldots, x_m$ are S-integers in T and by Lemma 6 we have the estimate (3.49). Then $n_T \leqq \delta!$,

$$D_T \leqq |D_f|^{n_T} \leqq (\delta^2\mathscr{F})^{(2\delta-2)\delta!}$$

(which follows from the proof of Lemma 5), (3.44) and (3.49) give (2.9) for $s=0$.

§ 3.6. Proof of the estimate (2.9) of Theorem 1 in the case $s>0$

In the proof of (2.9) it will be more useful to apply the absolute logarithmic height of algebraic numbers in place of the usual height.

When α is an algebraic number with defining polynomial

$$P(X) = a_0X^d + \ldots + a_d = a_0 \prod_{j=1}^{d} (X - \alpha_j)$$

over $\mathbf{Z}$, then

$$h(\alpha) = \frac{1}{d} \log \left(a_0 \prod_{j=1}^{d} \max \{1, |\alpha_j|\} \right) \tag{3.52}$$

can be regarded as the definition of the absolute logarithmic height of α (cf. [70]). For a more natural definition of $h(\alpha)$ we refer the reader to the books [35] and [70]. It is known (cf. [35], [70] and [71]) that

$$\log H(\alpha) \geqq dh(\alpha) - \log d \tag{3.53}$$

and

$$h(\alpha^n) = |n|h(\alpha) \tag{3.54}$$

for any non-zero rational integer n. Furthermore, if $\alpha_1, \ldots, \alpha_k$ are any algebraic numbers, then

$$h(\alpha_1 \ldots \alpha_k) \leqq h(\alpha_1) + \ldots + h(\alpha_k) \tag{3.55}$$

and

$$h(\alpha_1 + \ldots + \alpha_k) \leqq h(\alpha_1) + \ldots + h(\alpha_k) + \log k. \tag{3.56}$$

PROOF OF (2.9) IN THE CASE $s>0$. Let $(x_1, \ldots, x_m) \in R'^m$ be an arbitrary but fixed solution of (1) with $x_q \neq 0$. Let

$$x_i = \frac{P_{0i} + P_{1i}y + \ldots + P_{\delta-1,i}y^{\delta-1}}{Q_i}, \quad i = 1, \ldots, m, \tag{3.57}$$

be the representation of x_i in the form (2.4) where $P_{0i}, \ldots, P_{\delta-1,i}$ and Q_i are relatively prime elements of $\mathbf{Z}[z_1, \ldots, z_s]$. As we showed in § 3.3, for $\max_{1 \leqq i \leqq m} \operatorname{Deg} x_i$ (2.8)

holds. The estimate (2.8) enables us to reduce the equation (1) to equations considered over number fields by using specializations having the properties specified in Lemma 4. Applying Lemma 4, we have to choose the values t_0 in such a way that, after specializations, the new equations satisfy again the assumptions of Theorem 1 with $s=0$. Then we can apply Lemma 6 to estimate the length of the solutions.

Let a_1 and R'' be defined as in the relative case (cf. § 3.3). Then $R''\supseteq R'$, β and the coefficients of the l_k belong to R'', and (3.14) holds. Consider the determinant $\mathscr{D}$ and the elements τ_j defined in § 3.3. They are elements of R'', and by (3.25) we have

$$\max_j \operatorname{Deg}\tau_j \leqq m!\,(mA+(m-1)(\delta-1)D) = c_{17}.$$

The number of the τ_j is at most h. Using (2.5) and (2.6), we get $\operatorname{Deg}\mathscr{D}\leqq c_{17}$. Further, all the (non-zero) elements λ'_{i_e}, $\lambda''_{i_{e+1}}$, $\lambda_{i_e,e+1}$ in R'', considered in § 3.3, have Degrees not greater than $4A+2(\delta-1)D=c_{18}$. The number of these λ's is at most $3n$. Finally, we recall that the sC_1 occurring in (2.8) is an upper bound for $\operatorname{Deg} x_q$.

Let d be a rational integer satisfying

$$(3.58)\quad d > (4n+mn+\delta)(\delta-1)D+\delta\big((h+1)c_{17}+3nc_{18}+B+mnA+sC_1\big)+c_1 = c_{19}.$$

Let t_0 be a rational integer with $\mathscr{S}_{D_f}(t_0)\neq 0$ and $\mathscr{S}_{a_1}(t_0)\neq 0$. Let $y_1, \ldots, y_\delta$ be the roots of $f_{t_0}(X)=X^\delta+\mathscr{S}_{f_1}(t_0)X^{\delta-1}+\ldots+\mathscr{S}_{f_\delta}(t_0)$ in $\mathbf{C}$, and $T_{t_0}=\mathbf{Q}(y_1, \ldots, y_\delta)$ be the number field with the parameters specified in §3. 4. We recall that because of $\mathscr{S}_{D_f}(t_0)\neq 0$ the numbers $y_1, \ldots, y_\delta$ are distinct. For fixed j $(1\leqq j\leqq\delta)$, denote by α_j^* (cf. (3.36)) the image of an arbitrary element $\alpha\in R''$ in T_{t_0} under the specialization defined by $z_1\to t_0, \ldots, z_s\to t_0^{d^{s-1}}, y\to y_j$. Further, let l_{kj}^* and F_j^* be the image of l_k and F, respectively. Then $F_j^*\in T_{i_0}[X_1, \ldots, X_m]$ is decomposable, $F_j^*(\mathbf{X})=l_{1j}^*(\mathbf{X})\ldots l_{nj}^*(\mathbf{X})$ is its decomposition over T_{t_0} and $x_{1j}^*, \ldots, x_{mj}^*$ is a solution of the equation

$$(3.59)\qquad F_j^*(x_1, \ldots, x_m) = \beta_j^*.$$

Let $\mathscr{N}$ be a subset of $\mathbf{Z}$ which will be determined later. Suppose $|\mathscr{N}|>d^s-1$. By virtue of (3.58) and Lemma 4, there are at least $|\mathscr{N}|-(d^s-1)$ elements t_0 in $\mathscr{N}$ such that, for each of these t_0 and for each j, $1\leqq j\leqq\delta$, the images of D_f, a_1, $\mathscr{D}$, τ_j $(j=1, \ldots, h)$, λ'_{i_e}, $\lambda''_{i_{e+1}}$, $\lambda_{i_e,e+1}$, β, x_q and of the non-zero coefficients of the l_k $(k=1, \ldots, n)$ are all different from zero. Consequently, F_j^* and $l_{1j}^*, \ldots, l_{nj}^*$ satisfy all the conditions of Lemma 6. Further, the height of the images of the coefficients of the l_k does not exceed

$$\big(2\delta|t_0|^{d^s}\mathscr{A}(\delta!\mathscr{F})^{\delta-1}\big)^\delta = c_{20}$$

and

$$H(\beta^*)\leqq\big(2\delta|t_0|^{d^s}\mathscr{B}(\delta!\mathscr{F})^{\delta-1}\big)^\delta = c_{21}.$$

Since $x_i\in R'$, hence every prime factor of the denominator Q_i of x_i in (3.57) divides a. Let $p_1, \ldots, p_r$ denote the rational prime factors of a, and $G_1, \ldots, G_w$ its non-constant irreducible factors in the polynomial ring $\mathbf{Z}[z_1, \ldots, z_s]$. Consider the irreducible factorization

$$Q_i = p_1^{a_{1i}}\ldots p_r^{a_{ri}}G_1^{b_{1i}}\ldots G_w^{b_{wi}}$$

of Q_i with non-negative exponents. In view of (2.8) we have

$$b_{1i}+\ldots+b_{wi}\leqq sC_1.$$

$\mathcal{S}_{G_k}(t)$ is a polynomial in $\mathbf{Z}[t]$ with degree $\leqq d^s-1$ and with length $\leqq \mathcal{P}$. Thus, putting $H_i=G_1^{b_{1i}}\dots G_w^{b_{wi}}$ and using the notation of § 3.4 we have

$$|\mathcal{S}_{H_i}(t_0)| = \prod_{k=1}^{w} |\mathcal{S}_{G_k}(t_0)|^{b_{ki}} \leqq (\mathcal{P}|t_0|^{d^s})^{sC_1} = c_{22}, \quad i = 1, \dots, m.$$

Further, $\mathcal{S}_{G_k}(t_0)\neq 0$ because it divides $\mathcal{S}_{a_1}(t_0)\neq 0$ in $\mathbf{Z}$. So $\mathcal{S}_{H_i}(t_0)\neq 0$ for each i.

Let $x_{ij}^{*\prime}=\mathcal{S}_{H_i}(t_0)x_{ij}^*$ for $i=1, \dots, m$. Let $l_{kj}^{*\prime}$ denote the linear form obtained from l_{kj}^* by multiplying the coefficients of $X_1, \dots, X_m$ in l_{kj}^* by $\prod_{i=1}^{m} \mathcal{S}_{H_i}(t_0)$, and let

$$\beta_j^{*\prime} = \beta_j^* \left(\prod_{i=1}^{m} \mathcal{S}_{H_i}(t_0) \right)^n.$$

Then $c_{21}\cdot c_{22}^{mn\delta}$ and $c_{20}^{(m-1)\delta}$ are upper bounds of the height of $\beta_j^{*\prime}$ and of the heights of the coefficients of the $l_{kj}^{*\prime}$, respectively. Further, let S be the set of those valuations v of T_{t_0} for which $v(p_1\dots p_r)\neq 0$. The number of these valuations is at most $r\delta!$. Since $\mathbf{x}=(x_{1j}^{*\prime}, \dots, x_{mj}^{*\prime})$ is a solution of the equation

$$l_{1j}^{*\prime}(\mathbf{x})\dots l_{nj}^{*\prime}(\mathbf{x}) = \beta_j^{*\prime}$$

in S-integers of T_{t_0} with $x_{qj}^{*\prime}\neq 0$, by Lemmas 6 and 5 we have

$$\max_{1\leqq i\leqq m} H(x_{ij}^{*\prime}) < \exp\{c_{23}\}$$

where

$$c_{23} = c_{24}(mn)^3(\delta^2|t_0|^{d^s}\mathcal{F})^{\delta\ \delta!(r\delta!+7)} P^{\delta!}(\log P)^{r\delta!+9}\left(1+\delta mn\log(c_{21}c_{20}c_{22})\right)$$

and

$$c_{24} = \left(25(\delta!+r\delta!+2)\delta!\right)^{2\delta!(\cdot\delta!+11)+11r\delta!+23}.$$

In view of (3.53) we have

$$\max_{1\leqq i\leqq m} h(x_{ij}^{*\prime}) \leqq c_{23}+1. \tag{3.60}$$

Putting $h_i=p_1^{a_{1i}}\dots p_r^{a_{ri}}$, from (3.57) we get

$$x_{ij}^{*\prime} = \frac{\mathcal{S}_{P_{0i}}(t_0)+\mathcal{S}_{P_{1i}}(t_0)y_j+\dots+\mathcal{S}_{P_{\delta-1,i}}(t_0)y_j^{\delta-1}}{h_i}, \quad j = 1, \dots, \delta. \tag{3.61}$$

By virtue of (2.8) and (3.58), $d>\operatorname{Deg} x_i$. Since by definition $\operatorname{Deg} x_i\geqq\operatorname{Deg} P_{ki}$ $(k=0, \dots, \delta-1)$, $\operatorname{Deg} Q_i$, hence the polynomials $\mathcal{S}_{P_{ki}}(t)$ and $\mathcal{S}_{H_i}(t)$ are of degree $\leqq d^s-1$ and their coefficients coincide with those of P_{ki} and H_i, respectively. Further, in view of (3.60), (3.41), (3.55), (3.56) and (3.54) we get from (3.61)

$$h\left(\frac{\mathcal{S}_{P_{ki}}(t_0)}{h_i}\right) \leqq \delta!(c_{23}+2\delta^2\log c_{13}) = c_{25} \tag{3.62}$$

for $i=1, \dots, m$ and $k=0, \dots, \delta-1$.

Suppose that $\mathcal{N}$ consists of all non-zero rational integers having absolute value $\leqq d^s$. Then our argument above shows that (3.62) holds for at least d^s+1 rational integers t_0 with $|t_0|\leqq d^s$. We may choose in (3.58) $d=[c_{19}+1]$.

If $x_i=0$ then $L(x_i)=1$. Thus it suffices to consider the case $x_i\neq 0$. Then at least one of the polynomials $\mathscr{S}_{P_{ki}}(t)$ is different from zero. Suppose $h_i\neq\pm 1$. Since by assumption h_i and $P_{0i}, \ldots, P_{\delta-1,i}$ are relatively prime in $\mathbf{Z}[z_1, \ldots, z_s]$ hence, for each prime factor p of h_i, there is a polynomial $\mathscr{S}_{P_{ki}}(t)$ which has a coefficient, say u_{kip}, relatively prime to p. Considering now (3.62) for all possible values of t_0, we get

$$h\left(\frac{u_{kip}}{h_i}\right) \leqq (2c_{19})^{s(c_{19}+1)^s}\left((2c_{19})^{3s}+c_{25}\right) = c_{26}.$$

Using (3.52), this estimate implies

$$\log(p^{a_i}) \leqq h\left(\frac{u_{kip}}{h_i}\right) \leqq c_{26}$$

if $p^{a_i}|h_i$. Hence

$$h_i \leqq \exp\{rc_{26}\}. \tag{3.63}$$

By virtue of a well-known inequality (see e.g. [69], p. 29) we have

$$L(\mathscr{S}_{H_i}(t)) \leqq \left(L(\mathscr{S}_{G_1}(t))\right)^{b_{1i}} \ldots \left(L(\mathscr{S}_{G_w}(t))\right)^{b_{wi}}.$$

But $L(\mathscr{S}_{H_i}(t))=L(H_i)$ and $L(\mathscr{S}_{G_j}(t))=L(G_j)$ for all j, so we have $L(H_i)\leqq \mathscr{P}^{sC_1}$. This together with (3.63) implies

$$L(Q_i) \leqq \mathscr{P}^{sC_1}\exp\{rc_{26}\}. \tag{3.64}$$

From (3.62) we get

$$h\left(\mathscr{S}_{P_{ki}}(t_0)\right) \leqq c_{25}+rc_{26}$$

for all possible values of t_0. This implies that each coefficient of P_{ki} is at most

$$\exp\left\{c_{26}\left(1+r(2c_{19})\right)^{s(c_{19}+1)^s}\right\} = c_{27}$$

in absolute value. But $\operatorname{Deg} P_{ki}\leqq sC_1$, so

$$L(P_{ki}) \leqq (sC_1+1)c_{27}. \tag{3.65}$$

Finally, from (3.64) and (3.65) we get

$$\max_{1\leqq i\leqq m} L(x_i) \leqq \max\left\{(sC_1+1)c_{27},\ \mathscr{P}^{sC_1}\exp(rc_{26})\right\}.$$

But it is easily seen that this bound is less than

$$\exp\left\{\left((\delta+1)C_1\right)^{1+s(\delta+1)!(r\delta!+7)((\delta+1)C_1)^s}(mn)^3\cdot\right.$$

$$\left.\cdot\left((25(r+2)(\delta!)^2)^{10}\mathscr{F}^{2\delta}\right)^{\delta!(r\delta!+7)}P^{\delta!}(\log P)^{r\delta!+9}\left(1+n\log(\mathscr{A}\mathscr{B}\mathscr{P})\right)\right\},$$

hence (2.9) follows.

§ 3.7. Proofs of Theorems 2, 3, 4, 5 and 6 and of their corollaries

PROOF OF THEOREM 2. Since by hypothesis there are at least three pairwise non-proportional linear forms among the linear factors of F, hence these linear factors form a connected system. Further, it is easily seen that the other assumptions of Theorem 1 are also satisfied with the choice $m=2$ and $h=1$. Thus, Theorem 2 follows immediately from Theorem 1.

PROOF OF COROLLARY 2.1. Let $\Delta(y)$ denote the absolute value of the discriminant of y over $\mathbf{Q}$. In K there is an integral basis of the form

$$\left\{1, \frac{\psi_1(y)}{\Delta_1}, \ldots, \frac{\psi_{\delta-1}(y)}{\Delta_{\delta-1}}\right\}$$

where $\psi_j(y)=y^j+a_{j1}y^{j-1}+\ldots+a_{jj}$, $\Delta_j, a_{ji}\in\mathbf{Z}$ and $\Delta_j|\Delta(y)$, $|a_{ji}|\leqq\Delta(y)$ for all j and i (cf. [14]). Let a be the product of the distinct prime factors of $\Delta(y)$ and of all rational primes associated with the valuations belonging to S. It is clear that the ring $R'=\mathbf{Z}\left[\frac{1}{a}, y\right]$ contains the ring of S-integers of K. Further, a has at most $r+t$ distinct prime factors and these do not exceed $P+Q$. Using (3.50) and (3.51), from Theorem 2 we can now easily deduce (2.11).

Let k be a field of characteristic 0. Denote by R the polynomial ring $k[Z_1, \ldots, Z_s]$ over k, and let $K_0=k(Z_1, \ldots, Z_s)$. To deduce Corollary 2.2 from Theorem 2, we shall need the following

LEMMA 7. *Let*

$$h(X) = h_0X^N+\ldots+h_N = h_0\prod_{j=1}^{N}(X-\alpha_j)$$

be a polynomial with coefficients in R. Suppose that[15] $d(\alpha_j)\leqq D$ *and that* $K=K_0(\alpha_1, \ldots, \alpha_N)$ *has degree δ over K_0. Then there exists a primitive integral element y of K over R such that*

$$d(y) \leqq s\delta 2^{\delta}N^3D.$$

Further, $\operatorname{Deg}\alpha_j$, $j=1, \ldots, N$, *(with respect to the generating set $Z_1, \ldots, Z_s$, y of K over k) is not greater than*

$$s^2\delta^5(\delta!)^2 2^{\delta}N^3D.$$

PROOF. Let a_j (resp. n_j) be the leading coefficient (resp. the degree) of the defining polynomial of α_j over R. Then $\gamma_j=a_j\alpha_j$ is an integral element over R, and $d(\gamma_j)\leqq n_jD$. We can choose $c_1, \ldots, c_N$ in k such that

$$y = c_1\gamma_1+\ldots+c_N\gamma_N \tag{3.66}$$

generates K over K_0 and that the $N!$ elements

$$y_{\varphi} = c_1\gamma_{\varphi(1)}+\ldots+c_N\gamma_{\varphi(N)},$$

[15] We recall that $d(\alpha_j)$ denotes the maximum Degree of the coefficients of the defining polynomial of α_j over R (cf. § 2.4).

where φ runs through the permutations of $1, \ldots, N$, are distinct. The element y is obviously integral over R. Further, its minimal polynomial over K_0, say $f(X)= =X^\delta+f_1X^{\delta-1}+\ldots+f_\delta$, has coefficients in R and f is a divisor of the polynomial

$$g(X) = \prod_{\varphi} (X-y_\varphi)$$

which has coefficients in R.

Let

$$\alpha_j = \frac{P_{0j}+P_{1j}y+\ldots+P_{\delta-1,j}y^{\delta-1}}{Q_j}, \quad j = 1, \ldots, N,$$

be the representation of α_j in the form (2.4). For an arbitrary but fixed i, $1 \leqq i \leqq s$, consider the function field $M_i/\bar{k}_i$ defined in § 3.3. Using the notation of § 3.3, let $\Delta_i=[M_i: \bar{k}_i(Z)]$, and let f_l', P_{tj}', Q_j' denote the polynomials in $\bar{k}_i[Z]$ obtained from f_l, P_{tj}, Q_j by substituting Z for Z_i. By virtue of (3.4) and (3.2) we get

$$\mathbf{H}_{M_i}(\alpha_j) \leqq \Delta_i(N+1)D \tag{3.67}$$

and

$$\mathbf{H}_{M_i}(\gamma_j) \leqq \Delta_i N(N+1)D, \quad j = 1, \ldots, N.$$

Thus, from (3.66) we obtain

$$\mathbf{H}_{M_i}(y) \leqq 2\Delta_i N^3 D. \tag{3.68}$$

It is clear that the same estimates hold for each conjugate of α_j and y over K_0. This shows that

$$\mathbf{H}_{M_i}(f_l') \leqq 2l\binom{\delta}{l}\Delta_i N^3 D$$

whence, by (3.2),

$$\deg_z f_l' \leqq 2l\binom{\delta}{l} N^3 D, \quad l = 1, \ldots, \delta,$$

and so

$$\max_{1 \leqq l \leqq \delta} \operatorname{Deg} f_l = d(y) \leqq s\delta 2^\delta N^3 D. \tag{3.69}$$

Finally, by means of (3.67), (3.68), (3.69) and $\Delta_i \leqq \delta!$ we can derive the estimate

$$\operatorname{Deg} \alpha_j \leqq s^2\delta^5(\delta!)^2 2^\delta N^3 D, \quad j = 1, \ldots, N,$$

in the same way as we estimated $\max_{1 \leqq r \leqq m} \operatorname{Deg} x_r$ in § 3.3, starting from a bound obtained for the height of x_r and of its conjugates.

PROOF OF COROLLARY 2.2. Let

$$F(X_1, X_2) = a_0X_1^n+a_1X_1^{n-1}X_2+\ldots+a_nX_2^n.$$

First suppose $a_0 \neq 0$ and consider the factorization

$$F(X_1, X_2) = a_0(X_1-\alpha_1X_2)\ldots(X_1-\alpha_nX_2)$$

of F in a fixed algebraic closure of $K_0=k(Z_1, \ldots, Z_s)$. It is easily seen that $d(\alpha_j) \leqq H$ for $j=1, \ldots, n$. Thus, by Lemma 7 $K=K_0(\alpha_1, \ldots, \alpha_n)$ has a primitive integral

element y over $k[Z_1, \ldots, Z_s]$ with $d(y) \leqq s\delta 2^\delta n^3 H$. Further, Deg α_j (with respect to the generating set $Z_1, \ldots, Z_s, y$) is not greater than $s^2\delta^5(\delta!)^2 2^\delta n^3 H$. We can now apply Theorem 2 and (2.8) with $m=2$ gives (2.12).

If $a_0=0$, then by the assumption concerning the linear factors of F we have $a_1 \neq 0$ and we get (2.12) in a similar manner.

PROOF OF COROLLARY 2.3. In case $s=0$, (2.13) follows from Corollary 1.2 of [25]. Next we suppose $s>0$.

In view of Corollary 2.2 it suffices to prove the estimate (2.13). We could now derive (2.13) from Theorem 2 but with a larger bound. We shall get a better estimate by using Lemma 4 and reducing the assertion directly to the number field case.

Let

$$F(X_1, X_2) = a_0 X_1^n + a_1 X_1^{n-1} X_2 + \ldots + a_n X_2^n.$$

The discriminant D_F of F is different from zero and its Degree does not exceed $(2n-1)H$. Let x_1, x_2 be an arbitrary but fixed solution of (2.10) in $\mathbf{Z}[Z_1, \ldots, Z_s]$. By (2.12) we have

$$\max \{\text{Deg } x_1, \text{Deg } x_2\} < C_2$$

with the choice $\delta=n!$ in C_2. Apply Lemma 4 in this special situation to the elements $D_F, a_0, \ldots, a_n, \beta, x_1$ and x_2 of $\mathbf{Z}[Z_1, \ldots, Z_s]$ with $d=3C_2$. Clearly d is greater than the sum of the Degrees of these elements of $\mathbf{Z}[Z_1, \ldots, Z_s]$. Using the notation of § 3.4, the polynomials $\mathscr{S}_{x_1}(t), \mathscr{S}_{x_2}(t) \in \mathbf{Z}[t]$ have the same non-zero coefficients as $x_1, x_2 \in \mathbf{Z}[Z_1, \ldots, Z_s]$. Consider the specializations $\mathscr{S}_P(t_0)$ of the polynomials $P \in \mathbf{Z}[Z_1, \ldots, Z_s]$ of Degree $\leqq d$ into $\mathbf{Z}$. By Lemma 4 there are at least d^s+1 rational integers t_0 with absolute value at most $2d^s$ such that

$$0 < |\mathscr{S}_{a_i}(t_0)| \leqq (2d^s)^{d^s} \mathscr{H}$$

for each a_i different from zero,

$$0 < |\mathscr{S}_\beta(t_0)| \leqq (2d^s)^{d^s} \mathscr{B}$$

and $\mathscr{S}_{D_F}(t_0) \neq 0$, where $\mathscr{S}_{D_F}(t_0)$ coincides with the discriminant of the binary form

$$F^*(X_1, X_2) = \mathscr{S}_{a_0}(t_0) X_1^n + \ldots + \mathscr{S}_{a_n}(t_0) X_2^n.$$

Further, $x_i^* = \mathscr{S}_{x_i}(t_0) \neq 0$ provided that $x_i \neq 0$. Since x_1^*, x_2^* is a solution of the equation $F^*(x_1, x_2) = \mathscr{S}_\beta(t_0)$ in rational integers, we can apply Corollary 1.2 of [25] and get

$$\max_{i=1,2} |\mathscr{S}_{x_i}(t_0)| \leqq \exp\left\{(8N)^{36(N+1)}\left(4(6C_2+1)^{s(3C_2)^s}\mathscr{H}\right)^{10nN}(1+\log \mathscr{B})\right\} \tag{3.70}$$

with $N=n(n-1)(n-2)$. Since $\mathscr{S}_{x_i}(t) \in \mathbf{Z}[t]$ is a polynomial of degree $\leqq d^s-1$, considering (3.70) for at least d^s distinct t_0 having the above property we obtain (2.13).

PROOF OF THEOREM 3. Let $x_1, \ldots, x_m$ be an arbitrary but fixed solution of (2.14) in R', q the greatest integer for which $x_q \neq 0$, and $t=\max(q, 2)$. Put $M_t = K_0(\alpha_1, \ldots, \alpha_t)$, $n_t=[M_t: K_0]$ and $n_t'=[M: M_t]$. We may suppose without loss of generality that $\alpha_j^{(1)}, \ldots, \alpha_j^{(n_t)}$ are the images of α_j under the K_0-isomorphisms of

M_t into K. Then (2.14) may be written in the form

$$F_t(\mathbf{x}) = \left(\prod_{i=1}^{n_t} (\alpha_1^{(i)} x_1 + \ldots + \alpha_i^{(i)} x_t) \right)^{n_t'} = \beta.$$

Since $1, \alpha_2, \ldots, \alpha_t$ are linearly independent over K_0, among the linear factors of $F_t(\mathbf{X})$ there are t linearly independent over K. Divide the linear factors of $F_t(\mathbf{X})$ into subsystems so that two linear forms belong to the same subsystem if and only if in these forms the coefficients of X_j coincide for all j with $1 \leqq j \leqq t-1$. It is easily seen that all the conditions of Theorem 1 are satisfied, and so the assertion follows from Theorem 1.

PROOF OF THEOREM 4. It suffices to apply the above argument with $t=m$, and the assertion follows.

PROOF OF COROLLARY 3.1. By Lemma 7 there is a primitive integral element y of the normal closure of M over K_0 such that

$$d(y) \leqq sn! 2^{n!} ((m-1)n)^3 A'$$

and that the Degrees of the conjugates of $\alpha_1, \ldots, \alpha_m$ with respect to the generating set $Z_1, \ldots, Z_s, y$ do not exceed

$$s^2 (2(n!)^3)^{n!} ((m-1)n)^3 A'.$$

This shows that the assertion of Corollary 3.1 is an easy consequence of the estimate (2.8) of Theorem 1.

PROOF OF THEOREM 5. Let

$$l^{(i)}(X) = X_0 + \alpha_1^{(i)} X_1 + \ldots + \alpha_m^{(i)} X_m, \quad i = 1, \ldots, n,$$

be defined as in § 2.5. Putting

$$l_{ij}(\mathbf{X}) = l^{(i)}(\mathbf{X}) - l^{(j)}(\mathbf{X}),$$

(2.15) may be written in the form

$$\prod_{\substack{i,j=1 \\ i \neq j}}^{n} l_{ij}(\mathbf{x}) = (-1)^{n(n-1)/2} \beta. \tag{3.71}$$

Because of the linear independence of $1, \alpha_1, \ldots, \alpha_m$ over K_0, there are $m+1$ linearly independent forms among the linear factors $l^{(1)}, \ldots, l^{(n)}$. From this it follows that the system $\mathscr{L}$ of the linear forms l_{ij} has rank m over K. Further, we can easily see that $\mathscr{L}$ is connected. Hence the equation (3.71) satisfies all the conditions of Theorem 1.

By (2.5) the coefficients of the l_{ij} have Degrees $\leqq 2A$. Further, when $k=\mathbf{Z}$, representing the coefficients of the l_{ij} in the form (2.4) and using some well-known inequalities concerning the height and length of polynomials with coefficients in $\mathbf{Z}$ (cf. [69], pp. 26 and 110), we get $2\mathscr{A}^2 \exp\{3sA\}$ as an upper bound for the lengths of the coefficients of the l_{ij}. Now the assertion follows immediately from Theorem 1.

PROOF OF THEOREM 6. In view of (2.16) we can write (2.17) as

$$D(\omega_2 x_2 + \ldots + \omega_n x_n) = D(\omega_1, \ldots, \omega_n)\beta^2.$$

Since by (2.5) and (2.6) we have

$$\mathrm{Deg}\,(D(\omega_1, \ldots, \omega_n)\beta^2) \leqq 2(n+1)!(A+B+\delta D),$$

hence the assertion follows at once from Theorem 5.

PROOF OF COROLLARY 6.1. Using Lemma 7, we can deduce Corollary 6.1 from Theorem 6 in a similar manner as we deduced Corollary 3.1 from Theorem 3.

References

[1] G. Archinard, Extensions cubiques cycliques de **Q** dont l'anneau des entiers est monogène, *Enseignement Math.*, **20** (1974), 179—203.

[2] A. Baker, Contributions to the theory of Diophantine equations, *Phil. Trans. Roy. Soc. London*, A **263** (1968), 173—208.

[3] A. Baker, Bounds for the solutions of the hyperelliptic equation, *Proc. Cambridge Philos. Soc.* **65** (1969), 439—444.

[4] A. Baker, *Transcendental number theory*, 2nd ed. (Cambridge, 1979).

[5] Z. I. Borevich and I. R. Shafarevich, *Number theory*, 2nd ed., Academic Press (New York and London, 1967).

[6] J. Coates, An effective p-adic analogue of a theorem of Thue, *Acta Arith.*, **15** (1969), 279—305.

[7] J. Coates, An effective p-adic analogue of a theorem of Thue II, The greatest prime factor of a binary form, *Acta Arith.*, **16** (1970), 399—412.

[8] B. N. Delaunay (Delone), Über die Darstellung der Zahlen durch die binären kubischen Formen von negativer Diskriminante, *Math. Z.*, **31** (1930), 1—26.

[9] B. N. Delone and D. K. Faddeev, *The theory of irrationalities of the third degree*, AMS Transl. Math. Monographs Vol. 10 (Providence, 1964).

[10] M. Deuring, *Lectures on the theory of algebraic functions of one variable*, Springer Lecture Notes 314, 1972.

[11] N. I. Feldman, An effective refinement of the exponent in Liouville's theorem (in Russian). *Izv. Akad. Nauk SSSR*, **35** (1971), 973—990.

[12] M. N. Gras, Sur les corps cubiques cycliques dont l'anneau des entiers est monogène, *Ann. Sci. Univ. Besançon*, Fasc. **6** (1973).

[13] M. N. Gras, Lien entre le groupe des unités et la monogénéité des corps cubiques cycliques, *Publ. Math. Univ. Besançon*, Fasc. **1** (1975—76).

[14] K. Győry, Sur les polynômes à coefficients entiers et de discriminant donné II, *Publ. Math. Debrecen*, **21** (1974), 125—144.

[15] K. Győry, Sur les polynômes à coefficients entiers et de discriminant donné III, *Publ. Math. Debrecen*, **23** (1976), 141—165.

[16] K. Győry, On the greatest prime factors of decomposable forms at integer points, *Ann. Acad. Sci. Fenn. Ser. A I. Math.*, **4** (1978/1979), 341—355.

[17] K. Győry, On certain graphs composed of algebraic integers of a number field and their applications I, *Publ. Math. Debrecen*, **27** (1980), 229—242.

[18] K. Győry, Explicit upper bounds for the solutions of some diophantine equations, *Ann. Acad. Sci. Fenn. Ser. A I. Math.*, **5** (1980), 3—12.

[19] K. Győry, *Résultats effectifs sur la représentation des entiers par des formes décomposables*, Queen's Papers in Pure and Applied Math., No. **56** (Kingston, Canada, 1980).

[20] K. Győry, Sur certaines généralisations de l'équation de Thue-Mahler, *Enseignement Math.*, **26** (1980), 247—255.

[21] K. Győry, On the representation of integers by decomposable forms in several variables, *Publ. Math. Debrecen*, **28** (1981), 89—98.

[22] K. Győry, On certain graphs associated with an integral domain and their applications to diophantine problems, *Publ. Math. Debrecen*, **29** (1982), 79—94.

[23] K. Győry, On S-integral solutions of norm form, discriminant form and index form equations, *Studia Sci. Math. Hungar.*, **16** (1981), 149—161.
[24] K. Győry, Effective finiteness theorems for polynomials with given discriminant and integral elements with given discriminant over finnitely generated domains, *to appear.*
[25] K. Győry and Z. Z. Papp, Effective estimates for the integer solutions of norm form and discriminant form equations, *Publ. Math. Debrecen*, **25** (1978), 311—325.
[26] K. Győry and Z. Z. Papp, On discriminant form and index form equations, *Studia Sci. Math. Hungar.*, **12** (1977), 47—60.
[27] K. Győry and Z. Z. Papp, Norm form equations and explicit lower bounds for linear forms with algebraic coefficients, *Studies in Pure Mathematics* (*To the Memory of Paul Turán*), pp. 245—267. Akadémiai Kiadó (Budapest, 1983).
[28] K. Hensel, *Theorie der algebraischen Zahlen*, Teubner Verlag (Leipzig and Berlin, 1908).
[29] S. V. Kotov, The Thue-Mahler equation in relative fields (in Russian), *Acta Arith.*, **27** (1975), 293—315.
[30] S. V. Kotov, On diophantine equations of norm form type I, (in Russian), *Inst. Math. Akad. Nauk BSSR, Preprint* No. **9** (Minsk, 1980).
[31] S. V. Kotov, On diophantine equations of norm form type II, (in Russian), *Inst. Math. Akad. Nauk BSSR, Preprint* No. **10** (Minsk, 1980).
[32] S. V. Kotov and V. G. Sprindžuk, The Thue-Mahler equation in a relative field and approximation of algebraic numbers by algebraic numbers (in Russian), *Izv. Akad. Nauk SSSR*, **41** (1977), 723—751.
[33] S. Lang, Integral points on curves, *Inst. Hautes Études Sci. Publ. Math.*, **6** (1960), 27—43.
[34] S. Lang, *Diophantine geometry*, Interscience Tracts in Pure and Applied Mathematics, No. **11** (New York—London, 1962).
[35] S. Lang, *Elliptic curves, Diophantine analysis*, Springerà Verlag (Berlin—Heidelberg—New York, 1978).
[36] K. Mahler, *Lectures on diophantine approximations*, Notre Dame University (Ann Arbor, 1961).
[37] T. Nagell, *L'analyse indéterminée de degré supérieur*, Mémor. Sci. Math., Fasc. XXXIX (Paris, 1929).
[38] T. Nagell, Zur Theorie der kubischen Irrationalitäten, *Acta Math.*, **55** (1930), 33—65.
[39] T. Nagell, Sur les discriminants des nombres algébriques, *Arkiv för Mat.*, **7** (1967), 265—282.
[40] T. Nagell, Quelques propriétés des nombres algébriques du quatrième degré, *Arkiv för Mat.*, **7** (1968), 517—525.
[41] W. Narkiewicz, *Elementary and analytic theory of algebraic numbers* (Warszawa, 1974).
[42] C. F. Osgood, An effective lower bound on the diophantine approximation of algebraic functions by rational functions, *Mathematika*, **20** (1973), 4—15.
[43] C. F. Osgood, Effective bounds on the diophantine approximation of algebraic functions over fields of arbitrary characteristic and applications to differential equations, *Indag. Math.*, **37** (1975), 105—119.
[44] M. O. Rabin, Computable algebra, general theory and theory of computable fields, *Trans. Amer. Math. Soc.*, **95** (1960), 341—360.
[45] L. Rédei, *Algebra*, Akadémiai Kiadó (Budapest, 1967).
[46] H. P. Schlickewei, On norm form equations, *J. Number Theory*, **9** (1977), 370—380.
[47] W. M. Schmidt, Linearformen mit algebraischen Koeffizienten II, *Math. Ann.*, **191** (1971) 1—20.
[48] W. M. Schmidt, Approximation to algebraic numbers, *Enseignement Math.*, **17** (1971), 187—253.
[49] W. M. Schmidt, Applications of Thue's method in various branches of number theory, *Proc. Internat. Congr. Math.*, Vancouver, 1974, Vol. I, 177—185.
[50] W. M. Schmidt, On Osgood's effective Thue theorem for algebraic functions, *Commun. on Pure and Applied Math.*, **29** (1976), 759—773.
[51] W. M. Schmidt, Thue's equation over function fields, *J. Austral. Math. Soc. Ser. A*, **25** (1978), 385—422.
[52] W. M. Schmidt, Polynomial solutions of $F(x, y)=z^n$, *Proc. Queen's Number Theory Conf.*, (1979), pp. 33—65. *Queen's Papers in Pure and Applied Math.*, No. **54** (Kingston, Canada, 1980).
[53] W. M. Schmidt, *Diophantine approximation*, Springer Lecture Notes No. **785** (Berlin—Heidelberg—New York, 1980).
[54] A. Seidenberg, Constructions in algebra, *Trans. Amer. Math. Soc.*, **197** (1974), 273—313.
[55] T. N. Shorey. A. J. van der Poorten, R. Tijdeman and A. Schinzel, Applications of the Gelfond—

Baker method to Diophantine equations, *Transcendence Theory: Advances and Applications*, pp. 59—77. Academic Press (London—New York—San Francisco, 1977).
[56] C. L. Siegel, Approximation algebraischer Zahlen, *Math. Z.*, **10** (1921), 1973—213.
[57] C. L. Siegel, Über einige Anwendungen diophantischer Approximationen, *Abh. Preuss. Akad. Wiss.* (1929), 1—41.
[58] C. L. Siegel, Abschätzung von Einheiten, *Nachr. Göttingen* (1969), 71—86.
[59] T. Skolem, *Diophantische Gleichungen*, Springer Verlag (Berlin, 1938).
[60] V. G. Sprindžuk, A new application of p-adic analysis to representation of numbers by binary forms (in Russian), *Izv. Akad. Nauk SSSR*, **34** (1970), 1038—1063.
[61] V. G. Sprindžuk, On an estimate for solutions of Thue's equation (in Russian), *Izv. Akad. Nauk SSSR*, **36** (1972), 712—741.
[62] V. G. Sprindžuk, An effective analysis of the Thue and Thue-Mahler equations (in Russian), Current problems of analytic number theory, pp. 199—222 (Minsk, 1974).
[63] V. G. Sprindžuk, Representation of numbers by norm forms with two dominating variables, *J. Number Theory*, **6** (1974), 481—486.
[64] H. M. Stark, Effective estimates of solutions of some diophantine equations, *Acta Arith.*, **24** (1973), 251—259.
[65] H. M. Stark, Some effective cases of the Brauer—Siegel Theorem, *Inventiones Math.*, **23** (1974), 135—152.
[66] S. A. Stepanov, Diophantine equations over function fields (in Russian), *Mat. Sbornik*, **112** (1980), 86—93.
[67] K. B. Stolarsky, *Algebraic numbers and diophantine approximation*, M. Dekker (New York, 1974).
[68] L. A. Trelina, On the greatest prime factor of an index form (in Russian), *Dokl. Akad. Nauk. BSSR*, **21** (1977), 975—976.
[69] M. Waldschmidt, *Nombres Transcendants*, Springer Lecture Notes No. **402** (Berlin—Heidelberg—New York, 1974).
[70] M. Waldschmidt, *Nombres transcendants et groupes algébriques*, Astérisque **69—70**. Soc. Math. France, 1979.
[71] M. Waldschmidt, A lower bound for linear forms in logarithms, *Acta Arith.*, **37** (1980), 257—283.
[72] O. Zariski and P. Samuel, *Commutative Algebra*, Vols. I, II, Van Nostrand (Princeton, 1960).
[73] H. G. Zimmer, *Computational problems, methods, and results in algebraic number theory*, Springer Lecture Notes No. **262** (Berlin—Heidelberg—New York, 1972).

(Received November 9, 1981)

MATHEMATICAL INSTITUTE
KOSSUTH LAJOS UNIVERSITY
4010 DEBRECEN
HUNGARY

Acta Math. Hung.
42 (1—2) (1983), 81—95.

О РЕШЕТКАХ N-РАДИКАЛОВ, СТРОГИХ СЛЕВА РАДИКАЛОВ, НАСЛЕДСТВЕННЫХ СЛЕВА РАДИКАЛОВ

К. И. БЕЙДАР[1] (Москва) и К. САЛАВОВА (Братислава)

1. Введение

Основные определения и факты теории радикалов колец и алгебр могут быть найдены в книгах [2], [4] и [19].

В данной работе рассматриваются только ассоциативные кольца. Термин «класс колец» будет означать замкнутый относительно изоморфизмов класс колец. Пусть α — радикал, R — кольцо и $N \subseteq R$ — подмножество. Положим

$$\mathscr{S}(\alpha) = \{A | A \text{ — кольцо и } \alpha(A) = 0\},$$

$$r(R; N) = \{r \in R | Nr = 0\}, \quad l(R; N) = \{r \in R | rN = 0\}.$$

Обозначим через $\mathscr{U}\mathfrak{M}$ верхний радикал, а через $\mathscr{L}\mathfrak{M}$ — нижний радикал, порожденный классом колец $\mathfrak{M}$. Всюду в дальнейшем β — нижний нильрадикал Бэра. Выражение $L^l \lhd R$ ($L^r \lhd R$, $L \lhd R$) будет означать, что L — ненулевой левый (соответственно, правый, двухсторонний) идеал кольца R.

Определение Морита-контекста (сокращенно — М. к.) может быть найдено в работе [1]. С каждым М. к. (R, V, W, S) естественно связано кольцо «матриц»

$$H = \begin{bmatrix} R, & V \\ W, & S \end{bmatrix}$$

(см. [14]). Далее, М. к. (R, V, W, S) называется точным справа, если равенство $VsW = 0$ и включение $s \in S$ влекут равенство $s = 0$. Напомним, что радикал α называется:

а) наднильпотентным радикалом, если α — наследственный радикал (т. е. $\alpha(I) = \alpha(R) \cap I$ для всех $I \lhd R$) и $\alpha \geqq \beta$;

б) строгим слева радикалом, если из соотношений $I^l \lhd R$ и $\alpha(I) = I$ следует, что $I \subseteq \alpha(R)$;

в) наследственным слева радикалом, если из соотношений $I^l \lhd R$ и $I \subseteq \alpha(R)$ следует, что $\alpha(I) = I$;

г) N-радикалом, если α — строгий слева, наследственный слева радикал и $\alpha \geqq \beta$;

е) нормальным радикалом, если для каждого М. к. (R, V, W, S) выполнено соотношение $V\alpha(S)W \subseteq \alpha(R)$.

Определение и свойства нормального радикала можно найти в работах [7] и [14]. Определение и свойства N-радикала — в работах [13], [14] и [7]. Стро-

[1] Статья написана во время стажировки первого автора в Братиславе.

гие слева (справа) радикалы и наследственные слева (справа) радикалы изучались в работах [6], [9], [12] и [15].

Центральную роль в данной работе играют понятия специального и слабо специального классов колец, принадлежащие В. А. Андрунакиевичу и. Ю. М. Рябухину соответственно. Класс колец $\mathscr{P}$ называется специальным классом, если выполнены следующие условия: 1. Все кольца класса $\mathscr{P}$ первичны. 2. Если $I \lhd R$ и $R \in \mathscr{P}$, то $I \in \mathscr{P}$. 3. Ецли $I \lhd R$, $I \in \mathscr{P}$ и R — первичное кольцо, то $R \in \mathscr{P}$.

Класс колец $\mathscr{P}$ называется слабо специальным классом, если выполнены следующие условия: 4. Все кольца из класса $\mathscr{P}$ полупервичны. 5. Если $I \lhd R$ и $R \in \mathscr{P}$, то $I \in \mathscr{P}$. 6. Если $I \lhd R$, R — полупервичное кольцо, $l(R; I) = 0$ и $I \in \mathscr{P}$, то $R \in \mathscr{P}$.

Следуя [10], назовем класс колец $\mathscr{P}$ нормальным слабо специальным классом, если выполнены следующие условия: 7. Все кольца из класса $\mathscr{P}$ полупервичны. 8. Если $R \in \mathscr{P}$ и (R, V, W, S) — точный справа М. к., то $S \in \mathscr{P}$.

Класс колец $\mathscr{P}$ называется нормальным специальным классом, если $\mathscr{P}$ — нормальный слабо специальный класс и все кольца из класса $\mathscr{P}$ первичны.

Во втором параграфе данной работы наднильпотентные (специальные) строгие слева радикалы, наследственные слева и N-радикалы радикалы охарактеризованы как верхние радикалы слабо специальных (специальных) классов колец, удовлетворяющих некоторым дополнительным условиям. Отметим, что подобная характеризация специальных нормальных радикалов дана в работе [10].

Первый пример наднильпотентного неспециального радикала был построен Ю. М. Рябухиным (см. [2]). Им же было показано, что неспециальных радикалов «достаточно много»: каждый специальный радикал является объединением некоторого семейства наднильпотентных неспециальных радикалов (см. [2, стр. 252, Теорема 5]).

Первый пример наднильпотентного неспециального N-радикала был приведен Егерманом и Сандсом в работе [7]. В третьем параграфе данной работы для «решеток» строгих слева радикалов, наследственных слева радикалов и N-радикалов доказываются аналоги теоремы Ю. М. Рябухина о представлении специального радикала в виде объединения некоторого семейства наднильпотентных радикалов.

В четвертом параграфе дается приложение полученных результатов к теории колец и модулей частных. Показывается, что если R — полупервичное кольцо и (R, V, W, S) — точный справа М. к., то $Z(R) \supseteq VZ(S)W$, где $Z(R)$ — правый (или левый) сингулярный идеал кольца R (напомним, что $Z(R) = \{x \in R | r(R, x)$ — существенный правый идеал кольца $R\}$).

Перейдем к изложению вспомогательных результатов.

Теорема 1 (см. [14, Теоремы 1 и 8]). *Пусть (R, V, W, S) — Морита-контекст и α — N-радикал. Тогда:*

$$\alpha\left(\begin{bmatrix} R, & V \\ W, & S \end{bmatrix}\right) = \begin{bmatrix} A, & B \\ C, & D \end{bmatrix},$$

где $A = \alpha(R)$, $D = \alpha(S)$, $B = \{v \in V | Wv \subseteqq \alpha(S)\} = \{v \in V | vW \subseteqq \alpha(R)\}$,

$$C = \{w \in W | Vw \subseteqq \alpha(R)\} = \{w \in W | wV \subseteqq \alpha(S)\}.$$

Теорема 2 (см. [15], [6], [9] и [12]). *Имеют место следующие утверждения:* а) *если* α_i, $i\in I$ — *строгие слева (наследственные слева) радикалы, то* $\bigcap_{i\in I}\alpha_i$ — *строгий слева (соответственно, наследственный слева) радикал;* б) *если* α_i, $i\in I$ — *наследственные слева радикалы, то* $\bigcup_{i\in I}\alpha_i$ — *наследственный слева радикал.*

Отметим, что в этой теореме пересечение и объединение радикалов берется в «решетке» всех радикалов. Далее, в работе [1] показано, что β — N-радикал. Поэтому из ([14, Теорема 5, Замечание]) и из [5] вытекает следующая лемма.

Лемма 3. *Пусть* R — *кольцо,* $L^l\lhd R$ *и* $M\lhd R$. *Тогда:* 1) *если* P — *первичный (полупервичный) идеал кольца* L, *то существует первичный (соответственно, полупервичный) идеал* Q *кольца* R *такой, что* $P\supseteq L\cap Q\supseteq P^\alpha$; 2) *если* R — *первичное (полупервичное) кольцо то* $\beta(L)=r(L;L)$ *и* $L/r(L;L)$ — *первичное (соответственно, полупервичное) кольцо;* 3) *если* P — *полупервичный идеал кольца* I, *то* P — *идеал кольца* Q.

2. Характеризация некоторых классов наднильпотентных радикалов

Теорема 4. *Пусть* α — *наднильпотентный радикал. Тогда следующие условия эквивалентны:* 1) α — *строгий слева радикал;* 2) *если* R — *кольцо,* $L^l\lhd R$ *и* $\alpha(L)^2\neq 0$, *то* $\alpha(R)\neq 0$.

При выполнении этих условий справедливо включение $\alpha(L)^2\subseteq\alpha(R)$.

Доказательство. 1) Пусть α — строгий слева радикал, R — кольцо, $L^l\lhd R$ и $\alpha(L)^2\neq 0$. Ясно, что $L\alpha(L)\supseteq\alpha(L)^2$. Поэтому $L\alpha(L)$ — ненулевой идеал α-радикального кольца $\alpha(L)$. Так как α — наднильпотентный радикал, то $L\alpha(L)$ — α-радикальное кольцо. Но $L\alpha(L)^l\lhd R$ и α — строгий слева радикал. Значит, $\alpha(L)^2\subseteq L\alpha(L)\subseteq\alpha(R)$ и условие 2) выполнено.

2) Пусть выполнено условие 2). Далее, пусть R — кольцо, $L^l\lhd R$ и $\alpha(L)=L$. Положим $\bar{R}=R/\alpha(R)$ и $\bar{L}=L/L\cap\alpha(R)$. Предположим, что $\bar{L}\neq 0$. Ясно, что $\alpha(\bar{L})=\bar{L}$, $\bar{L}^l\lhd\bar{R}$. Так как α — наднильпотентный радикал, то $\bar{R}$ — полупервичное кольцо и $\bar{L}^2\neq 0$. Значит, $\bar{L}^2=\alpha(\bar{L})^2\neq 0$. Но тогда, $\alpha(\bar{R})\neq 0$, что противоречит равенству $\bar{R}=R/\alpha(R)$. Итак, $\bar{L}=0$, $L\lhd\alpha(R)$ и α — строгий слева радикал.

Теорема 5. *Пусть* α — *наднильпотентный радикал. Тогда следующие условия эквивалентны:* 1) α — *наследственный слева радикал;* 2) *если* R — *полупервичное кольцо,* $L^l\lhd R$ *и* $\alpha(R)L\neq 0$, *то* $\alpha(L)^2\neq 0$.

При выполнении этих условий справедливо включение $\alpha(L)\supseteq\alpha(R)\cap L+\beta(L)$.

Доказательство. 1) Пусть α — наследственный слева радикал и R, L — как в пункте 2) нашей теоремы. Ясно, что $\alpha(R)L^l\lhd\alpha(R)$, $(\alpha(R)L)^2\neq 0$. Далее, так как α — наследственный слева радикал, то $\alpha(R)L$ — α-радикальное кольцо. Но $\alpha(R)L\lhd L$. Поэтому $L(R)L\subseteq\alpha(L)$. Теперь ясно, что $(\alpha(L))^2\neq 0$.

2) Пусть выполнено условие 2). Далее, пусть R — кольцо, $L^l\lhd R$ и $L\subseteq\alpha(R)$. Предположим, что $\alpha(L)\neq L$. Так как $\alpha\geqq\beta$, то $\alpha(L)$ — полупервичный идеал кольца L. Из леммы 3 следует, что $\alpha(L)\supseteq M\cap L\supseteq\alpha(L)^2$ для некоторого полупервичного идеала M кольца R. Положим $\bar{R}=R/M$, $\bar{L}=L/L\cap M$. Ясно, что $\alpha(\bar{R})\supseteq\alpha(R)+M/M$, $\alpha(\bar{L})=\alpha(L)/L\cap M$, $\alpha(\bar{L})^2=0$ и $\bar{L}\subseteq\alpha(\bar{R})$. Так как $L\neq\alpha(L)$

и $L \cap M \subseteq \alpha(L)$, то $\bar{L} \neq 0$. Учитывая, что $\bar{R}$ — полупервичное кольцо, получаем, что $\bar{L}^2 \neq 0$. Значит, $0 \neq \bar{L}^2 \subseteq \alpha(\bar{R})\bar{L}$. Из условия теперь следует, что $\alpha(\bar{L})^2 \neq 0$. Это противоречит доказанному выше равенству $\alpha(\bar{L})^2 = 0$.

Определение 6. Класс колец $\mathscr{P}$ называется строгим слева слабо специальным классом, если выполнены следующие условия:

1) $\mathscr{P}$ — класс полупервичных колец;

2) если $R \in \mathscr{P}$ и $L^l \lhd R$, то $L/r(L; L) \in \mathscr{P}$;

3) если R — полупервичное кольцо, $I \lhd R$, $l(R; I) = 0$, $I \in \mathscr{P}$, то $R \in \mathscr{P}$.

Определение 7. Класс колец $\mathscr{P}$ называется наследственным слева слабо специальным классом, если выполнены следующие условия:

1) $\mathscr{P}$ — класс полупервичных колец;

2) если $R \in \mathscr{P}$ и $I \lhd R$, то $I \in \mathscr{P}$;

3) если R — полупервичное кольцо, $L^l \lhd R$, $l(R; L) = 0$ и $L/r(L, L) \in \mathscr{P}$, то $R \in \mathscr{P}$.

Замечание 8. Ясно, что строгий слева слабо специальный класс $\mathscr{P}$ и наследственный слева слабо специальный класс Q являются слабо специальными классами колец.

Замечание 9. Непосредственно из определения следует, что пересечение и объединение любого множества наследственных слева (строгих слева) слабо специальных классов снова будет наследственным слева (соответственно, строгим слева) слабо специальным классом.

Теорема 10. *Пусть α — наднильпотентный радикал, $\mathscr{P}$ — строгий слева слабо специальный класс колец и $\mathscr{SP}(\alpha) = \{R | \alpha(R) = 0\}$. Тогда:* 1) *$\mathscr{UP}$ — строгий слева наднильпотентный радикал ;* 2) *радикал α является строгим слева радикалом тогда и только тогда, когда $\mathscr{SP}(\alpha)$ — строгий слева слабо специальный класс колец.*

Доказательство. 1) Ясно, что каждый ненулевой левый идеал любого кольца из класса $\mathscr{P}$ может быть гомоморфно отображен на ненулевое кольцо из класса $\mathscr{P}$. Поэтому класс $\mathscr{P}$ является строгим слева классом в смысле работы [13]. Из ([13, Теорема 3]) теперь следует, что $\mathscr{UP}$ — строгий слева радикал. Наднильпотентность радикала $\mathscr{UP}$ вытекает из ([2, стр. 169, Теорема 2 и Замечания 8]).

2) Пусть α — строгий слева радикал и $R \in \mathscr{SP}(\alpha)$. Из определения класса $\mathscr{SP}(\alpha)$ следует, что $\alpha(R) = 0$. В силу теоремы 4, $\alpha(L)^2 \subseteq \alpha(R) = 0$. Значит, $\alpha(L) \subseteq \beta(L)$. Так как $\alpha \cong \beta$, то $\alpha(L) = \beta(L) = r(L; L)$ и $\alpha(L/r(L; L)) = 0$. Поэтому $L/r(L; L) \in \mathscr{SP}(\alpha)$. Теперь ясно, что условия (1) и (2) определения 6 выполнены. Так как α — наднильпотентный радикал, то α — наследственный радикал. Поэтому условие (3) определения 6 тоже выполнено. Таким образом $\mathscr{SP}(\alpha)$ — строгий слева слабо специальный класс. Наоборот, предположим, что $\mathscr{SP}(\alpha)$ — строгий слева слабо специальный класс. Ясно, что $\mathscr{SP}(\alpha)$ — класс всех α-полупростых колец. Поэтому $\alpha = \mathscr{USP}(\alpha)$. Из утверждения доказанного в пункте 1) нашей теоремы теперь следует, что α — строгий слева наднильпотентный радикал.

Теорема 11. *Пусть α — наднильпотентный радикал, $\mathscr{P}$ — наследственный слева слабо специальный класс и $\mathscr{SP}(\alpha)=\{R|\alpha(R)=0\}$. Тогда*: (1) *$\mathscr{UP}$ — наследственный слева наднильпотентный радикал ;* (2) *радикал α является наследственным слева радикалом тогда и только тогда, когда $\mathscr{SP}(\alpha)$ — наследственный слева слабо специальный класс колец.*

Доказательство. 1) Как и при доказательстве теоремы 10 убеждаемся, что $\gamma=\mathscr{UP}$ — наднильпотентный радикал. Пусть R — полупервичное кольцо, $L^l\lhd R$ и $L\subseteqq\gamma(R)$. Предположим, что $\gamma(L)\neq L$. Тогда существует идеал P кольца L такой, что $0\neq L/P\in\mathscr{P}$. Ясно, что P — полупервичный идеал. Поэтому $P\supseteqq L\cap Q\supseteqq P^2$ для некоторого полупервичного идеала Q кольца R (см. лемму 3). Положим $\bar{R}=R/Q$, $\bar{L}=L/L\cap Q$, $\bar{P}=P/L\cap Q$. Ясно, что

$$\gamma(\bar{R})\supseteqq\gamma(R)+Q/Q,\quad \bar{L}\subseteqq\gamma(\bar{R}),\quad \bar{P}^2=0\quad\text{и}\quad \bar{L}/\bar{P}=L/P\in\mathscr{P}.$$

В частности, $\bar{L}/\bar{P}$ — ненулевое полупервичное кольцо. Значит,

$$\bar{L}\neq 0,\quad r(\bar{L};\bar{L})=\beta(\bar{L})=\bar{P}\quad\text{и}\quad \bar{L}/r(\bar{L};\bar{L})\in\mathscr{P}.$$

Положим $M=r(\bar{R},l(\bar{R},\bar{L}))$. Ясно, что $M\supseteqq\bar{L}$. Из полупервичности кольца $\bar{R}$ следует, что $l(M,\bar{L})=0$. Так как M — идеал кольца $\bar{R}$, то M — полупервичное кольцо. Учитывая, что $\mathscr{P}$ — наследственный слева слабо специальный класс колец и $\bar{L}/r(\bar{L};\bar{L})\in\mathscr{P}$ получаем, что $M\in\mathscr{P}$. Значит, $\gamma(M)=0$. Но γ — наследственный радикал. Поэтому $0=\gamma(M)=\gamma(\bar{R})\cap M\supseteqq\bar{L}$ и $\bar{L}=0$. Это противоречит доказанному выше неравенству $\bar{L}\neq 0$. Следовательно $\gamma(L)=L$ и γ — наследственный слева радикал.

2) Пусть α — наследственный слева радикал, R — полупервичное кольцо, $L^l\lhd R$, $l(R;L)=0$ и $L/r(L;L)\in\mathscr{SP}(\alpha)$. Предположим, что $\alpha(R)\neq 0$. Так как $l(R;L)=0$, то $\alpha(R)L\neq 0$. Из теоремы 5 теперь следует, что $\alpha(L)^2\neq 0$. Значит, $\alpha(L)\nsubseteqq r(L;L)$ и $\alpha(L/r(L;L))\supseteqq\alpha(L)+r(L;L)/r(L;L)\neq 0$, что противоречит включению $L/r(L;L)\in\mathscr{SP}(\alpha)$. Следовательно $\alpha(R)=0$, $R\in\mathscr{SP}(\alpha)$ и класс $\mathscr{SP}(\alpha)$ удовлетворяет условию (3) определения 7. Так как идеал полупростого (в смысле любого радикала) кольца сам является полупростым кольцом, то условие (2) определения 7 тоже выполнено. Теперь ясно, что $\mathscr{SP}(\alpha)$ — наследственный слева слабо специальный класс колец. Оставшаяся часть пункта 2 теоремы 11 докозывается точно также, как и соответствующая часть пункта 2 теоремы 10.

Определение 12. Класс колец $\mathscr{P}$ называется слабо специальным N-классом, если выполнены следующие условия: (1) $\mathscr{P}$ — класс поупервичных колец; (2) если $R\in\mathscr{P}$ и $L^l\lhd R$, то $L/r(L;L)\in\mathscr{P}$; (3) если R — полупервичное кольцо, $L^l\lhd R$, $l(R;L)=0$ и $L/r(L;L)\in\mathscr{P}$, то $R\in\mathscr{P}$.

Замечание 13. Ясно, что класс колец является слабо специальным N-классом тогда и только тогда, когда он является строгим слева наследственным слева слабо специальным классом колец.

Из теорем 10 и 11 непосредственно вытекает следующая теорема.

Теорема 14. *Пусть α — наднильпотентный радикал, $\mathscr{P}$ — слабо специальный N-класс колец и $\mathscr{SP}(\alpha)=\{R|\alpha(R)=0\}$. Тогда:* 1) *$\mathscr{UP}$ — N-радикал;* 2) *радикал α является N-радикалом тогда и только тогда, когда $\mathscr{SP}(\alpha)$ — слабо специальный N-класс колец.*

Теорема 15. *Следующие утверждения эквивалентны:* 1) $\mathscr{P}$ *— слабо специальный N-класс;* 2) $\mathscr{P}$ *— слабо специальный нормальный класс.*

Доказательство. 1) Пусть $\mathscr{P}$ — слабо специальный N-класс. Далее, пусть (R, V, W, S) — точный справа М. к. и $R\in\mathscr{P}$. Из работы [1] следует, что S — полупервичное кольцо. Положим

$$H=\begin{bmatrix} R, & V \\ W, & S \end{bmatrix}.$$

В силу теоремы 1 имеем:

$$\beta(H)=\begin{bmatrix} 0, & V_0 \\ W_0, & 0 \end{bmatrix},$$

где

$$V_0=\{v\in V\,|\,Wv\subseteqq\beta(S)=0\}=\{v\in V\,|\,vW\subseteqq\beta(R)=0\}$$

и

$$W_0=\{w\in W\,|\,Vw\subseteqq\beta(R)=0\}=\{w\in W\,|\,wV\subseteqq\beta(S)=0\}.$$

Положими $\overline{V}=V/V_0$, $\overline{W}=W/W_0$. Тогда

$$\overline{H}=H/\beta(H)=\begin{bmatrix} R, & \overline{V} \\ \overline{W}, & S \end{bmatrix}$$

и $\overline{W}v\neq 0\neq v\,\overline{W}$, $\overline{V}w\neq 0\neq w\,\overline{V}$ для всех $0\neq v\in\overline{V}$, $0\neq w\in\overline{W}$. Пусть

$$L=\begin{bmatrix} R, & 0 \\ \overline{W}, & 0 \end{bmatrix}.$$

Ясно, что $L^l\lhd\overline{H}$,

$$(*)\qquad r(L;\,L)=\begin{bmatrix} 0, & 0 \\ \overline{W}, & 0 \end{bmatrix}\quad\text{и}\quad L/r(L;\,L)=\begin{bmatrix} R, & 0 \\ 0, & 0 \end{bmatrix}\cong R\in\mathscr{P}.$$

Предположим, что

$$\begin{pmatrix} r, & v \\ w, & s \end{pmatrix}\in r(\overline{H};\,L).$$

Тогда

$$0=\begin{pmatrix} r, & v \\ w, & s \end{pmatrix}\begin{bmatrix} R, & 0 \\ \overline{W}, & 0 \end{bmatrix}=\begin{bmatrix} rR+v\overline{W}, & 0 \\ wR+s\overline{W}, & 0 \end{bmatrix}$$

и $rR=0$, $v\,\overline{W}=0$, $wR=0$, $s\,\overline{W}=0$. Из доказанного выше и из полупервичности кольца R следует, что $r=0$ и $v=0$. Так как $\overline{V}s\,\overline{W}=0$, то $VsW=0$. Но (R, V, W, S) — точный справа М. к. Поэтому $s=0$. Далее, $\overline{V}w$ — левый идеал кольца R и $(\overline{V}w)R=\overline{V}(wR)=0$. Поэтому $\overline{V}w=0$ и $w=0$. Итак, $r(\overline{H};\,L)=0$. Так как $\mathscr{P}$ — наследственный слева слабо специальный класс колец, то из соотношений $(*)$ теперь следует, что $\overline{H}\in\mathscr{P}$. Но $\mathscr{P}$ — строгий слева слабо специальный класс колец. Поэрому $M/r(M;\,M)\in\mathscr{P}$, где

$$M=\begin{bmatrix} 0, & \overline{V} \\ 0, & S \end{bmatrix}.$$

Как и выше можно показать, что $M/r(M;\,M)\cong S$. Следовательно $S\in\mathscr{P}$ и $\mathscr{P}$ — нормальный слабо специальный класс колец.

2) Пусть $\mathscr{P}$ — нормальный слабо специальный класс, R — полупервичное кольцо и $L^l \lhd R$. Рассмотрии М. к. (R, L, R, L) и кольцо $H=\begin{bmatrix} R, & L \\ R, & L \end{bmatrix}$.

В силу теоремы 1, леммы 3 и полупервичности кольца R имеем:

$$\beta(H) = \begin{bmatrix} 0, & 0 \\ r(R; L), & r(L; L) \end{bmatrix}.$$

Следовательно

$$\overline{H} = H/\beta(H) = \begin{bmatrix} R, & L \\ R/r(R; L), & L/r(L; L) \end{bmatrix}$$

Ясно, что $(R, L, R/r(R; L), L/r(L; L))$ — точный справа М. к. Предположим, что $R \in \mathscr{P}$. Так как $\mathscr{P}$ — нормальный слабо специальный класс, то $L/r(L; L) \in \mathscr{P}$.

Предположим теперь, что $L/r(L; L) \in \mathscr{P}$ и $l(R; L)=0$. Тогда $(L/r(L; L), R/r(R, L), L, R)$ — точный справа М. к. Поэтому $R \in \mathscr{P}$. Следовательно $\mathscr{P}$ — слабо специальный N-класс.

Как отметил Сандс, наднильпотентный радикал является N-радикалом тогда и только тогда, когда он является нормальным радикалом. Теорема 15 показывает, что аналогичное утверждение справедливо и для классов колец.

Следующая теорема доказывается тем же методом, что и теорема 5.

Теорема 16. *Пусть α — специальный радикал. Тогда следующие условия эквивалентны:* 1) *α — наследственный слева радикал;* 2) *если R — первичное кольцо, $L^l \lhd R$ и $\alpha(R) \neq 0$, то $\alpha(L)^2 \neq 0$.*

Определение 16. Класс колец $\mathscr{P}$ называется специальным строгим слева классом (наследственным слева классом, N-классом), если $\mathscr{P}$ — класс первичных колец, являющийся слабо специальным строгим слева классом (соответственно, наследственным слева классом, N-классом).

Следующая теорема доказывается тем же методом, что и теоремы 10, 11 и 14.

Теорема 18. *Пусть α — специальный радикал, $\mathscr{P}$ — специальный строгий слева класс (наследственный слева класс, N-класс) и $\mathscr{P}(\alpha)=\{R | R$ — первичное кольцо и $\alpha(R)=0\}$. Тогда:*

1) *$\mathscr{U}\mathscr{P}$ — специальный строгий слева радикал (соответственно, наследственный слева радикал, N-радикал);*

2) *радикал α является строгим слева радикалом (наследственным слева радикалом, N-радикалом) тогда и только тогда, когда $\mathscr{P}(\alpha)$ — специальный строгий слева класс (соответственно, наследственный слева класс, N-класс) колец.*

Следующая теорема доказывается точно также, как и теорема 15.

Теорема 19. *Следующие условия эквивалентны:*

1) *$\mathscr{P}$ — специальный N-класс колец;*

2) *$\mathscr{P}$ — специальный нормальный класс колец.*

Из ([10, предложение 4]) теперь вытекает следующее следствие.

Следствие 20. *Класс $\mathscr{P}_\varphi$ всех первичных подпрямо неразложимых колец является N-классом.*

Пусть α — наднильпотентный радикал. Положим

$$S\mathscr{P}(\alpha) = \{R|\alpha(R) = 0\};$$

$$S\mathscr{P}^*(\alpha) = \{R|\beta(R) = 0,\ l(R; \alpha(R)) = 0\};$$

$$\mathscr{P}(\alpha) = \{R|R \text{ — первичное кольцо } \alpha(R)=0\};$$

$$\mathscr{P}^*(\alpha) = \{R|R \text{ — первичное кольцо } \alpha(R)\neq 0\};$$

$$\mathscr{P}_\varphi(\alpha) = S\mathscr{P}(\alpha)\cap\mathscr{P}_\varphi,\quad \mathscr{P}^*_\varphi(\alpha) = S\mathscr{P}^*(\alpha)\cap\mathscr{P}_\varphi.$$

Теорема 21. *Пусть α — наднильпотентный радикал. Предположим, что α — строгий слева радикал (наследственный слева радикал, N-радикал). Тогда:*

1) *$S\mathscr{P}^*(\alpha)$ — слабо специальный наследственный слева класс (соответсвенно, строгий слева класс, N-класс);*

2) *$\mathscr{P}_\varphi(\alpha)$ — специальный строгий слева класс (соответсвенно, наследственный слева класс, N-класс);*

3) *$\mathscr{P}^*(\alpha)$ и $\mathscr{P}^*_\varphi(\alpha)$ — специальные наследственные слева классы (соответственно, строгие слева классы, N-классы).*

Доказательство. 1) Пусть α — строгий слева радикал. Ясно, что класс $S\mathscr{P}^*(\alpha)$ удовлетворяет условию (1) определения 7. Так как α — наследственный радикал, то условие (2) тоже выполнено. Пусть R — полупервичное кольцо, $L^l\lhd R$, $l(R; L)=0$ и $L/r(L; L)\in S\mathscr{P}^*(\alpha)$. Значит, $l(L/r(L; L); \alpha(L/r(L, L)))=0$. Далее, пусть M — прообраз идеала $\alpha(L/r(L; L))$ в кольце L. Так как $r(L; L)=\beta(L)\leqq\alpha(L)$, то $\alpha(M)=M$. Очевидно, что $l(L; M^2)\leqq r(L; L)$. Пусть $N=l(R; M^2)$. Так как $LM^2\leqq M^2$, то $NLM^2=0$. Значит, $NL\leqq l(L; M^2)\leqq r(L; L)$. Поэтому $(NL)^2=0$. Но R — полупервичное кольцо и NL — левый идеал кольца R. Следовательно $NL=0$ и $N\leqq l(R; L)=0$. Таким образом $l(R; M^2)=0$. В силу теоремы 4, $\alpha(R)\supseteq\alpha(L)^2\supseteq M^2$. Значит, $l(R; \alpha(R))=0$, $R\in S\mathscr{P}^*(\alpha)$ и $S\mathscr{P}^*(\alpha)$ — слабо специальный наследственный слева класс.

Предположим теперь, что α — наследственный слева радикал. Ясно, что условия (1) и (3) определения 6 выполнены для класса $S\mathscr{P}^*(\alpha)$. Пусть $R\in S\mathscr{P}^*(\alpha)$, $L^l\lhd R$, $M=\alpha(R)L$, $N=r(L; L)$ и π: $L\to L/N=\bar{L}$ — канонический гомоморфизм колец. Далее, пусть $T=\pi^{-1}(l(\bar{L}; \pi(M)))$. Тогда $TM\leqq N$. Значит, $LTM=0$ и $(MT)^2=MTMT\leqq LTMT=0$. Так как R — полупервичное кольцо, то $MT=0$. Поэтому $\alpha(R)LT=0$ и $(LT\alpha(R))^2=0$. Теперь ясно, что $LT\alpha(R)=0$. Из равенства $l(R; \alpha(R))=0$ получаем, что $LT=0$, $T\leqq N$ и $l(\bar{L}; \pi(M))=0$. В силу теоремы 5, $\alpha(L)\supseteq M$. Поэтому $\alpha(\bar{L})\supseteq\pi(M)$ и $l(\bar{L}; \alpha(\bar{L}))=0$. Следовательно $\bar{L}\in S\mathscr{P}^*(\alpha)$ и $S\mathscr{P}^*(\alpha)$ — слабо специальный строгий слева класс.

Тот случай, когда α является N-радикалом, вытекает из доказанного выше.

2) Пересечение специального строго слева класса со слабо специальным строгим слева классом, очевидно, будет специальным строгим слева классом. Первое утверждение пункта 2) нашей теоремы теперь вытекает из следствия 20 и теоремы 4. Оставшиеся утверждения теоремы доказываются аналогично.

Следствие 22. *Сохраним условия и обозначения теоремы* 21 *и положим $\alpha^*_s=\mathscr{U}S\mathscr{P}^*(\alpha)$, $\alpha^*=\mathscr{U}\mathscr{P}^*(\alpha)$, $\alpha^*_\varphi=\mathscr{U}\mathscr{P}^*_\varphi(\alpha)$ и $\alpha_\varphi=\mathscr{U}\mathscr{P}_\varphi(\alpha)$. Тогда:* 1) *$\alpha^*_s$ — над-*

нильпотентный наследственный слева радикал (соответственно, строгий слева радикал, N-радикал); 2) α^* *и* α^*_φ *— специальные наследственные слева радикалы (соответственно, строгие слева радикалы, N-радикалы);* 3) α_φ *— специальный строгий слева радикал (соответственно, наследственный слева радикал, N-радикал).*

Теорема 23. *Пусть α — антипростой радикал. Имеют место следующие утверждения:* 1) $(\alpha^*_\varphi)^*_\varphi=\alpha$; 2) *радикал α является строгим слева радикалом тогда и только тогда, когда α^*_φ — наследственный слева радикал;* 3) *радикал α является наследственным слева радикалом тогда и только тогда, когда α^*_φ — строгий слева радикал;* 4) *радикал α является N-радикалом тогда и только тогда, когда α^*_φ — N-радикал.*

Доказательство. 1) По определению $\mathscr{P}^*_\varphi(\alpha)=S\mathscr{P}^*(\alpha)\cap\mathscr{P}_\varphi$ и $\alpha^*_\varphi=\mathscr{U}\mathscr{P}^*_\varphi(\alpha)$. По условию теоремы $\alpha=\mathscr{U}\mathscr{P}$, где $\mathscr{P}$ — специальный класс подпрямо неразложимых колец. Покажем, что $\mathscr{P}^*_\varphi(\alpha)=\mathscr{P}_\varphi\setminus\mathscr{P}$. Действительно, $\mathscr{P}^*_\varphi(\alpha)\subseteq\mathscr{P}_\varphi\setminus\mathscr{P}$. Пусть $R\in\mathscr{P}_\varphi\setminus\mathscr{P}$. Предположим, что $\alpha(R)=0$. Тогда R — подпрямое произведение колец из класса $\mathscr{P}$. Но R — подпрямо неразложимое кольцо. Значит, $R\in\mathscr{P}$. Получили противоречие с выбором кольца R. Поэтому $\alpha(R)\neq 0$. Так как R — первичное кольцо, то $l(R;\alpha(R))=0$. Следовательно $R\in\mathscr{P}^*_\varphi(\alpha)$ и $\mathscr{P}^*_\varphi(\alpha)=\mathscr{P}_\varphi\setminus\mathscr{P}$. Применяя доказанную формулу к радикалу $\alpha^*_\varphi=\mathscr{U}\mathscr{P}^*_\varphi(\alpha)$, получаем, что $\mathscr{P}^*_\varphi(\alpha^*_\varphi)=\mathscr{P}_\varphi\setminus\mathscr{P}^*_\varphi(\alpha)=\mathscr{P}_\varphi\setminus(\mathscr{P}_\varphi\setminus\mathscr{P})=\mathscr{P}$. Следовательно $(\alpha^*_\varphi)^*_\varphi=\alpha$.

2) Утверждения пунктов 2), 3) и 4) вытекают из доказанного выше и из следствия 22.

Замечание. В определении слабо специального нормального класса фигурирует термин «точный справа Морита-контекст». Поэтому это определение кажется на первый взгляд лево-право несимметричным. Однако это не так. Действительно, пусть (R, V, W, S) — точный справа М. к. Тогда (S, W, V, R) — точный слева М. к. (т. е. из равенства $VsW=0$ вытекает равенство $s=0$). Теперь ясно, что класс полупервичных колец $\mathscr{P}$ является нормальным слабо специальным классом тогда и только тогда, когда для любого точного слева М. к. (S, W, V, R) из включения $R\in\mathscr{P}$ следует включение $S\in\mathscr{P}$. Следовательно определение слабо специального нормального класса право-лево симметрично. Далее, использованное в данной работе понятие слабо специального N-класса мы, временно, заменим понятием слабо специального левого N-класса. Аналогично вводится понятие слабо специального правого N-класса (в определении 12 заменить левые идеалы на правые, левые аннуляторы — на правые, а правые — на левые). В силу теоремы 15 оба эти понятия совпадают с понятием слабо специального нормального. Как было показано выше последнее понятие право-лево симметрично. Следовательно класс колец $\mathscr{P}$ является слабо специальным левым N-классом тогда и только тогда, когда $\mathscr{P}$ — слабо специальный правый N-класс. Поэтому использование термина «слабо специальный N-класс» оправдано.

3. Теорема Рябухина для некоторых решеток наднильпотентных радикалов

Так как совокупность всех радикалов не является множеством, то мы будем заключать термин решетка (по отношению к тем или иным классам радикалов) в кавычки. Напомним, что решетка называется полной, если в ней содержатся объединения и пересечения любого семейства элементов.

Лемма 24. 1) *Совокупность всех наднильпотентных строгих слева радикалов образует «полную подрешетку» «полной решетки» всех радикалов;* 2) *Совокупность всех наднильпотентных наследственных слева радикалов образует «полную подрешетку» «полной решетки» всех радикалов;* 3) *Совокупность всех N-радикалов образует «полную подрешетку» «полной решетки» всех радикалов;* 4) *для каждой из перечисленных выше решеток справедливо равенство*

$$\gamma\wedge\left(\bigvee_{i\in\Gamma}\alpha_i\right)=\bigvee_{i\in\Gamma}(\gamma\wedge\alpha_i).$$

Доказательство. Утверждение пункта 3) доказано в работе [7, Теорема 8]. Утверждение пункта 4) непосредственно вытекает из [2, стр. 129, Теорема 3]. Пусть α_i, $i\in\Gamma$ — наднильпотентные радикалы и $\mathscr{R}(\alpha_i)=\{R|\alpha_i(R)=R\}$. Из [2, стр. 129, Теорема 3] следует, что

$$\mathscr{R}\left(\bigwedge_{i\in\Gamma}\alpha_i\right)=\bigcap_{i\in\Gamma}\mathscr{R}(\alpha_i),\quad S\mathscr{P}\left(\bigvee_{i\in\Gamma}\alpha_i\right)=\bigcap_{i\in\Gamma}S\mathscr{P}(\alpha_i).$$

Из первого равенства вытекает, что $\bigwedge\limits_{i\in\Gamma}\alpha_i$ — наследственный слева радикал, если α_i, $i\in\Gamma$ — наследственные слева радикалы. Так как пересечение любого семейства слабо специальных строгих слева классов снова будет слабо специальным строгим слева классом, то из второго равенства и теоремы 10 следует, что $\bigvee\limits_{i\in\Gamma}\alpha_i$ — строгий слева радикал, если α_i, $i\in\Gamma$ — строгие слева радикалы. Аналогично показывается, что $\bigvee\limits_{i\in\Gamma}\alpha_i$ — наследственный слева радикал, если α_i, $i\in\Gamma$ — наследственные слева радикалы. Наконец, из [15, Теорема 2] следует, что $\bigwedge\limits_{i\in\Gamma}\alpha_i$ — строгий слева радикал, если α_i, $i\in\Gamma$ — строгие слева радикалы.

Пусть S — кольцо, M — правый S-модуль, а N — левый S-модуль. Всюду в дальнейшем $M\otimes N=M\otimes_S N$.

Лемма 25. *Пусть* (R, V, W, S) *и* (S, X, Y, T) — *точные справа Морита-контексты. Тогда* $(R, V\otimes X, Y\otimes W, T)$ — *точный справа Морита-контекст.*

Доказательство. Непосредственная проверка показывает, что отображения $V\otimes X\times Y\otimes W\to R$ и $Y\otimes W\times V\otimes X\to T$, задаваемые правилами $(v\otimes x, y\otimes w)\mapsto v(xy)w$ и $(y\otimes w, v\otimes x)\mapsto y(wv)x$ соответственно, где $v\otimes x\in V\otimes X$ и $y\otimes w\in Y\otimes W$, превращают набор объектов $(R, V\otimes X, Y\otimes W, T)$ в Морита-контекст. Пусть $0\neq t\in T$, $0\neq v\otimes x\in V\otimes X$ и $0\neq y\otimes w\in Y\otimes W$. Тогда $v\neq 0$, $x\neq 0$, $y\neq 0$, $w\neq 0$. Так как (S, X, Y, T) — точный справа М. к., то $xty\neq 0$. Аналогично показывается, что $v(xty)w\neq 0$. Но $(v\otimes x)t(y\otimes w)=v(xty)w$. Следовательно $(R, V\otimes X, Y\otimes W, T)$ — точный справа М. к.

Напомним некоторые конструкции и результаты из книги [2, стр. 247—250]. Пусть L — линейно упорядоченное множество и A — кольцо. Превратим L в полугруппу, полагая $xy=\min\{x, y\}$ для всех $x, y\in L$. Обозначим через AL полугрупповое кольцо. Далее, пусть $\aleph$ — бесконечное кардинальное число и $\omega=w(\aleph)$ — наименьшее порядковое число такое, что мощность отрезка $[1, w]$ порядковых чисел равна $\aleph$. Превратим множество $\Omega=\Omega(\aleph)$ всех последовательностей $\{a_\gamma|1\leqq\gamma<\omega\}$, где a_γ — натуральные числа, в линейно упорядоченное множество, вводя порядок лексикографически. Всюду в дальнейшем символом $\Omega(\aleph)$ будет обозначаться именно такое линейно упорядоченное множество. Обозначим через $|X|$ мощность множества X.

Лемма 26 [2, стр. 247, Лемма 6]. *Пусть A — произвольное ненулевое кольцо, L — произвольное линейно упорядоченное множество. Тогда верны утверждения:* 1) *кольцо AL является подпрямым произведением колец, изоморфных A;* 2) *если Q — первичный гомоморфный образ кольца AL, то кольцо Q изоморфно первичному гомоморфному образу кольца A.*

Лемма 27 [2, стр. 249, Лемма 7]. *Пусть $\aleph$ — бесконечное кардинальное число, $\Omega=\Omega(\aleph)$ — соответствующее ему линейно упорядоченное множество. Если $x, y\in\Omega$ и $x<y$, то отрезок $[x, y]=\{z\in\Omega|x\leqq z\leqq y\}$ равномощен всему множеству Ω. А именно $|[x, y]|=|\Omega|=2^\aleph>\aleph$.*

Лемма 28. *Пусть A — ненулевое полупервичное кольцо, Z — кольцо целых чисел, $B=Z/r(Z; A)$, $\aleph$ — бесконечное кардинальное число, $\Omega=\Omega(\aleph)$ и I — ненулевой идеал кольца $A\Omega$. Тогда $|B\Omega/r(B\Omega; I)|=2^\aleph$.*

Доказательство. Пусть $0\neq u\in I$. Ясно, что элемент u представим в виде $u=\sum_{i=1}^{n} a_i x_i$, где $a_1, a_2, \ldots, a_n$ — ненулевые элементы кольца A и $x_1<x_2<\ldots<x_n$ — элементы множества Ω. Пусть $y_1, y_2\in[x_{n-1}, x_n]$ и $y_1\neq y_2$. Тогда

$$u(y_1-y_2)=\sum_{i=1}^{n-1} a_i x_i+a_n y_1-\sum_{i=1}^{n-1} a_i x_i-a_n y_2=a_n y_1-a_n y_2\neq 0.$$

Следовательно никакие два различных элемента множества $[x_{n-1}, x_n]$ не сравнимы по модулю идеала $r(B\Omega; I)$. Значит,

$$|B\Omega/r(B\Omega; I)|\geqq|[x_{n-1}, x_n]|=2^\aleph$$

(см. Лемму 27). С другой стороны, очевидно, что $|B\Omega|=|\Omega|=2^\aleph$. Теперь ясно, что $|B\Omega/r(B\Omega; I)|=2^\aleph$.

Определение 29. Будем говорить, что в «полной подрешетке» $\mathcal{M}$ «полной решетки» всех радикалов выполнена теорема Рябухина, если для любых наднильпотентных радикалов $\alpha, \gamma\in\mathcal{M}$ таких, что $\alpha>\gamma$ существует такое семейство наднильпотентных неспециальных радикалов $\alpha_i\in\mathcal{M}$, $i\in\Gamma$, лежащих между радикалами α и γ, что $\alpha=\bigvee_{i\in\Gamma}\alpha_i$.

Теорема 30. *В следующих «полных подрешетках» «полной решетки» всех радикалов выполнена теорема Рябухина:* 1) *в «подрешетке» всех наднильпотент-*

ных строгих слева радикалов; 2) *в «подрешетке» всех наднильпотентных наследственных слева радикалов;* 3) *в «подрешетке» всех N-радикалов.*

Доказательство. Пусть $\mathscr{M}$ — «подрешетка» всех наднильпотентных строгих слева радикалов, $\alpha, \gamma \in \mathscr{M}$ и $\alpha > \gamma$. Тогда существует кольцо Q такое, что $\alpha(Q)=Q$ и $\gamma(Q)\neq Q$. Положим $A=Q/\gamma(Q)$. Ясно, что A — ненулевое полупервичное кольцо, $\alpha(A)=A$ и $\gamma(A)=0$.

Выберем бесконечный кардинал $\aleph$ так, чтобы $\aleph \geqq |A|$. Пусть $\Omega=\Omega(\aleph)$, Z — кольцо целых чисел, $B=Z/r(Z; A)$ и $\mathscr{D}=\{(b, a)|b\in B\Omega, a\in A\Omega\}$. Превратим множество $\mathscr{D}$ в кольцо, полагая $(a, b)+(c, d)=(a+c, b+d)$, $(a, b)(c, d)=(ac, ad+bc+bd)$ для всех $(a, b), (c, d)\in\mathscr{D}$. Ясно, что $\{(0, a)|a\in A\Omega\}$ — идеал кольца $\mathscr{D}$, изоморфный кольцу $A\Omega$. Отождествим кольцо $A\Omega$ с этим идеалом кольца $\mathscr{D}$. Аналогично отождествим кольцо $B\Omega$ с подкольцом $\{(b, 0)|b\in B\Omega\}$ кольца $\mathscr{D}$. Из леммы 26 следует что $A\Omega$ и $B\Omega$ — полупервичные кольца. Значит, $\mathscr{D}$ — полупервичное кольцо. Пусть $\mathscr{L}$ — наименьший слабо специальный нормальный класс колец, содержащий кольцо $A\Omega$. Из определения слабо специального нормального класса и леммы 25 следует что класс $\mathscr{L}$ состоит из таких полупервичных колец S что $(A\Omega, V, W, S)$ — точный справа Морита-контекст для некоторых бимодулей V, W. Покажем, что $|S|\geqq 2^{\aleph}$ для всех $0\neq S\in\mathscr{L}$. Пусть $0\neq S\in\mathscr{L}$ и $(A\Omega, V, W, S)$ — точный справа М. к. Так как $(\mathscr{D}, \mathscr{D}, A\Omega, A\Omega)$ — точный справа М. к., то $(\mathscr{D}, X, Y, S)$ — точный справа М. к., где $X=\mathscr{D}\otimes_{A\Omega}V$, $Y=W\otimes_{A\Omega}A\Omega$ (см. лемму 25). Предположим, что $YX=0$. Тогда $(XY)^2=0$. Но XY — идеал полупервичного кольца $\mathscr{D}$. Следовательно $XY=0$ и $XsY=0$ для всех $s\in S$. Это противоречит точности справа М. к. $(\mathscr{D}, X, Y, S)$. Итак, $yx\neq 0$ для некоторых $x\in X$, $y\in Y$. Определим отображение φ: $B\Omega\to S$, полагая $\varphi(a)=yax$ для всех $a\in B\Omega$. Предположим, что $a\in\ker\varphi$. Тогда $yax=0$ и $X(yax)Y=0$. Так как $Xy, Yx\in\mathscr{D}$ и $B\Omega$ — подкольцо центра кольца $\mathscr{D}$, то $0=X(yax)Y=(Xy)a(xY)=XyxYa$. Далее, $yx\neq 0$ и $(\mathscr{D}, X, Y, S)$ — точный справа М. к. Поэтому $XyxY$ — ненулевой идеал кольца $\mathscr{D}$ и $a\in r(B\Omega; XyxY)$. Из леммы 28 теперь следует, что $|S|\geqq|\varphi(B\Omega)|\geqq|B\Omega/r(B\Omega; XyxY)|=2^{\aleph}$. Тем самым доказано, что включение $S\in\mathscr{L}$ и неравенство $S\neq 0$ влекут неравенство $|S|\geqq 2^{\aleph}$.

Пусть $\mathscr{L}_0$ наименьший слабо специальный строгий слева класс, содержащий кольцо $A\Omega$. Так как слабо специальный нормальный класс является слабо специальным N-классом, то он является и слабо специальным строгим слева классом (см. теорему 15). Поэтому $\mathscr{L}_0\subseteqq\mathscr{L}$ и $|S|\geqq 2^{\aleph}$ для всех $0\neq S\in\mathscr{L}_0$. Далее, пусть $\mathscr{R}(\alpha)=\{R|\alpha(R)=R\}$ и $S\mathscr{P}(\alpha)=\{R|\alpha(R)=0\}$. Так как $\alpha>\gamma$, то $S\mathscr{P}(\alpha)\subseteqq S\mathscr{P}(\gamma)$. По построению $A\in S\mathscr{P}(\gamma)$. Из леммы 26 следует, что $A\omega\in S\mathscr{P}(\gamma)$. Так как $S\mathscr{P}(\gamma)$ — слабо специальный строгий слева класс, то $\mathscr{L}_0\subseteqq S\mathscr{P}(\gamma)$ (см. теорему 10). Поэтому $S\mathscr{P}(\alpha)\subseteqq\mathscr{L}_0\cup S\mathscr{P}(\alpha)\subseteqq S\mathscr{P}(\gamma)$ и $\alpha\geqq\alpha_Q\geqq\gamma$, где $\alpha_Q=\mathscr{U}(\mathscr{L}_0\cup S\mathscr{P}(\alpha))$. Так как $\mathscr{L}_0\cup S\mathscr{P}(\alpha)$ — слабо специальный строгий слева класс, то α_Q — наднильпотентный строгий слева радикал (см. теорему 10). По построению $\alpha_Q(A\Omega)=0$. Покажем, что $\alpha_Q(A)=A$. Действительно, так как $\alpha(A)=A$, то кольцо A не может быть гомоморфно отображено ни на какое ненулевое кольцо из класса $S\mathscr{P}(\alpha)$. Далее, $|A|\leqq\aleph$ и $|S|\geqq 2^{\aleph}\geqq\aleph$ для всех $0\neq S\in\mathscr{L}_0$. Поэтому кольцо A не может быть гомоморфно отображено ни на какое ненулевое кольцо из класса $\mathscr{L}_0$. Теперь ясно, что $\alpha_Q(A)=A$. В силу леммы 26, каждый первичный гомоморфный образ H кольца $A\Omega$ изоморфен

первичному гомоморфному образу кольца A. Поэтому $\alpha_Q(H)=H$. Из [2, стр. 232, Предложение 3] следует, что α_Q — неспециальный радикал. Согласно доказанному $\alpha_Q \geqq \gamma$ и $\alpha_Q(A)=A$, $A=Q/\gamma(Q)$. Поэтому $\alpha_Q(Q)=Q$. Теперь ясно, что $\alpha=V\{\alpha_Q | Q\in\mathscr{R}(\alpha)\setminus\mathscr{R}(\gamma)\}$. Тем самым утверждение 1) теоремы 30 доказано.

Утверждения 2) и 3) доказываются аналогично.

4. Связь с теорией колец и модулей частных

Основные понятия и результаты теории колец и модулей частных можно найти в книгах [8] и [17]. Пусть A — кольцо. Напомним, что левым сингулярным идеалом $Z_l(A)$ кольца A называется множество $\{a\in A | l(A; a)$ — существенный левый идеал кольца $A\}$. Аналогично определяется правый сингулярный идеал $Z_r(A)$. Далее, левый сингулярный идеал кольца A является идеалом (см. [8], [17]).

Лемма 31 (К. И. Бейдар, А. В. Михалев). *Пусть A — полупервичное кольцо, $L^l\lhd A$, $N=r(L; L)$ и $\pi: L\to L/N=\bar{L}$ — канонический гомоморфизм колец. Тогда $\pi(Z_l(A)\cap L)\subseteqq Z_l(\bar{L})$.*

Доказательство. Пусть M — существенный левый идеал кольца A. Тогда $\pi(M\cap L)$ — существенный левый идеал кольца $\bar{L}$. Действительно, предположим, что $\pi(M\cap L)\cap K=0$ для некоторого $K^l\lhd\bar{L}$. Пусть $H=\pi^{-1}(K)$. Ясно, что $H^l\lhd L$ и $M\cap H=(M\cap L)\cap H\subseteqq N$. Так как $LH\supseteqq H$, то $M\cap LH\subseteqq N$. Следовательно $M\cap LH$ — нильпотентный левый идеал полупервичного кольца A. Значит, $M\cap LH=0$. Но M — существенный левый идеал. Поэтому $LH=0$, $H\subseteqq N$ и $K=\pi(H)=0$, что противоречит соотношению $K^l\lhd\bar{L}$. Итак, $\pi(M\cap L)$ — существенный левый идеал кольца $\bar{L}$.

Пусть $a\in L\cap Z_l(A)$. Из доказанного выше следует, что $\pi(l(A; a)\cap L)$ — существенный левый идеал кольца $\bar{L}$. Но $l(\bar{L}; \pi(a))\supseteqq\pi(l(A; a)\cap L)$. Значит, $l(\bar{L}; \pi(a))$ — существенный левый идеал кольца $\bar{L}$, $\pi(a)\in Z_l(\bar{L})$ и $\pi(Z_l(A)\cap L)\subseteqq Z_l(\bar{L})$.

Лемма 32. *Сохраним обозначения и предположения леммы* 31. *Тогда* $Z_l(A)\supseteqq L\pi^{-1}(Z_l(\bar{L}))$.

Доказательство. Пусть $a\in L$, $\pi(a)\in Z_l(\bar{L})$ и $M=\pi^{-1}(l(\bar{L}; \pi(a)))$. Тогда $Ma\subseteqq N$ и

$$(*) \qquad LMa=0.$$

Покажем, что LM — существенный A-подмодуль левого A-модуля L. Пусть $K\subseteqq L$ и $K^l\lhd A$. Так как A — полупервичное кольцо, то $0\neq K^2\subseteqq LK$, $K\not\subseteqq N$ и $\pi(K)\neq 0$. Значит, $\pi(K)\cap l(\bar{L}; \pi(a))\neq 0$. Поэтому найдутся такие элементы $b\in K\setminus N$ и $\alpha\in M\setminus N$, что $b-d\in N$. Но тогда $0\neq Lb=Ld\subseteqq K\cap LM$. Итак, LM — существенный подмодуль левого A-модуля L. Следовательно для любого элемента $x\in L$ левый идеал $(LM:x)=\{y\in A | yx\in LM\}$ является существенным. В силу равенства $(*)$ имеем для всех $x\in L$ следующее включение: $l(A; xa)\supseteqq(LM:x)$. Следовательно $l(A; xa)$ — существенный левый идеал кольца A, $xa\in Z_l(A)$ и $L\pi^{-1}(Z_l(\bar{L}))\subseteqq Z_l(A)$.

Теорема 33. *Пусть* $S\mathscr{P}_1=\{R | \beta(R)=0, l(R; Z_l(R))=0\}$ *и* $S\mathscr{P}_2=\{R | \beta(R)=$

$=0, Z_l(R)=0\}$. *Тогда* 1) $S\mathscr{P}_1$ *и* $S\mathscr{P}_2$ *— слабо специальные* N*-классы колец;* 2) $S\mathscr{P}_1$ *и* $S\mathscr{P}_2$ *— слабо специальные нормальные классы колец.*

Доказательство. Непосредственно из лемм 31 и 32 следует, что $S\mathscr{P}_1$ и $S\mathscr{P}_2$ — слабо специальные N-классы колец. Наша теорема теперь вытекает из теоремы 15.

Следствие 34. *Пусть* (R, V, W, S) *— точный справа Морита-контекст и* R *— полупервичное кольцо. Тогда* 1) $Z_l(R)\supseteq VZ_l(S)W$, $Z_l(S)\supseteq WZ_l(R)V$; 2) $Z_r(R)\supseteq VZ_r(S)W$, $Z_r(S)\supseteq WZ_r(R)V$; 3) *если* $Z_l(R)=0$ *то* $Z_l(S)=0$; 4) *если* $Z_r(R)=0$, *то* $Z_l(S)=0$.

Доказательство. Так как β — N-радикал, то $\beta(S)=0$. Рассмотрим кольцо

$$H=\begin{bmatrix} R, & V \\ W, & S \end{bmatrix}.$$

Из теоремы 1 следует, что

$$\beta(H)=\begin{bmatrix} 0, & B \\ C, & 0 \end{bmatrix},$$

где $C=\{w\in W\,|\,Vw=0\}=\{w\in W\,|\,wV=0\}$ и $B=\{v\in V\,|\,vW=0\}=\{v\in V\,|\,Wv=0\}$. Положим $\overline{V}=V/B$, $\overline{W}=W/C$. Тогда

$$\overline{H}=H/\beta(H)=\begin{bmatrix} R, & \overline{V} \\ \overline{W}, & S \end{bmatrix}.$$

Рассмотрим левые идеалы

$$M=\begin{bmatrix} R, & 0 \\ \overline{W}, & 0 \end{bmatrix} \quad N=\begin{bmatrix} 0, & \overline{V} \\ 0, & S \end{bmatrix}$$

кольца $\overline{H}$. Ясно, что

$$r(M;\,M)=\begin{bmatrix} 0, & 0 \\ \overline{W}, & 0 \end{bmatrix}, \quad r(N;\,N)=\begin{bmatrix} 0, & \overline{V} \\ 0, & 0 \end{bmatrix}$$

и $M/r(M;\,M)\cong R$, $N/r(N;\,N)\cong S$. Обозначим через π_1; $N\to S$, π_2: $M\to R$ канонические гомоморфизмы колец. Из леммы 32 следует, что

$$Z_l(\overline{H})\supseteq N\pi_1^{-1}(Z_l(S))=\begin{bmatrix} 0, & \overline{V} \\ 0, & S \end{bmatrix}\begin{bmatrix} 0, & \overline{V} \\ 0, & Z_l(S) \end{bmatrix}\supseteq\begin{bmatrix} 0, & \overline{V}Z_l(S) \\ 0, & 0 \end{bmatrix}.$$

В силу леммы 31 имеем

$$Z_l(R)\supseteq\pi_2(Z_l(\overline{H})M)\supseteq\pi_2\left(\begin{bmatrix} 0, & \overline{V}Z_l(S) \\ 0, & 0 \end{bmatrix}\begin{bmatrix} R, & 0 \\ \overline{W}, & 0 \end{bmatrix}\right)=\overline{V}Z_l(S)\overline{W}.$$

Осталось отметить, что $\overline{V}Z_l(S)\overline{W}=VZ_l(S)W$. Второе включение пункта 1) доказывается аналогично. Утверждение пункта 2) нашей теоремы доказывается с помощью «правых аналогов» лемм 31 и 32 и рассмотрения правых идеалов

$$\begin{bmatrix} R, & \overline{V} \\ 0, & 0 \end{bmatrix}\begin{bmatrix} 0, & 0 \\ \overline{W}, & S \end{bmatrix}$$

кольца $\overline{H}$. Утверждения пунктов 3) и 4) непосредственно следуют из пунктов 1) и 2).

Следствие 35. *Пусть L — левый идеал полупервичного кольца A, $N=r(L; L)$ и $\pi: L \to L/N = \bar{L}$ — канонический гомоморфизм колец. Тогда $Z_r(A) \supseteq L\pi^{-1}(Z_r(\bar{L}))A$, $Z_r(\bar{L}) \supseteq \pi(LZ_r(A)A)$.*

Доказательство. Как было показано при доказательстве теоремы 15 $(A, L, A/r(A; L), L/r(L; L))$ — точный справа М. к. Поэтому $LZ_r(\bar{L})[A/r(A; L)] \subseteq Z_r(A)$. Осталось заметить, что $LZ_r(\bar{L})[A/r(A; L)] = L\pi^{-1}(Z_r(\bar{L}))A$. Второе включение доказывается аналогично.

Напомним, что левый A-модуль V называется модулем без кручения, если для каждого $0 \neq v \in V$ существует элемент $f \in W = \mathrm{Hom}_A(V, A)$ такой, что $vf \neq 0$. Если V — левый A-модуль без кручения, то $(A, V, W, \mathrm{End}_A(V))$ — точный справа М. к. (см. [1]). Поэтому из следствия 34 вытекает следующее предложение.

Предложение 36. *Пусть A — полупервичное кольцо V — левый A-модуль без кручения, $S = \mathrm{Hom}_l(V; V)$ и $W = \mathrm{Hom}_A(V; A)$. Тогда* 1) $Z_l(S) \supseteq WZ_l(A)V$, $Z_r(S) \supseteq WZ_r(A)V$; 2) $Z_l(A) \supseteq VZ_l(A)W$, $Z_r(A) \supseteq VZ_r(S)W$; 3) *если* $Z_l(A)=0$, *то* $Z_r(S)=0$; 4) *если* $Z_r(A)=0$, *то* $Z_r(S)=0$.

Литература

[1] S. A. Amitsur, Rings of quotients and Morita contex, *J. Algebra*, **17** (1971), 273—298.
[2] В. А. Андрунакиевич, Ю. М. Рябухин, *Радикалы алгебр и структурная теория*, Наука, (Москва, 1979).
[3] K. I. Beidar, K. Salavová, Some examples of supernilpotent non-special radicals, *Acta Math. Acad. Sci. Hungar.* (to appear).
[4] N. Divinsky, *Rings and radicals.* University of Toronto (1965).
[5] J. W. Fisher, Structure of semiprime *PI*-rings, *Proc. Amer. Math. Soc.*, **39** (1973), 465—467.
[6] A. E. Hoffman and W. G. Leavitt, Properties inherited by the lower radical, *Portugal Math.*, **27** (1968), 63—66.
[7] A. Jaegerman, A. D. Sands, On normal radicals, N-radicals, and A-radicals, *J. Algebra*, **50** (1978), 337—349.
[8] И. Ламбек, *Кольца и модули*, Мир (Москва, 1971).
[9] W. G. Leavitt, Lower radical constructions, *Colloq. Math. Soc. Janos Bolyai*, **6** (1973), 319—324.
[10] W. K. Nicholson, J. F. Watters, Normal radicals and normal class of rings, *J. Algebra*, **59** (1979), 5—15.
[11] C. Procesi, *Rings with polynomial identities*, Marcel Dekker (New York, 1973).
[12] R. F. Rossa, More properties inherited by the lower radical, *Proc. Amer. Math. Soc.*, **33** (1972), 247—249.
[13] A. D. Sands, Strong upper radicals, *Quart. J. Math. Oxford*, **27** (1976), 21—24.
[14] A. D. Sands, Radical and Morita context, *J. Algebra*, **24** (1973), 335—345.
[15] A. D. Sands, On relations among radical properties, *Glasgow Math. Journal*, **18** (1977), 17—23.
[16] J. T. Stafford, A simple Noetherian ring is not Morita equivalent to a domain, *Proc. Amer. Math. Soc.*, **68** (1978), 159—160.
[17] B. Stenstrom, *Rings and modules of quotient*, Lect. Notes Math. 237, Springer (1971).
[18] F. Szász, Beiträge zur Radikaltheorie der Ringe, *Publ. Math. Debrecen*, **17** (1970), 267—271.
[19] F. Szász, *Radikale der Ringe*, VEB Deutscher Verlag der Wissenschaften (Berlin, 1973).

(Поступило 25. 11. 1981.)

DEPARTMENT OF MATHEMATICS
SLOVAK TECHNICAL UNIVERSITY
BRATISLAVA
RADLINSKEHO 11
CZECHOSLOVAKIA

Acta Math. Hung.
42 (1—2) (1983), 97—100.

DÉTERMINATION DES GÉODÉSIQUES DE CERTAINS ESPACES RIEMANNIENS SINGULIERS

E. GRECU (Bucharest)

1. Considérations générales sur les groupes semi-orthogonaux

On dénote par $R_p^{r,t}$ l'espace semi-euclidien à p dimensions, de l'indice r et de défaut $p-r-t$, dérivé de l'espace numérique réel à p dimensions R_p doué du produit scalaire (forme bilinéaire, symétrique, dégénérée)

$$(1) \qquad \langle x_1, x_2\rangle = -x_1^1x_1^2 - \ldots - x_1^rx_2^r + x_1^{r+1}x_2^{r+1} + \ldots + x_1^{r+t}x_2^{r+t} \quad (r+t<p)$$

où $x_1=(x_1^1, \ldots, x_1^p)$, $x_2=(x_2^1, \ldots, x_2^p)$ sont deux vecteurs arbitraire de R_p (voir [3]).

Il est facile de voir que le groupe linéaire homogène $x'^j=a_i^jx^i$ $(i,j=1, \ldots, p)$ laisse invariant le produit scalaire (1) si et seulement si la matrice $A=[a_i^j]$ vérifie la relation

$$(2) \qquad AE_p^{r,t}\tilde{A} = E_p^{r,t},$$

où $E_p^{r,t}$ est la matrice d'ordre p ayant la forme cellulaire

$$E_p^{r,t} = \begin{bmatrix} -E_r & 0 & 0 \\ 0 & E_t & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

et $\tilde{A}$ est la transposée de la matrice A. Nous avons dénoté par E_r la matrice d'unité d'ordre r et par 0 la matrice nulle.

De la relation (2) il résulte que la matrice A a la forme cellulaire

$$A = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix},$$

où A_1 est une matrice pseudo-orthogonale d'ordre $r+t$ et de l'indice r, A_2 est une matrice non-singulière d'ordre $p-r-t$ et A_3 est une matrice rectangulaire avec $r+t$ lignes et $p-r-t$ colonnes. Nous avons donc la relation $A_1E_{r+t}^{r,t}\tilde{A}_1=E_{r+t}^{r,t}$ où $E_{r+t}^{r,t}$ est une matrice d'ordre $r+t$ ayant la forme cellulaire

$$E_{r+t}^{r,t} = \begin{bmatrix} -E_r & 0 \\ 0 & E_t \end{bmatrix}.$$

Le groupe des matrices non-singulières A qui satisfont la relation (2) s'appelle le groupe semi-orthogonal en p variables, de l'indice r et de défaut $p-r-t$. On le dénote par $O^{r,t}(p)$.

Dans [1] nous avons montré que le groupe $O^{r,t}(p)$ a $n=p(p-r-t)+(r+t)\cdot$ $\cdot(r+t-1)/2$ paramètres et qu'il admet la représentation paramétrique rationnelle

$$A=\begin{bmatrix} A_1 & X_3 \\ 0 & X_2 \end{bmatrix}, \quad A_1=(E^{r,t}_{r+t}+X_1)^{-1}(E^{r,t}_{r+t}-X_1), \quad |E^{r,t}_{r+t}+X_1|\neq 0, \tag{3}$$

où X_1 est une matrice antisymétrique d'ordre $r+t$, X_2 est une matrice non-singulière d'ordre $p-r-t$, et X_3 est une matrice rectangulaire avec $r+t$ lignes et $p-r-t$ colonnes.

2. Définition de l'espace semi-riemannien V_n

L'espace vectoriel R_{p^2} des matrices réelles d'ordre p, doué du produit scalaire (forme bilinéaire, symétrique, dégénérée)

$$\langle A, B\rangle = \mathrm{Tr}(AE^{r,t}_p\tilde{B})$$

où $\mathrm{Tr}(\cdot)$ représente la trace de la matrice dans la parenthèse, est un espace semi-euclidien $R^{pr,pt}_{p^2}$. La métrique de l'espace $R^{pr,pt}_{p^2}$ est donc définie par la formule

$$ds^2=\mathrm{Tr}(dAE^{r,t}_p d\tilde{A}), \tag{4}$$

où dA est la différentielle de la matrice A.

Dans l'espace $R^{pr,pt}_{p^2}$, les équations qui résultent de l'égalité matricielle (2) définissent une variété algébrique V_n à n dimensions, qui est un espace semi-riemannien (espace de Riemann singulier ou espace riemannien à métrique dégénérée). La métrique de V_n est induite par la métrique (4) de l'espace $R^{pr,pt}_{p^2}$ dans lequel V_n est plongé, donc elle s'obtient en remplaçant dans (4) les éléments a^j_i de la matrice A par p^2 fonctions de n coordonnées. L'expression locale de la métrique de V_n relative à la carte (3) est

$$ds^2=-4\,\mathrm{Tr}\,[dX_1(E^{r,t}_{r+t}-X_1E^{r,t}_{r+t}X_1)^{-1}dX_1(E_{r+t}-E^{r,t}_{r+t}X_1+X_1E^{r,t}_{r+t}-X^2)^{-1}].$$

3. Détermination de certains sous-groupes à un paramètre du groupe $O^{r,t}(p)$

Dans ce qui suit, nous nous proposons de montrer que le chemin $A\colon R\to O^{r,t}(p)$ de l'espace semi-riemannien $O^{r,t}(p)$

$$A(u)=\begin{bmatrix} A_1(u) & A_3(u) \\ 0 & A_2(u) \end{bmatrix}, \quad A_1(u)=\exp\,[u(E^{r,t}_{r+t}Y_1+Y_1E^{r,t}_{r+t})] \tag{5}$$

est un sous-groupe à un paramètre du groupe semi-orthogonal $O^{r,t}(p)$. Ici Y_1 est une matrice antisymétrique d'ordre $r+t$, $A_2(u)$ est une matrice non-singulière d'ordre $p-r-t$, $A_3(u)$ est une matrice rectangulaire avec $r+t$ lignes et $p-r-t$ colonnes, et

$$\exp B=E_p+\frac{1}{1!}B+\frac{1}{2!}B^2+\ldots+\frac{1}{m!}B^m+\ldots. \tag{6}$$

Il est facile à voir que $Z_1=E^{r,t}_{r+t}Y_1+Y_1E^{r,t}_{r+t}$ est une matrice antisymétrique d'ordre $r+t$.

En introduisant les matrices

$$M_1(u) = \exp(uE^{r,t}_{r+t}Y_1), \quad N_1(u) = \exp(uY_1E^{r,t}_{r+t}) \tag{7}$$

on peut écrire la relation

$$A_1(u) = M_1(u)N_1(u). \tag{8}$$

En tenant compte de (6), (7) nous avons les formules

$$M_1(u) = E_p + \frac{u}{1!}(E^{r,t}_{r+t}Y_1) + \ldots + \frac{u^m}{m!}(E^{r,t}_{r+t}Y_1)^m + \ldots$$

$$N_1(u) = E_p + \frac{u}{1!}(Y_1E^{r,t}_{r+t}) + \ldots + \frac{u^m}{m!}(Y_1E^{r,t}_{r+t})^m + \ldots$$

d'où il résulte les relations

$$M_1(u)E^{r,t}_{r+t} = E^{r,t}_{r+t}N_1(u), \quad \tilde{M}_1(u) = N_1(-u). \tag{9}$$

De (9) nous avons immédiatement les formules

$$M_1(u)E^{r,t}_{r+t}\tilde{M}_1(u) = E^{r,t}_{r+t}, \quad E^{r,t}_{r+t}M_1(u)E^{r,t}_{r+t} = N_1(u).$$

qui nous montrent que $M_1(u)$ et $N_1(u)$ sont des matrices pseudo-orthogonales. Donc la matrice $A_1(u)$ donnée par la formule (8) est pseudo-orthogonale, par conséquent

$$A_1(u)E^{r,t}_{r+t}\tilde{A}_1(u) = E^{r,t}_{r+t} \tag{10}$$

c'est-à-dire le chemin $A_1\colon R \to O^r(r+t)$ de l'espace pseudo-riemannien $O^r(r+t)$

$$A_1(u) = \exp(uZ_1), \quad Z_1 = E^{r,t}_{r+t}Y_1 + Y_1E^{r,t}_{r+t} \tag{11}$$

est un sous-groupe à un paramètre du groupe pseudo-orthogonal $O^r(r+t)$ en $r+t$ variables, de l'indice r.

En tenant compte de (5), (10) il est facile à voir que nous avons la relation

$$A(u)E^{r,t}_{p}\tilde{A}(u) = E^{r,t}_{p} \tag{12}$$

donc le chemin (5) est un sous-groupe à un paramètre du groupe semi-orthogonal $O^{r,t}(p)$.

On peut énoncer le

THÉORÈME. *Les chemins* (5) *de l'espace riemannien singulier* $O^{r,t}(p)$ *sont des sous-groupes à un paramétre du groupe Lie* $O^{r,t}(p)$.

Nous voulons enfin démontrer que les chemins (5) sont des géodésiques de l'espace semi-riemannien $O^{r,t}(p)$.

En effet, si nous dérivons la relation (12) par rapport à u, nous avons

$$A'(u)E^{r,t}_{p}\tilde{A}(u) + A(u)E^{r,t}_{p}\tilde{A}'(u) = 0. \tag{13}$$

Cette relation constitue l'équation de la variété linéaire tangente au point $A(u)$ à l'espace semi-riemannien $O^{r,t}(p)$. En posant $C = A'E^{r,t}_{p}\tilde{A}$ on peut écrire l'équation (13) sous la forme $C + \tilde{C} = 0$.

Pour démontrer que le chemin (5) est géodésique dans l'espace $O^{r,t}(p)$ il est suffisant de démontrer que la vecteur $A''(u)$ est orthogonal à chaque vecteur A' qui satisfait la relation (13) pour $A=A(u)$, donc que nous avons $\langle A', A''(u)\rangle=0$. On sait [5] qu'ils résultent de (11) les relations

$$A_1'(u) = Z_1 A_1(u), \quad A_1''(u) = Z_1^2 A_1(u).$$

Il est facile à voir que la relation (13) implique la relation

$$A_1' E_{r+t}^{r,t} \tilde{A}_1 + A_1 E_{r+t}^{r,t} \tilde{A}_1' = 0. \tag{14}$$

Si nous posons $C_1=A_1' E_{r+t}^{r,t} \tilde{A}_1$, la relation (14) s'écrit sous la forme $C_1+\tilde{C}_1=0$. On obtient alors que

$$\langle A', A''(u)\rangle = \operatorname{Tr}\left(A' E_p^{r,t} \tilde{A}''(u)\right) = \operatorname{Tr}\left(A_1' E_{r+t}^{r,t} \tilde{A}_1''(u)\right) =$$
$$= \operatorname{Tr}\left(A_1' E_{r+t}^{r,t} \tilde{A}_1(u) Z_1^2\right) = \operatorname{Tr}(C_1 Z_1^2) = 0$$

puisque la matrice C_1 est antisymétrique, Z_1^2 est symétrique et le produit d'une matrice symétrique par une matrice antisymétrique est une matrice ayant la trace nulle. Nous avons donc le

THÉORÈME. *Les sous-groupes à un paramétre* (5) *du groupe Lie* $O^{r,t}(p)$ *sont des géodésiques de l'espace de Riemann singulier* $O^{r,t}(p)$.

Nous mentionnons que les sous-groupes à un paramètre du groupe orthogonal en p variables $O(p)$, ainsi que les géodésiques de l'espace riemannien $O(p)$ ont été déterminés par C. Teleman [5]. Autres considérations sur les espaces riemanniens singuliers sont dues à Gr. Moisil [2], B. A. Rozenfeld et L. M. Karpova [4], G. Vrânceanu [6].

Bibliographie

[1] E. Grecu, Sur certaines variétés de groupes, *Bull. Math. de la Soc. Sci. Math. de la R.S.R.* (sous press).
[2] Gr. Moisil, Sur les géodésiques des espaces de Riemann singuliers, *Bull. Math. de la Soc. Roum. des Sci.*, **42** (1941), 33—52.
[3] B. A. Rozenfeld, *Espaces non-euclidiens* (Moscou, 1974) (en russe).
[4] B. A. Rozenfeld, L. M. Karpova, Espaces semi-riemanniens symétriques, *Izv. Vyss. Učeb. Zaved., Matematika,* **38** (1964), 100—116 (en russe).
[5] C. Teleman, *Geometrie diferențială locală si globală.* Editura tehnică (Bucureşti, 1974), 246—249, 305—338.
[6] G. Vranceanu, Sur les invariants des espaces de Riemann singuliers, *Disq. Math. et Phys.*, **II**, 2—4 (1942), 17—48.

(Reçu le 25 novembre 1981)

INSTITUT POLYTECHNIQUE DE BUCAREST
ROUMANIE

Acta Math. Hung.
42 (1—2) (1983), 101—109.

ON NON-OSCILLATORY SOLUTIONS OF SECOND ORDER DIFFERENTIAL INEQUALITIES

N. PARHI (Berhampur)*

1. In this paper, we are concerned with the behaviour of solutions of the following second-order differential inequalities:

$$(r(t)y')'-q(t)(y')^{\alpha}-p(t)y^{\gamma}>0, \tag{1.1}$$

and

$$(r(t)y')'-q(t)(y')^{\alpha}-p(t)y^{\gamma}\geqq 0, \tag{1.2}$$

where p, q and r are real-valued continuous functions on $[0, \infty)$ such that $p(t)\geqq 0$, $r(t)>0$, for $t\in[0, \infty)$ and each of $\alpha>0$ and $\gamma>0$ is the ratio of odd integers.

By a solution of (1.1) or (1.2) we mean a continuously differentiable function $y(t)$ such that $r(t)y'(t)$ is continuously differentiable and (1.1) or (1.2) is satisfied. We restrict our considerations to those solutions of (1.1) or (1.2) which exist on the half line $[T, \infty)$, where T depends on the particular solution. Results in this paper generalize and unify some of the results in [2].

We classify solutions of (1.1) or (1.2) as follows: a solution $y(t)$ is said to be non-oscillatory if there exists a $t_1\geqq T$ such that $y(t)\neq 0$ for $t\geqq t_1$; $y(t)$ is said to be oscillatory if for any $t_1\geqq T$ there exist t_2 and t_3 satisfying $t_1<t_2<t_3$ such that $y(t_2)>0$ and $y(t_3)<0$; and it is said to be a Z-type solution if it has arbitrarily large zeros but is ultimately non-negative or non-positive. $y(t)$ is said to be disconjugate if it has at most one zero.

2. In this section, we show that all solutions of (1.1) or (1.2) are non-oscillatory.

THEOREM 2.1. *If* $q(t)\geqq 0$, *then all solutions of* (1.1) *are non-oscillatory.*

PROOF. Let $y(t)$ be an oscillatory or a non-negative Z-type solution of (1.1) with consecutive zeros at a and b $(T<a<b)$ such that $y'(a)\geqq 0$, $y'(b)\leqq 0$ and $y(t)>0$ for $t\in(a, b)$. So there exists a $c\in(a, b)$ such that $y'(c)=0$ and $y'(t)>0$ for $t\in(a, c)$. Integrating (1.1) from a to c, we get

$$0\geqq -r(a)y'(a)>\int_a^c q(t)(y'(t))^{\alpha}\,dt+\int_a^c p(t)y^{\gamma}(t)\,dt>0,$$

a contradiction.

If possible, let $y(t)$ be a non-positive Z-type solution of (1.1) with consecutive double zeros at a and b $(T<a<b)$. So there exist a $c_1\in(a, b)$ such that $y'(t)>0$ for $t\in[c_1, b)$ and a $c_2\in(a, b)$ such that $y(t)>-1$ for $t\in[c_2, b]$. Let $c=\max(c_1, c_2)$.

* This work was supported by the University Grants Commission, India.

Hence, for $t\in[c, b)$, $y'(t)>0$ and $y(t)>-1$. Integrating (1.1) from t to b, $t\in[c, b)$, we obtain

$$-r(t)y'(t) > \int_t^b q(s)(y'(s))^\alpha\, ds + \int_t^b p(s)y^\gamma(s)\, ds \geqq$$

$$\geqq \int_t^b p(s)y^\gamma(s)\, ds \geqq -\int_t^b p(s)\, ds,$$

that is,

$$\int_t^b p(s)\, ds > r(t)y'(t).$$

Since

$$\lim_{t\to b} \frac{1}{r(t)y'(t)} \int_t^b p(s)\, ds = \lim_{t\to b} \frac{-p(t)}{(r(t)y'(t))},$$

and from (1.1) $(r(t)y'(t))'|_{t=b}>0$, we have

$$\lim_{t\to b} \frac{1}{r(t)y'(t)} \int_t^b p(s)\, ds \leqq 0,$$

a contradiction. Hence the theorem.

The following examples illustrate Theorem 2.1.

EXAMPLE (i)

$$y'' - (y')^3 - \frac{1}{t^3} y^{1/3} > 0, \quad t \geqq 1.$$

In particular, $y(t)=\frac{1}{t}$ is a non-oscillatory solution.

EXAMPLE (ii) $y''-y^3>0$, $t\geqq 1$. $y(t)=t^{-2}$ and $z(t)=t^{-1}$ are non-oscillatory solutions.

REMARK 2.2. A solution $y(t)$ of (1.1) may have two zeros of the type $y(a)=0=y(b)$, $y'(a)<0$, $y'(b)>0$ and $y(t)<0$, $t\in(a, b)$ or $y(a)=0=y'(a)$ and $y(t)>0$, $t\neq a$. This suggests that all solutions of (1.1) are non-oscillatory but not necessarily disconjugate. In the following, we give an example to illustrate this remark.

EXAMPLE.

$$y'' - \frac{1}{t^2} y > 0, \quad t \geqq 2/3.$$

From Theorem 2.1 it follows that all solutions of the above inequality are non-oscillatory. In particular, $y(t)=(t-1)(t-2)$, $t\in[2/3, \infty)$, is a non-oscillatory solution of the inequality. Note that $y(t)$ has two zeros, viz, $t=1$ and $t=2$. Clearly, $y'(1)<0$, $y'(2)>0$ and $y(t)<0$ for $t\in(1,2)$. Again $y(t)=(t-1)^2$, $t\in[2/3, \infty)$ is a non-oscillatory solution of the inequality and it has a double zero at $t=1$, that is, $y(1)=0=y'(1)$. Clearly, $y(t)>0$ for $t\neq 1$.

THEOREM 2.3. *If* $q(t)\geqq 0$ *and* $r(t)$ *is differentiable, then all solutions of* (1.2) *are non-oscillatory.*

PROOF. In case $y(t)$ is an oscillatory or a non-negative Z-type solution of (1.2), we proceed as in Theorem 2.1 to get a contradiction.

If possible, let $y(t)$ be a non-positive Z-type solution of (1.2) with consecutive double zeros at a, b and $a'(a<b<a')$. So $y(a)=y(b)=y(a')=0$, $y'(a)=y'(b)=$ $=y'(a')=0$, $y(t)<0$, for $t\in(a, b)$ and $y(t)<0$ for $t\in(b, a')$. Consequently, there exist $c\in(a, b)$ and $d\in(b, a')$ such that $y''(t)<0$ for $t\in(c, d)$. In particular, $y''(b)<0$. Since $r(t)$ is differentiable, (1.2) takes the form

$$r(t)y''(t)+r'(t)y'(t)\geqq q(t)\left(y'(t)\right)^{\alpha}+p(t)y^{\gamma}(t).$$

Hence $y''(b)\geqq 0$, a contradiction.

This completes the proof of the theorem.

REMARK 2.4. Let $q(t)\geqq 0$. If, in (1.1), (i) $\alpha>0$ is the ratio of even and odd integers and $\gamma>0$ is the ratio of odd integers, or (ii) $\alpha>0$ is the ratio of odd integers and $\gamma>0$ is the ratio of even and odd integers, or (iii) each of $\alpha>0$ and $\gamma>0$ is the ratio of even and odd integers, then all solutions of (1.1) are non-oscillatory. Indeed, in Case (i), (1.1) reduces to $\left(r(t)y'(t)\right)'>p(t)y^{\gamma}(t)$ and we proceed as in Theorem 2.1. In Case (ii), (1.1) takes the form $\left(r(t)y'(t)\right)'>q(t)\left(y'(t)\right)^{\alpha}$. If $y(t)$ is oscillatory or non-negative Z-type, we proceed as in Theorem 2.1. If $y(t)$ is of non-positive Z-type with consecutive double zeros at a and b $(a<b)$, there exists a $c\in(a, b)$ such that $y'(t)>0$ for $t\in(c, b)$. We get a contradiction by integrating the above inequality from c to b. In case (iii), (1.1) reduces to $\left(r(t)y'(t)\right)'>0$. If $y(t)$ is of Z-type with consecutive double zeros at a and b $(a<b)$, we integrate the above inequality from a to b to get a contradiction. If $y(t)$ is oscillatory with consecutive zeros at a and b $(a<b)$ such that $y(t)>0$ for $t\in(a, b)$, we integrate the above inequality from a to b again to get a contradiction.

If $q(t)\geqq 0$ and $r(t)$ is differentiable, then the above remark is true for (1.2).

THEOREM 2.5. *If* $q(t)\leqq 0$, *then all solutions of* (1.1) *are non-oscillatory.*

PROOF. If $y(t)$ is an oscillatory or a non-negative Z-type solution of (1.1) with consecutive zeros at a and b $(a<b)$ such that $y'(a)\geqq 0$, $y'(b)\leqq 0$ and $y(t)>0$, $t\in(a, b)$, then there exists a $c\in(a, b)$ such that $y'(c)=0$ and $y'(t)<0$ for $t\in(c, b)$. We get a contradiction by integrating (1.1) from c to b.

Further, if $y(t)$ is a non-positive Z-type solution of (1.1) with consecutive double zeros at a and b $(a<b)$, there exists a $c\in(a, b)$ such that $y(t)>-1$ and $y'(t)<0$ for $t\in(a, c]$. Integrating (1.1) from a to t, $t\in(a, c]$, and proceeding as in Theorem 2.1, we get the necessary contradiction.

Hence the theorem.

THEOREM 2.6. *If* $q(t)\leqq 0$ *and* $r(t)$ *is differentiable, then all solutions of* (1.2) *are non-oscillatory.*

PROOF. When $y(t)$ is oscillatory or of non-negative Z-type, we proceed as in Theorem 2.5 to get a contradiction and if $y(t)$ is of non-positive Z-type, then a contradiction is obtained by proceeding as in the second part of Theorem 2.3.

REMARK 2.7. Let $q(t) \leqq 0$. If, in (1.1), $\alpha > 0$ is the ratio of odd integers and $\gamma > 0$ is the ratio of even and odd integers, then all solutions of (1.1) are non-oscillatory. Indeed, in this case (1.1) reduces to $(r(t)y'(t))' > q(t)(y'(t))^{\alpha}$. If $y(t)$ is oscillatory or non-negative Z-type, we proceed as in Theorem 2.5. If $y(t)$ is of non-positive Z-type with consecutive double zeros at a and b $(a<b)$, then there exists a $c \in (a, b)$ such that $y'(c)=0$ and $y'(t)<0$ for $t \in (a, c)$. We get a contradiction by integrating the above inequality from a to c. Note that in this case we do not require the differentiability of $r(t)$.

If $q(t) \leqq 0$, then the above remark holds for (1.2).

3. In this section, we study asymptotic behaviour of solutions of

$$(r(t)y')' - p(t)y^{\gamma} > 0 \tag{3.1}$$

and

$$(r(t)y')' - p(t)y^{\gamma} \geqq f(t), \tag{3.2}$$

where p, r and γ are the same as in (1.1) and f is a positive real-valued continuous function on $[0, \infty)$. From the discussion in Section 2, it follows that all solutions of (3.1) or (3.2) are non-oscillatory.

THEOREM 3.1. *If* $\int_0^{\infty} \frac{dt}{r(t)} = \infty$ *and* $\int_0^{\infty} p(t)\,dt = \infty$, *then ultimately positive bounded solutions of* (3.1) *tend to zero as* $t \to \infty$.

PROOF. Let $y(t)$ be an ultimately positive bounded solution of (3.1) such that $y(t)>0$ for $t \geqq t_0 \geqq T$. From (3.1) it is clear that $y'(t)$ has at most one zero in $[t_0, \infty)$. So there exists a $t_1 > t_0$ such that $y'(t) \neq 0$ in $[t_1, \infty)$. We claim that $y'(t)<0$ in $[t_1, \infty)$. If not, $y'(t)>0$, $t \in [t_1, \infty)$. Since $(r(t)y'(t))'>0$ for $t \geqq t_1$, $y'(t) > r(t_1)y'(t_1)\frac{1}{r(t)}$. Integrating this inequality from t_1 to t, we get

$$y(t) > y(t_1) + r(t_1)y'(t_1)\int_{t_1}^{t} \frac{ds}{r(s)}.$$

From the assumption on $r(t)$ it follows that $y(t)$ is unbounded. Hence $y'(t)<0$ for $t \geqq t_1$, that is, $y(t)$ is decreasing in $[t_1, \infty)$. So $\lim_{t\to\infty} y(t)$ exists. If possible, let $0 < k = \lim_{t\to\infty} y(t)$. So, for $0<\varepsilon<k$, there exists a $t_2 > t_1$ such that $t \geqq t_2$ implies that $k - \varepsilon < y(t)$. Integrating (3.1) from t_2 to t, we get

$$r(t)y'(t) > r(t_2)y'(t_2) + \int_{t_2}^{t} p(s)y^{\gamma}(s)\,ds > r(t_2)y'(t_2) + (k-\varepsilon)^{\gamma}\int_{t_2}^{t} p(s)\,ds.$$

Since $\int_0^{\infty} p(t)\,dt = \infty$, from the above inequality it follows that $r(t)y'(t)>0$ for sufficiently large t, that is, $y'(t)>0$ for sufficiently large t. This contradiction completes the proof of the theorem.

COROLLARY 3.2. *Let*

$$\int_0^\infty \frac{dt}{r(t)} = \infty \quad \text{and} \quad \int_0^\infty p(t)\,dt = \infty.$$

Then all ultimately positive solutions $y(t)$ *of* (3.1) *which do not tend to zero tend to* ∞. *Further, if* $\lim_{t\to\infty} r(t) < \infty$, *then* $\lim_{t\to\infty} \frac{y(t)}{t} = \infty$.

PROOF. $y(t) \not\to 0$ as $t \to \infty$ implies that $y(t)$ is unbounded and $y'(t) > 0$ for large t, This in turn implies that $y(t) \to \infty$ as $t \to \infty$. Further, if $\lim_{t\to\infty} r(t) = 0$, then $\lim_{t\to\infty} y'(t) = \infty$, because $y'(t) > r(t_1)y'(t_1)\frac{1}{r(t)}$ and $y'(t_1) > 0$ for large t_1. Let $0 < \lim_{t\to\infty} r(t) < \infty$. Since $y(t) \to \infty$ as $t \to \infty$, for $M > 0$ there exists a $t_2 > 0$ such that $t \geqq t_2$ implies that $y(t) > M$. From (3.1), we get

$$r(t)y'(t) > r(t_2)y'(t_2) + M^\gamma \int_{t_2}^t p(s)\,ds,$$

that is, $\lim_{t\to\infty} r(t)\,y'(t) = \infty$ and hence $\lim_{t\to\infty} y'(t) = \infty$. So

$$\lim_{t\to\infty} \frac{y(t)}{t} = \lim_{t\to\infty} y'(t) = \infty.$$

This proves the corollary.

THEOREM 3.3. *If* $\int_0^\infty \frac{1}{r(t)}\left(\int_0^\infty p(s)\,ds\right)dt = \infty$ *and* $\int_0^\infty p(t)\,dt = \infty$, *then ultimately positive bounded solutions of* (3.1) *tend to zero as* $t \to \infty$.

The proof is the same as that of Theorem 3.1.

THEOREM 3.4. *If* (3.1) *admits an ultimately positive, increasing, bounded solution, then*

$$\lim_{t\to\infty} \int_0^t \frac{1}{r(s)}\left(\int_0^s p(\theta)\,d\theta\right)ds < \infty.$$

PROOF. Let $y(t)$ be a solution of (3.1) such that $y(t) > 0$ for $t \geqq t_0 \geqq T$, increasing and bounded. Integrating (3.1) from t_0 to t, we get

$$r(t)y'(t) > r(t_0)y'(t_0) + \int_{t_0}^t p(s)y^\gamma(s)\,ds > r(t_0)y'(t_0) + y^\gamma(t_0)\int_{t_0}^t p(s)\,ds.$$

Dividing by $r(t)$ and integrating the resulting inequality from t_0 to t, we get

$$y(t) > y(t_0) + r(t_0)y'(t_0)\int_{t_0}^t \frac{ds}{r(s)} + y^\gamma(t_0)\int_{t_0}^t \frac{1}{r(s)}\left(\int_{t_0}^s p(\theta)\,d\theta\right)ds.$$

The result follows from the boundedness of $y(t)$.

The following example illustrates Theorem 3.4.

EXAMPLE. Consider

$$(t^2y')'-\frac{1}{t^2}y>0, \quad t\geqq a, \tag{3.3}$$

where $a>e$ is such that $\log t-1>0$ for $t\geqq a$. Clearly, $y(t)=t^{-1/t}$, $t\geqq a$, is a positive solution of (3.3) such that $y'(t)=\frac{y(t)}{t^2}(\log t-1)$. So $y(t)$ is increasing. Further, since $\lim_{t\to\infty} t^{-1/t}=1$ (cf. Widder [3], p. 270), $y(t)$ is bounded. It is easy to see that

$$\int_a^t \frac{1}{r(s)}\left(\int_a^s p(\theta)\,d\theta\right)ds<\frac{1}{a^2}.$$

THEOREM 3.5. *Let* $h'(t)\geqq 0$, *where* $h(t)=\frac{1}{r(t)}\int_0^t p(s)\,ds$, $t\geqq 0$. *If* (3.1) *has an ultimately positive, increasing, bounded solution, then*

$$\lim_{t\to\infty}\frac{1}{r(t)}\int_0^t p(s)\,ds<\infty.$$

PROOF. We claim that $h(t)$ is bounded. If not, for every $M>0$ there exists a $t_1>0$ such that $h(t_1)>M$. Since $h(t)$ is monotone increasing $h(t)>M$ for $t\geqq t_1$. Now, for $t\geqq t_1$,

$$\int_0^t \frac{1}{r(s)}\left(\int_0^s p(\theta)\,d\theta\right)ds=\int_0^{t_1}\frac{1}{r(s)}\left(\int_0^s p(\theta)\,d\theta\right)ds+\int_{t_1}^t\frac{1}{r(s)}\left(\int_0^s p(\theta)\,d\theta\right)ds\geqq$$

$$\geqq\int_{t_1}^t\frac{1}{r(s)}\left(\int_0^s p(\theta)\,d\theta\right)ds=\int_{t_1}^t h(s)\,ds>M(t-t_1).$$

So $\lim_{t\to\infty}\int_0^t\frac{1}{r(s)}\left(\int_0^s p(\theta)\,d\theta\right)ds=\infty$, a contradiction to Theorem 3.4. Consequently, $\lim_{t\to\infty} h(t)$ exists and is finite.

THEOREM 3.6. *Consider* (3.1) *with* $\gamma=1$. *If, for any ultimately positive increasing solution* $y(t)$ *of* (3.1), $r(t)y'(t)$ *is bounded, then*

$$\lim_{t\to\infty}\int_0^t p(s)\left(\int_0^s\frac{d\theta}{r(\theta)}\right)ds<\infty.$$

PROOF. Let $y(t)>0$ for $t\geqq t_0\geqq T$. So $r(t)y'(t)$ is increasing in $[t_0,\infty)$ and hence

$$y'(t)>r(t_0)y'(t_0)\frac{1}{r(t)}, \quad t>t_0.$$

Integrating the above inequality from t_0 to t, we get

$$y(t) > r(t_0)y'(t_0)\int_{t_0}^{t}\frac{ds}{r(s)}.$$

So, from (3.1), we have

$$\big(r(t)y'(t)\big)' > r(t_0)y'(t_0)p(t)\int_{t_0}^{t}\frac{ds}{r(s)}$$

which on integration yields

$$r(t)y'(t) > r(t_0)y'(t_0)\left[1+\int_{t_0}^{t}p(s)\left(\int_{t_0}^{s}\frac{d\theta}{r(\theta)}\right)ds\right].$$

Hence the result follows from the boundedness of $r(t)y'(t)$.

THEOREM 3.7. *Consider* (3.1) *with* $0<\gamma\leqq 1$. *Let*

$$\lim_{t\to\infty}\int_{0}^{t}\frac{1}{r(s)}\left(\int_{0}^{s}p(\theta)\,d\theta\right)ds < \infty.$$

If $y(t)$ *is an ultimately negative solution of* (3.1), *then* $\overline{\lim_{t\to\infty}}\,|y(t)<\infty$.

PROOF. Let $y(t)$ be an ultimately negative solution of (3.1) such that $y(t)<0$, $t\geqq t_0\geqq T$. Dividing (3.1) through by $y^{\gamma}(t)$ and integrating the resulting inequality from t_0 to t, we get

$$[r(s)y'(s)y^{-\gamma}(s)]_{t_0}^{t}+r\int_{t_0}^{t}r(s)\big(y'(s)\big)^2y^{-(\gamma+1)}(s)\,ds < \int_{t_0}^{t}p(s)\,ds.$$

Hence

$$y^{-\gamma}(t)r(t)y'(t) < y^{-\gamma}(t_0)r(t_0)y'(t_0)+\int_{t_0}^{t}p(s)\,ds,$$

that is,

$$y^{-\gamma}(t)y'(t) < y^{-\gamma}(t_0)r(t_0)y'(t_0)\frac{1}{r(t)}+\frac{1}{r(t)}\int_{t_0}^{t}p(s)\,ds.$$

If $0<\gamma<1$, further integration from t_0 to t yields

$$\frac{y^{1-\gamma}(t)}{1-\gamma} < \frac{y^{1-\gamma}(t_0)}{1-\gamma}+y^{-\gamma}(t_0)r(t_0)y'(t_0)\int_{t_0}^{t}\frac{ds}{r(s)}+ \tag{3.4}$$

$$+\int_{t_0}^{t}\frac{1}{r(s)}\left(\int_{t_0}^{t}p(\theta)\,d\theta\right)ds.$$

If $\gamma=1$, we get

$$\log|y(t)| < \log|y(t_0)|+y^{-1}(t_0)r(t_0)y'(t_0)\int_{t_0}^{t}\frac{ds}{r(s)}+\int_{t_0}^{t}\frac{1}{r(s)}\left(\int_{t_0}^{t}p(\theta)\,d\theta\right)ds. \tag{3.5}$$

Let $c\in(t_0, t)$ be fixed. So

$$\int_{t_0}^{t}\frac{1}{r(s)}\left(\int_{t_0}^{s}p(\theta)\,d\theta\right)ds=\int_{t_0}^{c}\frac{1}{r(s)}\left(\int_{t_0}^{s}p(\theta)\,d\theta\right)ds+$$

$$+\int_{c}^{t}\frac{1}{r(s)}\left(\int_{t_0}^{s}p(\theta)\,d\theta\right)ds\geqq L+\left(\int_{c}^{t}\frac{ds}{r(s)}\right)\left(\int_{t_0}^{c}p(\theta)\,d\theta\right),$$

where $L=\int_{t_0}^{c}\frac{1}{r(s)}\left(\int_{t_0}^{s}p(\theta)\,d\theta\right)ds$. Hence the given condition implies that

$$\lim_{t\to\infty}\int_{0}^{t}\frac{ds}{r(s)}<\infty.$$

From (3.4) and (3.5) we get, for $0<\gamma\leqq 1$, $\overline{\lim_{t\to\infty}}\,|y(t)|<\infty$ and hence the theorem.

REMARK 3.8. For $\gamma>1$, the condition

$$\lim_{t\to\infty}\int_{0}^{t}\frac{1}{r(s)}\left(\int_{0}^{s}p(\theta)\,d\theta\right)ds<\infty$$

is not sufficient to ensure the boundedness of all ultimately negative solutions of (3.1). This is clear from the following example.

EXAMPLE. $(t^3y')'-y^3>0$, $t>3$.

Clearly,

$$\lim_{t\to\infty}\int_{3}^{t}\frac{1}{r(s)}\left(\int_{3}^{s}p(\theta)\,d\theta\right)ds<\infty.$$

But the above inequality admits a negative unbounded solution $y(t)=-t$.

THEOREM 3.9. *Let* $f(t)\geqq p(t)$, $t\in[0,\infty)$ *and* $f(t)\not\equiv p(t)$ *on any subinterval of* $[0,\infty)$. *Then every ultimately negative solution* $y(t)$ *of* (3.2) *with* $|y(t)|<1$ *for large* t *is ultimately increasing or ultimately decreasing.*

PROOF. Let $y(t)<0$ and $|y(t)|<1$ for $t\geqq t_0\geqq T$. We claim that there exists a $t^*>t_0$ such that $y'(t)>0$ or <0 for $t\geqq t^*$. If this is not true, $y'(t)$ is weakly oscillatory (that is, oscillatory or Z-type) on $[t_0,\infty)$. Let $\langle t_n\rangle$ be a sequence of zeros of $y'(t)$ in $[t_0,\infty)$ such that $t_n\to\infty$ as $n\to\infty$. Integrating (3.2) from t_1 to t_n, we get

$$0\geqq\int_{t_1}^{t_n}f(t)\,dt+\int_{t_1}^{t_n}p(t)y^{\gamma}(t)\,dt>\int_{t_1}^{t_n}(f(t)-p(t))\,dt>0.$$

This contradiction proves the theorem.

THEOREM 3.10. *If* $\int_{0}^{\infty}f(t)dt=\infty$, $\int_{0}^{\infty}\frac{d(t)}{r(t)}=\infty$ *and* $\int_{0}^{\infty}p(t)\,dt<\infty$, *then no ultimately negative solution of* (3.2) *is bounded.*

PROOF. If possible, let (3.2) admit an ultimately negative bounded solution $y(t)$. Suppose that $y(t)<0$, $t\geqq t_0\geqq T$ and $|y(t)|\leqq M$, $t\geqq t_0$. Integrating (3.2) from t_0 to t, we get

$$r(t)y'(t) \geqq r(t_0)y'(t_0)+\int_{t_0}^{t} f(s)\,ds+\int_{t_0}^{t} p(s)y^{\gamma}(s)\,ds \geqq$$

$$\geqq r(t_0)y'(t_0)+\int_{t_0}^{t} f(s)\,ds-M^{\gamma}\int_{t_0}^{t} p(s)\,ds.$$

From the assumptions on f and p, it follows that there exists a $t_1\geqq t_0$ such that $r(t)y'(t)\geqq 1$ for $t\geqq t_1$. Integrating this inequality from t_1 to t and using the condition on $r(t)$, we get that $y(t)$ is positive for large t, a contradiction.

Hence the theorem.

THEOREM 3.11. *Consider* (3.1) *and*

$$y''-q(t)y^{\gamma}<0, \tag{3.6}$$

where γ *is the same as in* (3.1) *and* $q(t)\geqq 0$ *is a real-valued continuous function on* $[0,\infty)$. *Let* $y_p(t)$ *and* $y_q(t)$ *be solutions of* (3.1) *and* (3.6) *on* $[t_p,\infty)$ *and* $[t_q,\infty)$, *respectively, such that* $y_p(t_0)=y_q(t_0)$ *and* $y'_p(t_0)>y'_q(t_0)$, *where* $t_p\geqq 0$, $t_q\geqq 0$ *and* $t_0\geqq\max(t_p,t_q)$. *If* $p(t)\geqq q(t)$, *then* $y_p(t)>y_q(t)$, $y'_p(t)>y'_q(t)$ *and* $y''_p(t)>y''_q(t)$, *for* $t\geqq t_0$, *and* $\lim_{t\to\infty}(y_p(t)-y_q(t))=\infty$.

The proof is straightforward and hence is omitted.

Acknowledgement

The author is grateful to Professor I. Bihari for his many valuable remarks which helped to improve the paper.

References

[1] M. Marini and P. Zezza, On the asymptotic behaviour of the solution of a class of second-order linear differential equations, *J. Diff. Eqs.*, **28** (1978), 1—17.

[2] N. Parhi and S. K. Nayak, On non-oscillatory behaviour of solutions of nonlinear differential equations, *Atti Della Acad. Nazionale Dei Lincei*, **LXV** (1978), 58—62.

[3] D. V. Widder, *Advanced calculus*, Prentice Hall (1961).

(Received November 25, 1981)

DEPARTMENT OF MATHEMATICS
BERHAMPUR UNIVERSITY
BERHAMPUR-760 007
ORISSA
INDIA

Acta Math. Hung.
42 (1—2) (1983), 111—119.

REFINED EXTENSIONS OF SYNTOPOGENOUS STRUCTURES AND QUASI-UNIFORMITIES

K. MATOLCSY (Debrecen)

Introduction

In [3] conditions were given under which a syntopogenous structure $\mathscr{S}$ can be extended onto a set Y so that this extension $\mathscr{S}'$ induces a prescribed topology $\mathscr{T}'$ (which is necessarily an extension of the topology of $\mathscr{S}$). If this is possible, then $\mathscr{S}'$ is said to be *compatible* with the pair $(\mathscr{S}, \mathscr{T}')$. The similar problem was investigated for quasi-uniformities in [2].

The present paper introduces an operation $\cdot * \mathfrak{L}$ for syntopogenous structures, where $\mathfrak{L}$ is a system of the subsets of the fundamental set of these structures ($\mathscr{S} * \mathfrak{L}$ will be called a *refinement* of $\mathscr{S}$). As an application of this operator, from known compatible extensions we shall obtain new ones with the same family ot trace filters.

1. Refinements of syntopogenous structures

Let $<$ be a semi-topogenous order on a set E and $L \subset E$. We consider a relation $< * L$ on 2^E as follows:

$$A(< * L)B \text{ iff there exists a set } H \subset E \text{ such that}$$

$$A < H \quad \text{and} \quad A \cup (H \cap L) \subset B.$$

LEMMA 1.1. *$< * L$ is a semi-topogenous order on E satisfying the following conditions*:

(1.1.1) *If* $^a = ^q$, ^{p} *or* ^{b}, *then* $(< * L)^a = <^a * L$.

(1.1.2) *For a semi-topogenous order* $<_1$ *on* E, $< \subset <_1$ *and* $L_1 \subset L$ *imply* $< * L \subset <_1 * L_1$.

(1.1.3) $<^2 * L \subset (< * L)^2$.

(1.1.4) $< \subset < * L$.

PROOF. $\varnothing < \varnothing$ and $\varnothing \cup (\varnothing \cap L) = \varnothing$ imply $\varnothing (< * L) \varnothing$. $E < E$ and $E \cup (E \cap L) = E$ give $E (< * L) E$. If $A (< * L) B$, then we have $A \subset A \cup (H \cap L) \subset B$ for a set $H \subset E$. Suppose $A_1 \subset A (< * L) B \subset B_1$. Then $A \cup (H \cap L) \subset B$ for some H such that $A < H \subset E$, thus we get $A_1 < H$ and $A_1 \cup (H \cap L) \subset A \cup (H \cap L) \subset B \subset B_1$, i.e. $A_1 (< * L) B_1$.

(1.1.1): Suppose that $A (< * L)^a B$. By [1], (3.12), (4.4) and (5.9) this means

$$A = \bigcup_{i \in I} A_i, \quad B = \bigcap_{j \in J} B_j, \quad A_i (< * L) B_j \quad \text{for each} \quad (i, j) \in I \times J,$$

where if $^a=^b$ ($^a=^q$), then I and J are arbitrary (finite), or I is arbitrary and J consists of one element, provided $^a=^p$. There exists $H_{ij}\subset E$ such that $A_i<H_{ij}$ and $A_i\cup(H_{ij}\cap L)\subset B_j$ for any $(i,j)\in I\times J$. Then $A<^a\bigcup_{i\in I}H_{ij}=H_j$ for every $j\in J$, and because of $^{aa}=^a$ we get $A<^a\bigcup_{j\in J}H_j=H$. An elementary calculation shows that $A\cup(H\cap L)\subset B$, thus $A(<^a*L)B$. Conversely, assume $A(<^a*L)B$. Then $A<^aH$, $A\cup(H\cap L)\subset B$ for some $H\subset E$. Following the above train of thought, we have

$$A=\bigcup_{i\in I}A_i,\quad H=\bigcap_{j\in J}H_j,\quad A_i<H_j\quad\text{for any}\quad(i,j)\in I\times J,$$

where I and J are sets of indices chosen for the corresponding case of $^a=^q$, ^{p} or ^{b}. Then $A_i(<*L)A_i\cup(H_j\cap L)$ for every $(i,j)\in I\times J$, therefore $A(<*L)^aA\cup\cup(H_j\cap L)$, finally by $^{aa}=^a$, the relations

$$A(<*L)^a\bigcap_{j\in J}(A\cup(H_j\cap L))=A\cup(\bigcap_{j\in J}H_j\cap L)=A\cup(H\cap L)\subset B$$

hold, so that $A(<*L)^aB$.

(1.1.2): In the case of $<\subseteq<_1$ and $L_1\subset L$ the inequalities $A<H$, $A\cup(H\cap L)\subset B$ imply $A<_1H$ and $A\cup(H\cap L_1)\subset B$, thus from $A(<*L)B$ the relation $A(<_1*L_1)B$ follows.

(1.1.3): Suppose $A(<^2*L)B$. Then $A<K<H$ and $A\cup(H\cap L)\subset B$. Consider the set $C=A\cup(K\cap L)$. $A(<*L)C$ is trivial. At the same time $C\subset K$, hence $C<H$, further $C\cup(H\cap L)=A\cup(K\cap L)\cup(H\cap L)=A\cup(H\cap L)\subset B$, thus $C(<*L)B$. We got $A(<*L)^2B$.

(1.1.4): $A<B$ implies $A\subset B$, and from $B\cap L\subset B$ the inclusion $A\cup(B\cap L)\subset B$ follows, i.e. $A(<*L)B$. Q.E.D.

REMARK. Let us observe that for every semi-topogenous order $<$ on E, we have $<*\varnothing=\subset$ and $<*E=<$.

Further by $\mathfrak{L}$ we shall denote a non-empty system of the subsets of E satisfying the following condition:

(L) *$L_1, L_2\in\mathfrak{L}$ implies $L\subset L_1\cap L_2$ for some $L\in\mathfrak{L}$.*

If $\mathfrak{L}$ does not contain the empty set, then it is a filter base.

THEOREM 1.2. *For any syntopogenous structure $\mathscr{S}$ on E*

$$\mathscr{S}*\mathfrak{L}=\{<*L:\ <\in\mathscr{S},\ L\in L\}$$

is also a syntopogenous structure on E finer than $\mathscr{S}$. $(\mathscr{S}*\mathfrak{L})^t=(\mathscr{S}^t*\mathfrak{L})^t$, and for $^a=^p$ or ^{b} we have $(\mathscr{S}*\mathfrak{L})^a=\mathscr{S}^a*\mathfrak{L}$, consequently $\mathscr{S}*\mathfrak{L}$ is perfect or biperfect, provided $\mathscr{S}$ is so.

PROOF. Each element of $\mathscr{S}*\mathfrak{L}$ is topogenous, because by (1.1.1) $(<*L)^q=$ $=<^q*L=<*L$ for every $<\in\mathscr{S}$ and $L\in\mathfrak{L}$.

If $<_1, <_2 \in \mathscr{S}$, $L_1, L_2 \in \mathfrak{L}$ and $<_1 \mathbf{U} <_2 \subset < \in \mathscr{S}$, $L_1 \cap L_2 \supset L \in \mathfrak{L}$, then $(<_1 * L_1) \mathbf{U} (<_2 * L_2) \subset < * L$.

For $< \in \mathscr{S}$, $L \in \mathfrak{L}$ there is $<_1 \in \mathscr{S}$ with $< \subset <_1^2$. Then from (1.1.3): $< * L \subset \subset <_1^2 * L \subset (<_1 * L)^2$.

By (1.1.4) $\mathscr{S} < \mathscr{S} * \mathfrak{L}$.

In view of (1.1.1) $(\mathscr{S} * \mathfrak{L})^a = \mathscr{S}^a * \mathfrak{L}$, where $^a = ^p$ or ^{b}, therefore $(\mathscr{S} * \mathfrak{L})^a = = \mathscr{S} * \mathfrak{L}$, provided $\mathscr{S}^a = \mathscr{S}$.

For $\{<_0\} = \mathscr{S}^t$ we obtain

$$(\mathscr{S} * \mathfrak{L})^t = \mathbf{U}\{< * L : < \in \mathscr{S}, L \in \mathfrak{L}\} = \mathbf{U}\{<_0 * L : L \in \mathfrak{L}\} = (\mathscr{S}^t * \mathfrak{L})^t$$

(cf. [1], (8.38)). Q.E.D.

REMARK. For any syntopogenous structure $\mathscr{S}$ on E, $\mathscr{S} * \{\varnothing\}$ is discrete and $\mathscr{S} * \{E\} = \mathscr{S}$.

THEOREM 1.3. *Let $\mathscr{S}, \mathscr{S}'$ be syntopogenous structures on E. Then $\mathscr{S} * \mathfrak{L} < \mathscr{S}'$ is valid iff $\mathscr{S} < \mathscr{S}'$ and for each $L \in \mathfrak{L}$*

(1.3.1) *there exists $<' \in \mathscr{S}'$ such that $L <' L$ and $<' | (E - L) = \subset | (E - L)$.*

PROOF. Suppose $\mathscr{S} * \mathfrak{L} < \mathscr{S}'$. Then $\mathscr{S} < \mathscr{S} * \mathfrak{L} < \mathscr{S}'$ by 1.2. Let L be an element of $\mathfrak{L}$, $< \in \mathscr{S}$, and suppose $< * L \subset <' \in \mathscr{S}'$. It will be proved that $<'$ satisfies (1.3.1). Indeed, $L < E$, $L \cup (E \cap L) = L$ imply $L (< * L) L$, therefore $L <' L$. The inequality $<' | (E - L) \subset \subset | (E - L)$ is obvious. Conversely, assume $A (\subset | (E - L)) B$. Then $A \subset B \subset E - L$. $A < E$ and $A \cup (E \cap L) = A \cup L$, that is $A (< * L) A \cup L \subset B \cup L$, hence by [1], (6.22) we have $A (<' | (E - L)) B$.

Assume that $\mathscr{S} < \mathscr{S}'$ and $\mathscr{S}'$ has property (1.3.1) for every $L \in \mathfrak{L}$. If $< \in \mathscr{S}$ and $L \in \mathfrak{L}$, we can find orders $<_1, <_2$ of $\mathscr{S}'$ such that $< \subset <_1$, $L <_2 L$ and $<_2 | (E - L) = \subset | (E - L)$. Put $<_1 \mathbf{U} <_2 \subset <' \in \mathscr{S}'$. If $A (< * L) B$, then $A < H$ $A \cup (H \cap L) \subset B$ for some $H \subset E$. We obtain $A <' H$, $L <' L$, thus

$$A \cap L <' H \cap L. \tag{1.3.2}$$

$A \cap (E - L) (\subset | (E - L)) A \cap (E - L)$, thus from the choice of $<_2$

$$A \cap (E - L) <_2 (A \cap (E - L)) \cup L = A \cup L$$

follows (see [1], (6.22)), so that $A <' A \cup L$. $A \cap (E - L) \subset A <' H$ implies

$$A \cap (E - L) <' H \cap (A \cup L) \subset A \cup (H \cap L) \subset B,$$

hence

$$A \cap (E - L) <' B. \tag{1.3.3}$$

On the basis of (1.3.2)—(1.3.3):

$$A = (A \cap L) \cup (A \cap (E - L)) <' (H \cap L) \cup B = B,$$

consequently $A <' B$. We verified $< * L \subset <'$, and considering the arbitrary choice of $<$ and L, $\mathscr{S} * \mathfrak{L} < \mathscr{S}'$. Q.E.D.

As the following corollary of 1.3 shows, condition (1.3.1) is suitable to define the structure $\mathscr{S} * \mathfrak{L}$.

COROLLARY 1.4. *$\mathscr{S} * \mathfrak{L}$ is the coarsest of all syntopogenous structures $\mathscr{S}'$ such that $\mathscr{S} < \mathscr{S}'$ and* (1.3.1) *is satisfied for any $L \in \mathfrak{L}$.*

Since $\mathscr{S} * \mathfrak{L}$ is finer than $\mathscr{S}$, it will be called the *refinement* of $\mathscr{S}$ (by the system $\mathfrak{L}$).

In extension theoretical application we examine only refinements of a syntopogenous structure by systems of special subsets of the fundamental set. Among such refinements coarser than a given syntopogenous structure $\mathscr{S}'$ there is a finest one.

THEOREM 1.5. *Suppose $\emptyset \neq \mathfrak{A} \subset 2^E$, $A_1, A_2 \in \mathfrak{A} \Rightarrow A_1 \cap A_2 \in \mathfrak{A}$, and let us consider, for a syntopogenous structure $\mathscr{S}'$ finer than $\mathscr{S}$, the system $\mathfrak{L}'$ of those elements of $\mathfrak{A}$, which have property* (1.3.1). *Then $\mathfrak{L}'$ satifies the condition* (L), *and $\mathscr{S} * \mathfrak{L}'$ is finest among the refinements $\mathscr{S} * \mathfrak{L}$ such that $\mathscr{S} * \mathfrak{L} < \mathscr{S}'$ and $\mathfrak{L} \subset \mathfrak{A}$. In particular, if $\mathscr{S}'$ is a refinement of $\mathscr{S}$ by a subsystem of $\mathfrak{A}$, then $\mathscr{S}' \sim \mathscr{S} * \mathfrak{L}'$.*

PROOF. It will be shown that $\mathfrak{L}'$ has the property (L). Assume $L_1, L_2 \in \mathfrak{L}'$. Then there exist $<_1', <_2' \in \mathscr{S}'$ such that $L_i <_i' L_i$ and $<_i' | (E-L) = \subset | (E-L)$ $(i=1, 2)$. Putting $L = L_1 \cap L_2$, we prove $L <' L$ and $<' | (E-L) = \subset | (E-L)$ for $<_1' \cup <_2' \subset \subset <' \in \mathscr{S}'$. In fact, $L_i <' L_i$ $(i=1, 2)$ implies $L <' L$. Suppose $A \subset E-L$. One can decompose A into three disjoint sets so that $A = A_1 \cup A_2 \cup A'$, where $A_1 = A \cap L_1$, $A_2 = A \cap L_2$ and $A' = A \cap (E-L_1) \cap (E-L_2)$. Because of $A_1 \subset E-L_2$ we have $A_1 <_2' A_1 \cup L_2$ (cf. [1], (6.21)). Since $A_1 \subset L_1 <_1' L_1$, $A_1 <' L_1 \cap (A_1 \cup L) = A_1 \cup (L_1 \cap L_2)$ is obtained. In the same way $A_2 <' A_2 \cup (L_1 \cap L_2)$. $A' \subset E-L_i$ implies $A' <_i' A' \cup L_i$ $(i=1, 2)$, therefore $A' <' (A' \cup L_1) \cap (A' \cup L_2) = A' \cup (L_1 \cap L_2)$. From the topogenity of $<'$ we get $A <' A \cup (L_1 \cap L_2)$, that is $A (<' | (E-L)) A$, which means $\subset | (E-L) = <' | (E-L)$.

From 1.4 it follows that $\mathscr{S} * \mathfrak{L}' < \mathscr{S}'$. If $\mathfrak{L}$ is a subsystem of A for which $\mathscr{S} * \mathfrak{L} < \mathscr{S}'$, then by 1.3 we obtain $\mathfrak{L} \subset \mathfrak{L}'$, thus $\mathscr{S} * \mathfrak{L} < \mathscr{S} * \mathfrak{L}'$.

As any $L \in \mathfrak{L}'$ satisfies (1.3.1), from $\mathscr{S}' = \mathscr{S} * \mathfrak{L}$ and from the above inequality $\mathscr{S}' * \mathfrak{L}' < \mathscr{S}' = \mathscr{S} * \mathfrak{L} < \mathscr{S} * \mathfrak{L}' < \mathscr{S}' * \mathfrak{L}'$. Q.E.D.

THEOREM 1.6. *Let f be a mapping of a set X into E, and define $f^{-1}(\mathfrak{L}) = \{f^{-1}(L) : L \in \mathfrak{L}\}$. Then we have*

$$f^{-1}(\mathscr{S} * L) < f^{-1}(\mathscr{S}) * f^{-1}(\mathfrak{L}).$$

If f is an injection, then

$$f^{-1}(\mathscr{S} * \mathfrak{L}) = f^{-1}(\mathscr{S}) * f^{-1}(\mathfrak{L})$$

holds.

PROOF. The statement will be verified by the proof of the inequality

$$f^{-1}(< * L) \subset f^{-1}(<) * f^{-1}(L),$$

which is valid in general for every semi-topogenous order on E and for any set $L \subset E$.

Assume $A, B \subset X$. $Af^{-1}(< * L)B$ means that $f(A)(< * L)E - f(X-B)$, that is $f(A) < H$ for some $H \subset E$ with $f(A) \cup (H \cap L) \subset E - f(X-B)$. From here $Af^{-1}(<) f^{-1}(H)$ (see [1], (6.4)), and $A \cup (f^{-1}(H) \cap f^{-1}(L)) \subset f^{-1}(f(A) \cup (H \cap L)) \subset$ $\subset f^{-1}(E - f(X-B)) \subset B$.

Suppose that f is an injection. Then

$$f^{-1}(<) * f^{-1}(L) \subset f^{-1}(< * L)$$

is also true. For $A(f^{-1}(<) * f^{-1}(L))B$ there exists $H \subset X$ such that $Af^{-1}(<)H$, $A \cup (H \cap f^{-1}(L)) \subset B$. Put $H' = E - f(X-H)$. Then $f(A) < H'$ and $f(A) \cup$ $\cup (H' \cap L) \subset E - f(X-B)$, therefore $Af^{-1}(< * L)B$. Q.E.D.

REMARK. Let $\mathscr{T}$ be a topology on E, and let us denote by $\mathfrak{b}(x)$ a $\mathscr{T}$-neighbourhood base for every point $x \in E$. Then the system

$$\mathfrak{b}'(x) = \{\{x\} \cup (V \cap L) \colon V \in \mathfrak{b}(x), \quad L \in \mathfrak{L}\} \tag{1.7.1}$$

is a neighbourhood base of x in the topology $(\mathscr{T} * \mathfrak{L})^{tp}$.

If $\mathscr{U}$ is a quasi-uniformity on E associated with the biperfect syntopology $\mathscr{S}$ (in the sense of [1]), then the quasi-uniformity corresponding to $\mathscr{S} * \mathfrak{L}$ consists of the reflexive relations $U * L$ defined as follows:

$$(x, y) \in U * L \Leftrightarrow \begin{cases} (x, y) \in U & \text{for} \quad y \in L, \\ x = y & \text{for} \quad y \notin L, \end{cases} \tag{1.7.2}$$

where U and L run over all elements of $\mathscr{U}$ and $\mathfrak{L}$.

2. Applications of refinements in extension theory

In [3] the following problem was studied:

Let $[X, \mathscr{S}]$ be a syntopogenous space, $\mathscr{T} = \mathscr{S}^{tp}$, $X \subset Y$, and suppose that $\mathscr{T}'$ is a topology on Y which is an *extension* of $\mathscr{T}$, i.e. $[X, \mathscr{T}]$ is a dense subspace of $[Y, \mathscr{T}']$. Does there exist a syntopogenous structure $\mathscr{S}'$ on Y with the conditions

$$\mathscr{S}'|X \sim \mathscr{S} \quad \text{and} \quad \mathscr{S}'^{tp} = \mathscr{T}'?$$

Such a structure $\mathscr{S}'$ is called *compatible with the pair* $(\mathscr{S}, \mathscr{T}')$. A similar question was examined for quasi-uniformities in [2]. (Let $\mathscr{U}$ be a quasi-uniformity on X, $\mathscr{T}$ the topology of $\mathscr{U}$, and $\mathscr{T}'$ an extension of $\mathscr{T}$ on Y. $\mathscr{U}'$ is a quasi-uniformity on Y compatible with the pair $(\mathscr{U}, \mathscr{T}')$ iff $\mathscr{U}'|X \sim \mathscr{U}$ and the topology of $\mathscr{U}'$ agrees with $\mathscr{T}'$.)

Further let $\mathfrak{L}$ be a system satisfying (L) of the subsets of Y such that $X \subset L$ for any $L \in \mathfrak{L}$.

Then we have the following lemma:

LEMMA 2.1. *Let $\mathscr{S}'$ be a syntopogenous structure on Y. Then* $(\mathscr{S}' * \mathfrak{L})|X = \mathscr{S}'|X$.

PROOF. Considering the embedding injection f of X into Y, we get

$$(\mathscr{S}' * \mathfrak{L})|X = f^{-1}(\mathscr{S}' * \mathfrak{L}) = f^{-1}(\mathscr{S}') * f^{-1}(\mathfrak{L}) = (\mathscr{S}'|X) * f^{-1}(\mathfrak{L})$$

(see th. 1.6 and p. 101 of [1]). For any $L\in\mathfrak{L}$ the equalities $f^{-1}(L)=L\cap X=X$ hold, therefore $(\mathscr{S}'*\mathfrak{L})|X=(\mathscr{S}'|X)*\{X\}$. But for every semi-topogenous order $\prec$ on X, $\prec*X=\prec$ is obvious, thus $(\mathscr{S}'*\mathfrak{L})|X=\mathscr{S}'|X$. Q.E.D.

If $\mathscr{T}'$ is a topology on Y which is an extension of a topology $\mathscr{T}$ on X, then denoting by $\mathfrak{v}(x)$ the $\mathscr{T}'$-neighbourhood filter of $x\in Y$, the filters

$$\mathfrak{s}(x)=\{V\cap X\colon V\in\mathfrak{v}(x)\}\quad(x\in Y)$$

are called the *trace filters* of $\mathscr{T}'$.

COROLLARY 2.2. *Let $\mathscr{T}_0'$ be a topology on Y, $\mathscr{T}$ a topology on X, and suppose that $\mathscr{T}_0'$ is an extension of $\mathscr{T}$. Then $\mathscr{T}'=(\mathscr{T}_0'*\mathfrak{L})^{tp}$ is an extension of $\mathscr{T}$, the trace filters of which are identical with those of $\mathscr{T}_0'$.*

PROOF. From 2.1 $\mathscr{T}'|X=(\mathscr{T}_0'*\mathfrak{L})^{tp}|X=((\mathscr{T}_0'*\mathfrak{L})|X)^{tp}=(\mathscr{T}_0'|X)^{tp}=\mathscr{T}^{tp}=\mathscr{T}$. If $\mathfrak{b}(x)$ is a $\mathscr{T}_0'$-neighbourhood base of $x\in Y$, then by (1.7.1)

$$\mathfrak{b}'(x)=\{\{x\}\cup(V\cap L)\colon V\in\mathfrak{b}(x),\quad L\in\mathfrak{L}\}$$

is a $\mathscr{T}'$-neighbourhood base of x, and because of $X\subset L$ we have $(\{x\}\cup(V\cap L))\cap X=$ $=(\{x\}\cap X)\cup(V\cap L\cap X)=(\{x\}\cap X)\cup(V\cap X)=(\{x\}\cup V)\cap X=V\cap X$. Q.E.D.

THEOREM 2.3. *Let $\mathscr{S}_0'$ be a syntopogenous structure on Y compatible with the pair $(\mathscr{S},\mathscr{T}_0')$, and put $\mathscr{T}'=(\mathscr{T}_0'*\mathfrak{L})^{tp}$, $\mathscr{S}'=\mathscr{S}_0'*\mathfrak{L}$. Then $\mathscr{S}'$ is compatible with $(\mathscr{S},\mathscr{T}')$.*

PROOF. By 2.1 we have $\mathscr{S}'|X=(\mathscr{S}_0'*\mathfrak{L})|X=\mathscr{S}_0'|X\sim\mathscr{S}$. It is easy to see that $\mathscr{S}'^{tp}=(\mathscr{S}_0'^{t}*\mathfrak{L})^{tp}=(\mathscr{S}_0'^{t}*\mathfrak{L})^{ptp}=(\mathscr{S}_0'^{tp}*\mathfrak{L})^{tp}=(\mathscr{T}_0'*\mathfrak{L})^{tp}=\mathscr{T}'$ by 1.2 and [1], (8.50). Q.E.D.

The above theorem has a particular importance, when $\mathscr{T}_0'$ is a *strict extension* of $\mathscr{T}$ (i.e. the system

$$\{s(G)\colon G\subset X\ \text{ is open in }\ \mathscr{T}\}$$

is a base of the open sets of $\mathscr{T}_0'$, where denoting by $\mathfrak{s}(x)$ the trace filters of this extension,

$$s(G)=\{x\in Y\colon G\in\mathfrak{s}(x)\}).$$

In this case a number of constructions of compatible structures are known, and what is more, these can be perfect or biperfect, provided $\mathscr{S}$ is of this kind, and some simple conditions are satisfied by the trace filters (cf. [3], (3.1), (3.4), (6.3) and [2], (5.1), (6.3)).

We can formulate two corollaries of the results of [2], [3] and Theorem 2.3 of the present paper, while following the convention introduced in [3], we do not distinguish a topology from the associated "classical topology". The definition of the notion of a *round*, *pseudo-Cauchy* or a *stable* filter of a quasi-uniform space can be found in [2], Ch. 1 and 4.

THEOREM 2.4. *Let $\mathscr{U}$ be a quasi-uniformity on X, and $\mathscr{T}_0'$ be a strict extension of the topology $\mathscr{T}$ of $\mathscr{U}$ onto Y with the trace filters $\mathfrak{s}(x)$ $(x\in Y)$, further assume that $\mathscr{T}'=(\mathscr{T}_0'*\mathfrak{L})^{tp}$, and let $\mathfrak{v}(x)$ denote the $\mathscr{T}'$-neighbourhood filter of $x\in Y$.*

(2.4.1) *If every trace filter $\mathfrak{s}(x)$ $(x\in Y-X)$ is round and stable in $\mathscr{U}$, then there exists a quasi-uniformity compatible with $(\mathscr{U},\mathscr{T}')$.*

(2.4.2) *If* $Y-X=\bigcup_{i=1}^{n} Z_i$, *where* $x, y\in Z_i$ *implies* $\mathfrak{v}(x)=\mathfrak{v}(y)$ $(1\leqq i\leqq n)$, *then there exists a quasi-uniformity compatible with* $(\mathscr{U}, \mathscr{T}')$, *provided* $\mathfrak{s}(x)$ $(x\in Y-X)$ *is round and pseudo-Cauchy in* $\mathscr{U}$.

PROOF. First of all we need to see that our condition in (2.4.2) implies that $x, y\in Z_i \Rightarrow \mathfrak{v}_0(x)=\mathfrak{v}_0(y)$, where $\mathfrak{v}_0(x)$ is the $\mathscr{T}_0'$-neighbourhood filter of $x\in Y$. Suppose that $\mathfrak{v}(x)=\mathfrak{v}(y)$ for some $x, y\in Y$. If $W\in\mathfrak{v}_0(x)$, then there exists a $\mathscr{T}$-open set $S\in\mathfrak{s}(x)$ such that $s(S)\subset W$. Since $s(S)$ is also a $\mathscr{T}_0'$-neighbourhood of x, $L\in\mathfrak{L}$ implies $V=\{x\}\cup(s(S)\cap L)\in\mathfrak{v}(x)=\mathfrak{v}(y)$, therefore $S=s(S)\cap X=V\cap X\in\mathfrak{s}(y)$ (see 2.2), thus $y\in s(S)$, and from here $W\in\mathfrak{v}_0(y)$. We got $\mathfrak{v}_0(x)\subset\mathfrak{v}_0(y)$. The inequality $\mathfrak{v}_0(y)\subset\mathfrak{v}_0(x)$ can be obtained in the same manner, therefore $\mathscr{T}_0'$ satisfies the conditions of Theorem 5.1 in [2].

In view of [2], (5.3) and (5.1), both in the case of (2.4.1) and in that of (2.4.2) there is a quasi-uniformity $\mathscr{U}_0'$ on Y compatible with $(\mathscr{U}, \mathscr{T}_0')$. Turning to the associated biperfect syntopologies $\mathscr{S}$ and $\mathscr{S}_0'$, $\mathscr{S}_0'$ is compatible with $(\mathscr{S}, \mathscr{T}_0')$. Therefore by 2.3 and 1.2 $\mathscr{S}'=\mathscr{S}_0'*\mathfrak{L}$ is a biperfect structure compatible with $(\mathscr{S}, \mathscr{T}_0')$, and denoting by $\mathscr{U}'$ the quasi-uniformity corresponding to $\mathscr{S}'$, $\mathscr{U}'$ is compatible with $(\mathscr{U}, \mathscr{T}')$. Q.E.D.

In certain cases we need to decide, whether a topology $\mathscr{T}'$ can be written in the form $(\mathscr{T}_0'*\mathfrak{L})^{tp}$, where $\mathscr{T}_0'$ is a strict extension of $\mathscr{T}$ giving the trace filters $\mathfrak{s}(x)$, and $\mathfrak{L}$ is an arbitrary system in Y (satisfying (L)) with $X\subset L$ for any $L\in\mathfrak{L}$. Such a topology will be said to be a *refined strict extension* of $\mathscr{T}$ corresponding to the trace filters $\mathfrak{s}(x)$. For finite extensions we have the following result:

THEOREM 2.5. *Let the topology* $\mathscr{T}'$ *on* Y *be an extension of the topology* $\mathscr{T}$ *on* X *giving the trace filters* $\mathfrak{s}(x)$ *so that* $Y-X=\bigcup_{i=1}^{n} Z_i$, *where* $\mathfrak{v}(x)=\mathfrak{v}(y)$ *for any* $x, y\in Z_i$ $(1\leqq i\leqq n)$. *Then there exists a largest* $\mathscr{T}'$*-closed and* $\mathscr{T}'$*-discrete set* F *in* $Y-X$. $\mathscr{T}'$ *is a refined strict extension of* $\mathscr{T}$ *iff* $\mathfrak{s}(x)\subset\mathfrak{s}(y)$ *implies* $\mathfrak{v}(x)\subset\mathfrak{v}(y)$ *for any* $x\in Y$ *and* $y\in(Y-F)-X$. *In this case*

$$\mathscr{T}'=\mathscr{T}_0'*\{Y-F\},$$

where $\mathscr{T}_0'$ *is the strict extension of* $\mathscr{T}$ *corresponding to the filters* $\mathfrak{s}(x)$ $(x\in Y)$.

PROOF. Suppose that F_0 is a closed discrete set in $Y-X$. Then $x\in F_0$ implies $Z_i=\{x\}$ for some index $1\leqq i\leqq n$. In fact, if $y\in Z_i$, then $y\in\overline{F}_0=F_0$. Since there is a neighbourhood V of x such that $V\cap F_0=\{x\}$, $y\in V\cap F_0$ implies $y=x$. This shows that the elements of F_0 can be chosen only from a finite number of points of $Y-X$, thus in $Y-X$ there exists only a finite number of closed discrete sets, and their union is also closed and discrete.

Assume that $x\in Y$, $y\in(Y-F)-X$, $\mathfrak{s}(x)\subset\mathfrak{s}(y)$ imply $\mathfrak{v}(x)\subset\mathfrak{v}(y)$. We shall show that in $\mathscr{T}'$ the system

$$\{\{x\}\cup(s(S)\cap(Y-F)):\ S\in\mathfrak{s}(x)\}$$

is a neighbourhood base of $x\in Y$, that is

$$\mathscr{T}'=(\mathscr{T}_0'*\{Y-F\})^{tp}=\mathscr{T}_0'*\{Y-F\}$$

(see 1.2), where $\mathscr{T}_0'$ is the strict extension of $\mathscr{T}$ corresponding to the filters $\mathfrak{s}(x)$ (cf. (1.7.1)). Indeed, let us consider an open set $V\in\mathfrak{v}(x)$ for some $x\in Y$. Let the set G be defined as follows: $G=\bigcap_{i=1}^{n} G_i$, where the $\mathscr{T}$-open set G_i is selected so that $G_i=X$, if $\mathfrak{s}(x)\subset\mathfrak{s}(y)$ for $y\in Z_i$, and $G_i\in\mathfrak{s}(x)$, $G_i\notin\mathfrak{s}(y)$, provided $\mathfrak{s}(x)\not\subset\mathfrak{s}(y)$ for $y\in Z_i$. Then $G\in\mathfrak{s}(y)$ clearly implies $\mathfrak{s}(x)\subset\mathfrak{s}(y)$. Putting $S=V\cap G$, we have $S\in\mathfrak{s}(x)$ and $W\subset V$ for $W=\{x\}\cup(s(S)\cap(Y-F))$. The first statement is obvious. Let y be a member of W. Because of $x\in V$ we can suppose $x\neq y$, consequently $y\notin F$. If $y\in X$, then $y\in S\subset V$. If $y\in(Y-F)-X$, then $S\in\mathfrak{s}(y)$, thus $G\in\mathfrak{s}(y)$, and this implies $\mathfrak{s}(x)\subset\mathfrak{s}(y)$. From our condition $\mathfrak{v}(x)\subset\mathfrak{v}(y)$ follows, so that $y\in V'$. Conversely, let $S\in\mathfrak{s}(x)$ and $W=\{x\}\cup(s(S)\cap(Y-F))$. $s(S)$ and $Y-F$ are $\mathscr{T}'$-open. For $x\in Y-F$ we have $x\in s(S)\cap(Y-F)$, therefore $W\in\mathfrak{v}(x)$. If $x\in F$, then one can find a $V\in\mathfrak{v}(x)$ such that $V\cap F=\{x\}$ and $V\subset s(S)$. From here

$$V=(V\cap F)\cup(V\cap(Y-F))\subset W,$$

thus $W\in\mathfrak{v}(x)$.

Suppose that $\mathscr{T}'$ is a refined strict extension of $\mathscr{T}$, for example $\mathscr{T}'=(\mathscr{T}_0'*\mathfrak{L})^{tp}$, where $\mathscr{T}_0'$ is the strict extension of $\mathscr{T}$ corresponding to the filters $\mathfrak{s}(x)$ $(x\in Y)$, further $\mathfrak{L}$ is a system satisfying (L) and $X\subset L$ for any $L\in\mathfrak{L}$. It is easy to verify that $Y-L$ is closed and discrete in $\mathscr{T}'$ for every $L\in\mathfrak{L}$ (see (1.3.1)). $Y-L\subset Y-X$. therefore $Y-L$ lies in the largest closed discrete subset F of $Y-X$, i.e. $Y-F\subset L$. By (1.1.2) $<_0'*L\sqsubset<_0'*(Y-F)$ for $\{<_0'\}=\mathscr{T}_0'$. At the same time $Y-F<'Y-F$ and $<'|F=\subset|F$ for $\{<'\}=\mathscr{T}'$, thus from 1.3 we get $\mathscr{T}_0'*\mathfrak{L}<\mathscr{T}_0'*\{Y-F\}<\mathscr{T}'$, hence $\mathscr{T}'=(\mathscr{T}_0'*\mathfrak{L})^{tp}<(\mathscr{T}_0'*\{Y-F\})^{tp}<\mathscr{T}'$, i.e. $\mathscr{T}'=(\mathscr{T}_0'*\{Y-F\})^{tp}$ as we stated.

It will be proved that $x\in Y$, $y\in(Y-F)-X$ and $\mathfrak{s}(x)\subset\mathfrak{s}(y)$ imply $\mathfrak{v}(x)\subset\mathfrak{v}(y)$. In fact, if $V\in\mathfrak{v}(x)$, then there exists $S\in\mathfrak{s}(x)$ for which $\{x\}\cup(s(S)\cap(Y-F))\subset V$. $S\in\mathfrak{s}(y)$ implies $y\in s(S)\cap(Y-F)$. Since $s(S)$ and $Y-F$ are open in $\mathscr{T}'$, $V\in\mathfrak{v}(y)$, and in general $\mathfrak{v}(x)\subset\mathfrak{v}(y)$. Q.E.D.

EXAMPLE 2.6 There exists a finite extension, which is not a refined strict extension. Such one is the topology $\mathscr{T}'$ on the set $Y=\{0, 1, 2\}$, in which the open sets are $\varnothing$, $\{0\}$, $\{0, 2\}$ and Y. Suppose that $X=\{0, 1\}$ and $\mathscr{T}=\mathscr{T}'|X$. Then $\mathscr{T}'$ is a finite extension of $\mathscr{T}$. $Y-X=\{2\}$, and it is not closed, because $1\in\overline{\{2\}}$. Thus with the notation of 2.5 $F=\varnothing$, therefore if $\mathscr{T}'$ is a refined strict extension of $\mathscr{T}$, then $\mathscr{T}'=\mathscr{T}_1'*\{Y\}=\mathscr{T}_0'$ (that is $\mathscr{T}'$ is strict). This is impossible, since $\mathfrak{s}(0)=\mathfrak{s}(2)=\{\{0\}, X\}$, $\{0\}\in\mathfrak{v}(0)$, and $V\in\mathfrak{s}(0)$ implies $2\in s(V)$, so that $s(V)\not\subset\{0\}$.

REMARK. Let us observe that the *loose* extension $\mathscr{T}'$ of $\mathscr{T}$ on Y corresponding to the prescribed trace filters $\mathfrak{s}(x)$ $(x\in Y)$ can be identified with $(\mathscr{T}_0'*\{X\})^{tp}=\mathscr{T}_1'*\{X\}=\mathscr{T}_2'$, where $\mathscr{T}_1'$ is an arbitrary (topological) extension of $\mathscr{T}$ on Y giving the trace filters $\mathfrak{s}(x)$. In fact, in this case the $\mathscr{T}_2'$-neighbourhood base of a point $x\in Y$ is of the form

$$\{\{x\}\cup(V\cap X):\ V\in\mathfrak{b}_1(x),\} \tag{2.7.1}$$

where $\mathfrak{b}_1(x)$ is a $\mathscr{T}_1'$-neighbourhood base of x (see (1.7.1)), and it is obvious that the system (2.7.1) is equivalent to

$$\{\{x\}\cup S:\ S\in\mathfrak{s}(x)\}$$

(see [3], (0.6)).

References

[1] Á. Császár, *Foundations of General Topology* (Budapest—Oxford—London—New York—Paris, 1963).

[2] Á. Császár, Extensions of quasi-uniformities, *Acta Math. Acad. Sci. Hungar.*, **37** (1981), 121—145.

[3] Á. Császár and K. Matolcsy, Syntopogenous extensions for prescribed topologies, *Acta Math. Acad. Sci. Hungar.*, **37** (1981), 59—75.

(Received November 27, 1981)

KOSSUTH LAJOS UNIVERSITY
MATHEMATICAL INSTITUTE
DEBRECEN
H-4010

Acta Math. Hung.
42 (1—2) (1983), 121—129.

ON THE EIGENFUNCTION EXPANSIONS ASSOCIATED WITH THE SCHRÖDINGER OPERATOR HAVING SPHERICALLY SYMMETRICAL POTENTIAL

Š. A. ALIMOV (Moscow) and I. JOÓ (Budapest)

The aim of the present paper is to give a necessary and sufficient condition for the Lebesgue summability of eigenfunction expansions associated with the Schrödinger operator having spherically symmetrical potential, for functions from the class $\mathring{W}_2^1$ in three dimensions.

Let Ω be a bounded domain in $\mathbf{R}^3$, $x_0 \in \Omega$ and $0 < q(r)$ a function defined on $(0, \infty)$ such that

$$\|q\| \stackrel{\text{def}}{=} \int_0^\infty q^2(r) r^2 dr < \infty. \tag{1}$$

Consider the Schrödinger operator

$$Lu(x) = [-\Delta + q(|x-x_0|)]u(x) \quad (u \in C_0^\infty(\Omega),\ x \in \Omega \setminus \{x_0\})$$

with potential $q(|x-x_0|)_\Omega$. (We use also the notation $q(x)$ for this function.) According to a well known theorem of K. O. Friedrichs [7], the operator L has a positive selfadjoint extension $\hat{L}$ with discrete spectrum; let $0 < \lambda_1 \leqq \lambda_2 \leqq \ldots$ be the sequence of the eigenvalues of $\hat{L}$ and $\{u_n\}_1^\infty$ the corresponding complete orthonormal system of eigenfunctions in $L_2(\Omega)$, i.e.

$$(-\Delta + q)u_n(x) = \lambda_n u_n(x) \quad (x \in \Omega;\ n = 1, 2, \ldots).$$

The expansion of $f \in L_2(\Omega)$ with respect to the system $\{u_n\}$,

$$f = \sum_{n=1}^\infty f_n u_n, \quad f_n = (f, u_n)$$

is said to be Lebesgue summable to a at the point $x \in \Omega$ if

$$\lim_{r \to +0} \sum_{n=1}^\infty \frac{\sin \sqrt{\lambda_n} r}{\sqrt{\lambda_n} r} f_n u_n(x) = a$$

holds (cf. [1]).

We shall prove the following

THEOREM. *Let $f \in \mathring{W}_2^1(\Omega)$ be an arbitrary function. The expansion of f is Lebesgue summable at the point x_0 if and only if*

$$\lim_{r \to 0} \int_\theta f(x_0 + r\theta) d\theta \quad \textit{exists.} \tag{2}$$

Here $\int_\theta f(x_0+r\theta)\,d\theta$ *denotes the integral of* f *on the sphere of radius* r *with respect to the normalized Lebesgue measure (i.e.* $\int_\theta 1\,d\theta=1$*).*

PROOF. For the proof of the Theorem we need some lemmas. Define

$$\omega(t) = \min\left\{1, \frac{1}{t}\right\}, \quad > t0; \quad b^2(t) = \int_0^t \omega^2(s)ds.$$

Obviously

$$b^2(t) \leqq \min\{2, t\}, \quad t > 0. \tag{3}$$

Consider the sequence of functions

$$\begin{cases} w_0(r, \mu) = \dfrac{\sin r\mu}{r\mu}, \\ w_k(r, \mu) = \dfrac{1}{r\mu}\displaystyle\int_0^r w_{k-1}(t, \mu)\sin\mu(t-r)q(t)t\,dt; \quad k = 1, 2, \ldots. \end{cases} \tag{4}$$

It is easy to see that

$$\left|\frac{\sin r\mu}{r\mu}\right| = |w_0(r, \mu)| \leqq \omega(r\mu), \quad \left|\frac{\sin\mu(t-r)}{r\mu}\right| \leqq \omega(r\mu), \tag{5}$$

and hence

$$|w_k(r, \mu)| \leqq \omega(r\mu)\int_0^r |w_{k-1}(t, \mu)|q(t)t\,dt \leqq \tag{6}$$

$$\leqq \omega(r\mu)\left(\int_0^r |w_{k-1}(t, \mu)|^2 dt\right)^{1/2}\left(\int_0^r q^2(t)t^2 dt\right)^{1/2} \leqq$$

$$\leqq \omega(r\mu)\|q\|\left(\int_0^r |w_{k-1}(t, \mu)|^2 dt\right)^{1/2}.$$

LEMMA 1. *We have*

$$|w_k(r, \mu)| \leqq \omega(r\mu)\left[\frac{\|q\|\,b(r\mu)}{\sqrt{\mu}}\right]^k \quad (r > 0;\ \mu > 0;\ k = 0, 1, 2, \ldots). \tag{7}$$

PROOF. For $k=0$ the estimate (7) follows from (5). Suppose it is fulfilled for $k-1$ in place of k. Then, using (6) and the just mentioned induction hypothesis it follows by the Schwartz inequality

$$|w_k(r, \mu)| \leqq \omega(r\mu)\|q\|\left[\frac{\|q\|\,b(r\mu)}{\sqrt{\mu}}\right]^{k-1}\left(\int_0^r \omega^2(t\mu)dt\right)^{1/2} = \omega(r\mu)\left[\frac{\|q\|\,b(r\mu)}{\sqrt{\mu}}\right]^k.$$

Lemma 1 is proved.

Now define

$$\alpha(r, \mu) = \sum_{k=1}^{\infty} w_k(r, \mu).$$

LEMMA 2. *For the function* $\alpha(r, \mu)$ *the estimate*

$$|\alpha(r, \mu) \leqq \text{const}\, \omega(r, \mu) \frac{b(r\mu)}{\sqrt{\mu}} \|q\| \quad (r > 0, \mu > 4\|q\|^2) \tag{8}$$

holds.

PROOF. The estimate (8) follows from (3) and (7) immediately.

LEMMA 3. *If* $\mu > 4\|q\|^2$, *then for the only solution* $v(r, \mu)$ *of the integral equation*

$$v(r, \mu) = \frac{\sin r\mu}{r\mu} + \frac{1}{r\mu} \int_0^r v(t, \mu) \sin [\mu(t-r)]\, q(t)\, t\, dt \tag{9}$$

of Volterra type the following equality holds

$$v(r, \mu) = \frac{\sin r\mu}{r\mu} + \alpha(r, \mu) \quad (r > 0). \tag{10}$$

PROOF. Define

$$v_0(r, \mu) = w_0(r, \mu),$$

$$v_{k+1}(r, \mu) = \frac{\sin r\mu}{r\mu} + \frac{1}{r\mu} \int_0^r v_k(t, \mu) \sin \mu(t-r)\, q(t)\, t\, dt \quad (k = 0, 1, \ldots).$$

For the functions $v_k - v_{k-1}$ the equations (4) hold and according to (7) it follows the existence of the limit

$$v(r, \mu) \overset{\text{def}}{=} \lim_{k \to \infty} v_k(r, \mu).$$

The function $v(r, \mu)$ is the only continuous solution of the integral equation (9). On the other hand,

$$v_k(r, \mu) = w_k(r, \mu) + w_{k-1}(r, \mu) + \ldots + w_0(r, \mu)$$

i.e.

$$v(r, \mu) = w_0(r, \mu) \sum_{k=1}^{\infty} w_k(r, \mu) = \frac{\sin r\mu}{r\mu} + \alpha(r, \mu).$$

Lemma 3 is proved.

LEMMA 4. *For every n such that* $\mu_n > 4\|q\|^2$ *the equation*

$$\int_\theta u_n(x_0 + r\theta)\, d\theta = \left[\frac{\sin r\mu_n}{r\mu_n} + \alpha(r, \mu_n)\right] u_n(x_0), \quad \mu_n \overset{\text{def}}{=} \sqrt{\lambda_n} \tag{11}$$

is fulfilled.

PROOF. The Titchmarsh formula states (cf. [9], p. 232) that

$$\int_\theta u_n(x_0+r\theta)\,d\theta = \frac{\sin r\mu_n}{r\mu_n}u_n(x_0)+\int_0^r\left(\int_\theta q(t)u_n(x_0+t\theta)\,d\theta\right)\cdot\frac{1}{\mu_n}\frac{t}{r}\sin\mu_n(t-r)\,dt=$$

$$=\frac{\sin r\mu_n}{r\mu_n}u_n(x_0)+\frac{1}{r\mu_n}\int_0^r\left(\int_0 u_n(x_0+t\theta)\,d\theta\right)\sin\mu_n(t-r)\,q(t)\,t\,dt.$$

It follows that the function

$$\tilde v(t,\mu_n)=\int_\theta u_n(x_0+t\theta)\,d\theta$$

satisfies the integral equation

$$\tilde v(r,\mu_n)=\frac{\sin r\mu_n}{r\mu_n}u_n(x_0)+\frac{1}{r\mu_n}\int_0^r\tilde v(t,\mu_n)\sin\mu_n(t-r)q(t)\,t\,dt,$$

and hence, taking into consideration Lemma 3 and (10), we obtain

$$\tilde v(r,\mu_n)=v(r,\mu_n)u_n(x_0),$$

i.e.

$$\tilde v(r,\mu_n)=\frac{\sin r\mu_n}{r\mu_n}u_n(x_0)+\alpha(r,\mu_n)u_n(x_0).$$

Lemma 4 is proved.

LEMMA 5. *We have*

$$\sum_{|\mu_n-\mu|\leqq 1}|u_n(x_0)|^2\leqq c(x_0)\mu^2\quad(\mu\geqq 1). \tag{12}$$

PROOF. Consider the function

$$d(r,\mu)=\begin{cases}\mu\dfrac{\sin r\mu}{r} & \text{if}\quad R<r<2R\\ 0 & \text{if}\quad r\notin(R,2R)\end{cases}$$

where $0<2R<\operatorname{dist}(x_0,\partial\Omega)$; and calculate its Fourier coefficients with respect to the system $\{u_n\}$. Using (11) we get

$$d_n=\int_\Omega d(|x-x_0|,\mu)u_n(x)\,dx=\left(\frac{\mu}{\mu_n}\int_R^{2R}\sin r\mu_n\sin r\mu\,dr\right)u_n(x_0)+$$

$$+\left(\mu\int_R^{2R}r\sin r\mu\alpha(r,\mu_n)\,dr\right)u_n(x_0).$$

On the other hand, obviously

$$\int_R^{2R}\sin r\mu_n\sin r\mu\,dr\geqq\frac{R}{2}\cos 2R>0,$$

and it follows by (8) that

$$\left|\mu \int_R^{2R} r\alpha(r,\mu_n)\sin r\mu\, dr\right| \leqq c\mu \int_R^{2R} \frac{\omega(r\mu_n)}{\sqrt{\mu_n}}\, b(r\mu_n)\, r\, dr \| q\| \leqq$$

$$\leqq c\mu \int_R^{2R} \frac{1}{\sqrt{\mu_n}}\,\omega(r\mu_n)\, r\, dr \leqq c\mu \int_R^{2R} \frac{1}{\sqrt{\mu_n}}\cdot\frac{1}{r\mu_n}\, r\, dr \leqq c\frac{1}{\sqrt{\mu}}$$

for $\mu \geqq \mu_0$ and $|\mu_n - \mu| \leqq 1$. (c denotes everywhere in this work an absolute constant.)

Summarizing our estimates:

$$|d_n| \geqq c|u_n(x_0)|\left[1+O\left(\frac{1}{\sqrt{\mu}}\right)\right] \geqq c|u_n(x_0)|,$$

and hence (12) follows by Bessel inequality, using also the simple fact:

$$\int_\Omega |d(|x-x_0|, \mu)|^2 dx \leqq c\mu^2.$$

Lemma 5 is proved.

LEMMA 6. *Let* $f \in \mathring{W}_2^1(\Omega)$ *be arbitrary, then*

$$\sum_{n=1}^{\infty} |f_n|^2 \mu_n^2 \leqq c\|f\|_{W_2^1}. \tag{13}$$

holds.

PROOF. Let $f \in C_0^\infty(\Omega)$ be arbitrary. Then obviously

$$f_n = (f, u_n) = \int_\Omega f(x)u_n(x)dx = \frac{1}{\lambda_n}\int_\Omega f(x)[-\Delta + q(x)]u_n(x)dx =$$

$$= \frac{1}{\lambda_n}\int_\Omega f(x)(-\Delta u_n(x))dx + \frac{1}{\lambda_n}\int_\Omega f(x)q(x)u_n(x)dx = -\frac{1}{\lambda_n}(\Delta f)_n + \frac{1}{\lambda_n}(fq)_n$$

i.e.

$$\lambda_n f_n = (fq)_n - (\Delta f)_n.$$

For any $f \in C_0^\infty(\Omega)$ and $g \in L_2(\Omega)$ the Parseval equality gives

$$\sum_{n=1}^{\infty} f_n g_n = \int_\Omega f(x)g(x)dx,$$

and hence, for the special case $g(x) = f(x)\cdot q(x) - \Delta f(x)$ we obtain

$$\sum_{n=1}^{\infty} f_n^2 \lambda_n = \int_\Omega f(x)[-\Delta f(x)]dx + \int_\Omega |f(x)|^2 q(x)dx = \int_\Omega |\nabla f(x)|^2 dx +$$

$$+ \int_\Omega |f(x)|^2 q(x)dx = \|\nabla f\|^2_{L_2(\Omega)} + \|q|f|^2\|_{L_1(\Omega)} \leqq c\|f\|_{W_2^1(\Omega)} + \|q\|_{L_2(\Omega)}\cdot\|f\|^2_{L_4(\Omega)} \leqq$$

$$\leqq c\|f\|_{W_2^1(\Omega)}.$$

In the last step we have used the well known imbedding theorems [6]. It follows (13) immediately from our estimates if we take into consideration the fact that $\mathring{W}_2^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the space $W_2^1(\Omega)$. Lemma 6 is proved.

LEMMA 7. *Let*

$$\varphi(t) \leqq c \min\{t^\varepsilon, t^{-\varepsilon}\} \quad (t > 0,\ \varepsilon > 0).$$

Then we have the estimate

$$\sum_{n=1}^{\infty} \varphi(r\mu_n) \frac{|u_n(x_0)|^2}{\mu_n^3} \leqq c, \tag{14}$$

uniformly in $0<r<1$.

PROOF. The estimate (14) follows from the calculations below:

$$\sum_{r\mu_n<1} \varphi(r\mu_n) \frac{|u_n(x_0)|^2}{\mu_n^3} \leqq cr^\varepsilon \sum_{\mu_n<\frac{1}{r}} \frac{|u_n(x_0)|^2}{\mu_n^{3-\varepsilon}} \leqq cr^\varepsilon \sum_{k=1}^{\left[\frac{1}{r}\right]} \sum_{k=\mu_n<k+1} \frac{|u_n(x_0)|^2}{k^{3-\varepsilon}} \leqq$$

$$\leqq cr^\varepsilon \sum_{k=1}^{\left[\frac{1}{r}\right]} \frac{1}{k^{1-\varepsilon}} \leqq cr^\varepsilon \cdot \frac{1}{r^\varepsilon} \leqq c;$$

$$\sum_{r\mu_n\geqq 1} \varphi(r\mu_n) \frac{|u_n(x_0)|^2}{\mu_n^3} \leqq cr^{-\varepsilon} \sum_{\mu_n\geqq\frac{1}{r}} \frac{|u_n(x_0)|^2}{\mu_n^{3+\varepsilon}} \leqq$$

$$\leqq cr^{-\varepsilon} \sum_{k=\left[\frac{1}{r}\right]}^{\infty} \sum_{k\leqq\mu_n\leqq k+1} \frac{|u_n(x_0)|^2}{k^{3+\varepsilon}} \leqq cr^{-\varepsilon} \sum_{k=\left[\frac{1}{r}\right]}^{\infty} \frac{1}{k^{3+\varepsilon}} \leqq cr^{-\varepsilon} \cdot r^\varepsilon \leqq c.$$

Lemma 7 is proved.

LEMMA 8. *For any* $f\in\mathring{W}_2^1(\Omega)$ *the estimate*

$$\sum_{n=1}^{\infty} |\alpha(r, \mu_n) u_n(x_0) f_n| \leqq c\|f\|_{W_2^1(\Omega)} \tag{15}$$

holds uniformly in $0<r<1$.

PROOF. Using (13) and applying the Schwartz inequality we get

$$\sum_{n=1}^{\infty} |\alpha(r, \mu_n) u_n(x_0) f_n| \leqq \left(\sum_{n=1}^{\infty} \alpha^2(r, \mu_n) \frac{|u_n(x_0)|^2}{\mu_n^2} \right)^{1/2} \|f\|_{W_2^1(\Omega)},$$

so it is enough to prove the estimate

$$\sum_{n=1}^{\infty} \left| \alpha(r, \mu_n) \frac{u_n(x_0)}{\mu_n} \right|^2 \leqq c.$$

To this end it suffices to recall (8), taking into consideration the estimates

$$\mu_n^2\alpha^2(r,\mu_n) \leqq c\mu_n\frac{\omega^2(r\mu_n)b^2(r\mu_n)}{\mu_n} \leqq c\omega^2(r\mu_n)b^2(r\mu_n);$$

$$\omega(t) \leqq \frac{1}{t}, \quad b(t) \leqq \sqrt{2} \quad (t \geqq 1),$$

$$\omega(t) \leqq 1, \quad b(t) \leqq \sqrt{t} \quad (0 \leqq t \leqq 1);$$

further the identity

$$\sum_{n=1}^{\infty}\left|\alpha(r,\mu_n)\frac{u_n(x_0)}{\mu_n}\right|^2 = \sum_{n=1}^{\infty}[\mu_n\alpha^2(r,\mu_n)]\frac{|u_n(x_0)|^2}{\mu_n^3}.$$

Lemma 8 is proved.

LEMMA 9. *Let $f\in\mathring{W}_2^1(\Omega)$ be arbitrary. Then it is possible to change the definition of f on a set of Lebesgue measure zero such that for the resulting function (we denote it also by f according to the fact that these functions both represent the same element of $W_2^1(\Omega)$) its angular integral*

$$\varphi(r) \overset{\text{def}}{=} \int_\theta f(x_0+r\theta)\,d\theta$$

be continuous for $r>0$ and the following equality holds:

$$\int_\theta f(x_0+r\theta)\,d\theta = \sum_{n=1}^{\infty} f_n u_n(x_0)\frac{\sin r\mu_n}{r\mu_n} + O(1)\,\|f\|_{W_2^1(\Omega)} \quad (r>0). \tag{16}$$

PROOF. Consider the function

$$g(x) = \begin{cases} 1 & \text{if} \quad |x-x_0| \leqq R \\ 0 & \text{if} \quad |x-x_0| > R. \end{cases}$$

Then

$$(g,\,u_n) = u_n(x_0)\int_0^R \frac{\sin r\mu_n}{r\mu_n}\,r^2\,dr + u_n(x_0)\int_0^R \alpha(r,\mu_n)r^2\,dr \quad \text{(cf. (11))},$$

and by the Parseval formula we get

$$\int_{|x-x_0|\leqq R} f(x)\,dx = \sum_{n=1}^{\infty} f_n\cdot u_n(x_0)\int_0^R \frac{\sin r\mu_n}{r\mu_n}\,r^2\,dr + \sum_{n=1}^{\infty} f_n u_n(x_0)\int_0^R \alpha(r,\mu_n)r^2\,dr.$$

We may apply the differentiation on the right hand side for $R>0$, because, according to (13), the resulting series converges for $R>0$ (uniformly on $R\geqq\varepsilon_0>0$; cf. [2]):

$$\frac{d}{dr}\int_{|x-x_0|\leqq r} f(x)\,dx = \sum_{n=1}^{\infty} f_n u_n(x_0)\frac{\sin r\mu_n}{r\mu_n}\,r^2 + \sum_{n=1}^{\infty} f_n u_n(x_0)\alpha(r,\mu_n)r^2. \tag{17}$$

On the other hand, according to the imbedding theorem

$$W_2^1(\Omega) \to W_2^{1/2}(S_R) \quad (S_R = \{x \in \mathbf{R}^3 \colon |x| = R\})$$

(cf. [6]) we have the estimate

$$\|f\|_{L_2(S_R)} \leqq C_\varepsilon \|f\|_{W_2^1(\Omega)}, \quad R \geqq \varepsilon > 0.$$

This means, that for the function $\varphi(r)$ the estimate

$$|\varphi(r)| \leqq C_\varepsilon \|f\|_{W_2^1(\Omega)}$$

holds for almost every $r \in [\varepsilon, R_0]$, where $R_0 \overset{\text{def}}{=} \operatorname{dist}(x_0, \partial\Omega)$. Hence, by Fubini's theorem we get

$$\int_{|x-x_0| \leqq R} f(x)\,dx = \int_0^R \varphi(r) r^2\,dr$$

and by Lebesgue's theorem it follows

$$\frac{d}{dr} \int_{|x-x_0| \leqq r} f(x)\,dx = \varphi(r) r^2, \tag{18}$$

for almost every $r > 0$. From (17) and (18) we obtain the following equality for almost every $r > 0$:

$$\varphi(r) = \sum_{n=1}^{\infty} f_n u_n(x_0) \frac{\sin r\mu_n}{r\mu_n} + \sum_{n=1}^{\infty} f_n u_n(x_0) \alpha(r, \mu_n).$$

Since the right hand side of the last equality is a continuous function on $r > 0$, it is possible to change the function $\varphi(r)$ on a set of measure zero, so that the resulting function is continuous for every $r > 0$; hence the statement of Lemma 9 follows by Fubini's theorem.

PROOF OF THE THEOREM. The statement of the Theorem follows from that of Lemma 9 immediately.

REMARK. During this work we used the ideas of the papers [3], [4], [5] and that of the books [6], [7], [8], [9].

References

[1] G. Alexits, *Convergence problems of orthogonal series*, Akadémiai Kiadó (Budapest, 1961).
[2] В. А. Ильин, Э. Г. Позняк, *Математический анализ*, Volume **2**, Nauka (Moscow, 1980).
[3] В. А. Ильин, Проблемы локализации и сходимости для рядов Фурье по фундаментальным системам функций оператора Лапласа, *Успехи Математических Наук*, **13** (1958), 87—180.
[4] В. А. Ильин, Ш. А. Алимов, Условия сходимости спектральных разложений, отвечающих самосопряженным расширениям эллиптических операторов, V, *Дифференциальные Уравнения*, **10** (1974).
[5] Б. М. Левитан, О разложении по собственным функциям уравнения $\Delta u + \{\lambda - q(x_1, \ldots, x_n)\}u = 0$, *Изв. АН., сер. матем.*, **20** (1956), 437—468.

[6] С. М. Никольский, *Приближение функций многих переменных и теоремы вложения*, Nauka (Moscow, 1969).
[7] F. Riesz and B. Sz-Nagy, *Leçons d'analyse fonctionnelle*, Akadémiai Kiadó (Budapest, 1952).
[8] С. Л. Соболев, *Некоторые применения функционального анализа в математической физике*, Изд-во Ленинград ун-та, (Leningrad, 1950).
[9] E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations*, Clarendon Press (Oxford, 1958).

(Received December 9, 1981)

MOSCOW STATE UNIVERSITY
117 234 MOSCOW
LENINSKIE GORY
USSR

EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT II OF ANALYSIS
1088 BUDAPEST, MÚZEUM KRT. 6—8
HUNGARY

Acta Math. Hung.
42 (1—2) (1983), 131—138.

ON THE DISTANCE BETWEEN CONSECUTIVE PRIME IDEAL NUMBERS IN SECTORS

M. MAKNYS (Vilnius)

1. Introduction

One of the important questions in number theory is the estimation of the distance between consecutive primes p_{n+1} and p_n. Huxley [3] proved that

$$p_{n+1}-p_n \ll p_n^{7/12+\varepsilon}. \tag{1.1}$$

Results of such type for primes in progressions were obtained by Tatuzava [14], Jutila [5] and others.

In this paper we shall investigate a similar question in sectors of an algebraic quadratic immaginary number field.

Thus we consider an algebraic quadratic immaginary number field $\mathbf{Q}(\sqrt{d})$ ($d<0$ is square-free number) which is widened to an ideal number system K [1]. Let $\alpha, \ldots$ denote ideal numbers from K; $\mathfrak{p}, \ldots$ are prime ideal numbers, $N\alpha$ is the norm of α, $\mu\neq(0)$ is an integer ideal from $\mathbf{Q}(\sqrt{d})$, g' is the number of units of $\mathbf{Q}(\sqrt{d})$, g is the number of units mod μ, h is the class number of our field, $\varphi(\mu)$ is Euler's function.

The function

$$\lambda^m(\alpha) = e^{igm \arg \alpha} \tag{1.2}$$

will be called a Hecke's character of the first kind with exponent m, $m\in\mathbf{Z}$. Let χ be a character of multiplicative group of a reduced residue system mod μ. If for all units η mod μ we have

$$\chi(\eta)\lambda^m(\eta) = 1, \tag{1.3}$$

then the function

$$\Xi(\alpha) = \chi(\alpha)\lambda^m(\alpha) \tag{1.4}$$

is Hecke's character of the second kind (or Groessen character) mod μ with exponent m.

Hecke's Z-function with character $\Xi(\alpha)$ for $\mathbf{Q}(\sqrt{d})$ is defined by the series

$$\sum_{(\alpha)\neq(0)}^{*} \Xi(\alpha)N\alpha^{-s} = Z(s, \Xi), \quad s=\sigma+it, \tag{1.5}$$

where * denotes that α runs over a set of non-associated numbers from K. $Z(s, \Xi)$ is an entire function except the case of trivial character and has a functional equation of Dirichlet type [1].

As usual, $\Lambda(\alpha)$ denotes Mangoldt's function and for our field this function is

$$\Lambda(\alpha) = \begin{cases} \ln Np & \text{if } \alpha \text{ is associated with } p^k,\ k\in\mathbf{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

For our purpose we investigate the function

$$\psi(x;\ \nu,\ \mu,\ \varphi_1,\ \varphi_2) = \sum_{\substack{N\alpha \leqq x \\ \alpha \equiv \nu \bmod \mu \\ \varphi_1 < \arg \alpha \leqq \varphi_2}} \Lambda(\alpha) \tag{1.7}$$

where $x>2$, φ_1 and φ_2 are real numbers and $0<\varphi_2-\varphi_1\leqq 2\pi$, ν is an integer ideal number of K, $(\nu, \mu)=1$.

The finite difference $\psi(x+x^\theta;\ \nu,\ \mu,\ \varphi_1,\ \varphi_2)-\psi(x;\ \nu,\ \mu,\ \varphi_1,\ \varphi_2)$ gives us an information on the distance between consecutive prime ideal numbers in some ring sector.

By using zero-density theorems for nontrivial zeros of $Z(s, \Xi)$ [9, 11] we shall prove the following

THEOREM. *For sufficiently large x and fixed ideal μ we have*

$$\psi(x+x^\theta;\ \nu,\ \mu,\ \varphi_1,\ \varphi_2)-\psi(x;\ \nu,\ \mu,\ \varphi_1,\ \varphi_2) \sim \frac{x^\theta g'(\varphi_2-\varphi_1)}{2\pi h\varphi(\mu)} \tag{1.8}$$

with $\theta=\dfrac{11}{16}+\varepsilon_1$ *for each sector satisfying* $\varphi_2-\varphi_1>x^{\theta-1+\varepsilon_2}$.

Hence we get the following

COROLLARY. *In each ring sector with angle* $\varphi_2-\varphi_1>x^{\theta-1+\varepsilon_3}$ *bounded by two concentric circles with radii* $r_1=\sqrt{x+x^\theta}$ *and* $r_2=\sqrt{x}$, *there exists at least one ideal prime number* $\mathfrak{p}\equiv\nu \bmod \mu$.

REMARK. To get a result of type (1.1) we can put $\nu=(0)$, $\mu=(1)$, $\varphi_2-\varphi_1=2\pi$ in (1.8). Then it follows that

$$\Pi_k(x+x^\theta)-\Pi_k(x) \sim \frac{x^{\theta+\varepsilon}}{\ln x} \tag{1.9}$$

with $\theta=\dfrac{11}{16}$. Here $\Pi_k(x)$ denotes the number of prime ideals $\mathfrak{p}$ of first degree with the condition $N\mathfrak{p}\leqq x$, $x>2$, $\mathfrak{p}\in K$. The exponent $\theta=\dfrac{11}{16}$ is better than 7/10 which can be obtained by classical way using Tichmarsh result on Epstein's zeta-function [16], but weaker than $\dfrac{5}{8}$ obtained by Heath-Brown [2].

We use $\varepsilon_1, \varepsilon_2, \ldots$ to denote positive numbers which may be chosen arbitrarily small, $c_1, c_2, \ldots$ are absolute constants which depend on the field, the ideal μ and ε_i, but do not depend on φ_1, φ_2, m and t. The same is true for the symbols O and $\ll$.

2. Lemmas

To obtain (1.8) we shall use some lemmas.

LEMMA 1. *The zeros of* $Z(s, \Xi)$ *lie in* $\sigma<0$ *and*

$$\left|\sigma-\frac{1}{2}\right| \leqq \frac{1}{2}-c_1 \ln\left[((|m|+2)(|t|+2))\ln\ln((|m|+2)(|t|+2))\right]^{-\frac{3}{4}}. \tag{2.1}$$

Furthermore, the critical zeros from $0<\sigma<1$ *are symmetric to the line* $\sigma=\frac{1}{2}$.

LEMMA 2. *The number of zeros* $\varrho=\beta+i\gamma$ *of* $Z(s, \Xi)$ *with* $t\leqq\gamma\leqq t+1$, $t>1$ *is*

$$\ll \ln\left((|m|+2)(|t|+2)\right). \tag{2.2}$$

LEMMA 3. *Let* $x>3$, $T>2$, $M>2$,

$$\psi(x, \Xi) = \sum_{N\alpha\leqq x}^{*} \Xi(\alpha)\Lambda(\alpha), \tag{2.3}$$

$$E(\Xi) = \begin{cases} 1 \text{ if } \Xi = \Xi_0 \text{ is a trivial character,} \\ 0 \text{ otherwise.} \end{cases}$$

Then

$$\psi(x, \Xi) = E(\Xi)x - \sum_{|\gamma|\leqq T} \frac{x^{\varrho}}{\varrho} + O\left(\frac{x}{T}\ln^2 MT\right), \tag{2.4}$$

where the sum in (2.4) *is taken over all zeros* $\varrho=\beta+i\gamma$ *of* $Z(s, \Xi)$ *from the region* $0<\beta<1$, $0<|\gamma|\leqq T$.

Lemmas 1, 2 and 3 were proved by Kubilius [8].

LEMMA 4. *Let* $\Omega>0$, $0<\Delta<\frac{1}{2}\Omega$, φ_1 *and* φ_3 *real numbers and* $0<\varphi_2-\varphi_1\leqq$ $\leqq 2\pi$. *Then there exists a periodic function* $f(\omega)$ *with period* Ω *satisfying the conditions*

$$\begin{cases} f(\omega) = 1, \quad \omega\in[\varphi_1, \varphi_2], \\ 0\leqq f(\omega)\leqq 1, \quad \omega\in[\varphi_1-\Delta, \varphi_1], \quad \omega\in[\varphi_2, \varphi_2+\Delta], \\ f(\omega) = 0, \quad \omega\in[\varphi_2+\Delta, \varphi_1+\Omega-\Delta], \end{cases} \tag{2.5}$$

and $f(\omega)$ *has the Fourier expansion*

$$f(\omega) = \sum_{-\infty}^{\infty} a_m e^{\frac{2\pi i}{\Omega}m\omega},$$

where

$$a_0 = \frac{1}{\Omega}(\varphi_2-\varphi_1), \tag{2.6}$$

$$|a_m| \leqq \begin{cases} \frac{1}{\Omega}(\varphi_2-\varphi_1), \\ 2(\pi|m|)^{-1}, \quad m\neq 0, \\ \frac{2}{\pi|m|}\left(\frac{k\Omega}{\pi|m|\Delta}\right)^k, \quad m\neq 0, \quad k\in\mathbf{N}. \end{cases} \tag{2.7}$$

This is a lemma due to I. M. Vinogradoff [17].

LEMMA 5. *Let* $x>3$ *be sufficiently large,* $x<y\ll x$, $T>2$, $M>2$. *Then*

$$\psi(x+y;\ \nu,\ \mu,\ \varphi_1,\ \varphi_2)-\psi(x;\ \nu,\ \mu,\ \varphi_1,\ \varphi_2)= \tag{2.8}$$

$$=\frac{g'(\varphi_2-\varphi_1)}{2\pi h\varphi(\mu)}\,y\left\{1+O\left(\sum_{0<|m|\leqq M}^{\Xi}\ \sum_{|\gamma|\leqq T} x^{\beta-1}\right)+O\left(\frac{xMh\varphi(\mu)}{g'yT}\ln^2 MT\right)\right\}+R(x,M),$$

and for a fixed ideal $\mu\neq(0)$

$$\psi(x+y;\ \nu,\ \mu,\ \varphi_1,\ \varphi_2)-\psi(x;\ \nu,\ \mu,\ \varphi_1,\ \varphi_2)= \tag{2.9}$$

$$=\frac{g'(\varphi_2-\varphi_1)}{2\pi h\varphi(\mu)}\,y\left\{1+O\left(\sum_{0<|m|\leqq M}^{\Xi}\ \sum_{|\gamma|\leqq T} x^{\beta-1}\right)+O\left(\frac{xM}{yT}\ln^2 MT\right)\right\}+R(x,M),$$

where

$$R(x,M)=\sum_{\substack{N\alpha\leqq x\\ \alpha\equiv\nu \bmod \mu}}\Lambda(\alpha)\sum_{|m|>M}\lambda^m(\alpha)a_m$$

and the inner sum in the O *term is taken over the zeros of* $Z(s,\Xi)$ *from the region* $\frac{1}{2}<\beta<1$, $|\gamma|\leqq T$, *and summation runs over all characters* Ξ *for the exponents of which* $|m|\leqq M$ *holds.* Δ *is some parameter from Lemma* 4 *that will be chosen later,* a_m *are coefficients from Lemma* 4.

PROOF. Put $\Omega=\frac{2\pi}{g}$, $\omega=\arg\alpha$ in Lemma 4. Then from this lemma we have

$$\tilde{\varphi}(x;\ \nu,\ \mu,\ \varphi_1,\ \varphi_2)=\sum_{\substack{N\alpha\leqq x\\ \alpha\equiv\nu \bmod \mu}}^{*} f(\arg\alpha)\Lambda(\alpha)=\sum_{\substack{N\alpha\leqq\alpha\\ \alpha\equiv\nu \bmod \mu}}^{*}\Lambda(\alpha)\sum_{m}\lambda^m(\alpha)a_m= \tag{2.10}$$

$$=\sum_{|m|\leqq M} a_m\sum_{\substack{N\alpha\leqq x\\ \alpha\equiv\nu \bmod \mu}}^{*}\lambda^m(\alpha)\Lambda(\alpha)+R(x,M).$$

For $\alpha\equiv\nu \bmod \mu$, $(\nu,\mu)=1$

$$\sum_{\substack{N\alpha\leqq x\\ \alpha\equiv\nu \bmod \mu}}^{(\mu)*}\lambda^m(\alpha)\Lambda(\alpha)=\frac{1}{h\varphi(\mu)}\sum_{\chi}\bar{\chi}(\nu)\sum_{N\alpha\leqq x}^{(\mu)*}\chi(\alpha)\lambda^m(\alpha)\Lambda(\alpha).$$

Here (μ) denotes summation over ideal numbers α non-associated mod μ. If $E(\mu,\alpha)=\{\eta_1,\eta_2,\ldots,\eta_{l(\mu)}\}$ is the collection of units mod μ, $e(\mu)=\frac{g'}{g}$, then from the definition (1.4) and (1.3) of characters $\Xi(\alpha)$ it follows that

$$\sum_{N\alpha\leqq x}^{(\mu)*}\chi(\alpha)\lambda^m(\alpha)\Lambda(\alpha)=\sum_{N\alpha\leqq x}^{*}\chi(\alpha)\lambda^m(\alpha)\Lambda(\alpha)\sum_{\eta\in E(\mu,\alpha)}\chi(\eta)\lambda^m(\eta)=e(\mu)\sum_{N\alpha\leqq x}^{*}\Xi(\alpha)\Lambda(\alpha).$$

Since

$$\sum_{\substack{N\alpha\leqq x\\ \alpha\equiv\nu \bmod \mu}}^{(\mu)*}\lambda^m(\alpha)\Lambda(\alpha)=\frac{e(\mu)}{h\varphi(\mu)}\sum_{\chi}\bar{\chi}(\nu)\sum_{N\alpha\leqq x}^{*}\Xi(\alpha)\Lambda(\alpha),$$

by (2.10) and Lemma 3 we conclude

$$(2.11)\quad \psi(x;\ \nu, \mu, \varphi_1, \varphi_2) = \frac{e(\mu)}{h\varphi(\mu)} \sum_{\chi} \bar{\chi}(\nu) \sum_{|m| \leqq M} \left\{ E(\Xi)x - \sum_{|\gamma| \leqq T} \frac{x^{\varrho}}{\varrho} + O\left(\frac{x}{T} \ln^2 MT\right)\right\} +$$

$$+ R(x, M) = \frac{e(\mu) a_0 x}{h\varphi(\mu)} - \frac{e(\mu)}{h\varphi(\mu)} \sum_{\substack{\Xi \\ 0<|m| \leqq M}} a_m \bar{\chi}(\nu) \left(\sum_{|\gamma| \leqq T} \frac{x^{\varrho}}{\varrho} + O\left(\frac{x}{T} \ln^2 MT\right)\right) +$$

$$+ R(x, M) = \frac{(\varphi_2 - \varphi_1) g' x}{2\pi h\varphi(\mu)} + O\left(\frac{\varphi_2 - \varphi_1}{h\varphi(\mu)} \sum_{\substack{\Xi \\ 0<|m| \leqq M}} \sum_{|\gamma| \leqq T} \frac{x^{\varrho}}{\varrho} \right) +$$

$$+ O\left(\frac{Mx(\varphi_2 - \varphi_1)}{T} \ln^2 MT \right) + R(x, M).$$

By using the expression (2.11) with $x+y$ instead of x and the relations

$$\psi(x, \nu, \mu, \varphi_1, \varphi_2) = \tilde{\psi}(x, \nu, \mu, \varphi_1, \varphi_2) - \theta(\psi(x, \nu, \mu, \varphi_1 - \Delta, \varphi_1 + \Delta) +$$

$$+ \psi(x, \nu, \mu, \varphi_2 - \Delta, \varphi_2 + \Delta), \quad 0 \leqq \theta \leqq 1,$$

$$\frac{(x+y)^{\varrho} - x^{\varrho}}{\varrho} = \int_x^{x+y} u^{\varrho - 1} du \leqq y x^{\beta - 1}$$

we get (2.8) and (2.9) immediately.

LEMMA 6. *For* $|t| < T$, $|m| < M$, $T > 2$, $M > 2$ *we have*

$$(2.12)\qquad \left| Z\left(\frac{1}{2} + it, \Xi\right) \right| \ll (M+T)^{1/3} \ln^{c_2}(M+T).$$

For the proof see [6].

LEMMA 7. *Let* $N(\sigma, T, \Xi)$ *denote the number of zeros* $\varrho = \beta + i\gamma$ *of* $Z(s, \Xi)$ *in the region* $\sigma \leqq \beta \leqq 1$, $|\gamma| < T$, $\frac{1}{2} \leqq \sigma \leqq 1$, $T > 2$, $M > 2$. *Then*

$$(2.13)\qquad \sum_{\substack{\Xi \\ 0<|m| \leqq M}} N(\sigma, T, \Xi) \ll \begin{cases} [MT(M+T)]^{\frac{2(1-\sigma)}{2-\sigma}} \ln^{c_3} MT, & \frac{1}{2} \leqq \sigma \leqq 1, \\ [M+T]^{\frac{2(16\sigma-11)(1-\sigma)}{3\sigma(4\sigma-3)}} \ln^{c_4} MT, & \frac{3}{4} < \sigma \leqq 1. \end{cases}$$

PROOF. (2.13) is proved in [9], (2.14) follows from our estimate given for $N(\sigma, T, \Xi)$ depending on $Z\left(\frac{1}{2} + it, \Xi\right)$ (cf. [11]) and Lemma 6.

3. Proof of Theorem

After proving that

$$W = \sum_{0<|m|\leqq M}^{\Xi} \sum_{|\gamma|\leqq T} x^{\beta-1} = o(1)$$

our theorem will follow from (2.9) immediately.

For sufficiently large x we put in Lemma 5

$$M = x^{1-\Theta-\frac{\varepsilon}{2}}, \quad T = x^{2(1-\theta)}, \quad \Delta = x^{\theta-1+\varepsilon}, \quad \theta = \frac{11}{16}, \quad y = x^{\theta+\varepsilon_5}.$$

It is easy to verify that for such parameters the second O term in (2.9) is $o(1)$ and

$$R(x, M) \ll x^{-\varepsilon_6}$$

if we estimate $|a_m|$ by (2.7) with $k=[\ln x]$.

We denote

$$\sigma_0 = c_1(\ln x \ln\ln x)^{-3/4}, \quad l = [(1-\sigma_0-\theta)\ln x],$$

$$q = \left[\frac{\ln T}{\ln 2}\right], \quad T^{(n)} = \frac{T}{2^n}, \quad n = 0, 1 \ldots, N, N = [\ln x],$$

$$\sigma_j = \theta + \frac{j}{\ln x}, \quad j = 0, 1, \ldots, l.$$

It is enough to evaluate W only in the region $\sigma_0 \leqq \operatorname{Re}\varrho = \beta \leqq 1-\sigma_0$ (cf. Lemma 1). As in [8] we rewrite W in the form

$$W = \sum_{0<|m|\leqq M}^{\Xi} \sum_{|\gamma|\leqq T}^{\beta<\theta} x^{\theta-1} + \sum_{n=1}^{q} \sum_{0<|m|\leqq M}^{\Xi} \sum_{T^{(n)}<|\gamma|<T^{(n-1)}}^{\beta<\theta} x^{\theta-1} +$$

$$+ \sum_{j=0}^{l} \sum_{0<|m|\leqq M}^{\Xi} \sum_{\substack{|\gamma|\leqq T \\ \beta\leqq 0{,}98 \\ \sigma_j<\beta\leqq\sigma_{j+1}}} x^{\sigma_j-1} + \sum_{j=0}^{l} \sum_{0<|m|\leqq M}^{\Xi} \sum_{\substack{|\gamma|\leqq T \\ \beta>0{,}98 \\ \sigma_j<\beta\leqq\sigma_{j+1}}} x^{\sigma_j-1} = W_1+W_2+W_3+W_4.$$

From Lemmas 1 and 2 we have

$$W_1+W_2 \ll x^{\theta-1}\ln^2 x(\ln\ln x)^{3/4}M + \sum_{n=1}^{q} x^{\theta-1}M\ln x = o(1).$$

For W_3 and W_4 we put

$$T_0 = \begin{cases} T^{1/48}, & \sigma_j \leqq 0{,}98, \\ \ln(T+2), & 0{,}98 < \sigma_j \leqq 1-\sigma_0, \end{cases}$$

$$P = \left[\frac{\ln T - \ln T_0}{2}\right] + 1$$

and apply (2.13) and (2.14) respectively.

We have

$$W_3 = \sum_{j=0}^{l} x^{\sigma_j - 1} \ln^{c_5} x \left\{ [MT_0(M+T_0)]^{\frac{2(1-\sigma_j)}{2-\sigma_j}} + \right.$$

$$\left. + \sum_{p=0}^{p} [2^p MT_0(M+T_0)]^{\frac{2(1-\sigma_j)}{2-\sigma_j}} (T_0 2^p)^{-1} \right\} \ll \sum_{j=0}^{l} \ln^{c_6} x \left[x^{\frac{4-4\theta-4\varepsilon+1/12-1/12\theta-2+j/\ln x}{2-\sigma_j}} \right]^{1-\sigma_j} =$$

$$= o(1)$$

and

$$W_4 = \sum_{j=0}^{l} x^{\sigma_j - 1} \sum_{\substack{\Xi \\ 0<|m|\leqq M}} \left\{ \sum_{\substack{\beta>0{,}98 \\ |\gamma|\leqq T_0 \\ \sigma_j<\beta\leqq\sigma_{j+1}}} 1 + \sum_{p=0}^{p} \sum_{\substack{\beta>0{,}98 \\ T_0 2^p<|\gamma|\leqq T_0 2^{p+1} \\ \sigma_j<\beta\leqq\sigma_{j+1}}} 1 \right\} \ll$$

$$\ll \sum_{j=0}^{l} x^{\sigma_j - 1} \left\{ (M+T_0)^{\frac{2(16\sigma_j-11)(1-\sigma_j)}{3\sigma_j(4\sigma_j-3)}} + \sum_{p=0}^{p} (T_0 2^p)^{-1} (T_0 2^p + M2^p)^{\frac{2(16\sigma_j-11)(1-\sigma_j)}{3\sigma_j(4\sigma_j-3)}} \right\} \ln^{c_7} x \ll$$

$$\ll \sum_{j=0}^{l} x^{\sigma_j + 2\left(1-\theta-\frac{\varepsilon}{2}\right)\frac{(16\sigma_j-11)(1-\sigma_j)}{3\sigma_j(4\sigma_j-3)} - 1} \ln^{c_8} x = o(1).$$

By this the proof has been completed.

4. Concluding remarks

Our theorem is not uniform in μ. By using zero-density theorem for $Z(s, \Xi)$ [7] we proved [10] that uniformly for $N\mu \ll x^{1/8-\varepsilon}$ the relation

$$\psi(x+x^\theta;\ \nu, \mu, \varphi_1, \varphi_2) - \psi(x;\ \nu, \mu, \varphi_1, \varphi_2) \sim \frac{x^\theta g'(\varphi_2-\varphi_1)}{2\pi h\varphi(\mu)} \tag{4.2}$$

holds for $\theta = \frac{3}{4} + \varepsilon$ and each fixed sector $0 < \varphi_2 - \varphi_1 \leqq 2\pi$.

Results uniform in μ are better if we consider a mean value of the difference

$$\psi(x;\ \nu, \mu, \varphi_1, \varphi_2) - \frac{g' x(\varphi_2-\varphi_1)}{2\pi h\varphi(\mu)}. \tag{4.2}$$

Using his zero-density theorem uniform in $N\mu$ and m, Johnson [4] obtained the Bombieri—Vinogradoff analogue of the mean value of (4.2) uniformly for

$$N\mu \ll x^{1/2}(\ln x)^{-B}, \quad B>0,$$

and we proved [12] analogous results for the mean value of the square of (4.2) not using zero-density theorems by the method of large sieve, uniformly for $N\mu \ll x^{1/2-\varepsilon}$.

References

[1] E. Hecke, Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen I, *Math. Z.*, **1** (1918), 357—376; II, *Math. Z.*, **6** (1920), 11—51.
[2] D. R. Heath-Brown, On the density of zeros of the Dedekind zetafunction, *Acta Arith.*, **33** (1977), 169—181.
[3] M. N. Huxley, On the difference between consecutive primes, *Invent. Math.*, **15** (1972), 164—170.
[4] D. Johnson, Mean values of Hecke L-functions, *J. für Math.*, **305** (1979), 195—205.
[5] M. A. Jutila, A statistical density theorem for L-functions with applications, *Acta Arith.*, **16** (1969), 207—216.
[6] R. M. Kaufman, On the short equations of A. F. Lavrik, *Zap. Nauč. Sem. Len. otd. Mat. inst. AN SSSR*, **76** (1978), 124—158 (in Russian).
[7] F. B. Kovalčik, Density theorems and distributions of primes in sectors and progressions, *Dokl. AN SSSR*, **219** (1974), 31—34 (in Russian).
[8] J. Kubilius, On some problems on geometry of primes, *Mat. Sb.*, **31** (1952), 507—542 (in Russian).
[9] M. Maknys, Zero-density theorems for Hecke's Z-functions and distribution of primes in algebraic quadratic immaginary number field, *Litov. Mat. Sb.*, **16** (1976), 173—180 (in Russian).
[10] M. Maknys, On finite differences of $\psi(x; v, \mu, \varphi_1, \varphi_2)$, *Litov. Mat. Sb.*, **17** (1977), 117—118 (in Russian).
[11] M. Maknys, On the density of zeros of Hecke Z-functions, *Vsesojuz. škola po teorii čisel* (Dushanbe, 1977), 83—84 (in Russian).
[12] M. Maknys, Large sieve in quadratic fields, *Litov. Mat. Sb.*, **20** (1980), 79—86 (in Russian).
[13] K. Prachar, *Primzahlverteilung* (Berlin, 1957).
[14] A. V. Sokolovskij, A theorem on the zeros of Dedekinds zeta function and the distance between "neighbouring" prime ideals, *Acta Arith.*, **13** (1968), 321—334 (in Russian).
[15] T. Tatuzava, On the zeros of Dirichlets L-functions, *Proc. Jap. Acad.*, **26** (9) (1950), 3—13.
[16] E. C. Titchmarsh, On Epsteins zeta-function, *Proc. Lond. Math. Soc.*, **36** (1934), 485—500.
[17] I. M. Vinogradoff, *The method of trigonometric sums in the theory of numbers* (Moscow, 1971) (in Russian).

(Received December 9, 1981)

V. KAPSUKAS UNIVERSITY
DEPARTMENT OF PROBABILITY THEORY AND NUMBER THEORY
232 006 LITHUANIA SSR, VILNIUS, PARTIZANU 24

AND

EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT FOR COMPUTER SCIENCES
1088 BUDAPEST, MÚZEUM KRT. 6—8

Acta Math. Hung.
42 (1—2) (1983), 139—141.

ON THE KRULL INTERSECTION THEOREM

O. A. S. KARAMZADEH (Ahvaz)

Introduction

The Krull intersection theorem is one of the basic results in the theory of commutative Noetherian rings. The object of this note is to prove this theorem and some of its consequences for a class of non commutative not necessarily Noetherian rings.

The proof that we are going to present shortly is elementary and worth while even in the commutative case.

Throughout this paper by a ring R we mean an associative one with the identity element such that every one-sided ideal is two-sided (duo ring), i.e., $xR=Rx$, $\forall x\in R$. Nontrivial examples of duo rings are aboundant, see [1], [2]. If x is an element of a ring R, then $r(x)=\{r\in R\colon xr=0\}$ and $l(x)=\{r\in R\colon rx=0\}$ are the right and the left annihilators of x in R, respectively.

LEMMA 1. *If A is a finitely generated ideal in a ring R and B is an ideal of R containing some power of every element of A, then $A^n\subseteqq B$ for some integer n.*

PROOF. Evident.

DEFINITION. A ring R is said to be weakly Noetherian if for an arbitrary ideal A of R and any $\bar{x}=x+A$ in R/A there exists an integer n such that $r(\bar{x}^n)=r(\bar{x}^m)$; $l(\bar{x}^n)=l(\bar{x}^m)$, $\forall m\geqq n$. It is clear that every Noetherian ring is weakly Noetherian, but the converse is not true, for take R to be an infinite Boolean ring.

LEMMA 2. *Let B be an ideal in a weakly Noetherian ring R. Now if $x\in R$ and $A_n=\{a\in R\colon x^nax^n\in B\}$, then there exists an integer k such that $A_k=A_m$, $\forall m\geqq k$.*

PROOF. Put $A'_n=\{a'\colon ax^n=x^na'$ for some $a\in A_n\}$, then clearly A'_n is an ideal of R and $A'_n=\{a'\in R\colon x^{2n}a'\in B\}$. Now since R is weakly Noetherian, there exists an integer n such that $A'_m=A'_n$, $\forall m\geqq n$. We then claim that $\frac{A_{n+r}}{A_n}=l(x^r+A_n)$ in R/A_n for any integer $r\geqq 1$ and therefore there exists an integer q such that $A_{n+m}=$ $=A_{n+q}$, $m\geqq q$. To see this, we note that $l(x^r+A_n)\subseteqq\frac{A_{n+r}}{A_n}$ is clear and to show the converse we proceed by induction on r. For $r=1$, let $a\in A_{m+1}$, where $m\geqq n$, then $ax^{m+1}=x^{m+1}a'$ for some $a'\in A'_{m+1}$. But $a'\in A'_m$ and so $x^ma'=bx^m$ for some $b\in A_m$. Hence $ax^{m+1}=x^{m+1}a'=xbx^m$ and therefore $x^max^{m+1}=xx^mbx^m\in B$ implies $ax\in A_m$. Now assume that $\frac{A_{n+r-1}}{A_n}\subseteqq l(x^{r-1}+A_n)$ and let $b\in A_{n+r}$, then we have shown that $bx\in A_{n+r-1}$ and therefore $bx^r\in A_n$.

LEMMA 3. (Artin—Rees property.) *If A is a finitely generated ideal in a weakly Noetherian ring R, then for any ideal B of R there exists an integer n such that $A^n \cap B \subseteq AB \cap BA$.*

PROOF. Suppose that F is the set of all ideals C such that $C \cap B \subseteq AB \cap BA$. It is clear that F is nonempty, and then by Zorn's lemma F has a maximal element M. But $M + AB \cap BA \in F$ implies either $M \supseteq AB \cap BA$ or $B = AB \cap BA$. Now if $B = AB \cap BA$ then $A^n \cap B \subseteq AB \cap BA$, $\forall n$ and we are through. Therefore we may assume that $M \cap B = AB \cap BA$. Then to complete the proof it suffices, by Lemma 1, to show that some power of every element of A is contained in M. To this end let $x \in A$ be any element of A and put $A_r = \{a \in R: x^r a x^r \in M\}$, then by Lemma 2, there exists an integer k such that $A_k = A_m$, $\forall m \geqq k$. We now put $M' = x^k R x^k + M$ and want to show that $M' \cap B = AB \cap BA$. Clearly, $AB \cap BA \subseteq M' \cap B$. Conversely, suppose that $y \in M' \cap B$, then $y = x^k a x^k + m$, where $a \in R$, $m \in M$. Then $xyx = x^{k+1} a x^{k+1} + xmx \in AB \cap BA \subseteq M$. Hence $x^{k+1} a x^{k+1} \in M$ and therefore $x^k a x^k \in M$, whence $y \in M$ and $y \in M \cap B \subseteq AB \cap BA$. Now by the maximal property of M, we have $M = M'$, i.e., $x^{2k} \in M$.

REMARK. If A, B are two ideals in a ring R, then we may have $AB \neq BA$, see [1]. We claim that one can prove most of the basic results on commutative rings for duo rings in which the ideals commute with each other, see [2].

COROLLARY 1. (Krull intersection theorem.) *Let R be a weakly Noetherian ring and I a finitely generated ideal of R, then* $\bigcap_{n=1}^{\infty} I^n = \{r \in R: r(1-a) = 0 \text{ for some } a \in I\} = \{r \in R: (1-b)r = 0 \text{ for some } b \in I\}$.

PROOF. If either $r(1-a) = 0$ or $(1-b)r = 0$ for some $a, b \in I$, then $r = ra^m$ or $r = b^m r$ for all m, and clearly in both cases we have $r \in \bigcap_{n=1}^{\infty} I^n$. Conversely, let $x \in \bigcap_{n=1}^{\infty} I^n$, then by Lemma 3, there exists an integer m such that $I^m \cap xR \subseteq xI \cap Ix$. But $x \in I^m$ implies that $x = xa = bx$ for some $a, b \in I$. Hence $x(1-a) = (1-b)x = 0$ for some $a, b \in I$.

COROLLARY 2. *If R is a weakly Noetherian ring and I a finitely generated ideal contained in the Jacobson radical of R, then* $\bigcap_{n=1}^{\infty} I^n = (0)$.

COROLLARY 3. *Let R be a weakly Noetherian domain. Then any prime principal ideal (a) is a minimal prime ideal of R.*

COROLLARY 4. *If R is a weakly Noetherian ring with the unique maximal ideal $M = xR$, then every nonzero ideal of R is a power of M and M is the only nonzero prime ideal in R.*

PROOF. By Corollary 2, we have $\bigcap_{n=1}^{\infty} M^n = (0)$. Hence for any nonzero ideal A of R, there exists an integer m such that $A \subseteq M^m$, $A \nsubseteq M^{m+1}$. Now let $a \in A$ be

such that $a \in x^m R$, $a \notin x^{m+1} R$. Hence $a = x^m r$, where $r \notin M$. Therefore r is a unit and $x^m \in A$ implies that $A = x^m R$. It is now clear that M is the only non-zero prime ideal.

References

[1] H. H. Brungs, Three questions on duo rings, *Pacific J. Math.*, **52** (1975), 345—349.
[2] O. A. S. Karamzadeh, Ph. D. thesis (Exeter, 1974).
[3] D. Rees, Two classical theorems of ideal theory, *Proc. Cam. Phil. Soc.*, **52** (1956), 12—16.

(Received December 11, 1981)

MATHEMATICS DEPARTMENT
JUNDI-SHAPUR UNIVERSITY,
AHVAZ
IRAN

Acta Math. Hung.
42 (1—2) (1983), 143—151.

WEAK CONVERGENCE OF MARTINGALES WITH RANDOM INDICES TO INFINITELY DIVISIBLE LAWS

K. S. KUBACKI and D. SZYNAL (Lublin)

1. Introduction

Let $(X_{nk})_{n,k\in\mathbf{N}}$ be a doubly infinite array (DIA) of random variables (r.vs) defined on a common probability space $(\Omega, \mathscr{F}, P)$. Assume that (X_{nk}) is adapted to an array $(\mathscr{F}_{nk})_{n\in\mathbf{N},k\in\mathbf{N}\cup\{0\}}$ of row-wise increasing sub-σ-fields of $\mathscr{F}$, i.e. X_{nk} is $\mathscr{F}_{nk}$-measurable and $\mathscr{F}_{n,k-1}\subset\mathscr{F}_{nk}$. $\mathscr{F}_{n0}$ need not be the trivial σ-field $\{\varnothing, \Omega\}$. Then (X_{nk}) is called a martingale doubly infinite array (MDIA) if $E|X_{nk}|<\infty$, and $E(X_{nk}|\mathscr{F}_{n,k-1})=0$ almost surely (a.s.) for all $n, k\in\mathbf{N}$.

Now let $\{N_n, n\in\mathbf{N}\}$ be a sequence of positive integer-valued r.vs defined on the same probability space $(\Omega, \mathscr{F}, P)$. Let us denote

$$S_{nN_n}=\sum_{k=1}^{N_n}X_{nk},\quad \varphi_{nk}(t)=E\left(\exp\left(itX_{nk}\right)|\mathscr{F}_{n,k-1}\right),$$

$$f_{nN_n}(t)=\prod_{k=1}^{N_n}\varphi_{nk}(t),\quad \sigma_{nk}^2=E(X_{nk}^2|\mathscr{F}_{n,k-1}),$$

$$V_{nN_n}^2=\sum_{k=1}^{N_n}\sigma_{nk}^2,\quad b_{nN_n}=\max_{1\leqq k\leqq N_n}\sigma_{nk}^2.$$

Throughout, $I(A)$ denotes the indicator function of the set A, and the various kinds of convergence, with probability 1 (almost sure), in L_p norm, in probability, and weak (in distribution) are denoted by $\xrightarrow{\text{a.s.}}$, $\xrightarrow{L_p}$, $\xrightarrow{P}$ and $\xrightarrow{\mathscr{D}}$, respectively. We will also abuse notation slightly in the interests of brevity denoting $E(Y|\mathscr{F}_{nk})$ by E_kY, where Y is some variable (e.g., $=X_{ni}$) taken from the n-th row.

2. The main results

To begin with considerations we see that if N_n is $\mathscr{F}_{n0}$-measurable and $\mathscr{F}_{n0}$ is the trivial σ-field, then there exists a sequence $\{k_n\}$ of positive integer numbers such that $P(N_n=k_n)=1$ for all $n\in\mathbf{N}$. So, if $N_n\xrightarrow{P}\infty$, then $k_n\to\infty$. On the other hand, if $N_n, X_{n1}, X_{n2}, \ldots$ are independent random variables, then $Ef_{nN_n}(t)=$ $=E\exp\left(itS_{nN_n}\right)$. Thus, for a fixed real t, $f_{nN_n}(t)\xrightarrow{P}A_t$ as $n\to\infty$, if and only if $E\exp\left(itS_{nN_n}\right)\to A_t$ as $n\to\infty$, since $|f_{nN_n}(t)|\leqq 1$.

The following routine lemma will be useful throughout this paper.

LEMMA 1. *Let* $\{N_n\}$ *be a sequence of positive integer-valued random variable such that*

(1) $$f_{nN_n}(t) \xrightarrow{P} A_t \quad as \quad n \to \infty, \quad for\ some \quad t$$

implies

(2) $$E\left(\exp\left(itS^*_{nN_n}\right)\left(f^*_{nN_n}(t)\right)^{-1}\right) \to 1 \quad as \quad n \to \infty$$

where A_t *is a complex number such that* $|A_t| \neq 0$ ($S^*_{nN_n}$ *and* $f^*_{nN_n}$ *are defined in the proof*). *Then* $E\exp\left(itS_{nN_n}\right) \to A_t$ *as* $n \to \infty$.

PROOF. For each fixed n, let $Y_{nk} = X_{nk} I(A_{nk})$, where $A_{nk} = \{|f_{nk}(t)| \geqq |A_t|/2\}$. If we write $f^*_{nk}(t) = \prod\limits_{1 \leqq j \leqq k} E_{j-1} \exp\left(itY_{nj}\right)$, then

$$|f^*_{nN_n}(t)| = \prod_{k=1}^{N_n} |E_{k-1} \exp\left(itY_{nk}\right)| =$$

$$= \prod_{k=1}^{N_n} |I(A_{nk}) E_{k-1} \exp\left(itX_{nk}\right) + I(A^c_{nk})| \geqq |A_t|/2 \quad \text{a.s. for all} \quad n,$$

and, by (1),

$$P\left(\bigcup_{k=1}^{N_n} \{X_{nk} \neq Y_{nk}\}\right) = P\left(\bigcup_{k=1}^{N_n} A^c_{nk}\right) = P(A^c_{nN_n}) \to 0 \quad \text{as} \quad n \to \infty.$$

Moreover, by (1),

$$P\left(\bigcup_{k=1}^{N_n} \{f_{nk}(t) \neq f^*_{nk}(t)\}\right) = P\left(\bigcup_{k=1}^{N_n} \bigcup_{j=1}^{k} A^c_{nj}\right) = P(A^c_{nN_n}) \to 0 \quad \text{as} \quad n \to \infty,$$

and then

(1′) $$f^*_{nN_n}(t) \xrightarrow{P} A_t \quad \text{as} \quad n \to \infty.$$

Thus, it suffices to show that $E\exp\left(itS^*_{nN_n}\right) \to A_t$ as $n \to \infty$, where $S^*_{nN_n} = \sum\limits_{k=1}^{N_n} Y_{nk}$.

Now we have

$$|E\exp\left(itS^*_{nN_n}\right) - A_t| \leqq \left|E\exp\left(itS^*_{nN_n}\right)\{1 - A_t(f^*_{nN_n}(t))^{-1}\}\right| +$$
$$+ |A_t| \cdot |E\exp\left(itS^*_{nN_n}\right)\left(f^*_{pN_n}(t)\right)^{-1} - 1| \leqq 2/|A_t| E|f^*_{nN_n}(t) - A_t| +$$
$$+ |A_t| \cdot |E\exp\left(itS^*_{nN_n}\right)\left(f^*_{nN_n}(t)\right)^{-1} - 1| \to 0 \quad \text{as} \quad n \to \infty,$$

by (1′), (2) and the dominated convergence theorem ([6], p. 125).

REMARK 1. Note that if $f_{nk_n}(t) \xrightarrow{\text{a.s.}} A_t$, $|A_t| \neq 0$, $k_n \to \infty$, $n \to \infty$, and $N_n \xrightarrow{P} \infty$, $n \to \infty$, then (1) holds (see e.g. Lemma 2 in [4]). Taking into account that in our case $f_{nk_n}(t) \xrightarrow{P} A_t$ is equivalent to the almost sure convergence (since $P(\liminf\limits_{n \to \infty} A_{nk_n}) = 1$), we see that (1) takes place if for instance $f_{nk_n}(t) \xrightarrow{P} A_t$, $k_n \to \infty$, $n \to \infty$, and $N_n \xrightarrow{P} \infty$.

Now we are going to extend one of the results of [2] and [7]. We will assume that (X_{nk}) is a MDIA such that there exists a finite constant C for which

(3) $$\lim_{n \to \infty} P(V^2_{nN_n} > C) = 0,$$

and

(4) $$b_{nN_n} \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty.$$

PROPOSITION 1. *Let $\{N_n\}$ be a sequence of positive integer-valued random variables and suppose (X_{nk}) is a MDIA such that* (3) *and* (4) *hold. Then*

$$f_{nN_n}(t) \xrightarrow{P} \exp\left(\int_{-\infty}^{+\infty} (e^{itx}-1-itx)x^{-2}\, dG(x)\right) \quad as \quad n \to \infty,$$

where G is a bounded nondecreasing function, if and only if for all continuity points a, b of G

$$\sum_{k=1}^{N_n} E_{k-1}X_{nk}^2 I(a < X_{nk} \leqq b) \xrightarrow{P} G(b)-G(a) \quad as \quad n \to \infty. \tag{5}$$

PROOF. We may assume that

$$V_{nN_n}^2 \leqq C \quad \text{a.s. for all} \quad n \in \mathbf{N}. \tag{6}$$

Obviously, there is no loss of generality, because if X_{nk} does not satisfy (6), then we can set $Y_{nk}=X_{nk}I(V_{nk}^2 \leqq C)$. Then $\{Y_{nk}, \mathcal{F}_{nk}, k \in \mathbf{N}\}$ will form a martingale difference sequence and, by (3),

$$P\left(\bigcup_{k=1}^{N_n} \{X_{nk} \neq Y_{nk}\}\right) = P\left(\bigcup_{k=1}^{N_n} \{V_{nk}^2 > C\}\right) = P(V_{nN_n}^2 > C) \to 0 \quad \text{as} \quad n \to \infty. \tag{7}$$

Furthermore, (5) holds with X_{nk} replaced by Y_{nk}. Because of (7), it will suffice to prove Proposition 1 with Y_{nk} replacing X_{nk}, making possible to use property (6). Alternatively, we will prove the Proposition 1 as it stands and assume also that (6) holds. Routine estimates give us

$$|\varphi_{nk}(t)-1| \leqq \frac{1}{2}t^2\sigma_{nk}^2 \quad \text{a.s. for all} \quad n, \quad k \in \mathbf{N},$$

and

$$\sum_{k=1}^{N_n} |\log(1+z_{nk})-z_{nk}| \leqq \sum_{k=1}^{N_n} |z_{nk}|^2 \leqq \max_{1 \leqq k \leqq N_n} |z_{nk}| \sum_{k=1}^{N_n} |z_{nk}|.$$

This with $z_{nk}=\varphi_{nk}(t)-1$, by (4), yields for fixed t

$$\sum_{k=1}^{N_n} |\log \varphi_{nk}(t)+1-\varphi_{nk}(t)| \leqq t^4 b_{nN_n} V_{nN_n}^2/4 \leqq t^4 b_{nN_n} C/4 \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty.$$

Thus,

$$\log f_{nN_n}(t) = \sum_{k=1}^{N_n} \log \varphi_{nk}(t) = \sum_{k=1}^{N_n} (\varphi_{nk}(t)-1)+A_{nN_n}(t),$$

where for each t, $A_{nN_n}(t) \xrightarrow{P} 0$ as $n \to \infty$.

Next, for fixed t, let us denote

$$g_t(x) = \begin{cases} (e^{itx}-1-itx)/x^2 & \text{for} \quad x \neq 0 \\ -t^2/2 & \text{for} \quad x = 0. \end{cases}$$

For arbitrary positive ε_1, ε_2 and ε_3, choose an integer m sufficiently large and a subdivision $x_0<x_1<\ldots<x_m$, all continuity points of G, so that

$$|B_m(n)| = \left| \sum_{j=1}^{m} g_t(x_{j-1})\,(G(x_j)-G(x_{j-1})) - \int_{-\infty}^{+\infty} g_t(x)\,dG(x) \right| < \varepsilon_1,$$

$$\max_{1\leqq j\leqq m} (x_j-x_{j-1}) < \varepsilon_2, \quad |g_t(x)| < \varepsilon_3 \quad \text{for} \quad x \leqq x_0 \quad \text{or} \quad x > x_m.$$

Then, using (6), it follows that

$$|C_m(n)| = \left| \sum_{k=1}^{N_n} \sum_{j=1}^{m} E_{k-1} g_t(X_{nk}) X_{nk}^2 I(x_{j-1} < X_{nk} \leqq x_j) - \right.$$

$$\left. - \sum_{k=1}^{N_n} \sum_{j=1}^{m} g_t(x_{j-1}) E_{k-1} X_{nk}^2 I(x_{j-1} < X_{nk} \leqq x_j) \right| \leqq M_t C \varepsilon_2,$$

where $M_t=\sup_x |dg_t(x)/dx|<\infty$, and

$$|D_m(n)| = \left| \sum_{k=1}^{N_n} E_{k-1} g_t(X_{nk}) X_{nk}^2 - \sum_{k=1}^{N_n} \sum_{j=1}^{m} E_{k-1} g_t(X_{nk}) X_{nk}^2 I(x_{j-1} < X_{nk} \leqq x_j) \right| \leqq C\varepsilon_3.$$

Thus, by the relation given above, we obtain

$$\log f_{nN_n}(t) = \sum_{k=1}^{N_n} (\varphi_{nk}(t)-1) + A_{nN_n}(t) = \sum_{k=1}^{N_n} E_{k-1} g_t(X_{nk}) X_{nk}^2 + \tag{8}$$

$$+ A_{nN_n}(t) = \int_{-\infty}^{+\infty} g_t(x)\,dG(x) + B_m(n) +$$

$$+ \sum_{k=1}^{m} g_t(x_{j-1}) \left\{ \sum_{k=1}^{N_n} E_{k-1} X_{nk}^2 I(x_{j-1} < X_{nk} \leqq x_j) - (G(x_j)-G(x_{j-1})) \right\} +$$

$$+ C_m(n) + D_m(n) + A_{nN_n}(t).$$

Now we see that the convergence in probability of $f_{nN_n}(t)$ to a characteristic function of an infinitely divisible limit law with finite variance is equivalent to the convergence in probability to zero of the third term of (8), as $n\to\infty$ (for a sufficiently large and fixed m). The proof is complete.

Although it makes the issue somewhat simplified, the assumption that the rows of the array form a martingale difference sequence is not essential. As a trivial extension of Proposition 1, we can obtain

PROPOSITION 2. *Let $\{N_n\}$ be a sequence of positive integer-valued random variables, and let (U_{nk}) be a DIA with $X_{nk}=U_{nk}-E_{k-1}U_{nk}$, $n, k\in\mathbf{N}$, such that $X_{n1}, X_{n2}, \ldots$ satisfies* (3) *and* (4). *Then*

$$\tilde{f}_{nN_n}(t) := \prod_{k=1}^{N_n} E_{k-1} \exp(itU_{nk}) \xrightarrow{P} f(t) \quad \text{as} \quad n \to \infty,$$

where

$$\log f(t) = it\gamma + \int_{-\infty}^{+\infty} (e^{itx} - 1 - itx)x^{-2}dG(x), \tag{9}$$

with G a bounded nondecreasing function, if and only if

$$\sum_{k=1}^{N_n} E_{k-1}U_{nk} \xrightarrow{P} \gamma \quad as \quad n \to \infty, \tag{10}$$

$$\sum_{k=1}^{N_n} E_{k-1}X_{nk}^2 I(a < X_{nk} \leqq b) \xrightarrow{P} G(b) - G(a) \tag{11}$$

as $n \to \infty$, for all contiunity points a, b of G.

Now we are able to give the sufficient conditions for the weak convergence of S_{nN_n} to an infinitely divisible limit law with finite variance.

THEOREM 1. *Let $\{N_n\}$ be a sequence of positive integer-valued random variables as in Lemma* 1, *and let (U_{nk}) be a DIA with $X_{nk} = U_{nk} - E_{k-1}U_{nk}$, $n, k \in \mathbf{N}$, such that $X_{n1}, X_{n2}, \ldots$ satisfies* (3) *and* (4). *In order that S_{nN_n} converge weakly to an infinitely divisible limit law, whose characteristic function $f(t)$ is given by* (9), *it is sufficient that* (10) *and* (11) *hold.*

The proof is easily based on Proposition 2 and Lemma 1 and is not detailed here. Theorem 1 generalizes the main result of Brown and Eagleson ([2], Theorem 1).

COROLLARY 1. *Assume that (U_{nk}) is a DIA such that*

$$\sum_{k=1}^{N_n} E_{k-1}(U_{nk} - E_{k-1}U_{nk})^2 \xrightarrow{P} \lambda \quad as \quad n \to \infty, \quad \lambda > 0,$$

$$\sum_{k=1}^{N_n} E_{k-1}U_{nk} \xrightarrow{P} \lambda \quad as \quad n \to \infty,$$

$$\max_{1 \leqq k \leqq N_n} E_{k-1}(U_{nk} - E_{k-1}U_{nk})^2 \xrightarrow{P} 0 \quad as \quad n \to \infty,$$

and

$$\sum_{k=1}^{N_n} E_{k-1}(U_{nk} - E_{k-1}U_{nk})^2 I(|U_{nk} - E_{k-1}U_{nk} - 1| \geqq \varepsilon) \xrightarrow{P} 0$$

as $n \to \infty$, for all $\varepsilon > 0$. Then $S_{nN_n} \xrightarrow{\mathcal{D}} \mathcal{P}(\lambda)$, where $\mathcal{P}(\lambda)$ denotes the Poisson law with parameter λ.

COROLLARY 2. *Assume that (U_{nk}) is a DIA such that*

$$\sum_{k=1}^{N_n} E_{k-1}(U_{nk} - E_{k-1}U_{nk})^2 \xrightarrow{P} \lambda/\Theta^2 \quad as \quad n \to \infty, \quad \lambda > 0, \quad \Theta > 0,$$

$$\sum_{k=1}^{N_n} E_{k-1}U_{nk} \xrightarrow{P} \lambda/\Theta \quad as \quad n \to \infty,$$

$$\max_{1 \leqq k \leqq N_n} E_{k-1}(U_{nk} - E_{k-1}U_{nk})^2 \xrightarrow{P} 0 \quad as \quad n \to \infty,$$

and

$$\sum_{k=1}^{N_n} E_{k-1}(U_{nk}-E_{k-1}U_{nk})^2 I(U_{nk}-E_{k-1}U_{nk} \geqq a) \xrightarrow{P} \lambda(1+\Theta a)\Theta^{-2} I(a \geqq 0)e^{-\Theta a}$$

as $n \to \infty$. *Then* $S_{nN_n} \xrightarrow{\mathcal{D}} \Gamma(\lambda, \Theta)$, *where* $\Gamma(\lambda, \Theta)$ *denotes the gamma law with parameters* λ, Θ.

REMARK 2. Setting $\lambda = \nu/2$, $\Theta = 1/2$, we have the chi-square law with parameter ν, where ν is the number of degrees of freedom.

As a consequence of Proposition 2 we have

PROPOSITION 3 (random central limit theorem). *Let* $\{N_n\}$ *be a sequence of positive integer-valued random variables, and let* $\{s_n\}$ *be a sequence of positive real numbers. Suppose that* (U_{nk}) *is a DIA with* $X_{nk} = U_{nk} - E_{k-1}U_{nk}$, $n, k \in \mathbf{N}$, *such that*

$$s_n^{-2} \sum_{k=1}^{N_n} E_{k-1}X_{nk}^2 \xrightarrow{P} \sigma^2 \quad as \quad n \to \infty. \tag{12}$$

Then

$$\tilde{f}_{nN_n}(t/s_n) \xrightarrow{P} \exp(it\gamma - t^2\sigma^2/2) \quad as \quad n \to \infty,$$

for all real t, *and*

$$s_n^{-2} \max_{1 \leqq k \leqq N_n} E_{k-1}X_{nk}^2 \xrightarrow{P} 0 \quad as \quad n \to \infty$$

if and only if

$$s_n^{-1} \sum_{k=1}^{N_n} E_{k-1}U_{nk} \xrightarrow{P} \gamma \quad as \quad n \to \infty, \tag{13}$$

and for all $\varepsilon > 0$,

$$s_n^{-2} \sum_{k=1}^{N_n} E_{k-1}X_{nk}^2 I(|X_{nk}| \geqq \varepsilon s_n) \xrightarrow{P} 0 \quad as \quad n \to \infty. \tag{14}$$

REMARK 3. Rychlik [7] states the same result for martingale difference sequences $\{X_k\}$, but he also assumes that for all n, N_n is a stopping time. Setting $E_{k-1}U_{nk} = 0$ a.s. for all n, $k \in \mathbf{N}$ (then $\gamma = 0$), and $s_n^2 = E(U_{n1}^2 + \ldots + U_{nN_n}^2)$ $(= B_n^2$ in his notice) we have Rychlik's result ([7], Theorem 1) for a MDIA. But in our assumptions N_n need not be a stopping time.

Sketch of the proof of Proposition 3. As in the proof of Proposition 1, assume that (U_{nk}) is a MDIA (then $\gamma = 0$). We see that the assumption (3) is satisfied, i.e., $\lim_{n \to \infty} P(s_n^{-2}V_{nN_n}^2 > C) = 0$, where C is a constant greater than σ^2, and that by (12) the random Lindeberg condition (14) is equivalent to

$$s_n^{-2} \sum_{k=1}^{N_n} E_{k-1}X_{nk}^2 I(|X_{nk}| < \varepsilon s_n) \xrightarrow{P} \sigma^2 \quad \text{as} \quad n \to \infty,$$

which becomes (5) with $G(x) = \sigma^2 I(x \geqq 0)$. Moreover, by (14),

$$s_n^{-2} \max_{1 \leqq k \leqq N_n} E_{k-1}X_{nk}^2 \leqq \varepsilon^2 + s_n^{-2} \sum_{k=1}^{N_n} E_{k-1}X_{nk}^2 I(|X_{nk}| \geqq \varepsilon s_n) \xrightarrow{P} \varepsilon^2 \quad \text{as} \quad n \to \infty,$$

and as $\varepsilon>0$ is arbitrarily small, the random Lindeberg condition (14) implies $s_n^{-2}b_{nN_n}\xrightarrow{P}0$, $n\to\infty$. We also have $A_{nN_n}(t)\xrightarrow{P}0$ if $s_n^{-2}b_{nN_n}\xrightarrow{P}0$. So, taking into account the considerations given in the proof of Proposition 1, we obtain an equality similarly to (8).

Proposition 3 combined with Lemma 1 yields

THEOREM 2. *Assume that* $\{N_n\}$, $\{s_n\}$, (U_{nk}) *and* (X_{nk}) *are as in Proposition* 3, *i.e.* (12) *holds. In order that* $s_n^{-1}S_{nN_n}$ *converge weakly to the normal law* $N(\gamma,\sigma^2)$, *it is sufficient that* (13) *and* (14) *hold.*

Theorem 2 generalizes the main result of Brown [1].

Rychlik's result ([7], Theorem 2) and the Lindeberg—Feller's central limit theorem ([6], p. 295) can be extended to a DIA as follows.

THEOREM 3. *Let* $\{N_n\}$ *be a sequence of positive integer-valued random variables, and let* (X_{nk}) *be a MDIA such that for all* n, $N_n, X_{n1}, X_{n2}, \ldots$ *are independent random variables, and* $EX_{nk}=0$, $EX_{nk}^2=\sigma_{nk}^2$, $k\in\mathbf{N}$. *Assume that* $\{s_n\}$ *is a sequence of positive real numbers* $\left(e.g.,\ s_n^2=E\left(\sum_{k=1}^{N_n}X_{nk}^2\right)\right)$, *such that*

$$s_n^{-2}\sum_{k=1}^{N_n}\sigma_{nk}^2\xrightarrow{P}\sigma^2\quad as\quad n\to\infty.$$

Then

$$s_n^{-1}\sum_{k=1}^{N_n}X_{nk}\xrightarrow{\mathcal{D}}N(0,\sigma^2)\quad and\quad \lim_{n\to\infty}E\Big(\max_{1\le k\le N_n}\sigma_{nk}^2s_n^{-2}\Big)=0$$

if and only if for every $\varepsilon>0$,

$$\lim_{n\to\infty}s_n^{-2}E\sum_{k=1}^{N_n}EX_{nk}^2I(|X_{nk}|\ge\varepsilon s_n)=0.$$

3. Applications

Now we are discussing conditions on a MDIA (X_{nk}) and a sequence $\{N_n\}$ of positive integer-valued random variables under which (3) and (4) hold, implying the statements of Section 2.

First we note that if $V_{nk_n}^2\le C$ a.s. for all n, and

$$b_{nk_n}\xrightarrow{\text{a.s.}}0,\quad k_n\to\infty,\quad n\to\infty,\quad \text{and}\quad N_n\xrightarrow{P}\infty\quad\text{as}\quad n\to\infty,$$

then (3) and (4) hold (see e.g. Lemma 2 in [4]).

The following result generalizes the one of Gaenssler, Strobel and Stute ([5], Theorem 6), and at the same time a result of Csörgő ([3], Theorem 6). We use the notations

$$k_n\wedge N_n=\min(k_n,N_n),\quad k_n\vee N_n=\max(k_n,N_n).$$

THEOREM 4. *Let* (X_{nk}) *be a MDIA and suppose that* $\{N_n\}$ *is a sequence of positive integer-valued random variables as in Lemma* 1 *such that there exists a sequence* $\{k_n\}$

of positive real numbers, a finite constant C and a bounded nondecreasing function G for which

$$P(N_n > k_n) \to 0 \quad as \quad n \to \infty, \tag{15}$$

$$P(V^2_{nk_n} > C) \to 0 \quad as \quad n \to \infty, \tag{3'}$$

$$b_{nk_n} \xrightarrow{P} 0 \quad as \quad n \to \infty, \tag{4'}$$

and

$$\sum_{k=1}^{k_n \wedge N_n} E_{k-1} X^2_{nk} I(a < X_{nk} \leqq b) \xrightarrow{P} G(b) - G(a) \tag{5'}$$

as $n \to \infty$, *for all continuity points a, b of G. Then* $\sum_{k=1}^{N_n} X_{nk}$ *converges weakly as* $n \to \infty$, *to an infinitely divisible limit law whose characteristic function* $f(t)$ *is given by*

$$\log f(t) = \int_{-\infty}^{+\infty} (e^{itx} - 1 - itx) x^{-2} dG(x).$$

PROOF. Note that

$$\sum_{k=1}^{N_n} X_{nk} = \sum_{k=1}^{N_n \wedge k_n} X_{nk} + \left(\sum_{k=1}^{k_n \vee N_n} X_{nk} - \sum_{k=1}^{k_n} X_{nk} \right),$$

where the last term tends in probability to zero, since (15) holds. So it can be omitted in our considerations (see e.g. Lemma 1 in [4]). Further, we see that

$$P(V^2_{nN_n} > C) = \sum_{k=1}^{k_n} P(V^2_{nk} > C, N_n = k) + \sum_{k=k_n+1}^{\infty} P(V^2_{nk} > C, N_n = k) \leqq \tag{16}$$

$$\leqq \sum_{k=1}^{k_n} P(V^2_{nk_n} > C, N_n = k) + \sum_{k=k_n+1}^{\infty} P(N_n = k) \leqq P(V^2_{nk_n} > C) +$$

$$+ P(N_n > k_n) \to 0 \quad \text{as} \quad n \to \infty,$$

because of (15) and (3'), which is (3). In a similar way we see that (15), (16) and (4') imply (4). This is so, because

$$\max_{1 \leqq k \leqq N_n} E_{k-1} X^2_{nk} \leqq \max_{1 \leqq k \leqq k_n} E_{k-1} X^2_{nk} + I(N_n > k_n) \max_{k \leqq N_n} E_{k-1} X^2_{nk} \leqq$$

$$\leqq \max_{1 \leqq k \leqq k_n} E_{k-1} X^2_{nk} + I(N_n > k_n) V^2_{nN_n} \quad \text{a.s. for all} \quad n.$$

We also see that (15) and (5') imply (5). Thus, by Proposition 1, we obtain the desired result.

COROLLARY 3. *Let* (X_{nk}) *be a MDIA and suppose* $\{N_n\}$ *is a sequence of positive integer-valued random variables such that there exist sequences* $\{k_n\}$ *and* $\{s_n\}$ *of posi-*

tive real numbers for which

$$P(N_n > k_n) \to 0 \quad as \quad n \to \infty, \tag{15}$$

$$s_n^{-2} \sum_{k=1}^{k_n \wedge N_n} E_{k-1} X_{nk}^2 \xrightarrow{P} \sigma^2 \quad as \quad n \to \infty, \tag{3''}$$

$$s_n^{-2} \sum_{k=1}^{k_n} E_{k-1} X_{nk}^2 I(|X_{nk}| \geqq \varepsilon s_n) \xrightarrow{P} 0 \quad as \quad n \to \infty. \tag{5''}$$

Then $s_n^{-1} \sum_{k=1}^{N_n} X_{nk} \xrightarrow{\mathcal{D}} N(0, \sigma^2)$.

REMARK 4. Gaenssler, Strobel and Stute ([5], Theorem 6), give the same result for martingale difference sequences $\{X_k\}$, but they assume that for all n, N_n is a stopping time.

COROLLARY 4. *Let* (X_{nk}) *be a MDIA and suppose* $\{N_n\}$ *is a sequence of positive integer-valued random variables fulfilling*

$$\frac{N_n}{n} \xrightarrow{P} \Theta \quad as \quad n \to \infty, \tag{15'}$$

$$E_{k-1} X_{nk}^2 = \sigma_n^2 \quad for\ all \quad k \in \mathbf{N}, \quad and \quad \sigma_n^2 \xrightarrow{P} \sigma^2 \quad as \quad n \to \infty, \tag{3''}$$

$$n^{-1} \sum_{k=1}^{[n\Theta]} E_{k-1} X_{nk}^2 I(|X_{nk}| \geqq \varepsilon n^{1/2}) \xrightarrow{P} 0 \quad as \quad n \to \infty. \tag{5''}$$

Then

$$n^{-1/2} \sum_{k=1}^{N_n} X_{nk} \xrightarrow{\mathcal{D}} N(0, \Theta\sigma^2) \quad and \quad N_n^{-1/2} \sum_{k=1}^{N_n} X_{nk} \xrightarrow{\mathcal{D}} N(0, \sigma^2).$$

REMARK 5. Corollary 4 generalizes the result of Csörgő ([3], Theorem 6), where he considers an absolutely fair sequence $\{X_k\}$ of random variables satisfying $EX_1^2 = \sigma_1^2 > 0$, $E(X_k^2 | \mathscr{F}_{k-1}) = \sigma_1^2$ a.s. for $k \geqq 2$.

References

[1] B. M. Brown, Martingale central limit theorems, *Ann. Math. Statist.*, **42** (1971), 59—66.

[2] B. M. Brown and G. K. Eagleson, Martingale convergence to infinitely divisible limit laws with finite variances, *Trans. Amer. Math. Soc.*, **162** (1971), 449—453.

[3] M. Csörgő, On the strong law of large numbers and the central limit theorem for martingales, *Trans. Amer. Math. Soc.*, **131** (1968), 259—275.

[4] M. Csörgő and R. Fischler, Departure from independence: the strong law, standard and random-sum central limit theorems, *Acta Math. Acad. Sci. Hungar.*, **21** (1970), 105—114.

[5] P. Gaenssler, J. Strobel and W. Stute, On central limit theorems for martingale triangular arrays, *Acta Math. Acad. Sci. Hungar.*, **31** (1978), 205—216.

[6] M. Loeve, *Probability theory*, 3rd ed., Van Nostrand (New York, 1963).

[7] Z. Rychlik, Martingale random central limit theorems, *Acta Math. Acad. Sci. Hungar.*, **34** (1979), 129—139.

(Received December 15, 1981)

INSTITUTE OF APPLIED MATHEMATICS
ACADEMY OF AGRICULTURE
AKADEMICKA 13
20-834 LUBLIN
POLAND

INSTITUTE OF MATHEMATICS
MARIA SKŁODOWSKA UNIVERSITY
NOWOTKI 10
20-031 LUBLIN
POLAND

Acta Math. Hung.
42 (1—2) (1983), 153—164.

РЕШЕНИЕ ОДНОЙ ЭКСТРЕМАЛЬНОЙ ЗАДАЧИ ТЕОРИИ ОПЕРАТОРОВ

Д. Л. БЕРМАН (Ленинград)

1. Введение

Ради простоты все результаты формулируются и доказываются для пространства C 2π-периодических непрерывных функций с нормой $\|f\|=\operatorname{Max}|f(x)|$. k и n — натуральные числа. $f^{(k)}(x)$ — k-я производная от $f(x)$. T_n — множество всех тригонометрических полиномов порядка $\leqq n$. Через $\Omega_n^{(k)}$ обозначим класс всех линейных операторов $U_n(f, x)$ из C в C, обладающих свойством $U_n(f, x)=f^{(k)}(x)$, если $f\in T_n$. Положим $\varrho_n^{(k)}=\inf\limits_{U_n\in\Omega_n^{(k)}}\|U_n\|$. Будем говорить, что оператор $\overline{U}_n\in\Omega_n^{(k)}$ экстремальный в классе $\Omega_n^{(k)}$, если $\|\overline{U}_n\|=\varrho_n^{(k)}$. Возникает естественная задача о нахождении экстремальных операторов и о вычислении их нормы. Задача эта была решена в [1]. А именно, было доказано, что при любых k и n оператор

$$(1)\qquad V_n(f, x)=\sum_{j_1, j_2, \ldots, j_k} f(x+x_{j_1}+x_{j_2}+\ldots+x_{j_k})r_{j_1}r_{j_2}\ldots r_{j_k},$$

$$r_{j_s}=\frac{(-1)^{j_s-1}}{4n\sin^2 x_{j_s}/2};\quad x_{j_s}=\frac{(2j_s-1)\pi}{2n},\quad 1\leqq j_s\leqq 2n,\quad s=1, 2, \ldots, k,$$

принадлежит $\Omega_n^{(k)}$ и $\|V_n\|=\varrho_n^{(k)}$.

Класс операторов $\Omega_n^{(k)}$ весьма обширный и поэтому возникает вопрос о нахождении экстремальных операторов более простой конструкции, чем оператор (1). В частности, возникает задача о построении при произвольных k и n экстремального оператора из $\Omega_n^{(k)}$ с помощью однократного суммирования. Настоящая статья, в основном, посвящена решению этой задачи. Ее полное решение дается теоремой 1, исходя из которой при $1\leqq k\leqq 8$ для всех n получены сравнительно простые формулы для экстремальных операторов. Упомянутые результаты составляют содержание § 2. В § 3 изложен другой метод построения экстремальных операторов. В основе этого метода лежат интерполяционные тождества типа тождества М. Рисса [2]. В § 4, с помощью полученных формул для экстремальных операторов, выводится ряд элементарных тригонометрических тождеств. Эти тождества являются аналогами известного тождества М. Рисса [2]. В § 5 получены неравенства, относящиеся к системам узлов

$$(m_1)\qquad \theta_k=\frac{(2k-1)\pi}{2n},\quad k=1, 2, \ldots, 2n;$$

$$(m_2)\qquad \gamma_k=\frac{(k-1)\pi}{n},\quad k=1, 2, \ldots, 2n.$$

2. Построение экстремальных операторов

Пусть $\varphi \in T_n$. Очевидно, что φ можно представить в виде

$$\varphi(t) = \sum_{k=0}^{n} r_k \sin(kt+\alpha_k), \tag{2}$$

где r_k и α_k — вещественные числа, причем $r_k \geqq 0$; $k=0, 1, \ldots, n$. По заданному полиному $\varphi(t)$ можно построить полином

$$\tilde{\varphi}(t) = r_n + 2\sum_{k=0}^{n-1} r_k \cos[(n-k)t+\alpha_n-\alpha_k],$$

который будем называть ассоциированным с полиномом φ. Положим

$$\sigma(f, x) = \int_0^{2\pi} f(x+t)\varphi(t)\,dt.$$

Обозначим через Ω_n^{φ} множество всех линейных операторов U из C в C, обладающих тем свойством, что $U(f_n)=\sigma(f_n)$, если $f_n \in T_n$. Положим $\varrho^{\varphi} = \inf_{U \in \Omega_n^{(\varphi)}} \|U\|$. Будем говорить, что оператор $\overline{U} \in \Omega_n^{\varphi}$ экстремальный в классе Ω_n^{φ}, если $\|\overline{U}\| = \varrho_n^{\varphi}$. Понятно, что класс операторов $\Omega_n^{(k)}$, есть частный случай класса Ω_n^{φ}, когда $\varphi(t) = \frac{1}{\pi} D_n^{(k)}(t)$, где $D_n(t)$ есть ядро Дирихле

$$D_n(t) = \frac{1}{2} + \cos t + \cos 2t + \ldots + \cos nt = \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}. \tag{3}$$

В 1966 году [3], [4] нами была доказана следующая теорема.

Теорема. *Если*

$$\tilde{\varphi}\left(\theta_r - \frac{\alpha_n}{n}\right) \geqq 0, \quad \theta_r = \frac{(2r-1)\pi}{2n}, \quad r = 1, 2, \ldots, 2n,$$

то в классе Ω_n^{φ} *наименьшей нормой обладает оператор*

$$\overline{U}(f, x) = \sum_{r=1}^{2n} \lambda_r f\left(x+\theta_r-\frac{\alpha_n}{n}\right); \quad \lambda_r = (-1)^{r-1}\frac{\pi}{2n}\tilde{\varphi}\left(\theta_r-\frac{\alpha_n}{n}\right).$$

Положим $\varrho^{\varphi} = \inf_{U \in \Omega_n^{\varphi}} \|U\|$, *тогда* $\varrho^{\varphi} = \|\overline{U}\| = \pi r_n$.

Ядро Дирихле D_n обладает тем интересным свойством, что ассоциированные полиномы для производных от этого ядра неотрицательные на всей числовой оси, то есть

$$\tilde{D}_k^{(k)}(t) \geqq 0, \quad k = 1, 2, \ldots, -\infty < t < \infty. \tag{4}$$

Доказательство неравенств (4) находится в [5]. Поэтому в силу ранее сформулированной теоремы имеет место следующее следствие.

Следствие. *Пусть* $k \geqq 1$. *Тогда в классе* $\Omega_n^{(k)}$ *наименьшей нормой обладает оператор*

$$\overline{U}_n^{\langle k\rangle}(f, x) = \frac{1}{2n} \sum_{r=1}^{2n} f\left(x+\theta_r-\frac{\alpha_n}{n}\right)(-1)^{r-1}\tilde{D}_n^{(k)}\left(\theta_r-\frac{\alpha_n}{n}\right). \tag{5}$$

При этом $\varrho_n^{(k)} = n^k$.

С помощью этого следствия доказывается

Теорема 1. *Оператор*

$$U_n^{\langle 2m-1\rangle}(f, x) = \frac{(-1)^{m+1}}{2n} \sum_{k=1}^{2n} f\left(x+\frac{(2k-1)\pi}{2n}\right) \cdot \tag{6}$$

$$\cdot\left[(-1)^{k-1} n^{2m-1} + 2 \sum_{\nu=1}^{n-1} \nu^{2m-1} \sin \nu \frac{(2k-1)\pi}{2n}\right]$$

является экстремальным в классе $\Omega_n^{(2m-1)}$, $m=1, 2, \ldots$. *Оператор*

$$U_n^{\langle 2m\rangle}(f, x) = \frac{(-1)^{m+1}}{2n} \sum_{k=1}^{2n} f\left(x+\frac{(k-1)\pi}{n}\right) \cdot \tag{7}$$

$$\cdot\left[(-1)^{k-1} n^{2m} + 2 \sum_{\nu=1}^{n=1} \nu^{2m} \cos \nu \frac{(k-1)\pi}{n}\right]$$

является экстремальным в классе $\Omega_n^{(2m)}$, $m=1, 2, \ldots$. *При любом* k, $\varrho_n^{(k)} = n$.

Доказательство. В силу (3) имеем, что

$$D_n^{(k)}(t) = \sum_{\nu=1}^{n} \nu^k \sin\left(\nu t + \frac{(k+1)\pi}{2}\right), \quad k = 1, 2, \ldots.$$

Поэтому $\tilde{D}_n^{(k)} = n^k + 2 \sum_{\nu=0}^{n-1} \nu^k \cos(n-\nu)t$, $k=1, 2, \ldots$ и в соответствии с (2)

$$\alpha_1 = \alpha_2 = \ldots = \frac{k+1}{2}\pi, \quad \theta_r - \frac{\alpha_n}{n} = \frac{\pi}{2n}(2r-k-2).$$

Стало быть, формула (5) принимает вид

$$U_n^{\langle k\rangle}(f, x) = \frac{1}{2n} \sum_{r=1}^{2n} f\left(x+\frac{(2r-k-2)\pi}{2n}\right)(-1)^r \cdot \tag{8}$$

$$\cdot\left[n^k + 2 \sum_{\nu=1}^{n-1} \nu^k \cos(n-\nu) \cdot \frac{(2r-k-2)\pi}{2n}\right].$$

Заметим, что справедливы равенства

$$\cos(n-\nu)\frac{(2r-k-2)\pi}{2n} = (-1)^{r-1}\cos\left(\frac{k\pi}{2}+\frac{(2r-k-2)\pi}{2n}\right), \tag{9}$$

$$\cos\left(\frac{k\pi}{2}+\alpha\right) = \begin{cases}(-1)^{k/2}\cos\alpha, & k\equiv 0 \pmod 2;\\ (-1)^{\frac{k+1}{2}}\sin\alpha, & k\equiv 1 \pmod 2.\end{cases}$$

В силу (8), (9) выводим, что

$$U_n^{\langle 2m\rangle}(f,x) = -\frac{1}{2n}\sum_{r=1}^{2n} f\left(x+\frac{(r-m-1)\pi}{n}\right)\cdot \tag{10}$$

$$\cdot\left[(-1)^{r-1}n^{2m}+2(-1)^m\sum_{\nu=1}^{n-1}\nu^{2m}\cos\nu\frac{r-m-1}{n}\pi\right];$$

$$U_n^{\langle 2m-1\rangle}(f,x) = -\frac{1}{2n}\sum_{r=1}^{2n} f\left(x+\frac{(2r-2m-1)\pi}{2n}\right)\cdot$$

$$\cdot\left[(-1)^{r-1}n^{2m-1}+2(-1)^m\sum_{\nu=1}^{n-1}\nu^{2m-1}\sin\nu\frac{2r-2m-1}{2n}\pi\right].$$

Положим $r-m=k$, тогда из (10) получим, что

$$U_n^{\langle 2m\rangle}(f,x) = \frac{(-1)^{m+1}}{2n}\sum_{k=1-m}^{2n-m} f\left(x+\frac{(k-1)\pi}{n}\right)\cdot \tag{11}$$

$$\cdot\left[(-1)^{k-1}n^{2m}+2\sum_{\nu=1}^{n-1}\nu^{2m}\cos\nu\frac{(k-1)\pi}{n}\right].$$

Поскольку $f(x)$ и $\cos x$ 2π-периодические функции, то равенство (11) эквивалентно равенству (7). Формула (6) выводится аналогичными рассуждениями.

Теорема 2. *Оператор*

$$U_n^{\langle 1\rangle}(f,x) = \sum_{k=1}^{2n} f\left(x+\frac{(2k-1)\pi}{2n}\right)r_k, \quad r_k = \frac{(-1)^{k+1}}{4n\sin^2\frac{(2k-1)\pi}{4n}}, \tag{12}$$

из $\Omega_n^{(1)}$ *и является в этом классе экстремальным* $\|U_n^{\langle 1\rangle}\| = \varrho_n^{(1)} = n$.

Теорема 3. *Оператор*

$$U_n^{\langle 2\rangle}(f,x) = -\frac{2n^2+1}{6}f(x)-\frac{1}{2}\sum_{k=2}^{2n}(-1)^{k+1}f(x+\gamma_k)\operatorname{cosec}^2\frac{\gamma_k}{2}, \tag{13}$$

$$\gamma_k = \frac{(k-1)\pi}{n}, \quad k = 1, 2, \ldots, 2n,$$

из $\Omega_n^{(2)}$ *и является экстремальным в этом классе*

$$\varrho_n^{(2)} = \|U_n^{\langle 2\rangle}\| = n^2. \tag{13'}$$

Теорема 4. *Оператор*

$$U_n^{\langle 3\rangle}(f,x) = \frac{1}{2n}\sum_{k=1}^{2n}(-1)^k f(x+\theta_k)a_k, \quad a_k = \frac{3n^2+1}{2\sin^2\frac{\theta_k}{2}} - \frac{3}{4\sin^4\frac{\theta_k}{2}},$$

из класса $\Omega_n^{(3)}$ *и является экстремальным в этом классе* $\varrho_n^{(3)} = \|U_n^{\langle 3\rangle}\| = n^3$.

Теорема 5. *Оператор*

$$U_n^{\langle 4\rangle}(f,x) = \left[\frac{1}{30}(n-1)(2n-1)(3n^2-3n-1)+\frac{n^3}{2}\right]f(x)+ \tag{14}$$

$$+\frac{1}{2}\sum_{k=2}^{2n}(-1)^{k+1}f(x+\gamma_k)\operatorname{cosec}^2\frac{\gamma_k}{2}\left[2(n^2+1)-3\operatorname{cosec}^2\frac{\gamma_k}{2}\right]$$

из класса Ω_n *и является экстремальным в этом классе* $\varrho_n^{(4)} = \|U_n^{\langle 4\rangle}\| = n^4$.

Теорема 6. *Оператор*

$$U_n^{\langle 5\rangle}(f,x) = \frac{1}{2n}\sum_{k=1}^{2n}(-1)^k f(x+\theta_k)b_k,$$

$$b_k = \frac{15}{4}(2n^2+1)\operatorname{cosec}^4\frac{\theta_k}{2} - \frac{1}{2}(5n^4+10n^2+1)\operatorname{cosec}^2\frac{\theta_k}{2} - \frac{15}{4}\operatorname{cosec}^6\frac{\theta_k}{2},$$

из класса $\Omega_n^{(4)}$ *и является экстремальным в этом классе* $\varrho_n^{(5)} = \|U_n^{\langle 5\rangle}\| = n^5$.

Теорема 7. *Оператор*

$$U_n^{\langle 6\rangle}(f,x) = -\left[\frac{1}{42}(n-1)(2n-1)\left(3(n-1)^4+6(n-1)^3-3(n-1)+1\right)+\frac{n^5}{2}\right]f(x)+ \tag{15}$$

$$+\frac{1}{2}\sum_{k=2}^{2n}(-1)^{k+1}f(x+\gamma_k)\operatorname{cosec}^2\frac{\gamma_k}{2}\left[5n^2\left(3\operatorname{cosec}^2\frac{\gamma_k}{2}-2\right)-\right.$$

$$\left.-\frac{3}{2}\left(15\operatorname{cosec}^4\frac{\gamma_k}{2}-15\operatorname{cosec}^2\frac{\gamma_k}{2}+2\right)-3n^4\right]$$

из класса $\Omega_n^{(6)}$ *и является экстремальным в этом классе* $\varrho_n^{(6)} = \|U_n^{\langle 6\rangle}\| = n^6$.

Теорема 8. *Оператор*

$$U_n^{\langle 7\rangle}(f,x) = \frac{1}{2n}\sum_{j=1}^{2n}(-1)^j f(x+\theta_j)C_j, \quad \theta_j = \frac{(2j-1)\pi}{2n},$$

$$C_j = -\frac{315}{8}\operatorname{cosec}^8\frac{\theta_j}{2}+\frac{1}{4}(210+315n^2)\operatorname{cosec}^6\frac{\theta_j}{2}-$$

$$-\frac{1}{4}(63+315n^2+105n^4)\operatorname{cosec}^4\frac{\theta_j}{2}+\frac{1}{2}(1+21n^2+35n^4+7n^6)\operatorname{cosec}^2\frac{\theta_j}{2},$$

из класса $\Omega_n^{\langle 7\rangle}$ *и является экстремальным оператором в этом классе* $\varrho_n^{(7)} = \|U_n^{\langle 7\rangle}\| = n^7$.

По поводу доказательств теорем 2—8 заметим здесь лишь, что эти теоремы непосредственно вытекают из теоремы 1, но для этого нужно вычислить суммы $\sum_{\nu=1}^{n-1} \nu^k \cos \nu\alpha$, $\sum_{\nu=1}^{n-1} \nu^k \sin \nu\alpha$. Это можно сделать последовательным дифференцированием тождества (3). Вычисления эти элементарные, но громоздкие и мы их опускаем.

3. Метод М. Рисса вычисления экстремальных операторов

М. Рисс [2] доказал, что при $t \in T_n$ имеет место равенство

$$t_n'(\theta) = \sum_{k=1}^{2n} t(\theta+\theta_k) r_k, \quad r_k = \frac{(-1)^{k+1}}{4n \sin^2 \theta_k/2}. \tag{16}$$

Если в (16) положить $t(\theta) = \sin n\theta$, $\theta = 0$, то получим тождество М. Рисса [2]

$$\frac{1}{4n} \sum_{k=1}^{2n} \frac{1}{\sin^2 \frac{\theta_k}{2}} = n. \tag{17}$$

Исходя из формулы (16) введем оператор

$$U_n^{\langle 1\rangle}(f, x) = \sum_{k=1}^{2n} f(x+\theta_k) r_k.$$

В силу (16) имеем, что $U_n^{\langle 1\rangle} \in \Omega_n^{(1)}$. Для любого $U_n \in \Omega_n^{(1)}$, $U_n(\sin n\theta, x) = n \cos nx$. Поэтому $\|U_n\| \geqq n$. С другой стороны, в силу (17), $\|U_n^{\langle 1\rangle}\| \leqq n$. Стало быть, $\|U_n^{\langle 1\rangle}\| = \varrho_n^{(1)} = n$. Итак, оператор (12) экстремальный в $\Omega_n^{(1)}$. Итак, дано другое доказательство теоремы 2. Докажем этим же методом теорему 3. Известно [6], что при $t \in T_n$

$$t(\theta) = \frac{1}{2n} \sum_{k=1}^{2n} (-1)^{k+1} t(\gamma_k) \sin n\theta \operatorname{ctg} \frac{\theta-\gamma_k}{2} + C \sin n\theta,$$

где $\gamma_k = \dfrac{(k-1)\pi}{n}$ и C — некоторая константа. Продифференцируем дважды это равенство и положим $\theta = 0$. Получим

$$t''(0) = \frac{1}{2n} \sum_{k-1}^{2n} (-1)^{k+1} t(\gamma_k) \left(\sin n\theta \operatorname{ctg} \frac{\theta-\gamma_k}{2} \right)''_{\theta=0}.$$

Применим это тождество к сдвинутому полиному $t(\theta+x)$. Тогда, после простых вычислений, получим тождество

$$t''(x) = -\frac{2n^2+1}{6} t(x) - \frac{1}{2} \sum_{k=2}^{2n} \frac{(-1)^{k+1}}{\sin^2 \gamma_k/2} t(x+\gamma_k). \tag{18}$$

Поэтому оператор (13) из $\Omega_n^{(2)}$. Положим в (18) $t(x)=\cos nx$ и затем положим $x=0$, тогда имеем, что

$$\sum_{k=2}^{2n} \operatorname{cosec}^2 \frac{(k-1)\pi}{2n} = \frac{1}{3}(4n^2-1). \tag{19}$$

Отсюда и из (13) следует, что $\|U_n^{\langle 2\rangle}\| \leqq n^2$. С другой стороны, из равенства $U_n(\sin n\theta, x)=-n^2 \sin nx$ следует, что при $U_n \in \Omega_n^{(2)}$, $\|U_n\| \geqq n^2$. Стало быть, выполняется (13′). Итак, теорема 3 доказана.

Этот метод вычисления экстремальных операторов естественно называть методом. М. Рисса [2], поскольку он идентичен тому методу с помощью которого М. Рисс доказал свое тождество (16). Покажем как с помощью этого метода можно доказать теорему 4. Согласно интерполяционной формуле М. Рисса [2]

$$t(\theta) = \alpha_n \cos n\theta + \sum_{j=1}^{2n} t(\theta_j) q_n(\theta-\theta_j), \quad q_n(x) = \frac{1}{2n} \sin nx \operatorname{ctg} \frac{x}{2}, \quad t \in T_n, \tag{20}$$

где $t(\theta)=\alpha_n \cos n\theta+\beta_n \sin n\theta+\ldots+\alpha_0$ и $\theta_j=\dfrac{(2j-1)\pi}{2n}$, $j=1, 2, \ldots, 2n$. Продифференцируем (20) три раза и используем метод М. Рисса, тогда получим для любого $t \in T_n$ тождество

$$t^{(3)}(x) = \frac{1}{2n} \sum_{k=1}^{2n} (-1)^k t(x+\theta_k) a_k, \quad a_k = \frac{3n^2+1}{2\sin^2 \frac{\theta_k}{2}} - \frac{3}{4\sin^4 \frac{\theta_k}{2}}. \tag{21}$$

Поэтому $U_n^{\langle 3\rangle}$ из $\Omega_n^{(3)}$. Заметим, что при $\alpha \in \left(0, \dfrac{\pi}{4}\right)$, $\sin \alpha > \dfrac{2\sqrt{2}}{\pi}\alpha$. Значит,

$$\operatorname{cosec}^2 \frac{\pi}{4n} \leqq 2n^2, \quad n = 1, 2, \ldots. \tag{22}$$

Стало быть,

$$a_k \geqq 0, \quad k = 1, 2, \ldots, 2n. \tag{23}$$

Положим в (21) $t(x)=\sin nx$, и затем положим $x=0$, тогда получим, что

$$n^3 = \frac{1}{2n} \sum_{k=1}^{2n} a_k. \tag{24}$$

Из (23) и (24) вытекает, что $\|U_n^{\langle 3\rangle}\| \leqq n^3$. С другой стороны, всегда при $U_n \in \Omega_n^{(3)}$, $\|U_n\| \geqq n^3$. Итак, теорема 4 доказана. Аналогичным образом доказывается теорема 5. Надо только учесть, что

$$3 \operatorname{cosec}^2 \frac{\gamma_k}{2} \leqq 2n^2+2, \quad k = 2, 3, \ldots, 2n. \tag{25}$$

Действительно, согласно (22)

$$\frac{1}{\sin^2 \frac{\pi}{4n}} \leqq 2n^2. \tag{22}$$

Положим $2n=m$, тогда из (22) следует, что $4\operatorname{cosec}^2\frac{2\pi}{2m}\leqq 2m^2$, а значит тем более выполняются неравенства (25). Итак, в силу неравенств (25), все квадратные скобки из (14) положительные. Поэтому применим метод М. Рисса. Доказательство теоремы 6 методом М. Рисса затруднено тем обстоятельством, что трудно обнаружить, что все b_k одного знака. Поэтому в этом случае лучше пользоваться теоремой 1 для нахождения $U_n^{\langle 5\rangle}$. Доказательство теоремы 7 методом М. Рисса легко проходит, ибо из неравенства $45x^2-15(2n^2+3)x+2(3n^4+10n^2+3)>0$, $-\infty<x<\infty$ следует, что в сумме из правой части (15) все квадратные скобки одного знака. Методом М. Рисса может быть также доказана

Теорема 9. *Оператор*

$$U_n^{\langle 8\rangle}(f,x)=\left(\frac{n^8}{9}+\frac{2n^6}{3}-\frac{7n^4}{15}+\frac{2n^2}{9}-\frac{1}{30}\right)f(x)+\frac{1}{2}\sum_{k=2}^{2n}f(x+\gamma_k)d_k,$$

$$d_k=\operatorname{cosec}^2\frac{\gamma_k}{2}\left[-315\operatorname{cosec}^2\frac{\gamma_k}{2}+210(n^2+2)\operatorname{cosec}^4\frac{\gamma_k}{2}-42(n^4+5n^2+3)\operatorname{cosec}^2\frac{\gamma_k}{2}+\right.$$

$$\left.+4(n^6+7n^4+1)\right],\quad \gamma_k=\frac{(k-1)\pi}{n},$$

из класса $\Omega_n^{(8)}$ *и является экстремальным в этом классе* $\varrho_n^{(8)}=\|U_n^{\langle 8\rangle}\|=n^8$.

4. Тригонометрические тождества

Операторы $U_n^{\langle k\rangle}(f,x)$, $3\leqq k\leqq 8$ могут быть, в частности, использованы для получения следующих тождеств.

Теорема 10. *Справедливы следующие тождества*

(26) $$\sum_{k=2}^{2n}\operatorname{cosec}^4\frac{(k-1)\pi}{2n}=\frac{1}{45}(16n^4+40n^2-11);$$

(27) $$\sum_{k=2}^{2n}\operatorname{cosec}^6\frac{(k-1)\pi}{2n}=\frac{1}{945}(128n^6+348n^4-58n^3+770n^2-68n-175);$$

(28) $$\sum_{k=2}^{2n}\operatorname{cosec}^8\frac{(k-1)\pi}{2n}=\frac{1}{14\,175}(768n^8+2680n^6-580n^5+5924n^4-$$
$$-1840n^3+10\,760n^2-1360n-2177);$$

(29) $$\sum_{k=1}^{2n}\operatorname{cosec}^4\frac{(2k-1)\pi}{4n}=\frac{8n^2}{3}(2n^2+1);$$

(30) $$\sum_{k=1}^{2n}\operatorname{cosec}^6\frac{(2k-1)\pi}{4n}=\frac{16n^2}{15}(8n^4+5n^2+2);$$

(31) $$\sum_{k=1}^{2n}\operatorname{cosec}^8\frac{(2k-1)\pi}{4n}=\frac{8n^2}{315}(544n^6+448n^4+196n^2+72).$$

Доказательство. Докажем тождества (26), (27), (28). Тождества (29), (30), (31) доказываются аналогично. Так как оператор $U_n^{\langle 4\rangle}$ из класса $\Omega_n^{(4)}$, то справедливо равенство $U_n^{\langle 4\rangle}(\cos n\theta, x)=n^4\cos nx$. Стало быть,

$$U_n^{\langle 4\rangle}(\cos n\theta, 0) = n^4 \tag{32}$$

Это равенство, с помощью формулы (14), записывается так

$$n^4 = \frac{1}{30}(n-1)(2n-1)(3n^2-3n-1)+\frac{n^3}{2}+(n^2+1)\sum_{k=2}^{2n}\operatorname{cosec}^2\frac{\gamma_k}{2}-$$

$$-\frac{3}{2}\sum_{k=2}^{2n}\operatorname{cosec}^4\frac{\gamma_k}{2},\quad \gamma_k=\frac{(k-1)\pi}{n}.$$

Отсюда и из (19) следует (26). Далее, таким же рассуждением, получим, что $U_n^{\langle 6\rangle}(\cos n\theta, 0)=-n^6$. Отсюда, с помощью формулы (15), следует равенство

$$-n^6 = -\left[\frac{1}{42}(n-1)(2n-1)\left(3(n-1)^4+6(n-1)^3-3(n-1)+1\right)+\frac{n^5}{2}\right]+ \tag{33}$$

$$+\frac{1}{2}\sum_{k=2}^{2n}\operatorname{cosec}^2\frac{\gamma_k}{2}\left[5n^2\left(3\operatorname{cosec}^2\frac{\gamma_k}{2}-2\right)-\frac{3}{2}\left(15\operatorname{cosec}^4\frac{\gamma_k}{2}-15\operatorname{cosec}^2\frac{\gamma_k}{2}+2\right)-3n^4\right].$$

Из (19), (26), (33) вытекает формула (27). Далее, имсем, что $U_n^{\langle 8\rangle}(\cos n\theta, 0)=n^8$. Поэтому из (19), (26), (27) вытекает (28).

5. Неравенства, относящиеся к узлам (m_0) и (m_2)

Прежде всего докажем следующую теорему

Теорема 11. a) *Писть* $\theta_k=\dfrac{(2k-1)\pi}{2n}$, $1\leqq k\leqq 2n$ *и пусть операгор*

$$A_n(f, x) \equiv A_n = \sum_{k=1}^{2n} f(x+\theta_k)(-1)^{k-1}p_k, \tag{34}$$

где $\{p_k\}_{k=1}^{2n}$ *— вещественные числа, из класса* $\Omega_n^{(2m-1)}$, $m=1, 2, \ldots$. *Для того чтобы оператор* (34) *был экстремальным в классе* $\Omega_n^{(2m-1)}$ *необходимо и достаточно, чтобы все отличные от нуля* p_k *были одного знака.*

b) *Пусть* $\gamma_k=\dfrac{(k-1)\pi}{n}$, $1\leqq k\leqq 2n$ *и пусть оператор*

$$B_n(f, x) \equiv B_n = \sum_{k=1}^{2n} f(x+\gamma_k)(-1)^{k-1}q_k, \tag{35}$$

где $\{q_k\}_{k=1}^{2n}$ *— вещественные числа, из класса* $\Omega_n^{(2m)}$, $m=1, 2, \ldots$. *Для того чтобы оператор* (35) *был экстремальным в классе* $\Omega_n^{(2m)}$ *необходимо и достаточно, чтобы все отличные от нуля* q_k *были одного знака.*

Доказательство. Докажем первую часть теоремы. Вторая часть теоремы доказывается аналогичным образом. Прежде всего заметим, что из формулы (34) легко следует, что

$$\|A_n\| = \sum_{k=1}^{2n} |p_k|. \tag{36}$$

Так как оператор (34) из класса $\Omega_n^{(2m-1)}$, то $A_n(\sin n\theta, x) = \sin^{(2m-1)} nx$, или

$$\sin^{(2m-1)} nx = \sum_{k=1}^{2n} \sin n(x+\theta_k)(-1)^{k-1} p_k.$$

Положим в этом равенстве $x=0$, тогда получим, что

$$\sum_{k=1}^{2n} p_k = (-1)^{m-1} n^{2m-1}. \tag{37}$$

Докажем сперва необходимость. Допустим, что оператор (34) экстремальный в классе $\Omega_n^{(2m-1)}$. Тогда согласно теореме 1 его норма $\|A_n\| = n^{2m-1}$. Отсюда и из (36), (37) следует, что

$$\sum_{k=1}^{2n} |p_k| = \left| \sum_{k=1}^{2n} p_k \right|. \tag{38}$$

Это равенство возможно лишь тогда, когда все отличные от нуля p_k одного знака. Итак, необходимость доказана. Докажем достаточность. Если все отличные от нуля p_k одного знака, то выполняется равенство (38), а тогда из (36), (37) следует, что $\|A_n\| = n^{2m-1}$. Отсюда и из теоремы 1 следует, что оператор (34) экстремальный в классе $\Omega_n^{(2m-1)}$.

Теорема 12. *Для узлов* (m_1) *и* (m_2) *выполняются неравенства*

(39)
$$n^{2m-1} + 2(-1)^{k-1} \sum_{\nu=1}^{n-1} \nu^{2m-1} \sin \nu \frac{(2k-1)\pi}{2n} \geqq 0, \quad k = 1, 2, \ldots, 2n, \quad m = 1, 2 \ldots,$$

(40)
$$n^{2m} + 2(-1)^{k-1} \sum_{\nu=1}^{n-1} \nu^{2m} \cos \nu \frac{(k-1)\pi}{n} \geqq 0, \quad k = 1, 2, \ldots, 2n, \quad m = 1, 2, \ldots.$$

Доказательство. Докажем неравенство (39). (40) доказывается аналогичным образом. Согласно теореме 1 оператор (6) из класса $\Omega_n^{(2m-1)}$ и является в этом классе экстремальным. Очевидно, что этот оператор можно записать в виде

$$\begin{cases} U_n^{\langle 2m-1\rangle}(f, x) = \dfrac{1}{2n} \displaystyle\sum_{k=1}^{2n} f(x+\theta_k) p_k^{(m)}, \\ p_k^{(m)} = (-1)^m \left(n^{2m-1} + 2(-1)^{k-1} \displaystyle\sum_{\nu=1}^{n-1} \nu^{2m-1} \sin \frac{\pi\nu(2k-1)}{2n} \right). \end{cases} \tag{41}$$

Стало быть, к оператору (41) применима теорема 11. Таким образом, числа $\{p_k^{(m)}\}_{k=1}^{2n}$ одного знака. Очевидно, что

$$p_1^{(m)} = (-1)^m \left(n^{2m-1} + 2 \sum_{\nu=1}^{n-1} \nu^{2m-1} \sin \frac{\pi \nu}{2n} \right).$$

Поэтому $\operatorname{sign} p_k^{(m)} = \operatorname{sign} p_1^{(m)} = \operatorname{sign} (-1)^m$. Стало быть, $\operatorname{sign} p_k^{(n)} = \operatorname{sign} (-1)^m$, $1 \leqq k \leqq 2n$. Допустим, что m — четное число, тогда $p_k^{(m)} \geqq 0$, $1 \leqq k \leqq 2n$. Итак, при четном m неравенства (39) доказаны. Если m — нечетное, то из наших рассуждений следует, что $p_k^{(m)} \leqq 0$, $k = 1, 2, \ldots, 2n$, что опять приводит к неравенствам (39).

6. Добавление

1. Теорема 1 может быть сформулирована в следующей эквивалентной форме. Операторы

$$U_n^{\langle 2m-1 \rangle}(f, x) = -\sum_{k=1}^{2n} f(x+\theta_k) q_n^{(2m-1)}(\theta_k); \quad U_n^{\langle 2m \rangle}(f, x) = -\sum_{k=1}^{2n} f(x+\gamma_k) q_n^{(2m)}(\gamma_k),$$

$$q_n(x) = \frac{1}{2n} \sin nx \operatorname{ctg} \frac{x}{2},$$

соотвенственно из классов $\Omega_n^{(2m-1)}$, $\Omega_n^{(2m)}$, $m = 1, 2, \ldots$, и являются экстремальными в этих классах.

Это следует из теоремы 1 и равенства

$$\sum_{\nu=1}^{n-1} (\cos \nu t)^{(k)} = \frac{1}{2} \left(\sin nt \operatorname{ctg} \frac{t}{2} \right)^{(k)} - \frac{1}{2} (\cos nt)^{(k)}, \quad k \geqq 1.$$

2. Формула А. Х. Турецкого [6], которой пользовались в § 3, допускает следующее уточнение. Пусть $t(\theta) = a_n \cos n\theta + b_n \sin n\theta + \ldots + a_0$, тогда

$$t(\theta) = \sum_{k=1}^{2n} t(\gamma_k) q_n(\theta - \gamma_k) + b_n \sin n\theta. \tag{42}$$

Для доказательства (42) пользуемся известным методом (см. С. М. Никольский, Приближение функций многих переменных и теоремы вложения, «Наука», М. 1969). Положим $t_1(\theta) = \sum_{k=1}^{2n} t(\gamma_k) q_n(\theta + \gamma_k)$, тогда

$$t(\theta) - t_1(\theta) = C \sin n\theta, \tag{43}$$

где C — некоторая константа. Заметим, что $\int_0^{2\pi} q_n(\theta - \gamma_k) \sin n\theta \, d\theta = 0$. Поэтому из (43) следует, что $C = b_n$.

Таким же образом доказывается, что если $a_n^2+b_n^2>0$, то

$$t(\theta)=\sum_{k=1}^{2n} t\left(\frac{k-1}{n}\pi-\frac{\varphi_n}{n}\right) q_n\left(\theta-\frac{k-1}{n}\pi-\frac{\varphi_n}{n}\right)+\sqrt{a_n^2+b_n^2}\sin(n\theta+\varphi_n),$$

где $\operatorname{tg}\varphi_n=\dfrac{a_n}{b_n}$.

3. Из теорем 1 и 11 непосредственно следует, что полином $q_n(x)=\dfrac{\sin nx\operatorname{ctg}\dfrac{x}{2}}{2n}$ обладает следующими свойствами

$$\operatorname{sign} q^{(2m-1)}(\theta_{k+1})=-\operatorname{sign} q^{(2m-1)}(\theta_k);\quad \operatorname{sign} q^{(2m)}(\gamma_{k+1})=-\operatorname{sign} q^{(2m)}(\gamma_k),$$

$$k=1,2,\ldots,(2n-1),\quad m=1,2,\ldots$$

Литература

[1] Д. Л. Берман, Экстремалтные задачи теории полиномиалтных операторов, *Матем. сборник*, **60 (102)** (1963).

[2] M. Riesz, Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome, *Jahresber. d. Deutsch. Math. Verein.*, **23** (1914), 354—368.

[3] Д. Л. Берман, Экстремальные задачи теории полиномиальных операций, *Доклады АН СССР*, **170** (1966).

[4] Д. Л. Берман, Обобщение тождества М. Рисса и некоторые применения обобщенного тождества, *Известия выс. учебных заведений, Математика*, **2** (1966), 26—31.

[5] Д. Л. Берман, Ассоциированные по Ф. Риссу тригонометрические полиномы и их роль в теории линейных полиномиальных операций, *Учение записки Казанского университета*, **127** (1967), 17—34.

[6] А. Х. Турецкий, *Теория интерполирования в задачах* (Минск, 1968).

(Поступило 15. 12. 1981.)

ЛЕНИНГРАД, 19 223в
БАССЕЙНАЯ 68/31, КВ. 90
СССР

Acta Math. Hung.
42 (1—2) (1983), 165—169.

SEMIGROUPS WITH PRIMARY IDEALS OF PRIME POWER

R. K. JAIN (Amritsar)

Introduction

Commutative rings in which every primary ideal is prime power were studied by Butts and Gilmer in [1]. They called such rings with property (α). On the similar lines we define a commutative semigroup S to satisfy property (α) if every primary ideal in S is prime power. The fact that for a prime ideal P of a commutative ring R, R/P is a domain, played a vital role while studying the rings satisfying (α). But we are deprived of this property in semigroups. Thus we mainly study here the cancellative semigroups satisfying (α). The main result proved here is the following: If S is a cancellative monoid satisfying (α) in which the ascending chain condition for prime ideals holds then S is a Prüfer monoid. In particular a noetherian cancellative monoid is a Dedekind monoid if and only if it satisfies (α).

Throughout, semigroup means a commutative semigroup. If a is an element of a semigroup S then (a) denotes the smallest ideal containing a, namely $(a)=\{a\}\cup Sa$. If A is an ideal of S then $S/A=\bar{S}=[\bar{s}: s\in S]$ where $\bar{s}=\bar{t}$ if and only if either $s=t$ or s and t are in A. For any $x\in A$, $\bar{x}$ is denoted by o or $\bar{o}$. The mapping $\varphi: S\to\bar{S}$ with $\varphi(s)=\bar{s}$ for every $s\in S$ is an epimorphism.

If P is a minimal prime divisor of an ideal I then the intersection of all P-primary ideals containing I is called the isolated P-primary component of I. A set T such that for each x, y in T, xy is in T is called a multiplicative set. If each element of T is cancellable then we define $S_T=\{s/t: s\in S, t\in T\}$. S_T becomes a monoid with natural multiplication. The mapping $\varphi: S\to S_T$ defined by $\varphi(s)=st/t$ is a monomorphism. For any ideal A of S, A^e denotes the ideal $A_T=\{a/t: a\in A, t\in T\}$ and for any ideal B of S_T, B^c denotes the ideal $\varphi^{-1}(B)$ of S. If P is a proper prime ideal of S and $T=S-P$ has only cancellable elements then $A^e=A_T$ is written as A_P. Moreover for any positive integer n, $P^{(n)}$ denotes $(P^n)^{ec}$. For any ideal $A\subseteq P$, A^{ec} is P-primary.

If S is a monoid then the set of all non-unit elements form an ideal and this is the unique maximal ideal of S. S is said to be a Prüfer monoid if every two principal ideals are comparable. Equivalently, every finitely generated ideal is invertible. S is said to be a Dedekind monoid if every ideal is invertible.

For a semigroup S without identity, S^1 will denote the semigroup S with identity adjoined. If S has a zero element o such that $s\cdot o=o$ for every $s\in S$ and all elements of S, except o, are cancellable then S is said to be an almost cancellative semigroup. If $P_1<P_2$ are prime ideals and there is no prime ideal P' such that $P_1<P'<P_2$ then we say that P_1 lies directly below P_2.

We start by discussing the homomorphic images of semigroups satisfying (α).

LEMMA 1. *If S is a cancellative semigroup satisfying (α) then every homomorphic*

image of S satisfies (α). *In particular, for any ideal I of S, S/I is a semigroup satisfying* (α).

PROOF. Straightforward.

LEMMA 2. *If S is a cancellative semigroup satisfying* (α) *then for any proper prime ideal P, S_P satisfies* (α).

PROOF. Let Q be a prime ideal of S_P. Let A be any Q-primary ideal. Then A^c is Q^c-primary ideal in S and hence $A^c=(Q^c)^m$ for some integer m. Now $A=A^{ce}=$ $=((Q^c)^m)^e=(Q^{ce})^m=Q^m$.

REMARK 3. It is obvious that if S has a zero element and is almost cancellative then Lemmas 1 and 2 still hold.

LEMMA 4. *Let S be a cancellative semigroup satisfying* (α). *Let $P_0<P$ be proper prime ideals of S such that P is a minimal prime divisor of $P_0\cup(x)$ for some $x\in P-P_0$. Then the powers of P properly descend, $P_0\subseteqq\bigcap_1^\infty P^n$ and $\bigcap_1^\infty P^n$ is the intersection of all P-primary ideals of S.*

PROOF. Clearly P is a minimal prime divisor of $I_m=P_0\cup(x^m)$ for every positive integer m. Let $Q_m=I_m^{ec}$ with respect to the prime ideal P. Thus each Q_m is the isolated P-primary component of I_m. We claim that $Q_1>Q_2>Q_3>\ldots$. Clearly each Q_m contains Q_{m+1} but $x^m\in Q_m-Q_{m+1}$. For, if $x^m\in Q_{m+1}$ then $tx^m/t\in I_{m+1}^e$ for each $t\in T=S-P$. Let $tx^m/t=a/v$ where $a\in I_{m+1}$ and $v\in T$. Since S is cancellative, $vx^m\in P_0$ or (x^{m+1}). Then $vx^m=x^{m+1}$ or $sx^m=x^{m+1}$ for some $s\in S$, since v and x^m are not in P_0. Thus $v=x$ or $sx\in P$, which is impossible. Each Q_i is, thus, a power of P. Suppose $Q_i=P^{n_i}$, $n_1<n_2<\ldots$. Thus powers of P properly descend. Let A denote the intersection of all P-primary ideals. Then $A\subseteqq\bigcap_1^\infty P^{n_i}$ since each $P^{n_i}=Q_i$ is P-primary. Also $P_0\subseteqq Q_i$ for each i and hence $P_0\subseteqq\bigcap_1^\infty P^{n_i}\subseteqq\bigcap_1^\infty P^n$. As there are infinitely many different P-primary ideals and they are all powers of P, their intersection equals $\bigcap_1^\infty P^n$.

REMARK 5. Obviously Lemma 4 is valid even if S is almost cancellative instead of cancellative.

THEOREM 6. *Let S be a cancellative semigroup satisfying* (α) *and $P_0<P$ be proper prime ideals of S. Then $P_0\subseteqq\bigcap_1^\infty P^n$.*

PROOF. Let $x\in P-P_0$ be any element. We can find a minimal prime divisor P_1 of $P_0\cup(x)$ such that $P_1\subseteqq P$. By Lemma 4, $P_0\subseteqq\bigcap_1^\infty P_1^n$ and hence $P_0\subseteqq\bigcap_1^\infty P^n$.

COROLLARY 7. *Let S be a cancellative semigroup satisfying* (α) *and P be any proper prime ideal of S. Then $P^2=P$ if and only if P is the union of an ascending chain of prime ideals properly contained in P.*

PROOF. Suppose $P=\bigcup_1^\infty P_i$ where each prime ideal $P_i<P$ and $P_1\subseteq P_2\subseteq P_3\ldots$. By Theorem 6, $P_i\subseteq\bigcup_1^\infty P^n$ for each i. Thus $P=\bigcup_1^\infty P_i\subseteq\bigcap_1^\infty P^n$ implying $P=P^2$.

Conversely suppose P is not a union of an ascending chain of prime ideals properly contained in P. We may assume that P is not a minimal prime ideal, for, otherwise we can argue with S^0, the semigroup obtained by adjoining zero element to S, in which (o) is a prime ideal properly contained in P. Applying Zorn's lemma, we can find a prime ideal P_1 maximal with the property $P_1<P$ i.e. P_1 lies directly below P. If $x\in P-P_1$ then P is a minimal prime divisor of $P_1\cup(x)$ and Lemma 4 implies that $P^2<P$.

THEOREM 8. *Let S be a cancellative semigroup satisfying (α) and $P_0<P$ be proper prime ideals of S. If $P\neq P^2$ then $\bigcap_1^\infty P=P_1$ (say) is the unique prime ideal lying directly below P and powers of P are primary.*

PROOF. Let Σ be the set of all prime ideals properly contained in P. Then P_0 is in Σ and by Corollary 7, every ascending chain of prime ideals properly contained in P has a least upper bound from Σ. Zorn's lemma guarantees the existence of a maximal element P_1 in Σ. Thus P_1 lies directly below P. In the quotient semigroup S_p, denote A^e the extension of any ideal A of S. Let $\bar{S}=S_p/P_1^e$ and $\bar{P}=P^e/P_1^e$. By Lemmas 1 and 2, $\bar{S}$ satisfies (α). Since P_1 lies directly below P, P_1^e lies directly below P^e in S_p and so $\bar{Q}=\bar{P}_1=P_1^e/P_1^e$ lies directly below $\bar{P}$ which is the unique maximal ideal of $\bar{S}$. Moreover S cancellative implies that S_p is cancellative and hence $\bar{S}-(\bar{Q})$ is a cancellative semigroup. Since $\bar{P}$ is the unique non-zero prime ideal of the monoid $\bar{S}$, every non-zero ideal of $\bar{S}$ is $\bar{P}$-primary and thus is a power of $\bar{P}$. It immediately yields that $\bar{S}-(\bar{o})$ is a Dedekind monoid. Thus $\bigcap_1^\infty(\bar{P}-(\bar{o}))^n=\varnothing$ and hence $\bigcap_1^\infty\bar{P}^n=(\bar{o})$ in $\bar{S}$. We get $P_1^e=\bigcap_1^\infty(P^n)^e$ and $P_1=P_1^{ec}=\bigcap_1^\infty(P^n)^{ec}=\bigcap_1^\infty P^{(n)}$. Now $P_1<P$ implies that P^n is not contained in P_1. Thus $P^{(n)}$ is not contained in P_1 and hence $P^{(n)}\neq P^{(n+1)}$ for any n. We claim that $P^{(n)}=P^n$ for every n. It being true for $n=1$, we assume that $P^k=P^{(k)}$. Then $P^k=P^{(k)}>P^{(k+1)}\supseteq P^{k+1}$ and we get $P^{(k+1)}=P^{k+1}$ since $P^{(k+1)}$ being P-primary is a power of P. Thus for each n, $P^n=P^{(n)}$ is P-primary and $P_1=\bigcap_1^\infty P^{(n)}=\bigcap_1^\infty P^n$. Hence $\bigcap_1^\infty P^n$ is a prime ideal and is the unique prime ideal lying directly below P.

COROLLARY 9. *If S is a cancellative semigroup satisfying (α) and P any proper prime ideal of S then $\bigcap_1^\infty P^n$ is a prime ideal. If $P\neq P^2$ then $\bigcap_1^\infty P^n$ is the unique prime ideal lying directly below P. Moreover prime power ideals are primary.*

PROOF. Denote S^0 the semigroup obtained by adjoining zero to S. Then S^0 is almost cancellative. In S^0, (o) is a prime ideal. Taking $P_0=(o)$ in Theorem 8, we get the conclusion of the Corollary.

THEOREM 10. *If S is a cancellative semigroup satisfying* (α) *then for any proper prime ideal P, $P \subseteqq \bigcap_1^\infty S^n$.*

PROOF. Obvious, if S has an identity element. In the opposite case it directly follows from Corollary 9 applied to S^1.

THEOREM 11. *Let S be a cancellative monoid with maximal ideal M. If S satisfies* (α) *then the following are equivalent:*

(i) *S has no idempotent prime ideal,*

(ii) *S satisfies ascending chain condition on prime ideals.*

Moreover S is a Prüfer monoid.

PROOF. (i)$\rightarrow$(ii). It follows immediately from Corollary 7.

(ii)$\rightarrow$(i). If there is a prime ideal P with $P=P^2$ then in S^0, the semigroup S with zero adjoined, $(o)<P$ and Corollary 7 implies that P is the union of an infinite ascending chain of prime ideals properly contained in P, contradicting (ii).

To prove that S is a Prüfer monoid, we follow the following two steps:

STEP I. *Any two prime ideals of S are comparable.*

STEP II. *Any two principal ideals of S are comparable.*

PROOF OF STEP I. Let P_1, P_2 be incomparable prime ideals. Then $P=P_1 \cup P_2$ is prime, too, and $Q=P_1 \cup P_2^2$ is P-primary. Now $P^2 \subseteqq Q \subseteqq P$, but $Q \neq P$, because otherwise $P_1 \supseteqq P_2 - P_2^2 \neq \varnothing$ whence $P_1 \supseteqq P_2$ which contradicts the assumption. On the other hand, if $Q=P^2=P_1^2 \cup P_1P_2 \cup P_2^2$ then $P_1 - P_1^2 \subseteqq Q - P_1^2 \subseteqq P_1P_2 \cup P_2^2 \subseteqq P_2$ and we obtain the same contradiction.

PROOF OF STEP II. We shall use the notation $x \| y$ to say that (x), (y) are incomparable. If $x \| y$ then, by cancellativity, $ax \| ay$ for every $a \in S$. Thus, the set $P = \{x : x \| y \text{ for some } y\}$, if non-empty, is an ideal; we are going to show that it is prime.

As $(1)=S$ is comparable with every ideal, $P \neq S$. Let $a, b \notin P$, $r \in S$ arbitrary. We have to show that $ab \nparallel r$. By assumption, $a \nparallel r$. If $a \in (r)$, then $ab \in (r)$. If $r=ax$, then $b \nparallel x$ and since $ax=r \notin (ab)$, $x \notin (b)$ and so $b=xy$ for some y. Then $ab=axy=ry \in (r)$. This proves the assertion.

Let $x \in P-P^2$ and put $A=\{y \in P-P^2 : y \nparallel x\}$, $B=\{y \in P-P^2 : y \| x\}$. If $A=P-P^2$, then $z \| x$ implies $z \in P^2$. Let $z=yz'$, $y \in P=P^2$, $z' \in P$. As $y \nparallel x$, we have $x=ya$ for some $a \in S$ ($y=xa$ would imply $z=xaz'$). Here $a \notin P$ since $x \notin P^2$. Therefore $a \nparallel z'$ and $z'=az''$ whence $z=yaz''=xz''$, contrary to the assumption. The contradiction shows that $B \neq \varnothing$.

Next we show that $y_1 \nparallel y_2$ holds for any two elements $y_1, y_2 \in A$ and $y \| z$ for $y \in A$, $z \in B$. If $(y_1) \subseteqq (x) \subseteqq (y_2)$ or $(y_2) \subseteqq (x) \subseteqq (y_1)$ then the comparability is obvious. If $x=y_1a=y_2b$ then $a, b \notin P$ so that $a \nparallel b$, say $a=bc$. Then $y_1bc=y_2b$ and by cancellativity $y_1c=y_2$. If, on the other hand, $y_1=xa$, $y_2=xb$ then, as before, $a \nparallel b$ and $a=bc$ gives $y_1=xbc=y_2c$. Thus, $x \nparallel y_1$, $x \nparallel y_2$ imply $y_1 \nparallel y_2$ for any $x, y_1, y_2 \in P-P^2$, i.e. $\nparallel$ is an equivalence relation on $P-P^2$. Hence, $y \| z$ for $y \in A$, $z \in B$.

Now consider the ideal $Q=(A)\cup(B)^2$. Obviously, $Q=P-B<P$ and $Q=$ $=P^2\cup A>P^2$ which shows that Q is not a power of P. On the other hand, Q is P-primary. Indeed $a\notin Q$, $b\notin P$ imply that either $a\notin P$ and so $ab\notin P$ whence $a\notin Q$, or $a\in P-Q=B$, but then $ab\in P-P^2$ and $ab\nmid a$ which imply $ab\in B$, $ab\notin Q$. Hence Q is primary which is a contradiction to (α). This completes the proof.

Acknowledgement

I am indebted to the referee, Professor G. Pollák for various comments, particularly the proof of Theorem 11, which has considerably shortened my original proof.

References

[1] H. S. Butts and R. W. Gilmer, Primary ideals and prime power ideals, *Can. J. Maths.*, **18** (1966), 1183—1195.
[2] M. P. Dorofeeva, Hereditary and semi-hereditary monoids, *Semigroup Forum*, **4** (1972), 301—311.
[3] M. P. Dorofeeva, V. L. Mannepalli and M. Satyanarayana, Prüfer and Dedekind monoids, *Semigroup Forum*, **9** (1975), 294—309.
[4] R. W. Gilmer and J. Ohm, Primary ideals and valuation ideals, *Trans. Amer. Math. Soc.*, **117** (1965), 237—250.
[5] R. K. Jain, Structure of cancellative semigroups with semiprimary ideals primary, *AJSE*, Dhahran, to appear.
[6] R. K. Jain, Structure of cancellative semigroups with semiprimary ideals primer power, *AJSE*, Dhahran, to appear.
[7] M. D. Larsen and P. J. McCarthy, *Multiplicative theory of ideals*, Academic Press (New York and London, 1971).
[8] V. L. Mannepalli and M. Satyanarayana, Monoids of Dedekind type, *Semigroup Forum*, **9** (1974), 19—27.
[9] O. Zariski and P. Samuel, *Commutative Algebra*, Vol. I, Van Nostrand (Princeton — New York, 1958).

(Received December 16, 1981; revised April 30, 1982)

DEPARTMENT OF MATHEMATICS
GURU NANAK DEV UNIVERSITY
AMRITSAR-143 005
INDIA

Acta Math. Hung.
42 (1—2) (1983), 171—175.

ON THE DISTRIBUTION OF THE EIGENVALUES OF AN ORTHONORMAL SYSTEM, CONSISTING OF EIGENFUNCTIONS OF HIGHER ORDER OF A LINEAR DIFFERENTIAL OPERATOR

V. KOMORNIK (Budapest)

Let $G=[a, b]$ be a compact interval and consider the formal differential operator

$$Lu = u^{(n)} + p_2 u^{(n-2)} + \ldots + p_{n-1} u' + p_n u, \tag{1}$$

where $p_2, \ldots, p_n \in L^1(G)$ are arbitrary complex functions. Let λ be a complex number. The function $u_{-1}: G \to \mathbf{C}$, $u_{-1} \equiv 0$ is called an eigenfunction of order -1 of the operator L with the eigenvalue λ. A function $u_i: G \to \mathbf{C}$, $u_i \not\equiv 0$ $(i=0, 1, \ldots)$, as usual, is said to be an eigenfunction of order i of the operator L with the eigenvalue λ if u_i and its first $n-1$ derivatives are absolutely continuous on G, and if for almost all $x \in G$ the equation

$$(Lu_i)(x) = \lambda u_i(x) + u_{i-1}(x) \tag{2}$$

holds, where u_{i-1} is an eigenfunction of order $i-1$ with the eigenvalue λ.

Let now $(u_i) \subset L^2(G)$ be an arbitrary orthonormal system, consisting of eigenfunctions of order $\leqq m$ of the operator L, where m is a fixed natural number. Denote by λ_i the eigenvalue of u_i. Then the following assertion is true (we emphasize the fact that the functions u_i need not satisfy any boundary conditions):

THEOREM. *Let* $\lambda_i = \varrho_i^n$, $0 \leqq \arg \varrho_i < 2\pi/n$. *There exists a constant* $\mathcal{M}$, *depending on* n, m, $|b-a|$, $\|p_2\|_1, \ldots, \|p_n\|_1$ *such that for all* $\varrho \in \mathbf{C}$,

$$\sum_{|\varrho_i - \varrho| \leqq 1} 1 < \mathcal{M}(1 + |\operatorname{Re} \varrho|) \quad \text{if } n = 2, \tag{3}$$

$$\sum_{|\varrho_i - \varrho| \leqq 1} 1 < \mathcal{M}(1 + |\varrho|) \quad \text{if } n > 2. \tag{3*}$$

Consequently, $|\lambda_i| \to \infty$ $(i \to \infty)$.

REMARK. The first results of this type were proved in [1], [2] for the case $n=2$, $m=0$ when $p_2 \in L^q(G)$ for some $q>1$. That proof did not work for the case $n>2$. Later on, the case $m=0$ of the present theorem was proved in [4], using a general formula and some connections between the norms of the eigenfunctions and their derivatives, valid for the operator (1). The case $n=4$, $m=0$, $p_2=p_3=0$ was proved also by A. Sövegjártó [7]. His proof is a development of the ideas of [8], i.e. he uses a formula of Titchmarsh type. However, for large n, this formula becomes complicated.

Our proof follows the line of our previous paper [4] but we shall need some properties of the eigenfunctions of higher order, proved in [3], [5] and [6].

The proof of the theorem will be based on the following results, being special cases of Theorems 1 and 2 in [5] (for (5) see [3] or [6]):

PROPOSITION 1. *There exist entire functions h_r and functions f_k holomorph in the region $|z|<\pi$ $(1\leqq k\leqq N\equiv n(m+1),\ 0\leqq r\leqq m)$ such that if v_m is an eigenfunction of order $\leqq m$ of the operator* (1) *with the eigenvalue $\lambda=\varrho^n$, $0\leqq\arg\varrho<2\pi/n$, and the functions v_j $(j<m)$ are defined by* (2), *then we have for all x, $x+Nt\in G$,*

$$v_m(x)=\sum_{k=1}^{N}f_k(\varrho t)v_m(x+kt)+ \tag{4}$$

$$+\sum_{k=1}^{N}\sum_{r=0}^{m}\sum_{s=2}^{n}f_k(\varrho t)\int_x^{x+kt}(x+kt-\tau)^{n(r+1)-1}h_r\big(\varrho(x+kt-\tau)\big)p_s(\tau)v_{m-r}^{(n-s)}(\tau)\,d\tau.$$

PROPOSITION 2. *There exists a constant D depending on $n, m, |b-a|, \|p_2\|_1, \ldots$ $\ldots, \|p_n\|_1$ such that given any eigenfunction v_m of order $\leqq m$ of the operator* (1) *with the eigenvalue $\lambda=\varrho^n$, we have for all $0\leqq j\leqq m$, $0\leqq i<n$,*

$$\|v_{m-j}^{(i)}\|_\infty\leqq D(1+|\operatorname{Re}\varrho|)^{j+\frac{1}{p}}(1+|\varrho|)^{j+i}\|v_m\|_p \quad\text{if}\quad n=2,\quad 1\leqq p\leqq\infty, \tag{5}$$

$$\|v_{m-j}^{(i)}\|_\infty\leqq D(1+|\varrho|)^{nj+i+\frac{1}{p}}\|v_m\|_p \quad\text{if}\quad n>2,\quad 1\leqq p\leqq\infty. \tag{5*}$$

First we prove the theorem for the case $n=2$. Fix a constant C with the properties

$$|f_k(x)|\leqq C \quad\text{if}\quad |\operatorname{Re}z|\leqq 1, \tag{6}$$

$$|f_k(z_1)-f_k(z_2)|\leqq C|z_1-z_2| \quad\text{if}\quad |\operatorname{Re}z_i|\leqq 1,\quad i=1,2, \tag{7}$$

$$|h_r(z)|\leqq C\min(1,|z|^{-r-1}) \quad\text{if}\quad |\operatorname{Re}z|\leqq N. \tag{8}$$

The possibility of this choice is shown in [6].

Let now $\varrho\in\mathbf{C}$ be fixed arbitrarily and define

$$\mathcal{K}\equiv 4\cdot 2^{m+1}N^2C^2D^2\sqrt{1+|\operatorname{Re}\varrho|}\sqrt{b-a}, \tag{9}$$

$$R\equiv\min\left(\frac{b-a}{2N},\ \frac{1}{N(2+|\operatorname{Re}\varrho|)},\ \frac{1}{2NCD\sqrt{b-a}\sqrt{1+|\operatorname{Re}\varrho|}},\ \frac{1}{\mathcal{K}\|p_2\|_1}\right). \tag{10}$$

Choose an arbitrary u_i such that

$$|\varrho_i-\varrho|\leqq 1, \tag{11}$$

and introduce the notations $u_{i,m}\equiv u_i$, $u_{i,j-1}\equiv Lu_{i,j}-\lambda u_{i,j}$ $(j\leqq m)$. Then we have by (4) and (10) for all $a\leqq x\leqq\frac{a+b}{2}$, $0\leqq t\leqq R$,

$$u_i(x)=\sum_{k=1}^{N}f_k(\varrho_i t)u_i(x+kt)+\sum_{k=1}^{N}\sum_{r=0}^{m}f_k(\varrho_i t)\int_x^{x+kt}(x+kt-\tau)^{2r+1}\times$$

$$\times h_r\big(\varrho_i(x+kt-\tau)\big)p_2(\tau)u_{i,m-r}(\tau)\,d\tau,$$

whence, integrating by t from 0 to R, we obtain

$$(12)\quad Ru_i(x) = \sum_{k=1}^{N} \int_0^R f_k(\varrho t) u_i(x+kt)\,dt + \sum_{k=1}^{N} \int_0^R [f_k(\varrho_i t) - f_k(\varrho t)] u_i(x+kt)\,dt +$$

$$+ \sum_{k=1}^{N} \sum_{r=0}^{m} \int_0^R f_k(\varrho_i t) \int_x^{x+kt} (x+kt-\tau)^{2r+1} h_r\big(\varrho_i(x+kt-\tau)\big) p_2(\tau) u_{i,m-r}(\tau)\,d\tau\,dt \equiv$$

$$\equiv I_1 + I_2 + I_3.$$

Introduce the function $w\colon G \to \mathbf{C}$,

$$w(t) \equiv \sum_{k=1}^{N} k^{-1} \overline{f_k\big(\varrho k^{-1}(t-x)\big)}\, \varkappa_{\{x \leqq t \leqq x+kR\}}(t).$$

From (6) and (10) it follows that

$$(13)\qquad w \in L^2(G), \quad \|w\|_2 \leqq NC\sqrt{R}.$$

Moreover,

$$(14)\qquad I_1 = \langle u_i, w\rangle \equiv \int_G u_i(t)\overline{w(t)}\,dt.$$

Now we estimate the expressions I_2 and I_3. By (7), (10) and (11)

$$(15)\qquad |I_2| \leqq N \int_0^R Ct\,dt\,\|u_i\|_\infty = \frac{NC}{2} R^2 \|u_i\|_\infty \leqq (1/4) R \|u_i\|_2 \frac{1}{\sqrt{b-a}},$$

furthermore, by (6), (8), (5), (11), (9) and (10)

(16)

$$|I_3| \leqq N \sum_{r=0}^{m} RC^2 (NR)^r \min\,(NR, |\varrho_i|^{-1})^{r+1} \|p_2\|_1 D (1+|\operatorname{Re}\varrho_i|)^r (1+|\varrho_i|)^r \|u_i\|_\infty \leqq$$

$$\leqq (NR)^2 C^2 D \|p_2\|_1 \|u_i\|_\infty \sum_{r=0}^{m} [NR(2+|\operatorname{Re}\varrho|)]^r [\min\,(NR, |\varrho_i|^{-1})\,(1+|\varrho_i|)]^r \leqq$$

$$\leqq (NR)^2 C^2 D \|p_2\|_1 \|u_i\|_\infty \sum_{r=0}^{m} 2^r \leqq [2^{m+1} N^2 C^2 D \|p_2\|_1] R^2 \|u_i\|_\infty \leqq$$

$$\leqq (1/4) R \|u_i\|_2 \frac{1}{\sqrt{b-a}}.$$

From (12), (14), (15) and (16) we conclude

$$R|u_i(x)| \leqq |\langle u_i, w\rangle| + (1/2) R \|u_i\|_2 \frac{1}{\sqrt{b-a}},$$

whence

$$(17)\qquad R^2 |u_i(x)|^2 \leqq 2|\langle u_i, w\rangle|^2 + (1/2) R^2 \|u_i\|_2^2 \frac{1}{b-a}.$$

Let now I be an arbitrary finite index set such that $|\varrho_i - \varrho| \leqq 1$ for all $i \in I$. Then by (17)

$$R^2 \sum_{i\in I} |u_i(x)|^2 \leqq 2 \sum_{i\in I} |\langle u_i, w\rangle|^2 + (1/2) R^2 \sum_{i\in I} \|u_i\|_2^2 \frac{1}{b-a}.$$

Applying the Bessel inequality and (13), we obtain

$$R^2 \sum_{i\in I} |u_i(x)|^2 \leqq 2N^2C^2R + (1/2)R^2 \sum_{i\in I} \|u_i\|_2^2 \frac{1}{b-a}.$$

This is true for all $a \leqq x \leqq \frac{a+b}{2}$, and by symmetry, also for all $\frac{a+b}{2} \leqq x \leqq b$. Integrating both sides on G, we obtain in view of $\|u_i\|_3 = 1$

$$|I| \leqq 4N^2C^2R^{-1}(b-a).$$

Here the left side expression denotes the number of elements of I, while the right side number does not depend on I. Hence (3) follows by (9), (10) and the definition of I.

The case $n>2$ of the theorem can be proved similarly by repeating word for word the above argument and replacing (5), (6), (7), (8), (9), (10), (12) and (16) by the following ones (cf. [4]):

(5*) is formulated in Proposition 2;

(6*) $$|f_k(z)| \leqq C \quad \text{if} \quad |z| \leqq 1;$$

(7*) $$|f_k(z_1) - f_k(z_2)| \leqq C|z_1 - z_2| \quad \text{if} \quad |z_1| \leqq 1, \quad |z_2| \leqq 1;$$

(8*) $$|h_r(z)| \leqq C \quad \text{if} \quad |z| \leqq N;$$

(9*) $$K \equiv 4N^3C^2D^2\sqrt{1+|\varrho|}\sqrt{b-a},$$

(10*) $$R \equiv N^{-1} \min\left(\frac{b-a}{2}, \frac{1}{2+|\varrho|}, \frac{1}{2CD\sqrt{b-a}\sqrt{1+|\varrho|}}, \frac{1}{K\|p_2\|_1}, \sqrt{\frac{1}{K\|p_3\|_1}}, \ldots, \sqrt[n-1]{\frac{1}{K\|p_n\|_1}}\right);$$

(12*) $$Ru_i(x) = \sum_{k=1}^{N} \int_0^R f_k(\varrho t) u_i(x+kt)\, dt + \sum_{k=1}^{N} \int_0^R [f_k(\varrho_i t) - f_k(\varrho t)] u_i(x+kt)\, dt +$$

$$+ \sum_{k=1}^{N} \sum_{r=0}^{m} \sum_{s=2}^{n} \int_0^R f_k(\varrho_i t) \int_x^{x+kt} (x+kt-\tau)^{n(r+1)-1} h_r(\varrho_i(x+kt-\tau)) \times$$

$$\times p_s(\tau) u_{m-r}^{(n-s)}(\tau)\, d\tau\, dt \equiv I_1 + I_2 + I_3;$$

(16*) $$|I_3| \leqq N \sum_{r=0}^{m} \sum_{s=2}^{n} RC(NR)^{n(r+1)-1} C\|p_s\|_1 D(1+|\varrho_i|)^{nr+n-s} \|u_i\|_\infty \leqq$$

$$\leqq N^2C^2DR \sum_{r=0}^{m} \sum_{s=2}^{n} [NR(2+|\varrho|)]^{nr+n-s} [(NR)^{s-1}\|p_s\|_1] \|u_i\|_\infty \leqq$$

$$\leqq \left[N^2C^2D \sum_{r=0}^{m} \sum_{s=2}^{n} (NR)^{s-1}\|p_s\|_1\right] R\|u_i\|_\infty \leqq (1/4)R\|u_i\|_2 \frac{1}{\sqrt{b-a}}.$$

The theorem is proved.

REMARKS. Properly speaking, we need not assume the orthonormality of the system (u_i) in the theorem. Our proof equally works for the more general case when (u_i) is a normed Bessel system and, consequently, for any Riesz basis in $L^2(G)$, too.

The theorem and the proof are also valid in case $n=1$, i.e. for the operator $Lu=u'$; in this case estimate (3) is valid.

References

[1] В. А. Ильин, И. Йо, Равномерная оценка собственных функций и оценка сверху числа собственных значений оператора Штурма—Лиувилля с потенциалом из класса, *Дифференциальные уравнения*, **15** (1979), 1164—1174.
[2] I. Joó, *Spectral investigation of the Schrödinger operator*, Thesis, 1979 (in Hungarian).
[3] I. Joó, Upper estimates for the eigenfunctions of the Schrödinger operator, *Acta Sci. Math. Szeged*, **44** (1982), 87—93.
[4] V. Komornik, Upper estimate for the eigenvalues of an orthonormal system, consisting of eigenfunctions of a linear differential operator, *Studia Sci. Math. Hung.*
[5] V. Komornik, Upper estimates for the eigenfunctions of higher order of a linear differential operator, *Acta Sci. Math. Szeged.*
[6] V. Komornik, Generalization of a theorem of Joó, *Annales Univ. Sci. Budapest, Sect. Math.*
[7] А. Шевегярто, Равномерная оценка собственных функций и оценка сверху числа собственных значений оператора четвертого порядка, *Доклады Академии Наук* (to appear).
[8] И. Йо, Теорема о равносходимости, *Доклады Узб. Акад. Наук* (to appear).

DEPARTMENT II OF ANALYSIS
LORÁND EÖTVÖS UNIVERSITY
H-1445 BUDAPEST 8, PF. 323

(Received December 27, 1981)

PRINTED IN HUNGARY
Szegedi Nyomda, Szeged

Instructions for authors. Manuscripts should be typed on standard size papers (25 rows; 50 characters in each row). Mathematical symbols (except sin, log, etc.) will be set in italics, so *use handwritten letters, or indicate them by simple underlining*. The text of theorems, lemmas, corollaries are to be printed also in italics. The words THEOREM, LEMMA, COROLLARY, PROOF, REMARK, PROBLEM, etc. (or their German and French equivalents) should be typed in capital letters. Sub-headings will be set in boldface lower case; indicate them by red underlining. Other special types of symbols (such as German, Greek, script, etc.) should also be clearly identified. Mark footnotes by consecutive superscript numbers. When listing references, please follow the following pattern:

[1] G. Szegő, *Orthogonal polynomials*, AMS Coll, Publ. Vol. XXIII (Providence, 1939).
[2] A. Zygmund, Smooth functions, *Duke Math. J.*, **12** (1945), 47—76.

For abbreviation of names of journals follow the Mathematical Reviews. After the references give the author's affiliation.

Do not send abstract and second copy of the manuscript. Authors will receive only galley-proofs (one copy). Manuscripts will not be sent back to authors (neither for the purpose of proof-reading nor when rejecting a paper).

Authors obtain 100 reprints free of charge. Additional copies may be ordered from the publisher.

Periodicals of the Hungarian Academy of Sciences are obtainable at the following addresses:

AUSTRALIA
C.B.D. LIBRARY AND SUBSCRIPTION SERVICE
Box 4886, G.P.O., *Sydney N.S.W. 2001*
COSMOS BOOKSHOP, 145 Ackland Street
St. Kilda (Melbourne), Victoria 3182

AUSTRIA
GLOBUS, Höchstädtplatz 3, *1206 Wien XX*

BELGIUM
OFFICE INTERNATIONAL DE LIBRAIRIF
30 Avenue Marnix, *1050 Bruxelles*
LIBRAIRIE DE MONDE ENTIER
162 rue du Midi, *1000 Bruxelles*

BULGARIA
HEMUS, Bulvar Ruszki 6, *Sofia*

CANADA
PANNONIA BOOKS, P.O. Box 1017
Postal Station „B", *Toronto, Ontario M5T 2T8*

CHINA
CNPICOR, Periodical Department, P.O. Box 50 *Peking*

CZECHOSLOVAKIA
MAD'ARSKÁ KULTURA, Národní třida 22
115 66 Praha
PNS DOVOZ TISKU, Vinohradská 46, *Praha 2*
PNS DOVOZ TLAČE, *Bratislava 2*

DENMARK
EJNAR MUNKSGAARD, Norregade 6
1165 Copenhagen K

FEDERAL REPUBLIC OF GERMANY
KUNST UND WISSEN ERICH BIEBER
Postfach 46, *7000 Stuttgart 1*

FINLAND
AKATEEMINEN KIRJAKAUPPA, P.O. Box 128
SF-00101 Helsinki 10

FRANCE
DAWSON-FRANCE S. A., B. P. 40, *91121 Palaiseau*
EUROPÉRIODIQUES S. A., 31 Avenue de Versailles, *78170 La Celle St. Cloud*
OFFICE INTERNATIONAL DE DOCUMENTATION ET LIBRAIRIE, 48 rue Gay-Lussac *75240 Paris Cedex 05*

GERMAN DEMOCRATIC REPUBLIC
HAUS DER UNGARISCHEN KULTUR
Karl Liebknecht-Straße 9, *DDR-102 Berlin*
DEUTSCHE POST ZEITUNGSVERTRIEBSAMT
Straße der Pariser Kommüne 3-4, *DDR-104 Berlin*

GREAT BRITAIN
BLACKWELL'S PERIODICALS DIVISION
Hythe Bridge Street, *Oxford OXI 2ET*
BUMPUS, HALDANE AND MAXWELL LTD.
Cowper Works, *Olney, Bucks MK46 4BN*
COLLET'S HOLDINGS LTD. Denington Estate *Wellingborough, Northants NN8 2QT*
WM. DAWSON AND SONS LTD., Cannon House *Folkstone, Kent CT19 5EE*
H. K. LEWIS AND CO., 136 Gower Street
London WC1E 6BS

GREECE
KOSTARAKIS BROTHERS INTERNATIONAL
BOOKSELLERS, 2 Hippokratous Street, *Athens-143*

HOLLAND
MEULENHOFF-BRUNA B.V, Beulingstraat 2, Amsterdam
MARTINUS NIJHOFF B.V.
Lange Voorhout 9–11, *Den Haag*
SWETS SUBSCRIPTION SERVICE
347b Heereweg, *Lisse*

INDIA
ALLIED PUBLISHING PRIVATE LTD., 13/14
Asaf Ali Road, *New Delhi 110001*
150 B-6 Mount Road, *Madras 600002*
INTERNATIONAL BOOK HOUSE PVT. LTD.
Madame Cama Road, *Bombay 400039*
THE STATE TRADING CORPORATION OF INDIA LTD.,
Books Import Division, Chandralok
36 Janpath, *New Delhi 110001*

ITALY
INTERSCIENTIA, Via Mazzé 28, *10149 Torino*
LIBRERIA COMMISSIONARIA SANSONI, Via Lamarmora 45, *50121 Firenze*
SANTO VANASIA, Via M. Macchi 58 *20124 Milano*
D. E. A., Via Lima 28, *00198 Roma*

JAPAN
KINOKUNIYA BOOK-STORE CO. LTD.
17-7 Shinjuku 3 chome, Shinjuku-ku, *Tokyo 160-91*
MARUZEN COMPANY LTD., Book Department,
P.O. Box 5050 Tokyo International, *Tokyo 100-31*
NAUKA LTD. IMPORT DEPARTMENT
2-30-19 Minami Ikebukuro, Toshima-ku, *Tokyo 171*

KOREA
CHULPANMUL, *Phenjan*

NORWAY
TANUM-TIDSKRIFT-SENTRALEN A.S.,
Karl Johansgatan 41–43, *1000 Oslo*

POLAND
WĘGIERSKI INSTYTUT KULTURY,
Marszalkowska 80, *00-517 Warszawa*
CKP IW, ul. Towarowa 28, *00-958 Warszawa*

ROUMANIA
D. E. P., *Bucureşti*
ILEXIM, Calea Grivitei 64–66, *Bucureşti*

SOVIET UNION
SOJUZPECHAT — IMPORT, *Moscow*
and the post offices in each town
MEZHDUNARODNAYA KNIGA, *Moscow G-200*

SPAIN
DIAZ DE SANTOS, Lagasca 95, *Madrid 6*

SWEDEN
GUMPERTS UNIVERSITETSBOKHANDEL AB
Box 346, *401 25 Göteborg 1*

SWITZERLAND
KARGER LIBRI AG., Petersgraben 31, *4011 Basel*

USA
EBSCO SUBSCRIPTION SERVICES
P.O. Box 1943, *Birmingham, Alabama 35201*
F.W. FAXON COMPANY, INC.
15 Southwest Park, *Westwood Mass. 02090*
READ-MORE PUBLICATIONS, INC.
140 Cedar Street, *New York, N.Y. 10006*

YUGOSLAVIA
JUGOSLAVENSKA KNJIGA, Terazije 27, *Beograd*
FORUM, Vojvode Mišića 1, *21000 Novi Sad*

Index: 26.014

FORMERLY ACTA MATHEMATICA ACADEMIAE SCIENTIARUM HUNGARICAE

Acta Mathematica Hungarica

VOLUME 42, NUMBERS 3–4, 1983

EDITOR-IN-CHIEF

K. TANDORI

DEPUTY EDITOR-IN-CHIEF

J. SZABADOS

EDITORIAL BOARD

Á. CSÁSZÁR, P. ERDŐS, L. FEJES TÓTH, A. HAJNAL, I. KÁTAI,
L. LEINDLER, L. LOVÁSZ, A. PRÉKOPA, A. RAPCSÁK, P. RÉVÉSZ,
E. SZEMERÉDI, B. SZ.-NAGY

Akadémiai Kiadó, Budapest

ACTA MATH. HUNG. HU ISSN 0001—5954

ACTA MATHEMATICA
HUNGARICA

Acta Mathematica publishes papers on mathematics in English, German, French and Russian

Acta Mathematica is published in two volumes of four issues a year by

AKADÉMIAI KIADÓ
Publishing House of the Hungarian Academy of Sciences
H-1054 Budapest, Alkotmány u. 21.

Manuscripts and editorial correspondence should be addressed to

Acta Mathematica, H-1395 Budapest, P.O.B. 428

Orders may be placed with

KULTURA Foreign Trading Company
H-1389 Budapest P.O. Box 149

or with its representatives abroad.

Acta Mathematica is indexed in Current Contents

CONTENTS

Füredi, Z., An intersection problem with 6 extremes 177
Цао Цзя-Дин, Об операторах типа Канторовича и интегральных сплайнах Шенберга 189
Móricz, F., Lebesgue functions and multiple function series. III 205
Leawitt, W. G., Upper radicals of regular classes 213
Alon, N., A note on the decomposition of graphs into isomorphic matchings 221
Gut, A., Complete convergence and convergence rates for randomly indexed partial sums, with an application to some first passage times 225
Bácsó, S., Eine Charakterisierung der Finslersche Räume von konstanter Krümmung 233
Yoneda, K., Fourier transforms and convolutions of generalized measures defined on the dyadic field 237
Pal, B. K. and *Mukhopadhyay, S. N.*, The Cesaro—Denjoy—Bochner scale of integration ... 243
Yahya, S. M., On an isomorphism 257
Collier, M. G. and *Kelingos, J. A.*, Periodic Beurling distributions 261
Oswald, P., On some convergence properties of Haar—Fourier series in the classes $\varphi(L)$..... 279
Kátai, I., Multiplicative functions with regularity properties. I 295
Kroó, A., On the unicity of best L_1-approximation by polynomials of several variables 309
Csörgő, S., Tandori, K. and *Totik, V.*, On the strong law of large numbers for pairwise independent random variables 319
Sebestyén, Z., A moment problem with respect to C^*-algebras 331
Garay, B. M., Metrization and Liapunov functions. I 337
Huhn, A. P., A reduced free product of distributive lattices. I 349
Petruska, G., Derivatives take every value on the set of approximate continuity points 355

Acta Math. Hung.
42 (3—4) (1983), 177—187.

AN INTERSECTION PROBLEM WITH 6 EXTREMES

Z. FÜREDI (Budapest)

I. Introduction

Let X be a finite set of n elements. A family $\mathcal{F}$ of the subsets of X is *intersecting* if any two members of $\mathcal{F}$ intersect.

Erdős, Ko and Rado [3] proved that if $\mathcal{F}$ is an intersecting set-system of r-tuples of X and $n \geqq 2r$ then $|\mathcal{F}| \leqq \binom{n-1}{r-1}$. Equality holds in the case $n > 2r$ if and only if the members of $\mathcal{F}$ have a common element.

Let c be a real number, $0 < c \leqq 1$. The *degree* of the point (that is an element) x in the set-system $\mathcal{F}$ is denoted by $d_{\mathcal{F}}(x)$ or simply $d(x) =: |\{F : x \in F \in \mathcal{F}\}|$.

Erdős, Rotschild and Szemerédi [5] raised the following question: How large can be the intersecting set-system $\mathcal{F}$ of r-tuples of X if each point has degree at most $c|\mathcal{F}|$? For the case $c = 2/3$, $n > n_0(r)$ they proved that

$$|\mathcal{F}| \leqq |\mathcal{F}_{3,2}| \tag{1}$$

where $\mathcal{F}_{3,2} = \{F \subset X : |F| = r,\ |F \cap D| \geqq 2\},\ |D| = 3$.

Frankl [6] proved that (1) holds for any $2/3 \leqq c < 1$ if n is large enough $(n > n_0(r, c))$, and he solved the cases $3/7 < c \leqq 3/5$ as well, proving the conjectures of Erdős—Rotschild—Szemerédi. The aim of this paper is to settle the missing case $3/5 < c < 2/3$.

II. Results

For a finite set-system $\mathcal{H}$ the underlying set of which is a subset of X (i.e. $\cup \mathcal{H} \subset X$), we write $\mathcal{F}(\overline{\mathcal{H}}) = \{F \subset X : |F| = r$ and there exists an $H \in \mathcal{H}$ such that $H \subset F\}$, $\mathcal{F}(\mathcal{H}) = \{F \subset X : |F| = r$ and $(F \cap (\cup \mathcal{H})) \in \mathcal{H}\}$. Evidently, $\mathcal{F}(\mathcal{H})$ and $\mathcal{F}(\overline{\mathcal{H}})$ are intersecting set-systems if $\mathcal{H}$ is intersecting, and $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{F}(\overline{\mathcal{H}})$.

P. Frankl [6] proved that if $1/2 < c \leqq 3/5$ and $n > n_0(r, c)$ then

$$|\mathcal{F}| \leqq 10\binom{n-6}{r-3} + 15\binom{n-6}{r-4} + 6\binom{n-6}{r-5} + \binom{n-6}{r-6} =$$

$$= 10\binom{n-5}{r-3} + 5\binom{n-5}{r-4} + \binom{n-5}{r-5}.$$

If equality holds here then there exists a 3-uniform, 5-regular, intersecting set-system $\mathscr{H}_1$ on a 6-element set such that $\mathscr{F}=\mathscr{F}(\mathscr{H}_1)$. There exists exactly one such $\mathscr{H}_1$ (see Figure 1).

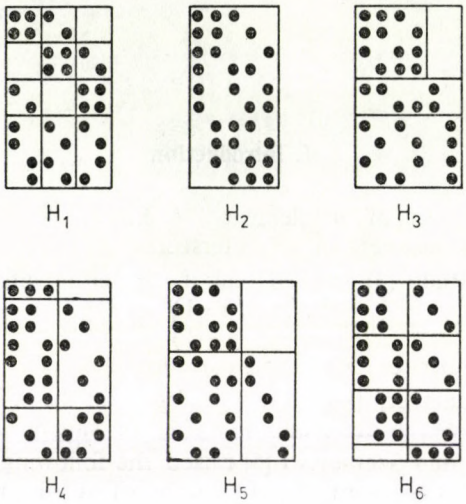


Fig. 1

We describe 6 hypergraphs. (The elements of the underlying set are denoted by positive integers, see also Figure 1.)

$\mathscr{H}_1=\{123, 124, 345, 346, 156, 256, 135, 146, 236, 245\}$,
$\mathscr{H}_2=\binom{[5]}{3}=\{123, 124, 125, 134, 135, 145, 234, 235, 245, 345\}$,
$\mathscr{H}_3=\{123, 124, 134, 234, 125, 345, 136, 246, 146, 236\}$,
$\mathscr{H}_4=\{123, 124, 125, 134, 136, 235, 236, 156, 246, 345\}$,
$\mathscr{H}_5=\{123, 124, 134, 234, 125, 345, 136, 246, 147, 237\}$,
$\mathscr{H}_6=\{124, 125, 126, 134, 135, 136, 234, 235, 236, 456\}$.

THEOREM 1. *Let $\mathscr{F}$ be an intersecting family consisting of r-element subsets of X, $|X|=n$. Suppose that for some $3/5<c<2/3$, for $n>n_0(r, c)$ and for every $x\in X$, $d_{\mathscr{F}}(x)\leqq c|\mathscr{F}|$ holds. Then*

$$|\mathscr{F}| \leqq 10\binom{n-5}{r-3}+5\binom{n-5}{r-4}+\binom{n-5}{r-5}. \tag{2}$$

Furthermore equality holds in (2) *iff $\mathscr{F}\equiv\mathscr{F}(\overline{\mathscr{H}_i})$ for some $1\leqq i\leqq 6$.*

Furthermore, if $c=1/2$ and $n>n_0(r)$ then

$$|\mathscr{F}| \leqq 10\binom{n-6}{r-3}, \tag{3}$$

and equality holds here if and only if $\mathscr{F}=\mathscr{F}(\mathscr{H}_1)$.

So the cardinality of a maximum $\mathcal{F}$ is constant on the whole interval $(1/2, 2/3)$. Our theorem differs from the theorem of Frankl because in case $3/5<c<2/3$ five more extremal systems are allowed. So we have non-isomorphic optimal families. This phenomenon is not rare in combinatorics even in the Erdős—Ko—Rado type theorems, cf. the theorem of Hilton—Milner for $r=3$ (see [6] or [8]).

The following is a consequence of Theorem 1.

THEOREM 2. *Let $\mathcal{F}$ be a family of intersecting r-subsets of X, $|X|=n$. Suppose that $|\mathcal{F}|>(10+\varepsilon)\binom{n-3}{r-2}$ where $\varepsilon>0$ is a positive constant. Then for $n>n_0(r,\varepsilon)$ there exists an $x\in X$ such that $d(x)>(2/3-\varepsilon)|\mathcal{F}|$.*

(This is also an improvement of a theorem of Frankl. He proved the lower bound $3/5+\min(0.01, 0.01\varepsilon)$ instead of $2/3-\varepsilon$.)

III. Definitions and lemmas

Define an *edge-contraction* as the following operation on a set-system $\mathcal{H}$: we substitute an edge $E\in\mathcal{H}$ by a smaller, nonempty $E'\subsetneq E$, and thus we get the set-system $\mathcal{H}-\{E\}\cup\{E'\}$. An intersecting set-system is *v-critical* if it has no multiple edges and the hypergraph obtained by contracting any of its edges is non-intersecting. That is

(4) For all $E\in\mathcal{H}$, $x\in E$ there exists an $F\in\mathcal{H}$ such that $E\cap F=\{x\}$.

Every v-critical intersecting set-system is a Sperner-family, that is

(5) If $E\in\mathcal{H}$ and $F\subsetneq E$ then $F\notin\mathcal{H}$.

Erdős and Lovász proved the following theorem [4]:

(6) If $\mathcal{H}$ is a v-critical intersecting set-system and $\max\{|E|: E\in\mathcal{H}\}=k$, then $|\mathcal{H}|\leqq k^k$.

We can get a v-critical intersecting set-system from any intersecting set-system $\mathcal{F}$ by contracting its edges as far as possible and deleting all but one copy of the appearing multiple edges. This $\mathcal{H}$ is called the *nucleus* of the set-system $\mathcal{F}$. Split $\mathcal{H}$ according to the cardinality of its members: $\mathcal{H}=\mathcal{H}^1\cup\mathcal{H}^2\cup\ldots\cup\mathcal{H}^r$ where $E\in\mathcal{H}^i$ implies $|E|=i$.

Denote by $\mathcal{B}$ the nucleus of $\mathcal{H}^1\cup\mathcal{H}^2\cup\mathcal{H}^3$. In what follows $\mathcal{B}$ is called the *nucleus of rank 3 of* $\mathcal{F}$. Of course, $\mathcal{B}$ is not unique, but this is not important.

(7) If $\mathcal{F}$ is an r-uniform, v-critical intersecting set-system with underlying set X, $|X|=n$, then there exists a set-system $\mathcal{B}$ such that

(a) $\mathcal{B}$ is v-critical, intersecting and for all $B\in\mathcal{B}$, $|B|\leqq 3$ (possibly $\mathcal{B}=\varnothing$);

(b) $|\mathcal{F}-\mathcal{F}(\overline{\mathcal{B}})|\leqq r^r\binom{n-4}{r-4}$.

Indeed, applying (6) we get

$$|\mathcal{F}-\mathcal{F}(\overline{\mathcal{B}})|\leqq\sum_{i=4}^{r}|\mathcal{F}\cap\mathcal{F}(\overline{\mathcal{H}^i})|\leqq\sum_{i=4}^{r}|\mathcal{F}(\overline{\mathcal{H}^i})|\leqq|\mathcal{H}|\binom{n-4}{r-4}\leqq r^r\binom{n-4}{r-4}.$$

Q.E.D.

IV. The first part of the proof of Theorem 1. The main lemma

We shall consider the whole interval [1/2, 2/3), thus we will prove the above mentioned theorem of Frankl at the same time. So let $1/2 \leqq c < 2/3$ be fixed and let $\mathcal{F}$ be an r-uniform intersecting set-system on X with $\max\{d_{\mathcal{F}}(x): x \in X\} \leqq c|\mathcal{F}|$. We are looking for $\mathcal{F}$ with maximal cardinality, hence we may suppose $|\mathcal{F}| \geqq |\mathcal{F}(\mathcal{H}_1)| = 10\binom{n-6}{r-3}$.

Write $\mathcal{B}$ for the nucleus of rank 3 of $\mathcal{F}$. For each $F \in \mathcal{F} \cap \mathcal{F}(\overline{\mathcal{B}})$ let us choose a $B \in \mathcal{B}$ with $B \subset F$. Let $\mathcal{F}_B$ denote the set of those members of $\mathcal{F}$ for which B is chosen. Thus $|\mathcal{F}| = \sum_{B \in \mathcal{B}} |\mathcal{F}_B| + |\mathcal{F} - \mathcal{F}(\overline{\mathcal{B}})|$. Define a weight $w(B)$ of B by $w(B) = |\mathcal{F}_B| \Big/ \binom{n-3}{r-3}$. Since by (7b) $|\mathcal{F} - \mathcal{F}(\overline{\mathcal{B}})| \leqq r^r \binom{n-4}{r-4}$ we get

$$\sum_{B \in \mathcal{B}} w(B) \geqq \frac{10\binom{n-6}{r-3}}{\binom{n-3}{r-3}} - \frac{r^r \binom{n-4}{r-4}}{\binom{n-3}{r-3}} > 9.9 \tag{8}$$

provided n is large enough ($n > 10r^{r+1}$). Moreover for any $x \in X$ we have

$$\sum_{B \ni x} w(B) < \frac{2}{3} \sum w(B). \tag{9}$$

Indeed

$$\frac{\sum_{B \ni x} w(B)}{\sum w(B)} = \frac{\sum_{B \ni x} w(B)\binom{n-3}{r-3}}{\sum w(B)\binom{n-3}{r-3}} = \frac{\sum_{B \ni x} |\mathcal{F}_B|}{\sum |\mathcal{F}_B|} \leqq \frac{d_{\mathcal{F}}(x)}{|\mathcal{F} \cap \mathcal{F}(\overline{\mathcal{B}})|} \leqq \frac{c|\mathcal{F}|}{|\mathcal{F}| - O\left(\binom{n-4}{r-4}\right)} < \frac{2}{3}$$

provided $n > n_0(r, c)$ $\left(n > \dfrac{1}{(2/3)-c} 10r^{r+1}\right)$.

The following lemma is the crucial point of the proof.

MAIN LEMMA. *Suppose that $\mathcal{B}$ is a v-critical, intersecting set-system of rank* 3. *Suppose further that there exists a non-negative weight function $w: \mathcal{B} \to \mathbf{R}$ such that* (8) *and* (9) *hold and $w(B) \leqq 1$ if $|B| = 3$. Then $\mathcal{B} \equiv \mathcal{H}_i$ for some $1 \leqq i \leqq 6$* (see Figure 1).

By (6) $|\mathcal{B}| \leqq 27$ thus the proof of this lemma is reduced to the investigation of finitely many "small" set-systems. After this lemma the proof of Theorem 1 is not hard. But we cannot hope for a simple proof of the lemma because its conclusion is somewhat complicated, and any proof must yield a description of the structures of the $\mathcal{H}_i$'s.

The following two parts of this paper (Chapters V and VI) contain only the proof of the Main Lemma. If the reader believies that the author has examined all (finitely many) cases of the v-critical intersecting set-systems of rank 3, then he or she can continue reading Chapter VII.

V. The first part of the proof of the lemma. The nucleus of rank 3 of a maximal $\mathscr{F}$ is 3-uniform

The formula (9) yields that $\mathscr{B}$ has no member with 1 element, since then $|\mathscr{B}|=1$. We will show that

(10) $\mathscr{B}$ *is 3-uniform.*

We will prove this by way of contradiction. Denote by $\mathscr{B}^2$ the members of $\mathscr{B}$ with 2-elements. In what follows the points of the underlying set of $\mathscr{B}$ will be denoted by the positive integers.

If $|\mathscr{B}^2|\geqq 4$ then its edges have a common point since $\mathscr{B}^2$ is intersecting. E.g. $B_1=\{1, 2\}$, $B_2=\{1, 3\}$, $B_3=\{1, 4\}$, $B_4=\{1, 5\}$. (See Figure 2.) By (4) there exists an edge B_5 not containing the point 1. But B_5 intersects B_1, B_2, B_3 and B_4 thus $\{2, 3, 4, 5\}\subset B_5$. This is a contradiction.

If $|\mathscr{B}^2|=3$ *and* the edges of $\mathscr{B}^2$ have a common point then let this be e.g. the point 1 and $B_1=\{1, 2\}$, $B_2=\{1, 3\}$, $B_3=\{1, 4\}$. (See Figure 3.) Since there exists an edge B_4 not containing the point 1 we get that $B_4=\{2, 3, 4\}$. There are no other edges of $\mathscr{B}$ which do not contain the point 1. Moreover there is no other edge of $\mathscr{B}$ which contains 1 because it would contain some B_i $(1\leqq i\leqq 3)$ contradicting (5). Thus in this case $\mathscr{B}=\{B_1, B_2, B_3, B_4\}$. Considering (9) at point 1 we have $(w_1+w_2+w_3)<\frac{2}{3}(w_1+w_2+w_3+w_4)$. This and the inequality $w_4\leqq 1$ give that $\sum w_i<3$. This contradicts (8) $(w_i=w(B_i))$.

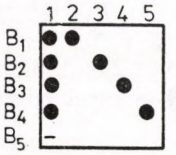


Fig. 2

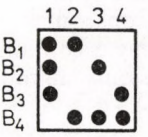


Fig. 3

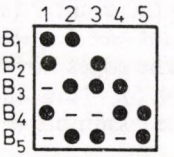
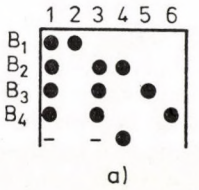
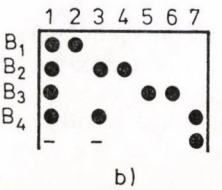


Fig. 4





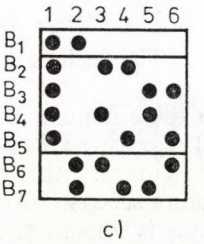


Fig. 5

If $|\mathscr{B}^2|=3$ *and* the edges of $\mathscr{B}^2$ have no a common point then they form a triangle, i.e. $B_1=\{1, 2\}$, $B_2=\{1, 3\}$, $B_3=\{2, 3\}$. The set-system $\mathscr{B}=\{B_1, B_2, B_3\}$, similarly to the above mentioned cases, is a maximal v-critical intersecting system. (I.e. if $\mathscr{B}'$ is a v-critical intersecting set-system and $\mathscr{B}\subset\mathscr{B}'$ then $\mathscr{B}=\mathscr{B}'$.) However for the triangle (9) does not hold.

If $|\mathscr{B}^2|=2$ then let $B_1=\{1, 2\}$, $B_2=\{1, 3\}$ (see Figure 4). By (4) there exists an edge B_3 not containing the point 1, $B_3=\{2, 3, 4\}$. There exists an edge B_4 meeting B_3 only in the point 4, i.e. $2, 3 \notin B_4$, $4 \in B_4$, thus $B_4=\{1, 4, 5\}$. There exists an edge B_5 which meets B_4 only in the point 5, i.e. $1, 4 \notin B_5$, $5 \in B_5$, thus $B_5=\{2, 3, 5\}$. The set-system $\mathscr{B}$ has no further edges containing the point 1, and it has no further edges not containing the point 1. Hence the set-system obtained above is maximal v-critical, i.e. $\mathscr{B}=\{B_1, \ldots, B_5\}$. Applying (9) at the point 1 we have $(w_1+w_2+w_4)<\frac{2}{3}(w_1+w_2+w_3+w_4+w_5)$. Moreover $w_3, w_4, w_5 \leqq 1$ hence $\sum w_i<6$, but that contradicts (8).

Finally if $|\mathscr{B}^2|=1$ then let $B_1=\{1, 2\}$. Applying (9) at the points 1 and 2 and summing we get $2w_1+(\sum_{i>1} w_i)<\frac{4}{3}(\sum w_i)$. From this $w_1<\frac{1}{2}(\sum_{i>1} w_i)$ and $(\sum w_i)<\frac{3}{2}(\sum_{i>1} w_i)$. By (8) $\sum w_i \geqq 9$ hence $(\sum_{i>1} w_i)>6$. Consequently at least 7 members of $\mathscr{B}$ with 3 elements intersect B_1. Thus at least 4 edges $(B_2 \ldots B_5)$ contain the point 1 (and by (5) they do not contain the point 2). There are no three sets from $B_2 \setminus \{1\}, B_3 \setminus \{1\}, B_4 \{1\}, B_5 \setminus \{1\}$ which have a common point because if we suppose on the contrary that (see Figure 5a) $B_2=\{1, 3, 4\}$ $B_3=\{1, 3, 5\}$, $B_4=\{1, 3, 6\}$ then we get a contradiction applying (4) to the edges B_2 at the point 4. Consequently among the sets $B_i-\{1\}$ $(2 \leqq i \leqq 5)$ there are two disjoint, e.g. $B_2=\{1, 3, 4\}$, $B_3=$ $=\{1, 5, 6\}$ (see Figure 5b and 5c). If the edge B_4 (or B_5) would contain a further point (say 7) then we immediately get a contradiction applying (4) to the edge B_4 (or B_5) at the point 7 (see Figure 5b). Thus $B_4 \setminus \{1\}, B_5 \setminus \{1\} \subset (3, 4, 5, 6)$ and they are disjoint (see Figure 5c). Consequently there is no further edge containing the point 1. The edges not containing 1 contain the point 2 and intersect (3, 4, 5, 6) in (3, 6) or (4, 5). Thus there are only at most six 3-element edges of $\mathscr{B}$ which contradicts the assumption $(\sum_{i>1} w_i)>6$.

Consequently $\mathscr{B}^2=\varnothing$ and this completes the proof of (10).

VI. Proof of the main lemma (last part). The nucleus of rank 3 of a maximal $\mathscr{F}$ is $\mathscr{H}_i$ $(1 \leqq i \leqq 6)$

By (9) there is no point contained in all the edges of $\mathscr{B}$. Further there is no pair $\{x, y\}$ covering all the edges of $\mathscr{B}$ (i.e. $\forall B \in \mathscr{B}$, $\{x, y\} \cap B \neq \varnothing$). Indeed, if we suppose the contrary then joining the edge $\{x, y\}$ with weight 0 to $\mathscr{B}$ we get an intersecting set-system which satisfies the assumptions of the Lemma. But (10) says that this is impossible. Consequently

(11) *For all the points* x, y *there exists an edge* $B \in \mathscr{B}$ *such that* $B \cap \{x, y\}=\varnothing$.

Since $(\sum w_i)>9$ we get $|\mathscr{B}| \geqq 10$. Then we can apply a theorem of Deza [1] which in this case states: If at least 8 3-element sets are given so that any two of them intersect in exactly 1 element, then all the sets have a common point. Consequently $\mathscr{B}$ has two edges (B_1 and B_2) intersecting in 2 elements. E.g. $B_1=\{1, 2, 3\}$, $B_2=$ $=\{1, 2, 4\}$ (see Fig. 6).

Firstly, suppose that

(12) there do not exist $B', B'', B''' \in \mathcal{B}$ such that $|B' \cap B'' \cap B'''| = 2$.

We will prove that in this case $\mathcal{B} = \mathcal{H}_1$.

By (11) there exists an edge B_3 such that $1, 2 \notin B_3$, say $B_3 = \{3, 4, 5\}$. We will show that $d_{\mathcal{B}}(1) \leqq 5$ (and similarly $d_{\mathcal{B}}(2) \leqq 5$). Indeed, if $d_{\mathcal{B}}(1) \geqq 6$, then there exists at least 4 edges (B_4, B_5, B_6, B_7) which contain the point 1 and by (12) do not contain the point 2. Also by (12) there are no two of them which contain the point 3 (or the point 4) (see Figure 6a). Thus there are two edges (B_6, B_7) which do not contain the points 3, 4 (and 2) e.g. $B_6 = \{1, 5, 6\}$, $B_7 = \{1, 5, 7\}$, see Fig. 6b. Then there is no edge not containing the points $\{1, 5\}$; however it meets all the edges B_1, B_2, B_3, B_6, B_7. This contradicts (11).

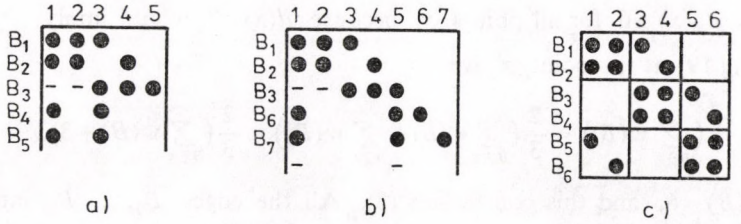


Fig. 6

Thus, by $|\mathcal{B}| \geqq 10$ and $d(1), d(2) \leqq 5$, there are at least (and by (12) at most) two edges which are disjoint from the points $\{1, 2\}$. They contain the points 3 and 4 (see Figure 6c) $B_3 = \{3, 4, 5\}$, $B_4 = \{3, 4, 6\}$. Since $|B_3 \cap B_4| = 2$ we again can say that $d(3) \leqq 5$, $d(4) \leqq 5$ and there exists exactly two further edges (B_5, B_6) which are disjoint from $\{3, 4\}$. Consequently $B_5 = \{5, 6, 1\}$, $B_6 = \{5, 6, 2\}$. Then the minimal number of points covering the system $\{B_1 \ldots B_6\}$ is 3, hence the set $\{1, 2, \ldots, 6\}$ contains all the edges of $\mathcal{B}$. Thus $d(1) = \ldots = d(6) = 5$ and $|\mathcal{B}| = 10$. For each pair of points from $\{1, \ldots, 6\}$ there is an edge B_i $(1 \leqq i \leqq 6)$ containing it, thus by (12) all the intersections $B_\alpha \cap B_\beta$ $(7 \leqq \alpha, \beta \leqq 10)$ have only 1 point in common. Moreover B_α $(7 \leqq \alpha \leqq 10)$ intersects each of the sets $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$ in one point only. This last two properties (up to isomorphisms) uniquely define the edges $B_7, \ldots, B_{10}$. (Since the edges $B_7, \ldots, B_{10}$ and the sets $\{1, 2, x\}$, $\{3, 4, x\}$, $\{5, 6, x\}$ form the finite projective plane of order 2.) So we get $\mathcal{H}_1$.

Now we suppose that (12) does not hold, i.e. $|B_1 \cap B_2 \cap B_3| = 2$, $B_1 = \{1, 2, 3\}$, $B_2 = \{1, 2, 4\}$, $B_3 = \{1, 2, 5\}$. (See Figure 7.) By (11) there exists an edge B_4 which does not contain 1 and 2, thus $B_4 = \{3, 4, 5\}$. An arbitrary other edge B of $\mathcal{B}$

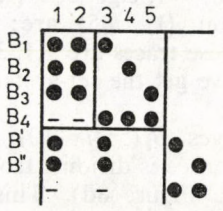


Fig. 7

is called *inner* or *outer* according to whether it is contained in $\{1, 2, 3, 4, 5\}$ or not. I.e. either $|B\cap\{1, 2\}|=1$ and $|B\cap\{3, 4, 5\}|=2$ or $|B\cap\{1, 2\}|=|B\cap\{3, 4, 5\}|=$ $=|B\setminus\{1, \ldots, 5\}|=1$, respectively.

First of all we show that there are no two outer edges B', B'' which intersect $\{1, \ldots, 5\}$ at the same points. Suppose on the contrary that $B'=\{1, 3, x\}$, $B''=$ $=\{1, 3, y\}$ $(x, y>5)$ then applying (11) to the points 1 and 3 we get a contradiction (see Fig. 7). Hence the number of outer edges (and the number of inner edges, too) is at most 6. These 6 and 6 sets form 6 complementary pairs (e.g. the complement of the outer edge $\{1, 4, x\}$ is the inner edge $\{2, 3, 5\}$). Naturally, $\mathscr{B}$ contains at most one member of each complementary pair, thus $|\mathscr{B}-\{B_1, B_2, B_3, B_4\}|\leqq 6$. It follows that $|\mathscr{B}|\leqq 10$, thus

$$|\mathscr{B}| = 10. \tag{13}$$

This yields $d_{\mathscr{B}}(x)\leqq 6$ for all points x, because $d(x)\geqq 7$ would imply $\sum_{B\not\ni x} w(B)\leqq 3$ thus writing (9) at the point x we get

$$\Big(\sum_{B\ni x} w(B)\Big) < \frac{2}{3}\Big(\sum_{B\ni x} w(B)+\sum_{B\not\ni x} w(B)\Big)\leqq \frac{2}{3}\Big(\sum_{B\ni x} w(B)+3\Big),$$

i.e. $\sum_{B\ni x} w(B)<6$, and this contradicts (8). All the edges $B_5, \ldots, B_{10}$ intersect the set $\{1, 2\}$ in (exactly) one element, hence we get $d_{\mathscr{B}}(1)=d_{\mathscr{B}}(2)=6$. Let us denote the number of outer edges containing the points 1 and 2 by α and β, respectively. By (13) one member of each complementary pair (consisting of one outer and inner set) belongs to $\mathscr{B}$. Thus there are exactly $3-\beta$ outer sets containing 1 belonging to $\mathscr{B}$. Hence $6=d_{\mathscr{B}}(1)=3+\alpha+(3-\beta)$, consequently $\alpha=\beta$.

If $\alpha=\beta=0$, i.e. every edge of $\mathscr{B}$ is inner then we get $\mathscr{H}_2$, the set-system of all 3-tuples of the underlying-set $\{1, 2, \ldots, 5\}$ (see Fig. 1).

If $\alpha=\beta=1$ then let the unique outer edge containing the point 1 be B_5 (see Fig. 8a), e.g. $B_5=\{1, 3, 6\}$. By (4) there is an edge B_6 such that $1, 3\notin B_6$, $6\in B_6$, i.e. $B_6=\{2, 4, 6\}$. All the other edges are inner and determined uniquely. Hence we get $\mathscr{H}_3$.

If $\alpha=\beta=2$ then there are only two inner edges (B_5 and B_6, see Fig. 8bc). As $\alpha=\beta$ the edges B_5 and B_6 intersect $\{1, 2\}$ in different points. There are two cases. First case: $|B_5\cap B_6|=2$, e.g. $B_5=\{1, 3, 4\}$, $B_6=\{2, 3, 4\}$, see Fig. 8b. Then the traces of the outer edges of $\mathscr{B}$ on the set $\{1, \ldots, 5\}$ are $\{1, 3\}$, $(2, 4\}$, $\{2, 3\}$ and $\{1, 4\}$. Here the first two traces are disjoint, and the last two are disjoint, too. Thus they have a common outer point, i.e. $B_7=\{1, 3, x\}$, $B_8=\{2, 4, x\}$ and $B_9=$ $=\{2, 3, y\}$, $B_{10}=\{1, 4, y\}$. If $x=y$ then we get $\mathscr{H}_3$ again and if $x\neq y$ we get $\mathscr{H}_5(x, y>5)$. Second case: $|B_5\cap B_6|=1$, e.g. $B_5=\{1, 3, 4\}$, $B_6=\{2, 3, 5\}$, see Fig. 8c. The traces of the outer edges on $\{1, \ldots, 5\}$ are: $\{1, 3\}$, $\{2, 4\}$, $\{2, 3\}$, $\{1, 5\}$. The outer edges corresponding to these traces are $\{1, 3, x\}$, $\{2, 4, x\}$, $\{2, 3, y\}$, $\{1, 5, y\}$. Hence if x coincides with y we get the set-system $\mathscr{H}_4$, and if they are different $(x>y>5)$ we get again $\mathscr{H}_5$.

If $\alpha=\beta=3$, i.e. all the edges B_i $(5\leqq i\leqq 10)$ are outer then we can order the 6 traces in such a way that any trace is disjoint from its successor e.g. $\{1, 3\}$, $\{2, 4\}$, $\{1, 5\}$, $\{2, 3\}$, $\{1, 4\}$, $\{2, 5\}$ (see Figure 8d). This implies that all B_i $(5\leqq i\leqq 10)$ have a common outer point. This gives $\mathscr{H}_6$.

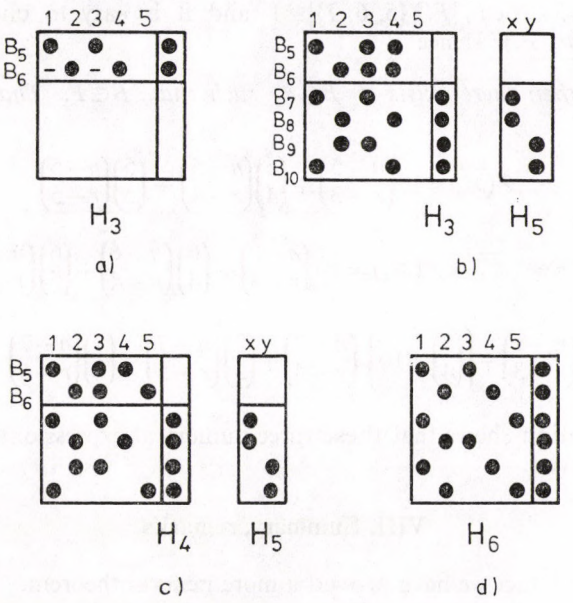


Fig. 8

VII. Proof of Theorem 1 (last part)

The Main Lemma implies that if $|\mathscr{F}| \geqq 10\binom{n-6}{r-3}$ and $n > n_0(r, c)$ then the v-critical nucleus B of rank 3 of $\mathscr{F}$ is $\mathscr{H}_i$ (for some $1 \leqq i \leqq 6$). As $\sum w(B) > 9.9$ by (8) we have that $w(B) > 0.9$ for each $B \in \mathscr{H}_i$. This means that $|\{F \in \mathscr{F}: F \supset B\}| \geqq$ $\geqq |\mathscr{F}_B| > 0.9\binom{n-3}{r-3}$. Let us choose $F_0 \in \mathscr{F}$ arbitrarily. If $F_0 \cap B = \varnothing$ for some $B \in \mathscr{H}_i$ then since $\mathscr{F}$ is intersecting we get

$$0.9\binom{n-3}{r-3} < |\{F \in \mathscr{F}: F \supset B\}| \leqq \sum_{x \in F_0} |\{F \in \mathscr{F}: F \supset B \cup \{x\}\}| \leqq r\binom{n-4}{r-4}.$$

This leads to a contradiction if $n > n_0(r)$. Hence

(14) *If* $B \in \mathscr{H}_i$ and $F \in \mathscr{F}$ *then* $B \cap F \neq \varnothing$.

It is easy to see (it follows from the constructions described in Chapter VI) that if a set F intersects all the edges of $\mathscr{H}_i$ then it contains one of them. (This fact is trivial for $\mathscr{H}_2$. Let $i = 1, 3, 4$ or 6, i.e. $\mathscr{H}_i$ is a set-system on 6 points. If F does not contain edges of $\mathscr{H}_i$ then $(\cup \mathscr{H}_i) \setminus F$ meets all the edges of $\mathscr{H}_i$, too. Hence $|F \cap (\cup \mathscr{H}_i)| = |(\cup \mathscr{H}_i) \setminus F| = 3$. However one of these two sets is an edge of $\mathscr{H}_i$. This leads to a contradiction. Finally if F meets all the edges of $\mathscr{H}_5$ (see Fig. 1) then $|F \cap \{1, 2, 3, 4\}| \geqq 2$. If $|F \cap \{1, 2, 3, 4\}| > 2$ then we are ready, and if

$|F\cap\{1, 2, 3, 4\}|=2$ then $|F\cap\{5, 6, 7\}|\geqq 1$ and it is easy to check that there is a $B\in\mathscr{H}_5$ with $B\subset F$.) Hence

(15) *If* $F\in\mathscr{F}$ *then there exists a* $B\in\mathscr{H}_i$ *such that* $B\subset F$. *That is* $\mathscr{F}\subset\mathscr{F}(\overline{\mathscr{H}_i})$.

Clearly

$$\mathscr{F}(\overline{\mathscr{H}_2})=10\binom{n-5}{r-3}+\binom{5}{4}\binom{n-5}{r-4}+\binom{5}{5}\binom{n-5}{r-5}$$

and

$$\mathscr{F}(\overline{\mathscr{H}_1})=\mathscr{F}(\overline{\mathscr{H}_3})=\mathscr{F}(\overline{\mathscr{H}_4})=\mathscr{F}(\overline{\mathscr{H}_6})=10\binom{n-6}{r-3}+\binom{6}{4}\binom{n-6}{r-4}+\binom{6}{5}\binom{n-6}{r-5}+\binom{6}{6}\binom{n-6}{r-6}$$

and

$$\mathscr{F}(\overline{\mathscr{H}_5})=10\binom{n-7}{r-3}+\left(\binom{7}{4}-10\right)\binom{n-7}{r-4}+\binom{7}{5}\binom{n-7}{r-5}+\binom{7}{6}\binom{n-7}{r-6}+\binom{7}{7}\binom{n-7}{r-7}.$$

An easy computation shows that these three numerical expressions are equal. Thus Theorem 1 is proved.

VIII. Summary, remarks

As a matter of fact we have proved a more general theorem.

THEOREM 3. *Let* $\mathscr{F}$ *be a family of intersecting r-subsets of* X, $|X|=n$. *Suppose that for some* $1/2\leqq c<2/3$ *and for every* $x\in X$, $d_{\mathscr{F}}|x|\leqq c|\mathscr{F}|$, *and that*

$$|\mathscr{F}|\geqq 9\binom{n-3}{r-3}+\frac{20}{(2/3)-c}r^{r+1}\binom{n-4}{r-4}.$$

Then $\mathscr{F}\subset\mathscr{F}(\overline{\mathscr{H}_i})$ *for some* $1\leqq i\leqq 6$ *(See Fig. 1).*

If $c=1/2$ then the extremal set-system is $\mathscr{F}(\mathscr{H}_1)$. If c is a little bit greater than $1/2$ then for the extremal set-system $\mathscr{F}$ we get: $\mathscr{F}(\mathscr{H}_1)\subset\mathscr{F}\subset\mathscr{F}(\overline{\mathscr{H}_1})$. However if $c>1/2+r/n$ then the extremal set-system is the whole $\mathscr{F}(\overline{\mathscr{H}_1})$ since $(\max_{x\in X} d_{\mathscr{F}(\overline{\mathscr{H}_1})}(x))/|\mathscr{F}(\overline{\mathscr{H}_1})|<1/2+r/n$ if $n>n_0(r)$. $\mathscr{F}(\mathscr{H}_1)$ is the unique extremum as long as $c\leqq\frac{3}{5}$, and there are five further extrema only if $c>3/5$.

In fact the proof presented above is a slight improvement of a method due to P. Frankl. The crucial observation is that the nucleus $\mathscr{B}$ of $\mathscr{F}$ is v-critical in this proof. P. Frankl used a different nucleus which was not v-critical. Indeed, in Chapters IV—VI we built up the set-systems $\mathscr{H}_1, \ldots, \mathscr{H}_6$ using (4). Further results can be found in the paper [7].

References

[1] M. Deza, Solution of a problem of Erdős—Lovász, *J. Comb. Theory, B* **16**, 166—167.
[2] P. Erdős, Problems and results in combinatorial analysis, *Teorie Combinatorie* (held in Roma 1973) Vol. II. pp. 3—17, Roma 1976.
[3] P. Erdős, Chao Ko, R. Rado, Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford*, (Ser. 2) **12** (1961), 313—320.

[4] P. Erdős, L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, *in "Infinite and Finite sets"* (A. Hajnal, R. Rado and V. T. Sós, Eds.) Proc. Coll. Math. Soc. J. Bolyai, Vol. 10. pp. 609—627, North-Holland (Amsterdam, 1974).
[5] P. Erdős, B. Rotschild, E. Szemerédi, (unpublished) see [2] or [6].
[6] P. Frankl, On intersecting families of finite sets, *J. Comb. Theory, A*, **24** (1978), 146—161.
[7] Z. Füredi, Erdős—Ko—Rado type theorems with upper bounds on the maximal degree, *Algebraic methods in graph theory*, Proc. Coll. Math. Soc. J. Bolyai, Vol. 26. pp. 177—207. North-Holland (Amsterdam, 1981).
[8] A. J. W. Hilton, E. C. Milner, Some intersection theorems for systems of finite sets, *Quart. J. Math. Oxford* (Ser. 2), **18** (1967), 369—384.

(Received March 30, 1981; revised October 28, 1982)

MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
H—1053 BUDAPEST
REÁLTANODA U. 13—15, HUNGARY

Acta Math. Hung.
42 (3—4) (1983), 189—203.

ОБ ОПЕРАТОРАХ ТИПА КАНТОРОВИЧА И ИНТЕГРАЛЬНЫХ СЛАЙНАХ ШЕНБЕРГА

ЦАО ЦЗЯ-ДИН (Шанхай)

§ 1.

Пусть $f(x)\in C[0, 1]$, n и k — натуральные числа. Шенберг [1] ввел метод приближения сплайн-функциями $S_{n,k}(f)$, являющимися обобщением полиномов Бернштейна [2]. В 1977 г. Мюллер [3] ввел интегральные сплайны Шенберга $T_{n,k}(f)$, являющиеся обобщением полиномов Канторовича [4], и изучил порядок L^p приближения при помощи $T_{n,k}(f)$.

В статьях [5], [6] автор ввол понятие обобщенных полиномов Канторовича. В этой статье вводится понятие общего оператора A типа Канторовича (обозначим его через $A\in K$), обобщающего $T_{n,k}(f)$. Посредством анализа ядер интегральниых операторов и тщательных вычислений, будет доказано одно интересное равенство (Теорема 1). Найдены оценки приближений классов функций W^1L и W^1L^p $(1<p<\infty)$ положительными линейными операторами $A\in K$ (Теоремы 2, 3). Для фиксированного k, дается обобщение теорем Мюллера [3] о порядке L^p приближения при помощи $T_{n,k}(f)$ (Теорема 4). Для фиксированного n, обобщается две теоремы Грундмана [7] и Мюллера [8] о полиномах Канторовича (Теорема 6).

§ 2.

Пусть $f(t)$ есть непрерывная функция на $[a, b]$, то есть $f\in C[a, b]$, $\|f\|_C=\max\limits_{a\leqq x\leqq b}|f(x)|$. Пусть $|f(t)|^p$ $(1\leqq p<\infty)$ интегрируема в смысле Лебега $(f\in L^p[a, b])$,

$$\|f\|_p=\|f\|_{L^p}=\left\{\int_a^b |f(t)|^p\,dt\right\}^{\frac{1}{p}},\quad L[a, b]\overset{\text{def}}{=}L^1[a, b],\ L^\infty[a, b]\overset{\text{def}}{=}C[a, b],$$

$$\|f\|_\infty=\|f\|_{L^\infty}=\|f\|_C \quad [9].$$

Следующие определения даны в [3]:
Пусть Δ_n: $0=x_0<x_1\ldots<x_n=1$ есть конечное разбиение $[0, 1]$,

$$x_{-k}=\ldots=x_{-1}=0,\quad \text{и}\quad x_{n+1}=\ldots=x_{n+k}=1,$$

$$\Delta_{n,k}\overset{\text{def}}{=}\{x_i\}_{i=-k}^{n+k}.\quad |\Delta_n|\overset{\text{def}}{=}\max_i(x_{i+1}-x_i).$$

Пусть

$$M(x, t)=(k+1)(t-x)_+^k=\begin{cases}(k+1)(t-x)^k, & t\geqq x\\ 0 & t<x,\end{cases}$$

$$M_{j,k}(x)\overset{\text{def}}{=}M(x, x_j, x_{j+1}, \ldots, x_{j+k+1})$$

есть B-сплайн, то есть $M_{j,k}(x)$ есть $(k+1)$-е разностное отношение (как функция от t, для фиксированного x) $M_{j,k}(x)$ есть сплайн-функция степени k. Далее

$$N_{j,k}(x) \stackrel{\text{def}}{=\!=} \frac{x_{j+k+1}-x_j}{k+1} \cdot M_{j,k}(x), \tag{2.1}$$

где $N_{j,k}(x)$ — нормированный B-сплайн [3].

Пусть $F(u) \in C[0, 1]$,

$$S_{n,k}(F(u), x) \stackrel{\text{def}}{=\!=} \sum_{j=-k}^{n-1} F(\xi_{j,k})\, N_{j,k}(x), \tag{2.2}$$

где

$$\xi_{j,k} = \frac{x_{j+1}+x_{j+2}+\ldots+x_{j+k}}{k} \quad (-k \leqq j \leqq n-1), \tag{2.3}$$

$$0 = \xi_{-k,k} < \xi_{-k+1,k} < \ldots < \xi_{n-1,k} = 1, \tag{2.4}$$

$S_{n,k}(F(u), x)$ есть сплайн-функция степени k ([1], [2]).

Известно [2], [3], [10—12], что

$$N_{j,k}(x) \geqq 0, \quad 0 \leqq x \leqq 1. \tag{2.5}$$

Пусть supp $N_{j,k}(x)$ есть замыкание множества $E\{x|\, N_{j,k}(x) \neq 0\}$, то есть

$$\operatorname{supp} N_{j,k}(x) = [x_j, x_{j+k+1}], \tag{2.6}$$

$$S_{n,k}(1, x) \equiv \sum_{j=-k}^{n-1} N_{j,k}(x) \equiv 1, \tag{2.7}$$

$$S_{n,k}(u, x) \equiv \sum_{j=-k}^{n-1} \xi_{j,k} N_{j,k}(x) \equiv x, \tag{2.8}$$

$$\int_0^1 M_{j,k}(x)\, dx = 1. \tag{2.9}$$

Обозначим через $S_{n,1}(F)$ линейное интерполирование [3], $S_{1,k}(F)$ есть полином Бернштейна $B_k(F)$ степени k [3], [4].

Пусть $f(t) \in L[0, 1]$, $F(u) = \int_0^u f(t)\, dt$, интегральные сплайны Шенберга [3] определяются посредством

$$T_{n,k}(f(t), x) \stackrel{\text{def}}{=\!=} \frac{d}{dx} S_{n,k+1}(F(u), x) = \sum_{j=-k}^{n-1} \frac{N_{j,k}(x)}{\xi_{j,k+1}-\xi_{j-1,k+1}} \int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}} f(t)\, dt, \tag{2.10}$$

$T_{n,k}(f)$ есть сплайн-функция степени k.

Из (2.1) и

$$\xi_{j,k+1}-\xi_{j-1,k+1} = \frac{x_{j+k+1}-x_j}{k+1}$$

следует, что

$$T_{n,k}(f(t), x) = \sum_{j=-k}^{n-1} M_{j,k}(x) \int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}} f(t)\,dt, \quad 0 \leqq x \leqq 1, \tag{2.11}$$

где $T_{1,k}(f) = P_k(f)$ есть полином Канторовича степени k [3], [4].

§ 3.

Пусть $1 \leqq p \leqq \infty$, $L_k^p[a, b] \stackrel{\text{def}}{=} \{f: f^{(k-1)}$ абсолютно непрерывна на $[a, b]$, и $f^{(k)} \in L^p[a, b]\}$, $C_k[a, b] \stackrel{\text{def}}{=} L_k^\infty[a, b]$, $L_k[a, b] \stackrel{\text{def}}{=} L_k^1[a, b]$, $W^k ML^p \stackrel{\text{def}}{=} \{f: f \in L_k^p[a, b]$ и $\|f^{(k)}\|_p \leqq M\}$, $W^k MC \stackrel{\text{def}}{=} W^k ML^\infty$, $W^k L^p \stackrel{\text{def}}{=} W^k 1L^p$, $W^k C \stackrel{\text{def}}{=} W^k L^\infty$.

Пусть $B[a, b]$ есть множество ограниченных измеримых функций на $[a, b]$, $\|f\|_B = \sup_{a \leqq x \leqq b} |f(t)|$. Пусть H есть линейный ограниченный оператор $L^p[a, b] \Rightarrow L^p[a, b]$, обозначим норму оператора через $\|H\|_{L^p \to L^p}$, аналогично определим $\|H\|_{B \to C}$.

Если A есть линейный оператор $B[a, b] \Rightarrow C[a, b]$, то

$$\mathscr{E}(W^1 C; x) \stackrel{\text{def}}{=} \sup_{f \in W^1 C} \left|A(f(t), x) - f(x) \cdot A(1, x)\right|.$$

Пусть $\theta(t) = 1$ $(t \geqq 0)$, $\theta(t) = 0$ $(t < 0)$.

Лемма 1. *Если A есть положительный линейный оператор $B[a, b] \Rightarrow C[a, b]$* [13], [14]
то

$$\mathscr{E}(W^1 C; x) = \int_a^b |D_1(x, z)|\,dz = A(|t-x|, x), \tag{3.1}$$

где

$$D_1(x, z) = A(\theta(t-z), x) - A(1, x) \cdot \theta(x-z). \tag{3.2}$$

Доказательство. Пусть $f(t) \in L_1[a, b]$, так как

$$\int_a^b f'(z)\theta(t-z)\,dz = \int_a^t f'(z)dz = f(t) - f(a),$$

то

$$f(t) = \int_a^b f'(z)\theta(t-z)\,dz + f(a). \tag{3.3}$$

Если A есть положительный линейный оператор на $B[a, b]$, то для $f(t) \in B[a, b]$ имеем

$$-\|f(t)\|_B \leqq f(t) \leqq \|f(t)\|_B,$$

$$-\|f(t)\|_B \cdot A(1, x) \leqq A(f(t), x) \leqq \|f(t)\|_B \cdot A(1, x),$$

$$\|A(f(t), x)\|_C \leqq \|A(1, x)\|_C \cdot \|f(t)\|_B.$$

Поэтому $\|A\|_{C\to C}\leqq\|A\|_{B\to C}\leqq\|A(1,x)\|_C$. Так как $\|A(1,x)\|_C=\dfrac{\|A(1,x)\|_C}{\|1\|_C}\leqq\|A\|_{C\to C}$, то

$$\|A\|_{C\to C}=\|A\|_{B\to C}=\|A(1,x)\|_C. \tag{3.4}$$

Фиксируем $x\in[a,b]$, $A(f(t),x)$ есть линейный непрерывный функционал, поэтому [9]

$$A(f(t),x)=\int_a^b f'(z)A(\theta(t-z),x)\,dz+f(a)A(1,x). \tag{3.5}$$

Из (3.3) следует, что

$$f(x)A(1,x)=\int_a^b f'(z)\theta(x-z)A(1,x)\,dz+f(a)A(1,x), \tag{3.6}$$

поэтому

$$A(f(t),x)-f(x)\cdot A(1,x)=\int_a^b f'(z)D_1(x,z)\,dz, \tag{3.7}$$

$$\mathscr{E}(W^1C;x)=\int_a^b|D_1(x,z)|\,dz. \tag{3.8}$$

При $a\leqq z\leqq x$,

$$D_1(x,z)=A(\theta(t-z),x)-A(1,x)\theta(x-z)= \tag{3.9}$$

$$=A(\theta(t-z),x)-A(1,x)=A(\theta(t-z)-1,x)\leqq 0,$$

а при $x<z\leqq b$.

$$D_1(x,z)=A(\theta(t-z),x)-A(1,x)\cdot 0=A(\theta(t-z),x)\geqq 0. \tag{3.10}$$

Так как

$$\varphi(z)\overset{\text{def}}{=}\frac{d}{dz}(|z-x|)=\begin{cases}1, & x<z\leqq b\\ -1, & a\leqq z<x,\end{cases}$$

то $|z-x|\in L_1[a,b]$. Применяя (3.7), положим $f(t)=|t-x|$. Тогда

$$A(|t-x|,x)=-\int_a^x D_1(x,z)\,dz+\int_x^b D_1(x,z)\,dz=\int_a^b|D_1(x,z)|\,dz,$$

из (3.8) следует, что

$$\mathscr{E}(W^1C;x)=\int_a^b|D_1(x,z)|\,dz=A(|t-x|,x).$$

§ 4.

Пусть $B(F(u),x)$ есть линейный оператор $C[a,b]\Rightarrow C_1[a,b]$. Для $f(t)\in L[a,b]$, положим $F(u)=\int_a^u f(t)\,dt$, $A(f(t),x)\overset{\text{def}}{=}\dfrac{d}{dx}B(F(u),x)$. Предположим, что выполнены следующие граничные условия: Для $g(u)\in C[a,b]$

$$B(g(u),a)=g(a),\quad B(g(u),b)=g(b). \tag{4.1}$$

В этом случае A называется оператором типа Канторовича (обозначим через $A\in K$), $A(f(t), x)$ есть линейный оператор $L[a, b]\Rightarrow C[a, b]$.

Теорема 1. *Пусть A есть оператор типа Канторовича и A есть положительный линейный оператор. Положим*

$$R(z)_L \overset{\text{def}}{=\!=} \int_a^b |D_1(x, z)|\, dx,$$

$$\varrho(z) \overset{\text{def}}{=\!=} B(|u-z|,\ z)+(z-a)\big(1-B(1, z)\big). \tag{4.2}$$

тогда

$$R(z)_L = \varrho(z). \tag{4.3}$$

Доказательство. Из (3.9), (3.10) следует, что

$$R(z)_L = \int_a^b |D_1(x, z)|\, dx = \int_a^z A\big(\theta(t-z), x\big)\, dx + \int_z^b [A(1, x)-A\big(\theta(t-z), x\big)]\, dx. \tag{4.4}$$

Так как

$$A\big(f(t), x\big) = \frac{d}{dx} B\big(F(u), x\big).$$

то

$$\int_z^b A(1, x)\, dx = B(u-a, b)-B(u-a, z). \tag{4.5}$$

Пусть $(u)_+ = \max(u, 0)$, легко проверить, что

$$\int_a^u \theta(t-z)\, dt = \max(u-z, 0) = (u-z)_+ . \tag{4.6}$$

Поэтому

$$A\big(\theta(t-z), x\big) = \frac{d}{dx} B\Big(\int_a^u \theta(t-z)\, dt, x\Big) = \frac{d}{dx} B\big((u-z)_+, x\big). \tag{4.7}$$

Из (4.4) следует, что

$$R(z)_L = B\big((u-z)_+, z\big)-B\big((u-z)_+, a\big)+B(u-a, b)-B(u-a, z)- \tag{4.8}$$
$$-B\big((u-z)_+, b\big)+B\big((u-z)_+, z\big).$$

Так как $a\leqq z\leqq b$, то применяя граничные свойства (4.1) имеем

$$B\big((u-z)_+, a\big) = (a-z)_+ = 0, \quad B\big((u-z)_+, b\big) = (b-z)_+ = b-z.$$

Так как $2(u-z)_+ = |u-z|+u-z$, то

$$R(z)_L = 2B\big((u-z)_+, z\big)+B(u-a, b)-B(u-a, z)-(b-z) =$$
$$= B(|u-z|, z)+B(u, z)-z\cdot B(1, z)+B(u, b)-a\cdot B(1, b)-B(u, z)+$$
$$+a\cdot B(1,\ z)-b+z.$$

Из (4.1) следует, что $B(u, b)=b$, $B(1, b)=1$, поэтому

$$R(z)_L = B(|u-z|, z)-z\cdot B(1, z)+b-a+a\cdot B(1, z)-b+z =$$
$$= B(|u-z|, z)+(z-a)\big(1-B(1, z)\big) = \varrho(z).$$

§ 5.

Лемма 2. *Пусть $K(x, t)$ на $a\leqq t\leqq b, a\leqq x\leqq b$ есть двумерная измеримая функция. Определим интегральный оператор H:*

$$f(x) = \int_a^b K(x, t)\varphi(t)\,dt \quad (f = H\varphi). \tag{5.1}$$

Для того, чтобы H был линейным ограниченным оператором $L[a, b]\Rightarrow L[a, b]$ необходимо и достаточно, чтобы

$$D = \operatorname{vrai}\sup_{a\leqq t\leqq b}\int_a^b |K(x, t)|\,dx < +\infty, \tag{5.2}$$

и норма оператора

$$\|H\|_{L\to L} = \sup_{\|\varphi\|_1\leqq 1}\|H\varphi\|_1 = D. \tag{5.3}$$

(См. [15], гл. 8, § 3. 3.16).

Теорема 2. *Пусть A есть оператор типа Канторовича и A есть положительный линейный оператор. Тогда*

$$\mathscr{E}(W^1L)_L = \sup_{f\in W^1L}\big\|A\big(f(t), x\big)-f(t)\cdot A(1, x)\big\|_1 = \operatorname{vrai}\sup_{a\leqq z\leqq b}\varrho(z). \tag{5.4}$$

Доказательство. Пусть $f\in L_1[a, b]$, $\varphi(z)=f'(z)$, то есть $\varphi(z)\in L[a, b]$. Из (3.7) следует, что

$$A\big(f(t), x\big)-f(x)\cdot A(1, x) = \int_a^b f'(z)D, (x, z)\,dz = \int_a^b \varphi(z)D_1(x, z)\,dz. \tag{5.5}$$

Применяя лемму 2 имеем

$$\mathscr{E}(W^1L)_L = \sup_{\|\varphi\|_1\leqq 1}\int_a^b\Big|\int_a^b \varphi(z)D_1(x, z)\,dz\Big|\,dx = \operatorname{vrai}\sup_{a\leqq z\leqq b}\int_a^b |D_1(x, z)|\,dx. \tag{5.6}$$

Из теоремы 1 следует, что

$$\mathscr{E}(W^1L)_L = \operatorname{vrai}\sup_{a\leqq z\leqq b} p(z).$$

Теорема 2 дает точное значение отклонения L приближения.

Лемма 3. *Пусть $K(x, t)$ на $a\leqq t\leqq b$, $a\leqq x\leqq b$ есть двумерная измеримая функция. Рассмотрим интегральный оператор H:*

$$f(x) = \int_a^b K(x, t)\varphi(t)\,dt \quad (f = H\varphi). \tag{5.7}$$

Если

$$\operatorname{vrai}\sup_{a\leqq x\leqq b}\int_a^b |K(x,t)|\,dt = M, \tag{5.8}$$

$$\operatorname{vrai}\sup_{a\leqq t\leqq b}\int_a^b |K(x,t)|\,dx = N, \tag{5.9}$$

то H есть линейный оператор $L^p[a,b]\Rightarrow L^p[a,b]$ $(1\leqq p<\infty)$ и

$$\|H\|_{L^p\to L^p}\leqq M^{1-\frac{1}{p}}\cdot N^{\frac{1}{p}}. \tag{5.10}$$

Доказательство. Лемма 3 при $p=1$ содержится в Лемме 2. Лемма 3 есть частный случай теоремы выпуклости Рисса—Торина (см. [17], том 2, стр. 95, или [18], стр. 54).

Теорема 3. *Пусть A есть оператор типа Канторовича и A есть положительный линейный оператор. Пусть далее*

$$\lambda = \max_{a\leqq x\leqq b} A(|t-x|, x), \tag{5.11}$$

$$\varrho = \operatorname{vrai}\sup_{a\leqq z\leqq b}\varrho(z), \tag{5.12}$$

и $1\leqq p\leqq\infty$. Тогда

$$\mathscr{E}(W^1L^p)_{L^p} \overset{\text{def}}{=\!=} \sup_{f\in W^1L^p}\left\|A\big(f(t),x\big)-f(x)\cdot A(1,x)\right\|_p \leqq \lambda^{1-\frac{1}{p}}\cdot\varrho^{\frac{1}{p}}. \tag{5.13}$$

Доказательство. Так как A есть положительный линейный оператор, очевидно, что $A(|t-x|, x)$ есть непрерывная функция от x [19]. Из леммы 1 следует, что

$$\mathscr{E}(W^1L^\infty)_{L^\infty} = \max_{a\leqq x\leqq b}\int_a^b |D_1(x,z)|\,dz = \max_{a\leqq x\leqq b} A(|t-x|, x) = \lambda, \tag{5.14}$$

поэтому (5.13) имеет место при $p=\infty$.

Пусть теперь $1\leqq p<\infty$, $f(z)\in L_1^p[a,b]\subseteq L_1[a,b]$, $\varphi(z)=f'(z)$, тогда $\varphi(z)\in L^p[a,b]\subseteq L[a,b]$, и равенство (5.5) выполнено.

Так как $A\in K$ есть положительный линейный оператор, применяя Теорему 1 имеем

$$\operatorname{vrai}\sup_{a\leqq z\leqq b}\int_a^b |D_1(x,z)|\,dx = \operatorname{vrai}\sup_{a\leqq z\leqq b}\varrho(z) = \varrho. \tag{5.15}$$

Из (5.5) и Леммы 3 следует

$$\mathscr{E}(W^1L^p)_{L^p} \overset{\text{def}}{=\!=} \sup_{f\in W^1L^p}\left\{\int_a^b A\big(f(t),x\big)-f(x)\cdot A(1,x)|^p\,dx\right\}^{\frac{1}{p}} = \tag{5.16}$$

$$= \sup_{\|\varphi\|_p\leqq 1}\left\{\int_a^b\left|\int_a^b \varphi(z)D_1(x,z)\,dz\right|^p dx\right\}^{\frac{1}{p}} \leqq \lambda^{1-\frac{1}{p}}\cdot\varrho^{\frac{1}{p}}.$$

§ 6.

Пусть $f\in L^p[a, b]$, $0\leqq\delta\leqq b-a$.

$$\omega_1(f, \delta)_{L^p} = \omega_1(f, \delta)_{L^p[a, b]} = \sup_{0\leqq t\leqq\delta}\left\{\int_a^{b-t} |f(x+t)-f(x)|^p\, dx\right\}^{\frac{1}{p}}$$

есть модуль непрерывности порядка 1.

Лемма 4. *Пусть* $1\leqq p\leqq\infty$, H *есть линейный оператор* $L^p[a, b]\Rightarrow L^p[a, b]$. *Для* $f\in L^p[a, b]$ *имеем*

$$\|Hf\|_p \leqq M_0\|f\|_p. \tag{6.1}$$

Пусть $0\leqq\varepsilon\leqq b-a$, $M_1\geqq 1$ *и для* $f\in L_1^p[a, b]$ *имеем*

$$\|Hf-f\|_p \leqq M_1\varepsilon\|f'\|_p. \tag{6.2}$$

Тогда для $f\in L^p[a, b]$

$$\|Hf-f\|_p \leqq M_2\omega_1(f, \varepsilon)_{L^p}, \tag{6.3}$$

где M_2 *зависит только от* M_0 *и* M_1.

Доказательство. Случай $\varepsilon>0$, $[a, b]=[-1, 1]$, рассмотрен в [20] (стр. 128). Применяя K-функциональный метод, можно аналогично доказать (6.3) для общего интервала $[a, b]$. Если $\varepsilon=0$, то для $f\in L_1^p[a, b]$ имеем

$$0\leqq\|Hf-f\|_p \leqq 0\|f'\|_p < \delta\|f'\|_p \quad (\delta>0).$$

Поэтому для $f\in L^p[a, b]$

$$0\leqq\|Hf-f\|_p \leqq M_2\omega_1(f, \delta)_{L^p},$$

полагая $\delta\to+0$ получим

$$0\leqq Hf-f\|_p \leqq 0 = M_2\omega_1(f, 0)_{L^p} = M_2\omega_1(f, \varepsilon)_{L^p},$$

т. е. (6.3) опять имеет место (мы имеем $\|Hf-f\|_p=0$, поэтому $Hf=f$ почти всюду).

Теорема 4. *Пусть* $1\leqq p\leqq\infty$, A *есть оператор типа Канторовича и* A *есть положительный линейный оператор,* $A(1, x)\equiv 1$. *Тогда для* $f\in L_1^p[a, b]$ *имеем*

$$\|A(f(t), x)-f(x)\|_p \leqq \lambda^{1-\frac{1}{p}}\varrho^{\frac{1}{p}}\|f'\|_p. \tag{6.4}$$

Далее пусть для $f\in L[a, b]$

$$\|A(f, x)\|_1 \leqq M_3\|f\|_1, \tag{6.5}$$

где M_3 *не зависит от* f. *Пусть*

$$0\leqq\lambda\leqq b-a, \quad 0\leqq\varrho\leqq b-a. \tag{6.6}$$

Тогда для $f\in L^p[a, b]$ *имеем*

$$\big\|A(f(t), x)-f(x)\big\|_p \leqq M_4\omega_1\big(f, \lambda^{1-\frac{1}{p}}\varrho^{\frac{1}{p}}\big)_{L^p}, \tag{6.7}$$

где M_4 *не зависит от* f, p.

Доказательство. Пусть $1 \leqq p \leqq \infty$, так как $A(1, x) \equiv 1$, то из Теоремы 3 получим

$$\sup_{f \in W^1 ML^p} \|A(f(t), x) - f(x)\|_p = M \sup_{f \in W^1 L^p} \|A(f(t), x) - f(x)\|_p \leqq M\lambda^{1-\frac{1}{p}} \varrho^{\frac{1}{p}}. \tag{6.8}$$

Пусть $f \in L_1^p[a, b]$, $M = \|f'\|_p$, тогда

$$\|A(f(t), x) - f(x)\|_p \leqq \lambda^{1-\frac{1}{p}} \varrho^{\frac{1}{p}} \|f'\|_p, \tag{6.9}$$

поэтому (6.4) имеет место.

Из (6.5) имеем

$$\|A\|_{L \to L} \leqq M_5, \tag{6.10}$$

где $M_5 = \max(M_3, 1)$. Так как A есть положительный линейный оператор на $C[a, b]$, из (3.4) следует

$$\|A\|_{C \to C} = \|A(1, x)\|_C = 1 \leqq M_5, \tag{6.11}$$

поэтому применяя теорему выпуклости Рисса—Торина (см. [17]) получаем

$$\|A\|_{L^p \to L^p} \leqq M_5. \tag{6.12}$$

То есть для $f \in L^p[a, b]$ имеем

$$\|A(f)\|_p \leqq M\|f\|_p. \tag{6.13}$$

Так как $0 \leqq \lambda^{1-\frac{1}{p}} \varrho^{\frac{1}{p}} \leqq (b-a)^{1-\frac{1}{p}+\frac{1}{p}} = b-a$, то из (6.13), (6.4) применяя Лемму 4 и полагая $\varepsilon = \lambda^{1-\frac{1}{p}} \varrho^{\frac{1}{p}} \geqq 0$, $M_1 = 1$, получим (6.7).

Замечание 1. В Теореме 1 при $B(1, z) = 1$, имеем $\varrho(z) = B(|u-z|, z)$. Далее пусть B есть положительный линейный оператор и фиксируем $z \in [a, b]$. Применяя одно неравенство из [13] (глава 1V, § 4) имеем

$$B(|u-z|, z) \leqq \sqrt{B((u-z)^2, z) B(1, z)} = \sqrt{B((u-z)^2, z)}.$$

Далее пусть $B(u, z) = z$, тогда

$$B((u-z)^2, z) = B(u^2, z) - 2zB(u, z) + z^2 B(1, z) = B(u^2, z) - z^2.$$

Поэтому

$$0 \leqq \varrho(z) \leqq \sqrt{B(u^2, z) - z^2}. \tag{6.14}$$

Теорема 4 ($p = 1$) есть обобщение результата из [16].

Лемма 5 (Марсден [2]). *Пусть* $E_{n,k}(x) = S_{n,k}(u^2, x) - x^2$, *тогда*

$$0 \leqq E_{n,k}(x) \leqq \min\left\{\frac{1}{2k}, \frac{k|\Delta_n|^2}{2}\right\}. \tag{6.15}$$

Лемма 6. *Имеет место неравенство*

$$0 \leqq \max_{0 \leqq x \leqq 1} T_{n,k}(|t-x|, x) \leqq (k+1)|\Delta_n|. \tag{6.16}$$

Доказательство. Из (2.11) следует

$$T_{n,k}(|t-x|, x) = \sum_{j=-k}^{n-1} M_{j,k}(x) \int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}} |t-x|\, dt.$$

В [3] доказано, что

(6.17) $$0 \leqq \max_{0 \leqq x \leqq 1} T_{n,k}(|t-x|, x) \leqq (k+1)|\Delta_n|.$$

Следствие 1. *Пусть* $1 \leqq p \leqq \infty$, *тогда для* $f \in L_1^p[0, 1]$ *имеем*

(6.18) $$\|T_{n,k}(f) - f\|_p \leqq (k+1)|\Delta_n| \|f'\|_p.$$

Доказательство. Для $g(u) \in C[0, 1]$ (см. [2], стр. 12)

$$S_{n,k}(g(u), 0) = g(0), \quad S_{n,k}(g(u), 1) = g(1).$$

Так как k есть натуральное число, то $S_{n,k+1}(F)$ есть сплайн-функция степени $k+1$. Поэтому $S_{n,k+1}(F) \in C_1[a, b]$, и из (2.10), (2.4), (2.5) получаем, что $T_{n,k}(f(t), x) = \dfrac{d}{dx} S_{n,k+1}(F(u), x)$ есть оператор типа Канторовича, и есть положительный оператор. Из леммы 6 имеем

(6.19) $$0 \leqq \lambda = \max_{0 \leqq x \leqq 1} T_{n,k}(|t-x|, x) \leqq (k+1)|\Delta_n|.$$

Согласно (2.2) и (2.5), $S_{n,k+1}(F)$ есть положительный линейный оператор. Из (2.7), (2.8) следует, что

(6.20) $$S_{n,k+1}(1, z) = 1, \quad S_{n,k+1}(u, z) = z.$$

Применяя Замечание 1 имеем $\varrho(z) = S_{n,k+1}(|u-z|, z)$, и

(6.21) $$0 \leqq \varrho = \max_{0 \leqq z \leqq 1} \varrho(z) \leqq \sqrt{\max_{0 \leqq z \leqq 1} (S_{n,k+1}(u^2, z) - z^2)}.$$

Из Леммы 5 следует, что

(6.22) $$0 \leqq \varrho \leqq \sqrt{\frac{k+1}{2}} |\Delta_n|^2 \leqq (k+1)|\Delta_n|.$$

Пусть $1 \leqq p \leqq \infty$, тогда

(6.23) $$0 \leqq \lambda^{1-\frac{1}{p}} \varrho^{\frac{1}{p}} \leqq (k+1)^{1-\frac{1}{p}} |\Delta_n|^{1-\frac{1}{p}} (k+1)^{\frac{1}{p}} |\Delta_n|^{\frac{1}{p}} = (k+1)|\Delta_n|.$$

Согласно (2.7) и (2.10)

(6.24) $$T_{n,k}(1, x) \equiv \sum_{j=-k}^{n-1} N_{j,k}(x) \equiv 1.$$

Применияя (6.4) в Теореме 4 получаем, что для $f \in L_1^p[0, 1]$

$$\|T_{n,k}(f) - f\|_p \leqq (k+1)|\Delta_n| \|f'\|_p.$$

Следствие 1 есть теорема Мюллера [3].

Следствие 2. *Пусть* $1 \leqq p \leqq \infty$. *Тогда для* $f \in L^p[0, 1]$

$$\|T_{n,k}(f) - f\|_p \leqq M_6 \omega_1(f, |\Delta_n|)_{L^p}, \tag{6.25}$$

где M_6 *не зависит от* f, n, p [3].

Доказательство. Так как

$$T_{n,k}(f(t), x) = \int_0^1 H_{n,k}(x, t) f(t)\, dt \quad [3], \tag{6.26}$$

где

$$H_{n,k}(x, t) = \sum_{j=-k}^{n-1} M_{j,k}(x) I_{j-1,k+1}(t), \tag{6.27}$$

$I_{j-1,k+1}(t)$ есть характеристическая функция множества $[\xi_{j-1,k+1}, \xi_{j,k+1}]$ на $[0, 1]$, то для $0 \leqq t \leqq 1$, $0 \leqq x \leqq 1$ имеем

$$H_{n,k}(x, t) \geqq 0 \quad [3]. \tag{6.28}$$

Из доказательства Следствия 1 мы знаем, что $T_{n,k}(f(t), x)$ есть оператор типа Канторовича, и есть положительный линейный оператор, согласно (6.24), $T_{n,k}(1, x) = 1$. По Лемме 2 и [3]

$$\|T_{n,k}\|_{L \to L} = \operatorname*{vrai\,sup}_{0 \leqq t \leqq 1} \int_0^1 |H_{n,k}(x, t)|\, dx = \operatorname*{vrai\,sup}_{0 \leqq t \leqq 1} \int_0^1 H_{n,k}(x, t)\, dx = 1. \tag{6.29}$$

Применяя Теорему 4 и оценки (6.19), (6.22), (6.23) получаем, что Следствие 2 имеет место.

Следствие 3. *Пусть* $1 \leqq p \leqq \infty$. *Тогда для* $f \in L^p[0, 1]$ *имеем*

$$\lim_{|\Delta_n| \to 0} \|T_{n,k}(f) - f\|_p = 0 \tag{6.30}$$

(k фиксированно).

Доказательство. Полагая $\lim_{\delta \to +0} \omega_1(f, \delta)_{L^p} = 0$, из Следствия 2 получим Следствие 3.

Следствия 2, 3 есть теоремы Мюллера [3].

§ 7.

Лемма 7. *Пусть* $c < d$, *тогда имеет место неравенство*

$$\int_c^d |t - \alpha|\, dt \leqq \max(|c - \alpha|, |d - \alpha|)(d - c). \tag{7.1}$$

Доказательство. Пусть $c \leqq \alpha \leqq d$, тогда

$$\int_c^d |t - \alpha|\, dt = \int_c^\alpha (\alpha - t)\, dt + \int_\alpha^d (t - \alpha)\, dt =$$

$$= \frac{(\alpha - c)^2}{2} + \frac{(d - \alpha)^2}{2} \leqq \left[\frac{\alpha - c}{2} + \frac{d - \alpha}{2}\right](d - c) \leqq$$

$$\leqq \max(|c - \alpha|, |d - \alpha|)(d - c).$$

Если $\alpha > d$, то

$$\int_c^d |t-\alpha|\,dt = \int_c^d (\alpha-t)\,dt \leqq (\alpha-c)\int_c^d dt = (\alpha-c)(d-c) \leqq$$

$$\leqq \max(|c-\alpha|, |d-\alpha|)\cdot(d-c),$$

если $\alpha < c$, аналогично можем доказать (7.1).

Лемма 8. *Имеет место неравенство*

$$0 \leqq \max_{0\leqq x\leqq 1} T_{n,k}(|t-x|, x) \leqq \frac{2}{\sqrt{k+1}}. \tag{7.2}$$

Доказательство. Рассмотрим $\xi_{j,k+1}-\xi_{j-1,k+1}$, $|\xi_{j,k+1}-\xi_{j,k}|$, $|\xi_{j,k}-\xi_{j-1,k+1}|$. Во-первых

$$0 \leqq x_i \leqq 1 \quad (i = -k, \ldots, n+k). \tag{7.3}$$

Далее x_i монотонно неубывают, и из (2.3) имеем

$$\xi_{j,k} = \frac{x_{j+1}+x_{j+2}+\ldots+x_{j+k}}{k} \quad (-k \leqq j \leqq n-1). \tag{7.4}$$

Поэтому

$$\xi_{j,k+1}-\xi_{j-1,k+1} = \frac{x_{j+k+1}-x_j}{k+1}, \quad \xi_{j,k+1}-\xi_{j-1,k+1} > 0, \tag{7.5}$$

$$\xi_{j,k+1}-\xi_{j,k} = \tag{7.6}$$

$$= \frac{1}{k(k+1)}\{k(x_{j+1}+x_{j+2}+\ldots+x_{j+k+1})-(k+1)(x_{j+1}+x_{j+2}+\ldots+x_{j+k})\} =$$

$$= \frac{1}{k(k+1)}\{kx_{j+k+1}-(x_{j+1}+x_{j+2}+\ldots+x_{j+k})\} \leqq \frac{kx_{j+k+1}}{k(k+1)} = \frac{x_{j+k+1}}{k+1} \leqq \frac{1}{k+1}.$$

Из (7.6) следует

$$\xi_{j,k+1}-\xi_{j,k} = \frac{1}{k(k+1)}\{(x_{j+k+1}-x_{j+1})+(x_{j+k+1}-x_{j+2})+\ldots+(x_{j+k+1}-x_{j+k})\} \geqq 0,$$

поэтому

$$|\xi_{j,k+1}-\xi_{j,k}| \leqq \frac{1}{k+1}. \tag{7.7}$$

$$\xi_{j,k}-\xi_{j-1,k+1} = \tag{7.8}$$

$$= \frac{1}{k(k+1)}\{(k+1)(x_{j+1}+x_{j+2}+\ldots+x_{j+k})-k(x_j+x_{j+1}+\ldots+x_{j+k})\} =$$

$$= \frac{1}{k(k+1)}\{(x_{j+1}+x_{j+2}+\ldots+x_{j+k})-kx_j\} \leqq$$

$$\leqq \frac{x_{j+1}+x_{j+2}+\ldots+x_{j+k}}{k(k+1)} \leqq \frac{k}{k(k+1)} = \frac{1}{k+1}.$$

Согласно (7.8)

$$\xi_{j,k}-\xi_{j-1,k+1}=\frac{1}{k(k+1)}\{(x_{j+1}-x_j)+(x_{j+2}-x_j)+\ldots+(x_{j+k}-x_j)\}\geqq 0,$$

поэтому

$$|\xi_{j,k}-\xi_{j-1,k+1}|\leqq\frac{1}{k+1}. \tag{7.9}$$

Из (7.5), (7.7), (7.9) и Леммы 7 следует

$$\int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}}|t-\xi_{j,k}|\,dt\leqq\max\left(|\xi_{j,k+1}-\xi_{j,k}|,\,|\xi_{j,k}-\xi_{j-1,k+1}|\right)\cdot(\xi_{j,k+1}-\xi_{j-1,k+1})\leqq \tag{7.10}$$

$$\leqq\frac{1}{k+1}(\xi_{j,k+1}-\xi_{j-1,k+1}).$$

Так как

$$T_{n,k}(|t-x|,x)=\sum_{j=-k}^{n-1}\frac{N_{j,k}(x)}{\xi_{j,k+1}-\xi_{j-1,k+1}}\int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}}|t-x|\,dt,$$

$$S_{n,k}(|u-x|,x)=\sum_{j=-k}^{n-1}|\xi_{j,k}-x|N_{j,k}(x),$$

то из (7.10), (2.5), (2.7) имеем, что для $x\in[0,1]$

$$|T_{n,k}(|t-x|,x)-S_{n,k}(|u-x|,x)|\leqq$$

$$\leqq\sum_{j=-k}^{n-1}\frac{N_{j,k}(x)}{\xi_{j,k+1}-\xi_{j-1,k+1}}\int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}}\big||t-x|-|\xi_{j,k}-x|\big|dt\leqq$$

$$\leqq\sum_{j=-k}^{n-1}\frac{N_{j,k}(x)}{\xi_{j,k+1}-\xi_{j-1,k+1}}\int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}}|t-\xi_{j,k}|\,dt\leqq\frac{1}{k+1}\sum_{j=-k}^{n-1}N_{j,k}(x)=\frac{1}{k+1}.$$

Поэтому

$$|T_{n,k}(|t-x)|,x)|\leqq S_{n,k}(|u-x|,x)+\frac{1}{k+1},\quad x\in[0,1]. \tag{7.11}$$

Из (2.2), (2.5) следует, что $S_{n,k}(F)$ есть положительный линейный оператор. Из (2.7) и (2.8) имеем $S_{n,k}(1,x)=1$, $S_{n,k}(u,x)=x$. Применяя Замечание 1 получаем

$$0\leqq S_{n,k}(|u-x|,x)\leqq\sqrt{S_{n,k}(u^2,x)-x^2}. \tag{7.12}$$

Из Леммы 5 следует $0\leqq S_{n,k}(u^2,x)-x^2\leqq\frac{1}{2k}$, поэтому

$$0\leqq S_{n,k}(|u-x|,x)\leqq\frac{1}{\sqrt{2k}},\quad x\in[0,1]. \tag{7.13}$$

Из (7.11) следует, что для $x\in[0, 1]$

$$0 \leqq T_{n,k}(|t-x|, x) \leqq \frac{1}{\sqrt{2k}}+\frac{1}{k+1} \leqq \frac{1}{\sqrt{k+1}}+\frac{1}{k+1} \leqq \frac{2}{\sqrt{k+1}}.$$

Поэтому

$$0 \leqq \max_{0\leqq x\leqq 1} T_{n,k}(|t-x|, x) \leqq \frac{2}{\sqrt{k+1}}.$$

Теорема 5. *Пусть* $1 \leqq p \leqq \infty$. *Тогда для* $f\in L_1^p[0, 1]$ *имеем*

$$\|T_{n,k}(f(t), x)-f(x)\|_p \leqq \frac{2}{\sqrt{k+1}}\|f'\|_p. \tag{7.14}$$

Доказательство. Из доказательства Следствия 1 мы знаем, что $T_{n,k}(f(t), x)=\frac{d}{dx} S_{n,k+1}(F(u), x)$ есть оператор типа Канторовича, и есть положительный линейный оператор. Из Леммы 8 следует

$$0 \leqq \lambda = \max_{0\leqq x\leqq 1} T_{n,k}(|t-x|, x) \leqq \frac{2}{\sqrt{k+1}}. \tag{7.15}$$

Согласно (7.13)

$$0 \leqq \varrho = \max_{0\leqq z\leqq 1} \varrho(z) = \max_{0\leqq z\leqq 1} S_{n,k+1}(|u-z|, z) \leqq \frac{1}{\sqrt{2(k+1)}} \leqq \frac{1}{\sqrt{k+1}}. \tag{7.16}$$

Пусть $1 \leqq p \leqq \infty$, тогда

$$0 \leqq \lambda^{1-\frac{1}{p}} \cdot \varrho^{\frac{1}{p}} \leqq \left(\frac{2}{\sqrt{k+1}}\right)^{1-\frac{1}{p}} \left(\frac{1}{\sqrt{k+1}}\right)^{\frac{1}{p}} \leqq \frac{2}{\sqrt{k+1}}. \tag{7.17}$$

Учитывая (6.24) и применяя (6.4) получаем, что (7.14) имеет место.

Теорема 6. *Пусть* $1 \leqq p \leqq \infty$. *Тогда для* $f\in L^p[0, 1]$ *имеем*

$$\|T_{n,k}(f)-f\|_p \leqq M_7 \omega_1\left(f, \frac{1}{\sqrt{k+1}}\right)_{L^p} \tag{7.18}$$

где M_7 *не зависит от* f, k, n, p.

Доказательство. Из доказательства Следствия 1 мы знаем, что $T_{n,k}(f(t), x)\in K$ есть положительный линейный оператор. Согласно (6.24) и (6.29), из (7.17) получаем $0 \leqq \lambda^{1-\frac{1}{p}} \varrho^{\frac{1}{p}} \leqq \frac{2}{\sqrt{k+1}}$. Тогда из Теоремы 4 следует, что Теорема 6 имеем место.

Следствие 4. *Пусть* $1 \leqq p \leqq \infty$. *Тогда для* $f\in L^p[0, 1]$ *имеем*

$$\lim_{k\to\infty} \|T_{n,k}(f)-f\|_p = 0 \tag{7.19}$$

(*n фиксированно*).

Доказательство. Так как $\lim_{\delta \to +0} \omega_1(f, \delta)_{L^p}=0$, то из Теоремы 6 получим Следствие 4.

Полагая в Теореме 6 $n=1$ и учитывая, что $T_{1,k}(f)=P_k(f)$, получим теорему Грундмана [7] ($p=1$) и теорему Мюллера [8] ($1<p\leqq\infty$). (О следствии 4 см. [3].)

Литература

[1] I. J. Schoenberg, On variation diminishing approximation, in *On numerical approximation* MRC, Symposium (R. E. Langer, ed.), Univ of Wisconsin Press (Madison, 1959), 249—274.

[2] M. J. Marsden, An identity for spline functions with application to variation-diminishing spline approximation, *J. Approx. Th.*, **3** (1970), 7—49.

[3] M. W. Müller, Degree of L_p-approximation by integral Schoenberg splines, *J. Approx. Th.*, **21** (1977), 385—393.

[4] G. G. Lorentz, *Bernstein polynomials* (Toronto, 1953).

[5] Цао Цзя-дин, О обобщенных полиномах Л. В. Канторовича и их асимптотических поведениях, *A Monthly J. of Science*, **25** (1980) 621 (in Russian).

[6] Cao Jia-ding, On generalized polynomials of L. V. Kantorovitch and their asymptotic behaviour, *Chinese Annals Math.*, **2** (1981), 243—256 (Chinese).

[7] A. Grundmann, Güteabschätzungen für den Kantorovič-Operator in der L_1-Norm, *Math. Zeit.*, **150** (1976), 45—47.

[8] M. W. Müller, Die Güte der L_p-Approximation durch Kantorovič-Polynome, *Math Zeit.*, **151** (1976), 243—247.

[9] Цао Цзя-дин, О линейных методах приближения (К-и-Т), *Acta Sc. Nat. Univ. Fudan*, **9** (1964), 43-52; О линейных методах приближения (II) (К-и-т), *Acta Sc. Nat. Univ. Fudan*, **10** (1965), 19-32.

[10] H. B. Curry, I. J. Schoenberg, On Pólya frequency functions, IV. The fundamental spline functions and their limits, *J. Analyse Math.*, **17** (1966), 71—107.

[11] C. de Boor, On calculating with B-spline, *J. Approx. Th.*, **6** (1972), 50—62.

[12] I. J. Schoenberg, On spline functions, in *Inequalities*, Symposium at Wright—Patterson, Air Force Base (O. Shisha, ed.), Academic Press (New York, 1967), 255—291.

[13] П. П. Коровкин, *Линейные операторы и теория приближений* (Москва, 1959).

[14] R. A. DeVore, *The Approximation of Continuous Functions by Positive Linear Operators*, Springer-Verlag (1972).

[15] Л. В. Канторович, Б. З. Вулих, А. Г. Пинскер, *Функциональный анализ в полуупорядоченных пространствах*, Гостехиздат (1950).

[16] Cao Jia-ding, Approximation of non-periodic functions in L_p space by the singular integrals, *Acta Math. Sinica*, **24** (1981), 9—25 (Chinese).

[17] A. Zygmund, *Trigonometric Series* (Cambridge, 1959).

[18] С. Г. Крейн, *Функциональный анализ*, Справочная математическая библиотека (Москва, 1964).

[19] Р. Г. Мамедов, *Доклады АН СССР*, **128** (1959), 674.

[20] R. A. DeVore, Degree of Approximation, in *Approximation Theory II* (G. G. Lorentz, C. K. Chui, L. L. Schumaker, eds.), Academic Press, INC (1976), 117—161.

(Поступило 28. 09. 1981.)

МАТЕМАТИЧЕСКИЙ ФАКУЛЬТЕТ
УНИВЕРСИТЕТ ФУДАН
ШАНХАЙ, КНР

Acta Math. Hung.
42 (3—4) (1983), 205—212.

LEBESGUE FUNCTIONS AND MULTIPLE FUNCTION SERIES. III

F. MÓRICZ (Szeged)*

1.

This paper is a continuation of paper [4] having the same title. Some notations occurring there will be used here without any further explanation. Now our main intention is to extend some recent results of Tandori [5, 6] from single series to multiple ones.

Let d be a fixed positive integer and consider the d-multiple series

$$\sum_{k\in Z_+^d} a_k\varphi_k(x) = \sum_{k_1=1}^{\infty}\dots\sum_{k_d=1}^{\infty} a_{k_1,\dots,k_d}\varphi_{k_1,\dots,k_d}(x), \tag{1}$$

where $Z_d^+ = \{k=(k_1, \dots, k_d): k_j=1, 2, \dots \text{ for each } j=1, \dots, d\}$, $\varphi=\{\varphi_k(x): k\in Z_d^+\}\subset$ $\subset L^1=L^1(X, \mathscr{A}, \mu)$, X is a finite measure space, and $\{a_k: k\in Z_+^d\}$ is a d-multiple sequence of real numbers (coefficients).

The following variants of the so-called Lebesgue functions, wellknown in the theory of single orthogonal series (see, e.g. [1, Ch. III]), were introduced for multiple series in [4]. Given a d-tuple $\varepsilon=(\varepsilon_1, \dots, \varepsilon_d)$ with coordinates $\varepsilon_j=0$ or 1, let

$$L_m^{\varepsilon}(x)=\int_X \Big(\max_{1\leqq p_{j_1}\leqq m_{j_1}}\dots\max_{1\leqq p_{j_e}\leqq m_{j_e}} |K_{\varepsilon p+(1-\varepsilon)m}(x, y)|\Big)\,d\mu(y),$$

where $m=(m_1, \dots, m_d)$, $p=(p_1, \dots, p_d)$, $\varepsilon_j=1$ if and only if $j=j_1, \dots, j_e$, $0\leqq e\leqq d$, and

$$K_p(x, y) = \sum_{k_1=1}^{p_1}\dots\sum_{k_d=1}^{p_d} \varphi_k(x)\,\varphi_k(y).$$

It is clear that for every ε we have

$$L_m^{\varepsilon}(x)\leqq L_m^*(x) = \int_X \Big(\sum_{k_1=1}^{m_1}\dots\sum_{k_d=1}^{m_d} |\varphi_k(x)\,\varphi_k(y)|\Big)\,d\mu(y) \quad (m\in Z_+^d;\ x\in X).$$

2.

The author has proved the following

THEOREM A ([4, Theorems 2 and 3]). *Assume that* $\varphi\subset L^1$ *is such that*

$$L_m^{\varepsilon}(x)\leqq C \quad (m\in Z_+^d;\ x\in X) \tag{2}$$

for all those ε *that contain at most* $d-1$ *coordinates equal to* 1, *where* C *is*

* This research was conducted while the author was on leave from Szeged University and a Visiting Professor at Ulm University, West Germany.

a positive constant. If

$$\sum_{k\in Z_+^d} a_k^2 < \infty, \tag{3}$$

then series (1) *regularly converges a.e.*

Condition (2) is obviously implied by

$$L_m^*(x) \leqq C \quad (m\in Z_+^d;\ x\in X). \tag{4}$$

First we show that condition (3) in Theorem A cannot be weakened even in the orthonormal case. The following theorem is an extension of a theorem of Tandori [5] from $d=1$ to $d\geqq 2$.

THEOREM 1. (i) *If*

$$\sum_{k\in Z_+^d} a_k^2 = \infty, \tag{5}$$

then there exists an orthonormal system φ *on the interval* $X=(0, 1)$ *such that* (4) *is satisfied, but*

$$\limsup_{\max(m_1,\ldots,m_d)\to\infty} \left|\sum_{k_1=1}^{m_1}\ldots\sum_{k_d=1}^{m_d} a_k\varphi_k(x)\right| = \infty \quad (0<x<1). \tag{6}$$

(ii) *If for every* $m\in Z_+^d$ *we also have*

$$\sum_{k\in Z_+^d \,\&\, k\leqq m} a_k^2 = \infty,$$

then there exists an orthonormal system φ *on* $(0, 1)$ *such that* (4) *is satisfied and*

$$\limsup_{\min(m_1,\ldots,m_d)\to\infty} \left|\sum_{k_1=1}^{m_1}\ldots\sum_{k_d=1}^{m_d} a_k\varphi_k(x)\right| = \infty \quad (0<x<1). \tag{7}$$

It is clear that for $d=1$ statements (i) and (ii) coincide. In the cases $d\geqq 2$, statement (6) implies that series (1) does not converge regularly at any point x, while (7) implies that series (1) does not converge even in Pringsheim's sense.

PROOF OF THEOREM 1. Since the proof is similar to that of Tandori's theorem mentioned above we sketch it only in the case $d=2$. We may write (k, l) instead of (k_1, k_2) and assume that $0\leqq a_k\leqq 1$.

(i) Regarding to (5) we can choose the squares $R_i=\{(k, l)\in Z_+^2:\ k\leqq r_i$ and $l\leqq r_i\}$ so large that

$$A_i^2 = \sum_{(k,l)\in R_i\setminus R_{i-1}} a_{kl}^2 \geqq 2^{2i} \quad (i=1, 2, \ldots;\ R_0=\varnothing).$$

Let $P_i=\{(k, l)\in R_i\setminus R_{i-1}: a_{kl}=0\}$. For a given i, with the lattice points $(k, l)\in(R_i\setminus R_{i-1})\setminus P_i$ we can associate two systems of disjoint intervals $I_{kl}\subset[2^{-2i}, 1)$ and $J_{kl}\subset[2^{-2i-1}, 2^{-2i})$ having length proportional to a_{kl}^2/A_i^2 such that

$$\bigcup_{(k,l)\in(R_i\setminus R_{i-1})\setminus P_i} I_{kl} = \left[\frac{1}{2^{2i}}, 1\right)$$

and

$$\bigcup_{(k,l)\in(R_i\setminus R_{i-1})\setminus P_i} J_{kl} = \left[\frac{1}{2^{2i+1}}, \frac{1}{2^{2i}}\right).$$

Further, set for $(k, l)\in(R_i\setminus R_{i-1})\setminus P_i$

$$|\varphi_{kl}(x)| = \begin{cases} \dfrac{A_i}{2^{i/2}a_{kl}}, & \text{if } x\in I_{kl}, \\ \left[\left(1-\dfrac{1}{2^{2i}}+\dfrac{1}{2^{3i}}\right)/|J_{kl}|\right]^{1/2}, & \text{if } x\in J_{kl}, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Let p_i be the number of the lattice points contained in P_i. In case $p_i>0$ divide $[2^{-2i-2}, 2^{-2i-1})$ into p_i subintervals of equal length, say H_{kl}, where (k, l) runs over P_i. For such (k, l) we define

$$|\varphi_{kl}(x)| = \begin{cases} [1/|H_{kl}|]^{1/2}, & \text{if } x\in H_{kl}, \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

The system $\varphi=\{\varphi_{kl}(x): (k, l)\in Z_+^2\}$ of functions, whose absolute values are just defined, is clearly normed. It is a routine to determine the signs of the values of the step functions $\varphi_{kl}(x)$, $(k, l)\in R_i\setminus R_{i-1}$, on the subintervals which are the halves of the constancy intervals of the preceding step functions $\varphi_{kl}(x)$, $(k, l)\in R_{i-1}$ — by recurrence with respect to i — in such a manner that φ becomes an orthogonal system of step functions. The verification of the fulfilment of (4) and (6) is also an easy exercise.

(ii) Now we can choose the squares $Q_i=\{(k, l)\in Z_+^2: r_{i-1}<k,\ l\leqq r_i\}$ in such a way that

$$\tilde{A}_i^2 = \sum_{(k,l)\in Q_i} a_{kl}^2 \geqq 2^{2i} \quad (i = 1, 2, \ldots).$$

We define the step functions $\varphi_{kl}(x)$ for $(k, l)\in Q_i\setminus P_i$ by (8) using $\tilde{A}_i$ instead of A_i, while for $(k, l)\in(R_i\setminus R_{i-1})\setminus(Q_i\setminus P_i)$ by (9). We do not enter into further details.

3.

The extension of [1, Theorem 3.1.2] was proved only under restrictive conditions.

THEOREM B ([4, Theorem 4]). *Assume that*

$$(X, \mathcal{A}, \mu) = \bigtimes_{j=1}^{d} (X_j, \mathcal{A}_j, \mu_j)$$

is a product space,

$$\varphi_k(x) = \sum_{j=1}^{d} \varphi_{k_j}^{(j)}(x_j)$$

is a product system such that each $\{\varphi_{k_j}^{(j)}(x_j): k_j=1, 2, \ldots\}$ *is orthonormal on* $(X_j, \mathcal{A}_j, \mu_j)$ *and satisfies the condition*

$$L_{m_j}^{(j)*}(x_j) = \int_{X_j} \left(\max_{1 \leqq p_j \leqq m_j} \left| \sum_{k_j=1}^{p_j} \varphi_{k_j}^{(j)}(x_j) \varphi_{k_j}^{(j)}(y_j) \right| \right) d\mu(y_j) \leqq C_j \lambda_{m_j}^{(j)}$$

$$(m_j = 1, 2, \ldots; x_j \in X_j),$$

where each $\{\lambda_{m_j}^{(j)}: m_j=1, 2, \ldots\}$ *is a nondecreasing sequence of positive numbers, and*

$$\lambda_m = \prod_{j=1}^{d} \lambda_{m_j}^{(j)}. \tag{10}$$

If

$$\sum_{k \in Z_+^d} a_k^2 \lambda_k < \infty,$$

then series (1) *regularly converges a.e.*

It is an unsolved problem that the assumption of orthonormality or the factorizable property of φ is essential in Theorem B or not. From now on we assume only the fulfilment of (10) with the supplementary condition

$$1 \leqq \lambda_{m_j}^{(j)} \uparrow \infty \quad (m_j \to \infty). \tag{11}$$

Following Tandori [6], for each $j=1, \ldots, d$ define a partition of the natural numbers: $Z_j(l)=\{m_j: 2^l \leqq \lambda_{m_j}^{(j)} < 2^{l+1}\}$ $(l=0, 1, \ldots)$. Let $l_1^{(j)} < \ldots < l_p^{(j)} < \ldots$ be those numbers for which $Z_j(l)$ is not void and denote by $v_j(p)+1, v_j(p)+2, \ldots, v_j(p+1)$ the elements of $Z_j(l_p^{(j)})$ in their consecutive order.

THEOREM 2. *Assume that* (10) *and* (11) *are satisfied. If*

$$L_m^{\varepsilon}(x) \leqq C\lambda_m \quad (m \in Z_+^d;\ x \in X) \tag{12}$$

for all those ε *that contain at most* $[d/2]$ *coordinates equal to* 1, *and if*

$$\sum_{p_1=1}^{\infty} \ldots \sum_{p_d=1}^{\infty} \Lambda_{p_1, \ldots, p_d} < \infty, \tag{13}$$

where

$$\Lambda_{p_1, \ldots, p_d}^2 = \sum_{k_1=v_1(p_1)+1}^{v_1(p_1+1)} \ldots \sum_{k_d=v_d(p_d)+1}^{v_d(p_d+1)} a_{k_1, \ldots, k_d}^2 \lambda_{k_1, \ldots, k_d},$$

then series (1) *regularly converges a.e.*

PROOF. In order to make the substance of the proof clearer we present it again for $d=2$. Let

$$\delta_{pq}(x) = \max_{v_1(p) < m \leqq v_1(p+1)} \max_{v_2(q) < n \leqq v_2(q+1)} \left| \sum_{k=v_1(p)+1}^{m} \sum_{l=v_2(q)+1}^{n} a_{kl} \varphi_{kl}(x) \right|$$

and

$$E_{pq}^{(i)} = \{x \in X: \delta_{pq}(x) = \max_{\nu_1(p)<m\leqq\nu_1(p+1)} \max_{\nu_2(q)<n\leqq\nu_2(q+1)} (-1)^i \sum_{k=\nu_1(p)+1}^{m} \sum_{l=\nu_2(q)+1}^{n} a_{kl}\varphi_{kl}(x)\}$$

$$(i = 0 \quad \text{or} \quad 1; \quad p, q = 1, 2, \ldots).$$

For $x \in E_{pq}^{(0)}$ let $(m(x), n(x))$ be determined by the condition that $\sum_{k=\nu_1(p)+1}^{m(x)} \sum_{l=\nu_2(q)+1}^{n(x)} a_{kl}\varphi_{kl}(x)$ equals the double maximum occurring in the definition of $E_{pq}^{(0)}$. If there exist more than one such pairs $(m(x), n(x))$, then choose those of them for which $m(x)$ is the smallest integer, and then choose that of the selected pairs for which $n(x)$ is minimal. Imitating wellknown techniques (see, e.g. [6, Satz I]) we can obtain that

(14)
$$\int_{E_{pq}^{(0)}} \delta_{pq}(x)\, d\mu(x) \leqq$$

$$\leqq \Lambda_{pq} \left\{ \int_X \int_X \left| \sum_{k=\nu_1(p)+1}^{\min(m(x),m(y))} \sum_{l=\nu_2(q)+1}^{\min(n(x),n(y))} \frac{\varphi_{kl}(x)\varphi_{kl}(y)}{\lambda_k^{(1)}\lambda_l^{(2)}} \right| d\mu(x)\, d\mu(y) \right\}^{1/2}.$$

The double integral on the right hand side of (14) can be estimated from above only in the trivial way, i.e. it does not exceed the following quantity:

$$\int_X \int_X \left\{ \left| \sum_{k=\nu_1(p)+1}^{m(x)} \sum_{l=\nu_2(q)+1}^{n(x)} \frac{\varphi_{kl}(x)\varphi_{kl}(y)}{\lambda_k^{(1)}\lambda_l^{(2)}} \right| + \left| \sum_{k=\nu_1(p)+1}^{m(x)} \sum_{l=\nu_2(q)+1}^{n(y)} \cdots \right| + \right.$$

$$\left. + \left| \sum_{k=\nu_1(p)+1}^{m(y)} \sum_{l=\nu_2(q)+1}^{n(x)} \cdots \right| + \left| \sum_{k=\nu_1(p)+1}^{m(y)} \sum_{l=\nu_2(q)+1}^{n(y)} \cdots \right| \right\} d\mu(x)\, d\mu(y).$$

Let us deal, for example, with the second sum. We begin by performing a double Abel transformation (see [2] and also [3]):

$$\sum_{k=\nu_1(p)+1}^{m(x)} \sum_{l=\nu_2(q)+1}^{n(y)} \frac{\varphi_{kl}(x)\varphi_{kl}(y)}{\lambda_k^{(1)}\lambda_l^{(2)}} =$$

$$= \sum_{r=\nu_1(p)+1}^{m(x)-1} \sum_{s=\nu_2(q)+1}^{n(y)-1} \left(\Delta_{11} \frac{1}{\lambda_r^{(1)}\lambda_s^{(2)}} \right) \left(\sum_{k=\nu_1(p)+1}^{r} \sum_{l=\nu_2(q)+1}^{s} \varphi_{kl}(x)\varphi_{kl}(y) \right) -$$

$$- \sum_{r=\nu_1(p)+1}^{m(x)-1} \left(\Delta_{10} \frac{1}{\lambda_r^{(1)}\lambda_{n(y)}^{(2)}} \right) \left(\sum_{k=\nu_1(p)+1}^{r} \sum_{l=\nu_2(q)+1}^{n(y)} \varphi_{kl}(x)\varphi_{kl}(y) \right) -$$

$$- \sum_{s=\nu_2(q)+1}^{n(y)-1} \left(\Delta_{01} \frac{1}{\lambda_{m(x)}^{(1)}\lambda_s^{(2)}} \right) \left(\sum_{k=\nu_1(p)+1}^{m(x)} \sum_{l=\nu_2(q)+1}^{s} \varphi_{kl}(x)\varphi_{kl}(y) \right) +$$

$$+ \frac{1}{\lambda_{m(x)}^{(1)}\lambda_{n(y)}^{(2)}} \left(\sum_{k=\nu_1(p)+1}^{m(x)} \sum_{l=\nu_2(q)+1}^{n(y)} \varphi_{kl}(x)\varphi_{kl}(y) \right),$$

where

$$\Delta_{11}\frac{1}{\lambda_r^{(1)}\lambda_s^{(2)}}=\left(\frac{1}{\lambda_r^{(1)}}-\frac{1}{\lambda_{r+1}^{(1)}}\right)\left(\frac{1}{\lambda_s^{(2)}}-\frac{1}{\lambda_{s+1}^{(2)}}\right),$$

$$\Delta_{10}\frac{1}{\lambda_r^{(1)}\lambda_{n(y)}^{(2)}}=\left(\frac{1}{\lambda_r^{(1)}}-\frac{1}{\lambda_{r+1}^{(1)}}\right)\frac{1}{\lambda_{n(y)}^{(2)}}\quad\text{etc.}$$

Hence it follows immediately that

$$\text{(15)}\qquad \left|\sum_{k=\nu_1(p)+1}^{m(x)}\sum_{l=\nu_2(q)+1}^{n(y)}\frac{\varphi_{kl}(x)\varphi_{kl}(y)}{\lambda_k^{(1)}\lambda_l^{(2)}}\right|\leqq$$

$$\leqq\sum_{r=\nu_1(p)+1}^{\nu_1(p+1)-1}\sum_{s=\nu_2(q)+1}^{\nu_2(q+1)-1}\left(\Delta_{11}\frac{1}{\lambda_r^{(1)}\lambda_s^{(2)}}\right)\left|\sum_{k=\nu_1(p)+1}^{r}\sum_{l=\nu_2(q)+1}^{s}\varphi_{kl}(x)\varphi_{kl}(y)\right|+$$

$$+\sum_{r=\nu_1(p)+1}^{\nu_1(p+1)-1}\frac{1}{\lambda_{\nu_2(q)+1}^{(2)}}\left(\Delta_{10}\frac{1}{\lambda_r^{(1)}}\right)\left|\sum_{k=\nu_1(p)+1}^{q}\sum_{l=\nu_2(q)+1}^{n(y)}\varphi_{kl}(x)\varphi_{kl}(y)\right|+$$

$$+\sum_{s=\nu_2(q)+1}^{\nu_2(q+1)-1}\frac{1}{\lambda_{\nu_1(p)+1}^{(1)}}\left(\Delta_{01}\frac{1}{\lambda_s^{(2)}}\right)\left|\sum_{k=\nu_1(p)+1}^{m(x)}\sum_{l=\nu_2(q)+1}^{s}\varphi_{kl}(x)\varphi_{kl}(y)\right|+$$

$$+\frac{1}{\lambda_{\nu_1(p)+1}^{(1)}\lambda_{\nu_2(q)+1}^{(2)}}\left|\sum_{k=\nu_1(p)+1}^{m(x)}\sum_{l=\nu_2(q)+1}^{n(y)}\varphi_{kl}(x)\varphi_{kl}(y)\right|.$$

Taking into account that e.g.

$$\left|\sum_{k=\nu_1(p)+1}^{m(x)}\sum_{l=\nu_2(q)+1}^{n(y)}\varphi_{kl}(x)\varphi_{kl}(y)\right|\leqq|K_{m(x),n(y)}(x,y)|+$$

$$+|K_{m(x),\nu_2(q)}(x,y)|+|K_{\nu_1(p),n(y)}(x,y)|+|K_{\nu_1(p),\nu_2(q)}(x,y)|,$$

we find — applying an integration and using (12) — that

$$\int_X\int_X\left|\sum_{k=\nu_1(p)+1}^{m(x)}\sum_{l=\nu_2(q)+1}^{n(y)}\varphi_{kl}(x)\varphi_{kl}(y)\right|d\mu(x)\,d\mu(y)\leqq$$

$$\leqq\int_X L_{m(x),n(x)}^{(0,1)}(x)\,d\mu(x)+\int_X L_{m(x),\nu_2(q)}^{(0,0)}(x)\,d\mu(x)+$$

$$+\int_X L_{\nu_1(p),n(y)}^{(0,0)}(y)\,d\mu(y)+\int_X L_{\nu_1(p),\nu_2(q)}^{(0,0)}(x)\,d\mu(x)\leqq$$

$$\leqq C\mu(X)\{\lambda_{\nu_1(p+1)}^{(1)}\lambda_{\nu_2(q+1)}^{(2)}+\lambda_{\nu_1(p+1)}^{(1)}\lambda_{\nu_2(q)}^{(2)}+\lambda_{\nu_1(p)}^{(1)}\lambda_{\nu_2(q+1)}^{(2)}+\lambda_{\nu_1(p)}^{(1)}\lambda_{\nu_2(q)}^{(1)}\}\leqq$$

$$\leqq 4C\mu(X)\lambda_{\nu_1(p+1)}^{(1)}\lambda_{\nu_2(q+1)}^{(2)}.$$

Finally making three similar estimations, we get from (15) that

$$\int_X \int_X \left| \sum_{k=\nu_1(p)+1}^{m(x)} \sum_{l=\nu_2(q)+1}^{n(y)} \frac{\varphi_{kl}(x)\varphi_{kl}(y)}{\lambda_k^{(1)}\lambda_l^{(2)}} \right| d\mu(x)\, d\mu(y) \leqq$$

$$\leqq 4C\mu(X)\lambda^{(1)}_{\nu_1(p+1)}\lambda^{(2)}_{\nu_2(q+1)} \left\{ \sum_{r=\nu_1(p)+1}^{\nu_1(p+1)-1} \sum_{s=\nu_2(q)+1}^{\nu_2(q+1)-1} \Delta_{11} \frac{1}{\lambda_r^{(1)}\lambda_s^{(2)}} + \right.$$

$$+ \sum_{r=\nu_1(p)+1}^{\nu_1(p+1)-1} \frac{1}{\lambda^{(2)}_{\nu_2(q)+1}} \left(\Delta_{10} \frac{1}{\lambda_r^{(1)}} \right) + \sum_{s=\nu_2(q)+1}^{\nu_2(q+1)-1} \frac{1}{\lambda^{(1)}_{\nu_1(p)+1}} \left(\Delta_{01} \frac{1}{\lambda_s^{(2)}} \right) +$$

$$\left. + \frac{1}{\lambda^{(1)}_{\nu_1(p)+1}\lambda^{(2)}_{\nu_2(q)+1}} \right\} \leqq 64C\mu(X).$$

Summing up the considerations beginning from (14) we obtain the estimate

$$\int_{E^{(0)}_{p,q}} \delta_{pq}(x)\, d\mu(x) \leqq 256C\mu(X)\Lambda_{pq}.$$

An analogous inequality is valid for the integral of $\delta_{pq}(x)$ over $E^{(1)}_{pq}$, too. By Beppo Levi's theorem (13) implies

$$\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \delta_{pq}(x) < \infty \quad \text{a.e.} \tag{16}$$

Taking into account that

$$\max_{\nu_1(p)<m_1\leqq m_2\leqq\nu_1(p+1)} \max_{\nu_2(q)<n_1\leqq n_2\leqq\nu_2(q+1)} \left| \sum_{k=m_1}^{m_2} \sum_{l=n_1}^{n_2} a_{kl}\varphi_{kl}(x) \right| \leqq 4\delta_{pq}(x),$$

we can establish by (16) the a.e. regular convergence of series (1).

4.

Finally, we point out that condition (13) cannot be relaxed.

THEOREM 3. *If*

$$\sum_{p_1=1}^{\infty} \cdots \sum_{p_d=1}^{\infty} \Lambda_{p_1,\ldots,p_d} = \infty,$$

then there exists a system φ *of step functions on the interval* $X=(0, 1)$ *satisfying the condition*

$$L_m^*(x) \leqq C\lambda^{(1)}_{m_1} \ldots \lambda^{(d)}_{m_d} \quad (m = (m_1, \ldots, m_d) \in Z_+^d,\ 0 < x < 1),$$

but for the rectangular partial sums of series (1) *we have*

$$\lim_{\min(m_1,\ldots,m_d)\to\infty} \sum_{k_1=1}^{m_1} \cdots \sum_{k_d=1}^{m_d} a_{k_1,\ldots,k_d}\varphi_{k_1,\ldots,k_d}(x) = \infty \quad (0 < x < 1).$$

The proof can be carried out in a similar manner as the corresponding theorem was proved by Tandori for single series [6, Satz II]. We only give here the definition of the functions $\varphi_k(x)$.

For each $p=(p_1, \ldots, p_d)\in Z_+^d$ let $\{I_r(p)\colon r=(r_1, \ldots, r_d)$, where $r_j=v_j(p_j)+1, \ldots, v_j(p_j+1)$ for each $j=1, \ldots, d\}$ be disjoint intervals with the following properties:

$$\bigcup_{r_1=v_1(p_1)+1}^{v_1(p_1+1)} \cdots \bigcup_{k_d=v_d(p_d)+1}^{v_d(p_d+1)} I_r(p) = (0, 1)$$

and the length

$$|I_r(p)| = \frac{a_{r_1, \ldots, r_d}^2}{\sum_{k_1=v_1(p_1)+1}^{v_1(p_1+1)} \cdots \sum_{k_d=v_d(p_d)+1}^{v_d(p_d+1)} a_{k_1, \ldots, k_d}^2}.$$

In case $a_{r_1, \ldots, r_d}=a_r\neq 0$ set

$$\varphi_r(x) = \begin{cases} A_p/a_r, & \text{if } x\in I_r(p), \\ 0, & \text{otherwise;} \end{cases}$$

while in case $a_r=0$ set $\varphi_r(x)\equiv 0$.

References

[1] G. Alexits, *Convergence problems of orthogonal series*, Pergamon (New York—Oxford—Paris, 1961).

[2] G. H. Hardy, On the convergence of certain multiple series, *Proc. London Math. Soc.*, Ser. 2, **1** (1903—1904), 124—128.

[3] F. Móricz, The Kronecker lemmas for multiple series and some applications, *Acta Math. Acad. Sci. Hungar.*, **37** (1981), 39—50.

[4] F. Móricz, Lebesgue functions and multiple function series. I, *Acta Math. Acad. Sci. Hungar.*, **37** (1981), 481—496.

[5] K. Tandori, Bemerkung zu einem Satz von S. Kaczmarz, *Acta Sci. Math. (Szeged)*, **42** (1980), 171—173.

[6] K. Tandori, Über einen Satz von Alexits und Sharma, *Acta Sci. Math. (Szeged)*, **42** (1980), 175—182.

(Received November 23, 1981)

JÓZSEF ATTILA UNIVERSITY
BOLYAI INSTITUTE
SZEGED, ARADI VÉRTANÚK TERE 1
H—6720, HUNGARY

Acta Math. Hung.
42 (3—4) (1983), 213—220.

UPPER RADICALS OF REGULAR CLASSES

W. G. LEAVITT (Lincoln)

1. Preliminaries

For a class M of rings we write as usual

$$UM=\{R \mid \text{every } 0\neq R/I\notin M\}, \quad SM=\{R \mid I\notin M \text{ for every } I\lhd R\},$$

$$M_k=\{R \mid I\overset{e}{\lhd} R \text{ for some } I\in M, \text{ where } I\overset{e}{\lhd} R \text{ means } I \text{ is an essential ideal of } R\}.$$

We call I *accessible* in R if $I_1\lhd\ldots\lhd I_n=R$ for some $n\geqq 1$ and *essentially accessible* if each $I_i\overset{e}{\lhd} I_{i+1}$. The *essential closure* of a class M is

$$M^c=\{R \mid I \text{ is essentially accessible in } R \text{ for some } I\in M\}.$$

The class M is called *regular* if $M\subseteqq SUM$, or equivalently if UM is the largest radical class relative to which every $R\in M$ is semisimple. A radical class UM is said to have the *intersection property* relative to a class N if for every ring R the radical $UM(R)=\cap(I \mid R/I\in N)$.

A radical class is called *hereditary* if $I\lhd R\in UM$ implies $I\in UM$, and in a recent paper several criteria were derived equivalent to UM hereditary [5; Theorem 4]. In the case that M has the property

(E) Whenever $J\lhd I\lhd R$ with $0\neq I/J\in M$ then there exist ideals V, U of R such that $U\subset V\subseteqq I$ and $0\neq V/U\in M$,

these criteria were shown to specialize to:

THEOREM 1. *If a class M has property* (E) *and UM is a radical class then the following are equivalent:*

(1) *UM is hereditary.*
(2) *$UM=UM_k$.*
(3) *$UM\cap M_k=0$.*
(4) *Every $0\neq R\in M_k$ has some image $0\neq R/I\in SUM$.*
(5) *UM has the intersection property relative to M_k.*

As an immediate consequence we have:

COROLLARY 1 [5; Theorem 5]. *Theorem* 1 *applies to any regular class of semiprime rings, or any regular class containing all the zero rings (rings with trivial multiplication).*

Remark that Corollary 1 contains several previous results as special cases. These include [4; Theorem 7], [6; Theorem 7], and [7; Proposition 10].

In the present paper we will construct examples to show that Theorem 1 does not apply to regular classes in general. That is, an example will be constructed of a regular class M with property (4) of Theorem 1 but for which UM is not hereditary. We will show that for a regular class M we must replace M_k by M'_k where in general M' is somewhat larger than M. We will also show that property (E) is not necessary. That is, we will construct a regular class M which does not have property (E) but for which UM is hereditary, and has the intersection property relative to M.

We first recall some results from [5]. Let Z^0 be the zero ring on the additive group of the integers, and Z_p^0 the zero ring on the integers modulo the prime p.

PROPOSITION 1 [5; Remark preceding Theorem 10]. *For an arbitrary class M, if UM is radical and M contains a nilpotent ring then either $Z^0 \in M$ or some $Z_p^0 \in M$.*

Also as an immediate consequence of [5; Proposition 2]

PROPOSITION 2. *Let I and J be ideals of R such that $0 \neq I/J \in M$. Then there exists $C \lhd R$ such that $I \subseteq C$ and $R/C \in M_k$.*

2. Hereditary radicals for regular classes

Let S_0 be the class of all direct sums of copies of Z^0, and S_p the class of all direct sums of copies of Z_p^0. Then

LEMMA 1. *Let M be a regular class and suppose that for some $J \lhd I \lhd R$ we have $I/J \in M$ with $J \overline{\lhd} R$. Then there exist ideals V, U of R with $U \subset V \subseteq I$ and such that either:*

(a) *$V/U \in S_p$ for some p such that $Z_p^0 \in M$, or*

(b) *$V/U \in (S_0)_k$ where V/U is a zero ring.*

PROOF. Let $J' \lhd R$ be generated by J. Then $J' \neq J$ and by the Andrunakievitch lemma $(J')^3 \subseteq J$. Thus either $0 \neq J'/J$ is a zero ring, or if not then $0 \neq (J')^2 + J/J$ is a zero ring. In either case we have a zero ring contained as an ideal of $I/J \in M \subseteq \subseteq SUM$. Thus SUM contains a nilpotent ring (in fact a zero ring) so M contains a nilpotent ring and by Proposition 1 either $Z^0 \in M$ or some $Z_p^0 \in M$. Note that if any $Z_p^0 \in M$ then automatically $Z^0 \in SUM$ so that in any case $Z^0 \in SUM$. Since one or the other of J'/J or $(J')^2/((J')^2 \cap J)$ is a zero ring we have, in either case an ideal $H \lhd R$ with an image $0 \neq H/A \in SUM$ where H/A is a zero ring and $H \subseteq I$.

Now $H^2 \subseteq A$ and let T/H^2 be the torsion ideal of H/H^2. Note that $T \lhd R$, and if $T \nsubseteq A$ then there will be some $x \in T$ with $x \notin A$ having additive order p modulo H^2 for some prime p. Then from the map H/H^2 onto H/A we obtain an ideal $Z_p^0 \lhd H/A$ generated by x. Thus $Z_p^0 \in SUM$ so $Z_p^0 \in M$. Also every element of the ideal $(x) \lhd R$ has additive order p modulo H^2 so $(x) + H^2/H^2$ is a vector space over Z_p and is thus a direct sum of copies of Z_p^0, that is it is in S_p.

Finally if $T \subseteq A$ then $H \neq T$ and $0 \neq H/T$ is torsion-free. Then any non-zero $x \in H/T$ generates an ideal of H/T isomorphic with Z^0 and if J is the ideal of

H/T generated by any maximal independent subset of H/T then $J \in S_0$ and clearly $J \overset{e}{\lhd} H/T$. Thus $H/T \in (S_0)_k$ and is, of course, a zero ring.

Remark that if $Z_p^0 \in M$ then even if $T \subseteq A$ in the above proof it still might happen that $0 \neq V/U \in S_p$ for some ideals V, U of R such that $U \subset V \subseteq I$. This would be the case, for example, if for some $x \in I$ we have $x \notin (px)+(x)^2$.

Now write $W = \{R \in (S_0)_k \mid R^2 = 0\}$ and $P = \{p \mid Z_p^0 \in M\}$. Define

$$M' = \begin{cases} M & \text{if } M \text{ is a class of semiprime rings,} \\ M \cup W \cup \bigcup_{p \in P} S_p & \text{otherwise.} \end{cases}$$

PROPOSITION 3. *Let M be a regular class. For an arbitrary ring R if $J \lhd I \lhd R$ with $0 \neq I/J \in M$ then there exists $C \lhd R$ such that $I \nsubseteq C$ and $R/C \in M_k'$.*

PROOF. If $J \lhd R$ then Proposition 2 gives $C \lhd R$ with $I \nsubseteq C$ and $R/C \in M_k \subseteq M_k'$. Otherwise Lemma 1 yields ideals V and U of R such that $U \subset V \subseteq I$ and $V/U \in M'$. Again Proposition 2 gives an ideal $C \lhd R$ with $V \nsubseteq C$ (hence $I \nsubseteq C$) and such that $R/C \in M_k'$.

We can now generalize Corollary 1 as follows:

THEOREM 2. *If M is a regular class then the following are equivalent:*

(1) *UM is hereditary.*
(2) *$UM = UM_k'$.*
(3) *$UM \cap M_k' = 0$.*
(4) *Every $0 \neq R \in M_k'$ has an image $0 \neq R/I \in SUM$.*
(5) *$M_k' \subseteq SUM$.*
(6) *UM has the intersection property relative to M_k'.*

PROOF. In [5; Theorem 2] we showed the equivalence of (2), (3), and (4). Also (4)→(1) since if UM were not hereditary we would have $0 \neq I \lhd R \in UM$ with $I \notin UM$, that is some $0 \neq I/J \in M$. But then by Proposition 3 we would have some $0 \neq R/C \in M_k'$ contradicting (4). Clearly (5)→(4) and to show (1)→(5) note that if $Z^0 \in M$ then $S_0 \subseteq SUM$, and if any $Z_p^0 \in M$ then the corresponding $S_p \subseteq SUM$. From [1] it is known that UM hereditary implies $(SUM)_k = SUM$. Thus if UM is hereditary we have in all cases that $M' \subseteq SUM$. But then $M_k' \subseteq (SUM)_k = SUM$.

Now (6)→(5) [see 5; Theorem 7] since if $R \in M_k'$ then by the intersection property $UM(R) = \cap (I \mid R/I \in M_k') = 0$, so $R \in SUM$. Then to show (5)→(6) first let $R \in SUM$ [see 5; Theorem 8] and suppose we had $0 \neq K = \cap (I \mid R/I \in M_k')$. Then $K \in SUM$ so some image $0 \neq K/J \in M$. But then Proposition 3 yields the contradiction: $R/C \in M_k'$ for some C such that $K \nsubseteq C$.

Thus for an arbitrary ring R we at least know that $UM(R) = \cap (I \mid R/I \in M_k'$ and $UM(R) \subseteq I)$. Thus it suffices to consider the case $UM(R) \nsubseteq I$. But then we would have $0 \neq (UM(R)+I)/I \cong UM(R)/(I \cap UM(R)) \in UM$ contradicting $(UM(R)+I)/I \lhd R/I \in M_k' \subseteq SUM$.

It is well known that UM can be radical without M being regular. In fact we have

PROPOSITION 4 [2; Theorem 1]. *For M an arbitrary class, UM is a radical class if and only if every $0 \neq R \in M$ has an image $0 \neq R/I \in SUM$.*

In [5; Theorems 6 and 7] we have a generalization of Theorem 2 to an arbitrary class M replacing M'_k by a certain larger class $\tilde{M}_k$. Here we will use Theorem 2 to generalize in a different way. For a given class M let $H(M)$ be the homomorphic closure of M and write $N=H(M)\cap SUM$. Then define

$$\overline{N}=\begin{cases}N \text{ if } M \text{ is a class of semiprime rings,}\\ N\cup W\bigcup_{p\in P}S_p \text{ otherwise.}\end{cases}$$

THEOREM 3. *Let M be an arbitrary class of rings. If UM is radical then Theorem 2 applies provided M'_k is replaced by $\overline{N}_k$.*

PROOF. Since $N\subseteq SUM$ we have $UM=USUM\subseteq UN$. But also if $0\neq R\notin UM$ then R has some $0\neq R/I\in M$. From Proposition 4 it follows that R has a non-zero image in N, that is $R\notin UN$, and so $UM=UN$. Clearly $Z_p^0\in M$ if and only if $Z_p^0\in N$. Also if $Z_p^0\notin M$ but $Z^0\in M$ then $Z^0\in N$ and conversely. Thus in fact $\overline{N}=N'$ and since N is regular, Theorem 2 applies to $AN=UM$ replacing M'_k with $N'_k=\overline{N}_k$.

Remark that in Theorem 3 we could use any class $N\subseteq SUM$ which is such that every $0\neq R\in M$ has an image $0\neq R/I\in N$.

3. Constructions

For future use we will need certain rings A with the property $J\lhd I\lhd A$ but $J\not\lhd A$ and the simplest way to construct such a ring is:

PROPOSITION 5. *For a given field F there exists a ring A with a single proper ideal B such that $A/B\cong F$ and $B\cong F^0$, the zero ring on the additive group of F.*

PROOF. Simply take $A=F[x]/(x)^2$ with $B=(x)/(x)^2$.

COROLLARY 2. *For any prime p there exists a ring A with unit and a single proper ideal B such that $B\in S_p$ but $B\ncong Z_p^0$.*

PROOF. In Proposition 5 take F to be a field of degree n over Z_p for any (finite or infinite) $n\geqq 2$.

COROLLARY 3. *There exists a ring A with unit and a single proper ideal B such that $B\in(S_0)_k$ but $B\notin\{Z^0\}_k$.*

PROOF. In Proposition 5 take F a field of any degree $n\geqq 2$ over the rationals. Since F has dimension n over the rationals it has a basis $\{x_1, x_2, \ldots\}$ of n elements which is a maximal independent set over Z. Thus it generates an ideal $J\overset{e}{\lhd}B=F^0$ which is a direct sum of copies of Z^0. Thus $B\in(S_0)_k$ and suppose $K\overset{e}{\lhd}B$ where $K\cong Z^0$, say K is generated over Z by the element u. Then $K\cap\cap(x_1)\neq 0$ and $K\cap(x_2)\neq 0$, so that $k_1u=n_1x_1$ and $k_2u=n_2x_2$ for integers k_1, k_2, n_1, n_2. But then we have the contradiction $0\neq k_1k_2u=k_2n_1x_1=k_1n_2x_2$.

Note that in Corollary 2 we have a direct sum of copies of Z_p^0 essential in A but not Z_p^0 itself. However we can have a ring in which Z_p^0 is essential even when other members of S_p are also essential. In fact we have

PROPOSITION 6. *If* $R \in \{Z_p^0\}^C$ *then* R *is subdirectly irreducible with heart* $H \in S_p$ *and if also* $Z_p^0 \lhd R$ *then* $H = Z_p^0$.

PROOF. If $0 \neq I \lhd R$ then since Z_p^0 is essentially accessible in R it follows inductively that $Z_p^0 \cap I \neq 0$. Thus $Z_p^0 \cap I = Z_p^0$ so $Z_p^0 \subseteq I$. Thus R is subdirectly irreducible and let H be its heart. Then $Z_p^0 \subseteq H$ so $Z_p^{0\prime} = H$ where $Z_p^{0\prime}$ is the ideal of R generated by Z_p^0. By an extension of the proof used for the Andrunakievitch lemma it is easy to show that some power say $H^n \subseteq Z_p^0$ sor $H^{2n} = 0$. Thus $H^2 \neq H$ so $H^2 = 0$. Also since every member of $H = Z_p^{0\prime}$ has additive order p it follows that $H \in S_p$. Also if $Z_p^0 \lhd R$ then $H = Z_p^{0\prime} = Z_p^0$.

We will illustrate with the following construction:

PROPOSITION 7. *For any cardinality* $n \geqq 2$ *there exists a ring* A *such that* $Z_p^0 \overset{e}{\lhd} A$ *and* $Z_p^0 \subset J \lhd A$ *where* J *is a direct sum of* n *copies of* Z_p^0.

PROOF. Let V be a vector space over Z_p of dimension n with (well-ordered) basis $\{y_1, y_2, \ldots, y_\alpha, \ldots\}$ and let R be the ring of linear transformations on V generated by the set $\{x_2, \ldots, x_\alpha, \ldots\}$ of cardinality $n-1$ defined by $x_\alpha(y_\alpha) = y_1$ for all α and $x_\alpha(y_\beta) = 0$ for all $\alpha \neq \beta$. We then let A be the ring generated over Z_p by the elements of V and R with multiplication defined by

$$xy = yx = \begin{cases} \text{multiplication in } R & \text{if } x, y \in R, \\ 0 & \text{if } x, y \in V, \\ x(y) & \text{if } x \in R \text{ and } y \in V. \end{cases}$$

Then $(y_1) \cong Z_p^0$ and it is easy to check that $(y_1) \subseteq I$ for all $0 \neq I \lhd A$. Thus $(y_1) \overset{e}{\lhd} A$ and $(y_1) \subseteq J = (y_1, y_2, \ldots) \lhd A$.

Remark that if we take the y's as a basis for a Z-module and the x's as Z-endomorphisms with all relations as above, then we have

PROPOSITION 7′. *For any cardinality* $n \geqq 2$ *there exists a ring* A *such that* $Z^0 \overset{e}{\lhd} A$ *and* $Z^0 \subset J \lhd A$ *where* J *is a direct sum of* n *copies of* Z^0.

We now introduce a type of construction which we will use to show that Theorem 2 is sharp in the sense that there exists a regular class M such that $M_k \subseteq SUM$ but for which UM is not hereditary. Another construction of this type will answer (affirmatively) a question raised in [5] as to whether or not there exists a class M not having property (E) but for which UM is hereditary.

For a given class M and a given prime p we define

$$F(M) = \{R \mid R \text{ is accessible in some } A \in M \text{ and } R \text{ does not contain } Z_p^0 \text{ as a proper direct summand}\}.$$

We begin the construction with any class M_1 containing at least one non-zero ring. Then define

$$M_{i+1} = F(M_i^C) \quad \text{for all} \quad i \geqq 1, \quad \text{and} \quad M = \bigcup_1^\infty M_i.$$

We will need

LEMMA 2. *Let* $R=K\oplus H$ *where* K *is either simple or some* Z_p^0. *If* $I\overset{e}{\lhd}R$ *the* $I=K\oplus I\cap H$.

PROOF. Since $I\overset{e}{\lhd}R$ we have $I\cap K\neq 0$. Thus $I\cap K=K$ so $K\subseteq I$. If $x\in I$ then $x=k+h$ for some $k\in K$ and $h\in H$. But then $h=x-h\in I\cap H$ so $I=K\oplus I\cap H$.

LEMMA 3. *If no ring in* M_1 *has* Z_p^0 *as a proper direct summand then the same is true of* M.

PROOF. This follows from the definition of F.

LEMMA 4. *If no ring in* M_1 *has* Z_p^0 *as a proper direct summand then* $M=M_k$.

PROOF. If $A\in M_k$ then $R\overset{e}{\lhd}A$ for some $R\in M$. Then $R\in M_i$ for some i and so $A\in M_i^c$. Therefore $A\in F(M_i^c)=M_{i+1}\subseteq M$ unless A has Z_p^0 as a proper direct summand, that is $A=Z_p^0\oplus H$ for some $0\neq H\lhd A$. But then by Lemma 2 we would have $R=Z_p^0\oplus R\cap H$ where $R\cap H\neq 0$, contradicting Lemma 3.

LEMMA 5. *If* $Z_p^0\in M_1$ *then* $M\subseteq SUM$.

PROOF. Let $0\neq I\lhd R\in M$ so $R\in M_i$ for some i. Then $R\in M_i^c$ so $I\in F(M_i^c)=$ $=M_{i+1}\subseteq M$ unless $I=Z_p^0\oplus H$ for some $H\neq 0$. But then $I/H=Z_p^0\in M$. Thus in either case all non-zero ideals of R have non-zero images in M, that is $R\in SUM$.

We now have a class M (provided M_1 satisfies the hypotheses of Lemmas 4 and 5) such that $M=M_k\in SUM$. We will also need

LEMMA 6. *If* K *has a unit and is accessible in* R *then* $K\lhd R$ *so* $R=K\oplus H$ *for some* $H\lhd R$.

PROOF. Let $K\lhd I\lhd W$ and let e be the identity of K. Then for any $k\in K$ and $w\in W$ we have $kw=kew\in KI\subseteq K$. Similarly $wk\in K$ so $K\lhd W$ and by induction $K\lhd R$. (Note that for this result we only need $K^2=K$.)

LEMMA 7. *Let* $R=K\oplus H$ *where* K *has a unit. If* $I\lhd R$ *then* $I=I\cap K\oplus I\cap H$.

PROOF. Let e be the identity of K. If $x\in I$ then $x=k+h$ for some $k\in K$ and $h\in H$. But then $xe=ke=k\in I\cap K$. Thus $h=x-k\in I\cap H$.

We can now show that Theorem 2 is sharp, namely

THEOREM 4. *There exists a class* M *such that* $M=M_k\subseteq SUM$ *but* UM *is not hereditary.*

PROOF. We use the above construction with $M_1=\{Z_p^0\}$. By Lemmas 4 and 5 we have $M=M_k\subseteq SUM$. Then let A be one of the rings of Corollary 2. Since $Z_p^0\in M$ we have $S_p\subseteq SUM$ so the single proper ideal B of A is in SUM. Thus $A\in(SUM)_k$ and if UM were hereditary it would follow that $A\in SUM$. But then either $A\in M$ or the field $F=A/B\in M$.

Now for all rings H we have $A\oplus H\notin M_1$ and we wish to show that $A\oplus H\notin M$ for all H (so in particular $A\notin M$). Thus suppose the contrary, then there would be some i such that $A\oplus H\notin M_i$ for all H but some $A\oplus H\in M_{i+1}=F(M_i^c)$. Then we would have $A\oplus H$ accessible (so A accessible) in some $R\in M_i^c$. By Lemma 6 we would then have $R=A\oplus Q$ for some $Q\lhd R$. Now $R\in M_i^c$ so there

would be some $I_1 \in M_i$ essentially accessible in R where I_1 does not have the form $A \oplus H$. Thus there would be some I not of form $A \oplus H$ such that $I \overset{e}{\lhd} A \oplus Q$ for some Q. By Lemma 7 we would have $I = I \cap A \oplus I \cap Q$ where $I \cap A \neq 0$. Since $A \cap I \neq A$ it would follow that $I \cap A = B$. But then $I = B \oplus I \cap Q = Z_p^0 \oplus Z_p^0 + \ldots \oplus \oplus I \cap Q = Z_p^0 \oplus H$ where $H \neq 0$, contradicting Lemma 3. Thus $A \oplus H \notin M$ for all H and so $A \notin M$. By a similar argument $F = A/B \notin M$ and so we conclude that $A \notin SUM$, that is UM is not hereditary.

In [5] we asked if M must have property (E) in order for UM to be hereditary, and we will use a variant of the above construction to show that the answer is negative. In fact we will show:

THEOREM 5. *There exists a regular class M such that UM is hereditary and has the intersection property relative to M, but M does not have property* (E).

PROOF. We will use the above construction for M, except that we will now define $F(M)$ by

$$F(M) = \{R | R \text{ is accessible in some } A \in M \text{ and } R \ncong Z_p^0 \oplus Z_p^0\}.$$

It is clear that if we replace $Z_p^0 \oplus H$ by $Z_p^0 \oplus Z_p^0$ then the statements and proofs of Lemmas 3, 4, and 5 remain valid. Thus if we now let

$$M_1 = S_0 \cup \{R \in S_p^C \mid R \ncong Z_p^0 \oplus Z_p^0\},$$

then $Z_p^0 \in M_1$ but $Z_p^0 \oplus Z_p^0 \notin M_1$ so from Lemmas 4 and 5 we have $M = M_k \subseteq SUM$. Now if I is essentially accessible in R and contains no elements of additive order q then the same is true of R. Thus no $R \in M_1$ contains an element of order $q \neq p$ and so, inductively the same is true of all M. Thus $Z_q^0 \notin M$ for all $q \neq p$ and so $M' = M \cup S_p$.

By Theorem 2 to show UM is hereditary it suffices to show that all rings in M_k' have images in SUM. Thus let $R \in M_k'$. If $R \in M_k$ then already $R \in M \subseteq SUM$ and the only remaining case is $R \in (S_p)_k$. But then $R \in M_1 \subseteq M$ unless $R = Z_p^0 \oplus Z_p^0$ and in this case $R/Z_p^0 \cong Z_p^0 \in M \subseteq SUM$.

We conclude that UM is hereditary and to show that UM has the intersection property relative to M note that by Theorem 2 UM at least has the intersection property relative to M_k'. Thus for an arbitrary ring R we have $UM(R) = \cap I \mid R/I \in M_k'$. Therefore $R/I \in M_k = M$ unless $R/I \in (S_p)_k$ and $R/I = Z_p^0 \oplus Z_p^0$. But then there exist ideals I_1 and I_2 of R such that $I_1/I \cong I_2/I \cong Z_p^0$ with $I = I_1 \cap I_2$ and each $(R/I)/(I_i/I) \cong Z_p^0 \in M$.

Thus $I = I_1 \cap I_2$ where $R/I_1 \cong R/I_2 \cong Z_p^0 \in M$, so for all cases $UM(R) = \cap I \mid R/I \in M_k = M$.

Finally to show that M does not have property (E) consider the ring A of Corollary 2 which has the single proper ideal $B = Z_p^0 \oplus Z_p^0$. We have $B \notin M$ but $B/Z_p^0 \cong Z_p^0 \in M$.

We now consider classes containing Z^0 but none of the Z_p^0. First we have

LEMMA 8. *If $R \in M^C$ and all rings in M are torsion-free then R is torsion-free. Thus any accessible subring of R is torsion-free.*

PROOF. Again if $I \in M$ is essentially accessible in R and R had elements

of order p then we would have the contradiction $0 \neq I \cap J$ where J is the ideal of all elements of R of order p.

We can now construct:

THEOREM 6. *There exists a regular class M such that UM is hereditary and has the intersection property relative to M, such that $S_0 \subseteq M$ but $S_p \subseteq UM$ for all primes p.*

PROOF. Let $M_1 = S_0$ and let M be defined as above with again $F(M) = \{R \mid R$ is accessible in some $A \in M\}$. As before it is easy to show that $M = M_k \subseteq SUM$. Also all rings in S_0 are torsion-free so by Lemma 8 this is also true of all rings in M. Thus no ring from any S_p is in M. Since $S_0 \subseteq M$ we have $(S_0)_k \subseteq M$ so $M' = M'_k \subseteq SUM$. By Theorem 2 UM is hereditary with the intersection property relative to M. Also if $R \in S_p$ then every $0 \neq R/I \in S_p$, that is every $0 \neq R/I \notin M$. Thus $R \in UM$ so $S_p \subseteq UM$.

Note that Proposition 7′ gives examples of rings in this class.

References

[1] E. P. Armendariz, Closure properties in radical theory, *Pac. Jour. Math.*, **26** (1968), 1—7.
[2] P. O. Enersen and W. G. Leavitt, The upper radical construction, *Publ. Math. Debrecen*, **20** (1973), 219—222.
[3] B. J. Gardner, Absolute retracts, relatively injective simple objects, and Brown—McCoy radicals, *Summer Res. Inst. of Austral. Math. Soc.* (Hobart, 1981).
[4] G. A. P. Heyman and C. Roos, Essential extensions in radical theory for rings, *Jour. Austral. Math. Soc.*, **23** (Ser. A) (1977), 340—347.
[5] W. G. Leavitt, Hereditary upper radicals, *Studia Sci. Math. Hungar*, **16** (1981), 15—23.
[6] H. J. Leroux, C. A. P. Heyman, and T. L. Jenkins, Essentially closed classes of rings and upper radicals, *Acta Math. Acad. Sci. Hungar*, **38** (1981), 63—68.
[7] Ju. M. Rjabuhin and R. Wiegandt, On special radicals, supernilpotent radicals, and weakly homomorphically closed classes, *Jour. Austral. Math. Soc.*, **31** (1981), 152—162.

(Received December 18, 1981)

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF NEBRASKA
LINCOLN, NEBRASKA 68588-0323
USA

Acta Math. Hung.
42 (3—4) (1983), 221—223.

A NOTE ON THE DECOMPOSITION OF GRAPHS INTO ISOMORPHIC MATCHINGS

N. ALON (Cambridge)

All graphs considered are finite, undirected, with no loops and no multiple edges. A graph H is said to have a G-decomposition if it is the union of pairwise edge-disjoint subgraphs each isomorphic to G. We denote this situation by $G|H$.

Many results are known about G-decomposition, for references see e.g. [1] and [6]. In this paper we establish some necessary and sufficient conditions for a graph H to have a tK_2-decomposition, where tK_2 is the graph consisting of t independent edges. Our result implies, as a very special case, the main result of Bialostocki and Roditty [3], that states that if G is a graph with e edges and maximum degree Δ, then, with a finite number of exceptions, $3K_2|G$ iff $3|e$ and $\Delta \leqq e/3$.

For every graph G, $E(G)$ is the set of edges of G and $e(G)=|E(G)|$. $\Delta(G)$ is the maximum degree of G and $\chi'(G)$ is the chromatic index (=edge-chromatic number) of G.

We begin with the following simple lemma, which is proved in [2]:

LEMMA 1. *Let G be a graph and let $M, N \subset E(G)$ be disjoint matchings of G with $|M|>|N|$. Then there are disjoint matchings M' and N' of G such that $|M'|=|M|-1$, $|N'|=|N|+1$ and $M' \cup N' = M \cup N$.* □

As an easy consequence we obtain

LEMMA 2. *For every graph G and every $t>1$, $tK_2|G$ iff*

$$t|e(G) \quad \text{and} \quad \chi'(G) \leqq e(G)/t. \tag{1}$$

PROOF. If $tK_2|G$ then obviously (1) holds. Conversely, if (1) holds put $r=e(G)/t$. Since $\chi'(G) \leqq r$, there are r disjoint matchings $F_1, \ldots, F_r$ of G that cover $E(G)$. By repeated application of Lemma 1 to pairs of these r matchings that differ in size by two or more we obtain r disjoint matchings $E_1, \ldots, E_r$ of G that cover $E(G)$ and $|E_i|=t$ for all $1 \leqq i \leqq r$. □

REMARK 1. König's Theorem (for proof see [4, p. 105]), asserts that for every bipartite graph G, $\chi'(G)=\Delta(G)$. This and Lemma 2 imply that for every bipartite graph G $tK_2|G$ iff

$$t|e(G) \quad \text{and} \quad \Delta(G) \leqq e(G)/t. \tag{2}$$

This result is stated as Lemma 3.2 of [5]. In Theorem 1 below we prove that the same result holds for every graph G, with a finite number of exceptions for every value of t.

REMARK 2. Vizing's Theorem (for proof see [4, pp. 107—108]) asserts that for every graph G, $\chi'(G) \leqq \Delta(G)+1$. This and Lemma 2 imply that if $t|e(G)$ and $\Delta(G) < e(G)/t$, then $tK_2|G$.

The following lemma is proved in [4, p. 119]:

LEMMA 3. *If G is a graph and $\chi'(G)=\Delta(G)+1$ then*

$$e(G) \geqq \frac{1}{8}\left(3(\Delta(G))^2+6\cdot\Delta(G)-1\right). \quad \square$$

Now we are ready to prove our main result:

THEOREM 1. *For every $t>1$ and for every graph G that satisfies*

$$e(G) > (8/3)t^2-2t, \tag{3}$$

the following two conditions are equivalent:

$$tK_2|G. \tag{4}$$

$$t|e(G) \quad \text{and} \quad \Delta(G) \leqq e(G)/t. \tag{5}$$

PROOF. Clearly (4) implies (5) (even if G does not satisfy (3)). Conversely, assuming G satisfies (3) and (5) let us prove (4). Put $\Delta=\Delta(G)$ and $e=e(G)$. If $\Delta < e/t$ then Remark 2 implies (4), and if $\chi'(G)=\Delta$ then Lemma 2 implies (4). Thus we are left with the case that $\chi'(G)=\Delta+1$ and $\Delta=e/t$. We shall show that this case contradicts (3). By Lemma 3

$$8e \geqq 3\Delta^2+6\Delta-1 = 3(e/t)^2+6(e/t)-1. \tag{6}$$

Since the left side of (6) is divisible by e/t, and (3) implies that $e/t>1$, (6) implies

$$8e \geqq 3(e/t)^2+6(e/t).$$

The last inequality implies $e \leqq (8/3)t^2-2t$, which contradicts (3). Thus $tK_2|G$ and the theorem is established. $\square$

REMARK 3. For $t>1$ let G_t be the disjoint union of K_{2t-1} (= the complete graph on $2t-1$ vertices), and any graph H with $t-1$ edges. Clearly $e(G_t)= =2t^2-2t$, $\Delta(G_t)=2t-2$ and $\chi'(G_t)=\chi'(K_{2t-1})=2t-1$. Thus G_t satisfies (5), and by Lemma 2, G_t does not satisfy (4). This shows that the lower bound for $e(G)$ in condition (3) is not very far from being best possible.

References

[1] J. C. Bermond and J. Schönheim, G-decomposition of K_n, where G has four vertices or less, *Discrete Math.*, **19** (1977), 113—120.

[2] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, The Macmillan Press Ltd. (London and Basingstoke, 1978), p. 97.

[3] A. Bialostocki and Y. Roditty, $3K_2$-decomposition of a graph, *Acta Math. Acad. Sci. Hungar.*, **40** (1982), 201—208.

[4] S. Fiorini and R. J. Wilson, Edge colorings of graphs, in *Selected Topics in Graph Theory* (ed. L. W. Beineke and R. J. Wilson) Academic Press (London, New York and San Francisco, 1978), pp. 103-126.

[5] J. Folkman and D. R. Fulkerson, Edge colorings in bipartite graphs, in *Combinatorial Mathematics and its Applications* (ed. R. C. Bose and T. A. Dowling), The University of North Carolina Press (1967), pp. 561-577.

[6] R. M. Wilson, Decomposition of complete graphs into subgraphs isomorphic to a given graph, *Proc. 5th British Combinatorial Conference* (Aberdeen, 1975), pp. 647-659.

(Received December 27, 1981)

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MATHEMATICS
CAMBRIDGE, MASSACHUSETTS 02139
USA

Acta Math. Hung.
42 (3—4) (1983), 225—232.

COMPLETE CONVERGENCE AND CONVERGENCE RATES FOR RANDOMLY INDEXED PARTIAL SUMS WITH AN APPLICATION TO SOME FIRST PASSAGE TIMES

A. GUT (Uppsala)

1. Introduction

DEFINITION. A sequence of random variables $\{U_n\}_{n=1}^{\infty}$ is said to *converge completely* to a constant c if

$$\sum_{n=1}^{\infty} P(|U_n - c| > \varepsilon) < \infty \quad \text{for all} \quad \varepsilon > 0.$$

This definition was introduced by Hsu and Robbins [11], who, as a complement to the strong law of large numbers proved that the sequence of arithmetic means of independent, identically distributed (i.i.d.) random variables converges completely to the expected value of the summands, provided the variance is finite. For the converse, see Erdős [7, 8].

Such a result can also be interpreted as a result on the rate of convergence in the law of large numbers. The following result on convergence rates generalizes the result of Hsu, Robbins and Erdős. The extensions are due to Spitzer [15], Katz [12], Baum and Katz [2] and Chow [3].

THEOREM 1.1. *Let* $\{X_n\}_{n=1}^{\infty}$ *be a sequence of i.i.d. random variables, set* $S_n = \sum_{k=1}^{n} X_k$, $n=1, 2, \ldots$ *and let* $\alpha r \geqq 1$, $\alpha > 1/2$. *The following statements are equivalent:*

(i) $E|X|^r < \infty$ *and, if* $r \geqq 1$, $EX = 0$

(ii) $\sum_{n=1}^{\infty} n^{\alpha r - 2} P(|S_n| > n^{\alpha}\varepsilon) < \infty$ *for all* $\varepsilon > 0$

(iii) $\sum_{n=1}^{\infty} n^{\alpha r - 2} P\left(\max_{k \leqq n} |S_k| > n^{\alpha}\varepsilon\right) < \infty$ *for all* $\varepsilon > 0$.

If $\alpha r > 1$, $\alpha > 1/2$ *the above are also equivalent to*

(iv) $\sum_{n=1}^{\infty} n^{\alpha r - 2} P\left(\sup_{k \geqq n} |S_k/k^{\alpha}| > \varepsilon\right) < \infty$ *for all* $\varepsilon > 0$.

For the case $\alpha r = 1$, $\alpha > 1/2$ we have

THEOREM 1.2. *Let* $\{X_n\}_{n=1}^{\infty}$ *be a sequence of i.i.d. random variables and set* $S_n = \sum_{k=1}^{n} X_k$. *The following are equivalent:*

(i) $E|X|^r \cdot \log^+|X| < \infty$ *and, if* $r \geqq 1$, $EX = 0$

(ii) $\sum_{n=1}^{\infty} n^{-1} \cdot \log n \cdot P(|S_n| > n^{1/r}\varepsilon) < \infty$ *for all* $\varepsilon > 0$

(iii) $\sum_{n=1}^{\infty} n^{-1} \cdot \log n \cdot P(\max_{k \geqq n} |S_k| > n^{1/r}\varepsilon) < \infty$ *for all* $\varepsilon > 0$

(iv) $\sum_{n=1}^{\infty} n^{-1} \cdot P(\sup_{k \geqq n} |S_k/k^{1/r}| > \varepsilon) < \infty$ *for all* $\varepsilon > 0$.

If $\alpha = 1/2$ none of the sums can converge in view of the central limit theorem and one has to use other normalizing sequences, see Davis [5, 6], Lai [13] and Gut [10]. For generalizations to random variables with multidimensional indices, see Smythe [14] and Gut [9, 10], where further references can be found. For convergence in Theorem 1.1 (ii) along subsequences of the natural numbers ($\alpha = 1$, $r = 2$), see Asmussen and Kurtz [1].

Various results for sums of random variables have been generalized to the case of *randomly indexed* partial sums. The complete convergence result of Hsu, Robbins and Erdős has been extended in this direction by Szynal [16]. A more general result is given in Csörgő and Révész [4], Theorem 7.1.1, page 252. An application of that theorem, see pages 253—254, provides a generalization of Szynal's result. However, the result of the application can be proved directly and, furthermore, with weaker assumptions. This will be our first task and it is carried out in Section 2.

Next, we consider a particular sequence of random indices, namely the first passage times of a summation process with positive drift. In Section 3 we state and prove a convergence rate result for these first passage times, which in Section 4 is applied to the results of Section 2, thus producing a result for the stopped process. The overshoot, i.e. the amount with which the process jumps above the level to be crossed, is also treated.

In the last section some further generalizations are mentioned.

2. Randomly indexed partial sums

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of i.i.d. random variables with mean zero, whenever the mean is finite, and let $\{N_n\}_{n=1}^{\infty}$ be a sequence of nonnegative, integer valued random variables. All random variables are defined on the same probability space.

THEOREM 2.1. a) *Let* $\alpha r > 1$ *and* $\alpha > 1/2$. *Suppose that* $E|X_1|^r < \infty$ *and that* $EX_1 = 0$ *if* $r \geqq 1$. *If, for some* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha r - 2} P(|N_n - Nn| > n\varepsilon) < \infty, \tag{2.1}$$

where N *is a positive random variable, such that* $P(N \geqq A) = 1$ *for some* $A > \varepsilon > 0$, *then*

$$\sum_{n=1}^{\infty} n^{\alpha r - 2} P(|S_{N_n}| > N_n^{\alpha}\varepsilon) < \infty. \tag{2.2}$$

b) *Let* $\alpha r=1$, $\alpha>1/2$. *Suppose that* $E|X_1|^r\log^+|X_1|<\infty$ *and that* $EX_1=0$ *if* $1\leqq r(<2)$. *If* (2.1) *holds with* N *as above, then* (2.2) *holds.*

However, if (2.1) *holds with* $P(A\leqq N\leqq B)=1$, *where* $0<\varepsilon<A\leqq B<\infty$, *then* $E|X_1|^r<\infty$ *and* $EX_1=0$ *for* $r\geqq 1$ *imply that* (2.2) *holds.*

c) *Let* $\alpha r\geqq 1$ *and* $\alpha>1/2$. *Suppose that* $E|X_1|^r<\infty$ *and that* $EX_1=0$ *if* $r\geqq 1$. *If* (2.1) *holds with* $P(N\leqq B)=1$ *for some* $B>0$, *then*

$$\sum_{n=1}^{\infty} n^{\alpha r-2}P(|S_{N_n}|>n^{\alpha}\varepsilon)<\infty. \tag{2.3}$$

REMARK 2.1. Szynal [16] proves that $EX_1^2<\infty$ and $EX_1=0$ and (2.1) with $P(A\leqq N\leqq B)=1$ imply (2.2), (i.e. the second half of b) with $r=2$, $\alpha=1$). The case $r\geqq 1$, $\alpha=1$ constitutes Example 7.1.1, pages 253—254 in Csörgő and Révész [4], where it is also assumed that N is bounded above and below and further that $E|X_1|^s<\infty$ for some $s>r+2$.

REMARK 2.2. To indicate the sharpness we recall that the finiteness of $\sum_{n=1}^{n} n^{\alpha r-2}P(|S_n|>n^{\alpha}\varepsilon)$ requires $E|X_1|^r<\infty$ (and $EX_1=0$ if $r\geqq 1$), cf. Theorem 1.1.

PROOF OF a). The idea is to exploit Theorem 1.1 (iv).

$$P(|S_{N_n}|>N_n^{\alpha}\varepsilon)\leqq P(\{|S_{N_n}|>N_n^{\alpha}\varepsilon\}\cap\{|N_n-Nn|\leqq n\varepsilon\})+P(|N_n-Nn|>n\varepsilon)\leqq$$

$$\leqq P\Big(\sup_{k\geqq n(A-\varepsilon)}|S_k/k^{\alpha}|>\varepsilon\Big)+P(|N_n-Nn|>n\varepsilon).$$

Thus,

$$\sum_{n=1}^{\infty} n^{\alpha r-2}P(|S_{N_n}|>N_n^{\alpha}\varepsilon)\leqq\sum_{n=1}^{\infty} n^{\alpha r-2}P\Big(\sup_{k\geqq n(A-\varepsilon)}|S_k/k^{\alpha}|>\varepsilon\Big)+$$

$$+\sum_{n=1}^{\infty} n^{\alpha r-2}P(|N_n-Nn|>n\varepsilon)$$

and the conclusion follows from Theorem 1.1 (iv).

PROOF OF b). The same computations in conjunction with Theorem 1.2 (iv) yield the first assertion.

For the second one we note that

$$P(\{|S_{N_n}|>N_n^{\alpha}\varepsilon\}\cap\{|N_n-Nn|\leqq n\varepsilon\})\leqq$$

$$\leqq P\Big(\max_{n(A-\varepsilon)\leqq k\leqq n(B+\varepsilon)}|S_k|>(n(A-\varepsilon))^{\alpha}\varepsilon\Big)\leqq P\Big(\max_{k\leqq 2nB}|S_k|>n^{\alpha}(A-\varepsilon)^{\alpha}\varepsilon\Big)$$

and thus that

$$\sum_{n=1}^{\infty} n^{\alpha r-2}P(|S_{N_n}|>N_n^{\alpha}\varepsilon)\leqq$$

$$\leqq\sum_{n=1}^{\infty} n^{\alpha r-2}P\Big(\max_{k\leqq 2nB}|S_k|>n^{\alpha}(A-\varepsilon)^{\alpha}\varepsilon\Big)+\sum_{n=1}^{\infty} n^{\alpha r-2}P(|N_n-Nn|>n\varepsilon)$$

and the conclusion follows from Theorem 1.1 (iii).

PROOF OF c). The proof here is almost the same as before, the difference being that we use the estimate

$$P(\{|S_{N_n}| > n^\alpha\varepsilon\} \cap \{|N_n - Nn| \leqq n\varepsilon\}) \leqq P\Big(\max_{k \leqq n(B+\varepsilon)} |S_k| > n^\alpha\varepsilon\Big).$$

This concludes the proof of the theorem.

3. First passage times

Throughout this section $\{X_n\}_{n=1}^\infty$ is a sequence of i.i.d. random variables with positive, finite expectation, μ. We define the first passage time

$$N(t) = \min\{n; S_n > t\}, \quad t \geqq 0, \tag{3.1}$$

but restrict it to t being positive, integer valued, i.e.

$$N_n = \min\{k; S_k > n\}, \quad n = 1, 2, \ldots . \tag{3.2}$$

THEOREM 3.1. *Let* $r \geqq 1$, $\frac{1}{2} < \alpha \leqq 1$ *and* $\alpha r \geqq 1$. *If* $E|X_1|^r < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{\alpha r - 2} P(|N_n - n/\mu| > n^\alpha\varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

REMARK 3.1. Note that, for $r=2$ and $\alpha=1$, the theorem states that $\{N_n/n\}_{n=1}^\infty$ converges completely to $1/\mu$. Note also that (2.1) is satisfied with $N=1/\mu$, since $\alpha \leqq 1$.

PROOF. It is clearly enough to prove the theorem for $\varepsilon < 1$.

Assume that $\mu=1$. This is only a matter of scaling and no restriction. Set $M_n = \max\{S_1, S_2, \ldots, S_n\}$ and observe that

$$S_n - n \leqq M_n - n \leqq \max_{1 \leqq k \leqq n} (S_k - k), \tag{3.3}$$

$$\{N_n > k\} = \{M_k < n\}. \tag{3.4}$$

To simplify the notation we pretend that $n+n^\alpha\varepsilon$ and $n-n^\alpha\varepsilon$ are integers. It follows that

$$\begin{aligned}
\{|N_n - n| > n^\alpha\varepsilon\} &= \{N_n > n + n^\alpha\varepsilon\} \cup \{N_n < n - n^\alpha\varepsilon\} = \{M_{n+n^\alpha\varepsilon} < n\} \cup \{M_{n-n^\alpha\varepsilon} > n\} = \\
&= \{M_{n+n^\alpha\varepsilon} - (n + n^\alpha\varepsilon) < -n^\alpha\varepsilon\} \cup \{M_{n-n^\alpha\varepsilon} - (n - n^\alpha\varepsilon) > n^\alpha\varepsilon\} \subset \\
&\subset \{S_{n+n^\alpha\varepsilon} - (n + n^\alpha\varepsilon) < -n^\alpha\varepsilon\} \cup \Big\{\max_{1 \leqq k \leqq n - n^\alpha\varepsilon} (S_k - k) > n^\alpha\varepsilon\Big\} \subset \\
&\subset \{|S_{n+n^\alpha\varepsilon} - (n + n^\alpha\varepsilon)| > n^\alpha\varepsilon\} \cup \Big\{\max_{1 \leqq k \leqq n - n^\alpha\varepsilon} |S_k - k| > n^\alpha\varepsilon\Big\} \subset \\
&\subset \Big\{\max_{1 \leqq k \leqq n + n^\alpha\varepsilon} |S_k - k| > n^\alpha\varepsilon\Big\} \subset \Big\{\max_{1 \leqq k \leqq n(1+\varepsilon)} |S_k - k| > n^\alpha\varepsilon\Big\}.
\end{aligned}$$

Thus,

$$\sum_{n=1}^{\infty} n^{\alpha r - 2} P(|N_n - n| > n^\alpha\varepsilon) \leqq \sum_{n=1}^{\infty} n^{\alpha r - 2} P\Big(\max_{1 \leqq k \leqq n(1+\varepsilon)} |S_k - k| > n^\alpha\varepsilon\Big),$$

which is finite in view of Theorem 1.1 (iii) and the proof is complete.

4. The stopped sum and the overshoot

It is now an easy matter to combine Theorems 2.1 and 3.1 to obtain results about the stopped sum $\{S_{N_n}\}_{n=1}^{\infty}$ with $\{N_n\}_{n=1}^{\infty}$ defined as in (3.2).

By choosing $N=1/\mu$ in (2.1) the following result emerges.

THEOREM 4.1. *Let* $r\geqq 1$, $1/2<\alpha\leqq 1$ *and* $\alpha r\geqq 1$. *Further, let* $\{X_n\}_{n=1}^{\infty}$ *be a sequence of i.i.d. random variables with positive, finite expectation,* μ, *and define* $N_n=\min\{n;\ S_k>n\}$ *for* $n=1, 2, \ldots$. *If* $E|X_1|^r<\infty$, *then*

(i) $$\sum_{n=1}^{\infty} n^{\alpha r-2}P(|S_{N_n}-N_n\mu|>N_n^{\alpha}\varepsilon)<\infty \quad \textit{for all} \quad \varepsilon>0$$

(ii) $$\sum_{n=1}^{\infty} n^{\alpha r-2}P(|S_{N_n}-N_n\mu|>n^{\alpha}\varepsilon)<\infty \quad \textit{for all} \quad \varepsilon>0.$$

In this context it is natural to investigate the overshoot or the so called excess over the boundary, which is defined as

$$R_n = S_{N_n}-n. \tag{4.1}$$

Further, since $n<S_{N_n}\leqq n+X_{N_n}$, we also have

$$0\leqq R_n\leqq X_{N_n}. \tag{4.2}$$

Thus, if $\mu=1$, $\varepsilon<1$, $r\geqq 1$, $1/2<\alpha\leqq 1$ and $\alpha r\geqq 1$, then

$$P(R_n>N_n^{\alpha}\varepsilon)\leqq P(X_{N_n}>N_n^{\alpha}\varepsilon)\leqq P(\{X_{N_n}>N_n^{\alpha}\varepsilon\}\cap\{|N_n-n|\leqq n\varepsilon\})+$$

$$+P(|N_n-n|>n\varepsilon)\leqq P\Big(\max_{n(1-\varepsilon)\leqq k\leqq n(1+\varepsilon)}|X_k|>(n(1-\varepsilon))^{\alpha}\varepsilon\Big)+P(|N_n-n|>n\varepsilon)\leqq$$

$$\leqq 2n\varepsilon P\big(|X_1|>n^{\alpha}(1-\varepsilon)^{\alpha}\varepsilon\big)+P(|N_n-n|>n^{\alpha}\varepsilon).$$

Consequently, if $E|X_1|^r<\infty$, then

$$\sum_{n=1}^{\infty} n^{\alpha r-2}P(R_n>N_n^{\alpha}\varepsilon)\leqq\sum_{n=1}^{\infty} n^{\alpha r-2}P(X_{N_n}>N_n^{\alpha}\varepsilon)\leqq$$

$$\leqq 2\varepsilon\sum_{n=1}^{\infty} n^{\alpha r-1}P\big(|X_1|>n^{\alpha}(1-\varepsilon)^{\alpha}\varepsilon\big)+\sum_{n=1}^{\infty} n^{\alpha r-2}P(|N_n-n|>n^{\alpha}\varepsilon)<\infty$$

by Theorem 3.1 and because of the fact that $\sum_{n=1}^{\infty} n^{\alpha r-1}P(|X_1|>n^{\alpha})<\infty$ if and only if $E|X_1|^r<\infty$.

A similar argument yields

$$P(R_n>n^{\alpha}\varepsilon)\leqq P(X_{N_n}>n^{\alpha}\varepsilon)\leqq 2n\varepsilon P(|X_1|>n^{\alpha}\varepsilon)+P(|N_n-n|>n^{\alpha}\varepsilon).$$

The following result is thus obtained:

THEOREM 4.2. *Under the hypotheses of Theorem* 4.1 *we have*

$$\text{(i)} \quad \sum_{n=1}^{\infty} n^{\alpha r-2} P(X_{N_n} > N_n^{\alpha}\varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 0$$

$$\text{(ii)} \quad \sum_{n=1}^{\infty} n^{\alpha r-2} P(X_{N_n} > n^{\alpha}\varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 0$$

$$\text{(iii)} \quad \sum_{n=1}^{\infty} n^{\alpha r-2} P(R_n > N_n^{\alpha}\varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 0$$

$$\text{(iv)} \quad \sum_{n=1}^{\infty} n^{\alpha r-2} P(R_n > n^{\alpha}\varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

5. Some further results

A. By using obvious modifications of the proofs above together with Lai [13], Theorem 3, Davis [6], Theorem 1 and Gut [10], Section 3 one can obtain results corresponding to the limiting case $\alpha = 1/2$. The proofs are omitted.

THEOREM 5.1. a) *Let* $r > 2$. *Suppose that* $EX_1 = 0$, $EX^2 = \sigma^2$ *and* $E|X|^r \cdot (\log^+ |X|)^{-r/2} < \infty$. *If, for some* $\varepsilon > \sigma\sqrt{r-2}$,

$$\sum_{n=1}^{\infty} n^{(r/2)-2} P(|N_n - Nn| > n\varepsilon) < \infty, \tag{5.1}$$

where N *is a positive random variable, such that* $P(N \leqq A) = 1$ *for some* $A > \varepsilon > 0$, *then*

$$\sum_{n=1}^{\infty} n^{(r/2)-2} P\left(|S_{N_n}| > \varepsilon \sqrt{N_n \log^+ N_n}\right) < \infty. \tag{5.2}$$

b) *Let* $r = 2$. *Suppose that* $EX_1 = 0$ *and* $EX^2 < \infty$. *If* (5.1) *holds with* N *as above, then* (5.2) *holds.*

c) *Let* $r \geqq 2$. *Suppose that* $EX_1^2 < \infty$ *and* $EX_1 = 0$. *If* (5.1) *holds with* $P(N \leqq B) = 1$ *for some* $B > 0$, *then*

$$\sum_{n=1}^{\infty} n^{(r/2)-2} P(|S_{N_n}| > \varepsilon \sqrt{n \log n}) < \infty. \tag{5.3}$$

REMARK 5.1. In view of Gut [10], Theorem 3.5 and the comments following that theorem nothing better than b) can be said for the case $P(A \leqq N \leqq B) = 1$ (cf. the second half of b) in Theorem 2.1 above and in Theorem 5.2 below).

B. A similar result can also be proved for the law of the iterated logarithm by applying results from Gut [10], Section 6.

THEOREM 5.2. a) *Suppose that* $EX_1 = 0$, $EX_1^2 = \sigma^2$ *and that*

$$EX_1^2 \cdot \frac{\log^+ |X_1|}{\log^+ \log^+ |X_1|} < \infty.$$

If, for some $\varepsilon>2\sigma$,

$$\sum_{n=1}^{\infty}\frac{1}{n}\cdot P(|N_n-Nn|>n\varepsilon)<\infty, \tag{5.4}$$

where N *is a positive random variable, such that* $P(N\geqq A)=1$ *for some* $A>\varepsilon>0$, *then*

$$\sum_{n=1}^{\infty}\frac{1}{n}\cdot P(|S_{N_n}|>\varepsilon\sqrt{N_n\log^+\log^+ N_n})<\infty. \tag{5.5}$$

However, if (5.4) *holds for some* $\varepsilon>\sigma\sqrt{2}$ *with* $P(A\leqq N\leqq B)=1$ *for* $0<\varepsilon<A\leqq B<\infty$, *then* $EX_1^2=\sigma^2<\infty$ *and* $EX_1=0$ *imply that* (5.5) *holds.*

b) *Suppose that* $EX_1^2=\sigma^2<\infty$ *and* $EX_1=0$. *If* (5.4) *holds for some* $\varepsilon>\sigma\sqrt{2}$ *with* $P(N\leqq B)=1$ *for some* $B>0$, *then*

$$\sum_{n=3}^{\infty}\frac{1}{n}\cdot P(|S_{N_n}|>\varepsilon\sqrt{n\log\log n})<\infty. \tag{5.6}$$

REMARK 5.2. Since sums of the type $\sum_{n=3}^{\infty}n^pP(|S_n|>\varepsilon\sqrt{n\log\log n})$ (where $EX_1=0$) can not converge for $p>-1$ (see Gut [10] and further references given there) there is no result in Theorem 5.2 corresponding to part a) of Theorems 2.1 and 5.1.

C. The application to first passage times is also possible in the cases considered in A and B of this section. This follows from the above results together with obvious modifications of Theorem 3.1 (recall however the comments made in Remark 5.1 concerning Theorem 5.1 b)). It follows, for example, that if $\{X_n\}_{n=1}^{\infty}$ are i.i.d with mean $\mu>0$ and finite variance σ^2, then, for $\varepsilon>\sigma\sqrt{2}$, we have

$$\sum_{n=3}^{\infty}\frac{1}{n}\cdot P(|N_n-n/\mu|>\varepsilon\sqrt{n\log\log n})<\infty \tag{5.5}$$

$$\sum_{n=1}^{\infty}\frac{1}{n}\cdot P(|S_{N_n}-N_n\mu|>\varepsilon\sqrt{N_n\log^+\log^+ N_n})<\infty \tag{5.6}$$

$$\sum_{n=3}^{\infty}\frac{1}{n}\cdot P(|S_{N_n}-N_n\mu|>\varepsilon\sqrt{n\log\log n})<\infty. \tag{5.7}$$

D. By defining $S(t)=S_{[t]}$ for $t\geqq 0$ (or by defining $S(t)=S_{[t]}$ if t is an integer and through linear interpolation otherwise) and by using the usual definition, (3.1), of first passage times, all results above involving first passage times can be reformulated for $N(t)$ and $S(t)$ with the aid of integrals (cf. e.g. Chow [3]). For example, under the conditions of Theorem 4.1 we have, for all $\varepsilon>0$,

$$\int_1^{\infty}t^{\alpha r-2}P(|N(t)-t/\mu|>t^{\alpha}\varepsilon)\,dt<\infty \tag{5.8}$$

$$\int_1^{\infty}t^{\alpha r-2}P(|S_{N(t)}-N(t)\mu|>t^{\alpha}\varepsilon)\,dt<\infty. \tag{5.9}$$

References

[1] S. Asmussen and T. G. Kurtz, Necessary and sufficient conditions for complete convergence in the law of large numbers, *Ann. Probability*, **8** (1980), 176—182.
[2] L. E. Baum and M. Katz, Convergence rates in the law of large numbers, *Trans. Amer. Math. Soc.*, **120** (1965), 108—123.
[3] Y. S. Chow, Delayed sums and Borel summability of independent, identically distributed random variables, *Bull. Inst. Math. Acad. Sinica*, (1973), 207—220.
[4] M. Csörgö and P. Révész, *Strong approximations in probability and statistics*. Akadémiai Kiadó (Budapest, 1981).
[5] J. A. Davis, Convergence rates for the law of the iterated logarithm, *Ann. Math. Statist.*, **39** (1968), 1479—1485.
[6] J. A. Davis, Convergence rates for probabilities of moderate deviations, *Ann. Math. Statist.*, **39** (1968), 2016—2028.
[7] P. Erdős, On a theorem of Hsu and Robbins, *Ann. Math. Statist.*, **20** (1949), 286—291.
[8] P. Erdős, Remark on my paper "On a theorem of Hsu and Robbins", *Ann. Math. Statist.*, **21** (1950), 138.
[9] A. Gut, Marcinkiewicz laws and convergence rates in the law of large numbers for random variables with multidimensional indices, *Ann. Probability*, **6** (1978), 469—482.
10] A. Gut, Convergence rates for probabilities of moderate deviations for sums of random variables with multidimensional indices, *Ann. Probability*, **8** (1980), 298—313.
[11] P. L. Hsu and H. Robbins, Complete convergence and the law of large numbers, *Proc. Nat. Acad. Sci. U.S.A.*, **33** (1947), 25—31.
[12] Melvin L. Katz, The probability in the tail of a distribution, *Ann. Math. Statist.*, **34** (1963), 312—318.
[13] T. L. Lai, Limit theorems for delayed sums, *Ann. Probability*, **2** (1974), 432—440.
[14] R. T. Smythe, Sums of independent random variables on partially ordered sets, *Ann. Probability*, **2** (1974), 906—917.
[15] F. Spitzer, A combinatorial lemma and its applications to probability theory, *Trans. Amer. Math. Soc.*, **82** (1956), 323—339.
[16] D. Szynal, On almost complete convergence for the sum of a random number of independent random variables, *Bull. Acad. Sci. Polon. Ser. Math. Astronom. Phys.*, **20** (1972), 571—574.

(Received December 27, 1981)

UPPSALA UNIVERSITY
DEPARTMENT OF MATHEMATICS
THUNBERGSVÄGEN 3
S-752 38 UPPSALA
SWEDEN

Acta Math. Hung.
42 (3—4) (1983), 233—236.

EINE CHARAKTERISIERUNG DER FINSLERSCHEN RÄUME VON KONSTANTER KRÜMMUNG

S. BÁCSÓ (Debrecen)

In der Theorie der Finslerschen Räume F_n $(n>2)$ ist es wohlbekannt, daß in einem Finslerschen Raum von skalarer Krümmung die Weylsche Projektivkrümmung verschwindet (vgl. [1], S. 776, [4] Kap. IV, § 8). (In dieser Arbeit nehmen wir durchweg $n>2$ an.) Als Umkehrung dieses Satzes hat L. Berwald folgendes bewiesen: Die Räume F_n von skalarer Krümmung sind unter den Finslerschen Räumen mit Projektivkrümmung Null durch

$$R_0{}^p{}_{0h\|p}-\frac{1}{n-1}R_0{}^p{}_{0p\|h}+R_0{}^p{}_{0p}l_h=0 \tag{1}$$

gekennzeichnet ([1] S. 776, [4] Kap. IV. § 8), wo $R_j{}^i{}_{kl}(x,\dot{x})$ der Krümmungstensor der Cartanschen Theorie ist und $R_0{}^p{}_{0h} \overset{\text{def}}{=} R_i{}^p{}_{jh}l^i l^j$, $l^i=\frac{\dot{x}^i}{L}$, $l_i=\frac{\partial L}{\partial \dot{x}^i}$ gelten. $L(x,\dot{x})$ ist die metrische Grundfunktion von F_n. Die Operation $\|_h$ ist durch $L\dot{\partial}_h$ definiert, wo $\dot{\partial}_h=\frac{\partial}{\partial \dot{x}^h}$ bedeutet.

Nach I. Z. Szabó [5], [6], M. Matsumoto [2] und P. N. Pandey [3] ist ein Finslerscher Raum F_n genau dann von skalarer Krümmung, wenn sein Weylscher Tensor $W^i{}_{jk}$ verschwindet. Für Finslersche Räume von skalarer Krümmung sind daher die Gleichungen (1) automatisch erfüllt.

Im folgenden geben wir eine Charakterisierung der Finslerschen Räumen von konstanter Krümmung unter den F_n von skalarer Krümmung. Der Satz hat eine ähnliche Struktur wie der erwähnte Satz von Berwald. Wir benutzen die Bezeichungen von H. Rund [4] Kap. IV. § 6.—8.

SATZ 1. *Ein Finslerscher Raum F_n von skalarer Krümmung ist genau dann konstanter Krümmung, wenn*

$$H_{\|j}+LH_j-(n+1)Hl_j=0, \tag{2}$$

wo $H^i{}_{jk}(x,\dot{x})$ der Grundtensor der affinen Krümmung und $H=\frac{1}{n-1}H_r{}^j{}_j\dot{x}^r$ sind.

BEWEIS. Der Grundtensor der Projektivkrümmung des Raumes F_n ist

$$W^i{}_{jk}=H^i{}_{jk}+\frac{\dot{x}^i}{n+1}H_r{}^r{}_{kj}+\frac{\delta^i_j}{n+1}(\dot{\partial}_k H+H_k)-\frac{\delta^i_k}{n+1}(\dot{\partial}_j H+H_j). \tag{3}$$

Nach Überschiebung der Gleichungen (3) mit dem Einheitsvektor l_i folgt wegen

$l_i H^i{}_{jk}=0$ (vgl. [4] Kap. IV. § 8. S. 149)

$$l_i W^i{}_{jk} = \frac{L}{n+1} H_r{}^r{}_{kj} + \frac{l_j}{n+1}(\dot\partial_k H + H_k) - \frac{l_k}{n+1}(\dot\partial_j H + H_j). \tag{4}$$

Setzen wir den Tensor $H_r{}^r{}_{jk}$ aus (4) in (3) ein, so bekommen wir

$$H^i{}_{jk} = \frac{\delta^i_k}{n+1}(\dot\partial_j H + H_j) - \frac{\delta^i_j}{n+1}(\dot\partial_k H + H_k) + \tag{5}$$

$$+ \frac{l^i l_j}{n+1}(\dot\partial_k H + H_k) - \frac{l_k l^i}{n+1}(\dot\partial_j H + H_j) + l^i l_r W^r{}_{jk} - W^i{}_{jk}.$$

Diese Gleichungen haben in einem Raum von skalarer Krümmung wegen $W^i{}_{jk}=0$ die Form

$$H^i{}_{jk} = \frac{1}{n+1}(\dot\partial_j H + H_j)(\delta^i_k - l^i l_k) - \frac{1}{n+1}(\dot\partial_k H + H_k)(\delta^i_j - l^i l_j). \tag{6}$$

Wenn die Gleichungen (2), die sich in der Form

$$L(\dot\partial_j H + H_j) = (n+1) H l_j \tag{7}$$

schreiben lassen, gültig sind, so bekommen wir

$$H^i{}_{jk} = \frac{H}{L}(l_j \delta^i_k - l_k \delta^i_j), \tag{8}$$

daß heißt F_n ist ein Raum von konstanter Krümmung (vgl. [4] Kap. IV. §7. (7.11)).

Umgekehrt, nehmen wir an, daß F_n ein Raum von konstanter Krümmung ist, d. h. die Gleichungen (8) gültig sind. Ein Raum von konstanter Krümmung ist auch von skalarer Krümmung. Daher sind die Gleichungen (6) auch in diesem Fall gültig. Durch Vergleich von (8) mit (6) erhalten wir (7), d. h. (2).

Unter Anwendung von Satz 1 können wir auch eine neue Charakterisierung der Finslerschen Räume von skalarer Krümmung angeben.

LEMMA 1. $l^i W^i_k = 0$ *gilt dann und nur dann, wenn*

$$W^i_k = H^i_k - H(\delta^i_k - l^i l_k), \tag{9}$$

wo $W^i_k = W^i{}_{jk}\dot x^j$, $H^i_k = H^i{}_{jk}\dot x^j$ *sind.*

BEWEIS. Es sei $l_i W^i_k = 0$. Nach Überschiebung der Gleichungen (5) mit dem Vektor $\dot x^j$ bekommen wir (9) unter Anwendung von $H_i \dot x^i = (n-1)H$ ([4], Kap. IV. (6.24)). Ungekehrt, gilt (9), so ist wegen $l_i H^i_k = 0$ (vgl. [4], Kap. IV. §8. 149) $l_i W^i_k = 0$ erfüllt.

LEMMA 2. $l_i W^i_k = 0$ *und* (1) *sind äquivalent.*

BEWEIS. Wegen $H_h{}^i{}_{jk}\dot x^h = R_h{}^i{}_{jk}\dot x^h$ (vgl. [4], Kap. IV. §6. (6.5)), wo $H_h{}^i{}_{jk} = \dot\partial_h H^i{}_{jk}$ ist, haben die Bedingungen die folgende Form:

$$H^p_{h\|p} - H_{\|h} + (n+1) H l_n = 0. \tag{10}$$

Der projektive Abweichungstensor hat die Gestalt

$$W_k^i = H_k^i - H\delta_k^i - \frac{1}{n+1}(\dot\partial_r H_k^r - \dot\partial_k H)\dot x^i \tag{11}$$

(vgl. [4], Kap. IV. §8. (8.10)). Wenn wir nun die Gleichungen (11) mit dem Einheitsvektor l_i überschieben, so erhalten wir, daß $l_i W_k^i = 0$ dann und nur dann gilt, wenn (10) erfüllt ist.

LEMMA 3. *$l_i W^i{}_{jk} = 0$ ist genau dann gültig, wenn $l_i W_k^i = 0$.*

BEWEIS. Bekanntlich gilt

$$\dot\partial_r H_k^r = (n-1)\dot\partial_k H - 3H_k$$

([4], IV. (6.21)). Somit bekommen wir aus den mit $l_i W_k^i = 0$ äquivalenten Gleichungen (10)

$$(n-2)L\dot\partial_k H + (n+1)Hl_k - 3LH_k = 0. \tag{12}$$

Wir leiten (12) nach $\dot x^j$ ab, und dann ergibt sich durch Vertauschung der Indizes k und j und nach Subtraktion der erhaltenen Gleichungen

$$l_j(\dot\partial_k H + H_k) - l_k(\dot\partial_j H + H_j) + LH_r^r{}_{kj} = 0,$$

woraus $l_i W^i{}_{jk} = 0$ wegen (4) folgt.

Die Umkehrung ist eine Konsequenz der Gleichungen $W_k^i = W^i{}_{jk}\dot x^j$.

SATZ 2. *Ein Finslerscher Raum F_n $(n>2)$ ist genau dann von skalarer Krümmung, wenn $l_i W_h{}^i{}_{jk} = 0$ gilt.*

BEWEIS. Es sei $l_i W_h{}^i{}_{jk} = 0$, und betrachten wir die Relation

$$\dot\partial_h(l_i W^i{}_{jk}) = l_i W_h{}^i{}_{jk} + \frac{1}{L} W^i{}_{jk}(g_{hi} - l_h l_i), \tag{13}$$

die sich durch die Ableitung von $l_i W^i{}_{jk}$ ergibt. Aus $l_i W_h{}^i{}_{jk} = 0$ erhalten wir $l_i W^i{}_{jk} = 0$ wegen $W_h{}^i{}_{jk}\dot x^h = W^i{}_{jk}$. Überschiebt man das einzige verbleibende Glied von (13) mit g^{hs}, so ergibt sich $W^s{}_{jk} = 0$, daß heißt F_n ist ein Raum von skalarer Krümmung.

Umgekehrt, wenn F_n von skalarer Krümmung ist, gilt $W_h{}^i{}_{jk} = 0$, so daß $l_i W_h{}^i{}_{jk} = 0$ ist.

Literaturverzeichnis

[1] L. Berwald, Über Finslersche und Cartansche Geometrie. IV, *Annals of Math.*, **48** (1947), 755—781.

[2] M. Matsumoto, Projective changes of Finsler metrics and projectively flat Finsler spaces, *Tensor N. S.*, **34** (1980), 303—315.

[3] P. N. Pandey, On a Finsler space of zero projective curvature, *Acta Math. Acad. Sci. Hungar.*, **39** (1982), 387—388.

[4] H. Rund, *The differential geometry of Finsler spaces.* Springer-Verlag (Berlin—Göttigen—Heidelberg, 1959).

[5] I. Z. Szabó, Ein Finslerscher Raum ist gerade dann von skalarer Krümmung, wenn seine Weylsche Projektivkrümmung verschwindet, *Acta Sci. Math. Szeged*, **39** (1977), 163—168.

[6] I. Z. Szabó, Finslersche Projektivgeometrie II, Über Finslersche Projektivbündel mit Weylscher Projektivkrümmung Null, *Acta Math. Acad. Sci. Hungar.*, **35** (1980), 97—108.

(Eingegangen am 3.Januar 1982.)

KOSSUTH LAJOS UNIVERSITY
MATHEMMTICAL INSTITUTE
DEBRECEN, HUNGARY
H—4010

Acta Math. Hung.
42 (3—4) (1983), 237—241.

FOURIER TRANSFORMS AND CONVOLUTIONS OF GENERALIZED MEASURES DEFINED ON THE DYADIC FIELD

K. YONEDA (Osaka)

1. Introduction

In this paper we shall define Fourier transforms and convolutions of generalized measures, so-called *dyadic measure* or *quasi-measure* defined on the dyadic field and give fundamental theorems. For details of dyadic measures and the dyadic field we refer to [3].

Let $w_y(x)$ be a Walsh function (see [1], [2]), namely $w_y(x)=(-1)^{i+\sum_{j=1}^{} t_i u_j}$ where $x=(\ldots, t_i, \ldots)$ and $y=(\ldots, u_j, \ldots)$. Walsh functions are characters of the dyadic field. When m is a dyadic measure, we shall define the Walsh Fourier transform of m by the following equation:

$$\hat{m}[p/2^n, (p+1)^-/2^n] = 1/2^n \int_0^{(2^n)^-} w_y(p/2^n)\, m(dy)$$

where $\int_I m(dy)=m(I)$. In this paper I and I' are dyadic intervals. It is easy to see that $\hat{m}$ is a dyadic measure. We shall call $\hat{m}$ the Walsh—Fourier transform of m. We can easily prove that $\hat{\hat{m}}=m$.

At first we shall prove the following theorems on Walsh—Fourier transforms of dyadic measures.

THEOREM 1. *A dyadic measure m satisfies*

$$\sum_{p=0}^{\infty} |m[p/2^n, (p+1)^-/2^n]|^2 < \infty \tag{1}$$

for $n=0, \pm 1, \ldots$ if and only if there exists a function $\hat{f}$ which is a locally L^2-integrable function and $\hat{m}=m_{\hat{f}}$.

THEOREM 2. *A dyadic measure m satisfies*

$$\sum_{p=0}^{\infty} |m[p/2^n, (p+1^-)/2^n]| < \infty \tag{2}$$

for $n=0, \pm 1, \ldots$ if and only if there exists a function $\hat{f}$ whose Walsh Fourier series on each dyadic interval $[0, (2^n)^-]$ converges absolutely and $\hat{m}=m_{\hat{f}}$.

We may define the convolution of two dyadic measures m and m'. When

$$\sum_{k=0}^{\infty} m[(p+k)/2^n, (p+k)+1^-/2^n]\, m'[k/2^n/(k+1^-)/2^n]$$

exists and is finite for all n and p, write $m*m'[p/2^n, (p+1^-)/2^n]$ for it and say that $m*m'$ is the convolution of m and m'. It is easy to see that $m*m'$ is a dyadic measure and $m*m'=m'*m$. This definition is a generalization of the classical definition of the convolution of functions.

THEOREM 3. *If two dyadic measures m and m' satisfy* (1) *respectively, then there exist two locally L^2-integrable functions $\hat{f}$ and $\hat{g}$ such that*

$$(m*m')\hat{}\,(I) = \int_I \hat{f}(x)\hat{g}(x)\,dx$$

for each dyadic interval I and $\hat{m}=m_{\hat{f}}$ and $\hat{m}'=m_{\hat{g}}$.

THEOREM 4. *If a dyadic measure m satisfies* (2) *and a dyadic measure m' satisfies* $\sup_x |m'(I_n(x))|<\infty$ *for each* $n=0, \pm1, \ldots$, *then* $(m*m')\hat{}\,(I)=\int_I \hat{f}(x)\hat{m}'(dx)$ *where $\hat{f}$ is a locally integrable function satisfying $\hat{m}=m_{\hat{f}}$.*

2. Proofs of Theorems 1 and 2

Since $\hat{\hat{m}}=m$ and

$$\hat{m}[p/2^n, (p+1^-)/2^n] = 1/2^n \int_0^{(2^n)^-} w_y(p/2^n)\,m(dy),$$

we have

$$(3) \qquad m[p/2^n, (p+1^-)/2^n] = 1/2^n \int_0^{(2^n)^-} w_y(p/2^n)\,\hat{m}(dy) = 1/2^n \int_0^{1^-} w_p(t)\,\hat{m}(2^n\,dt).$$

From the hypothesis, the last equation of (3) is the p-th Walsh—Fourier coefficient of an L^2-integrable function. Set

$$m[p/2^n(p+1^-)/2^n] = \int_0^{1^-} f_n(2^n t)\,w_p(t)\,dt$$

where f_n is an L^2-integrable function on $[0, (2^n)^-]$. We can easily prove that $f_n(x)=f_{n+1}(x)$ on $[0, (2^n)^-]$. Set $f(x)=f_n(x)$ on $[0, (2^n)^-]$. We have

$$m[p/2^n, (p+1^-)/2^n] = \int_0^{1^-} f(2^n t)\,w_p(t)\,dt$$

and f is a locally L^2-integrable function.

The converse is obvious. Theorem 2 can be proved similarly.

3. Proofs of Theorems 3 and 4

From the definition we have for $2^N \leqq p < 2^{N+1}$

$$(m * m')\hat{}\,[p/2^n, (p \dot{+} 1^-)/2^n] = 1/2^n \int_0^{2^{n-}} w_p(x/2^n)(m * m')(dx) =$$

$$= 1/2^n \int_0^{1^-} w_p(t)(m * m')(2^n dt) =$$

$$= 1/2^n \sum_{k=0}^{2^{N+1}-1} w_p(k/2^{N+1})(m * m')[k/2^{N-n+1}, (k \dot{+} 1)^-)/2^{N-n+1}] =$$

$$= 1/2^n \sum_{k=0}^{2^{N+1}-1} w_p(k/2^{N+1}) \Big(\sum_{i=0}^{\infty} m[(k \dot{+} i)/2^{N-n+1}, (k \dot{+} i \dot{+} 1^-)/2^{N-n+1}] \times$$

$$\times m'[i/2^{N-n+1}, (i \dot{+} 1^-)/2^{N-n+1}] \Big) =$$

$$= 1/2^n \sum_{i=0}^{\infty} m'[i/2^{N-n+1}, (i \dot{+} 1^-)/2^{N-n+1}] \times$$

$$\times \sum_{k=0}^{2^{N+1}-1} w_k(p/2^{N+1}) m[i/2^{N-n+1}, (i \dot{+} 1^-)/2^{N-n+1}] =$$

$$= 1/2^n \sum_{s=0}^{\infty} \sum_{i=s2^{N+1}}^{(s+1)2^{N+1}-1} m'[i/2^{N+n-1}, (i \dot{+} 1^-)/2^{N+n-1}] \times$$

$$\times \sum_{k=0}^{2^{N+1}-1} w_k(p/2^{N+1}) m[i/2^{N-n+1}, (i \dot{+} 1^-)/2^{N-n+1}] =$$

$$= 1/2^n \sum_{s=0}^{\infty} \Big(\sum_{i=s2^{N+1}}^{(s+1)2^{N+1}-1} m'[i/2^{N-n+1}, (i \dot{+} 1^-)/2^{N-n+1}] \times$$

$$\times w_s(p/2^{N+1}) \Big) \Big(\sum_{j=s2^{N+1}}^{(s+1)2^{N+1}-1} m[j/2^{N-n+1}, (j \dot{+} 1^-)/2^{N-n+1}] w_j(p/2^{N+1}) \Big).$$

On the other hand, we have

$$\sum_{i=s2^{N+1}}^{(s+1)2^{N+1}-1} m[i/2^{N-n+1}, (i \dot{+} 1^-)/2^{N-n+1}] w_i(p/2^{N+1}) =$$

$$= \sum_{i=s2^{N+1}}^{(s+1)2^{N+1}-1} 1/2^{N-n+1} \int_0^{(2^{N-n+1})^-} w_i(t/2^{N-n+1}) \hat{m}(dt) w_i(p/2^{N+1}) =$$

$$= 1/2^{N-n+1} \int_0^{(2^{N-n+1})^-} \Big(\sum_{i=0}^{2^{N+1}-1} w_i(t/2^{N-n+1} \dot{+} p/2^{N+1}) \Big) \hat{m}(dt) =$$

$$= 1/2^{N-n+1} \int_0^{(2^{N-n+1})^-} w_{s2^n}(t) \Big(\sum_{i=0}^{2^{N+1}-1} w_i(t/2^{N-n+1} \dot{+} p/2^{N+1}) \Big) \hat{m}(dt) =$$

$$= 2^n \int_{p/2^n}^{(p \dot{+} 1^-)/2^n} w_{s2^n}(t) \hat{m}(dt),$$

since

$$\sum_{i=0}^{2^{N+1}-1} w_i(t/2^{N-n+1} \dot{+} p/2^{N+1}) = \begin{cases} 2^{N+1} & \text{for } t \in [p/2^n, (p \dot{+} 1^-)/2^n] \\ 0 & \text{otherwise.} \end{cases}$$

Then we obtain

$$(m * m')\hat{}\,[p/2^n, (p \dot{+} 1^-)/2^n] =$$

$$= 2^n \sum_{s=0}^{\infty} \Big(\int_{p/2^n}^{(p \dot{+} 1)/2^n} w_{s2^n}(x)\, \hat{m}(dx) \Big) \Big(\int_{p/2^n}^{(p \dot{+} 1^-)/2^n} w_{s2^n}(y)\, \hat{m}'(dy) \Big) =$$

$$= 2^n \sum_{s=0}^{\infty} \Big(\int_{p/2^n}^{(p \dot{+} 1^-)/2^n} w_{s2^n}(x)\, \hat{f}(x)\, dx \Big) \Big(\int_{p/2^n}^{(p \dot{+} 1^-)/2^n} w_{s2^n}(y)\, \hat{g}(y)\, dy \Big) =$$

$$= \lim_{k \to \infty} 2^n \int_{p/2^n}^{(p \dot{+} 1^-)/2^n} \int_{p/2^n}^{(p \dot{+} 1^-)/2^n} \hat{f}(x)\, \hat{g}(y) \Big(\sum_{s=0}^{2^k-1} w_{s2^n}(x \dot{+} y) \Big)\, dx\, dy =$$

$$= \lim_{k \to \infty} 2^{n+k} \sum_{i=p2^k}^{(p+1)2^k-1} \int_{i/2^{n+k}}^{(i \dot{+} 1^-)/2^{n+k}} \hat{f}(x)\, dx \int_{i/2^{n+k}}^{(i \dot{+} 1^-)/2^{n+k}} \hat{g}(y)\, dy =$$

$$= \lim_{k \to \infty} \sum_{i=p2^k}^{(p+1)2^k-1} \int_{i/2^{n+k}}^{(i \dot{+} 1^-)/2^{n+k}} \hat{f}(x) \Big(2^{n+k} \int_{i/2^{n+k}}^{(i \dot{+} 1^-)/2^{n+k}} \hat{g}(y)\, dy \Big)\, dx.$$

We know that if $x \in [i/2^s, (i \dot{+} 1^-)/2^s]$, then $\hat{g}_s(x) \equiv 2 \int_{i/2^s}^{(i \dot{+} 1^-)/2^s} \hat{g}(y)\, dy$ is the 2^s-th partial sum of the Walsh—Fourier series of $\hat{g}$ and converges to $\hat{g}$ in the L^2-norm. Therefore we get

$$\Big| (m * m')\hat{}\,[p/2^n, (p \dot{+} 1^-)/2^n] - \int_{p/2^n}^{(p \dot{+} 1^-)/2^n} \hat{f}(x)\, \hat{g}(x)\, dx \Big| =$$

$$= \Big| \sum_{i=p2^k}^{(p+1)2^\kappa-1} \int_{i/2^{n+k}}^{(i \dot{+} 1^-)/2^{n+k}} \hat{f}(x) [\hat{g}_{n+k}(x) - \hat{g}(x)]\, dx \Big| \leqq$$

$$\leqq \sum_{i=p2^k}^{(p+1)2^k-1} \Big[\int_{i/2^{n+k}}^{(i \dot{+} 1^-)/2^{n+k}} |\hat{f}(x)|^2\, dx \Big]^{1/2} \Big[\int_{i/2^{n+k}}^{(i \dot{+} 1^-)/2^{n+k}} |\hat{g}_{n+k}(x) - \hat{g}(x)|^2\, dx \Big]^{1/2} \leqq$$

$$\leqq \Big[\sum_i \int_{i/2^{n+k}}^{(i \dot{+} 1^-)/2^{n+k}} |\hat{f}(x)|^2\, dx \Big]^{1/2} \Big[\sum_j \int_{j/2^{n+k}}^{(j \dot{+} 1^-)/2^{n+k}} |\hat{g}_{n+k}(x) - \hat{g}(x)|^2\, dx \Big]^{1/2} =$$

$$= \Big[\int_{p/2^n}^{(p \dot{+} 1^-)/2^n} |\hat{f}(x)|^2\, dx \Big]^{1/2} \Big[\int_{p/2^n}^{(p \dot{+} 1^-)/2^n} |\hat{g}_{n+k}(x) - \hat{g}(x)|^2\, dx \Big]^{1/2}.$$

The last quantity tends to zero as k tends to infinity. We get

$$(m * m')\hat{}\,[p/2^n, (p \dot{+} 1^-)/2^n] = \int_{p/2^n}^{(p\dot{+}1^-)/2^n} \hat{f}(x)\,\hat{g}(x)\,dx$$

for all n and p. Theorem 3 is proved.

The proof of Theorem 4 is similar to that of Theorem 3.

COROLLARY 1. *If m is a dyadic measure and m' is a Radon measure which has mass 1 at x_0, then*

$$(m * m')(I) = m(x_0 \dot{+} I) \quad \textit{and} \quad (m * m')\hat{}\,(I) = \int_I w_{x_0}(y)\,\hat{m}(dy)$$

where $x_0 \dot{+} I = \{x_0 \dot{+} y : y \in I\}$.

COROLLARY 2. *If f and g are integrable functions then* $(f * g)\hat{}\,(x) = \hat{f}(x)\hat{g}(x)$.

References

[1] N. J. Fine, On Walsh functions, *Trans. Amer. Math. Soc.*, **65** (1949), 372—414.
[2] N. J. Fine, The generalized Walsh functions, *ibid*, **69** (1950), 66—77.
[3] K. Yoneda, Positive definite generalized measures on the dyadic field, *Acta Math. Acad. Sci. Hungar.*, **40** (1982), 147—151.

(Received January 20, 1982; revised August 25, 1982)

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OSAKA PREFECTURE
SAKAI, OSAKA, JAPAN

Acta Math. Hung.
42 (3—4) (1983), 243—255.

THE CESARO—DENJOY—BOCHNER SCALE OF INTEGRATION

B. K. PAL and S. N. MUKHOPADHYAY (Burdwan)

1. Introduction

Following [1, 11, 12] we have introduced Denjoy—Bochner integral and established its integration by parts formula in [8]. In the present paper we introduce a scale of Cesaro—Denjoy—Bochner integrals, the C_nD_*B-integrals, such that the strength of the integral is increased with n, the C_0D_*B-integral being the special Denjoy—Bochner integral.

2. Notations and terminology

Throughout the paper, R is the real line, X is a real Banach space, $\|\cdot\|$ its norm, and X^* its conjugate space. We shall use the notation $o(h^n)$, h real, to mean a quantity (real or X-valued, which will be clear from the context), which when divided by h^n, tends to 0 or θ (the zero element of X), as $h\to 0$.

DEFINITION. Let $F:[a,b]\to X$ and let $t_0\in[a,b]$. Let n be a positive integer. If there are constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ depending on t_0 but not on h such that

$$F(t_0+h)-F(t_0)-h\alpha_1-\frac{h^2}{2!}\alpha^2-\ldots-\frac{h^n}{n!}\alpha_n=o(h^n)$$

then α_n is called the strong Peano derivative of F at t_0 of order n and is denoted by $F_{(n)}(t_0)$. It is easily seen that if $F_{(n)}(t_0)$ exists then $F_{(n)}(t_0)$ $(1\leqq k\leqq n)$ exists. In particular $F_{(1)}(t_0)$ is the strong derivative of F at t_0. We shall write for convenience $F_{(0)}(t_0)=F(t_0)$.

Let n be any non negative integer. If $F_{(n)}(t_0)$ exists, we write

$$\varepsilon_n(t_0,t)=\varepsilon_n(F;\,t_0,t)=\begin{cases}\dfrac{n!}{(t-t_0)^n}\left[F(t)-\sum_{k=0}^{n}\dfrac{(t-t_0)^k}{k!}F_{(k)}(t_0)\right], & t\neq t_0\\ \theta, & t=t_0.\end{cases}$$

Clearly then $\varepsilon_n(F;\,t_0,t)$ is strongly continuous. Let $F_{(n)}$ exist in $[a,b]$. Let

$$\omega_n(c,d)=\max\left\{\sup_{c<t<d}\|\varepsilon_n(c,t)\|,\ \sup_{c<t<d}\|\varepsilon_n(d,t)\|\right\},\quad a\leqq c\leqq d\leqq b.$$

Then F is said to be strongly AC_{n*} over a set $E\subset[a,b]$ if $F_{(n)}$ exists in $[a,b]$ and for arbitrary $\varepsilon>0$ there is a $\delta>0$ such that for all finite collection of non overlapping intervals $(a_1,b_1),(a_2,b_2)\ldots(a_m,b_m)$ with end points in E and $\sum_{r=i}^{m}(b_n-a_r)<\delta$ we have

$$\sum_{r=1}^{m}\omega_n(a_r,b_r)<\varepsilon.$$

The function F is said to be strongly AC_nG_* in $[a, b]$ if F is strongly continuous on $[a, b]$ and if we can write $[a, b]=\bigcup_i E_i$ where E_i's are non-overlapping closed sets and F is strongly AC_{n*} over each E_i. (Note that the continuity of F is necessary only for $n=0$. For $n\geqq 1$, it is a consequence of the existence of $F_{(n)}$.) The definition of AC_nG_* for real functions is as in [9; p. 231] or [4] according as $n=0$ or >0.

3. Preliminary lemmas

LEMMA 1. *If $F:[a, b]\to X$ is such that at a point $t_0\in[a, b]$, $F_{(n)}(t_0)$ exists then for each $x^*\in X^*$, $(x^*F)_{(n)}(t_0)$ exists and $(x^*F)_{(n)}(t_0)=x^*(F_{(n)}(t_0))$. If F is strongly AC_nG_* in $[a, b]$ then for all $x^*\in X^*$, x^*F is a numerical valued AC_nG_* function in $[a, b]$.*

Since $x^*\varepsilon_n(F; t_0, t)=\varepsilon_n(x^*F; t_0, t)$ for each $x^*\in X^*$, the first part is clear. Also since for each $x^*\in X^*$

$$|\varepsilon_n(x^*F;\ t_0, t)| = |x^*\varepsilon_n(F; t_0, t)| \leqq \|x^*\|\,\|\varepsilon_n(F; t_0, t)\|$$

we have

$$\omega_n(x^*F; c, d) \leqq \|x^*\|\omega_n(F; c, d)$$

and hence the results follows.

LEMMA 2. *Let $F:[a, b]\to X$ be strongly continuous and let $F_{(n)}$ exist in $[a, b]$ and let*

$$\varphi(t) = \int_a^t F(\xi)\,d\xi, \quad \text{for} \quad a\leqq t\leqq b.$$

Then

(i) $\varphi_{(n+1)}(\xi)=F_{(n)}(\xi)$ *for all* $\xi\in[a, b]$.
(ii) *If* $F_{(n+1)}$ *exists at a point* t_0 *then* $\varphi_{(n+2)}(t_0)$ *exists and equals* $F_{(n+1)}(t_0)$.
(iii) *If* F *is* AC_{n*} *over a set* $E\subset[a, b]$ *then* φ *is* AC_{n+1*} *over* E.

PROOF. Let t_0 and t_0+h belong to $[a, b]$. Now

$$\varphi(t_0+h)-\varphi(t_0) = \int_{t_0}^{t_0+h} F(\xi)\,d\xi = \tag{3.1}$$

$$= \int_{t_0}^{t_0+h}\left[F(t_0)+(\xi-t_0)F_{(1)}(t_0)+\ldots+\frac{(\xi-t_0)^n}{n!}F_{(n)}(t_0)+\frac{(\xi-t_0)^n}{n!}\varepsilon_n(F; t_0, \xi)\right] =$$

$$= hF(t_0)+\frac{h^2}{2!}F_{(1)}(t_0)+\ldots+\frac{h^{n+1}}{(n+1)!}F_{(n)}(t_0)+\int_{t_0}^{t_0+h}\frac{(\xi-t_0)^n}{n!}\varepsilon_n(F; t_0, \xi)\,d\xi.$$

Since $\|\varepsilon_n(F; t_0, t_0+h)\|\to 0$ as $h\to 0$, we have

$$\lim_{h\to 0}\frac{(n+1)!}{h^{n+1}}\int_{t_0}^{t_0+h}\frac{(\xi-t_0)^n}{n!}\varepsilon_n(F; t_0, \xi)\,d\xi = \theta.$$

So, by (3.1) $\varphi_{(n+1)}(t_0)$ exists and equals $F_{(n)}(t_0)$.

Exactly as above we can show that $\varphi_{(n+2)}$ exists and is equal to $F_{(n+1)}$ at those points t_0 where $F_{(n+1)}$ exists.

Now by (3.1) we have

$$\varepsilon_{n+1}(\varphi;\, t_0, t_0+h) = \frac{(n+1)!}{h^{n+1}} \int_{t_0}^{t_0+h} \frac{(\xi-t_0)^n}{n!}\, \varepsilon_n(F;\, t_0, \xi)\, d\xi$$

from which (iii) follows.

LEMMA 3. *Let $n>0$ and $F: [a, b]\to X$ be such that $F_{(n)}$ exists in $[a, b]$ and F be strongly AC_{n*} in a closed set $E\subset[a, b]$. Then F is strongly AC_{n-1*} in E.*

PROOF. It can be shown as in [10, Lemma 1] that if $a\leqq c\leqq d\leqq b$ then

$$\|F_{(n)}(d)-F_{(n)}(c)\| \leqq A\omega_n(F;\, c, d)$$

where A depends only on n. Hence if $c, d\in E$ then

$$\big|\|F_{(n)}(d)\|-\|F_{(n)}(c)\|\big| \leqq A\omega_n(F;\, c, d)$$

and therefore the real valued function $\|F_{(n)}\|$ is absolutely continuous on E. The set E being closed, $\|F_{(n)}\|$ is bounded on E. Let M be the upper bound of $\|F_{(n)}\|$ on E. Then since

$$\varepsilon_{n-1}(F;\, t_0, t_1) = \frac{t_1-t_0}{n}\,[F_{(n)}(t_0)+\varepsilon_n(F;\, t_0, t_1)],$$

we have

$$\omega_{n-1}(F;\, c, d) \leqq \frac{|d-c|}{n}\,[M+\omega_n(F;\, c, d)]$$

whenever $c, d\in E$. Thus F is strongly AC_{n-1*} on E.

COROLLARY. *If F is strongly AC_nG_* on $[a, b]$ then F is strongly $AG_{n-1}G_*$ on $[a, b]$.*

LEMMA 4. *If a function $\varphi: [a, b]\to X$ is strongly AC_nG_* and $\varphi_{(n+1)}=\theta$ almost everywhere then $\varphi_{(n)}$ is constant.*

PROOF. Let $x^*\in X^*$ be arbitrary. By Lemma 1, $x^*\varphi$ is a numerical valued AC_nG_* function and moreover

$$(x^*\varphi)_{(n+1)} = x^*\varphi_{(n+1)} = x^*\theta = 0$$

almost everywhere. So, $(x^*\varphi)_{(n)}$ is constant [4]. But $\varphi_{(n)}$ exists and $(x^*\varphi)_{(n)}=$ $=x^*\varphi_{(n)}$. Since x^* is arbitrary, the lemma is proved.

4. The C_nD_*B integral

DEFINITION. A function $f: [a, b]\to X$ is said to be C_nD_*B (Cesaro—Denjoy—Bochner) integrable if there exists a function $F: [a, b]\to X$ which is strongly AC_nG_* in $[a, b]$ and $F_{(n+1)}=f$ almost everywhere in $[a, b]$. Then $F_{(n)}$ is called an indefinite C_nD_*B-integral of f and $F_{(n)}(b)-F_{(n)}(a)$ is called its definite C_nD_*B-

integral in $[a, b]$ and is denoted by

$$(C_nD_*B)\int_a^b f(t)\,dt.$$

The definite integral of an integrable function is unique by Lemma 4. It can be verified that the class of all C_nD_*B integrable functions in $[a, b]$ is a linear space and the C_nD_*B integral is a linear operator from this linear space to X, and this operator is additive on abutting intervals.

In [8] we have defined a special Denjoy—Bochner-integral, the D_*B-integral, such that a function $f:[a, b]\to X$ is D_*B integrable in $[a, b]$ if there is a function $F:[a, b]\to X$ such that F is strongly AC_0G_* in $[a, b]$ and $AD_sF=f$ almost everywhere in $[a, b]$, where AD_sF denotes the strong approximate derivative of F. This definition of the D_*B integral is equivalent to that given in [12]. The definition of the C_0D_*B-integral here differs from that of D_*B in using the strong derivative $F_{(1)}$ instead of strong approximate derivative AD_sF. Thus the C_0D_*B-integral is less comprehensive than the D_*B-integral in [8]. Nevertheless, we shall see that the C_0D_*B-integral is strictly more general than the Lebesgue—Bochner integral. We remark in passing that the C_0D_*B-integral will coincide with the D_*B-integral in [8] (and hence with the special Denjoy—Bochner integral in [12]) if the indefinite D_*B-integral F satisfies the additional condition that for each $t_0\in[a, b]$ there is $M=M(t_0)$ such that

$$\|F(t)-F(t_0)\| \leqq M|t-t_0|$$

for all t in some neighbourhood of t_0. (See Lemma 3 of [2].) On the other hand, the C_0D_*B-integral is analogous to the DB-integral defined in [11; p. 45].

THEOREM 5. *If f is C_nD_*B-integrable then f is strongly measurable.*

PROOF. Let $x^*\in X^*$ be arbitrary. By hypothesis there is a function $F:[a,b]\to X$ which is strongly AC_nG_* and $F_{(n+1)}=f$ almost everywhere. So, by Lemma 1 x^*F is AC_nG_* and $(x^*F)_{(n+1)}=x^*f$ almost everywhere. Thus x^*f is C_nD_*-integrable (or, equivalently C_nP-integrable) scalar function. Hence x^*f is measurable (see [3]). Since x^* is arbitrary f is weakly measurable.

Now $F_{(1)}, F_{(2)}, \ldots, F_{(n)}$ exist everywhere and $F_{(n+1)}=f$ almost everywhere. Since F is continuous, the range of F is separable. Let X_0 be the closure of the space spanned by the range of F. So, X_0 is separable. Since

$$\lim_{h\to 0}\frac{F(t+h)-F(t)}{h} = F_{(1)}(t),$$

so $F_{(1)}(t)\in X_0$ for all $t\in[a, b]$. So, $\frac{2!}{h^2}[F(t+h)-F(t)-hF_{(1)}(t)]\in X_0$ for all $h\neq 0$. Hence its limit $F_{(2)}(t)\in X_0$ for all $t\in[a, b]$. After finite number of steps we can show that $F_{(3)}(t), F_{(4)}(t), \ldots, F_{(n)}(t)$ belong to X_0 for all $t\in[a, b]$. Therefore

$$\frac{(n+1)!}{h^{n+1}}\left[F(t+h)-\sum_{k=0}^{n}\frac{h^k}{k!}F_{(k)}(t)\right]\in X_0$$

for all $h \neq 0$ and for all $t \in [a, b]$. Hence the limit $F_{(n+1)}(t)$, if it exists, is in X_0. Since $F_{(n+1)}$ exists and equals f almost everywhere in $[a, b]$, $f(t) \in X_0$ for almost all $t \in [a, b]$. Thus $f(t)$ is essentially separably valued. Hence by [7; Theorem 3.5.3, p. 72] f is strongly measurable.

THEOREM 6. *If f is Lebesgue—Bochner integrable in $[a, b]$ then f is C_0D_*B-integrable in $[a, b]$ and the integrals are equal.*

If f is Lebesgue—Bochner integrable in $[a, b]$ then its indefinite integral F is strongly absolutely continuous in $[a, b]$ [7, Theorem 3.7.11, p. 83] and f is the strong derivative of F almost everywhere in $[a, b]$ [7, Corollary 2, p. 68]. Since a strongly absolutely continuous function is strongly AC_0G_* the result follows.

THEOREM 7. *If f is $C_{n-1}D_*B$-integrable in $[a, b]$ then it is C_nD_*B-integrable in $[a, b]$ and the integrals are equal.*

PROOF. By the $C_{n-1}D_*B$ integrability of f there is a function, say F, which is strongly $AC_{n-1}G_*$ and $F_{(n)} = f$ almost everywhere and $F_{(n-1)}$ is the $C_{n-1}D_*B$ indefinite integral of f. Since F is strongly continuous, writing

$$\varphi(t) = \int_a^t F(\xi)\, d\xi$$

we get Lemma 2, that the function φ is strongly AC_nG_* and $\varphi_{(n)}(t) = F_{(n-1)}(t)$ everywhere and $\varphi_{(n+1)}(t) = F_{(n)}(t) = f(t)$ almost everywhere in $[a, b]$. Thus f is C_nD_*B-integrable in $[a, b]$. The rest is clear.

THEOREM 8. *If $f : [a, b] \to X$ is C_nD_*B-integrable in $[a, b]$ and*

$$F(t) = (C_nD_*B) \int_a^t f(\xi)\, d\xi, \quad t \in [a, b]$$

then F is $C_{n-1}D_*B$-integrable in $[a, b]$.

PROOF. Let $\varphi : [a, b] \to X$ be the function which is strongly AC_nG_* and $\varphi_{(n+1)} = f$ almost everywhere in $[a, b]$. By definition $\varphi_{(n)}(t) - \varphi_{(n)}(a) = F(t)$ for all $t \in [a, b]$. Also by Lemma 3, φ is strongly $AC_{n-1}G_*$ in $[a, b]$. Hence $F(t) + \varphi_{(n)}(a)$ is $C_{n-1}D_*B$ integrable in $[a, b]$. Since $\varphi_{(n)}(a)$ is a constant, F is $C_{n-1}D_*B$-integrable in $[a, b]$.

5. Integration by parts for C_nD_*B integrals

LEMMA 9. *If n is any positive integer then*

$$\sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} r^i = \begin{cases} 0 & \text{for } i = 0, 1, \ldots n-1, \\ n! & \text{for } i = n; \end{cases} \tag{5.1}$$

$$\sum_{r=0}^{n-k} (-1)^r \binom{n}{r} \binom{n-r}{k} = \begin{cases} 0 & \text{for } k = 0, 1, \ldots, n-1, \\ 1 & \text{for } k = n. \end{cases} \tag{5.2}$$

PROOF. The relation (5.1) is well known. To prove (5.2) we have, when $0 \leqq k \leqq n$

$$I = \sum_{r=0}^{n-k} (-1)^r \binom{n}{r}\binom{n-r}{k} = \frac{1}{k!}\sum_{r=0}^{n-k} (-1)^r \binom{n}{r}(n-r)(n-r-1)\ldots(n-r-k+1) =$$

$$= \frac{1}{k!}\sum_{r=0}^{n} (-1)^r \binom{n}{r}(n-r)(n-r-1)\ldots(n-r-k+1) =$$

$$= \frac{1}{k!}\sum_{r=0}^{n} (-1)^{r+k} \binom{n}{r}[r^k + p_1 r^{k-1} + \ldots + p_k]$$

where $p_1, p_2, \ldots, p_k$ depend on n and k but not on r. So, by (5.1) we have

$$I = \frac{(-1)^{n-k}}{k!}\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}\left[r^k + \sum_{j=1}^{k} p_j r^{k-j}\right] =$$

$$= \frac{(-1)^{n-k}}{k!}\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r} r^k = \begin{cases} 0 & \text{if } k = 0, 1, \ldots, n-1, \\ 1 & \text{if } k = n. \end{cases}$$

LEMMA 10. *Let* $F : [a, b] \to X$ *and* $G : [a, b] \to R$ *be two functions such that* $F_{(n)}(\xi)$ *and* $G_{(n)}(\xi)$ *exist. Then the function* $H = GF$ *is such that* $H_{(n)}(\xi)$ *exists and*

$$H_{(r)}(\xi) = \sum_{p=0}^{n}\binom{r}{p} F_{(p)}(\xi) G_{(r-p)}(\xi), \quad r = 0, 1, \ldots, n.$$

PROOF. We may suppose $\xi = 0$. Since $F_{(n)}(0)$ and $G_{(n)}(0)$ exist, we have

$$F(h) = \sum_{r=0}^{n}\frac{h^r}{r!} F_{(r)}(0) + o(h^n) \quad \text{and} \quad G(h) = \sum_{r=0}^{n}\frac{h^r}{r!} G_{(r)}(0) + o(h^n)$$

and hence

$$H(h) = G(h)\,F(h) = \sum_{r=0}^{n} h^r \left[\sum_{p=0}^{r}\frac{1}{p!\,(r-p)!} F_{(p)}(0) G_{(r-p)}(0)\right] + o(h^n) =$$

$$= \sum_{r=0}^{n}\frac{h^r}{r!}\left[\sum_{p=0}^{r}\binom{r}{p} F_{(p)}(0) G_{(r-p)}(0)\right] + o(h^n)$$

and this proves the lemma.

LEMMA 11. *Let* n *be any positive integer and let* $F : [a, b] \to X$ *be strongly* AC_nG_* *and* $G : [a, b] \to R$ *be such that* $G^{(n)}$ *is bounded and* G *be* AC_nG_* *on* $[a, b]$. *Then* FG *is strongly* AC_nG_* *on* $[a, b]$.

PROOF. Since $F_{(n)}$ and $G^{(n)}$ exist, by Lemma 10 $(FG)_{(n)}$ exists. To prove the theorem we first prove the relation

$$\varepsilon_n(FG;\, c,\, t) = F(t)\,\varepsilon_n(G;\, c,\, t) + \sum_{r=0}^{n}\binom{n}{r} G_{(r)}(c)\,\varepsilon_{n-r}(F;\, c,\, t) \tag{5.3}$$

where

$$\varepsilon_0(F;\, c,\, t) = F(t) - F(c).$$

Applying Lemma 10 for $n=1$ we have

$$\varepsilon_1(FG;\, c,\, t) = \frac{1}{t-c}\{F(t)G(t)-F(c)G(c)-F_{(1)}(c)G(c)(t-c)-F(c)G_{(1)}(c)(t-c)\} =$$

$$= G(c)\varepsilon_1(F;\, c,\, t)+F(t)\varepsilon_1(G;\, c,\, t)+G_{(1)}(c)\,\varepsilon_0(F:\, c,\, t).$$

Thus (5.3) is true for $n=1$. Suppose that it is true for $n-1$. Now for any function $\varphi\colon [a, b]\to X$ having n-th strong Peano derivative we have from the definition of ε_n,

$$\varepsilon_n(\varphi;\, c,\, t) = \frac{n}{t-c}\varepsilon_{n-1}(\varphi;\, c,\, t)-\varphi_{(n)}(c) \tag{5.4}$$

where n is any positive integer. So applying (5.4), Lemma 10 and (5.3) for $n-1$ we get

$$\varepsilon_n(FG;\, c,\, t) = \frac{n}{t-c}\varepsilon_{n-1}(FG;\, c,\, t)-(FG)_{(n)}(c) = \tag{5.5}$$

$$= \frac{n}{t-c}\left\{F(t)\varepsilon_{n-1}(G;\, c,\, t)+\sum_{r=0}^{n-1}\binom{n-1}{r}G_{(r)}(c)\varepsilon_{n-1-r}(F;\, c,\, t)\right\}-$$

$$-\sum_{r=0}^{n}\binom{n}{r}F_{(n-r)}(c)G_{(r)}(c) =$$

$$= F(t)\varepsilon_n(G;\, c,\, t)+G(c)\varepsilon_n(F;\, c,\, t)+\frac{n}{t-c}\sum_{r=1}^{n-1}\binom{n-1}{r}G_{(r)}(c)\varepsilon_{n-1-r}(F;\, c,\, t)-$$

$$-\sum_{r=0}^{n-1}\binom{n}{r}G_{(r)}(c)F_{(n-r)}(c)+G_{(n)}(c)\varepsilon_0(F;\, c,\, t).$$

Now using (5.4)

$$\frac{n}{t-c}\binom{n-1}{r}G_{(r)}(c)\varepsilon_{n-1-r}(F;\, c,\, t)-\binom{n}{r}G_{(r)}(c)F_{(n-r)}(c) =$$

$$= \binom{n}{r}G_{(r)}(c)\left[\frac{n-r}{t-c}\varepsilon_{n-1-r}(F;\, c,\, t)-F_{(n-r)}(c)\right] = \binom{n}{r}G_{(r)}(c)\varepsilon_{n-r}(F;\, c,\, t).$$

Hence from (5.5)

$$\varepsilon_n(FG;\, c,\, t) = F(t)\varepsilon_n(G;\, c,\, t)+\sum_{r=0}^{n}\binom{n}{r}G_{(r)}(c)\varepsilon_{n-r}(F;\, c,\, t)$$

which shows that (5.3) is true for n. Thus by induction (5.3) is true for all n. Now, let

$$\sup_{a\leqq t\leqq b}\|F(t)\| = M, \quad \sup_{a\leqq t\leqq b}|G_{(r)}(t)| = M_r; \quad r = 0, 1, \ldots n.$$

Then using relation (5.3) and the definition of ω_n we get

$$\omega_n(FG;\, c,\, d) \leqq M\omega_n(G;\, c,\, d)+\sum_{r=0}^{n}\binom{n}{r}M_r\omega_{n-r}(F;\, c,\, d). \tag{5.6}$$

Now since F and G are strongly AC_nG_*, $[a, b]=\bigcup_{k=1}^{\infty} E_k$ such that F and G are AC_{n*} on each E_k. By Lemma 3, F is strongly AC_{n-r*} on each E_k for $r=1,2,\ldots,n$. Let $\varepsilon>0$ be arbitrary and E_k be fixed. Then there are positive numbers $\delta_0, \delta_1, \ldots, \delta_n$ and $\eta>0$ such that for every finite collection of non-over lapping intervals $\{[a_i, b_i]\}$ with end points in E_k we have

$$\sum_i \omega_n(G; a_i, b_i) < \varepsilon \quad \text{whenever} \quad \sum_i (b_i-a_i) < \eta$$

$$\sum_i \omega_{n-r}(F; a_i, b_i) < \varepsilon \quad \text{whenever} \quad \sum_i (b_i-a_i) < \delta_r$$

for $r=0, 1, 2, \ldots, n$. Thus choosing $\delta=\min[\delta_0, \delta_1, \ldots, \delta_n, \eta]$ we get from (5.6)

$$\sum_i \omega_n(FG; a_i, b_i) < \left[M+\sum_{r=0}^{n}\binom{n}{r} M_r\right]\varepsilon$$

whenever $\sum_i (b_i-a_i)<\delta$. Hence FG is strongly AC_{n*} on each E_k. Since FG is strongly continuous, this completes the proof of the lemma.

THEOREM 12. *Let n be any positive integer and let $F:[a, b]\to X$ be strongly AC_nG_* and $G:[a, b]\to R$ be such that $G^{(n)}$ is ACG_*. Then the function ψ defined by*

$$\psi(\xi) = F(\xi)G(\xi)+\sum_{r=1}^{n}(-1)^r\binom{n}{r}\frac{1}{(r-1)!}\int_a^{\xi}(\xi-t)^{r-1}F(t)G^{(r)}(t)\,dt$$

is strongly AC_nG_*.

PROOF. Since $G^{(n)}$ is ACG_*, $G^{(r)}$ is the $n-r$ fold integral of $G^{(n)}$. Hence by Lemma 2, $G^{(r)}$ is $AC_{n-r}G_*$ for $r=0, 1, 2, \ldots, n$. Moreover, $G^{(n)}$ is bounded and so $(G^{(r)})^{(n-r)}$ are bounded on $[a, b]$ for $r=0, 1, 2, \ldots, n$. Now since F is AC_nG_*, by Lemma 3 it is $AC_{n-r}G_*$ for $r=1, 2 \ldots n$. Thus by Lemma 11, $FG^{(r)}$ is $AC_{n-r}G_*$ for $r=0, 1, 2, \ldots, n$. Now, since $FG^{(r)}$ is strongly continuous, it is Lebesgue—Bochner-integrable and hence integrating by parts [8] it is seen that the integral

$$\frac{1}{(r-1)!}\int_a^{\xi}(\xi-t)^{r-t}F(t)G^{(r)}(t)\,dt$$

is the r-fold integral of $FG^{(r)}$, and hence by Lemma 2 it is strongly AC_nG_* for $r=1, 2, \ldots, n$. Thus every term of ψ is strongly AC_nG_* and so ψ is strongly AC_nG_* on $[a, b]$.

LEMMA 13. *Let n be any positive integer and let $F:[a, b]\to X$ and $G:[a, b]\to R$ be such that $F_{(n)}$ exists in $[a, b]$ and $G^{(n)}$ is continuous in $[a, b]$. Then the function $\psi:[a, b]\to X$ defined by*

$$\psi(\xi) = F(\xi)G(\xi)+\sum_{r=1}^{n}(-1)^r\binom{n}{r}\frac{1}{(r-1)!}\int_a^{\xi}(\xi-t)^{r-1}F(t)G^{(r)}(t)\,dt$$

is such that $\psi_{(n)}=F_{(n)}G$ in $[a, b]$. Moreover, if $F_{(n+1)}(t_0)$ and $G^{(n+1)}(t_0)$ exist

then $\psi_{(n+1)}(t_0)$ *exists and*

$$\psi_{(n+1)}(t_0) = F_{(n+1)}(t_0)\, G(t_0) + F_{(n)}(t_0)\, G^{(1)}(t_0).$$

PROOF. By Lemma 10, for fixed r, $1 \leqq r \leqq n$, the function $H = FG^{(r)}$ is such that $H_{(n-r)}$ exists in $[a, b]$. Let $t_0 \in [a, b]$. Then

$$H(t) = \sum_{i=0}^{n-r} \frac{(t-t_0)^i}{i!} H_{(i)}(t_0) + o\big((t-t_0)^{n-r}\big). \tag{5.7}$$

Now since H is continuous, integrating by parts [8] or [7, p. 63]

$$\frac{1}{(r-1)!} \int_{t_0}^{\xi} (\xi - t)^{r-1} H(t)\, dt = \int_{t_0}^{\xi} d\xi_1 \int_{t_0}^{\xi_1} d\xi_2 \ldots \int_{t_0}^{\xi_{r-1}} H(\xi_r)\, d\xi_r \tag{5.8}$$

and

$$\int_a^{t_0} (\xi - t)^{r-1} H(t)\, dt = \int_a^{t_0} [(\xi - t_0) + (t_0 - t)]^{r-1} H(t)\, dt = \tag{5.9}$$

$$= \sum_{i=0}^{r-1} \binom{r-1}{i} (\xi - t_0)^i \int_a^{t_0} (t_0 - t)^{r-1-i} H(t)\, dt.$$

From (5.7), (5.8) and (5. 9)

$$\frac{1}{(r-1)!} \int_a^{\xi} (\xi - t)^{r-1} H(t)\, dt = \frac{1}{(r-1)!} \sum_{i=0}^{r-1} \binom{r-1}{i} (\xi - t_0)^i \int_a^{t_0} (t_0 - t)^{r-1-i} H(t)\, dt +$$

$$+ \sum_{i=0}^{n-r} \frac{(\xi - t_0)^{i+r}}{(i+r)!} H_{(i)}(t_0) + o\big((\xi - t_0)^n\big).$$

Thus since t_0 is arbitrary

$$\left[\frac{1}{(r-1)!} \int_a^{\xi} (\xi - t)^{r-1} H(t)\, dt\right]_{(n)} = H_{(n-r)}$$

everywhere in $[a, b]$. Therefore by Lemma 10 and (5.2),

$$\psi_{(n)}(\xi) = \sum_{k=0}^{n} \binom{n}{k} F_{(k)}(\xi)\, G^{(n-k)}(\xi) + \sum_{r=1}^{n} (-1)^r \binom{n}{r} \sum_{k=0}^{n-r} \binom{n-r}{k} F_{(k)}(\xi)\, G^{n-k}(\xi) =$$

$$= \sum_{r=0}^{n} (-1)^r \binom{n}{r} \sum_{k=0}^{n-r} \binom{n-r}{k} F_{(k)}(\xi)\, G^{(n-k)}(\xi) =$$

$$= \sum_{k=0}^{n} \left[\sum_{r=0}^{n-k} (-1)^r \binom{n}{r} \binom{n-r}{k}\right] F_{(k)}(\xi)\, G^{(n-k)}(\xi) = F_{(n)}(\xi)\, G(\xi).$$

Also at a point t_0 where $F_{(n+1)}$ and $G^{(n+1)}$ exist, for fixed r, $1 \leqq r \leqq n$, the

function $H=FG^{(r)}$ is such that $H_{(n-r+1)}(t_0)$ exists and so one gets as above

$$\psi_{(n+1)}(t_0)=\sum_{k=0}^{n+1}\binom{n+1}{k}F_{(k)}(t_0)G^{(n-k+1)}(t_0)+$$

$$+\sum_{r=1}^{n}(-1)^r\binom{n}{r}\sum_{k=0}^{n-r+1}\binom{n-r+1}{k}F_{(k)}(t_0)G^{(n-k+1)}(t_0)=$$

$$=\sum_{r=0}^{n}(-1)^r\binom{n}{r}\sum_{k=0}^{n-r+1}\binom{n-r+1}{k}F_{(k)}(t_0)G^{(n-k+1)}(t_0)=$$

$$=\sum_{k=1}^{n+1}\left[\sum_{r=0}^{n-k+1}(-1)^r\binom{n}{r}\binom{n-r+1}{k}\right]F_{(k)}(t_0)G^{(n-k+1)}(t_0)+$$

$$+\sum_{r=0}^{n}(-1)^r\binom{n}{r}F(t_0)G^{(n+1)}(t_0)=$$

$$=\sum_{k=1}^{n}\left[\sum_{r=0}^{n-k+1}(-1)^r\binom{n}{r}\binom{n-r+1}{k}\right]F_{(k)}(t_0)G^{(n-k+1)}(t_0)+F_{(n+1)}(t_0)G(t_0).$$

Since

$$\binom{n}{r}=\binom{n+1}{r}-\binom{n}{r-1},$$

we have

$$\sum_{r=0}^{n-k+1}(-1)^r\binom{n}{r}\binom{n-r+1}{k}=\binom{n+1}{k}+\sum_{r=1}^{n-k+1}(-1^r)\left[\binom{n+1}{r}-\binom{n}{r-1}\right]\binom{n-r+1}{k}=$$

$$=\sum_{r=0}^{n-k+1}(-1)^r\binom{n+1}{r}\binom{n-r+1}{k}+\sum_{v=0}^{n-k}(-1)^v\binom{n}{v}\binom{n-v}{k}.$$

Hence by (5.2)

$$\psi_{(n+1)}(t_0)=F_{(n+1)}G(t_0)+F_{(n)}(t_0)G^{(1)}(t_0).$$

THEOREM 14. *Let $f:[a,b]\to X$ be C_nD_*B-integrable on $[a,b]$ and*

$$\varphi(t)=(C_nD_*B)\int_a^t f(\xi)\,d\xi.$$

Let $G:[a,b]\to R$ be such that $G^{(n)}$ is absolutely continuous on $[a,b]$. Then fG is C_nD_*B-integrable on $[a,b]$ and

$$(C_nD_*B)\int_a^b fG=[\varphi G]_a^b-(C_{n-1}D_*B)\int_a^b\varphi G^{(1)}.$$

PROOF. We prove the theorem for $n=1$. Let $F:[a,b]\to X$ be the function such that $F_{(1)}=\varphi$ and $F_{(2)}=f$ almost everywhere and F is strongly AC_1G_*. By Theorem 8 the function φ is C_0D_*B-integrable. Since $G^{(1)}$ is absolutely continuous, by [8] $\varphi G^{(1)}$ is D_*B-integrable in $[a,b]$ and hence by our earlier remark, it is C_0D_*B-integrable in $[a,b]$. By Theorem 12 and Lemma 13 there exists a func-

tion $\psi:[a,b]\to X$ which is strongly AC_1G_* and $\psi_{(1)}=F_{(1)}G=\varphi G$ everywhere and $\psi_{(2)}=F_{(1)}G^{(1)}+GF_{(2)}$ almost everywhere, that is $\psi_{(2)}=\varphi G^{(1)}+Gf$ almost everywhere. Hence $\varphi G^{(1)}+Gf$ is C_1D_*B-integrable and

$$\psi_{(1)}(b)-\psi_{(1)}(a)=\int_a^b[\varphi(t)G^{(1)}(t)+G(t)f(t)]\,dt$$

i.e.

$$\varphi(b)G(b)-\varphi(a)G(a)=\int_a^b[\varphi(t)G^{(1)}(t)+G(t)f(t)]\,dt.$$

Now since $\varphi G^{(1)}$ is C_0D_*B-integrable, by Theorem 7

$$[\varphi G]_a^b=(C_0D_*B)\int_a^b\varphi G^{(1)}+(C_1D_*B)\int_a^b fG$$

proving the theorem for $n=1$.

Now we suppose that the theorem is true for $m-1$ and prove it for $n=m$. The proof will then follow by induction. As above, let F be the function such that $F_{(m)}=\varphi$ and $F_{(m+1)}=f$ almost everywhere and F be strongly AC_mG_*. Since φ is $C_{m-1}D_*B$-integrable and $(G^{(1)})^{(m-1)}=G^{(m)}$ is absolutely continuous, by induction hypothesis $\varphi G^{(1)}$ is $C_{m-1}D_*B$-integrable. Applying Theorem 12 and Lemma 13 there exists a function $\psi:[a,b]\to X$ which is AC_mG_* and $\psi_{(m)}=F_{(m)}G=\varphi G$ everywhere and $\psi_{(m+1)}=F_{(m)}G^{(1)}+GF_{(m+1)}$ almost everywhere. So, $\varphi G^{(1)}+Gf$ is C_mD_*B-integrable and

$$\psi_{(m)}(b)-\psi_{(m)}(a)=\int_a^b[\varphi(t)G^{(1)}(t)+G(t)f(t)]\,dt.$$

Now since $\varphi G^{(1)}$ is $C_{m-1}D_*B$-integrable, by Theorem 7 it is C_mD_*B-integrable and hence

$$(C_mD_*B)\int_a^b fG=[\varphi G]_a^b-(C_{m-1}D_*B)\int_a^b\varphi G^{(1)}$$

proving the theorem for $n=m$. This completes the proof.

6. Examples

EXAMPLE 1. There exists a C_0D_*B-integrable function which is not Lebesgue—Bochner integrable.

Let f be an every where finite real valued function on $[0,1]$ which is D_*-integrable but not L-integrable. (For the existence of such a function see [9; p. 187].) Hence there exists a function F which is ACG_* and $F^{(1)}=f$ almost everywhere in $[0,1]$. Let $\{c_n\}\in l_2$ be fixed. Define $g:[0,1]\to l_2$ by $g(t)=\{c_nf(t)\}$, $t\in[0,1]$ and $G:[0,1]\to l_2$ by $G(t)=\{c_nF(t)\}$, $t\in[0,1]$. Then if $\sum c_n^2=k^2$, for each $t\in[0,1]$, $\|g(t)\|=|k|\,|f(t)|$. Since f is not Lebesgue integrable, $\|g(t)\|$ is not L-integrable in $[0,1]$ and hence by [7; Theorem 3.7.4, p. 80] the function

g is not Lebesgue—Bochner integrable on $[0, 1]$. Now

$$\|G(t_1)-G(t_2)\| = \left[\sum (c_n F(t_1)-c_n F(t_2))^2\right]^{\frac{1}{2}} = |k|\,|F(t_1)-F(t_2)|.$$

So, G is strongly AC_0G_*. Also if at $t=t_0$, $F^{(1)}(t_0)=f(t_0)$ then at $t=t_0$,

$$\left\|\frac{G(t_0+h)-G(t_0)}{h}-g(t_0)\right\| = \left[\sum c_n^2\left(\frac{F(t_0+h)-F(t_0)}{h}-f(t_0)\right)^2\right]^{1/2} =$$

$$= |k|\left|\frac{F(t_0+h)-F(t_0)}{h}-f(t_0)\right| \to 0$$

as $h\to 0$. Hence $G_{(1)}=g$ almost everywhere in $[0, 1]$. Thus g is C_0D_*B-integrable in $[0, 1]$ and

$$(C_0D_*B)\int_0^1 g(t)\,dt = G(1).$$

EXAMPLE 2. For each $n>0$ there exists a C_nD_*B-integrable function which is not $C_{n-1}D_*B$ integrable.

Let f be a real valued finite function in $[0, 1]$ which is C_nP-integrable but not $C_{n-1}P$-integrable. (For the definition of C_rP-integral see [6]. For the existence of such a function see [5]. In fact, in [5] a function is given which is CP-integrable but not D^*-integrable. The same method may be applied to construct a function which is C_nP-integrable but not $C_{n-1}P$-integrable.) So, there is a real valued function φ in $[0, 1]$ such that φ is AC_nG_* and $\varphi_{(n+1)}=f$ almost everywhere in $[0, 1]$, [10]. We may suppose $\varphi_{(n)}(0)=0$. Let $C=\{c_r\}\in l_2$ be fixed such that $\sum c_r^2=1$. Define the function g and ψ on $[0, 1]$ with values in l_2 such that

$$g(t) = Cf(t),\ t\in[0, 1],\quad \psi(t) = C\varphi(t),\ t\in[0, 1].$$

Then $\psi_{(i)}=C\varphi_{(i)}$, for $i=1, 2, \ldots, n+1$ and $\varepsilon_n(\psi; t_0, t)=C\varepsilon_n(\varphi; t_0, t)$. So, ψ is strongly AC_nG_* and $\psi_{(n+1)}=g$ almost everywhere. Hence g is C_nD_*B-integrable and

$$\psi_{(n)}(1) = (C_nD_*B)\int_0^1 g(t)\,dt.$$

If possible, let g be $C_{n-1}D_*B$-integrable. Then there is $\theta\colon[0, 1]\to l_2$ such that θ is strongly $AC_{n-1}G_*$ and $\theta_{(n)}=g$ almost everywhere in $[0, 1]$. By Lemma 2, the function

$$G(t) = \int_0^t \theta(\xi)\,d\xi$$

is strongly AC_nG_* and $G_{(n+1)}=\theta_{(n)}$ at each point where $\theta_{(n)}$ exists. Thus $\psi_{(n+1)}=$ $=G_{(n+1)}$ almost everywhere. Hence by Lemma 4, $\psi_{(n)}-G_{(n)}$ is a constant and hence $\psi_{(1)}-\theta$ is a polynomial of degree at most $n-1$. Hence

$$\varepsilon_{n-1}(\theta; t_0, t) = \varepsilon_{n-1}(\psi_{(1)}; t_0, t) = C\varepsilon_{n-1}(\varphi_{(1)}; t_0, t).$$

Therefore, since θ is strongly $AC_{n-1}G_*$, $\varphi_{(1)}$ is $AC_{n-1}G_*$. Also if $\theta_{(n)}(t_0)$ exists

and equals $g(t_0)$ then since $\psi_{(1)}$ and θ differ by a polynomial of degree at most $n-1$,

$$\left\|\frac{n}{t-t_0}\varepsilon_{n-1}(\theta;\, t_0,\, t)-g(t_0)\right\| = \left\|\frac{n}{t-t_0}\varepsilon_{n-1}(\psi_{(1)};\, t_0,\, t)-g(t_0)\right\| =$$

$$= \left\|\frac{n}{t-t_0}\varepsilon_{n-1}(\varphi_{(1)};\, t_0,\, t)-f(t_0)\right\|$$

which tends to 0 as $t\to t_0$, showing that $(\varphi_{(1)})_{(n)}(t_0)=f(t_0)$. Hence $(\varphi_{(1)})_{(n)}=f$ almost everywhere. Thus f is $C_{n-1}P$-integrable in [0, 1] (see [10]), which is a contradiction.

References

[1] A. Alexiewicz, On Denjoy integrals of abstract functions, *Soc. Sci. Lett. Varsovie, C. R. Cl. III. Sci. Math. Physics*, **41** (1948), 79—129.

[2] A. Alexiewicz, On differentiation of vector valued functions, *Studia Math.*, **11** (1950), 185—196.

[3] J. A. Bergin, A new characterisation of Cesaro—Perron integrals using Peano derivates, *Trans. Amer. Math. Soc.*, **228** (1977), 287—305.

[4] P. S. Bullen and S. N. Mukhopadhyay, Peano derivatives and general integrals, *Pacific J. Math.*, **47** (1973), 43—58.

[5] J. C. Burkill, The Cesaro—Perron integral, *Proc. London Math. Soc.*, **34** (1932), 314—322.

[6] J. C. Burkill, The Cesaro—Perron scale of integration, *Proc. London Math. Soc.*, **39** (1935), 541—552.

[7] E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Coll. Publication 31 (Providence, 1957).

[8] B. K. Pal, Integration by parts formulae for Denjoy—Bochner and Denjoy—Pettis integrals, to appear.

[9] S. Saks, *Theory of the integral*, Dover publications (New York, 1964).

[10] W. L. C. Sargent, On generalised derivatives and Cesaro—Denjoy integrals, *Proc. London Math. Soc.*, **52** (1951), 365—376.

[11] D. W. Solomon, *Denjoy integration in abstract spaces*, Memoirs of the Amer. Math. Soc. No. 85 (Providence, 1969).

[12] B. S. Thomson, Constructive definition for non-absolutely convergent integrals, *Proc. London Math. Soc.*, **20** (1970), 699—716.

(Received January 22, 1982)

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF BURDWAN
BURDWAN 713104, WEST BENGAL
INDIA

Acta Math. Hung.
42 (3—4) (1983), 257—260.

ON AN ISOMORPHISM

S. M. YAHYA (Dhahran)

1. Introduction

Let A, B, C be three modules over a Dedekind domain R. It follows from the Künneth theorem (see [2; p. 343]) that

$$\text{(1.1)} \qquad \mathrm{Tor}(A, B)\otimes C \oplus \mathrm{Tor}(A\otimes B, C) \cong A\otimes \mathrm{Tor}(B, C)\oplus \mathrm{Tor}(A, B\otimes C),$$

where Tor stands for Tor_1^R. The object of this paper is to determine the structure of each side of (1.1), providing thus a structural proof of the isomorphism and an insight into the structures of the modules involved. In the end we consider an application of this isomorphism, showing how to deduce from it the structure of the torsion submodule of the tensor product of two modules over a Dedekind domain.

Before we proceed further we establish the notation which will be used in this paper.

R	a Dedekind domain
$A, B, C, \ldots$	modules over R
Q	the quotient field of R
K	the module Q/R
A_t	the torsion submodule of A
$\tilde{A}$	the module A/A_t
P	a prime ideal of R
A_P	the P-primary component of A
$R(P^\infty)$	the P-primary component of K
$\bigoplus_\nu A$	the direct sum of ν copies of A
$\bigoplus_P$	the direct sum over all prime ideals of R.

2. Structure of (1.1)

Let U_P, V_P, W_P be basic submodules of A_P, B_P, C_P respectively, and let $A_P/U_P \cong \bigoplus_{\alpha_P} R(P^\infty)$, $B_P/V_P \cong \bigoplus_{\beta_P} R(P^\infty)$ and $C_P/W_P \cong \bigoplus_{\gamma_P} R(P^\infty)$. Let $\tilde{U}_P, \tilde{V}_P, \tilde{W}_P$ be P-basic submodules of $\tilde{A}, \tilde{B}, \tilde{C}$ respectively (see [4]) and let λ_P, μ_P, ν_P be their respective ranks.

For finding the structure of each side of (1.1) we need some lemmas.

2.1. LEMMA. $(A\otimes B)_t \cong \bigoplus_P [U_P\otimes V_P \oplus \bigoplus_{\mu_P} A_P \oplus \bigoplus_{\lambda_P} B_P]$.

PROOF. We know that $(A\otimes B)_t\cong A_t\otimes B_t\oplus A_t\otimes\tilde{B}\oplus\tilde{A}\otimes B_t$ (see [4; Corollary 3.10]). Now $A_t\otimes B_t\cong\bigoplus_P A_P\otimes B_P\cong\bigoplus_P U_P\otimes V_P$ (see [4; Section 3 (iv)]). Also $A_t\otimes\tilde{B}\cong\bigoplus_P A_P\otimes\tilde{B}\cong\bigoplus_P A_P\otimes\tilde{V}_P\cong\bigoplus_P\bigoplus_{\mu_P} A_P$ (see [4; Theorem 3.1]). Similarly, $\tilde{A}\otimes B_t\cong\bigoplus_P\bigoplus_{\lambda_P} B_P$. Hence the result.

2.2. LEMMA. $\operatorname{Tor}(U_P, B_P)+\operatorname{Tor}(A_P, V_P)\cong\operatorname{Tor}(U_P, V_P)\oplus\bigoplus_{\alpha_P} V_P\oplus\bigoplus_{\beta_P} U_P$.

PROOF. We have the following pure exact sequences (see [3; Corollary 3.7] or [1]).

$$0\to\operatorname{Tor}(U_P, V_P)\to\operatorname{Tor}(U_P, B_P)\to\operatorname{Tor}(U_P, B_P/V_P)\to 0, \tag{2.3}$$

$$0\to\operatorname{Tor}(U_P, V_P)\to\operatorname{Tor}(A_P, V_P)\to\operatorname{Tor}(A_P/U_P, V_P)\to 0. \tag{2.4}$$

Now $\operatorname{Tor}(U_P, B_P/V_P)\cong\bigoplus_{\beta_P}\operatorname{Tor}(U_P, R(P^\infty))\cong\bigoplus_{\beta_P} U_P$ (see [2; Lemma 8.14]), which is a direct sum of cyclic modules. Hence the pure exact sequence (2.3) splits. Similarly, $\operatorname{Tor}(A_P/U_P, V_P)\cong\bigoplus_{\alpha_P} V_P$, and the sequence (2.4) splits. We thus get the required result.

2.5. LEMMA. $\operatorname{Tor}(U_P, B_P)+\operatorname{Tor}(A_P, V_P)$ *is a basic submodule of* $\operatorname{Tor}(A_P, B_P)$.

PROOF. Since U_P, V_P are direct sums of cyclic P-modules, it follows that $\operatorname{Tor}(U_P, V_P)$ is a direct sum of cyclic P-modules. Hence we deduce from Lemma 2.2 that $\operatorname{Tor}(U_P, B_P)+\operatorname{Tor}(A_P, V_P)$ is a direct sum of cyclic P-modules. Let us now consider the following pure exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{Tor}(U_P, V_P) & \to & \operatorname{Tor}(U_P, B_P) & \to & \operatorname{Tor}(U_P, \bar{B}_P) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{Tor}(A_P, V_P) & \to & \operatorname{Tor}(A_P, B_P) & \to & \operatorname{Tor}(A_P, \bar{B}_P) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{Tor}(\bar{A}_P, V_P) & \to & \operatorname{Tor}(\bar{A}_P, B_P) & \to & \operatorname{Tor}(\bar{A}_P, \bar{B}_P) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\bar{A}_P=A_P/U_P$ and $\bar{B}_P=B_P/V_P$. It follows easily from the above diagram that the sequence

$$0\to\big(\operatorname{Tor}(U_P, B_P)+\operatorname{Tor}(A_P, V_P)\big)\to\operatorname{Tor}(A_P, B_P)\to\operatorname{Tor}(\bar{A}_P, \bar{B}_P)\to 0$$

is pure exact. Also $\operatorname{Tor}(\bar{A}_P, \bar{B}_P)\cong\operatorname{Tor}\big(\bigoplus_{\alpha_P} R(P^\infty), \bigoplus_{\beta_P} R(P^\infty)\big)\cong\bigoplus_{\beta_P}\operatorname{Tor}\big(R(P^\infty), R(P^\infty)\big)\cong\bigoplus_{\alpha_P\beta_P} R(P^\infty)$, so it is divisible. Hence the result follows.

2.6. LEMMA.

$$\operatorname{Tor}(A, B)\otimes C\cong$$

$$\cong\bigoplus_P\big[\operatorname{Tor}(U_P, V_P)\otimes W_P\oplus\bigoplus_{\alpha_P}(V_P\otimes W_P)\oplus\bigoplus_{\beta_P}(W_P\otimes U_P)\oplus\bigoplus_{\nu_P}\operatorname{Tor}(A_P, B_P)\big].$$

PROOF. Since $\operatorname{Tor}(A, B)$ is a torsion module it follows from [4; Theorem 3.4] that $\operatorname{Tor}(A, B)\otimes C\cong\operatorname{Tor}(A, B)\otimes C_t\oplus\operatorname{Tor}(A, B)\otimes\tilde{C}$. Now $\operatorname{Tor}(A, B)\otimes C_t\cong$ $\cong\bigoplus_P\operatorname{Tor}(A_P, B_P)\otimes C_P\cong\bigoplus_P[\operatorname{Tor}(U_P, V_P)\otimes W_P\oplus\bigoplus_{\alpha_P}(V_P\otimes W_P)\oplus\bigoplus_{\beta_P}(U_P\otimes W_P)]$ by Lemmas 2.2 and 2.5. Also $\operatorname{Tor}(A, B)\otimes\tilde{C}\cong\bigoplus_P\operatorname{Tor}(A_P, B_P)\otimes\tilde{C}\cong\bigoplus_P\operatorname{Tor}(A_P, B_P)\otimes$ $\otimes\tilde{W}_P$ by [4; Theorem 3.1] $\cong\bigoplus_P\bigoplus_{\nu_P}\operatorname{Tor}(A_P, B_P)$. Hence the lemma.

2.7. LEMMA.

$$\operatorname{Tor}(A\otimes B, C)\cong$$

$$\cong\bigoplus_P[\operatorname{Tor}(U_P\otimes V_P, W_P)\oplus\bigoplus_{\gamma_P}(U_P\otimes V_P)\oplus\bigoplus_{\lambda_P}\operatorname{Tor}(B_P, C_P)\oplus\bigoplus_{\mu_P}\operatorname{Tor}(C_P, A_P)].$$

PROOF. We know that $\operatorname{Tor}(A\otimes B, C)\cong\operatorname{Tor}((A\otimes B)_t, C_t)$. Hence, by Lemma 2.1, it follows that $\operatorname{Tor}(A\otimes B, C)\cong\bigoplus_P[\operatorname{Tor}(U_P\otimes V_P, C_P)\oplus\bigoplus_{\lambda_P}\operatorname{Tor}(B_P, C_P)\oplus$ $\oplus\bigoplus_{\mu_P}\operatorname{Tor}(A_P, C_P)]$. Also we have a pure exact sequence

$$0\to\operatorname{Tor}(U_P\otimes V_P, W_P)\to\operatorname{Tor}(U_P\otimes V_P, C_P)\to\operatorname{Tor}(U_P\otimes V_P, \bar{C}_P)\to 0,$$

where $\bar{C}_P=C_P/W_P$. Now $\operatorname{Tor}(U_P\otimes V_P, \bar{C}_P)\cong\operatorname{Tor}(U_P\otimes V_P, \bigoplus_{\gamma_P}R(P^\infty))\cong$ $\cong\bigoplus_{\gamma_P}(U_P\otimes V_P)$. Since $U_P\otimes V_P$ is a direct sum of cyclic modules, the above pure exact sequence splits. Hence $\operatorname{Tor}(U_P\otimes V_P, C_P)\cong\operatorname{Tor}(U_P\otimes V_P, W_P)\oplus\bigoplus_{\gamma_P}U_P\otimes V_P$, whence the lemma follows.

From Lemmas 2.6 and 2.7 we get the following theorem.

2.8. THEOREM.

$$\operatorname{Tor}(A, B)\otimes C\oplus\operatorname{Tor}(A\otimes B, C)\cong$$

$$\cong\bigoplus_P[\operatorname{Tor}(U_P, V_P)\otimes W_P\oplus\operatorname{Tor}(U_P\otimes V_P, W_P)\oplus\bigoplus_{\alpha_P}(V_P\otimes W_P)\oplus$$

$$\oplus\bigoplus_{\beta_P}(W_P\otimes U_P)\oplus\bigoplus_{\gamma_P}(U_P\otimes V_P)\oplus\bigoplus_{\lambda_P}\operatorname{Tor}(B_P, C_P)\oplus\bigoplus_{\mu_P}\operatorname{Tor}(C_P, A_P)\oplus\bigoplus_{\nu_P}\operatorname{Tor}(A_P, B_P)].$$

In the same way we can establish the following theorem.

2.9. THEOREM.

$$A\otimes\operatorname{Tor}(B, C)\oplus\operatorname{Tor}(A, B\otimes C)\cong$$

$$\cong\bigoplus_P[U_P\otimes\operatorname{Tor}(V_P, W_P)\oplus\operatorname{Tor}(U_P, V_P\otimes W_P)\oplus$$

$$\oplus\bigoplus_{\alpha_P}(V_P\otimes W_P)\oplus\bigoplus_{\beta_P}(W_P\otimes U_P)\oplus\bigoplus_{\gamma_P}(U_P\otimes V_P)\oplus$$

$$\oplus\bigoplus_{\lambda_P}\operatorname{Tor}(B_P, C_P)\oplus\bigoplus_{\mu_P}\operatorname{Tor}(C_P, A_P)\oplus\bigoplus_{\nu_P}\operatorname{Tor}(A_P, B_P)].$$

Since each of the modules U_P, V_P, W_P is a direct sum of cyclic P-modules, it follows immediately that $\operatorname{Tor}(U_P, V_P)\otimes W_P\oplus\operatorname{Tor}(U_P\otimes V_P, W_P)\cong U_P\otimes$ $\otimes\operatorname{Tor}(V_P, W_P)\oplus\operatorname{Tor}(U_P, V_P\otimes W_P)$. Thus we can conclude from Theorems 2.8 and 2.9 that the two sides of (1.1) are structurally isomorphic.

3. An application

It is interesting to note that from isomorphism (1.1) we can deduce the structure of the torsion submodule of the tensor product of two modules. Let us put $C=K$ in (1.1). We get (see [2; Theorem 8.17]) the following:

$$(A\otimes B)_t \cong A\otimes B_t\oplus \mathrm{Tor}\,(A, B\otimes K), \tag{3.1}$$

for $\mathrm{Tor}\,(A, B)\otimes K=0$, as $\mathrm{Tor}\,(A, B)$ is a torsion module and K is divisible. Again putting $B=K$ and $C=B_t$ in (1.1) we get

$$A_t\otimes B_t\oplus \mathrm{Tor}\,(A\otimes K, B_t)\cong A\otimes B_t. \tag{3.2}$$

From (3.1) and (3.2) we deduce that

$$(A\otimes B)_t \cong A_t\otimes B_t\oplus \mathrm{Tor}\,(A, B\otimes K)\oplus \mathrm{Tor}\,(A\otimes K, B_t)\cong \tag{3.3}$$

$$\cong A_t\otimes B_t\oplus \mathrm{Tor}\,(A_t, B\otimes K)\oplus \mathrm{Tor}\,(A\otimes K, B_t)$$

Now $B\otimes K\cong \tilde{B}\otimes K\cong \bigoplus_P \bigoplus_{\mu_P} R(P^\infty)$ (see [4; Theorem 3.1]). Hence $\mathrm{Tor}\,(A_t, B\otimes K)\cong$ $\cong \bigoplus_P \bigoplus_{\mu_P} \mathrm{Tor}\,(A_P, R(P^\infty))\cong \bigoplus_P \bigoplus_{\mu_P} A_P$. Similarly, $\mathrm{Tor}\,(A\otimes K, B_t)\cong \bigoplus_P \bigoplus_{\lambda_P} B_P$. Also, since $A_t\otimes B_t\cong \bigoplus_P U_P\otimes V_P$, we have

$$(A\otimes B)_t \cong \bigoplus_P [U_P\otimes V_P\oplus \bigoplus_{\mu_P} A_P\oplus \bigoplus_{\lambda_P} B_P]. \tag{3.4}$$

References

[1] L. Fuchs, Note on purity and algebraic compactness for modules, *Studies on Abelian Groups* (Sympos., Montpellier, 1967), Springer-Verlag (1968), 121—129.
[2] J. J. Rotman, *An Introduction to Homological Algebra*, Academic Press (1979).
[3] S. M. Yahya and Ali A. Al-Daffa, On cogenerators in modules over Dedekind domains, *Kobe Univ. Math. Seminar Notes*, **8** (1980), 91—102.
[4] S. M. Yahya, Tensor product of modules over Dedekind domains, *Acta Math. Acad. Sci. Hungar.*, **40** (1982), 85—91.

(Received January 25, 1982)

DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF PETROLEUM AND MINERALS
DHAHRAN
SAUDI ARABIA

Acta Math. Hung.
42 (3—4) (1983), 261—278.

PERIODIC BEURLING DISTRIBUTIONS

M. G. COLLIER (Fredericksburg) and J. A. KELINGOS (Nashville)

1. Introduction

In 1961 A. Beurling [1] presented the foundations for a certain more general theory of distributions, which in euclidean space encompasses the Schwartz theory as a special case. In 1965 G. Björck [3] published an extensive and energetic paper in which he develops and extends Beurling's ideas rather thoroughly, at least in euclidean n-space. (Beurling's original groundwork is valid in locally compact, sigma-compact, abelian groups.) In addition to elaborating on Beurling's ideas, Björck discusses the relationship between Beurling spaces and Denjoy—Carleman classes, introduces generalized tempered distributions, convolution, and Fourier transforms, proves Paley—Wiener theorems — all this in the introductory chapter. The main thrust of the work is toward applications to linear partial differential equations; existence, approximation, interior regularity of solutions. Beurling distributions have since appeared mainly in partial differential equations and functional analysis. In 1967 M. A. Dostal [5] investigated the singular supports of Beurling distributions and their connection with convolution operators. A regularity theorem for solutions to elliptic equations was proved by O. John [10]. In 1971 C. A. Berenstein and M. A. Dostal [2] showed that the spaces of Beurling distributions are analytically-uniform spaces, a notion introduced by L. Ehrenpreis in 1960. In their paper [2] the authors study the equivalence of different topologies in the space of Fourier transforms of Beurling test functions. In [7] O. Grudzinski considers certain Sobolev spaces of Beurling distributions, while in [17] O. Weingärtner and F. Constantinescu characterize the positive semidefinite Beurling distributions. In 1979 Grudzinski [8] published a rather extensive paper devoted entirely to tempered Beurling distributions.

While the above summary by no means exhausts the literature on Beurling distributions, the authors have nonetheless been unable to find any work done in the application of these distributions to complex analysis. We do not imply by this that complex analysis does not find its way into the study of Beurling distributions. After all, the Fourier transform of a distribution with compact support is an entire function, and the growth properties of these entire functions is what the Paley—Wiener theorems are all about (see [5] for some very intricate work with entire functions). On the other hand, complex analysts have apparently not yet found a use for Beurling distributions in the study of analytic or harmonic functions on the unit disc or on the half plane. This seems all the more remarkable in light of the fact that Beurling himself proved in [1] a theorem which characterizes those analytic functions in the unit disc whose boundary values are Beurling distributions. This theorem alone suggests many problems in function theory that ought to be investigated. B. Korenblum's interesting paper [11] appears to have relevance in this

direction, but even the connection with Schwartz distributions is not actually mentioned there.

The purpose of this paper is to lay the groundwork needed in order to look at some of these problems. The organization of the paper loosely follows chapter 11 of Zemanian's book [18]. Specifically, we investigate Beurling test spaces and (periodic) distributions on the unit circle. In order to take advantage of the machinery already developed for Beurling distributions in euclidean space (specifically R^1), we prove certain "correspondence" theorems in section 3, the main tool for which is the proof of the existence of *unitary functions*. In section 4 we consider Fourier series and convolutions of periodic distributions, and we characterize the periodic Beurling distributions by the growth rate of their Fourier coefficients. Also in section 4 we generalize results of Goldberg [6] concerning the connection between Fourier coefficients and Fourier transforms. Finally in section 5 we begin the study of harmonic and analytic functions in the unit disc with generalized boundary values by defining the generalized Poisson integral, and we characterize those harmonic functions with Beurling distributional boundary values in terms of the growth rate of their L_1 norms.

The authors would like to thank R. F. Arenstorf and R. R. Goldberg for helpful conversations in the preparation of this manuscript.

2. Notation and preliminary results

As we mentioned in the introduction, a thorough description of the Beurling test spaces and distributions in R^n is given in [3]. Briefly, the definitions are as follows. We deal only with R^1, and consider continuous real-valued functions ω which satisfy the following conditions:

(α) $$0 = \omega(0) \leqq \omega(\xi+\eta) \leqq \omega(\xi)+\omega(\eta) \quad \text{for all} \quad \xi, \eta \text{ in } R^1;$$

(β) $$\int_{R^1} \frac{\omega(\xi)}{1+\xi^2}\, d\xi < +\infty;$$

(γ) there exists real a and positive b such that for all ξ in R^1,

$$\omega(\xi) \geqq a + b \log(1+|\xi|);$$

(δ) ω is concave down on $[0, \infty)$ and is even.

This last condition is necessary for some of our results but still provides sufficient generality. Indeed, Beurling [3, p. 355] showed that a function satisfying (α)—(γ) is dominated by a function satisfying (α)—(δ). We observe that $\omega(0)=0$ and concavity imply the subadditivity of (α), and $\omega \geqq 0$ implies ω is increasing on $[0, \infty)$.

DEFINITION 2.1. For a given ω satisfying (α)—(δ), denote by $\mathscr{D}_\omega = \mathscr{D}_\omega(R^1)$ all complex-valued functions φ in $L^1(R^1)$ with compact support and such that for all $\lambda > 0$,

$$\|\varphi\|_\lambda = \|\varphi\|_\lambda^\omega = \int_{R^1} |\hat{\varphi}(\xi)|\, e^{\lambda\omega(\xi)}\, d\xi < +\infty,$$

where $\hat{\varphi}$ denotes the Fourier transform

$$\hat{\varphi}(\xi) = \int_{R^1} \varphi(x) e^{-i\xi x}\, dx.$$

For a subset K of R^1, let $\mathscr{D}_\omega(K) = \{\varphi \in \mathscr{D}_\omega : \operatorname{supp} \varphi \subseteqq K\}$. Let $\{K_i\}$, $i \geqq 1$, be an increasing sequence of compact subsets with union R^1. Then a locally convex topology is defined in $\mathscr{D}_\omega$ as the inductive limit of the topologies of $\mathscr{D}_\omega(K_i)$, $i = 1, 2, \ldots$, each of which is a Fréchet space with topology defined by the sequence of seminorms $\| \cdot \|_m^\omega$, $m = 1, 2, \ldots$. $\mathscr{D}_\omega$ is complete [3, p. 358] and a sequence of functions $\varphi_\nu \in \mathscr{D}_\omega$ converges to 0 in the topology of $\mathscr{D}_\omega$ if and only if the functions have common compact support and $\|\varphi_\nu\|_\lambda \to 0$ for each $\lambda > 0$.

DEFINITION 2.2. The space $\mathscr{D}'_\omega = \mathscr{D}'_\omega(R^1)$ of all continuous linear functionals on $\mathscr{D}_\omega$ is given the weak topology and is called the space of Beurling distributions on R^1.

A distribution then is a linear functional on $\mathscr{D}_\omega$ whose restriction to each $\mathscr{D}_\omega(K_i)$ is continuous. Equivalently, a linear functional f on $\mathscr{D}_\omega$ is continuous if and only if to each compact set K there exists $\lambda > 0$ and C such that

$$|\langle f, \varphi \rangle| \leqq C \|\varphi\|_\lambda, \quad \text{all} \quad \varphi \in \mathscr{D}_\omega(K). \tag{2.1}$$

In particular, a linear functional f on $\mathscr{D}_\omega$ is continuous if and only if it is sequentially continuous: $\varphi_\nu \to 0$ in $\mathscr{D}_\omega$ implies $\langle f, \varphi_\nu \rangle \to 0$. Convergence in $\mathscr{D}'_\omega$ means pointwise convergence: f_ν converges in $\mathscr{D}'_\omega$ if the sequence of complex numbers $\langle f_\nu, \varphi \rangle$ converges for each φ in $\mathscr{D}_\omega$. Furthermore, $\mathscr{D}'_\omega$ is closed under convergence.

We mention briefly some of the ramifications of conditions (α)—(γ). If ω satisfies (α), $\mathscr{D}_\omega$ is an algebra under pointwise multiplication and for each $\lambda > 0$ we have:

$$\|\varphi\psi\|_\lambda \leqq \frac{1}{2\pi} \|\varphi\|_\lambda \|\psi\|_\lambda, \quad \varphi, \lambda \in \mathscr{D}_\omega. \tag{2.2}$$

Assuming (α), then (β) is equivalent to the non-triviality of $\mathscr{D}_\omega$. In particular, $\mathscr{D}_\omega$ contains partitions of unity. When $\omega(\xi) = \log(1 + |\xi|)$, $\mathscr{D}_\omega$ is the Schwartz space $\mathscr{D} = C_0^\infty$. Condition (γ) implies $\mathscr{D}_\omega \subset \mathscr{D}$ is dense in $\mathscr{D}$ and is closed under differentiation.

Therefore $\mathscr{D}'_\omega \supset \mathscr{D}'$ and Beurling distributions are seen to be a proper generalization of Schwartz distributions. In addition Björck [3, pp. 364 and 365] has shown that condition (γ) implies supremum norms can be used instead of integral norms. Since this fact is important to our results, we list it as a lemma (see also [8], p. 299).

DEFINITION 2.3. If $\varphi \in L^1(R^1)$ and λ is real, set

$$|||\varphi|||_\lambda = |||\varphi|||_\lambda^\omega = \sup_{\xi \in R^1} |\hat{\varphi}(\xi)| e^{\lambda\omega(\xi)}.$$

LEMMA 2.4. *If ω satisfies (α)—(γ) and $\varphi \in L^1(R^1)$, then*

$$A |||\varphi|||_\lambda^\omega \leqq \|\varphi\|_\lambda^\omega \leqq B |||\varphi|||_{\lambda + 2/b}^\omega,$$

where b is the constant of condition (γ), B is a constant depending only on b, and A is a constant depending on the support of φ.

The definition of Beurling distributions on the unit circle Γ runs almost parallel to the above paragraphs. One begins by considering real-valued functions ω defined only on the integers $Z(=\hat{\Gamma})$ satisfying conditions (α)—(δ), where R^1 is replaced by the integers Z, $[0, \infty)$ by $0, 1, 2, \ldots$, and of course (β) is replaced by

$$(\beta) \qquad \sum_{n\in Z} \frac{\omega(n)}{1+n^2} < +\infty.$$

In the sequel we will always specify the domain of ω (either R^1 or Z) in order to discern to which sets of conditions (α)—(δ) we are referring, or we will say things like "condition (α) on R_1" or "(β) on Γ", etc.

DEFINITION 2.5. For a given ω on Z satisfying (α)—(δ), denote by $\mathscr{P}_\omega = \mathscr{P}_\omega(\Gamma)$ all complex-valued functions θ in $L^1(\Gamma)$ such that for all $\lambda > 0$,

$$\|\theta\|_\lambda = \|\theta\|_\lambda^\omega = \sum_{n\in Z} |\hat{\theta}(n)| e^{\lambda\omega(n)} < +\infty,$$

where $\hat{\theta}$ denotes the Fourier coefficients

$$\hat{\theta}(n) = \frac{1}{2\pi} \int_\Gamma \theta(\xi) e^{-in\xi} d\xi.$$

We topologize $\mathscr{P}_\omega$ directly by means of the sequence of seminorms $\|\cdot\|_m^\omega$, $m = 1, 2, \ldots$. $\mathscr{P}_\omega$ is then a Frechet space. The completeness of $\mathscr{P}_\omega$ will follow from the completeness of $\mathscr{D}_\omega$ (section 3). Again a sequence $\theta_\nu \in \mathscr{P}_\omega$ converges to 0 in the topology of $\mathscr{P}_\omega$ if and only if $\|\theta_\nu\|_\lambda \to 0$ for each $\lambda > 0$.

DEFINITION 2.6. The space $\mathscr{P}'_\omega = \mathscr{P}'_\omega(\Gamma)$ of all continuous linear functionals on $\mathscr{P}_\omega$ is given the weak topology and is called the space of Beurling distributions on Γ.

A linear functional on $\mathscr{P}_\omega$ is continuous if and only if it is sequentially continuous. Convergence in $\mathscr{P}'_\omega$ means point-wise convergence.

The space $\mathscr{P}_\omega$ can be thought of as a subset of the periodic functions defined on R^1 and belonging to $L^1(-\pi, \pi)$, and we will not distinguish between a function θ in $\mathscr{P}_\omega$ and its periodic extension to R^1. The space $\mathscr{P}'_\omega$ however is certainly related to *periodic* Beurling distributions on R^1, but it is the purpose of section 3 to set up an algebraic and topological equivalence between these two sets. (A *periodic* distribution on R^1 is a distribution $f \in \mathscr{D}'_\omega$ such that $\langle f(t), \varphi(t+2\pi)\rangle = \langle f(t), \varphi(t)\rangle$, all $\varphi \in \mathscr{D}_\omega$. Simply stated, $f(t+2\pi) = f(t)$.)

The number that $f \in \mathscr{P}'_\omega$ assigns to $\theta \in \mathscr{P}_\omega$ will be denoted by $f \cdot \theta$ to avoid confusion with the number $\langle f, \varphi\rangle$ that $f \in \mathscr{D}'_\omega$ assigns to $\varphi \in \mathscr{D}_\omega$. We will write $f \cdot \theta(t)$, $f(t) \cdot \theta(t)$, $\langle f, \varphi(t)\rangle$, or $\langle f(t), \varphi(t)\rangle$ if there is confusion as to the variable.

The significance of conditions (α)—(γ) on Γ should be mentioned again. First of all it is routine to show that if $\omega(n) = \log(1+|n|)$, then $\mathscr{P}_\omega = \mathscr{P}\,(= C^\infty(\Gamma))$, the Schwartz test space on the circle. Condition (α) has the same significance as for R^1. Condition (β) again is equivalent to the existence of *local units* in $\mathscr{P}_\omega$, or more generally, partitions of unity (which is *not* equivalent this time to the non-triviality of $\mathscr{P}_\omega$, since for example all (periodic) real analytic functions satisfy (β) on Γ). (A *local unit* is a function which is identically one on an interval and zero

off some larger interval.) The proof that, on Γ, (β) is equivalent to the existence of local units is given at the end of section 3.

A word should be said about the fact that sometimes ω is defined on R^1 and sometimes on Z. Since $\mathscr{D}_\omega$ and $\mathscr{P}_\omega$ will often be considered simultaneously, we note that if ω is defined and satisfies (α)—(δ) on R^1, then it is clear that ω/Z satisfies (α)—(δ) on Γ. Conversely, if ω is defined on Z and satisfies (α)—(δ) on Γ, the continuous piecewise linear extension of ω to R^1 will satisfy (α)—(δ) on R^1.

Finally, in section 4 we prove that a periodic distribution is *tempered,* and accordingly we define here the space of tempered Beurling distributions with which we will be dealing. The space defined here is that mentioned by Björck in [3]. Recently Grudzinski [8] has developed a more general theory of tempered Beurling distributions, but we will not use that generality here.

DEFINITION 2.7. For a given ω on R^1 satisfying (α)—(δ), denote by $\mathscr{S}_\omega = \mathscr{S}_\omega(R^1)$ the set of all functions $\varphi \in L^1(R^1)$ such that φ and $\hat{\varphi}$ are in C^∞, and for each integer $n \geqq 0$ and each real $\lambda \geqq 0$

$$p_{n,\lambda}(\varphi) = \sup_{\xi \in R^1} e^{\lambda\omega(\xi)} |\varphi^{(n)}(\xi)| < \infty,$$

and

$$\pi_{n,\lambda}(\varphi) = \sup_{\xi \in R^1} e^{\lambda\omega(\xi)} |\hat{\varphi}^{(n)}(\xi)| < \infty.$$

The topology of $\mathscr{S}_\omega$ is defined by the semi-norms $p_{n,\lambda}$ and $\pi_{n,\lambda}$. A continuous linear functional on $\mathscr{S}_\omega$ is called an ω-tempered distribution. The space of all ω-tempered distributions is given the weak topology and is called $\mathscr{S}'_\omega$.

Convergence in $\mathscr{S}_\omega$ means convergence in each semi-norm, and convergence in $\mathscr{S}'_\omega$ is again pointwise.

Just as in the classical case, it is true that $\mathscr{D}_\omega \subset \mathscr{S}_\omega \subset \mathscr{S}'_\omega \subset \mathscr{D}'_\omega$, each inclusion is a dense imbedding, and convergence in any of the four implies convergence in the spaces to the right.

It is clear from the definition that if φ is in $\mathscr{S}_\omega$, so is $\hat{\varphi}$. Accordingly, the Fourier transform $\hat{f}$ of a distribution f in $\mathscr{S}'_\omega$ is defined by

$$\langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle, \quad \varphi \in \mathscr{S}_\omega. \tag{2.3}$$

The functional $\hat{f}$ is again in $\mathscr{S}'_\omega$ if f is. If f has compact support, then $\hat{f}$ is a *regular* distribution given by the function

$$\hat{f}(\xi) = \langle f(t), e^{-i\xi t} \rangle, \tag{2.4}$$

where the right hand side is understood to be $\langle f(t), \lambda(t) e^{-i\xi t} \rangle$. Here $\lambda \in \mathscr{D}_\omega$ is identically equal to one over some neighborhood of the support of f.

Finally [8, pp. 316—319], if $f, g \in \mathscr{S}'_\omega$ and at least one has compact support, then the convolution of f and g can be defined in a way analogous to the classical Schwartz case, $f * g \in \mathscr{S}'_\omega$ and

$$\widehat{f * g} = \hat{f}\hat{g}. \tag{2.5}$$

3. Correspondence theorems

We begin with two preliminary results relating the Fourier coefficients of 2π-periodic functions and the Fourier transform of functions on R^1. More will be made of this in section 4. This lemma is a special case of results found in [6].

LEMMA 3.1. *Let* $\varphi\in L^1(R^1)$ *and* $\theta\in L^1(\Gamma)$. *Then* $\theta\varphi\in L^1(R^1)$ *and*

$$\widehat{\theta\varphi}(\xi)=\sum_{n\in Z}\hat{\varphi}(\xi-n)\hat{\theta}(n). \tag{3.1}$$

If $\varphi\in L^1(R^1)$ *and* θ *is defined by*

$$\theta(t)=\sum_{n\in Z}\varphi(t-2\pi n), \tag{3.2}$$

then $\theta\in L^1(\Gamma)$ *and*

$$2\pi\hat{\theta}(n)=\hat{\varphi}(n). \tag{3.3}$$

In order to set up algebraic and topological correspondences between $\mathscr{D}_\omega$ and $\mathscr{P}_\omega$, $\mathscr{D}'_\omega$ and $\mathscr{P}'_\omega$, we prove that *unitary* functions exist in $\mathscr{D}_\omega$.

DEFINITION 3.2. A function $\xi(t)$ in $\mathscr{D}_\omega$ is unitary if

$$\sum_{n\in Z}\xi(t-2\pi n)\equiv 1. \tag{3.4}$$

THEOREM 3.3. *Unitary functions exist in* $\mathscr{D}_\omega$.

PROOF. Fix $0<\delta<\pi$, choose a non-trivial function $\psi\geqq 0$ in $\mathscr{D}_\omega$ whose support is contained in $[-\delta,\delta]$, and define $\varphi(x)=(1/c)\psi(x-\pi)\psi(\pi-x)$ where c is a constant chosen so that $\int_{R^1}\varphi=1$. Then φ is non-negative, symmetric about π, has support in $[\pi-\delta,\pi+\delta]$, and is in $\mathscr{D}_\omega$ since $\mathscr{D}_\omega$ is closed under multiplication. Define

$$\xi(t)=\begin{cases}\int_{|t|}^{2\pi}\varphi & \text{if } |t|<2\pi\\ 0 & \text{if } |t|\geqq 2\pi.\end{cases}$$

Note that ξ is non-negative, even, has support in $[-\pi-\delta,\pi+\delta]$, and is identically one on $[-\pi+\delta,\pi-\delta]$.

To check (3.4) fix t and consider two cases. If t is a multiple of 2π, say $t=2\pi N$, there is only one integer n such that $|t-2\pi n|<2\pi$, namely $n=N$. In this case the left hand side of (3.4) is equal to $\int_0^{2\pi}\varphi=1$. If t is not a multiple of 2π, there are two consecutive integers such that $|t-2\pi n|<2\pi$, say n and $n+1$. In this case the left hand side of (3.4) becomes $\int_{t-2\pi n}^{2\pi}\varphi+\int_{(n+1)2\pi-t}^{2\pi}\varphi$. Making a linear change of variable in the second integral and using the symmetry of φ about π, the sum of these integrals reduces again to $\int_0^{2\pi}\varphi=1$.

To show that ξ is in $\mathscr{D}_\omega$, we consider

$$\hat{\xi}(t) = \int_{-2\pi}^{2\pi} \int_{|u|}^{2\pi} \varphi(x)\, dx e^{-iut}\, du.$$

Reversing the order of integration yields

$$\hat{\xi}(t) = \int_{0}^{2\pi} \int_{-x}^{x} \varphi(x) e^{-iut}\, du\, dx = (i/t)(\hat{\varphi}(t) - \hat{\varphi}(-t)).$$

Since the C^∞ function on the right has a removable singularity at the origin and since φ is in $\mathscr{D}_\omega$, it is clear that $\|\xi\|_\lambda \leqq c\|\varphi\|_\lambda < \infty$, where c is some constant. This completes the proof.

Before proving our first correspondence theorem, we show that each function in $\mathscr{P}_\omega$ is a *multiplier* for $\mathscr{D}_\omega$. In the notation of [3, p. 367], we have $\mathscr{P}_\omega \subset \mathscr{E}_\omega$, the space of multipliers for $\mathscr{D}_\omega$. Specifically, the situation here is as follows.

THEOREM 3.4. *If $\varphi \in \mathscr{D}_\omega$ and $\theta \in \mathscr{P}_\omega$, then $\theta\varphi \in \mathscr{D}_\omega$ and*

$$\|\theta\varphi\|_\lambda \leqq \|\theta\|_\lambda \|\varphi\|_\lambda, \tag{3.5}$$

where the three norms are in $\mathscr{D}_\omega, \mathscr{P}_\omega$ and $\mathscr{D}_\omega$ respectively.

The proof is a straightforward computation which uses (3.1) and Fubini's theorem twice.

THEOREM 3.5. *Each φ in $\mathscr{D}_\omega$ generates a unique θ in $\mathscr{P}_\omega$ through the expression* (3.2). *Conversely, for a given unitary ξ in $\mathscr{D}_\omega$, each θ in $\mathscr{P}_\omega$ can be generated by means of* (3.2) *from $\varphi = \theta\xi$.*

PROOF. The function θ in (3.2) is clearly periodic. Next observe that by condition (γ), $\omega(n) \geqq a + b \log(1+|n|)$, $b > 0$, hence

$$\sum_{n\in Z} \exp\left(-\frac{2}{b}\omega(n)\right) \leqq \exp(-2a/b) \sum_{n\in Z} \frac{1}{(1+|n|)^2} < \infty.$$

Using (3.3) we therefore have

$$\|\theta\|_\lambda = \sum_{n\in Z} |\hat{\theta}(n)| e^{\lambda\omega(n)} = \frac{1}{2\pi} \sum_{n\in Z} |\hat{\varphi}(n)| e^{\lambda\omega(n)} \leqq$$

$$\leqq \frac{1}{2\pi} \left(\sup_n |\hat{\varphi}(n)| \exp((\lambda + 2/b)\omega(n))\right) \sum_{n\in Z} \exp\left(-\frac{2}{b}\omega(n)\right) \leqq C|||\varphi|||_{\lambda+2/b} \leqq$$

$$\leqq C' \|\varphi\|_{\lambda+2/b} < \infty,$$

by Lemma 2.4, which proves the first half of the theorem.

For the converse, if θ is given in $\mathscr{P}_\omega$ and ξ is unitary in $\mathscr{D}_\omega$, then $\varphi = \theta\xi$ is in $\mathscr{D}_\omega$ by Theorem 3.4. But $\sum_{n\in z} \varphi(t - 2\pi n) \equiv \theta(t)$ from (3.4) and the periodicity of θ, which completes the proof.

The above proof and (2.2) yield the following estimates for two functions φ in $\mathscr{D}_\omega$ and θ in $\mathscr{P}_\omega$ related by (3.2).

$$C'\|\varphi\|_\lambda \leqq \|\theta\|_\lambda \leqq C\|\varphi\|_{\lambda+2/b}. \tag{3.6}$$

Here C depends only on b of condition (γ) and C' depends on the choice of unitary function ξ. An immediate consequence of (3.6) is the correspondence of convergence in $\mathscr{D}_\omega$ and $\mathscr{P}_\omega$.

THEOREM 3.6. *If $\varphi_\nu \to \varphi$ in $\mathscr{D}_\omega$, then $\theta_\nu \to \theta$ in $\mathscr{P}_\omega$, where each θ_ν and θ is generated by φ_ν and φ through* (3.2). *Conversely if ξ is unitary in $\mathscr{D}_\omega$ and $\theta_\nu \to \theta$ in $\mathscr{P}_\omega$, then $\varphi_\nu = \xi\theta_\nu \to \varphi = \xi\theta$ in $\mathscr{D}_\omega$.*

Since $\mathscr{D}_\omega$ is closed under convergence [3], the above theorem yields

COROLLARY 3.7. *$\mathscr{P}_\omega$ is closed under convergence.*

We turn now to the relationship between $\mathscr{P}'_\omega$ and periodic distributions in $\mathscr{D}'_\omega$. Let $\tilde{\mathscr{D}}'_\omega$ denote the set of periodic distributions in $\mathscr{D}'_\omega$.

THEOREM 3.8. *There exists an algebraic isomorphism between $\tilde{\mathscr{D}}'_\omega$ and $\mathscr{P}'_\omega$ which preserves convergence in these spaces.*

PROOF. Define $T : \tilde{\mathscr{D}}'_\omega \to \mathscr{P}'_\omega$ by $Tf \cdot \theta = \langle f, \xi\theta \rangle$, f in $\tilde{\mathscr{D}}'_\omega$, θ in $\mathscr{P}_\omega$, ξ unitary in $\mathscr{D}_\omega$, and define $S : \mathscr{P}'_\omega \to \tilde{\mathscr{D}}'_\omega$ by $\langle Sg, \varphi \rangle = g \cdot \theta$, g in $\mathscr{P}'_\omega$, φ in $\mathscr{D}_\omega$, where $\theta(t) = \sum_n \varphi(t - 2\pi n)$.

First of all, T is well defined, for if $\varkappa$ is any other unitary function in $\mathscr{D}_\omega$, we see that $\langle f, \xi\theta \rangle = \langle f(t), \sum_n \xi(t)\theta(t)\varkappa(t-2\pi n) \rangle$. Since ξ and $\varkappa$ have compact support, only finitely many functions in this sum are non-trivial, hence this equals

$$\sum_n \langle f(t), \xi(t)\theta(t)\varkappa(t-2\pi n) \rangle = \sum_n \langle f(t+2\pi n), \xi(t+2\pi n)\theta(t+2\pi n)\varkappa(t) \rangle =$$

$$= \sum_n \langle f(t), \theta(t)\varkappa(t)\xi(t+2\pi n) \rangle = \langle f(t), \theta(t)\varkappa(t) \sum_n \xi(t+2\pi n) \rangle = \langle f, \varkappa\theta \rangle.$$

If f is in $\tilde{\mathscr{D}}'_\omega$, it is easy to verify that Tf is a *linear* functional on $\mathscr{P}_\omega$. The continuity of Tf follows from Theorem 3.6. Similar remarks are valid for Sg where g is in $\mathscr{P}_\omega$. That Sg is *periodic* follows from the fact that if φ is in $\mathscr{D}_\omega$, both $\varphi(t)$ and $\varphi(t+2\pi)$ generate the same θ through (3.2).

All that remains is to show T and S are bijective. Toward this end we show ST is the identity on $\tilde{\mathscr{D}}'_\omega$ and TS is the identity on $\mathscr{P}'_\omega$. Let f be in $\tilde{\mathscr{D}}'_\omega$ and φ in $\mathscr{D}_\omega$. Then

$$\langle (ST)f, \varphi \rangle = Tf \cdot \theta = \langle f, \xi\theta \rangle = \langle f(t), \xi(t) \sum_n \varphi(t-2\pi n) \rangle =$$

$$= \sum_n \langle f(t), \xi(t)\varphi(t-2\pi n) \rangle = \sum_n \langle f(t+2\pi n), \xi(t+2\pi n)\varphi(t) \rangle =$$

$$= \sum_n \langle f(t), \xi(t+2\pi n)\varphi(t) \rangle = \langle f(t), \varphi(t) \sum_n \xi(t+2\pi n) \rangle = \langle f, \varphi \rangle.$$

Thus $(ST)f = f$. (The sums are once again finite.)

Similarly, if g is in $\mathscr{P}'_\omega$ and $\theta\in\mathscr{P}_\omega$,

$$(TS)g\cdot\theta = \langle Sg, \xi\theta\rangle = \langle Sg, \varphi\rangle.$$

But

$$\sum_n \varphi(t-2\pi n) = \sum_n \xi(t-2\pi n)\,\theta(t-2\pi n) = \theta(t),$$

since θ is periodic and ξ is unitary. So $\langle Sg, \varphi\rangle = g\cdot\theta$ and hence $(TS)g=g$.

The fact that T and S preserve convergence follows immediately from their definitions and Theorem 3.6, which concludes the proof.

As a corollary we show that $\mathscr{P}'_\omega$ is *closed* under convergence.

COROLLARY 3.9. *$\mathscr{P}'_\omega$ is closed under convergence.*

PROOF. Let g_ν be in $\mathscr{P}'_\omega$ and converge (point-wise). Define the functional g on $\mathscr{P}_\omega$ by $g\cdot\theta = \lim_{\nu\to\infty} g_\nu\cdot\theta$. Then g is clearly linear on $\mathscr{P}_\omega$. Now $Sg_\nu = f_\nu$ converges in $\mathscr{D}'_\omega$ to say f. But $\mathscr{D}'_\omega$ is closed under convergence and therefore $f\in\mathscr{D}'_\omega$. Since each f_ν is periodic, so is f. Hence Tf is in $\mathscr{P}'_\omega$. But $Tf\cdot\theta = \langle f, \xi\theta\rangle = \lim_{\nu\to\infty}\langle f_\nu, \xi\theta\rangle = \lim_{\nu\to\infty} g_\nu\cdot\theta = g\cdot\theta$. So $Tf=g$ and hence g is in $\mathscr{P}'_\omega$.

Henceforth no distinction will be made between elements of $\mathscr{P}'_\omega$ and periodic distributions in $\mathscr{D}'_\omega$. Both will be treated as functionals on $\mathscr{P}_\omega$ and on $\mathscr{D}_\omega$ as circumstances happen to demand.

As an application we observe that if f is a locally integrable, 2π-periodic function on R^1, and if we interpret f as a periodic *regular* distribution in $\mathscr{D}'_\omega$, $\langle f,\varphi\rangle = \int_{-\infty}^{\infty} f(t)\,\varphi(t)\,dt$, $\varphi\in\mathscr{D}_\omega$, we see that as a functional on $\mathscr{P}_\omega$, $f\cdot\theta = \langle f, \xi\theta\rangle = \int_0^{2\pi} f(t)\,\theta(t)\,dt$.

As another application of these results, we examine the connection between local units on R^1, local units on Γ, and condition (β) on R^1 and on Γ. We show that condition (β) on Γ is necessary and sufficient for $\mathscr{P}_\omega$ to be non-quasi-analytic.

THEOREM 3.10. *Suppose ω is defined on Z and satisfies (α) and (δ) on Γ. Then ω satisfies (β) on Γ,* i.e. $\sum_{n\in Z}\frac{\omega(n)}{1+n^2}<\infty$, *if and only if $\mathscr{P}_\omega$ contains local units.*

PROOF. Suppose first that $\sum_n \frac{\omega(n)}{1+n^2}<\infty$. We connect the points $(n, \omega(n))$ to form a piecewise linear, even, continuous $\omega(x)$ on R^1 which is then concave on $[0,\infty)$. Thus $\omega(x)/x^2$ is decreasing on $[0,\infty)$ and hence $\int_{R^1}\frac{\omega(x)}{1+x^2}\,dx<\infty$ by the integral test. Therefore local units exist in $\mathscr{D}_\omega$ (see section 2). It is clear that any desired local unit in $\mathscr{P}_\omega$ can be obtained from a corresponding local unit in $\mathscr{D}_\omega$ through (3.2).

Suppose next that $\sum_n \frac{\omega(n)}{1+n^2}=\infty$ and θ is in $\mathscr{P}_\omega$. Choosing $\lambda=1$ in Definition 2.5 we see $\hat\theta(n)=O(e^{-\omega(n)})$. By Theorem XV of [13] (see page 20, with $\lambda_n=n$ so that $D=1$), if θ vanishes on any interval, it vanishes identically, i.e. local units cannot exist.

COROLLARY 3.11. *If θ is in $\mathscr{P}_\omega$ and vanishes on an interval of Γ, then some truncation $\theta\chi_{[a,a+2\pi]}$ is in $\mathscr{D}_\omega$.*

PROOF. Without loss of generality, assume $\theta(t)=0$ if $\pi-\delta\leqq t\leqq\pi+\delta$. Let ξ be the unitary function in $\mathscr{D}_\omega$ constructed in Theorem 3.3. As pointed out there, ξ has support in $[-\pi-\delta, \pi+\delta]$ and is $\equiv 1$ on $[-\pi+\delta, \pi-\delta]$. Hence $\theta\chi_{[-\pi,\pi]}=\theta\xi$ is in $\mathscr{D}_\omega$ by Theorem 3.4.

REMARK. The above corollary gives a direct method of constructing local units in $\mathscr{D}_\omega$ from local units in $\mathscr{P}_\omega$ and hence it would seem that the second half of the proof of Theorem 3.10 could be proved directly in this way without using the very deep result of [13]. However the reasoning is circular in that one needs the existence of unitary functions in $\mathscr{D}_\omega$ which presumes the existence of local units in $\mathscr{D}_\omega$.

4. Fourier series and 2π-convolution

In this section we characterize periodic distributions by the growth rate of their Fourier coefficients. The main tool of this characterization is convolution. Since periodic distributions do not have compact support, their convolutions in $\mathscr{D}'_\omega$ do not exist. However a *finite* convolution, called 2π-convolution, can be defined in every case in $\mathscr{P}'_\omega$. The reader is referred to [18] where many of the details of this section can be found. We begin with a natural extension of the concept of *slow growth.*

DEFINITION 4.1. A sequence of constants $\{b_\nu\}$, $\nu\in Z$, is said to be of ω-slow growth if there exist constants M and a such that for all $\nu\in Z$,

$$|b_\nu|\leqq Me^{a\omega(\nu)}. \tag{4.1}$$

If f is in $\mathscr{P}'_\omega$, the *Fourier coefficients* of f are defined as usual to be

$$\hat{f}(\nu)=\frac{1}{2\pi}f(t)\cdot e^{-i\nu t},\quad \nu\in Z. \tag{4.2}$$

Since the Fourier series of a function $\theta\in\mathscr{P}_\omega$ converges to θ in $\mathscr{P}_\omega$, we have

$$f\cdot\theta=\sum_{n\in Z}\hat{f}(-n)\hat{\theta}(n). \tag{4.3}$$

The following result is a corollary of one of Beurling's original theorems [1, Theorem II]. In this simpler setting we give a direct proof using the results of Section 3.

THEOREM 4.2. *If f is in $\mathscr{P}'_\omega$, $\{\hat{f}(\nu)\}$ is ω-slow growth.*

PROOF. Let ξ be unitary in $\mathscr{D}_\omega$. Then from (2.1) and Theorem 3.4, there exists a $\lambda>0$ such that

$$|\hat{f}(\nu)|=\frac{1}{2\pi}|f(t)\cdot e^{-i\nu t}|=\frac{1}{2\pi}|\langle f(t),\xi(t)e^{-i\nu t}\rangle|\leqq$$

$$\leqq C\|\xi(t)e^{-i\nu t}\|_\lambda\leqq C\|\xi\|_\lambda\|e^{-i\nu t}\|_\lambda=Me^{\lambda\omega(-\nu)}=Me^{\lambda\omega(\nu)}.$$

On the path to a converse, we prove the following

THEOREM 4.3. *If* $\{b_\nu\}$ *is a sequence of* ω*-slow growth, then the trigonometric series* $\sum_{\nu\in Z} b_\nu e^{i\nu t}$ *converges in* $\mathscr{S}'_\omega$, *therefore in* $\mathscr{D}'_\omega$, *therefore in* $\mathscr{P}'_\omega$, *to a periodic Beurling distribution* f, *and* $\hat{f}(\nu)=b_\nu$, $\nu\in Z$.

PROOF. If $\varphi\in\mathscr{S}'_\omega$ (see Definition 2.7) then $\sum_n b_n\hat{\varphi}(-n)$ converges absolutely, for if b is the constant of condition (γ),

$$\sum|b_n||\hat{\varphi}(-n)| \leqq M\sum|\hat{\varphi}(n)|e^{(a+2/b)\omega(n)}e^{-(2/b)\omega(n)} \leqq M\pi_{0,a+2/b}\sum e^{-(2/b)\omega(n)} < \infty.$$

If we let $f_\nu(t)=\sum_{n=-\nu}^{\nu} b_n e^{int}$, then as regular distributions in $\mathscr{S}'_\omega$,

$$\langle f_\nu,\varphi\rangle = \int_{R^1} f_\nu(t)\varphi(t)\,dt = \sum_{n=-\nu}^{\nu} b_n \int_{R^1}\varphi(t)e^{int}\,dt \to \sum_n b_n\hat{\varphi}(-n) = \langle f,\varphi\rangle.$$

Since $\mathscr{S}'_\omega$ is closed under convergence, $f\in\mathscr{S}'_\omega\subset\mathscr{D}'_\omega$. Since each f_ν is periodic, so is f.

The proof that $\hat{f}(\nu)=b_\nu$ is exactly as in the classical case ([18], p. 332).

Before completing our characterization of elements of $\mathscr{P}'_\omega$ by proving the converse to theorem 4.3, we introduce convolution.

DEFINITION 4.4. Let f and g be arbitrary distributions in $\mathscr{P}'_\omega$. The 2π-convolution of f with g is the functional defined on $\mathscr{P}_\omega$ by

$$(f*g)\cdot\theta = f(t)\cdot[g(\tau)\cdot\theta(t+\tau)], \quad \theta\in\mathscr{P}_\omega. \tag{4.4}$$

This definition makes sense provided $g(\tau)\cdot\theta(t+\tau)$ is in $\mathscr{P}_\omega$. Assuming this to be true for the moment, we note that if f and g are locally integrable periodic functions (regular distributions), it follows as in the classical case ([18], p. 323) that $f*g$ is the regular distribution corresponding to the function $\int_0^{2\pi} f(\tau)\,g(t-\tau)\,d\tau$.

We prove two lemmas. The first is a "Fubini-type" theorem.

LEMMA 4.5. *If* $\theta\in\mathscr{P}_\omega$ *and* $g\in\mathscr{P}'_\omega$, *then*

$$g(t)\cdot\int_0^{2\pi}\theta(t+\tau)\,d\tau = \int_0^{2\pi} g(t)\cdot\theta(t+\tau)\,d\tau. \tag{4.5}$$

PROOF. First of all, the integrand on the right hand side of (4.5) is a continuous function of τ since if $\tau\to a$, then

$$\|\theta(t+\tau)-\theta(t+a)\|_\lambda = \sum_n|\hat{\theta}(n)||e^{i\tau n}-e^{ian}|e^{\lambda\omega(n)}$$

approaches 0 by the Lebesgue dominated convergence theorem, since the series is dominated by $2\sum_n|\hat{\theta}(n)|e^{\lambda\omega(n)}<\infty$. Hence $\theta(t+\tau)\to\theta(t+a)$ in $\mathscr{P}_\omega$ and therefore $g(t)\cdot\theta(t+\tau)\to g(t)\cdot\theta(t+a)$.

Next let $\alpha_\nu(t)$, $\nu=1, 2, \ldots$ be a sequence of Riemann sums for the integral $\int_0^{2\pi} \theta(t+\tau)\,d\tau=\alpha(t)$, which converges to this integral. Using the Lebesgue dominated convergence theorem, one easily shows $\alpha_\nu\to\alpha$ in $\mathscr{P}_\omega$. Hence $g\cdot\alpha_\nu\to g\cdot\alpha$. But $g\cdot\alpha_\nu$ is a Riemann sum for the right hand side of (4.5) and hence converges to that integral, which proves the result.

LEMMA 4.6. *If $\theta\in\mathscr{P}_\omega$ and $g\in\mathscr{P}'_\omega$, then $\varphi(t)=g(\tau)\cdot\theta(t+\tau)$ is in $\mathscr{P}_\omega$.*

PROOF. Applying Lemma 4.5 to the function $\theta(t+\tau)e^{-in(t+\tau)}$ and to the distribution $g(t)e^{itn}$ we see that

$$\hat{\varphi}(n)=\frac{1}{2\pi}\int_0^{2\pi} g(t)\cdot\theta(t+\tau)e^{-i\tau n}\,d\tau=$$

$$=\frac{1}{2\pi}\,g(t)\cdot\int_0^{2\pi}\theta(t+\tau)e^{-i\tau n}\,d\tau=g(t)\cdot\hat{\theta}(n)e^{itn}=2\pi\hat{\theta}(n)\,\hat{g}(-n).$$

Hence by Theorem 4.2,

$$\|\varphi\|_\lambda=2\pi\sum_n|\hat{\theta}(n)\hat{g}(-n)|e^{\lambda\omega(n)}\leqq M\sum_n|\hat{\theta}(n)|e^{(\lambda+a)\omega(n)}=M\|\theta\|_{\lambda+a}<\infty.$$

Thus $f*g$ is well defined for f and g in $\mathscr{P}'_\omega$.

THEOREM 4.7. *If f and g are in $\mathscr{P}'_\omega$, so is $f*g$.*

PROOF. The linearity of the functional $f*g$ is clear. If $\theta_\nu\to0$ in $\mathscr{P}_\omega$ then $\varphi_\nu(t)=g(\tau)\cdot\theta_\nu(t+\tau)\to0$ in $\mathscr{P}_\omega$ since $\|\varphi_\nu\|_\lambda\leqq M\|\theta_\nu\|_{\lambda+a}$ as above. Hence $f\cdot\varphi_\nu=(f*g)\cdot\theta_\nu\to0$.

We list now three properties of convolution. The properties of associativity, commutativity and continuity (left and right) must await the converse of Theorem 4.2 (namely that the Fourier series of a distribution converges to that distribution).

THEOREM 4.8. *Suppose f, g, f_ν, $\nu=1, 2, \ldots$, are in $\mathscr{P}'_\omega$. If $f_\nu\to f$ in $\mathscr{P}'_\omega$, then $f_\nu*g\to f*g$ in $\mathscr{P}'_\omega$ (left continuity). Furthermore*

$$\widehat{(f*g)}(n)=2\pi\hat{f}(n)\,\hat{g}(n). \tag{4.6}$$

Finally if $g\in\mathscr{P}_\omega$, then $f*g$ is a regular distribution given by

$$(f*g)(t)=f(\tau)\cdot g(t-\tau). \tag{4.7}$$

PROOF. The left continuity as well as (4.6) follows from the definitions. In order to prove (4.7) we let $\theta\in\mathscr{P}_\omega$ and consider

$$(f*g)(t)\cdot\theta(t)=f(\tau)\cdot[g(t)\cdot\theta(\tau+t)]=$$

$$=f(\tau)\cdot[g(t-\tau)\cdot\theta(t)]=f(\tau)\cdot\int_0^{2\pi}\theta(t)\,g(t-\tau)\,dt=f(\tau)\cdot\alpha(\tau).$$

As in the proof of Lemma 4.5 we let $\alpha_\nu(\tau)$ be a sequence of Riemann sums which converges to the integral $\alpha(\tau)$, verify that the convergence is in $\mathscr{P}_\omega$ (using the Lebesgue dominated convergence theorem), and conclude that $f(\tau)\cdot\alpha(\tau)=$ $=\lim_{\nu\to\infty} f(\tau)\cdot\alpha_\nu(\tau)$. But $f(\tau)\cdot\alpha_\nu(\tau)$ is a Riemann sum for

$$\int_0^{2\pi}\theta(t)f(\tau)\cdot g(t-\tau)\,dt=[f(\tau)\cdot g(t-\tau)]\cdot\theta(t),$$

which completes the proof.

In order to complete the characterization of distributions in terms of their Fourier series, we use a special case of right continuity of the convolution process. If $\delta_{2\pi}(t)$ denotes the Dirac-delta distribution in $\mathscr{P}'\subset\mathscr{P}'_\omega$: $\delta_{2\pi}(t)\cdot\theta(t)=\theta(0)$, $\theta\in\mathscr{P}_\omega$, then the Fourier series $\frac{1}{2\pi}\sum_n e^{int}$ for $\delta_{2\pi}$ converges to $\delta_{2\pi}$ in $\mathscr{P}'$ ([18], p. 333), and therefore in $\mathscr{P}'_\omega$. As usual $f*\delta_{2\pi}=f$, for all $f\in\mathscr{P}'_\omega$. If we denote the partial sums by $g_\nu(t)=\frac{1}{2\pi}\sum_{n=-\nu}^{n} e^{int}$, then we claim that

$$f*g_\nu\to f*\delta_{2\pi}=f,\quad\text{in }\mathscr{P}'_\omega\text{ as }\quad\nu\to\infty. \tag{4.8}$$

To see this, note first that if $\theta\in\mathscr{P}_\omega$, then as in the proof of Lemma 4.6 we have

$$\|[g_\nu(\tau)-\delta_{2\pi}(\tau)]\cdot\theta(t+\tau)\|_\lambda=2\pi\sum_n|\hat\theta(n)|\,|\hat g_\nu(-n)-\delta_{2\pi}(-n)|\,e^{\lambda\omega(n)}=$$

$$=\sum_n|\hat\theta(n)|\,|g_\nu(\tau)\cdot e^{in\tau}-\delta_{2\pi}(\tau)\cdot e^{in\tau}|\,e^{\lambda\omega(n)}.$$

But since $g_\nu\to\delta_{2\pi}$ in $\mathscr{P}'_\omega$, this series $\to0$ as $\nu\to\infty$ by the Lebesgue dominated convergence theorem, since the series is dominated by $\frac{1}{\pi}\|\theta\|_\lambda$. It follows that if $f\in\mathscr{P}'_\omega$, $(f*g_\nu)\cdot\theta=f(t)\cdot[g_\nu(\tau)\cdot\theta(t+\tau)]\to f(t)\cdot[\delta_{2\pi}(\tau)\cdot\theta(t+\tau)]=f\cdot\theta$, which proves (4.8).

The characterization of periodic distributions is finally given by Theorems 4.2, 4.3 and the following

THEOREM 4.9. *If $f\in\mathscr{P}'_\omega$, then $\sum_n\hat f(n)e^{int}$ converges in $\mathscr{P}'_\omega$ to f.*

PROOF. By Theorem 4.2 $\hat f(n)$ is a sequence of ω-slow growth, and by Theorem 4.3, $\sum_n\hat f(n)e^{int}$ converges in $\mathscr{P}'_\omega$ to some distribution $g\in\mathscr{P}'_\omega$. If $\theta\in\mathscr{P}_\omega$, then

$$f(t)\cdot\theta(t)=\big(f(t)*\delta_{2\pi}(t)\big)\cdot\theta(t)=\lim_{n\to\infty}\Big(f(t)*\frac{1}{2\pi}\sum_{\nu=-n}^{n}e^{i\nu t}\Big)\cdot\theta(t)=$$

$$=\lim_{n\to\infty}f(t)\cdot\Big[\frac{1}{2\pi}\sum_{\nu=-n}^{n}e^{i\nu\tau}\cdot\theta(t+\tau)\Big]=\lim_{n\to\infty}\frac{1}{2\pi}\sum_{\nu=-n}^{n}f(t)\cdot[e^{i\nu t}\cdot\theta(t+\tau)]=$$

$$=\lim_{n\to\infty}\frac{1}{2\pi}\sum_{\nu=-n}^{n}\big(f(t)*e^{i\nu t}\big)\cdot\theta(t)=\lim_{n\to\infty}\frac{1}{2\pi}\sum_{\nu=-n}^{n}[f(\tau)\cdot e^{i\nu(t-\tau)}]\cdot\theta(t)=$$

$$=\lim_{n\to\infty}\Big(\sum_{\nu=-n}^{n}\hat f(\nu)e^{i\nu t}\Big)\cdot\theta(t)=g(t)\cdot\theta(t).$$

Hence $g=f$.

COROLLARY 4.10. *If two distributions in $\mathscr{P}'_\omega$ have the same Fourier coefficients, they are identical.*

Using (4.6) we see that if f and g are in $\mathscr{P}'_\omega$, then

$$(f * g)(t) = 2\pi \sum_{n\in Z} \hat{f}(n)\,\hat{g}(n)e^{int}. \tag{4.9}$$

It follows that 2π-convolution is commutative, associative, and continuous in both arguments.

We remark for emphasis that Theorems 4.2, 4.3 and 4.9 imply that $\mathscr{P}'_\omega \subset \mathscr{S}'_\omega$, i.e. periodic Beurling distributions are ω-tempered.

We conclude this section by examing the relationship between Fourier transforms and Fourier coefficients of distributions.

THEOREM 4.11. *Suppose $f\in\mathscr{D}'_\omega$ and has compact support. Then*

$$g(t) = \sum_{n\in Z} f(t-2\pi n) \tag{4.10}$$

is in $\mathscr{P}'_\omega$ and $\hat{g}(v)=\hat{f}(v)$, $v\in Z$.

PROOF. Here $\hat{g}(v)$ are the Fourier coefficients of g and $\hat{f}(v)$ is the restriction to the integers of the Fourier transform of f (see (2.3) and (2.4)). The series in (4.10) obviously converges in $\mathscr{D}'_\omega$ since f has compact support, and hence for each $\varphi\in\mathscr{D}_\omega$, $\sum_n \langle f(t-2\pi n), \varphi(t)\rangle$ has only finitely many non-zero terms. It is clear that g, so defined, is periodic. Hence, if $\lambda(t)$ is a function in $\mathscr{D}_\omega$ which is identically equal to one on an interval containing the support of f, and if ξ is a unitary function in $\mathscr{D}_\omega$, we have

$$\hat{g}(v) = g(t)\cdot e^{-ivt} = \langle g(t), \xi(t)e^{-ivt}\rangle = \sum_n \langle f(t-2\pi n), \xi(t)e^{-ivt}\rangle =$$

$$= \sum_n \langle f(t), \xi(t+2\pi n)e^{-ivt}\rangle = \sum_n \langle f(t), \lambda(t)\xi(t+2\pi n)e^{-ivt}\rangle.$$

But again this sum has only finitely many non-zero terms and is therefore equal to

$$\langle f(t), \lambda(t)e^{-ivt}\sum_n \xi(t+2\pi n)\rangle = \langle f(t), \lambda(t)e^{-ivt}\rangle = \hat{f}(v).$$

The converse situation is more complicated, but has already been worked out in the classical case by Goldberg [6, Theorem H]. The statement and proof generalize to the Beurling setting with no difficulty, except that (2.5) needs to be invoked. We state the result without proof.

THEOREM 4.12. *Let $f\in\mathscr{P}'_\omega$ and let g be the piecewise linear function whose graph consists of the line segments successively joining the points $(v, \hat{f}(v))$, $v\in Z$. Then g is the Fourier transform of an ω-tempered distribution. Indeed, g is the Fourier transform of*

$$\frac{1-\cos t}{\pi t^2} f(t).$$

5. An application to harmonic functions

In this section we consider harmonic functions in the unit disc D: $|z|<1$, $z=re^{it}$, with Beurling distributional boundary values. The problem is to identify these harmonic functions by the growth rates of their L_1 norms on circles of radius $r<1$ as r approaches 1. If $u(r,t)=u_r(t)$ is harmonic in D and

$$\|u_r\|_{L_1}=\frac{1}{2\pi}\int_0^{2\pi}|u_r(t)|\,dt, \tag{5.1}$$

it is well known that the (regular distributions) $u_r(t)$ converge in $\mathscr{P}'$ as $r\to 1$ to a (Schwartz) distribution in $\mathscr{P}'$ if and only if there exists a $\lambda\geqq 0$ such that

$$\|u_r\|_{L_1}=O\left(\frac{1}{(1-r)^{\lambda}}\right), \quad \text{as} \quad r\to 1. \tag{5.2}$$

The proof of this uses the fact that a Schwartz distribution on the unit circle Γ is the nth order distributional derivative of a continuous function, for some n. This is not true for Beurling distributions. In Theorem 5.2 we give a necessary and sufficient condition on the growth rate of $\|u_r\|_{L_1}$ in order that the u_r converge in $\mathscr{P}'_\omega$ to a distribution in $\mathscr{P}'_\omega$. When $\omega(n)=\log(1+|n|)$, we obtain as a corollary an elementary proof of (5.2) which avoids the fact that a Schwartz distribution is the derivative of a continuous function.

We begin by considering the *Poisson kernel*,

$$P(r,t)=P_r(t)=\sum_{n\in Z} r^{|n|}e^{int}, \quad 0\leqq r<1. \tag{5.3}$$

As a function of $z=re^{it}$, it is *harmonic* in D, being the real part of $(1+z)/(1-z)$, and as a family of functions on Γ, indexed by r, they belong to every $\mathscr{P}_\omega$, providing ω satisfies conditions (α)—(δ) on Γ. For, since ω is concave, $\omega(n)/n$ is decreasing in n and hence approaches a constant $a\geqq 0$ as $n\to\infty$. But if $a>0$, then $\omega(n)/n^2\geqq a/n$, and therefore $\sum_n \omega(n)/n^2$ diverges, which contradicts condition (β). Thus for each $\lambda>0$, $[r^n e^{\lambda\omega(n)}]^{1/n}\to r<1$ as $n\to\infty$, and by the root test $\sum_n r^{|n|}e^{\lambda\omega(n)}=\|P_r\|_\lambda<\infty$.

Next if $f\in\mathscr{P}'_\omega$, we form the *generalized Poisson integral*

$$u(r,t)=u_r(t)=\frac{1}{2\pi}(f*P_r)(t)=\frac{1}{2\pi}f(\tau)\cdot P_r(t-\tau)=\sum_n \hat{f}(n)\,r^{|n|}e^{int}. \tag{5.4}$$

Since $|\hat{f}(n)|\leqq Me^{\lambda\omega(n)}$ for some positive constants M and λ, the series in (5.4) and the derived series obtained by differentiating term by term with respect to r or t are dominated by a constant times $\sum |n|^k r^{|n|}e^{\lambda\omega(n)}$, for some k, which by the root test again, converges uniformly in each subdisc $0\leqq r\leqq R<1$. Hence term by term differentiation is justified and we see that $u(r,t)$ is *harmonic* in D, since each term $r^{|n|}e^{int}$ is.

It is well known that as distributions, the P_r converge in $\mathscr{P}'$ and therefore in $\mathscr{P}'_\omega$ to the Dirac-delta distribution $2\pi\delta_{2\pi}$. Thus $u_r(t)=\frac{1}{2\pi}(f*P_r)(t)\to f(t)$

in $\mathscr{P}'_\omega$. To see this directly, let $\theta(t)=\sum_n \hat{\theta}(n)e^{int}\in\mathscr{P}_\omega$. Then

$$u_r(t)\cdot\theta(t)=\int_0^{2\pi} u_r(t)\theta(t)\,dt=\sum_n \hat{\theta}(n)\int_0^{2\pi} u_r(t)\,e^{int}\,dt=\sum_n \hat{\theta}(n)\hat{f}(-n)r^{|n|}.$$

But $\hat{f}(n)$ is ω-slow growth, so this series is dominated by $M\sum_n|\hat{\theta}(n)|e^{\lambda\omega(n)}=M\|\theta\|_\lambda<$ $<\infty$, and hence as $r\to 1$, $u_r\cdot\theta\to\sum_n\hat{\theta}(n)\hat{f}(-n)=f\cdot\theta$ by (4.3).

Conversely if $u(r,t)=u_r(t)$ is harmonic in D, and if $u_r(t)\to f(t)$ in $\mathscr{P}'_\omega$ as $r\to 1$, then $u(r,t)=(f*P_r)(t)$. For if one writes $u(r,t)=\sum_n a_n r^{|n|}e^{int}$ (see [9], p. 29), then

$$\hat{u}_r(n)=u_r(t)\cdot e^{-int}\to f(t)\cdot e^{-int}=\hat{f}(n)\quad\text{as}\quad r\to 1.$$

But $\hat{u}_r(n)=a_n r^{|n|}\to a_n$. Therefore

$$u(r,t)=\sum_n \hat{f}(n)\,r^{|n|}e^{int}=(f*P_r)(t).$$

We have proved

THEOREM 5.1. *A harmonic function $u(r,t)$ in the unit disc converges as $r\to 1$ in the sense of distributions to an $f\in\mathscr{P}'_\omega$ if and only if u is the generalized Poisson integral of f. This is the case if and only if the Laurent series coefficients of u are of ω-slow growth.*

In this case we will say $u(r,t)\in\mathscr{P}'_\omega$.

THEOREM 5.2. *Suppose $u(r,t)$ is harmonic in the unit disc, and ω satisfies conditions (α)—(δ) on Γ. Then $u\in\mathscr{P}'_\omega$ if and only if there exist positive constants λ and Λ such that*

$$\|u_r\|_{L_1}=O(e^{\Lambda\omega(X)}),\quad r\to 1, \tag{5.5}$$

where $X=X(r)$ is the unique solution to

$$\omega(X)/X=(1/2\lambda)\log 1/r. \tag{5.6}$$

PROOF. We begin by showing (5.5) is necessary. We are given $u(r,t)=$ $=\sum_n a_n r^{|n|}e^{int}$, where the a_n are ω-slow growth: $|a_n|\leqq M\exp\lambda\omega(n)$, $n\in Z$, $\lambda>0$, $M>0$. It follows that

$$\|u_r\|_{L_1}\leqq C\sum_{n=0}^{\infty} r^n e^{\lambda\omega(n)}=CS(r),\quad 0\leqq r<1, \tag{5.7}$$

and the problem is to decipher the asymptotic behavior of $S(r)$ as $r\to 1$.

First of all, since $\omega(x)$ is concave, (5.6) has solutions for each $r<1$ (and sufficiently close to 1), and since by condition (β), $\omega(x)$ cannot be linear for all x, the solution is unique (again for r sufficiently close to 1, since ω may start out linear).

St $r=e^{-t}$, $0<t<\infty$. Then for t small enough, $\lambda\omega(x)-tx\geqq 0$ on some interval $0\leqq x\leqq x_0$. If $\xi=\xi(t)$ is the first value of x where $\lambda\omega(x)-tx$ attains its maximum, and if $X=X(t)$ is the unique solution to $\dfrac{2\lambda\omega(X)}{X}=t=\log\dfrac{1}{r}$, then

$\xi(t)<x_0<X(t)$. Hence

$$S(r)=\sum_{n=0}^{\infty}e^{\lambda\omega(n)-tn}=\sum_{n=0}^{[\xi]-1}e^{\lambda\omega(n)-tn}+\sum_{n=[\xi]+2}^{\infty}e^{\lambda\omega(n)-tn}+$$

$$+e^{\lambda\omega([\xi])-t[\xi]}+e^{\lambda\omega([\xi]+1)-t([\xi]+1)}\leqq\int_0^{\infty}e^{\lambda\omega(x)-tx}\,dx+2e^{\lambda\omega(\xi)-t\xi}.$$

On the other hand,

$$\int_0^X e^{\lambda\omega(x)-tx}\,dx\leqq e^{\lambda\omega(X)}\int_0^X e^{-tx}\,dx<\frac{1}{t}e^{\lambda\omega(X)}.$$

Next if $x\geqq X$, $2\lambda\omega(x)/x\leqq t$ and therefore $\lambda\omega(x)-tx\leqq-\frac{1}{2}tx$. Hence

$$\int_X^{\infty}e^{\lambda\omega(x)-tx}\,dx\leqq\frac{2}{t}e^{-tX}e^{\frac{1}{2}tX}<\frac{2}{t}e^{\frac{1}{2}tX}=\frac{2}{t}e^{\lambda\omega(X)}.$$

Finally since $\xi(t)<X(t)$,

$$2e^{\lambda\omega(\xi)-t\xi}\leqq 2e^{\lambda\omega(\xi)}\leqq 2e^{\lambda\omega(X)}<\frac{1}{t}e^{\lambda\omega(X)}.$$

We have shown $\|u_r\|_{L_1}\leqq\frac{4}{t}e^{\lambda\omega(X)}$, $t=\log 1/r$. But by condition (γ): $\log(1+|x|)\leqq$ $\leqq b\omega(x)+a$, $b>0$, we have $2\lambda/t\leqq 1+2\lambda/t\leqq e^a\exp b\omega(2\lambda/t)\leqq e^a\exp b\omega(X)$, since by (5.6) $2\lambda/t=X/\omega(X)\leqq X$, since $\omega(X)\geqq 1$ for X large, again by condition (γ). Hence (5.5) follows with $\Lambda=\lambda+b$.

For the sufficiency of (5.5), suppose $u(r,t)=\sum_n a_n r^{|n|}e^{int}$ is harmonic in the unit disc. Then if X satisfies (5.6), we have

$$|a_n|=\frac{r^{-|n|}}{2\pi}\left|\int_0^{2\pi}u(r,t)e^{-int}\,dt\right|\leqq r^{-|n|}\|u_r\|_{L_1}\leqq cr^{-|n|}e^{\Lambda\omega(X)}.$$

But this is valid for all $n\in Z$ and all $0<r<1$. For a given n, choose $r=\exp(-2\lambda\omega(n)/|n|)$. Hence $r^{-|n|}=\exp 2\lambda\omega(n)$. Also

$$\omega(X)/X=(1/2\lambda)\log 1/r=\omega(n)/|n|.$$

So $X=|n|$ and $e^{\Lambda\omega(X)}=e^{\lambda\omega(n)}$. The conclusion is that $|a_n|\leqq ce^{(\Lambda+2\lambda)\omega(n)}$, i.e. the a_n are ω-slow growth and $u\in\mathscr{P}'_\omega$.

We conclude by considering two special cases. First of all, in the classical Schwartz case, $\omega(x)=\log(1+|x|)$, and although (5.6) cannot be solved explicitly for X, several applications of L'Hospital's rule show X is asymptotic to $(2\lambda/t)\log(1+2\lambda/t)$, $t=\log 1/r$, as $t\to 0$ $(r\to 1)$. Choosing $\Lambda=\lambda$ in (5.5) we see

from (5.6) that

$$\|u_r\|_{L_1} = O(e^{\lambda\omega(X)}) = O(e^{Xt/2}) = O\big(\exp\lambda\log(1+2\lambda/t)\big) =$$
$$= O(1+2\lambda/t^{\lambda}) = O(1/t^{\lambda}) = O\big(1/(1-r)^{\lambda}\big) \quad \text{as} \quad r \to 1,$$

which is the classical result (see (5.2)).

Secondly in the Gevrey case, $\omega(x)=x^{\alpha}$, $0<\alpha<1$, (5.6) can be solved explicitly for X and we obtain the following result.

COROLLARY 5.3. *Suppose* $u(r, t)=u_r(t)=\sum_n a_n r^{|n|} e^{int}$ *is harmonic in the unit disc, and* $0<\alpha<1$. *Then*

$$a_n = O(e^{\lambda|n|^{\alpha}}), \quad as \quad |n| \to \infty,$$

for some positive constant λ, *if and only if*

$$\|u_r\|_{L_1} = O(\exp \Lambda/(1-r)^{\alpha/(1-\alpha)}), \quad as \quad r \to 1,$$

for some positive constant Λ.

Bibliography

[1] A. Beurling, *Quasi-analyticity and general distributions*, Lectures 4 and 5, A. M. S. Summer institute, Stanford, 1961 (mimeographed).

[2] C. A. Berenstein and M. A. Dostal, Topological properties of analytically uniform spaces, *Trans. Amer. Math. Soc.*, **154** (1971), 493—513.

[3] G. Björck, Linear partial differential operators and generalized distributions, *Ark. Mat.*, **6** (1966), 351—407.

[4] James E. Brennan, Approximation in the mean by polynomials on non-Carathéodory domains, *Ark. Matematik*, **15** (1977), 117—168.

[5] M. A. Dostal, An analogue of a theorem of Vladimir Bernstein and its applications to singular supports of distributions, *Proc. London Math. Soc.*, **19** (1969), 553—576.

[6] R. R. Goldberg, Restrictions of Fourier transforms and extension of Fourier sequences, *J. Approx. Th.*, **3** (1970), 149—155.

[7] Olaf von Grudzinski, Spezielle Sobolev-Räume von Beurling-Distributionen, *Math. Z.*, **146** (1976), 173—187.

[8] Olaf von Grundzinki, Temperierte Beurling-Distributionen, *Math. Nachr.*, **91** (1979), 297—320.

[9] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall (New Jersey, 1962).

[10] O. John, Sulla regolarità delle soluzioni delle equazioni lineari ellitiche rielle classi di Beurling, *Boll. Un. Mat. Ital.*, **4** (1969), 183—195.

[11] B. Korenblum, An extension of the Nevanlinna theory, *Acta Math.*, **135** (1975), 187—219.

[12] Thomas Kriete, On the structure of certain $H^2(\mu)$ spaces, *Ind. Math. J.*, **29** (1979), 757—773.

[13] N. Levinson, *Gap and density theorems*, American Mathematical Society (New York, 1940).

[14] H. Millington, Nuclear spaces of generalized test functions, *Can. Math. Bull.*, **16** (1973), 269—273.

[15] R. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, American Mathematical Society (New York, 1934).

[16] W. Rudin, *Functional Analysis*, McGraw Hill (New York, 1973).

[17] Otto Weingärtner and Florin Constantinescu, Der Satz von Bochner—Schwartz für Ultra-distributionen, *Math. Z.*, **147** (1976), 175—180.

[18] A. H. Zemanian, *Distribution Theory and Transform Analysis*, McGraw Hill (New York, 1965).

(Received January 28, 1982)

MARY WASHINGTON COLLEGE
FREDERICKSBURG, VA.
USA

VANDERBILT UNIVERSITY
NASHVILLE, TENNESSEE 37235
USA

Acta Math. Hung.
42 (3—4) (1983), 279—293.

ON SOME CONVERGENCE PROPERTIES OF HAAR—FOURIER SERIES IN THE CLASSES $\varphi(L)$

P. OSWALD (Dresden)

1. Introduction

In the paper presented we will consider some properties of the ψ-convergence of the Haar—Fourier series of functions belonging to the class $\varphi(L)$. This topic is closely related to a problem of P. L. Ul'janov (cf. [16, p. 39/40] or [19, p. 815/816]).

First some definitions and notations are given. Let Φ be the set of functions $\varphi: R \to R$ with the properties

$$0 = \varphi(0) = \lim_{t \to 0} \varphi(t) < \varphi(t) = \varphi(-t) \leqq \varphi(\tau) < \lim_{t \to \infty} \varphi(t) = \infty,$$

where $0 < t < \tau < \infty$. In the following $\varphi, \psi \in \Phi$ will always be presumed. The class $\varphi(L)$ consists of all Lebesgue-measurable functions $f: [0, 1] \to R$ satisfying

$$\|f\|_\varphi = \int_0^1 \varphi(f(x))\, dx < \infty. \tag{1.1}$$

A sequence of measurable functions f_n, $n \geqq 1$, is said to be ψ-convergent to the measurable function f, if $f - f_n$ belongs to $\psi(L)$ for large n and

$$\lim_{n \to \infty} \|f - f_n\|_\psi = 0 \tag{1.2}$$

(this will be denoted by $f_n \xrightarrow{\psi} f$).

The basic properties of the classes $\varphi(L)$ and the ψ-convergence were studied in detail by P. L. Ul'janov (cf. [15—19]), see also the earlier paper of S. Mazur, W. Orlicz [4].

If φ is convex then $\varphi(L)$ is a so-called Orlicz class. These classes and the related Orlicz spaces were investigated by many authors (cf. [2], for selected topics also see [22]).

Let $H = \{\chi_j(x)\}$, $j \geqq 1$, be the Haar system defined by $\chi_1(x) = 1$, $x \in [0, 1]$, and

$$\chi_j(x) = \begin{cases} 2^{k/2}, & x \in ((l-1)2^{-k}, (2l-1)2^{-k-1}) \\ -2^{k/2}, & x \in [(2l-1)2^{-k-1}, l2^{-k}] \\ 0, & x \notin ((l-1)2^{-k}, l2^{-k}) \end{cases}$$

where $j = 2^k + 1$, $l = 1, \ldots, 2^k$, $k = 0, 1, \ldots$. Denote by

$$S_n f(x) = \sum_{j=1}^{n} a_j(f) \chi_j(x), \quad a_n(f) = \int_0^1 f(x) \chi_n(x)\, dx, \tag{1.3}$$

$n \geqq 1$, the partial sums of the Haar—Fourier series of a function $f \in L^1$.

In this paper we intend to study the conditions on $\varphi, \psi \in \Phi$ which guarantee $S_n f \xrightarrow{\psi} f$ for all $f \in \varphi(L)$. In this connection it is natural to assume

$$\varphi(L) \subset L^1. \tag{1.4}$$

Using the notation

$$H(\psi) = \{f \in L^1 : S_n f \xrightarrow{\psi} f\}, \tag{1.5}$$

the problems considered below can be stated as follows:

(I) Find necessary and sufficient conditions on φ satisfying the inclusion $\varphi(L) \subset H(\varphi)$.

(II) For given φ, find necessary and sufficient conditions on ψ yielding $\varphi(L) \subset H(\psi)$.

Obviously, the second problem is more general, but it is quite useful first to set about problem (I).

It should be mentioned that questions of this type may also be formulated (and solved) for the convergence of Fourier series or the representation by series with respect to other function systems (cf. the above cited papers of P. L. Ul'janov). Furthermore, it would be possible to consider analogous problems for various operators acting in spaces of measurable functions. In these directions there are always some results which we briefly comment on now.

a) *Representation by series with respect to the Faber—Schauder system.* Let

$$F = \{\varphi_0(x) = 1,\ \varphi_j(x) = \int_0^x \chi_j(t)\,dt,\ x \in [0, 1],\ j \geqq 1\}$$

be the Faber—Schauder system. The set of all measurable functions possessing a representation by some φ-convergent series with respect to F will be denoted by $F(\varphi)$. The complete answer to the question analogous to (I) was given by P. L. Ul'janov [15, 16]:

Let $\varphi \in \Phi$. The inclusion $\varphi(L) \subset F(\varphi)$ is satisfied iff

$$\varphi(t+1) = O(\varphi(t)), \quad t \to \infty \ (\omega\text{-condition}). \tag{1.6}$$

b) *Conjugate functions.*[1] Let $\tilde{\varphi}(L)$ be the class of all functions

$$\tilde{f}(x) = \text{v.p.} -\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cot \frac{x-t}{2}\,dt, \quad x \in (-\pi, \pi],$$

where $f \in \varphi(L)$ and (1.4) is satisfied. The relations

$$\tilde{\varphi}(L) \subset \varphi(L) \quad \text{and} \quad \tilde{\varphi}(L) \subset \psi(L), \text{ resp.}, \tag{1.7}$$

were investigated by O. D. Zereteli (cf. e.g. [20, 21]) under the assumption that

$$\varphi(2t) = O(\varphi(t)), \quad t \to \infty \ (\Delta_2\text{-condition}). \tag{1.8}$$

[1] Here and in c), all functions are assumed to be 2π-periodic, the modifications in the further definitions are obvious.

In this situation final results were obtained. In [9] the first inclusion in (1.7) was treated without additional assumptions on $\varphi \in \Phi$. The precise result reads as follows:

Let $\varphi \in \Phi$, where (1.4) is satisfied. Then the relation $\tilde{\varphi}(L) \subset \varphi(L)$ holds if and only if there exists a convex function $\varphi_c \in \Phi$ with the following properties

$\varphi(t) \asymp \varphi_c(t), \quad t \to \infty$, [2]

the convex function φ_c and its conjugate in the sense of Young satisfy the Δ_2-condition (1.8).

(In other words, the class $\varphi(L)$ has to be (topological) equivalent to a reflexive Orlicz space.)

c) *Convergence of trigonometric Fourier series and representation by trigonometric series.* In analogy with (1.5), let $T(\psi)$ be the set of all $f \in L^1$ whose Fourier series are ψ-convergent to f. In [9] it was proved that, (1.4) supposed, the inclusion $\varphi(L) \subset T(\varphi)$ holds iff $\varphi(L)$ is equivalent to a reflexive Orlicz space (for the description of this property, see b)). Partly, this statement is a consequence of results known for Orlicz spaces (cf. [3]). As far as we know, conditions for $\varphi(L) \subset T(\psi)$ were not explicitly stated anywhere.

Recently, V. I. Ivanov [1] has shown that for $\varphi \in \Phi$ satisfying (1.8), but not (1.4) there always exists a φ-convergent representation by trigonometric series for any $f \in \varphi(L)$. In [1] he also examined representation properties by power series in $\varphi(L)$. It should be mentioned that till then a result of A. A. Talaljan [10], [11] was known for general complete orthonormal systems G but only for a special class of φ:

If $\varphi \in \Phi$ satisfies the conditions

$$\varphi(t+\tau) \leqq \varphi(t)+\varphi(\tau), \quad 0 < t, \ \tau < \infty, \ \lim_{t \to \infty} \frac{\varphi(t)}{t} = 0, \tag{1.9}$$

then for any $f \in \varphi(L)$ there exists a series with respect to G representing f in the sense of φ-convergence.

d) *Unconditional convergence of Haar—Fourier series.* Here we only refer to the investigations of G. E. Tkebucava [12—14] and other authors concerning conditions for unconditional ψ-convergence of the Haar—Fourier series (and the Fourier series with respect to other spline systems) of $f \in \varphi(L)$, where further detailed information may be found.

Now we return to the contents of the present paper. The following section summarizes some preliminaries on the classes $\varphi(L)$ and the examples of functions used for proving the necessity in the theorems below.

In Section 3 a sufficient condition is stated for which the inclusion

$$\varphi(L) \subset H(\psi) \tag{1.10}$$

holds. This condition is a direct consequence of the results known for Orlicz classes (see [16]). Its necessity will be proved for ψ satisfying the Δ_2-condition (1.8). We

[2] $A \asymp B$ means both $A = O(B)$ and $B = O(A)$.

also show a simple second necessary condition: If (1.10) holds then for every $\alpha<2$ we must have

$$\varphi(\alpha\cdot t)=O(\varphi(t)),\quad t\to\infty. \tag{1.11}$$

These results already involve the complete answer to problem (I).

THEOREM 1. *Let $\varphi\in\Phi$ and $\varphi(L)\subset L^1$. Then the inclusion $\varphi(L)\subset H(\varphi)$ is valid if and only if φ satisfies the Δ_2-condition* (1.8) *and*

$$\varphi(t)\asymp\varphi_c(t),\quad t\to\infty, \tag{1.12}$$

where $\varphi_c(t)$ is the convex minorant with respect to $\varphi(t)$. These conditions are equivalent to the following one: The class $\varphi(L)$ can be identified with a separable Orlicz space.

The main results of Section 3 were contained in our preprint [7] (for some of their applications cf. [8]).

In Section 4, problem (II) is solved for arbitrary convex $\varphi\in\Phi$. It will be proved that the sufficient condition established in Section 3 (see [16]) can be sharpened for rapidly increasing convex functions $\varphi\in\Phi$. To illustrate this situation we formulate our result (see Theorem 3 in Section 4) for the special case $\varphi(t)=e^{|t|}-1$, i.e. $\varphi(L)=e^L$.

The following relations are equivalent ($\psi\in\Phi$)

$$e^L\subset H(\psi) \tag{1.13}$$

$$\psi(t)=O(\sqrt{t}e^{t/2}),\quad t\to\infty. \tag{1.14}$$

For comparison, from [16] it follows only that $\psi(t)=O(e^{t/2})$, $t\to\infty$, yields (1.13).

Acknowledgement

This study was completed during the author's stay at the Moscow University. I am indebted to Prof. P. L. Ul'janov and his colleagues for valuable discussions and kind hospitality.

2. Preliminaries

The first four assertions (Lemma 1—4) are well-known or obvious (cf. [4], [15—19], [7]).

LEMMA 1. *Let $\varphi,\psi\in\Phi$. The imbedding $\varphi(L)\subset\psi(L)$ holds iff $\psi(t)=O(\varphi(t))$, $t\to\infty$. In this case from $f_n\xrightarrow{\varphi}f$ it automatically follows that $f_n\xrightarrow{\psi}f$. Analogously, the equality $\varphi(L)=\psi(L)$ is equivalent to $\varphi(t)\asymp\psi(t)$, $t\to\infty$, and the notions of φ- and ψ-convergence coincide.*

REMARK 1. From Lemma 1 it becomes clear that the imbedding $\varphi(L)\subset H(\psi)$ does not depend on the behaviour of the functions $\varphi(t),\psi(t)$ for small t. In particular, it could be assumed that $\varphi(t)=|t|\varphi(1)$, $|t|\leqq 1$.

LEMMA 2. *If* $\varphi\in\Phi$ *does not satisfy the* Δ_2*-condition* (1.8) *then we have for arbitrary* $\alpha>1$

$$\varphi(\alpha\cdot t)\neq O(\varphi(t)),\quad t\to\infty\quad\left(\Leftrightarrow\varlimsup_{t\to\infty}\frac{\varphi(\alpha t)}{\varphi(t)}=\infty\right). \tag{2.1}$$

LEMMA 3. *If the* Δ_2*-condition* (1.8) *holds for* $\varphi\in\Phi$ *then there exist two positive constants* C_1, C_2 *for which we have*

$$\|f+g\|_\varphi\leqq C_1\{\|f\|_\varphi+\|g\|_\varphi\}+C_2,\quad f,\,g\in\varphi(L). \tag{2.2}$$

By

$$\varphi_c(t)=\inf\{\lambda\varphi(t_1)+(1-\lambda)\varphi(t_2)\colon t=\lambda t_1+(1-\lambda)t_2,\ \lambda\in[0,1],\ t_1,\,t_2\in R\} \tag{2.3}$$

we denote the convex minorant function with respect to $\varphi(t)$. It is evident (cf. Lemma 1) that in the case $\varphi(L)\subset L^1$ we have $\varliminf_{t\to\infty}\varphi(t)t^{-1}>0$ and, thus, $\varphi_0(t)\not\equiv 0$. According to Remark 1 it is always possible to change the values of $\varphi(t)$ for small t in order to obtain $\varphi_c(t)\in\Phi$. For the alternative situation when $\varliminf_{t\to\infty}\varphi(t)t^{-1}=0$ it follows that $\varphi_c(t)\equiv 0$. Furthermore, the relation

$$\varphi(t)=O(\varphi_c(t)),\quad t\to\infty, \tag{2.4}$$

will hold if and only if $\varphi(t)$ is equivalent to some convex function (or, in other words, if $\varphi(L)$ coincides with some Orlicz class). This fact is obvious by Lemma 1 and the trivial inequality $\varphi_c(t)\leqq\varphi(t)$, $t\in R$.

Let $f\in L^1$, and $\Delta\subset[0,1]$ be some interval. Then

$$m(f,\Delta)=|\Delta|^{-1}\int_\Delta f(x)\,dx,\quad m(f)=m(f,[0,1]) \tag{2.5}$$

denote the mean value of f on Δ or on $[0,1]$, resp. ($|E|$ denotes the measure of a Lebesgue-measurable set $E\subset[0,1]$.) Thus, the following Lemma is a consequence of Jensen's inequality.

LEMMA 4. *If* $\varphi\in\Phi$ *is convex then for arbitrary* $f\in\varphi(L)$

$$|\Delta|\varphi(m(f,\Delta))\leqq\|f\|_\varphi. \tag{2.6}$$

The following assertions contain the main idea for constructing suitable examples in the proof of necessity in the theorems of Sections 3 and 4.

LEMMA 5. *Let* $\varphi,\psi\in\Phi$, $\varphi(L)\subset L^1$, *and*

$$\psi(t)\neq O(\varphi_c(t)),\quad t\to\infty. \tag{2.7}$$

Then for arbitrarily given constants A, $B>0$ *there exists a non-negative step function* (x) *satisfying the inequalities*

$$\psi(m(f))>A\|f\|_\varphi>AB. \tag{2.8}$$

PROOF. According to the assumptions we have $\varphi_c(t)\in\Phi$ and, in particular, $\lim_{t\to\infty}\varphi_c(t)=\infty$. In virtue of (2.3) and (2.7) one can determine $\lambda\in[0,1]$, $t_1, t_2\geqq 0$, and $t=\lambda t_1+(1-\lambda)t_2$ in such a way that

$$B<\varphi_c(t)\leqq\lambda\varphi(t_1)+(1-\lambda)\varphi(t_2)\leqq\varphi_c(t)+1<A^{-1}\psi(t).$$

It remains to put

$$f(x)=\begin{cases}t_1, & x\in[0,\lambda],\\ t_2, & x\in(\lambda,1],\end{cases}$$

and the Lemma is proved.

LEMMA 6. *Let $\varphi,\psi\in\Phi$. If for some $\alpha<2$ we have the relation*

$$\psi(\alpha t)\neq O(\varphi(t)),\quad t\to\infty, \tag{2.9}$$

then for arbitrary $A, B>0$ there exists a step function $f(x)$ with the properties

$$\|f-m(f)\|_\psi>A\|f\|_\varphi>AB. \tag{2.10}$$

PROOF. Without any loss of generality, assume $1\leqq\alpha<2$ and set

$$f_t(x)=\begin{cases}t, & x\in[0,1-\alpha/2]\\ -t, & x\in(1-\alpha/2,1],\end{cases}\qquad t>0.$$

For these step functions we have the relations $\|f_t\|_\varphi=\varphi(t)$ and

$$\|f_t-m(f_t)\|_\psi\geqq\left(1-\frac{\alpha}{2}\right)\psi\left(t-\left(1-\frac{\alpha}{2}\right)t+\frac{\alpha}{2}t\right)=\left(1-\frac{\alpha}{2}\right)\psi(\alpha t).$$

The rest of the proof is obvious.

LEMMA 7. *If $\varphi\in\Phi$ and $f_k\in\varphi(L)$, $k\geqq 1$, satisfy the inequalities*

$$1<\|f_1\|_\varphi<\|f_2\|_\varphi<\dots \tag{2.11}$$

then one can define integers $1\leqq n_1<n_2<\dots$ for which the function

$$f(x)=\begin{cases}f_k\left(2^{n_k}(x-2^{-n_k})\right), & x\in(2^{-n_k},2^{-n_k+1}],\ k\geqq 1\\ 0, & \text{elsewhere in } [0,1]\end{cases} \tag{2.12}$$

belongs to $\varphi(L)$, more precisely, $\|f\|_\varphi\leqq 1$.

PROOF. We determine the numbers n_k with respect to

$$2^{-k-1}<2^{-n_k}\|f_k\|_\varphi\leqq 2^{-k},\quad k\geqq 1. \tag{2.13}$$

By (2.11) we have $1\leqq n_1<n_2<\dots$. Thus, the definition of $f(x)$ in (2.12) is correct, and

$$\|f\|_\varphi=\sum_{k=1}^{\infty}2^{-n_k}\|f_k\|_\varphi\leqq\sum_{k=1}^{\infty}2^{-k}=1,\quad \text{q. e. d.}$$

3. Some conditions concerning $\varphi(L)\subset H(\psi)$

In this section we shall prove a theorem which is more general than Theorem 1 stated in Section 1.

THEOREM 2. *Let* $\varphi, \psi\in\Phi$ *and* $\varphi(L)\subset L^1$.

a) *If*

$$\psi(2t)=O(\varphi_c(t)),\quad t\to\infty, \tag{3.1}$$

then

$$\varphi(L)\subset H(\psi). \tag{3.2}$$

b) *If* ψ *satisfies the* Δ_2*-condition* (1.8) *then the inverse assertion holds, i.e.* (3.2) *implies* (3.1).

c) *For* (3.2) *it is also necessary that*

$$\psi(\alpha t)=O(\varphi(t)),\quad t\to\infty,\quad \text{if}\quad \alpha<2. \tag{3.3}$$

PROOF. Assertion a) is a direct consequence of the results obtained by P. L. Ul'janov [16, cf. Theorems 6.3, 6.4, and Corollary 6.1].

For proving b), assume that ψ satisfies the Δ_2-condition and (3.1) is not fulfilled. Thus, we get

$$\psi(t)\neq O(\varphi_c(t)),\quad t\to\infty. \tag{3.4}$$

Now we make use of Lemma 5 with $A=4$, $B=1$; the resulting function is denoted by f_1. By induction we obtain the functions f_k, $k\geqq 2$, applying Lemma 5 with the parameters $A=2^k$ and $B=\|f_{k-1}\|_\varphi$. The sequence of functions f_k, $k\geqq 1$, satisfies the assumptions of Lemma 7. Therefore, the function f given by (2.12), (2.13) belongs to $\varphi(L)$ and has the property (see also (2.8))

$$\|S_{2^{n_k}}\|_\psi\geqq\int_{2^{-n_k}}^{2^{-n_k+1}}\psi(m(f_k))\,dx=2^{-n_k}\psi(m(f_k))> \tag{3.5}$$

$$>2^{2k}2^{-n_k}\|f_k\|_\varphi>2^{k-1},\quad k=1,2,\ldots.$$

From this we shall obtain $S_nf\overset{\psi}{\nrightarrow}f$. Indeed, if $f\notin\psi(L)$ then by Lemma 3 it follows that $f-S_nf\notin\psi(L)$ for all $n\geqq 1$. On the other hand, if $f\in\psi(L)$ then (2.2) and (3.5) yield

$$\|f-S_{2^{n_k}}f\|_\psi\geqq\frac{1}{C_1}\|S_{2^{n_k}}f\|_\psi-\|f\|_\psi-\frac{C_2}{C_1}>C_1^{-1}(2^{k-1}-C_2-C_1\|f\|_\psi)\to\infty$$

for $k\to\infty$. Thus, the desired result is established, we get $\varphi(L)\not\subset H(\psi)$, and b) is proved.

In case c), the construction is based on Lemma 4, with obvious changes the considerations are the same as above. Instead of (3.5) now we have

$$\|f-S_{2^{n_k}}f\|_\psi>2^{k-1},\quad k\geqq 1.$$

The details are left to the reader.

PROOF OF THEOREM 1 (cf. Section 1). The sufficiency was essentially proved by W. Orlicz [6] (cf. [2, p. 106]; obviously, it also follows from part a) of Theorem 2). The necessity of the Δ_2-condition of φ is a consequence of Theorem 2, c) and Lemma 2. Furthermore, if φ satisfies the Δ_2-condition then $\varphi(L)\subset H(\varphi)$ and Theorem 2, b) yields

$$\varphi(t)=O(\varphi(2t))=O(\varphi_c(t)),\quad t\to\infty,$$

and (1.12) is established, too. In [4] or [2, p. 75/76] it is proved that these properties of φ are equivalent to the coincidence of $\varphi(L)$ with a separable Orlicz space.

REMARK 2. In the proof of Theorem 2, b) and c) there were only used partial sums with $n=2^{n_k}$. Consequently, the same necessary conditions remain valid for the similar problem of the ψ-convergence of Walsh—Fourier series of functions belonging to $\varphi(L)$. In particular, from this and known results for the case of Orlicz spaces the analogue of Theorem 1 can be obtained for the Walsh system.

REMARK 3. As a corollary to Theorem 2 it should be mentioned that for convex $\varphi\in\Phi$ and with $\lambda(t)=2t$ we have

$$S_nf\xrightarrow{\varphi}f\quad\text{for all } f\in\varphi(\lambda(L)). \tag{3.6}$$

In this connection the problem arises whether there exists some function $\lambda\in\Phi$ (satisfying, say, $\lambda(t)/t\geqq 1$, $t>0$) for which (3.6) holds for arbitrary $\varphi\in\Phi$ with $\varphi(L)\subset L^1$. Obviously, for *each fixed* $\varphi\in\Phi$ such a λ can be obtained in dependence on φ (e.g. take some convex $\lambda\in\Phi$ with $\lambda(t)\geqq\varphi(2t)$, $t>0$). However, the above formulated question on the existence of a λ satisfying (3.6) *simultaneously for all* φ must be negated.

Indeed, for given λ we can first determine the sequences $t_0=1<t_1<\ldots\to\infty$ and $\delta_0=1<\delta_1<\ldots$ with respect to

$$\begin{cases} t_k>2^{2k+1}\{\lambda(t_{k-1})+t_{k-1}\}, \\ \delta_k=2^{2k+1}\delta_{k-1}+\lambda(t_k), \end{cases}\quad k\geqq 1. \tag{3.7}$$

Put

$$\varphi(t)=\begin{cases} |t|, & |t|\leqq 1 \\ 1, & 1<|t|\leqq\lambda(1) \\ \delta_k, & \lambda(t_{k-1})<|t|\leqq\lambda(t_k), \end{cases}\quad k\geqq 1 \tag{3.8}$$

and

$$f_k(x)=\begin{cases} t_{k-1}, & x\in[0,\,1-2^{-2k-1}], \\ t_k, & x\in(1-2^{-2k-1},\,1], \end{cases}\quad k\geqq 1.$$

By these definitions we obtain

$$\|f_k\|_{\varphi(\lambda(L))}=(1-2^{-2k-1})\varphi(\lambda(t_{k-1}))+2^{-2k-1}\varphi(\lambda(t_k))=$$

$$=(1-2^{-2k-1})\delta_{k-1}+2^{-2k-1}\delta_k$$

and, consequently,

$$2^{-2k-1}\delta_k<\|f_k\|_{\varphi(\lambda(L))}<2^{-2k}\delta_k,\quad k\geqq 1. \tag{3.9}$$

Furthermore,

$$t_k > m(f_k) = 2^{-2k-1}t_k + (1-2^{-2k-1})t_{k-1} > t_{k-1} + \lambda(t_{k-1}).$$

This yields

$$\|f_k - m(f_k)\|_\varphi = \delta_k, \quad k \geqq 1. \tag{3.10}$$

Now Lemma 7 can be applied to f_k, $k \geqq 1$, and $\varphi(\lambda(t))$ instead of $\varphi(t)$. Thus, we get a function $f \in \varphi(\lambda(L))$ satisfying (cf. (3.9), (3.10), and the construction given in (2.12), (2.13))

$$\|f - S_{2^{n_k}} f\|_\varphi > C \cdot 2^k \to \infty, \quad k \to \infty,$$

i.e. $\varphi(\lambda(L)) \not\subset H(\varphi)$. This is the desired result (by construction we also have $\varphi(L) \subset L^1$, see (3.7), (3.8)).

REMARK 4. Applying Theorem 1 and the methods developed by A. M. Olevskiĭ [5, ch. 3] we proved in [8] that for any complete orthonormal system $G = \{g_k\}$, $k \geqq 1$, with

$$g_k \in L^\infty, \quad k \geqq 1, \tag{3.11}$$

and arbitrary $\varphi \in \Phi$ satisfying the conditions (1.4), (1.8) the following result is valid:

If for all $f \in \varphi(L)$ the Fourier series with respect to G is φ-convergent to f then we have (1.12), i.e. $\varphi(L)$ necessarily coincides with some Orlicz space.

It would be desirable to weaken the assumptions (3.11) and (1.8).

4. Solution of problem (II) for Orlicz classes

In the present section convex $\varphi \in \Phi$ will be considered. In this case we obtain a sufficient and necessary condition on $\psi \in \Phi$ yielding the inclusion

$$\varphi(L) \subset H(\psi). \tag{4.1}$$

From the results of the preceding section it follows that

$$\psi(2t) = O(\varphi(t)), \quad t \to \infty, \tag{4.2}$$

is sufficient for (4.1), and

$$\psi(\alpha t) = O(\varphi(t)), \quad t \to \infty, \quad \alpha < 2, \tag{4.3}$$

is necessary. Obviously, there remains a gap between these conditions if φ does not satisfy (1.8). Now we state a sufficient and necessary condition which is intermediate to (4.2) and (4.3).

THEOREM 3. *Let $\varphi \in \Phi$ be convex. Set*

$$\varphi_0(t) = \inf\left\{\varphi(t-m) + \frac{t}{l}\varphi(m+l):\ 0 \leqq m, l \leqq t\right\}, \quad t > 0, \tag{4.4}$$

and $\varphi_0(t) = \varphi_0(-t)$, $t < 0$ ($\varphi_0(0) = 0$). Then for $\psi \in \Phi$ the relation (4.1) *holds if and only if*

$$\psi(t) = O(\varphi_0(t)), \quad t \to \infty. \tag{4.5}$$

PROOF. First it should be mentioned that

$$\varphi(t/2) \leqq \varphi_0(t) \leqq 2\varphi(t), \quad t \in R \tag{4.6}$$

(for estimating from above take $m=0$, $l=t$, for the lower bound consider the two cases $m \geqq t/2$ and $m<t/2$).

The proof of the necessity of (4.5) remains valid for arbitrary continuous and increasing $\varphi \in \Phi$ satisfying (1.4). Let

$$m_{t,l} = \inf\left\{m\colon \varphi(t-m) \leqq \frac{t}{l}\varphi(l+m),\ 0 \leqq m \leqq t\right\}, \tag{4.7}$$

where $0<l \leqq t$ (if $l=0$ we put $m_{t,0}=0$), and

$$m_t = \sup\{m_{t,l}\colon 0 \leqq l \leqq t\}, \quad 0 < t < \infty. \tag{4.8}$$

By the assumed properties of φ it is obvious that the function $m_{t,l}$ is continuous in t and in l, separately. Furthermore, if $m_{t,l}>0$ then $\varphi(t-m_{t,l})=\left(\frac{t}{l}\right)\cdot$ $\cdot\varphi(l+m_{t,l})$ and

$$\varphi(t-m_{t,l}) \leqq \varphi(t, l) \leqq 2\varphi(t-m_{t,l})$$

where

$$\varphi(t, l) = \inf\left\{\varphi(t-m)+\frac{t}{l}\varphi(l+m)\colon 0 \leqq m \leqq t\right\}. \tag{4.9}$$

Thus, in the case $m_t>0$ there exists some $l^* \in (0, t)$ with the properties $m_t=m_{t,l^*}$ and

$$\varphi_0(t) = \inf\{\varphi(t, l)\colon 0 \leqq l \leqq t\} \asymp \inf\{\varphi(t-m_{t,l})\colon 0 \leqq l \leqq t\} = \varphi(t-m_t). \tag{4.10}$$

Now, assume that (4.5) is not satisfied. Then there exists a sequence $t_1<$ $<t_2<\ldots\to\infty$ for which we have

$$\lim_{k\to\infty} \psi(t_k)/\varphi_0(t_k) = \infty. \tag{4.11}$$

First we consider the case of $m_{t_k}=0$, $k \geqq 1$. According to (4.7), (4.8) this gives

$$\varphi(t_k) \leqq (t_k/l)\varphi(l), \quad 0 < l \leqq t_k,$$

in particular, $\varphi(t_k)=O(t_k)$, $k\to\infty$. But from (1.4) and Lemma 1 we obtain $t_k=$ $=O(\varphi_0(t_k))$, $k\to\infty$. Therefore, (4.6) and (4.11) yield the relation $\psi(t)\neq O(\varphi(t))$, $t\to\infty$, and by Theorem 2, c) it follows that $\varphi(L) \not\subset H(\psi)$.

In the case $m_{t_k} \not\equiv 0$ we can suppose that $m_{t_k}>0$, $k \geqq 1$. If the m_{t_k} are bounded from above (i.e. $m_{t_k}<m_0<\infty$) then from (4.10), (4.11) we have

$$\varphi_0(t_k) \geqq \varphi(t_k-m_{t_k}) \geqq \varphi(t_k-m_0),$$

and for $\alpha>1$

$$\overline{\lim_{k\to\infty}} \frac{\psi(\alpha(t_k-m_0))}{\varphi(t_k-m_0)} \geqq \overline{\lim_{k\to\infty}} \frac{\psi(t_k)}{\varphi_0(t_k)} = \infty.$$

By Theorem 2, c) we again obtain $\varphi(L) \not\subset H(\psi)$.

Thus, it remains to consider the case $\lim_{k\to\infty} m_{t_k}=\infty$ $(m_{t_k}>0)$. We put (cf. (4.10))

$$f_k(x)=\begin{cases} t_k-m_{t_k,l_k^*}, & x\in[0,\lambda_k] \\ -(m_{t_k,l_k^*}+l_k^*), & x\in(\lambda_k,1], \end{cases} \qquad \lambda_k=\frac{l_k^*}{l_k^*+t_k},\ k\geqq 1.$$

It is easy to check that

$$\|f_k\|_\varphi=\lambda_k\varphi(t_k-m_{t_k,l_k^*})+(1-\lambda_k)\varphi(m_{t_k,l_k^*}+l_k^*)\asymp\lambda_k\varphi_0(t_k),$$

and

$$\|f_k-m(f_k)\|_\psi\geqq\lambda_k\psi(t_k-m_{t_k,l_k^*}-m(f_k))=\lambda_k\psi(t_k),$$

$k\geqq 1$. These two relations together with (4.11) show an analogy to the assertion (2.10) of Lemma 6 (the remaining property $\|f_k\|_\varphi\to\infty$, $k\to\infty$, follows from

$$\|f_k\|_\varphi\geqq C\varphi_0(t_k)\geqq C\varphi(t_k/2),$$

if $\lambda_k\geqq 1/2$, and $\|f_k\|_\varphi\geqq 1/2\varphi(m_{t_k})$, if $\lambda_k<1/2$). The further construction (via Lemma 7) is obvious, we get $\varphi(L)\not\subset H(\psi)$, and the proof of the necessity of (4.5) is complete.

The basic inequality needed in the proof of sufficiency reads as follows: Let $\varphi\in\Phi$ be convex. Then

$$\|f-m(f)\|_{\varphi_0}^{\natural}\leqq 12\|f\|_\varphi \tag{4.13}$$

holds for arbitrary $f\in\varphi(L)$.

If $m(f)=0$ then (4.13) is an immediate consequence of (4.6). Hence, without any loss of generality, let $m(f)<0$. Furthermore, it remains to consider an arbitrary non-increasing $f\in\varphi(L)$.

Let $f(x)\geqq 0$ for $x\in[0,\alpha)$, and $f(x)<0$ for $x\in(\alpha,1]$. Clearly, we have $\alpha<1$. In the case $\alpha=0$ form (4.6) and Lemma 4 it follows that

$$\|f-m(f)\|_{\varphi_0}\leqq 2\|f-m(f)\|_\varphi\leqq 2\|\max\{|f(x)|,|m(f)|\}\|_\varphi\leqq$$

$$\leqq 2\{\|f\|_\varphi+\|m(f)\|_\varphi\}\leqq 4\|f\|_\varphi.$$

Thus, suppose $0<\alpha<1$. Again according to (4.6) and Lemma 4 we obtain

$$\int_\alpha^1\varphi_0(f(x)-m(f))\,dx\leqq 2\Big\{\int_\alpha^1\varphi(f(x))\,dx+(1-\alpha)\|m(f)\|_\varphi\Big\}\leqq 4\|f\|_\varphi. \tag{4.14}$$

Now we turn to the main part of the proof of (4.13). We denote $m_0=-m(f)>0$ and

$$E_1=\{x\in[0,\alpha):\ f(x)\in U_1=\{t\geqq 0:\ \varphi_0(t+m_0)\leqq 4\varphi(t)\}\},$$

$$E_2=\{x\in[0,\alpha):\ f(x)\in U_2=\{t\geqq 0:\ \varphi_0(t+m_0)>4\varphi(t)\}\}.$$

Obviously,

$$\int_{E_1}\varphi_0(f(x)-m(f))\,dx\leqq 4\int_{E_1}\varphi(f(x))\,dx\leqq 4\|f\|_\varphi. \tag{4.15}$$

For $t\in U_2$ the definition (4.4) of $\varphi_0(t)$ yields

$$\varphi_0(t+m_0)\leqq\varphi(t)+\inf\left\{\frac{(t+m_0)}{l}\cdot\varphi(m_0+1):\ 0\leqq l\leqq t+m_0\right\},$$

i.e.

$$3\varphi(t)\leqq\inf\left\{\frac{(t+m_0)}{l}\varphi(m_0+l):\ 0\leqq l\leqq t+m_0\right\}=\varphi_1(t,m_0). \tag{4.16}$$

Let $U_3=\{t\geqq 0:\ 3\varphi(t)\leqq\varphi_1(t,m_0)\}$ and

$$t_0=\max\{t:\ t\in U_3\} \tag{4.17}$$

(the existence of the maximum follows from the continuity of φ and the inequality

$$3\varphi(t)\leqq\varphi_1(t,m_0)\leqq\frac{t+m_0}{t+m_0}\varphi(t),\ \text{i.e. } t\leqq 2m_0,\ \text{for } t\in U_3).$$

In particular,

$$t_0\leqq 2m_0. \tag{4.18}$$

Again using the continuity of φ we obtain $t_0\in U_1$ and

$$\int_{E_1}\varphi_0(f(x)-m(f))\,dx\leqq|E_2|\sup\{\varphi_0(t+m_0):\ t\in U_2(\subset U_3)\}\leqq$$

$$\leqq|E_2|\varphi_0(t_0+m_0)\leqq 4\alpha\varphi(t_0)\leqq\frac{4\alpha}{3}\inf\left\{\frac{t_0+m_0}{l}\varphi(m_0+l):\ 0\leqq l\leqq t_0+m_0\right\}.$$

Now, denote

$$m_1=-\frac{1}{1-\alpha}\int_\alpha^1 f(x)\,dx>m_0>0,$$

and put

$$l=m_1-m_0=\int_0^1 f(x)\,dx-\frac{1}{1-\alpha}\int_\alpha^1 f(x)\,dx\geqq\frac{-\alpha}{1-\alpha}\int_\alpha^1 f(x)\,dx\geqq\frac{\alpha m_0}{1-\alpha}$$

in the last relation. According to (4.18) and Lemma 4 this yields

$$\int_{E_2}\varphi_0(f(x)-m(f))\,dx\leqq\frac{4}{3}\frac{t_0+m_0}{m_0}\varphi(m_1)(1-\alpha)\leqq \tag{4.19}$$

$$\leqq 4(1-\alpha)\varphi(m_1)\leqq 4\int_\alpha^1\varphi(f(x))\,dx\leqq 4\|f\|_\varphi.$$

Thus, taking into account (4.14), (4.15), and (4.19), we finally obtain (4.13).

Now we are able to prove $\varphi(L)\subset H(\varphi_0)$ what is equivalent to the sufficiency in Theorem 3. Let $f\in\varphi(L)$. For fixed $n\geqq 1$ we denote by Δ_i, $i=1,\dots,n$, the corresponding dyadic intervals where the partial sum S_nf of the Haar—Fourier series takes constant values, i.e.

$$S_nf(x)=\frac{1}{|\Delta_i|}\int_{\Delta_i}f(t)\,dt=m_i,\quad x\in\Delta_i,\ i=1,\dots,n.$$

For any given $m_0>0$ we define $t_0>0$ with respect to (4.16), (4.17). If

$$E = \{x\in[0, 1]: |f(x)-S_nf(x)| \geqq t_0+m_0\}$$

then (cf. Remark 1 in Section 2)

$$\int_{[0,1]\setminus E} \varphi_0(f(x)-S_nf(x))\,dx \leqq \tag{4.20}$$

$$\leqq C(m_0, \varphi) \int_{[0,1]\setminus E} |f(x)-S_nf(x)|\,dx \leqq C(m_0, \varphi)\|f-S_nf\|_{L^1}.$$

The remaining integral will be estimated as follows:

$$\int_E \varphi_n(f(x)-S_nf(x))\,dx \leqq$$

$$\leqq \sum_{i:\,|m_i|<m_0} \int_{E\cap\Delta_i} \varphi_0(|f(x)|+m_0)\,dx + \sum_{i:\,|m_i|\geqq m_0} \int_{\Delta_i} \varphi_0(f(x)-m_i)\,dx \equiv I_1+I_2.$$

For the terms in the first sum I_1 from (4.16), (4.17) it follows as in the proof of (4.13) that

$$\int_{E\cap\Delta_i} \varphi_0(|f(x)|+m_0)\,dx \leqq 4 \int_{E\cap\Delta_i} \varphi(f(x))\,dx$$

(cf. (4.15), by $|f(x)-m_i|\geqq t_0+m_0$, $|m_i|<m_0$ we have $|f(x)|>t_0$, i.e. $|f(x)|\in U_1$). Therefore,

$$I_1 \leqq 4 \int_E \varphi(|f(x)|)\,dx. \tag{4.21}$$

On the other hand, the inequality (4.13) can be applied to each of the terms of I_2. This yields

$$I_2 \leqq 12 \sum_{i:\,|m_i|>m_0} \int_{\Delta_i} \varphi(f(x))\,dx = 12 \int_{E'} \varphi(f(x))\,dx, \tag{4.22}$$

where $E' = \bigcup_{i:\,|m_i|>m_0} \Delta_i$ has the property (cf. Lemma 4)

$$|E'| \leqq \varphi(m_0)^{-1} \sum_{i:\,|m_i|>m_0} \int_{\Delta_i} \varphi(m_i)\,dx \leqq \varphi(m_0)^{-1}\|f\|_\varphi. \tag{4.23}$$

Hence, the relations (4.20)—(4.22) give the inequality

$$\|f-S_nf\|_{\varphi_0} \leqq 12 \int_{E'} \varphi(f(x))\,dx + 4 \int_E \varphi(f(x))\,dx + C(m_0, \varphi)\|f-S_nf\|_{L^1}.$$

Here, for $f\in\varphi(L)$ the first two integrals will be small if $|E|$, $|E'|$ are so. But according to (4.23) $|E'|$ will be small, independent of $n\geqq 1$, if m_0 is taken sufficiently large. Now, if n tends to infinity, for this fixed m_0 we obtain $\lim_{n\to\infty} |E| = 0$ (convergence in measure of the Haar—Fourier series) and $\|f-S_nf\|_{L_1}\to 0$. Therefore, $S_nf \xrightarrow{\varphi_0} f$ for all $f\in\varphi(L)$. This finally proves Theorem 3.

COROLLARY 1. *If* $\varphi \in \Phi$, $\varphi(L) \subset L^1$, *and*

$$\psi(t) = O\big((\varphi_c)_0(t)\big), \quad t \to \infty, \tag{4.24}$$

then $\varphi(L) \subset H(\psi)$.

COROLLARY 2. *Let* $0 < \alpha < \infty$ *and*

$$e^{L^\alpha} = \Big\{ f \in L^1 : \int_0^1 e^{|f(x)|^\alpha}\, dx < \infty \Big\}.$$

Then for $\psi \in \Phi$ *the relation* $e^{L^\alpha} \subset H(\psi)$ *holds if and only if*

$$\psi(t) = O(t^{\alpha/2} e^{t/2}), \quad t \to \infty. \tag{4.25}$$

COROLLARY 3. *The class* $e^{(e^L)} = \Big\{ f \in L : \int_0^1 e^{(e^{|f(x)|})}\, dx < \infty \Big\}$ *belongs to* $H(\psi)$, $\psi \in \Phi$, *if and only if*

$$\psi(t) = O(\sqrt{t}\, e^{t/4} e^{e^t/2}), \quad t \to \infty. \tag{4.26}$$

Corollary 1 is an immediate consequence of Theorem 3 and Lemma 1. Concerning (4.25), (4.26) we only have to check $\varphi_0(t)$ with respect to (4.4) in the corresponding special cases. Since this is a technical but tedious verification the details will be omitted. We only consider the simplest case $\alpha = 1$ in (4.25). Let $\varphi(t) = e^{|t|} - 1$. Obviously,

$$\inf \{(t/l) e^{m+l} : 0 \leqq l \leqq t\} = t e^{m+1} \quad (l = 1).$$

Furthermore, define m_0 by the condition $e^{t-m_0} = t e^{m_0+1}$. As in the proof of the necessity of Theorem 3 (cf. (4.7)—(4.10)) we have

$$\varphi_0(t) \asymp e^{t-m_0} = t e^{m_0+1} = (e^{t-m_0} t e^{m_0+1})^{1/2} \asymp \sqrt{t}\, e^{t/2}$$

for $t \to \infty$.

In this connection it would be of some interest to obtain a more explicit formula describing $\varphi_0(t)$.

References

[1] В. И. Иванов, Представление измеримых функций кратными тригонометрическими рядами, *ДАН СССР*, **259** (1981), 279—282.

[2] M. A. Krasnosel'skii, Ya. B. Rutickii, *Convex functions and Orlicz spaces*, Noordhoff (Groningen, 1961).

[3] С. Г. Крейн, Ю. И. Петунин, Е. М. Семёнов, *Теория интерполяции операторов*, Наука (Москва, 1978).

[4] S. Mazur, W. Orlicz, On some classes of linear spaces, *Studia Math.*, **17** (1958), 97—119.

[5] A. M. Olevskii, *Fourier series with respect to general orthogonal systems*, Springer (Berlin—Heidelberg—New York, 1975).

[6] W. Orlicz, Über eine gewisse Klasse von Räumen vom Typus B, *Bull. Acad. Pol. Ser. A Cracovie*, **8** (1932), 207—220.

[7] P. Oswald, Über die Konvergenz von Haar-Reihen in $\varphi(L)$, *Preprint TU Dresden*, 07-35-78, 1978.

[8] P. Oswald, Über die Konvergenz von Orthogonalreihen in $\varphi(L)$, *Wiss. Zeitschr. TU Dresden*, **30** (1981), 117—119.

[9] П. Освальд, Ряды Фурье и сопряженная функция в $\varphi(L)$, *Analysis Math.*, **8** (1982), 287—303.

[10] A. A. Talaljan, The representation of functions in the classes L_p, $0<p<1$, by orthogonal series, *Acta Math. Acad. Sci. Hungar.*, **21** (1970), 1—9.
[11] А. А. Талалян, Вопросы представления и единственности в теории ортогональных рядов, *Итоги наук, Мат. анализ* 1970, ВИНИТИ (Москва, 1971), 5—64.
[12] Г. Е. Ткебучава, О рядах по системе Хаара, *Сообщ. АН Груз. ССР*, **69** (1973), 277—280.
[13] G. E. Tkebucava, Properties of some function spaces and Ciesielski systems, *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astr. Phys.*, **25** (1977), 457—460.
[14] Г. Е. Ткебучава, О базисах в пространствах Орлича, *Analysis Math.*, **7** (1981), 69—80.
[15] П. Л. Ульянов, Представление функций из класса $\varphi(L)$ рядами, *Труды МИАН СССР*, **112** (1971), 386—399.
[16] П. Л. Улянов, Представление функций рядами и классы $\varphi(L)$, *Успехи мат. наук*, **27** (1972), 3—52.
[17] П. Л. Улянов, О различных видах сходимости в классах $\varphi(L)$, *Труды МИАН СССР*, **134** (1975), 327—352.
[18] П. Л. Улянов, О счётной базе и сопряженном классе одного топологического пространства, *Мат. сборн.*, **100** (1976), 14—36.
[19] П. Л. Улянов, Замечание о сходимости в среднем, *Мат. заметки*, **21** (1977), 807—816.
[20] О. Д. Церетели, Об интегрируемости сопряженных функций, *Труды Тбил. мат. инст.*, **43** (1973), 149—168.
[21] О. Д. Церетели, Метрические свойства сопряженных функций. В: *Итоги науки и техники*, Совр. пробл. мат. 7, ВИНИТИ (Москва, 1975), 18—57.
[22] A. Zygmund, *Trigonometric series*, vol. 1—2, Cambr. Univ. Press (Cambridge, 1959).

(Received January 28, 1982)

SEKTION MATHEMATIK
TECHNISCHE UNIVERSITÄT DRESDEN
8027 DRESDEN
MOMMSENSTR. 13
GDR

Acta Math. Hung.
42 (3—4) (1983), 295—308.

MULTIPLICATIVE FUNCTIONS WITH REGULARITY PROPERTIES. I

I. KÁTAI (Budapest), corresponding member of the Academy

1.

A. Sárközy [1] has considered the class of complex-valued multiplicative functions $f(n)$ for which a linear recursion

$$\sum_{i=1}^{k} a_i f(n+i) = 0 \quad (n = 1, 2, \ldots) \tag{1.1}$$

is satisfied, and determined all $f(n)$ that is a solution of a suitable nontrivial ($a_0 \neq 0$, $a_k \neq 0$) recursion.

Now we are interested in the solution of

$$\sum_{i=0}^{k} a_i f(n+i) = o(1) \quad (n \to \infty). \tag{1.2}$$

Let us consider the simplest case

$$f(n+1)-f(n) = o(1) \quad (n \to \infty) \tag{1.3}$$

and assume that $f(n)$ is completely multiplicative. It is obvious that $f(n)$ is a solution if $f(n) \to 0$ is satisfied, and the latter is equivalent with $f(p) \to 0$, where p runs over the primes. Furthermore from (1.3) we get that $f(n)/n \to 0$. The most interesting case is $|f(n)|=1$ $(n=1, 2, \ldots)$.

J. L. Mauclaire and Leo Murata [2] proved that a multiplicative function $f(n)$ with the properties

(1) $$|f(n)| = 1 \quad (n = 1, 2, \ldots)$$

(2) $$\frac{1}{x} \sum_{n \leqq x} |f(n+1)-f(n)| \to 0 \quad (x \to \infty)$$

has to be completely multiplicative.

In [3] I proved that the conditions

(3) $$|f(n)| = 1 \quad (n = 1, 2, \ldots)$$

(4) $$|f(n+1)-f(n)|n^{\gamma} \to 0 \quad (\gamma > 0 \text{ constant})$$

involve that $f(n)=n^{i\tau}$, τ being a suitable constant. I hope that (3) and $f(n+1)-f(n) \to 0$ guarantee that $f(n)=n^{i\tau}$ but I am unable to prove it presently.

The operators E, Δ, Δ_B, I are defined as follows: $Ex_n = x_{n+1}$, $Ix_n = x_n$, $\Delta = E-I$, $\Delta_B = E^B - I$. Let $\Delta^k = (E-I)^k$, $\Delta_B^k = (E^B-I)^k$.

Let $f(n)$ be a completely multiplicative function, $P(z)=a_0+a_1z+\ldots+a_kz^k$ ($a_k=1$, $a_0\neq 0$) be a polynomial over the field of complex numbers for which

$$P(E)f(n)\to 0 \tag{1.4}$$

is satisfied.

Let A be the set of all polynomials $Q(z)$ for which

$$Q(E)f(n)\to 0 \tag{1.5}$$

holds.

It is obvious that A is an ideal. The constant polynomials belong to A if and only if $f(n)\to 0$.

Let us assume that $f(n)\not\to 0$. We shall see that there exist at most $k-1$ primes p_i for which $f(p_i)=0$. Assume in the contrary that $p_1, \ldots, p_k$ are distinct primes such that $f(p_i)=0$. Let N_0 be such a positive integer for which $N_0+i-1\equiv \equiv 0 \pmod{p_i}$ $(i=1, \ldots, k)$.

Then $f(n+j)=0$ for $j=0, \ldots, k-1$, $n\equiv N_0 \pmod B$, $B=p_1, \ldots, p_k$, and so $f(n+k)\to 0$. From (1.4) it follows immediately the following assertion: if there exist k residue classes mod B with consecutive residues $l=r, r+1, \ldots, r+k-1$, such that $f(n)\to 0$ for $n\equiv l \pmod B$, then $f(n)\to 0$ on the whole sequence of natural numbers.

Let now $m>1$ be an arbitrary integer. Let $P(z)=\prod_{i=1}^{k}(z-\theta_i)$ and $Q_m(z)$ be defined by

$$Q_m(z)=\prod_{i=1}^{k}(z-\theta_i^m). \tag{1.6}$$

Since $P(z)$ is a divisor of $Q_m(z^m)$, therefore $Q_m(z^m)$ belongs to A, and so $Q_m(E^m)f(n)\to 0$. Putting nm into n and by using the multiplicativity,

$$Q_m(E^m)f(nm)=f(m)Q_m(E)f(n). \tag{1.7}$$

Let us assume that P is a minimal degree polynomial in A. Assume furthermore that $f(m)\neq 0$. Then $Q_m(E)f(n)\to 0$, $Q_m(z)\in A$.

Since $\delta(z)=$ greatest common divisor of $(P(z), Q_m(z))\in A$, therefore $\deg\delta(z)\geqq k$ and so $P(z)=Q_m(z)$, consequently

$$\{\theta_1, \ldots, \theta_k\}=\{\theta_1^m, \ldots, \theta_k^m\}. \tag{1.8}$$

(1.8) holds for every m that does not contain a prime p with $f|p|=0$. Since the set of such primes is finite and $a_0=\theta_1 \ldots \theta_k(-1)^k\neq 0$, we get that $|\theta_j|=1$, and $\varphi_j=\dfrac{\arg\theta_j}{2\pi}$ are rational numbers. Let $\varphi_j=\dfrac{a_j}{B}$ with $(a_1, \ldots, a_k)=1$. B contains only such prime factors p for which $f(p)=0$.

So

$$P(z)=\prod_{j=1}^{k}\left(z-\exp\left(\frac{2\pi i a_j}{B}\right)\right)$$

and $Q_B(z^B)=(z^B-1)^k\in A$. We have proved the following

THEOREM 1.1. *Let $f(n)$ be a completely multiplicative function for which $f(n)\not\to 0$. Let P be a minimal degree polynomial for which* (1.4) *holds. Then*

$$(E^B-I)^k f(n)\to 0 \tag{1.9}$$

where $B=1$, or $B=p_1^{\alpha_1}\dots p_j^{\alpha_j}$ and $f(p_1)=\dots=f(p_j)=0$. There exist at most $k-1$ primes p for which $f(p)=0$.

Let us assume now that $f(n)$ is a multiplicative function for which (1.4) holds. The condition $f(n)\to 0$ is equivalent with $f(p^\alpha)\to 0$ when p^α runs over the prime powers. Assume that $f(n)\not\to 0$. We shall see that there exist at most $k-1$ distinct primes p_j for which $f(p_j^{\alpha_j})=0$ with suitable exponents α_j. Let N_0 be a suitable integer such that $p_j^{\alpha_j}\| N_0+(j-1)$ $(j=1,\dots,k)$.

If $n\equiv N_0 \pmod{B_1}$, $B_1=p_1^{\alpha_1+1}\dots p_k^{\alpha_k+1}$, then $f(n+j)=0$ $(j=0,\dots,k-1)$ and so $f(n+k)\to 0$ if $n\equiv N_0 \pmod{B_1}$. So there exist k consecutive residue classes mod B_1 on which $f(n)\to 0$, and so arguing as before, $f(n)\to 0$ if $n\to\infty$.

Let now $m=P$ be a prime, $P>2k+2$. Defining the ideal A as earlier we get that $Q_m(z^m)\in A$. Let $Q_m(z)=b_0+b_1z+\dots+b_kz^k$ $(b_k=1)$. Consequently $Q_m(E^m)f(n)\to 0$. We have

$$(Y_n=)Q_m(E^m)f(mn)=b_0f(mn)+b_1f\big(m(n+1)\big)+\dots+b_kf\big((m(n+k)\big).$$

By taking

$$\Delta(n;\,m)=\sum_{j=0}^{k}b_j\big\{f(m(n+j))-f(m)f(n+j)\big\},$$

we get

$$Y_n=\Delta(n;\,m)+f(m)Q_m(E)f(n).$$

Since $Y_n\to 0$ and $P(z)Q_m(z)\in A$, we have

$$P(E)\Delta(n;\,m)\to 0 \quad (n\to\infty). \tag{1.10}$$

Let now $n=P^\nu u$, $(m=P)$, $(u,\,P)=1$, $P>2k+2$. Then $(n+j+h,\,m)=1$ if $1\leqq j+h\leqq 2h$, consequently

$$P(E)\Delta(n;\,P)=a_0b_0\{f(P^{\nu+1})-f(P)f(P^\nu)\}f(u).$$

Since $P(E)\Delta(P^\nu u;\,P)\to 0$ as $u\to\infty$, $(u,\,P)=1$ and $f(u)\not\to 0$ as $(u,\,P)=1$, $u\to\infty$, we get that

$$f(P^{\nu+1})=f(P)f(P^\nu). \tag{1.11}$$

So we have proved that $f(P^\nu)=f^\nu(P)$ for $P>2k+2$.

Consequently (1.7) is valid if each prime factor of m is greater than $2k+2$, and so for $f(m)\neq 0$ we get (1.8).

We have proved the following

THEOREM 1.2. *Assume that $f(n)$ is a complex-valued multiplicative function for which $f(n)\not\to 0$. Let P be a minimal degree polynomial for which* (1.4) *holds. Then $f(P^\nu)=f(P)^\nu$ for every prime $P>2k+2$, and the relation*

$$(E^B-I)^k f(n)\to 0 \tag{1.12}$$

holds with a suitable integer B.

REMARK. If B is a suitable integer satisfying (1.12) and $B|C$, then $(E^C-I)^k f(n)\to 0$. Consequently we may assume that $f(P^\nu)=f(P)^\nu$, and $f(P)\neq 0$ if $(P, B)=1$ in Theorem 1.2.

2.

THEOREM 2.1. *Let $B\geqq 1$ be a fixed integer and let $f(n)$ satisfy the following conditions on the set $(n, B)=1$:*

$$f(n) \text{ is completely multiplicative,} \tag{2.1}$$

$$|f(n)| = 1, \tag{2.2}$$

$$f(n+B)-f(n) = O(n^{-\gamma}) \quad (n\to\infty), \tag{2.3}$$

with a positive constant γ.

Then $f(n)=\chi_B(n)n^{it}$ for $(n, B)=1$, where χ_B is a character mod B.

REMARK. This theorem for $B=1$ has been proved in [3]. I hope that the same assertion holds if we put $f(n+B)-f(n)\to 0$ instead of (2.3).

PROOF. For a real number z let $\|z\|$ denote its distance from the nearest integer. Let $f(n)=e^{2\pi i g(n)}$. Then $g(n)$ is determined mod 1 and we may assume that $g(n)$ is a completely additive function. The condition (2.3) is equivalent with

$$\|g(n+B)-g(n)\| = O(n^{-\gamma}), \quad (n, B) = 1. \tag{2.4}$$

We shall deduce from (2.4) that

$$g(n)-\tau\log n \equiv s(n) \pmod 1, \quad (n, B) = 1 \tag{2.5}$$

with a suitable τ, where

$$\begin{cases} s(m_1)\equiv s(m_2) \pmod 1 & \text{if} \quad m_1\equiv m_2 \pmod B, \\ s(m_1m_2)\equiv s(m_1)+s(m_2) \pmod 1, \\ s(1)\equiv 0 \pmod 1. \end{cases} \tag{2.6}$$

This involves Theorem 2.1, since from (2.6) it follows that $e^{2\pi i s(n)}$ is a character mod B.

Let I_n be the nearest integer to $g(n+B)-g(n)$,

$$\sigma(n) = g(n+B)-g(n)-I_n.$$

So we have $\sigma(n)\in[-1/2, 1/2]$, and from (2.4) that

$$n|\sigma(n)| \ll n^{1-\gamma}, \quad (n, B) = 1. \tag{2.7}$$

Let

$$T(x) = \max_{\substack{m\leqq x \\ (m, B)=1}} m|\sigma(m)|. \tag{2.8}$$

Since

$$\Delta_B g(n)-\Delta_B g(n+B)=g\big((n+B)^2\big)-g\big((n+B)^2-B^2\big)=$$

$$=\sum_{j=0}^{B-1}\Delta_B g\big((n+B)^2-B^2+jB\big),$$

therefore

$$\sigma(n+B)-\sigma(n)=-\sum_{j=0}^{B-1}\sigma\big((n+B)^2-B^2+jB\big)$$

if $n>n_0$. Applying this identity for $n+lB$ $(l=1, \ldots, H-1)$ instead of n, we have

$$\sigma(n+HB)-\sigma(n)=\sum_{l=1}^{H}\big(\sigma(n+lB)-\sigma\big(n+(l-1)B\big)\big)= \tag{2.8}'$$

$$-\sum_{l=1}^{H}\sum_{j=0}^{B-1}\sigma\big((n+lB)^2-B^2+jB\big)$$

and after summing for $H=0, 1, \ldots, R-1$,

$$\sum_{H=0}^{R-1}\sigma(n+HB)-R\sigma(n)=-\sum_{l=1}^{R}\sum_{j=0}^{B-1}\sigma\big((n+lB)^2-B^2+jB\big)(R-l). \tag{2.9}$$

Let $n=mR$ and observe that

$$\sum_{H=0}^{R-1}\sigma(mR+HB)=g(mR+Rb)-g(mR)-\sum_{H=0}^{R-1}I_{mR+HB}= \tag{2.10}$$

$$=\sigma(m)+\Big(I_m-\sum_{H=0}^{R-1}I_{mR+HB}\Big).$$

The left hand side of (2.10) is not greater than

$$\frac{RT\big((m+B)R\big)}{mR}=\frac{T\big((m+B)R\big)}{m} \tag{2.11}$$

that is less than $1/2$ if $m>m_0$ and R is not too large, exactly if

$$T\big((m+B)R\big)<\frac{m}{2}. \tag{2.12}$$

Consequently the left hand side of (2.10) is $\sigma(m)$, and so

$$\sigma(m)-R\sigma(mR)=-\sum_{l=1}^{R}\sum_{j=0}^{B-1}\sigma\big((mR+lB)^2-B^2+jB\big)(R-l). \tag{2.13}$$

The right hand side of (2.13) is majorized by

$$\frac{BR^2T\big((m+B)^2R^2\big)}{m^2R^2}. \tag{2.14}$$

Consequently

$$|m\sigma(m)-Rm\sigma(mR)| \leqq \frac{BT\left((m+B)^2R^2\right)}{m}. \tag{2.15}$$

Let us assume that $T(x)$ is bounded, $T(x)\leqq K$. Putting $m=N_1$, $R=N_2$ and $m=N_2$, $R=N_1$ into (2.15), and by using the triangle-inequality we get

$$|N_1\sigma(N_1)-N_2\sigma(N_2)| \leqq \frac{BK}{N_1}+\frac{BK}{N_2},$$

whenever $(N_1N_2, B)=1$.

This shows that $N\sigma(N)$ is a Cauchy-sequence, $N\sigma(N)\to\tau$, $(N, B)=1$, $N\to\infty$. Let $\sigma(m)=\frac{\tau}{m}+\frac{\varepsilon_m}{m}$, $\varepsilon_m\to 0$. Let furthermore p and q be arbitrary positive integers satisfying the conditions:

$$p\equiv q \pmod B,\ (p, B)=1,\ 1<q/p,\ |\tau|\log q/p<1/2. \tag{2.16}$$

Starting from the relation

$$g(q)-g(p)=\sum_{l=0}^{l_0}\left(g(pU+lB)-g\left(pU+(l-1)B\right)\right), \tag{2.17}$$

$$(U, B)=1, \quad l_0=\frac{(q-p)U}{B},$$

we observe that the right hand side of (2.17) is

$$\sum_{l=1}^{l_0}\sigma\left(pU+(l-1)B\right)+J(U), \tag{2.18}$$

where $J(U)$ is an integer that may depend on U. The sum in (2.18) is

$$\tau\sum_{l=1}^{l_0}\frac{1}{pU+(l-1)B}+\sum_{l=0}^{l_0}\frac{\varepsilon_{pU+(l-1)B}}{pU+(l-1)B}$$

as $U\to\infty$. So

$$g(q)-g(p)-\tau\log\frac{q}{p}=J(U)+O_U(1)$$

that gives that $J(U)$ is constant if $U>U_0(p, q)$ and by putting $U\to\infty$, we get

$$g(q)-g(p)-\tau\log q/p=\text{integer}. \tag{2.19}$$

Let $s(n)=g(n)-\tau\log n$. From (2.19) we get $s(n_1)\equiv s(n_2) \pmod 1$ if $n_1\equiv n_2 \pmod B$, $(n_1, B)=1$. By taking $m\equiv 1 \pmod B$, $q=pm$ into (2.19) we get $s(m)\equiv 0 \pmod 1$. Furthermore s being an additive function involves that $s(m_1m_2)\equiv s(m_1)+s(m_2) \pmod 1$.

The proof has been finished if $T(x)$ is bounded. It has remained to prove only that $T(x)$ is bounded. From (2.7) we know that $T(x)<Kx^{\delta}$ where δ and K are constants, $\delta<1$.

Let $q>1$ be a fixed integer, $(q, B)=1$. Assume that $T(q^{M+1})$ is attained in the interval $[q^M, q^{M+1}]$.

Let $N\in[q^M, q^{M+1}]$, $(N, B)=1$. We define l_1 to be the smallest non-negative integer for which $N-l_1B\equiv 0 \pmod q$ holds. Let N_1 be defined by $N=qN_1+l_1B$ $(0\leqq l_1\leqq q)$.

From (2.8)' we deduce that

$$|\sigma(qN_1+l_1B)-\sigma(qN_1)| \leqq l_1B\frac{T(N^2)}{(qN)^2}$$

and from (2.13) — by using it with $R=q$ — that

$$\left|\sigma(qN_1)-\frac{\sigma(N_1)}{q}\right| \leqq \frac{BT\big((N_1+B)^2q^2\big)}{N_1^2q}.$$

So we have

$$|N\sigma(N)| \leqq \frac{N|\sigma(N_1)|}{q}+\frac{l_1BNT(N^2)}{q^2N^2}+\frac{BNT\big((N_1+B)^2q^2\big)}{N_1^2q}. \tag{2.20}$$

Assume that $T(q^{M+1})$ is attained at N. From (2.20) we get

$$T(q^{M+1}) \leqq T(q^M)+c_1\frac{T(q^{2M+3})}{q^M}+c_2\frac{T(q^M)}{q^M} \tag{2.21}$$

with suitable positive constants c_1, c_2. Assume that $\delta>1/2$. From (2.21) we have

$$T(q^{M+1}) \leqq T(q^M)+c_3(q^M)^{(2\delta-1)}.$$

By using the monotonicity of T, we have $T(x)=O(x^{2\delta-1})$. So we have proved the following assertion. If $1/2<\delta<1$, and $T(x)\ll x^\delta$, then $T(x)\ll x^{2\delta-1}$. Repeating this argument for $\delta=\delta_1$, $2\delta_1-1=\delta_2$, ... after finitely many steps we get an exponent $\delta_1\in(0, 1/2)$, and so $T(x)\ll x^{\delta_1}\ll x^{1/2}$.

Assume now that $\delta=1/2$. Then from (2.21) we get

$$T(q^{M+1}) \leqq T(q^M)+c_4,$$

whence it follows that $T(x)\ll\log x$.

Let now m_1, m_2 be chosen such that

$$T\big((m_1+1)m_2\big)<\frac{1}{2}m_2, \quad T\big((m_2+1)m_1\big)<\frac{1}{2}m_1.$$

This involves (2.13). From (2.15) we deduce that

$$|m_1\sigma(m_1)-m_2\sigma(m_2)| \leqq K(\log m_1m_2)\left(\frac{1}{m_1}+\frac{1}{m_2}\right) \tag{2.22}$$

with a suitable constant K for each pair of m_1, m_2 that satisfy

$$(c_5<)\, m_1<m_2<e^{\sigma m_1} \tag{2.23}$$

with a small positive σ, and a positive constant c_5.

From (2.22) we get

$$\sigma(m_2)-\frac{m_1}{m_2}\sigma(m_1)=C\frac{\log m_1 m_2}{m_1 m_2} \tag{2.24}$$

for m_1, m_2 satisfying (2.23), where C is a bounded variable.

Let $m_1=U$, $(U, B)=1$, and m_2 run over the integers $m_2=pU+lB$ $(l=0,1,\ldots,$ $\ldots, l_0-1)$, $l_0=\frac{q-p}{p}$, where $q\equiv p \pmod B$, $(p, B)=1$, $p<q$. From (2.24) we get

$$f(q)-f(p)=J(U)+\sigma(U)U\sum_{l=1}^{l_0-1}\frac{1}{pU+lB}+O\left(\frac{\log U}{U}\right),$$

$J(U)$ being an integer. Furthermore

$$\sum_{l=0}^{l_0-1}\frac{1}{pU+lB}=\frac{1}{B}\log\frac{q}{p}+O\left(\frac{1}{U}\right).$$

So we have got that

$$f(q)-f(p)\equiv\frac{\sigma(U)U}{B}\log\frac{q}{p}+\varepsilon(U) \pmod 1 \tag{2.25}$$

where $\varepsilon(U)\to 0$ for $U\to\infty$. From (2.25) it follows that

$$U\sigma(U)\equiv\frac{\alpha B}{\log\frac{q}{p}}+\varepsilon'(U)\left(\mathrm{mod}\ \frac{B}{\log\frac{q}{p}}\right) \tag{2.26}$$

with a suitable α and $\varepsilon'(U)\to\infty$. But this is possible only if $U\sigma(U)$ is bounded, taking into account that $U\sigma(U)$ is a slowly varying function (see (2.22)).

By this the proof is complete.

3.

For $k=0, 1, \ldots$ let $\mathscr{E}_B^{(k)}$ denote all solutions of

$$(E^B-I)^k f(n)\to 0, \quad (n, B)=1, \ n\to\infty \tag{3.1}$$

where $f(n)$ is a completely multiplicative function on the set $(n, B)=1$.

It is obvious that $\mathscr{E}_B^{(k)}\subseteq\mathscr{E}_B^{(k+1)}$ $(k=0, 1, 2, \ldots)$. $f\in\mathscr{E}_B^{(0)}$ if and only if $f(n)\to 0$ as $(n, B)=1$, $n\to\infty$, that is equivalent with the condition: $f(p)\to 0$ as p runs over the primes.

Let $\mathscr{B}_B$ be that subset of $\mathscr{E}_B^{(1)}$ the element of which satisfy

$$|f(n)|=1, \quad (n, B)=1. \tag{3.2}$$

THEOREM 3.1. *Let $k\geqq 1$. f belongs to $\mathscr{E}_B^{(k)}$ if*

(a) $f\in\mathscr{E}_B^{(0)}$, *or*

(b) $f\in\mathscr{B}_B$, or

(c) $f(n)=n^{\sigma+i\tau}\chi_B(n)$, $(n, B)=1$, *where* $0<\sigma<k$, χ_B *is a character* mod B.

PROOF. First we assume that $k=1$. From (3.1) we get immediately that $f(n)=$ $=o(n)$, and so $|f(g)|<q$ for every q coprime to B. Let q be fixed, $(q, B)=1$. Let us consider

$$E(x) = \max_{\substack{n\leqq x \\ (n,B)=1}} |f(n)|.$$

For an integer N_0 let a_0 be the least nonnegative integer so that $N_0-a_0B\equiv 0 \pmod q$, and let N_1 be defined by $N_0=qN_1+a_0B$.

If $(N_0, B)=1$, then $(N_1, B)=1$, furthermore

$$f(N_0)=f(q)f(N_1)+o_{N_0}(1), \tag{3.3}$$

and hence we may immediately deduce that

$$E(qx)\leqq |f(q)|E(x)+o_x(1) \quad (x\to\infty). \tag{3.4}$$

Let $|f(q)|=q^\eta$, $\eta>0$. Then (3.4) involves that $E(x)\ll x^{\eta+\varepsilon}$ for every positive constant ε. On the other hand $E(q^N)\geqq |f(q)|^N=q^{N\eta}$, and so we have

$$\frac{\log E(x)}{\log x}\to\eta \ (=\eta_q), \tag{3.5}$$

i.e. that η may not depend on q if $|f(q)|>1$.

If there exists a q with $|f(q)|<1$, then from (3.4) first we deduce that $E(x)$ is bounded. By writing

$$\Delta(y) = \max_{n\geqq y} |f(n)|$$

and taking into account (3.3), we get

$$\Delta(qy+qB)\leqq |f(q)|\Delta(y)+\varepsilon_y \quad (\varepsilon_y\to 0,\ y\to\infty)$$

and so that $\Delta(y)\to 0$, i.e. that $f(n)\to 0$ as $(n, B)=1$. Consequently $f\in\mathscr{E}_B^{(0)}$.

For $|f(q)|=1$, from (3.4) we deduce that $E(x)\ll\log x$. In this case $f(n)\not\to 0$ on the set $(n, B)=1$, and there does not exist q_1 with $|f(q_1)|>1$, since $\lim \dfrac{\log E(x)}{\log x}$ is well defined. So, if $|f(q)|=1$ for at least one q, $(q, B)=1$, then $f\in\mathscr{B}_B$.

In the remaining case $\eta_q=\eta=$constant (>0) for all q, $(q, B)=1$. We rewrite $f(n)$ as $f(n)=n^\eta t(n)$, $|t(n)|=1$. Observing that $(n+B)^\eta=n^\eta+O(n^{\eta-1})$, and

$$\Delta_B f(n) = n^\eta \Delta_B t(n)+O(n^{\eta-1}),$$

we get $\Delta_B t(n)\ll n^{-\eta}$, that is t satisfies the conditions of Theorem 2.1, and so $t(n)=n^{i\tau}\chi_B(n)$. By this our theorem has been proved for $k=1$.

Let us assume that $k\geqq 2$, and that the theorem is true for $k-1$. The condition (3.1) involves that $f(n)=o(n^k)$. We consider

$$H(n) = (E^B-I)^{k-1}f(n). \tag{3.6}$$

It is obvious that

$$H(n+lB)-H(n)\to 0 \tag{3.7}$$

for bounded l, if $(n, B)=1$, $n\to\infty$.

Let

$$(1+z+\ldots+z^{q-1})^{k-1}=\alpha_0+\alpha_1 z+\ldots+\alpha_{(q-1)(k-1)}z^{(q-1)(k-1)}.$$

Then $\sum \alpha_j = q^{k-1}$. Furthermore

$$(E^{Bq}-I)^{k-1}f(qn) = (I+E^B+\ldots+E^{(q-1)})^{k-1}(E^B-I)^{k-1}f(qn) =$$

$$= \sum_{j=0}^{(q-1)(k-1)} \alpha_j H(qn+jB).$$

On the other hand the left hand side is $f(q)H(n)$. So, from (3.7) we get that

$$H(qn+lB)(\Sigma\alpha_j) = f(q)H(n)+\varepsilon_{n,l},$$

i.e. that

$$H(qn+lB) = \frac{f(q)}{q^{k-1}} H(n)+\varepsilon'_{n,l}, \tag{3.8}$$

where $\varepsilon'_{n,l}\to 0$ as $n\to\infty$, l being bounded.

Assume first that there exists a q, $(q, B)=1$, for which $|f(q)|<q^{k-1}$. From (3.8), we can deduce — repeating the argument used above — that $H(n)\to 0$ as $(n, B)=1$, $n\to\infty$. In this case $f\in\mathscr{E}_B^{(k-1)}$ and we are ready.

Let $|f(q)|=q^{k-1+\eta}$, $0\leqq\eta<1$. Then, for the function

$$E_1(x) = \max_{\substack{n\leqq x\\(n,B)=1}} |H(n)|,$$

starting from (3.8) we deduce that

$$\lim \frac{\log E_1(x)}{\log x} = \eta_q = \eta = \text{constant}.$$

Let now consider $f(n)$ in the form

$$f(n) = nv(n), \quad |v(n)| = n^{k-2+\eta}.$$

We observe that

$$\Delta_B^k f(n) = \sum_{l=0}^{k} (-1)^{k-l}\binom{k}{l}(n+lB)v(n+lB) =$$

$$= n\Delta_B^k v(n)+B\sum_{l=1}^{k}(-1)^{k-l}\binom{k}{l} lv(n+lB) = n\Delta_B^k v(n)+Bk\Delta_B^{k-1}v(n+B),$$

whence we have

$$n\Delta_B^k v(n)+Bk\Delta_B^{k-1}v(n+B) \to 0. \tag{3.9}$$

Let $g(n)=\Delta_B^{k-1}v(n)$. From (3.9) we have

$$(n+Bk)g(n+B)-ng(n) = \lambda_n, \quad \lambda_n\to 0. \tag{3.10}$$

Using this in the form

$$|g(n+B)| \leqq \frac{n}{n+Bk}|g(n)|+\frac{|\lambda_n|}{n+Bk},$$

we deduce first easily that $g(n)=O(\log n)$, and putting this into (3.9), we get $\Delta_B^k v(n)\to 0$.

Observing that $g(n)$ plays the same role for $v(n)$ as $H(n)$ does for $v(n)$, we have

$$g(qn+lB)=\frac{v(q)}{q^{k-1}}g(n)+\varepsilon''_{n,l},\quad \varepsilon''_{n,l}\to 0. \tag{3.11}$$

Starting from (3.11) and taking into account that $|v(q)|<q^{k-1}$, we have $g(n)\to 0$. This involves that $v\in\mathscr{E}_B^{(k-1)}$. If $0<\eta<1$, then by using the induction assumption we get $v(n)=\chi_B(n)n^{k-2+\eta+i\tau}$, whence $f(n)=\chi_B(n)n^{k-1+\eta+i\tau}$, that is a solution belonging to (c).

Let now consider the case $n=0$. Since $g(n)\to 0$, therefore (3.10) gives that

$$g(n+lB)-g(n)=\frac{\delta_n}{n}\quad(\delta_n\to 0)$$

if l varies boundedly, and so, instead of (3.12) we have

$$g(qn+lB)=\frac{v(q)}{q^{k-1}}g(n)+\frac{\delta_{n',l}}{n},\ \delta_{n,l}\to 0. \tag{3.12}$$

Since $|v(q)|=q^{k-2}$,

$$\Delta(x)=\max_{\substack{n\leqq x\\(n,B)=1}}|ng(n)|,$$

we deduce from (3.12) that

$$\Delta(qx)\leqq\Delta(x)+o_x(1)\quad(x\to\infty),$$

that involves that $\Delta(x)=o(\log x)$.

So we have

$$g(n)=o\left(\frac{\log n}{n}\right),\quad (n,B)=1,\ n\to\infty. \tag{3.13}$$

This will involve very soon that the only type of solution is $f(n)=\chi_B(\eta)n^{k-1+i\tau}$. To prove this we define

$$f(n)=nv_1(n),\ v_1(n)=nv_2(n),\ \ldots,\ v_{k-2}(n)=nv_{k-1}(n),$$

where $v_1\in\mathscr{E}_B^{(k-1)}$, $v_2\in\mathscr{E}_B^{(k-2)}$, ..., $v_{k-2}\in\mathscr{E}_B^{(2)}$, $v_{k-1}\in\mathscr{E}_B^{(1)}$, $|v_j(n)|=n^{k-1-j}$ $(j=1,\ldots,k-1)$. Applying the above argument for $v_{k-2}(n)=f(n)$, $\Delta v_{k-1}(n)=g(n)$, we deduce that

$$\Delta v_{k-1}(n)=o\left(\frac{\log n}{n}\right),\quad (n,B)=1,\ n\to\infty. \tag{3.14}$$

Since $|v_{k-1}(n)|=1$ for $(n,B)=1$, therefore the conditions of Theorem 2.1 are satisfied, consequently $v_{k-1}(n)=n^{i\tau}\chi_B(n)$, and so $f(n)=n^{k-1+i\tau}\chi_B(n)$ for $(n,B)=1$.

The proof is complete.

4.

We shall refine Theorem 2.1 for the case $B=1$.

THEOREM 4.1. *Let $\psi(n)$ be a positive function tending to 0 monotonically, satisfying*

$$\sum_{j=1}^{\infty} \psi(2^j) < \infty. \tag{4.1}$$

Let $f(n)$ be a multiplicative function such that $|f(n)|=1$ $(n=1, 2, \ldots)$,

$$|\Delta f(n)| \ll \psi(n) \quad (n = 1, 2, \ldots) \tag{4.2}$$

Then $f(n)=n^{i\tau}$.

PROOF. 1. We may assume that $\psi(n)>1/n$. In the opposite case we take $\psi_1(n)=\psi(n)+1/n$, the relation (4.1) holds with ψ_1 instead of ψ, too.

2. We may assume that $f(2)=1$. It is enough to observe only that $|\Delta f(n)\cdot \cdot n^{-i\alpha})| \ll \psi(n)$ for every real α.

We shall use the following

LEMMA. *Let $\beta=\wedge\alpha$, α be an irrational, β be a real number, with the following property: if $m_1<m_2<\ldots$ is an infinite sequence of natural numbers such that $\|m_\nu\alpha\| \to 0$, then $\|m_\nu\beta\| \to 0$.*

Then $\beta=k\alpha$, k being an integer, or β-integer.

If β is a noninteger rational number then the property obviously does not hold. Assume that β is an irrational number.

If β and α are rationally independent then the sequence $(m\alpha, m\beta)$ $(m=1,2,\ldots)$ is uniformly distributed and the property stated in the lemma cannot hold. Consequently $\beta=\dfrac{P}{Q}\alpha$, with suitable integers P, Q. If $Q=1$ or $P=0$, then we are ready. Assume that $(P, Q)=1$, and $Q>1$.

Let us consider the sequence $\dfrac{p_\nu}{q_\nu}$ of partial quotients to α. We have

$$\alpha = \frac{p_\nu}{p_\nu}+\frac{\theta_\nu}{q_\nu^2}, \quad |\theta_\nu| \leqq 1, \qquad \beta = \frac{P}{Q}\alpha = \frac{Pp_\nu}{Qq_\nu}+\frac{P\theta_\nu}{Q}\cdot\frac{1}{q_\nu^2}.$$

By choosing now $m_\nu=q_\nu$, we get

$$q_\nu\beta = \frac{P}{Q}p_\nu + o_\nu(1),$$

and so $\left\|\dfrac{P}{Q}p_\nu\right\| \to 0$. This may hold only if $Q|p_\nu$ for every large ν, that contradicts the wellknown relation $|p_\nu q_{\nu-1}-p_{\nu-1}q_\nu|=1$. The lemma is proved.

Let $N_0>N_1>\ldots>N_\nu$ be a finite sequence of positive integers such that

$$N_0 = qN_1+a_0, \quad N_1 = qN_2+a_1, \ \ldots, \ N_{\nu-1} = qN_\nu+a_{\nu-1},$$

and the digits a_j are bounded: $|a_k| \leqq K$.

By using (4.2) we get immediately that

$$|f(N_i)-f(q)f(N_{i+1})| \ll \psi(N_{i+1}) \quad (i=0, \ldots, \nu-1),$$

i.e.

$$|f(N_0)-f(q)^{\nu} f(N_\nu)| \ll \psi(N_1)+\ldots+\psi(N_\nu). \tag{4.3}$$

Let now $p>2$ be an arbitrary odd integer. We assume that $f(2)=1$. Let r_ν be an arbitrary sequence for which

$$\left\| r_\nu \frac{\log 2}{\log p} \right\| \to 0 \tag{4.4}$$

holds. Let s_ν be the nearest integer to $r_\nu \dfrac{\log 2}{\log p}$,

$$s_\nu = r_\nu \frac{\log 2}{\log p} \pm \left\| r_\nu \frac{\log 2}{\log p} \right\|.$$

So we have

$$|p^{s_\nu}-2^{r_\nu}| = 2^{r_\nu}|e^{s_\nu \log p - r_\nu \log 2}-1| < 2^{r_\nu - l_\nu},$$

where $l_\nu \to \infty$ $(\nu \to \infty)$.

Furthermore

$$p^{s_\nu} = 2^{r_\nu} \pm (b_0+b_1 2+\ldots+b_{r_\nu-l_{\nu-1}} 2^{r_\nu-l_\nu-1})$$

where $b_j=0$ or 1. By using (4.3) we get that

$$|f(p^{s_\nu})-1| \leqq \psi(2^{l_\nu})+\psi(2^{l_\nu+1})+\ldots+\psi(2^{r_\nu})$$

whence by (4.1), $|f(p^{s_\nu})-1| \to 0$.

Putting $f(p)=e^{2\pi i\gamma}$ we get that $\|s_\nu \gamma\| \to 0$ which under the condition (4.4) is equivalent with

$$\left\| r_\nu \gamma \frac{\log 2}{\log p} \right\| \to 0.$$

By using our lemma with $\alpha = \dfrac{\log 2}{\log p}$, $\beta = \gamma \dfrac{\log 2}{\log p}$ we get that γ or β is an integer. If γ=integer, then we may choose $\gamma=0$. If β=integer, then $\gamma = k(p)\dfrac{\log p}{\log 2}$, $k(p)$ is an integer. Observing that $f(pq)=f(p)f(q)$ for $(p, q)=1$, $(pq, 2)=1$ we get

$$k(pq)\frac{\log pq}{\log 2} = k(p)\frac{\log p}{\log 2}+k(q)\frac{\log q}{\log 2}+I_{p,q},$$

$I_{p,q}$ is an integer. This equality involves that

$$(pq)^{k(pq)} = p^{k(p)} q^{k(q)} 2^{I_{p,q}}$$

whence we get that $k(p)$=constant$=K$ for every odd p. We may take $f(2)=e^{2\pi i K}$, i.e.

$$f(n) = e^{2\pi i K \frac{\log n}{\log 2}}$$

for $n>1$.

By this the theorem is proved.

REMARK. Recently I could prove the following assertion. *If* $f(n)$ *is a multiplicative function satisfying* $|f(n)|=1$ $(n=1, 2, \ldots)$ *and*

$$\sum_{n=1}^{\infty} \frac{|\Delta f(n)|}{n} < \infty,$$

then $f(n)=n^{i\tau}$.

References

[1] A. Sárközy, On multiplicative arithmetic functions satisfying a linear recursion, *Studia Sci. Math. Hungar.* (in print).
[2] J. L. Mauclaire and Leo Murata, On the regularity of arithmetic multiplicative functions I, *Proc. of the Japan Academy*, **56** ser. A (1980), 438—440.
[3] I. Kátai, Additive functions with regularity properties, *Acta Sci. Math. Szeged* (in print).

(Received January 28, 1982)

EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE
BUDAPEST, MÚZEUM KRT. 6—8
H—1088

Acta Math. Hung.
42 (3—4) (1983), 309—318.

ON THE UNICITY OF BEST L_1-APPROXIMATION BY POLYNOMIALS OF SEVERAL VARIABLES

A. KROÓ (Budapest)

1. Introduction

In the present paper we shall consider certain problems of best mean approximation for functions of several variables. It is wellknown that there is an essential difficulty in the extension of the theory of best Chebyshev approximation to the multivariate case. The basic problem here is the absence of Haar subspaces of functions of more than one variable. However, this fact seems not to be an insuperable obstacle in developing the theory of best L_1-approximation of functions of several variables. Some first steps in this direction have been made in [1]. In particular, it was shown in [1] that the L_1-approximation of continuous functions of two variables by the tensor product of a two-dimensional Haar subspaces and an arbitrary Haar subspace is unique. This result gives some hope that the general problem of unicity of best L_1-approximation of continuous functions of several variables by tensor products of Haar subspaces can be solved positively. Nevertheless, this problem is still unsolved even in the case of ordinary polynomials. In the present paper we continue the study of uniqueness of multivariate L_1-approximation presenting another approach to this problem. We shall consider approximation by arbitrary multivariate polynomials but restrict ourselves only to approximating rational functions.

The main advantage of this approach is that it includes the study of the unicity of polynomials of several variables with fixed leading coefficients which have minimal L_1-deviation from zero. In case of Chebyshev approximation this problem was considered in [2], [6] and [7]. The results of our paper imply the unicity of the polynomials of least L_1-deviation having fixed leading coefficients. Thus we show that the solution of the Korkin—Zolotarjov type extremal problem is unique in the multivariate case, too. For rectangular regions this unique solution is given by the product of Chebyshev polynomials of second kind.

2. Some notations and the main theorem

Let $\mathbf{R}^n$ denote the n-dimensional real Euclidean space $(n \in \mathbf{N})$; $\mathbf{Z}_+^n = \{(\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n$: α_i is a nonnegative integer, $1 \leqq i \leqq n\}$. Let M be an arbitrary finite subset of $\mathbf{Z}_+^n$. Then by $P^n(M)$ we denote the linear space of polynomials of n variables $(x_1, \ldots, x_n) \in \mathbf{R}^n$ supported by M, i.e.

$$P^n(M) = \Big\{ \sum_{(\alpha_1, \ldots, \alpha_n) \in M} a_{\alpha_1, \ldots, \alpha_n} x_1^{\alpha_1} \ldots x_n^{\alpha_n} \colon a_{\alpha_1, \ldots, \alpha_n} \in \mathbf{R}^1 \Big\}.$$

Furthermore, P^n denotes the space of all polynomials of the variables $(x_1, \ldots, x_n) \in \mathbf{R}^n$. Thus $P^n = \bigcup P^n(M)$, where the union is taken over all finite subsets M in $\mathbf{Z}_+^n$.

The set of constant functions is denoted by P^0. The diameter of the finite subset $M \subset \mathbf{Z}_+^n$ is given by

$$d(M) = \max_{(\alpha_1, \ldots, \alpha_n) \in M} (\alpha_1 + \alpha_2 + \ldots + \alpha_n). \tag{1}$$

By M^* we denote the subset of "largest" points of M, i.e., $M^* = \{(\alpha_1^*, \ldots, \alpha_n^*) \in M : \alpha_1^* + \alpha_2^* + \ldots + \alpha_n^* = d(M)\}$. The following simple property of M will play a central role in our considerations.

DEFINITION. We say that the finite subset $M \subset \mathbf{Z}_+^n$ satisfies the S-property if and only if for any $p, q \in P^n \setminus \{0\}$ the inclusion $pq \in P^n(M)$ implies that both p and q belong to $P^n(M)$.

Remark, that if M satisfies the S-property then M contains the origin, i.e. $P^0 \subset P^n(M)$. In what follows we shall consider only sets M satisfying the S-property.

Let us give two examples.

EXAMPLE 1. Let $M = M(k) = \{(\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_+^n : \sum_{i=1}^{n} \alpha_i \leqq k\}$, where $k \in \mathbf{N}$. Evidently, $M(k)$ satisfies the S-property, $d(M(k)) = k$. We shall use the notation $P^n(M(k)) = P^n(k)$.

EXAMPLE 2. Let $(k_1, \ldots, k_n) \in \mathbf{Z}_+^n$ and set

$$M = M(k_1, \ldots, k_n) = \{(\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_+^n : 0 \leqq \alpha_i \leqq k_i,\ 1 \leqq i \leqq n\},$$

$$P^n(M(k_1, \ldots, k_n)) = P^n(k_1, \ldots, k_n).$$

We can easily show again, that $M(k_1, \ldots, k_n)$ satisfies the S-property. In this case $d(M(k_1, \ldots, k_n)) = \sum_{i=1}^{n} k_i$, $M^*(k_1, \ldots, k_n)$ consists of the single point $(k_1, \ldots, k_n)$.

Furthermore, let K be an open bounded connected subset of $\mathbf{R}^n$. By $L_1(K)$ we denote the space of real Lebesgue integrable functions on K endowed with the L_1-norm $\|f\| = \int_K |f|$. In what follows the L_1-norm will always be meant on K. We shall consider the best approximation in this norm by polynomials of $P^n(M)$. We say that $p^* \in P^n(M)$ is a best approximation of $f \in L_1(K)$ if and only if

$$\|f - p^*\| = \inf \{\|f - p\| : p \in P^n(M)\}.$$

Since $P^n(M)$ is finite-dimensional, the best approximation always exists. The question of unicity of best approximation is much more complicated. By a theorem of Krein (see [3], p. 230) the best approximation can not be unique for each $f \in L_1(K)$. Let us denote by $C_1(K)$ the subspace of continuous functions on K endowed with the L_1-norm. A natural question arises: does any $f \in C_1(K)$ possess a unique best approximation out of $P^n(M)$ if M satisfies the S-property? For functions of one variable this question is answered positively by the famous theorem of Jackson. In [1] we were able to verify this statement for $n=2$ only under further strong restriction on $P^n(M)$. In this paper we prove the above statement for a dense subset of $C_1(K)$ consisting of all rational functions. Set $Q^n(K) = \{p_1/p_2 : p_1, p_2 \in P^n,\ p_2 > 0 \text{ on } K\}$. Evidently, $P^n \subset Q^n(K)$, hence $Q^n(K)$ is dense in $C_1(K)$.

The following theorem is our principal result.

THEOREM 1. *If M satisfies the S-property then each $f\in Q^n(K)$ (in particular, each $f\in P^n$) possesses a unique best approximation out of $P^n(M)$.*

By this theorem we shall easily verify the uniqueness of the solution of a Korkin—Zolotarjov-type extremal problem for polynomials of several variables.

At first we shall investigate the structure of common zeros of relatively prime polynomials of P^n. It turns out that this question plays a crucial part in the study of multivariate L_1-approximation. Then the proof of Theorem 1 will be given. Finally, we shall apply the above unicity result for the study of Korkin—Zolotarjov type extremal problems.

3. On the structure of common zeros of relatively prime polynomials

Two polynomials $p, q\in P^n\setminus\{0\}$ are called *relatively prime* if each of their common polynomial divisors belongs to P^0, i.e. is a constant function. Furthermore, we say that $p\in P^n$ is *prime* if it has no polynomial divisors different from constants and itself. Let us denote by $Z(f_1, \ldots, f_s)$ the set of common zeros of the functions $f_1, f_2, \ldots, f_s$ in $\mathbf{R}^n$ ($s\geqq 1$). In this section we shall verify that if p and q are relatively prime then $K\setminus Z(p, q)$ is connected. First of all we give a characterization of relatively prime polynomials. Let us denote by P_i^{n-1} those polynomials of P^n which depend only on the variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ ($1\leqq i\leqq n$).

PROPOSITION 1. *The polynomials $p, q\in P^n\setminus\{0\}$ are relatively prime if and only if there exist $p_i, q_i\in P^n$, $t_i\in P_i^{n-1}\setminus\{0\}$ ($1\leqq i\leqq n$) such that*

$$p_i p+q_i q = t_i \quad (1\leqq i\leqq n). \tag{2}$$

PROOF. At first we shall verify that conditions (2) are necessary. Let $p, q\in P^n\setminus\{0\}$ be relatively prime and set $Q_i^{n-1}=\{s/t\colon s\in P_i^{n-1},\ t\in P_i^{n-1}\setminus\{0\}\}$ ($1\leqq i\leqq n$). Then p and q can be considered as elements of $Q_i^{n-1}[x_i]$ where $R[x]$ denotes the ring of polynomials of the variable x over the field R. Let us prove that p and q are relatively prime in the ring $Q_i^{n-1}[x_i]$, i.e. each of their common divisors in $Q_i^{n-1}[x_i]$ belongs to Q_i^{n-1}. Assume the contrary. Then for suitable $s, t\in Q_i^{n-1}[x_i]$, $r\in Q_i^{n-1}[x_i]\setminus Q_i^{n-1}$

$$p = rt, \quad q = rs. \tag{3}$$

Multiplying equalities (3) by proper polynomials $h_1, h_2\in P_i^{n-1}$ we obtain the representations

$$h_1 p = \tilde{r}\tilde{t}, \quad h_2 q = \tilde{r}\tilde{s}, \tag{4}$$

where $\tilde{t}, \tilde{s}\in P^n$, $\tilde{r}\in P^n\setminus P_i^{n-1}$. Let r^* be a prime divisor of $\tilde{r}$ such that $r^*\in P^n\setminus P_i^{n-1}$. Then by the unique prime representation theorem (see [5], p. 116) r^* is a divisor of either h_1 or p. But $h_1\in P_i^{n-1}$, hence r^* is a divisor of p. Analogously, (4) implies that r^* is a divisor of q. This is a contradiction, since r^* is not a constant function. Thus p and q are relatively prime in $Q_i^{n-1}[x_i]$. Then there exist $p_i^*, q_i^*\in Q_i^{n-1}[x_i]$ such that $p_i^*p+q_i^*q\in Q_i^{n-1}\setminus\{0\}$ (see [5], p. 73). Multiplying again by a proper polynomial of P_i^{n-1} we obtain that for some $p_i, q_i\in P^n$, $p_i p+q_i q\in P_i^{n-1}\setminus\{0\}$. Since this procedure can be done for each i, $1\leqq i\leqq n$, the necessity of conditions (2) is proved.

On the other hand, if p and q have a nontrivial common divisor $r \in P^n \setminus P^0$, then $r \in P^n \setminus P_j^{n-1}$ for some $1 \leqq j \leqq n$. This implies that for any $p^*, q^* \in P^n$, $p^*p + q^*q \in (P^n \setminus P_j^{n-1}) \cup \{0\}$. Thus condition (2) is violated. This completes the proof of the proposition.

For $a, b \in \mathbf{R}^n$ we denote by $[a, b]$ and $(a, b]$ the sets $\{\lambda a + (1-\lambda)b : 0 \leqq \lambda \leqq 1\}$ and $\{\lambda a + (1-\lambda)b : 0 \leqq \lambda < 1\}$, respectively.

PROPOSITION 2. *Let $p_1, \ldots, p_s \in P^n \setminus \{0\}$, $a \in K_1$ where $K_1 \subset \mathbf{R}^n$ is open. Then there exists $b \in K_1$ such that*

$$(a, b] \subset \bigcap_{i=1}^{s} (K_1 \setminus Z(p_i)). \tag{5}$$

PROOF. Since $p_1, \ldots, p_s \in P^n \setminus \{0\}$ there exists a straight line l in $\mathbf{R}^n$ such that $a \in l$ and $p_i|_l \in P^1 \setminus \{0\}$ $(1 \leqq i \leqq s)$. Then each p_i has a finite number of zeros on l, hence we can find a point $b \in l$ such that (5) holds.

Let now F_n be a rectangular region in $\mathbf{R}^n$, i.e. $F_n = \{(x_1, \ldots, x_n) \in \mathbf{R}^n : c_i < x_i < d_i, 1 \leqq i \leqq n\}$, where $c_i, d_i \in \mathbf{R}^1$, $c_i < d_i$ $(1 \leqq i \leqq n)$. Applying Propositions 1 and 2 we shall verify the needed statement for F_n.

LEMMA. *Let $p, q \in P^n \setminus \{0\}$ be relatively prime polynomials. Then $F_n \setminus Z(p, q)$ is connected.*

PROOF. If $n=1$ then $Z(p, q) = \emptyset$, hence the lemma is evident, Moreover, if $n=2$ then according to the Bezout theorem $Z(p, q)$ contains at most a finite number of points. Hence our statement again easily follows. We shall prove the lemma by induction on n. Assume that $F_n \setminus Z(p, q)$ is connected if $n = k-1$ $(k \geqq 2)$ and consider the case $n = k$. Let $a, b \in F_n \setminus Z(p, q)$ and let us prove that a and b can be connected by a continuous piecewise linear curve $\gamma \subset F_n \setminus Z(p, q)$. It follows from Proposition 1 that there exist $p_i, q_i \in P^k$ and $t_i \in P_i^{k-1} \setminus \{0\}$ $(1 \leqq i \leqq k)$ such that (2) holds. Then

$$Z(p, q) \subset Z(t_1, t_2, \ldots, t_k). \tag{6}$$

Using (6) and Proposition 2 we may assume without loss of generality that $a, b \in \bigcap_{i=1}^{k} (F_n \setminus Z(t_i))$. Set $a = (a_1, \ldots, a_k)$, $b = (b_1, \ldots, b_k)$, $b^* = (b_1, \ldots, b_{k-1}, a_k) \in F_n$. Since $b \in F_n \setminus Z(t_k)$ and $t_k \in P_k^{k-1} \setminus \{0\}$ we have by (6)

$$[b, b^*] \subset (F_n \setminus Z(t_k)) \subset (F_n \setminus Z(p, q)). \tag{7}$$

Set $F_n^* = \{(x_1, \ldots, x_k) \in F_n : x_k = a_k\}$; $p_i^* = p_i|_{F_n^*}$, $q_i^* = q_i|_{F_n^*}$, $p^* = p|_{F_n^*}$, $q^* = q|_{F_n^*}$, $t_i^* = t_i|_{F_n^*}$ $(1 \leqq i \leqq k)$. Then F_n^* is a rectangular region in $\mathbf{R}^{k-1}$, $p_i^*, q_i^*, p^*, q^*, t_i^* \in P^{k-1}$ $(1 \leqq i \leqq k)$. Moreover, since $a \in \bigcap_{i=1}^{k} (F_n \setminus Z(t_i))$ each t_i^* is nontrivial on F_n^*, i.e. $t_i^* \in P_i^{k-2} \setminus \{0\}$ $(1 \leqq i \leqq k-1)$. Using equalities (2) we obtain

$$p_i^* p^* + q_i^* q^* = t_i^* \quad (1 \leqq i \leqq k-1). \tag{8}$$

Thus by Proposition 1, $p^*, q^* \in P^{k-1}$ are relatively prime polynomials.

Furthermore, $a, b^* \in F_n^* \setminus Z(p^*, q^*)$, where $p^*, q^* \in P^{k-1}$ are relatively prime and F_n^* is a rectangular region in $\mathbf{R}^{k-1}$. By our assumption $F_n^* \setminus Z(p^*, q^*)$ is already connected. Thus there exists a continuous piecewise linear curve γ^* connecting a and b^* such that

$$\gamma^* \subset (F_n^* \setminus Z(p^*, q^*)) \subset (F_n \setminus Z(p, q)). \tag{9}$$

Finally, (7) and (9) imply that the curve $\gamma = [b, b^*] \cup \gamma^*$ is the needed one.

By the above lemma we can easily obtain the analogous statement for arbitrary open connected sets in $\mathbf{R}^n$.

COROLLARY 2. *Let $p, q \in P^n \setminus \{0\}$ be relatively prime. Then $K \setminus Z(p, q)$ is connected.*

Remark, that throughout this paper K denotes a bounded open and connected subset of $\mathbf{R}^n$ but the boundedness need not be assumed in the above corollary.

PROOF. Let $a, b \in K \setminus Z(p, q)$. By connectedness of K there exists a curve $\gamma = [a, c_1] \cup [c_1, c_2] \cup \ldots \cup [c_m, b]$ such that $\gamma \subset K$. Since K is open and $Z(p, q)$ is nowhere dense we may assume without loss of generality that $c_i \in K \setminus Z(p, q)$, $1 \leqq i \leqq m$. Each interval $[c_{j-1}, c_j]$ can be covered by a rectangular region lying in K. The rest of the proof follows immediately from the above lemma.

4. Proof of Theorem 1

We shall use the following characterization of best L_1-approximation (see [3], p. 46): $p_0 \in P^n(M)$ is a best approximation of $f \in L_1(K)$ if and only if

$$\left| \int_K p \operatorname{sgn}(f - p_0) \right| \leqq \int_{Z_K(f-p_0)} |p| \tag{10}$$

for each $p \in P^n(M)$. Here and in what follows we use the notation $Z_K(f) = Z(f) \cap K$.

Assume now that Theorem 1 is false. Then for some $f \in Q^n(K) \setminus P^n(M)$ there exist two different best approximations $p_1, p_2 \in P^n(M)$. Evidently, $(p_1 + p_2)/2 \in P^n(M)$ is also a best approximation of f hence

$$2 \int_K \left| f - \frac{p_1 + p_2}{2} \right| = \int_K (|f - p_1| + |f - p_2|). \tag{11}$$

Since f, p_1 and p_2 are continuous and K is open (11) implies that throughout the set K

$$|2f - p_1 - p_2| = |f - p_1| + |f - p_2|. \tag{12}$$

Let $f = t_1/t_2$, where $t_1, t_2 \in P^n$ and $t_2 > 0$ on K. Set $t = 2t_1 - t_2(p_1 + p_2) \in P^n \setminus \{0\}$; $q = p_1 - p_2 \in P^n(M) \setminus \{0\}$. Then $\operatorname{sgn}\left(f - \frac{p_1 + p_2}{2}\right) = \operatorname{sgn} t$, hence according to (10)

$$\int_K p \operatorname{sgn} t = 0 \quad (p \in P^n(M)). \tag{13}$$

(Here we used that the measure of $Z_K(t)$ is zero.) Moreover, it follows from (12), that

$$Z_K(t) \subset Z_K(q). \tag{14}$$

Let now

$$t = r_1^{\alpha_1} \dots r_s^{\alpha_s}$$

be the prime representation of $t, r_i \in P^n \setminus \{0\}$ is prime, $\alpha_i \in \mathbf{N}$, $1 \leqq i \leqq s$. We may assume that for $1 \leqq i \leqq s_0$, $r_i^{\alpha_i}$ does not change its sign on K, i.e.

$$t = t^* \prod_{i=s_0+1}^{s} r_i^{\alpha_i} \quad (s_0 \leqq s), \tag{15}$$

where $t^* \in P^n \setminus \{0\}$, $\operatorname{sgn} t^* = \gamma$ a.e. on K ($\gamma = 1$ or -1) and for each $s_0+1 \leqq i \leqq s$ α_i is odd and r_i possesses both positive and negative values on K. Evidently $s_0 < s$, since otherwise $\operatorname{sgn} t = \operatorname{sgn} t^* = \gamma$ a.e. on K, and setting in (13) $p = 1$ we arrive at a contradiction. Let us verify that for each $s_0+1 \leqq i \leqq s$, r_i is a divisor of q. Assume the contrary. Then for some $s_0+1 \leqq j \leqq s$, r_j and q are relatively prime. Thus by Corollary 2 $K \setminus Z(r_j, q)$ is connected. But it follows from (14) that $Z_K(r_j) \subset Z_K(t) \subset Z_K(q)$, hence $K \setminus Z(r_j, q) = K \setminus Z(r_j)$. Therefore $K \setminus Z(r_j)$ is connected. But a continuous function can not change its sign on a connected set where it has no zeros. This is a contradiction since r_j possesses both positive and negative values on K. Thus each r_i, $s_0+1 \leqq i \leqq s$ is a divisor of q, i.e.

$$q = q^* \prod_{i=s_0+1}^{s} r_i,$$

where $q^* \in P^n \setminus \{0\}$. Set $\tilde{g} = \prod_{i=s_0+1}^{s} r_i$. Then $q = q^* \tilde{g}$, where $q \in P^n(M) \setminus \{0\}$, $q^*, \tilde{g} \in P^n \setminus \{0\}$. The S-property of M yields that $\tilde{g} \in P^n(M)$. Moreover (15) implies that a.e. on K

$$\operatorname{sgn} t = \operatorname{sgn} t^* \prod_{i=s_0+1}^{s} \operatorname{sgn} r_i = \gamma \operatorname{sgn} \tilde{g}$$

holds. Since $\tilde{g} \in P^n(M) \setminus \{0\}$ this relation contradicts (13) if we set $p = \tilde{g}$ there. The proof of Theorem 1 is complete.

5. Korkin—Zolotarjov type extremal problems for polynomials of several variables

Consider the set of polynomials $P^n(M)$. The Korkin—Zolotarjov type extremal problem consists in minimizing the L_1-deviation of polynomials of $P^n(M)$ from zero while the coefficient of a given term of "highest degree" is fixed. (A term has highest degree if the sum of its exponents equals $d(M)$, see (1).) Thus we have the following

PROBLEM I. For a given $(\alpha_1^*, \dots, \alpha_n^*) \in M^*$ find the polynomials

$$p = \sum_{(\alpha_1, \dots, \alpha_n) \in M} a_{\alpha_1, \dots, \alpha_n} x_1^{\alpha_1} \dots x_n^{\alpha_n} \in P^n(M)$$

with $a_{\alpha_1^*, \dots, \alpha_n^*} = 1$ which have the minimal L_1-norm on K.

This problem is equivalent to the L_1-approximation of $x_1^{\alpha_1^*}\dots x_n^{\alpha_n^*}$ by polynomials from $P^n(\tilde{M})$, where $\tilde{M}=M\setminus\{(\alpha_1^*, \dots, \alpha_n^*)\}$. In order to apply Theorem 1 we have to show that the S-property of M is not violated after omitting a largest point.

PROPOSITION 3. *Let M satisfy the S-property, $(\alpha_1^*, \dots, \alpha_n^*)\in M^*$. Then the set $\tilde{M}=M\setminus\{(\alpha_1^*, \dots, \alpha_n^*)\}$ satisfies the S-property, too.*

PROOF. Let $q_1, q_2\in P^n\setminus\{0\}$ and assume that $q_1q_2\in P^n(\tilde{M})$. Since $\tilde{M}\subset M$, $P^n(\tilde{M})\subset P^n(M)$. Hence the S-property of M implies that $q_1, q_2\in P^n(M)$. Assume that one of these polynomials, say q_1, does not belong to $P^n(\tilde{M})$. Then $q_1=ax^{\alpha_1^*}\cdot\dots\cdot x^{\alpha_1^*}+p$, where $a\neq 0$, $p\in P^n(\tilde{M})$. Since $q_1q_2\in P^n(M)$ and $\alpha_1^*+\dots+\alpha_n^*=d(M)$, $q_2\in P^0$. Hence $q_1q_2=cq_1\notin P^n(\tilde{M})$ $(c\in\mathbf{R}^1\setminus\{0\})$. By this contradiction we obtain the needed statement.

Now Theorem 1 and Proposition 3 immediately imply the unicity of the solution of Problem I.

THEOREM 2. *Let $n\in\mathbf{N}$, K be an open bounded connected subset of $\mathbf{R}^n$. Then if M satisfies the S-property, Problem* I *possesses a unique solution for any $(\alpha_1^*, \dots, \alpha_n^*)\in M^*$.*

Thus under hypothesis of Theorem 2 the Korkin—Zolotarjov type extremal problem has a unique solution. For a rectangular region this solution can be given explicitly. Set

$$I^n = \{(x_1, \dots, x_n)\in\mathbf{R}^n: -1<x_i<1,\ 1\leqq i\leqq n\}.$$

Let $U_m(x)=2^{-m}\sin(m+1)\arccos x/\sqrt{1-x^2}$ be the Chebyshev polynomial of the second kind, $-1<x<1$. This polynomial is the solution of the Korkin—Zolotarjov extremal problem in the case when $n=1$, $M=\{0, 1, \dots, m\}$, $K=(-1, 1)$. The polynomial $U_m(x)$ satisfies the following wellknown orthogonality conditions:

$$\int_{-1}^{1} x^i \operatorname{sgn} U_m(x)\,dx = 0 \quad (0\leqq i\leqq m-1). \tag{16}$$

We shall need a simple fact on the structure of the sets M satisfying the S-property. If

$$a=(a_1, \dots, a_n),\quad b=(b_1, \dots, b_n)\in\mathbf{R}^n$$

then $a\leqq b$ denotes that $a_i\leqq b_i$ for each $1\leqq i\leqq n$.

PROPOSITION 4. *Let M satisfy the S-property, $\alpha\in M$. Then for each $\beta\in\mathbf{Z}_+^n$ such that $\beta\leqq\alpha$ we have $\beta\in M$.*

PROOF. Since $\alpha=(\alpha_1, \dots, \alpha_n)\in M$, $x_1^{\alpha_1}\dots x_n^{\alpha_n}\in P^n(M)$. Thus it follows from the S-property that for any $\beta=(\beta_1, \dots, \beta_n)\in\mathbf{Z}_+^n$ such that $\beta\leqq\alpha$, $x_1^{\beta_1}\dots x_n^{\beta_n}\in P^n(M)$. Therefore $\beta\in M$.

THEOREM 3. *Let $n\in\mathbf{N}$, $K=I^n$ and assume that M satisfies the S-property. Then for each $(\alpha_1^*, \dots, \alpha_n^*)\in M^*$ the polynomial $U_{\alpha_1^*, \dots, \alpha_n^*}(x_1, \dots, x_n)=\prod_{i=1}^{n} U_{\alpha_i^*}(x_i)$ is the unique solution of Problem* I.

PROOF. Proposition 4 implies that $U_{\alpha_1^*,\dots,\alpha_n^*}(x_1, \dots, x_n) \in P^n(M)$. Obviously, $a_{\alpha_1^*,\dots,\alpha_n^*}=1$. Furthermore, since $\alpha_1^*+\alpha_2^*+\dots+\alpha_n^*=d(M)$, for each $(\beta_1, \dots, \beta_n)\in M$, $(\beta_1, \dots, \beta_n)\neq(\alpha_1^*, \dots, \alpha_n^*)$ there exists an index $1\leqq j\leqq n$ such that $\beta_j<\alpha_j^*$. Therefore it follows from (16) that for any $(\beta_1, \dots, \beta_n)\in M$, $(\beta_1, \dots, \beta_n)\neq(\alpha_1^*, \dots, \alpha_n^*)$

$$\int_{I^n} x_1^{\beta_1}\dots x_n^{\beta_n} \operatorname{sgn} U_{\alpha_1^*,\dots,\alpha_n^*}(x_1, \dots, x_n) = \prod_{i=1}^{n} \int_{-1}^{1} x_i^{\beta_i} \operatorname{sgn} U_{\alpha_i^*}(x_i)\, dx_i = 0$$

holds. Using now the characterization theorem of best approximation (see (10)) we can easily derive that $U_{\alpha_1^*,\dots,\alpha_n^*}(x_1, \dots, x_n)$ is an extremal polynomial. The unicity of this solution of Problem I is an immediate consequence of Theorem 2.

Theorem 3 can be applied in particular to the polynomial spaces $P^n(k)$ and $P^n(k_1, \dots, k_n)$. The space $P^n(k_1, \dots, k_n)$ has a specific property that $M^*(k_1, \dots, k_n)$ consists of the single point $(k_1, \dots, k_n)$. For polynomial spaces of this type we can pose the Zolotarjov problem.

PROBLEM II. Let M^* consist of the single point $(\alpha_1^*, \dots, \alpha_n^*)$ and let $(\beta_1, \dots, \beta_n)\in M$, $\beta_1+\dots+\beta_n=d(M)-1$. Then among all polynomials $p=\sum_{(\alpha_1,\dots,\alpha_n)\in M} a_{\alpha_1,\dots,\alpha_n} x_1^{\alpha_1}\dots x_n^{\alpha_n}$ with $a_{\alpha_1^*,\dots,\alpha_n^*}=1$ and $a_{\beta_1,\dots,\beta_n}=-c$ $(c\in\mathbf{R}^1)$ find those which have the minimal L_1-norm on K.

In the case when $n=1$, $M=\{0, 1, \dots, m\}$, $K=(-1, 1)$ the unique solution of the Zolotarjov problem is given by the polynomial

$$\tilde{Z}_{m,c}(x) = \begin{cases} (x-c)\,U_{m-1}(x), & |c|\geqq 1; \\ U_m(x)-cU_{m-1}(x)+\dfrac{c^2}{4}\,U_{m-2}(x), & |c|<1 \end{cases}$$

(see [4], p. 99). This implies the following orthogonality conditions:

$$\int_{-1}^{1} x^i \operatorname{sgn} \tilde{Z}_{m,c}(x)\, dx = 0 \quad (0\leqq i\leqq m-2). \tag{17}$$

First of all we shall verify that the S-property implies the uniqueness of the solution of Problem II. Remark that by Proposition 4 the S-property insures the existence of $(\beta_1, \dots, \beta_n)\in M$ with $\sum_{i=1}^{n}\beta_i=d(M)-1$.

THEOREM 4. *Let $n\in\mathbf{N}$, K be an open bounded connected subset of $\mathbf{R}^n$. Furthermore assume that M satisfies the S-property and M^* consists of the single point $(\alpha_1^*, \dots, \alpha_n^*)$. Then for any $(\beta_1, \dots, \beta_n)\in M$ such that $\sum_{i=1}^{n}\beta_i=d(M)-1$, Problem* II *has a unique solution.*

PROOF. Problem II is equivalent to approximation of $x_1^{\alpha_1^*}\dots x_n^{\alpha_n^*}-cx_1^{\beta_1}\dots x_n^{\beta_n}$ by polynomials of $P^n(M_1)$, where $M_1=M\setminus\{(\alpha_1^*, \dots, \alpha_n^*), (\beta_1, \dots, \beta_n)\}$. Since

$(\alpha_1^*, \ldots, \alpha_n^*) \in M^*$, it follows from Proposition 3 that $\tilde{M} = M \setminus \{(\alpha_1^*, \ldots, \alpha_n^*)\}$ satisfies the S-property. Furthermore, using that $(\alpha_1^*, \ldots, \alpha_n^*)$ is the unique largest point in M we obtain that $d(\tilde{M}) = d(M) - 1$. Then $(\beta_1, \ldots, \beta_n)$ is a largest point in $\tilde{M}$, i.e. $(\beta_1, \ldots, \beta_n) \in \tilde{M}^*$. Hence according to Proposition 3 $M_1 = \tilde{M} \setminus \{(\beta_1, \ldots, \beta_n)\}$ satisfies the S-property. Now the needed statement follows immediately from Theorem 1.

The above theorem gives a sufficient condition for the uniqueness of the solution of the Zolotarjov type extremal problem in several variables. For rectangular regions we can again find this unique solution explicitly.

THEOREM 5. *Let $n \in \mathbf{N}$, $K = I^n$. Moreover assume that M satisfies the S-property and M^* consists of the single point $\alpha^* = (\alpha_1^*, \ldots, \alpha_n^*)$. Then for any $\beta = (\beta_1, \ldots, \beta_n) \in M$ such that $\beta_1 + \ldots + \beta_n = d(M) - 1$ and $\beta \leqq \alpha^*$ the polynomial $\prod_{\substack{i=1 \\ i \neq j}}^{n} U_{\alpha_i^*}(x_i) Z_{\beta_j+1, c}(x_j)$ is the unique solution of Problem* II. *The index j is uniquely determined by the relation $\beta_j = \alpha_j^* - 1$.*

PROOF. Since $\beta \leqq \alpha^*$ and $\beta_1 + \beta_2 + \ldots + \beta_n = d(M) - 1$ there exists exactly one j such that $\beta_j = \alpha_j^* - 1$ $(1 \leqq j \leqq n)$. Furthermore $\alpha_i^* = \beta_i$ for $1 \leqq i \leqq n$, $i \neq j$. Therefore

$$\prod_{\substack{i=1 \\ i \neq j}}^{n} U_{\alpha_i^*}(x_i) Z_{\beta_j+1, c}(x_j) = x_1^{\alpha_1^*} \ldots x_n^{\alpha_n^*} - \tag{18}$$

$$- c x_1^{\beta_1} \ldots x_n^{\beta_n} + \sum_{(\alpha_1, \ldots, \alpha_n) \in M_0} a_{\alpha_1, \ldots, \alpha_n} x_1^{\alpha_1} \ldots x_n^{\alpha_n},$$

where for arbitary $\alpha = (\alpha_1, \ldots, \alpha_n) \in M_0$, $\alpha \leqq \alpha^*$ and $\alpha_i < \beta_i$ for some $1 \leqq i \leqq n$. This implies that the above polynomial belongs to $P^n(M)$ and has the needed representation. Moreover the polynomial (18) satisfies the following orthogonality conditions: for each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_+^n$ such that $\alpha_i < \beta_i$ for some $1 \leqq i \leqq n$:

$$\int_{I^n} x_1^{\alpha_1} \ldots x_n^{\alpha_n} \operatorname{sgn} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n} U_{\alpha_i^*}(x_i) Z_{\beta_j+1, c}(x_j) \right) =$$

$$= \prod_{\substack{i=1 \\ i \neq j}}^{n} \int_{-1}^{1} x_i^{\alpha_i} \operatorname{sgn} U_{\alpha_i^*}(x_i)\, dx_i \int_{-1}^{1} x_j^{\alpha_j} \operatorname{sgn} Z_{\beta_j+1, c}(x_j)\, dx_j = 0.$$

(Here we applied relations (16) and (17).) This proves that the polynomial (18) is a solution of Problem II. The unicity of this solution follows from Theorem 4.

Theorem 5 can be applied to the polynomial space $P^n(k_1, \ldots, k_n)$. Moreover, Proposition 3 gives a method of generating S-sets from a given S-set, hence Theorems 2—5 possess a relatively wide field of applications.

Finally, we would like to mention some unsolved problems. The most exciting question is to verify the unicity of the best L_1-approximation for each $f \in C_1(K)$. Theorem 1 which states the unicity for a dense (with respect to uniform norm) subset of $C_1(K)$ is a step in this direction. Nevertheless it seems to be unlikely that the methods of this paper can be employed for the solution of the general problem. Another

interesting problem is connected with the determination of extremal polynomials. Theorem 2 states the unicity of the solution of the Korkin—Zolotarjov type extremal problem for any open bounded connected region K in $\mathbf{R}^n$. But we can present the solution explicitly only when K is a rectangular region. It would be of great interest to find the solution explicitly for other sets too, for example in the case when K is a ball in $\mathbf{R}^n$.

References

[1] A. Kroó, Some theorems on unicity of multivariate L_1-approximation, *Acta Math. Acad. Sci. Hungar.*, **40** (1982), 179—189.
[2] H. S. Shapiro, Some theorems on Čebysev approximation II, *J. Math. Anal. Appl.*, **17** (1967), 262—268.
[3] I. Singer, *Best approximation in normed linear spaces by elements of linear subspaces*, Springer (Berlin—Heidelberg—New York, 1970).
[4] V. M. Tihomirov, *Some questions of Approximation Theory*, Izdatelstvo MGU (Moscow, 1976) (in Russian).
[5] B. L. Van der Waerden, *Modern Algebra*, Nauka (Moscow, 1979) (in Russian).
[6] H. Ehlich, K. Zeller, Cebyshev-Polynome in mehreren Veränderlichen, *Math. Zeitschr.*, **93** (1966), 142—143.
[7] A. Kroó, On Chebyshev subspaces in the space of multivariate differentiable functions (to appear).

(Received February 18, 1982)

MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
H—1053 BUDAPEST, REÁLTANODA U. 13—15
HUNGARY

Acta Math. Hung.
42 (3—4) (1983), 319—330.

ON THE STRONG LAW OF LARGE NUMBERS FOR PAIRWISE INDEPENDENT RANDOM VARIABLES

S. CSÖRGŐ, K. TANDORI, member of the Academy, and V. TOTIK (Szeged)

Results and discussion

The basic idea of Etemadi's clever "elementary proof of the strong law of large numbers" for identically distributed random variables in [2] consists in the observation that it is enough to establish the law for the positive and negative parts of the underlying variables separately. Then the monotonicity of the corresponding partial sums enables one to avoid the use of the Kolmogorov inequality, and the only property required from these positive or negative part variables is that the variance of a sum from some truncated versions of them be equal to the sum of the termwise variances. The simplest way to ensure the latter requirement of lack of pairwise correlation is to assume that the original sequence consists of pairwise independent variables. Thus Etemadi was able to show that the averages of pairwise independent identically distributed random variables converge almost surely to the common mean of the variables assumed to be finite, thereby relaxing the assumption of total independence in Kolmogorov's classic law.

The aim of the present note is to point out that Etemadi's idea works for nonidentically distributed random variables as well. We derive analogues of Kolmogorov's other classical strong law for nonidentically distributed variables assumed to be pairwise independent only. Our sufficient conditions raise a few problems and some of these are also solved and discussed below.

Setting $D^2(X)=E(X-EX)^2$ for the variance of the random variable X, the principal result is

THEOREM 1. *If the pairwise independent random variables* $X_1, X_2, \ldots$ *satisfy the conditions*

$$\sum_{m=1}^{\infty} \frac{D^2(X_m)}{m^2} < \infty \tag{1}$$

and

$$\frac{1}{n} \sum_{m=1}^{n} E|X_m - EX_m| = O(1), \tag{2}$$

then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} (X_m - EX_m) = 0 \tag{3}$$

almost surely.

For a sequence $a=\{a_n\}_1^\infty$ of positive numbers set

$$Y_n(a) = \begin{cases} Y_n, & \text{if } |Y_n| \leqq a_n \\ 0, & \text{otherwise,} \end{cases}$$

where $Y_n=X_n-EX_n$, $E|X_n|<\infty$, $n=1, 2, \ldots$. The "truncated" version of Theorem 1 is

THEOREM 2. *If the random variables $X_1, X_2, \ldots$ are pairwise independent such that condition* (2) *is satisfied and the centered and truncated sequence satisfies*

$$\sum_{m=1}^{\infty} \frac{D^2(Y_m(a))}{m^2} < \infty \tag{4}$$

and

$$\sum_{m=1}^{\infty} P\{Y_m \neq Y_m(a)\} < \infty, \tag{5}$$

then for any sequence $\{b_n\}_1^\infty$ of constants satisfying

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m-1}^{n} (b_m - EY_m(a)) = 0$$

we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} (Y_m - b_m) = 0 \tag{6}$$

almost surely.

In general, the choice $b_m=0$, $m=1, 2, \ldots$ cannot be ensured under the given conditions. However, if, in addition, the variables $X_1, X_2, \ldots$ are identically distributed in Theorem 2, then (4) and (5) are known to hold by the classical proof of Kolmogorov for the truncating sequence $a_n=n$, $n=1, 2, \ldots$ (pairwise independence suffices here!), and it is easy to see that $EY_n(a)-EX_n=EY_n(a)-EX_1\to 0$, as $n\to\infty$, with this truncation. Therefore Etemadi's result follows from Theorem 1 in exactly the same classic way as Kolmogorov's theorem for identically distributed variables followed from his other theorem for non-identically distributed variables.

The latter Kolmogorov theorem tells us that condition (1) alone is sufficient for (3) to hold for totally independent random variables. It is well known that, in this case, this condition is best possible in the sense that if $\{\sigma_1^2\}_n^\infty$ is a sequence of positive numbers such that

$$\sum_{m=1}^{\infty} \frac{\sigma_m^2}{m^2} = \infty,$$

then there exists a sequence $X_1, X_2, \ldots$ of totally independent random variables not obeying (3) but for which $D^2(X_m)=\sigma_m^2$, $m=1, 2, \ldots$ (cf. Révész [6]). On the other hand, according to the equally classical result of Menšov and Rademacher (cf. again [6]), the stronger condition

$$\sum_{m=2}^{\infty} \frac{D^2(X_m)(\log m)^2}{m^2} < \infty \tag{7}$$

already implies (3) for any sequence $X_1, X_2, \ldots$ of pairwise uncorrelated (orthogonal) random variables. The latter condition is also best possible for uncorrelated variables in the above sense (Tandori [8]), provided that the corresponding σ_m^2/m^2 sequence is nonincreasing. We show that condition (1) alone is not enough to imply (3) when the variables are only pairwise independent.

THEOREM 3. *For every* ε, $0<\varepsilon<1$, *there exists a sequence* $X_1, X_2, \ldots$ *of pairwise independent random variables such that*

$$\sum_{m=2}^{\infty} \frac{D^2(X_m)\,(\log\log m)^{1-\varepsilon}}{m^2} < \infty \tag{8}$$

and

$$P\left\{\overline{\lim_{n\to\infty}} \frac{1}{n}\left|\sum_{m=1}^{\infty}(X_m - EX_m)\right| = \infty\right\} > 0. \tag{9}$$

Therefore some auxiliary conditions are necessary to assume beside (1) to have (3). Theorem 1 provides an example for such a condition. The role of this side-condition (2) is not entirely clear in general because, as simpe counterexamples show, (2) is not necessary for (3) even if we assume

$$\sum_{m=2}^{\infty} \frac{D^2(X_m)\,(\log m)^2}{m^2} = \infty$$

beside (1).

The construction in Theorem 3 relies on a construction of divergent Walsh series by Tandori [7]. His construction was later refined by Bočkarev [1] and Nakata [5]. Using the results of these authors, one can in fact strengthen Theorem 3 to obtain the following version of it.

THEOREM 3*. *For every* ε, $0<\varepsilon<1$, *there exists a sequence* $X_1, X_2, \ldots$ *of pairwise independent random variables such that*

$$\sum_{m=2}^{\infty} \frac{D^2(X_m)\,(\log m)^{1-\varepsilon}}{m^2} < \infty \tag{10}$$

and (9) *still holds true.*

The proof of this result is not given in this note because it would require very large space as compared to that of Theorem 3. Condition (10) is now quite close to condition (7). In this respect we mention the following conjecture of ours, an affirmative proof of which would mean that pairwise independent and pairwise uncorrelated random variables do not really differ from each other from the point of view of the strong law of large numbers if one is looking at the growth of the variances only.

CONJECTURE. *For pairwise independent random variables the condition*

$$\sum_{m=2}^{\infty} \frac{D^2(X_m)\,(\log m)^{2-\varepsilon}}{m^2} < \infty, \quad 0 < \varepsilon < 1,$$

is not enough in general to ensure (3) *to hold almost surely.*

On the other hand, pairwise independent and pairwise uncorrelated random variables do differ from each other from the point of view of the strong law of large numbers below the domain of (10), or that of the Conjecture, i.e., when we have only (1) and, necessarily, an extra assumption such as (2). Indeed, pairwise independence cannot be relaxed to pairwise orthogonality in Theorem 1. This is the content of our last result in this paper.

THEOREM 4. *There exists a sequence* $X_1, X_2, \ldots$ *of pairwise uncorrelated random variables such that conditions* (1) *and* (2) *are satisfied, but*

$$\overline{\lim_{n\to\infty}} \frac{1}{n}\left|\sum_{m=1}^{n}(X_m - EX_m)\right| = \infty$$

almost surely.

In fact the random variables constructed in the proof of Theorem 4 satisfy the condition

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n} E|X_m - EX_m| = 0.$$

Proofs

PROOF OF THEOREM 1. We may and do assume $EX_n=0$, $n=1, 2, \ldots$. Introduce the partial sums of the positive and negative parts of our variables $X_n=X_n^+-X_n^-$:

$$S_n^+ = X_1^+ + \ldots + X_n^+, \quad S_n^- = X_1^- + \ldots + X_n^-, \quad n = 1, 2, \ldots.$$

By assumption (2) there is a constant A such that the inequality

$$0 \leqq \frac{1}{n} ES_n^+ \leqq A$$

is satisfied for each n. Let $\alpha>1$, $\varepsilon>0$ and $L=[A/\varepsilon]$, the integer part of A/ε. For each pair of integers m and s, $m\geqq 0$, $s=0, \ldots, L$, put

$$k_s^-(m) = \inf\{k:\ \alpha^m \leqq k < \alpha^{m+1},\ \frac{1}{k} ES_k^+ \in [s\varepsilon, (s+1)\varepsilon\},$$

$$k_s^+(m) = \sup\{k:\ \alpha^m \leqq k < \alpha^{m+1},\ \frac{1}{k} ES_k^+ \in [s\varepsilon, (s+1)\varepsilon)\}$$

if the set on the right is not empty, and let $k_s^+(m)=k_s^-(m)=[\alpha^m]$ otherwise.

Since, obviously, $D^2(X_n^+)+D^2(X_n^-)\leqq D^2(X_n)$ for each n, we obtain, by the pairwise independence of $X_1^+, X_2^+, \ldots$, the estimate

$$(11)\qquad \sum_{m=0}^{\infty}\frac{1}{(k_s^{\pm}(m))^2} E(S^+_{k_s^{\pm}(m)} - ES^+_{k_s^{\pm}(m)})^2 = \sum_{m=0}^{\infty}\frac{1}{(k_s^{\pm}(m))^2}\sum_{j=1}^{k_s^{\pm}(m)} D^2(X_j^+) \leqq$$

$$\leqq \sum_{m=0}^{\infty}\frac{1}{(k_s^{\pm}(m))^2}\sum_{j=1}^{k_s^{\pm}(m)} D^2(X_j) = \sum_{j=1}^{\infty} D^2(X_j) \sum_{\{m:\, k_s^{\pm}(m)\geqq j\}}\frac{1}{(k_s^{\pm}(m))^2} \leqq$$

$$\leqq \sum_{j=1}^{\infty} D^2(X_j) \sum_{\{m:\,\alpha^m\geqq j\}}\frac{1}{(\alpha^m)^2} \leqq \frac{\alpha^2}{\alpha^2-1}\sum_{j=1}^{\infty}\frac{D^2(X_j)}{j^2} < \infty$$

for any $s=0, \ldots, L$, by condition (1). Hence the monotone convergence theorem implies that

$$\lim_{m\to\infty} \frac{1}{k_s^{\pm}(m)} (S^+_{k_s^{\pm}} - ES^+_{k_s^{\pm}(m)}) = 0 \quad \text{a.s.} \tag{12}$$

for each $s = 0, \ldots, L$.

Now for any natural number n there exists an $m=m(n)$ and an $s=s(n)$, $\lim_{n\to\infty} m(n)=\infty$, $0\leqq s(n)\leqq L$, such that $\alpha^m \leqq n < \alpha^{m+1}$ and $ES_n^+/n \in [s\varepsilon, (s+1)\varepsilon)$. By the definitions of $k_s^{\pm}(m)$ we have

$$k_s^-(m) \leqq n \leqq k_s^+(m), \quad \left| \frac{1}{k_s^{\pm}(m)} ES^+_{k_s^{\pm}} - \frac{1}{n} ES_n^+ \right| \leqq \varepsilon,$$

and so

$$-\varepsilon - \left(1-\frac{1}{\alpha}\right) A + \frac{1}{\alpha} \frac{1}{k_s^-(m)} (S^+_{k_s^-(m)} - ES^+_{k_s^-(m)}) \leqq$$

$$\leqq -\varepsilon - \left(1-\frac{1}{\alpha}\right) \frac{1}{k_s^-(m)} ES^+_{k_s^-(m)} + \frac{1}{\alpha} \frac{1}{k_s^-(m)} (S^+_{k_s^-(m)} - ES^+_{k_s^-(m)}) \leqq$$

$$\leqq \frac{1}{n} S^+_{k_s^-(m)} - \frac{1}{n} ES_n^+ \leqq \frac{1}{n} (S_n^+ - ES_n^+) \leqq \frac{1}{n} S^+_{k_s^+(m)} - \frac{1}{k_s^+(m)} ES^+_{k_s^+(m)} + \varepsilon \leqq$$

$$\leqq \frac{\alpha}{k_s^+(m)} (S^+_{k_s^+(m)} - ES^+_{k_s^+(m)}) + (\alpha-1)A + \varepsilon.$$

By (12) these inequalities yield

$$-\varepsilon - \left(1-\frac{1}{\alpha}\right) A \leqq \underline{\lim}_{n\to\infty} \frac{1}{n} (S_n^+ - ES_n^+) \leqq \overline{\lim}_{n\to\infty} \frac{1}{n} (S_n^+ - ES_n^+) \leqq (\alpha-1)A + \varepsilon$$

almost surely, and since this is true for any $\alpha>1$ and $\varepsilon>0$, we obtain that

$$\lim_{n\to\infty} \frac{1}{n} (S_n^+ - ES_n^+) = 0 \quad \text{a.s.}$$

It can be proved in the same way that

$$\lim_{n\to\infty} \frac{1}{n} (S_n^- - ES_n^-) = 0 \quad \text{a.s.},$$

and the theorem follows.

PROOF OF THEOREM 2. Clearly,

$$\frac{1}{n} \sum_{m=1}^{n} E|Y_m(a) - EY_m(a)| \leqq \frac{2}{n} \sum_{m=1}^{n} E|Y_m(a)| \leqq \frac{2}{n} \sum_{m=1}^{n} E|X_m - EX_m| = O(1),$$

and hence an application of Theorem 1 to $Y_1(a), Y_2(a), \ldots$ yields

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} (Y_m(a) - EY_m(a)) = 0 \quad \text{a.s.}$$

Condition (5) and the Borel—Cantelly lemma then give (6) with $b_m = Y_m(a)$, $m=1, 2, \ldots$.

PROOF OF THEOREM 3. Consider the probability space $(\Omega, \mathscr{A}, P)$, where $\Omega=(0, 1)$, $\mathscr{A}$ is the σ-algebra of the Lebesgue-measurable subsets of Ω and P is the Lebesgue measure on $\mathscr{A}$. Let $r_n(\omega)=\operatorname{sign}\sin(2^n\pi\omega)$, $n=1, 2, \ldots$; $\omega\in(0, 1)$, be the n-th Rademacher function. Consider the sequence $\{w_n(\omega)\}_0^\infty$ of Walsh functions, i.e., $w_0(\omega)\equiv 1$, and if n is a positive integer with the diadic representation $n=2^{\nu_1}+\ldots+2^{\nu_p}$, $\nu_1<\ldots<\nu_p$, then $w_n(\omega)=r_{\nu_1+1}(\omega)\ldots r_{\nu_p+1}(\omega)$, $\omega\in(0, 1)$. For any $n=1, 2, \ldots$ we have $Ew_n=0$ and $D(w_n)=1$. Set

$$\varphi(m)=2^{2^m} \quad \text{and} \quad \psi(m)=2^{2^{2^m}}, \quad m=1, 2, \ldots .$$

Tandori [7] proved that if $m>1$, then there is a rearrangement

$$w_{i_1(m)}(\omega), \ldots, w_{i_{\varphi(m)}(m)}(\omega)$$

of the functions $w_1(\omega), \ldots, w_{\varphi(m)}(\omega)$ and there can be given real numbers $a_1(m), \ldots$ $\ldots, a_{\varphi(m)}(m)$ such that

$$a_1^2(m)+\ldots+a_{\varphi(m)}^2(m)\leqq 5m$$

and if $\omega\in(0, 1/4)$ and is not diadic, then

$$\max_{1\leqq k\leqq\varphi(m)}\sum_{j=1}^{k} a_j(m)w_{i_j(m)}(\omega)\geqq\frac{m}{2}.$$

For any $m=1, 2, \ldots$, define the numbers

$$D_{\varphi(m)+n}^2=\frac{(\psi(m))^2 a_n^2(2^m)}{2^{m(2-\varepsilon/2)}}, \quad n=1, \ldots, \psi(m),$$

where $0<\varepsilon<1$, and introduce the random variables

$$X_{\psi(m)+n}(\omega)=D_{\psi(m)+n}r_{\varphi(m)+1}(\omega)w_{i_n(2^m)}(\omega), \quad n=1, \ldots, \psi(m).$$

For any other index k, let $X_k(\omega)\equiv 0$, $\omega\in(0, 1)$. By the definition of the Walsh functions it is obvious that $X_1, X_2, \ldots$ are pairwise independent, $EX_n=0$, $n=1, 2, \ldots$, and the definition also ensures that condition (8) is satisfied (implying (1)). Exactly as we saw it in (11), condition (1) alone implies that

$$\lim_{m\to\infty}\frac{1}{\psi(m)}\sum_{j=1}^{\psi(m)} X_j=0$$

almost everywhere in (0, 1). Nevertheless we show that

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\left|\sum_{j=1}^{n} X_j(\omega)\right|=\infty \tag{13}$$

almost everywhere in (0, 1/4).

For any integer $m\geqq 1$ let $\psi(m)<n\leqq 2\psi(m)$. Then

$$\frac{1}{n}\sum_{j=1}^{n} X_j(\omega)=\frac{1}{n}\sum_{j=\psi(m)+1}^{n} X_j(\omega)+\frac{\psi(m)}{n}\frac{1}{\psi(m)}\sum_{j=1}^{\psi(m)} X_j(\omega)=V_n(\omega)+Z_n(\omega),$$

say. We know that $\lim_{n\to\infty} Z_n=0$ almost everywhere in $(0, 1)$. But

$$V_n(\omega)=\frac{\psi(m)}{n2^{m(1-\varepsilon/4)}}r_{\varphi(m)+1}(\omega)\sum_{j=1}^{n-\psi(m)}a_j(2^m)w_{i_j(2^m)}(\omega),$$

and hence

$$\max_{\psi(m)<n\leqq 2\psi(m)}|V_n(\omega)|\geqq\frac{1}{2}\frac{1}{2^{m(1-\varepsilon/4)}}\max_{1\leqq k\leqq\psi(m)}\left|\sum_{j=1}^{k}a_j(2^m)w_{i_j(2^m)}(\omega)\right|\geqq\frac{1}{4}2^{m(\varepsilon/4)}$$

almost everywhere in $(0, 1/4)$. It now follows that we have (13) indeed, almost everywhere in $(0, 1/4)$.

PROOF OF THEOREM 4. Our point of departure is a function system of Kaczmarcz [3] which, in essence, is a simplified version of a system considered originally by Menšov [4]. Let $p\geqq 2$ be an integer and for $l=1, \ldots, 2p$, consider the functions

$$u_l(x)=\frac{1}{k-p-l-\frac{1}{2}},\quad x\in\left[\frac{k-1}{p},\frac{k}{p}\right),\quad k=1, \ldots, 4p.$$

We have

$$\int_0^4 u_l^2(x)\,dx=\frac{1}{p}\sum_{k=1}^{4p}\frac{1}{\left(k-p-l-\frac{1}{2}\right)^2},$$

whence

$$\frac{C_1}{p}\leqq\int_0^4 u_l^2(x)\,dx\leqq\frac{C_2}{p},\quad l=1, \ldots, 2p, \tag{14}$$

where $C_1, C_2, \ldots$ will denote positive absolute constants. If $i>j$, then

$$\alpha_{ij}=\int_0^4 u_i(x)\,u_j(x)\,dx=\frac{1}{p}\sum_{k=1}^{4p}\frac{1}{\left(k-p-i-\frac{1}{2}\right)\left(k-p-j-\frac{1}{2}\right)}=$$

$$=\frac{1}{p}\frac{1}{i-j}\sum_{k=1}^{4p}\left\{\frac{1}{k-p-i-\frac{1}{2}}-\frac{1}{k-p-j-\frac{1}{2}}\right\}=$$

$$=\frac{1}{p}\frac{1}{i-j}\left\{\sum_{k=1-p-i}^{3p-i}\frac{1}{k-\frac{1}{2}}-\sum_{k=1-p-j}^{3p-j}\frac{1}{k-\frac{1}{2}}\right\}=$$

$$=\frac{1}{p}\frac{1}{i-j}\left\{\sum_{k=1-p-i}^{-p-j}\frac{1}{k-\frac{1}{2}}-\sum_{k=3p-i+1}^{3p-j}\frac{1}{k-\frac{1}{2}}\right\},$$

and therefore

$$|\alpha_{ij}| \leqq \frac{1}{p}\frac{1}{i-j}\left\{\frac{i-j}{p+j+\frac{1}{2}} - \frac{i-j}{3p-i-\frac{1}{2}}\right\} \leqq \frac{C_3}{p^2}. \tag{15}$$

We extend the definition of these functions $u_l(x)$ to the interval $(4, 5]$ in such a way that they be orthogonal on the whole interval $[0, 5]$. Let us divide up the interval $(4, 5]$ into $N=2p(2p-1)$ pairwise disjoint subintervals I_{ij}, $i, j=1, \ldots, 2p$, $i \neq j$, of equal length and set

$$u_l(x) = \begin{cases} \sqrt{\frac{1}{2}N|\alpha_{ij}|}, & x \in I_{lj},\ j=1, \ldots, 2p,\ j \neq l, \\ -\sqrt{\frac{1}{2}N|\alpha_{ij}|}\ \operatorname{sign}\alpha_{ij}, & x \in I_{jl},\ j=1, \ldots, 2p,\ j \neq l, \\ 0, & \text{otherwise,} \end{cases}$$

for each $l=1, \ldots, 2p$. The functions $u_l(x)$ thus obtained are obviously simple (step) functions on $[0, 5]$ and they constitute an orthogonal system there. Furthermore

$$\int_0^5 u_l^2(x)\,dx = \int_0^4 u_l^2(x)\,dx + \frac{1}{2}\sum_{j=1}^{l-1}|\alpha_{lj}| + \frac{1}{2}\sum_{j=l+1}^{2p}|\alpha_{lj}|$$

and hence by (14) and (15),

$$\frac{C_4}{p} \leqq \int_0^5 u_l^2(x)\,dx \leqq \frac{C_5}{p}, \quad l=1, \ldots, 2p. \tag{16}$$

If $x \in [2, 3)$, then there is a non-negative integer $m(x)$ depending on x, $m(x)<p$, such that

$$x \in \left[\frac{2p+m(x)}{p}, \frac{2p+m(x)+1}{p}\right).$$

Then by definition, $u_1(x) \geqq 0, \ldots, u_{p+m(x)}(x) \geqq 0$, and

$$\sum_{l=1}^{p+m(x)} u_l(x) = \sum_{l=1}^{p+m(x)} \frac{1}{2p+m(x)+1+p-l-\frac{1}{2}} = \sum_{l=1}^{p+m(x)} \frac{1}{l-\frac{1}{2}},$$

and this implies that

$$\max_{1 \leqq m \leqq 2p}\left|\sum_{l=1}^{m} u_l(x)\right| \geqq C_6 \log p, \quad x \in [2, 3). \tag{17}$$

Moreover we have

$$\int_0^5 u_l^+(x)\,dx = \frac{1}{p}\sum_{k=p+l+1}^{4p}\frac{1}{k-p-l-\frac{1}{2}} + \frac{1}{\sqrt{2p(2p-1)}}\sum_{\substack{j=1 \\ j \neq l}}^{2p}\sqrt{|\alpha_{lj}|},$$

$$\int_0^5 u_l^-(x)\,dx = \frac{1}{p}\sum_{k=1}^{p+l}\frac{1}{k-p-l-\frac{1}{2}} + \frac{1}{\sqrt{2p(2p-1)}}\sum_{\substack{j=1 \\ j \neq l}}^{2p}\sqrt{|\alpha_{lj}|},$$

whence

$$\int_0^5 u_l^+(x)\,dx \leqq C_7 \frac{\log p}{p}, \quad \int_0^5 u_l^-(x)\,dx \leqq C_7 \frac{\log p}{p}. \tag{18}$$

Put

$$v_l(x) = u_l(x) \Big/ \sqrt{\int_0^5 u_l^2(x)\,dx}, \quad l = 1, \ldots, 2p.$$

These are again simple functions on $[0, 5]$ and constitute an orthonormal system there, and by (16), (17) and (18) we have

$$\max_{1 \leqq m \leqq 2p} \left| \sum_{l=1}^{m} v_l(x) \right| \geqq C_8 \sqrt{p} \log p, \quad x \in [2, 3), \tag{19}$$

$$\int_0^5 v_l^+(x)\,dx \leqq C_9 \frac{\log p}{\sqrt{p}}, \quad \int_0^5 v_l^-(x)\,dx \leqq C_9 \frac{\log p}{\sqrt{p}}. \tag{20}$$

Let

$$g_l(x) = \begin{cases} \sqrt{5} v_l(5x), & x \in (0, 1), \\ 0, & \text{otherwise}, \end{cases}$$

and for $l = 1, \ldots, 2p$, introduce

$$f_l(p; x) = \begin{cases} \dfrac{1}{\sqrt{2}} g_l(2x), & x \in \left(0, \dfrac{1}{2}\right), \\ -\dfrac{1}{\sqrt{2}} g_l\left(2\left(x - \dfrac{1}{2}\right)\right), & x \in \left(\dfrac{1}{2}, 1\right), \\ 0, & \text{otherwise}. \end{cases}$$

The latter functions are simple and they form an orthonormal system on $(0, 1)$ with

$$\int_0^1 f_l(p; x)\,dx = 0, \quad l = 1, \ldots, 2p, \tag{21}$$

and by (20) we have for any $l = 1, \ldots, 2p$ that

$$\int_0^1 f_l^+(p; x)\,dx \leqq C_{10} \frac{\log p}{\sqrt{p}}, \quad \int_0^1 f_l^-(p; x)\,dx \leqq C_{10} \frac{\log p}{\sqrt{p}}. \tag{22}$$

Furthermore, by (19) there is a simple set $A(p) \subseteqq (0, 1)$, i.e. $A(p)$ is the union of a finite number of intervals, such that

$$\operatorname{mes} A(p) \geqq \frac{1}{5} \tag{23}$$

and

$$\max_{1 \leqq m \leqq 2p} \left| \sum_{l=1}^{m} f_l(p; x) \right| \geqq C_{11} \sqrt{p} \log p, \quad x \in A(p). \tag{24}$$

Finally, for another parameter $a>1$, we introduce the functions

$$h_l(a, p; x) = \begin{cases} \sqrt{a}\, f_l(p; ax), & x \in \left(0, \frac{1}{a}\right), \\ 0, & \text{otherwise,} \end{cases}$$

$l=1, \ldots, 2p$. These functions are still simple on $(0, 1)$, they constitute an orthonormal system such that by (21) and (22) we have

$$\int_0^1 h_l(a, p; x)\, dx = 0, \quad l = 1, \ldots, 2p, \tag{25}$$

$$\int_0^1 h_l^+(a, p; x)\, dx \leqq C_{10} \frac{\log p}{\sqrt{ap}}, \quad \int_0^1 h_l^-(a, p; x)\, dx \leqq C_{10} \frac{\log p}{\sqrt{ap}}, \tag{26}$$

$l=1, \ldots, 2p$, and, by (23) and (24), there is a simple set $H(a, p) \subseteqq (0, 1)$ such that

$$\operatorname{mes} H(a, p) \geqq \frac{1}{5a}, \tag{27}$$

$$\max_{1 \leqq m \leqq 2p} \left| \sum_{l=1}^{m} h_l(a, p; x) \right| \geqq C_{11} \sqrt{ap} \log p, \quad x \in H(a, p). \tag{28}$$

After so much preparation let $(\Omega, \mathcal{A}, P)$ be again the probability space in the proof of Theorem 4. On this space we now define a sequence $\{X_n\}_1^\infty$ of random variables and a sequence $\{E_m\}_2^\infty$ of sets in $\mathcal{A}$ such that the following conditions will be satisfied. The X_n will be simple functions and they will be pairwise uncorrelated. The events E_m will be simple sets and totally independent, and for any $m \geqq 2$ we shall have

$$P\{E_m\} \geqq \frac{1}{5(m-1)}. \tag{29}$$

Moreover, for any $m \geqq 2$, the following relations will be satisfied:

$$D^2(X_n) = \frac{2^m}{(m-1)^2}, \quad 2^{m+1} < n \leqq 2^{m+2}, \tag{30}$$

$$\max_{1 \leqq k \leqq 2^{m+1}} \left| \frac{X_{2^{m+1}+1}(\omega) + \ldots + X_{2^{m+1}+k}(\omega)}{2^{m+1}+k} \right| \geqq C_{12} \sqrt{m-1}, \quad \omega \in E_m, \tag{31}$$

$$EX_l^+ \leqq C_{13} \frac{1}{\sqrt{m-1}}, \quad EX_l^- \leqq C_{13} \frac{1}{\sqrt{m-1}}, \quad 2^{m+1} < l \leqq 2^{m+2}. \tag{32}$$

To begin the construction, set $X_n(\omega) = r_n(\omega) = \operatorname{sign} \sin(2^n \pi \omega)$, $\omega \in (0, 1)$, $n=1, \ldots, 2^3$. These are simple and uncorrelated. Let now m_0 be an integer, not less than two, and assume that the random variables $X_1, \ldots, X_{2^{m_0}+1}$ and the sets $E_1, \ldots, E_{m_0}$ are already defined such that the variables are simple and pairwise uncorrelated, the events are simple and independent, and relations (29)—(32) are satisfied for $m=2, \ldots, m_0$. Then a sequence $I_1, \ldots, I_r$ of pairwise disjoint intervals

can be given such that $(0, 1)=\bigcup_{k=1}^{r} I_k$, each of the variables X_j, $j=1, \ldots, 2^{m_0+1}$, is constant on any of the intervals $I_1, \ldots, I_r$, and each of the sets $E_2, \ldots, E_{m_0}$ appears as the union of some of the intervals $I_1, \ldots, I_r$.

For a function $X(\omega)$ on $(0, 1)$ and an interval $I=(a, b)\subseteqq(0, 1)$ let

$$X(I;\ \omega)=\begin{cases} X\left(\dfrac{\omega-a}{b-a}\right), & \omega\in(a, b), \\ 0, & \text{otherwise.}\end{cases}$$

Also, for a set $H\subseteqq(0, 1)$, let $H(I)$ be the set contained in I which is obtained from H by the application of the transformation $x\to(b-a)x+a$.

Consider now the above construction of the $h_l(a, p;\ x)$ functions and $H(a, p)$ sets for $p=2^{m_0}$, $a=m_0$. Let us break up the interval I_k into two disjoint intervals of equal length: $I_k=I_k'\cup I_k''$, $k=1, \ldots, r$. We define the next block of variables as

$$X_{2^{m_0+1}+l}(\omega)=\sum_{k=1}^{r}\frac{\sqrt{2^{m_0}}}{m_0}h_l(m_0, 2^{m_0};\ I_k';\ \omega)-\sum_{k=1}^{r}\frac{\sqrt{2^{m_0}}}{m_0}h_l(m_0, 2^{m_0};\ I_k'';\ \omega),$$

$l=1, \ldots, 2^{m_0+1}$, and the next event as

$$E_{m_0+1}=\left(\bigcup_{k=1}^{r}H(m_0, 2^{m_0};\ I_k')\right)\cup\left(\bigcup_{k=1}^{r}H(m_0, 2^{m_0};\ I_k'')\right).$$

It is plain that the X_k, $k=2^{m_0+1}, \ldots, 2^{m_0+2}$, are simple functions and E_{m_0+1} is a simple set. The events $E_2, \ldots, E_{m_0+2}$ are obviously independent, and, by (27), inequality (29) is satisfied for $m=m_0+1$. Equation (25) implies that

$$EX_k=0, \tag{33}$$

$k=2^{m_0+1}, \ldots, 2^{m_0+2}$, and hence it is evident by construction that $X_1, \ldots, X_{2^{m_0+2}}$ are pairwise uncorrelated as well. Finally, (30), (31) and (32) also follow from (26) and (28) for $m=m_0+1$, on the basis of the construction.

Hence the required sequences $\{X_n\}_1^\infty$ and $\{E_m\}_2^\infty$ are obtained by induction, and (33) also holds for any $k\geqq1$. Now we claim that the constructed X_n's satisfy all the requirements of the theorem.

From (30) we get

$$\sum_{n=2^3+1}^{\infty}\frac{D^2(X_n)}{n^2}=\sum_{m=2}^{\infty}\sum_{n=2^{m+1}+1}^{2^{m+2}}\frac{D^2(X_n)}{n^2}\leqq$$

$$\leqq 2\sum_{m=2}^{\infty}\frac{1}{m^2}2^m\sum_{n=2^{m+1}+1}^{2^{m+2}}\frac{1}{n^2}\leqq 2\sum_{m=2}^{\infty}\frac{1}{m^2}<\infty.$$

This and (33) imply in the usual way that

$$\lim_{m\to\infty}\frac{X_1+\ldots+X_{2^{m+1}}}{2^{m+1}}=0 \tag{34}$$

almost surely. It follows from (32) that

$$\lim_{l\to\infty} EX_l^+ = \lim_{l\to\infty} EX_l^- = 0,$$

and hence

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} E|X_n| = 0.$$

Through the second Borel—Cantelli lemma (29) gives that

$$P\{\overline{\lim_{m\to\infty}} E_m\} = 1.$$

Therefore it follows by (31) that the inequality

$$\max_{2^{m+1}<k\leqq 2^{m+2}} \left|\frac{X_1+\ldots+X_k}{k}\right| \geqq \max_{2^{m+1}<k\leqq 2^{m+2}} \left|\frac{X_{2^m+1}+\ldots+X_k}{k}\right| - \left|\frac{X_1+\ldots+X_{2^{m+1}}}{2^{m+1}}\right| \geqq$$

$$\geqq C_{12}\sqrt{m} - \left|\frac{X_1+\ldots+X_{2^{m+1}}}{2^{m+1}}\right|$$

holds almost surely for infinitely many m. This and (34) imply that, indeed, almost surely

$$\overline{\lim_{n\to\infty}} \left|\frac{X_1+\ldots+X_n}{n}\right| = \infty.$$

Acknowledgement

We thank Pál Révész for asking the question which is answered in Theorem 4.

References

[1] S. V. Bočkarev, Rearrangements of Fourier—Walsh series, *Izv. Akad. Nauk SSS, RSer. Mat.* **43** (1979), 1025—1041 (Russian).
[2] N. Etemadi, An elementary proof of the strong law of large numbers, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, **55** (1981), 119—122.
[3] S. Kaczmarcz, Notes on orthogonal series. II, *Studia Math.*, **5** (1943), 103—106.
[4] D. Menchoff, Sur les séries de fonctions orthogonales (Première partie), *Fundamenta Math.*, **4** (1923), 82—105.
[5] S. Nakata, On the unconditional convergence of Walsh series, *Anal. Math.*, **5** (1979), 201—205.
[6] P. Révész, *The strong laws of large numbers*, Academic Press (New York, 1967).
[7] K. Tandori, Über die Divergenz der Walshschen Reihen, *Acta Sci. Math Szeged*, **27** (1966), 261—263.
[8] K. Tandori, Bemerkung zum Gesetzt der großen Zahlen, *Periodica Math. Hung.*, **2** (1972), 33—39.

BOLYAI INSTITUTE
JÓZSEF ATTILA UNIVERSITY
ARADI VÉRTANÚK TERE 1.
H-6720 SZEGED, HUNGARY

(Received March 17, 1982; revised April 5, 1982)

Acta Math. Hung.
42 (3—4) (1983), 331—335.

A MOMENT PROBLEM WITH RESPECT TO C^*-ALGEBRAS

Z. SEBESTYÉN (Budapest)

Dedicated to András Kósa on his 50th birthday

Introduction

Let A be a (complex) C^*-algebra with or without a unit, and let G be a multiplicative semigroup closed under involution (briefly *-semigroup) in A such that the span of G is norm dense in it. Given a complex valued function f on G it is natural to ask: under what condition does there exist a (necessarily unique) positive linear extension of f to the whole A? The answer is given in Theorem 1 below.

The situation described is a common generalization of the classical Hausdorff moment problem and the trigonometric moment problem known as Hergholz Theorem or Bochner Theorem in the case of an Abelian group. The C^*-algebra in question consists of the continuous complex valued functions on a compact Hausdorff space (the dual group of a discrete Abelian group in the last case). Our solution differs from the known answer of Hausdorff [1] and Riesz [2] (in the first case). Our result is more adequate in the sense that we have a solution also in the case when f is given only on a *-ideal of G (Theorem 2). This is not the case with the classical conditions. The last extension question is closely related to a recent theorem of the author [5] and a recent result of Szafraniec [3].

Our main objective is to construct a *-representation of the *-semigroup G on some Hilbert space from the given positive definite function f. This is a cyclic representation owing to the classical Riesz Representation Theorem of continuous linear functionals on Hilbert spaces. It turns out that the *-representation constructed in this way has a unique extension to a *-representation of the original C^*-algebra. The cyclic vector thus determines the desired positive linear functional.

We have thus a unique approach to a positive definite function on a group and a positive linear functional on a C^*-algebra. In particular, the involution and the multiplicative structure of a C^*-algebra determine the states (positive linear functionals) on it (Corollary 1).

The content of this paper is a part of the author's talk delivered on the Riesz—Fejér Memory Conference held at Budapest in August 1980.

THEOREM 1. *Let G be a *-semigroup in the C^*-algebra A which spans a norm dense *-subalgebra in it. A complex valued function f on A has a (necessarily unique) positive linear extension φ to A if and only if there exists a positive constant M such that*

$$\frac{1}{M}\left|\sum_g c_g f(g)\right|^2 \leqq \sum_{g,h} c_g \bar{c}_h f(h^* g) \leqq M \left\|\sum_g c_g g\right\|^2 \tag{1}$$

*holds for any finite sequence $\{c_g\}$ of complex numbers indexed by elements of the *-semigroup G.*

PROOF. The necessity of condition (1) is easy to prove, since if φ is a continuous linear functional extending f, we have

$$\left|\sum_g c_g f(g)\right|^2 = \left|\varphi\left(\sum_g c_g g\right)^2 \leqq \|\varphi\|\,\varphi\left(\left(\sum_g c_g g\right)^*\left(\sum_g c_g g\right)\right) =\right.$$

$$= \|\varphi\| \sum_{g,h} c_g \bar{c}_h \varphi(h^* g) \leqq \|\varphi\|^2 \left\|\sum_{g,h} c_g \bar{c}_h h^* g\right\| = \|\varphi\|^2 \left\|\sum_g c_g g\right\|^2$$

for any sequence $\{c_g\}$ in question.

To prove the sufficiency of (1), take the linear space F of all complex valued functions of finite support on G, the generating elements of which are functions δ_g taking value 1 in $g \in G$ and 0 otherwise, where g varies on G. By (1) the following relation defines a semi-inner product on F:

$$\left\langle \sum_h c_h \delta_h, \sum_k d_k \delta_k \right\rangle = \sum_{h,k} c_h \bar{d}_k f(k^* h) \tag{2}$$

for any two elements $\sum_h c_h \delta_h$ and $\sum_k d_k \delta_k$ of F, where $\sum$ denotes (as always in this note) a finite sum. Let H be the Hilbert space obtained from F by first factoring with respect to the nullspace of $\langle .,. \rangle$ and then by completing the arising pre-Hilbert space with respect to the norm obtained from the inner product on this factor space. For simplicity we denote the inner product by $\langle .,. \rangle$ and the image of a function δ_g $(g \in G)$ in H by δ_g, too. We have also by (1) a continuous linear functional given on a dense subset of H by

$$\sum_g c_g \delta_g \mapsto \sum_g c_g f(g). \tag{3}$$

This functional has a unique continuous extension on H and thus, by the Riesz Representation Theorem, has the form

$$\sum_g c_g f(g) = \left\langle \sum_g c_g \delta_g, x \right\rangle \tag{4}$$

with a unique element x in the Hilbert space H.

The shift operation S_g (for any g in G) given on F by

$$S_g\left(\sum_h c_h \delta_h\right) = \sum_h c_h \delta_{gh} \tag{5}$$

extends to a continuous linear operator on H. Indeed,

$$\left\|S_g\left(\sum_h c_h \delta_h\right)\right\|^2 = \sum_{h,k} c_h \bar{c}_k f(k^* g^* g h) =$$

$$= \left\langle S_{g^* g}\left(\sum_g c_h \delta_h\right), \sum_h c_h \delta_h \right\rangle \leqq \left\|S_{g^* g}\left(\sum_h c_h \delta_h\right)\right\| \left\|\sum_h c_h \delta_h\right\|$$

holds by the Schwarz inequality and, by a repeated use of this argument, we have by (1)

$$\|S_g(\sum_h c_h\delta_h)\|^{2^{n+2}} \leqq \|S_{(g^*g)^{2^{n-1}}}(\sum_h c_h\delta_h)\|^2 \|\sum_h c_h\delta_h\|^{2^{n+1}-2} =$$

$$= \sum_{h,k} c_h\bar{c}_k f(k^*(g^*g)^{2^n}h)\|\sum_h c_h\delta_h\|^{2^{n+1}-2} \leqq$$

$$\leqq M\|\sum_h c_h(g^*g)^{2^{n-1}}h\|^2\|\sum_h c_h\delta_h\|^{2^{n+1}-2} \leqq$$

$$\leqq M\|(g^*g)^{2^{n-1}}\|^2\|\sum_h c_h h\|^2\|\sum_h c_h\delta_h\|^{2^{n+1}-2} =$$

$$= M\|g\|^{2^{n+1}}\cdot\|\sum_h c_h h\|^2\|\sum_h c_h\delta_h\|^{2^{n+1}-2}$$

for $n=0, 1, 2, \ldots$, proving (by $n\to\infty$) that $\|S_g\|\leqq\|g\|$ for any g in G. A simple calculation shows that $S_g^*=S_{g^*}$ and $S_{gg'}=S_gS_{g'}$ for every g, g' in G. In other words $S: G\to B(H)$ is a *-representation of the *-semigroup G on the Hilbert space H, where $B(H)$ is the C^*-algebra of all bounded linear operators on H. We shall prove that

$$S_g x = \delta_g \tag{6}$$

for any g in G, and

$$\|\sum_g c_g S_g\| \leqq \|\sum_g c_g g\| \tag{7}$$

for any finite sequence $\{c_g\}$ indexed by elements of G. This implies that S has a unique continuous extension to A, which yields a *-representation of A on H which is cyclic by (6) with cyclic vector x in H.

Indeed, since

$$\langle\sum_h c_h\delta_h, S_g x\rangle = \langle S_g^*(\sum_h c_h\delta_h), x\rangle =$$

$$= \langle\sum_h c_h\delta_{g^*h}, x\rangle = \sum_h c_h f(g^*h) = \langle\sum_h c_h\delta_h, \delta_g\rangle$$

holds for any $\sum_h c_h\delta_h$ in H, a dense subset of H, (6) is proved. To prove (7) take and $y=\sum_h d_h\delta_h$ in H and an $a=\sum_g c_g g$ in A for which S_a denotes the operator $\sum_g c_g S_g$ on H. Then, similarly as before,

$$\|S_a y\|^2 = \langle S_{a^*a}y, y\rangle \leqq \|S_{a^*a}y\|\,\|y\|,$$

$$\|S_a y\|^{2^{n+1}} \leqq S_{(a^*a)^{2^{n+1}}}y\|^2\|y\|^{2^{n+1}-2} = \|\sum_{h,s} d_h\lambda_s\delta_{g_s h}\|^2\|y\|^{2^{n+1}-2} =$$

$$= \sum_{h,k,s,t} d_h\bar{d}_k\lambda_s\bar{\lambda}_t f(k^*g_t^*g_s h)\|y\|^{2^{n+1}-2} \leqq M\|\sum_{h,s} d_h\lambda_s g_s h\|^2\|y\|^{2^{n+1}-2} \leqq$$

$$\leqq M\|\sum_s \lambda_s g_s\|^2\|\sum_h d_h h\|^2\|y\|^{2^{n+1}-2} = M\|(a^*a)^{2^{n-1}}\|^2\|\sum_h d_h h\|^2\|y\|^{2^{n+1}-2} =$$

$$= M\|a\|^{2^{n+1}}\|\sum_h d_h h\|^2\|y\|^{2^{n+1}-2}$$

holds for $n=0, 1, 2, \ldots$, where $\sum_s \lambda_s g_s$ stands for $(a^*a)^{2^{n-1}}$. Thus we have, as $n \to \infty$, that $\|S_a y\| \leqq \|a\| \, \|y\|$, proving (7) because such y's are dense in H.

We have finally by (4) and (6) $f(g)=\langle \delta_g, x\rangle = \langle S_g x, x\rangle$ for any g in G, where $\varphi(a)=\langle S_a x, x\rangle$ $(a \in A)$ defines a positive linear functional on A, extending f. The proof is complete.

COROLLARY 1. *Let $f: A \to C$ be a complex valued function satisfying with a constant $M>0$ the inequalities*

$$\frac{1}{M}\left|\sum_a c_a f(a)\right|^2 \leqq \sum_{a,b} c_a \bar{c}_b f(b^*a) \leqq M \left\|\sum_a c_a a\right\|^2$$

for any finite sequence $\{c_a\}$ of complex numbers indexed by elements of A. f is then a positive linear functional, i.e. a state on A.

COROLLARY 2. *Let K be a compact subset of the real line $\mathbf{R}$ and let $\{\mu_n\}_{n=0}^{\infty}$ be a sequence of complex numbers. There is (a necessarily unique) positive measure μ on K such that*

$$\int_K t^n \, d\mu(t) = \mu_n \quad (n = 0, 1, 2, \ldots)$$

if and only if

$$0 \leqq \sum_{m,n} c_m \bar{c}_n \mu_{m+n} \leqq \mu_0 \max_{t \in K} \left|\sum_n c_n t^n\right|^2 \tag{8}$$

holds for any finite sequence $\{c_n\}$ of complex numbers.

PROOF. The necessity of (8) is clear. In order to verify the sufficiency we have to prove that

$$\left|\sum_n c_n \mu_n\right|^2 \leqq \mu_0 \sum_{m,n} c_m \bar{c}_n \mu_{m+n}$$

for any sequence $\{c_n\}$ in question. But the positive definiteness of $\{\mu_n\}_{n=0}^{\infty}$ in (8) gives this, due to the existence of an identity element, in the well-known manner. Here the *-semigroup $\{t^n\}_{n=0}^{\infty}$ $(t \in K)$ spans a norm-dense *-subalgebra in $C(K)$, the space of continuous functions on K, by the Stone—Weierstrass Theorem.

COROLLARY 3. *The sequence $\{\mu_n\}_{n=0}^{\infty}$ is a moment sequence for $[0, 1]$ if and only if*

$$0 \leqq \sum_{m,n} c_m \bar{c}_n \mu_{m+n} \leqq \mu_0 \max_{0 \leqq t \leqq 1} \left|\sum_n c_n t^n\right|^2$$

holds for any finite sequence $\{c_n\}$ of complex numbers.

COROLLARY 4. *Let K be a compact subset of the complex plane $\mathbf{C}$ and let $\{\mu_{m,n}\}_{m,n=0}^{\infty}$ be a double sequence of complex numbers. There exists a (necessarily unique) nonnegative measure μ on K with*

$$\int_K \lambda^m (\bar{\lambda})^n \, d\mu(\lambda) = \mu_{m,n} \quad (m, n = 0, 1, 2, \ldots)$$

if and only if

$$0 \leqq \sum_{i,j} c_i \bar{c}_j \mu_{m_i+n_j, n_i+m_j} \leqq \mu_{0,0} \max_{\lambda \in K} \left|\sum_i c_i \lambda^{m_i} (\bar{\lambda})^{m_i}\right|^2$$

holds for any finite sequence $\{c_i\}$ of complex numbers.

Assume that *-ideal I in G is given, that is a *-subsemigroup in G such that $ug, gu \in I$ holds for any g in I and u in G. Given a complex valued function f on I, it is natural to ask when an extension of f to a positive linear functional φ on A exists. The answer is the same as before when I was identical with G.

THEOREM 2. *The function $f: I \to \mathbf{C}$ has a positive linear extension φ on A if and only if* (1) *holds for any finite sequence $\{c_g\}$ of complex numbers indexed by elements of I.*

PROOF. We have only to prove the sufficiency of this condition. Going along the line of the proof of Theorem 1, just replacing G by I, we get a *-representation S of I on H with a cyclic vector x satisfying (4) and (6) for any g in I.

Now (5) gives a definition of S_g for any g in G, too (where h varies on I). The proof of $\|S_g\| \leqq \|g\|$ and then (7) remains also valid. Hence the proof of the theorem follows.

COROLLARY 5. *Given a sequence $\{\mu_n\}_{n=N}^{\infty}$ of complex numbers there exists a positive measure μ on* [0, 1] *such that*

$$\int_0^1 t^n \, d\mu(t) = \mu_n \quad (n = N, N+1, \ldots)$$

if and only if there is a positive constant M such that

$$\frac{1}{M} \Big| \sum_{n \geqq N} c_n \mu_n \Big|^2 \leqq \sum_{m,n \geqq N} c_m \bar{c}_n \mu_{m+n} \leqq M \max_{0 \leqq t \leqq 1} \Big| \sum_{n \geqq N} c_n t^n \Big|^2 \tag{9}$$

holds for any finite sequence $\{c_n\}_{n \geqq N}$ of complex numbers.

REMARK. The sequence $\{\mu_n\}_{n=1}^{\infty}$ where $\mu_n = 1/n$ $(n = 1, 2, \ldots)$ does not satisfy (9), since then

$$\sum_{m,n \geqq 1} c_m \bar{c}_n \mu_{m+n} = \int_0^1 \Big| \sum_{n \geqq 1} c_n t^n \Big|^2 \frac{dt}{t} \leqq M \max_{0 \leqq t \leqq 1} \Big| \sum_{n \geqq 1} c_n t^n \Big|^2$$

would be true. But since $\mu_n = \int_0^1 t^{n-1} \, dt$ $(n = 1, 2, \ldots)$, this seqŭence is a moment sequence in the sense of Hausdorff and Riesz.

References

[1] F. Hausdorff, Moment probleme für ein endliches Intervall, *Math. Z.*, **16** (1923), 220—248.

[2] F. Riesz—B. Sz.-Nagy, *Functional Analysis* (New York, 1960).

[3] F. H. Szafraniec, Sur les functions admettant une extension de type positif, *C. R. Acad. Sci. Paris.*

[4] F. H. Szafraniec, Boundedness of the shift operator related to positive definite forms: An application to moment problems, *Arkhiv för Math.*, **19** (1981), 251—9.

[5] Z. Sebestyén, On extendibility of *-representations from *-ideals, *Acta Sci. Math.* (*Szeged*), **40** (1978), 169—174.

(Received March 30, 1982)

DEPARTMENT OF MATH. ANALYSIS II
EÖTVÖS LORÁND UNIVERSITY
MUZEUM KÖRÚT 6—8.
1088 BUDAPEST, HUNGARY

Acta Math. Hung.
42 (3—4) (1983), 337—348.

METRIZATION AND LIAPUNOV FUNCTIONS. I

B. M. GARAY (Budapest)

I.

Let (X, d) be a meric space and $T_t: X \to X$, $t \in \mathbf{R}$, a dynamical system on X. For $\emptyset \neq Y \subset X$ and $x \in X$, let $d(x, Y) = \inf \{d(x, y) | y \in Y\}$. $d(x, Y)$ is the distance between x and Y. Let $\emptyset \neq M \subset X$ be an asymptotically stable closed invariant set. Denote by $A(M)$ its region of attraction. $A(M)$ is open. It is known [1] that there is a continuous function $V: A(M) \to \mathbf{R}^+$ satisfying the following conditions:

(i) $V(x) = 0$ for $x \in M$, and $V(x) > 0$ for $x \in A(M) \setminus M$,

(ii) for $x \in A(M)$, the function $k_x: \mathbf{R} \to \mathbf{R}^+$ defined by $k_x(t) = V(T_t x)$ is monotonically decreasing and tends to zero as t tends to infinity.

The construction of V is based on the metric d. V is called to be a monotone Liapunov function with respect to M. V can be chosen to satisfy the additional condition $V(x) \cong d(x, M)$ as well [1].

The following definition is implicitly contained in [5].

DEFINITION. Let ϱ be a metric defined on $A(M)$. We say that ϱ is a metric of Liapunov type if ϱ is topologically equivalent to $d|_{A(M)}$ and if the function $V: A(M) \to \mathbf{R}^+$ defined as $V(x) = \varrho(x, M)$ is a monotone Liapunov function with respect to M.

EXAMPLE. Suppose A is an $n \times n$ matrix and consider the differential equation $\dot{x} = Ax$. Assume that the eigenvalues of A satisfy $\operatorname{Re} \lambda < 0$. Let W be an $n \times n$ positive definite matrix. It is well-known that there is a positive definite matrix V satisfying the matrix equation $VA + A^*V = -W$ and that the function $\hat{V}: \mathbf{R}^n \to \mathbf{R}^+$ defined as $\hat{V}(x) = x^*Vx$ is a monotone Liapunov function with respect to the origin. It is clear that $\varrho: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^+$ defined by $\varrho(x, y) = (\hat{V}(x-y))^{1/2}$ is a metric of Liapunov type. (d is to be taken for the usual euclidean metric.) Thus, the classical theorem of Liapunov on quadratic forms can be interpreted as a theorem on metrics of Liapunov type.

The aim of the first part of this paper is to prove the following theorem on the existence of metrics of Liapunov type.

THEOREM 1. *Let (X, d) be a metric space and $T_t: X \to X$ a dynamical system on X. Let $\emptyset \neq M \subset X$ be an asymptotically stable closed invariant set. Assume that there is a retraction $r: A(M) \to M$ with the following property:*

(*) *for any $\{y_n\}_{n=1}^{\infty} \subset A(M)$, $x \in M$: $d(y_n, M) \to 0$ and $r(y_n) \to x$ imply $y_n \to x$.*

Finally, let V be a monotone Liapunov function on $A(M)$ with respect to M. Assume that for $x \in A(M)$, $V(x) \cong u(d(x, M))$ where $u: \mathbf{R}^+ \to \mathbf{R}^+$ is a continuous non-decreasing function, $u(0) = 0$ and $u(s) > 0$ for $s > 0$.

Then there exists a metric ϱ *of Liapunov type such that* $\varrho|_M = d|_M$ *and for* $x \in A(M)$, $V(x) = \varrho(x, M) = \varrho(x, r(x))$.

PROOF. The proof is an application of methods used in the proof of the Nagata—Smirnov—Bing metrization theorem [7]. For $n = 1, 2, 3, \ldots$ let

$$\mathscr{H}_n = \left\{ \left\{ y \in A(M) \middle| d(x, y) < \frac{1}{n} \right\} \middle| x \in A(M) \right\}.$$

$\mathscr{H}_n$ is an open covering of $A(M)$. Let $\mathscr{F}_n$ be a partition of unity subordinated to $\mathscr{H}_n$, i.e. $\mathscr{F}_n = \{f_\gamma^n : A(M) \to \mathbf{R}^+ \text{ continuous } | \gamma \in \Gamma_n\}$ where

(1) for any $x \in A(M)$ there is a neighbourhood $\mathscr{U}_n$ of x and there is a finite subset $\Delta_n \subset \Gamma_n$ such that for $\gamma \in \Gamma_n \setminus \Delta_n$, $\mathscr{U}_n \cap \operatorname{supp}(f_\gamma^n) = \varnothing$ i.e. $\mathscr{F}_n$ is locally finite.

(2) for any $\gamma \in \Gamma_n$ there is an $x \in A(M)$ such that

$$\operatorname{supp}(f_\gamma^n) \subset \left\{ y \in A(M) | d(x, y) < \frac{1}{n} \right\}$$

(3) for $x \in A(M)$, $\sum_{\gamma \in \Gamma_n} f_\gamma^n(x) = 1$.

Consequently, for $x \in A(M)$,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{\gamma \in \Gamma_n} f_\gamma^n(x) = 1$$

($\operatorname{supp}(f_\gamma^n)$ is defined as $\operatorname{cl}_{A(M)}\{y \in A(M) | f_\gamma^n(y) \neq 0\}$).

Let $g_\gamma^n : A(M) \to \mathbf{R}^+$ be defined as follows:

$$g_\gamma^n(x) = V(x) f_\gamma^n(x), \quad \gamma \in \Gamma_n, \quad n = 1, 2, 3, \ldots.$$

Now we are in a position to define ϱ. For $x, y \in A(M)$, let

$$\varrho(x, y) = d(r(x), r(y)) + \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{\gamma \in \Gamma_n} |g_\gamma^n(x) - g_\gamma^n(y)| +$$

$$+ \left| V(x) - V(y) - \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{\gamma \in \Gamma_n} (g_\gamma^n(x) - g_\gamma^n(y)) \right|.$$

It follows directly from (3) that $\varrho(x, y)$ is finite. By (2), $x \neq y$ implies $\varrho(x, y) \neq 0$. Therefore, ϱ is a metric on $A(M)$. For $x, y \in M$, $\varrho(x, y) = d(x, y)$. For $y \in M$, $x \in A(M)$, by (3), $\varrho(x, y) = d(r(x), y) + V(x)$. Taking infimum for $y \in M$, the infimum is attained if and only if $y = r(x)$. Therefore, $\varrho(x, r(x)) = \varrho(x, M) = V(x)$.

We prove that ϱ is topologically equivalent to $d|_{A(M)}$.

Fix an $x \in A(M)$ and let $\varepsilon > 0$ be arbitrarily small. By the continuity of V, there is a $\delta_1 > 0$ such that $d(x, y) < \delta_1$ implies $|V(x) - V(y)| < \varepsilon$. Further, there exists an $N_0 \in \mathbf{N}$ such that $d(x, y) < \delta_1$ implies

$$\sum_{n=N_0+1}^{\infty} \frac{1}{2^n} \sum_{\gamma \in \Gamma_n} |g_\gamma^n(x) - g_\gamma^n(y)| < \varepsilon.$$

Pick a $\delta_2>0$ such that $d(x, y)<\delta_2$ implies $y\in\bigcap_{n=1}^{N_0}\mathcal{U}_n$. It follows from (1) and from the continuity of the functions g_γ^k, $k=1, 2, \ldots, N_0$, $\gamma\in\Delta_k$ that there exists a $\delta_3<\delta_2$ such that for $y\in A(M)$, $d(x, y)<\delta_3$

$$\sum_{k=1}^{N_0}\frac{1}{2^k}\sum_{\gamma\in\Gamma_k}|(g_\gamma^k(x)-g_\gamma^k(y)|=\sum_{k=1}^{N_0}\frac{1}{2^k}\sum_{\gamma\in\Delta_k}|g_\gamma^k(x)-g_\gamma^k(y)|<\varepsilon.$$

Finally, there is a δ_4 such that $d(x, y)<\delta_4$ implies $d(r(x), r(y))<\varepsilon$. This is a consequence of the continuity of r.

Let $\delta=\min\{\delta_1, \delta_3, \delta_4\}$. Using the triangle inequality we obtain that $d(x, y)<\delta$ implies $\varrho(x, y)<6\varepsilon$.

Conversely, we show that for any $x\in A(M)$ and for any $\varepsilon>0$ there is a $\delta=$ $=\delta(x, \varepsilon)>0$ such that $\varrho(x, y)<\delta$ implies $d(x, y)<\varepsilon$. We distinguish two cases.

Case 1. Assume that $x\in A(M)\setminus M$. For $N\in\mathbb{N}\setminus\{0\}$, as a consequence of (2) and (3), $\sum_{\gamma\in\Delta_n}g_\gamma^N(x)=V(x)$ and for $\gamma\in\Delta_N$, $g_\gamma^N(y)=0$ provided that $d(x, y)>\frac{2}{N}$. Consequently, $d(x, y)>\frac{2}{N}$ implies

$$\varrho(x, y)\geqq\frac{1}{2^N}\sum_{\gamma\in\Delta_N}|g_\gamma^N(x)-g_\gamma^N(y)|=V(x)/2^N.$$

Therefore, $d(x, y)<2/N$ is implied by $\varrho(x, y)\leqq V(x)/2^N$.

Case 2. Assume that $x\in M$. Since $x\in M$, for $y\in A(M)$ $\varrho(x, y)=d(x, r(y))+$ $+V(y)$. Thus, $\varrho(x, y)<\delta$ implies $d(x, r(y))<\delta$ and $V(y)<\delta$. It follows easily from the conditions on V and r, precisely from property $(*)$ and from the inequality $V(x)\geqq u(d(x, M))$, that $d(x, r(y))<\delta$ and $V(y)<\delta$ imply $d(x, y)<\varepsilon$ if δ is sufficiently small. Q. e. d.

We emphasize the fact already mentioned that the existence of a function V satisfying the conditions of Theorem 1 is guaranteed (u can be taken for $\mathrm{id}_{\mathbf{R}^+}$).

REMARK. It follows directly from the definition of compactness and from a simple continuity argument that any retraction $r: A(M)\to M$ has property $(*)$ if M is compact.

Let $X=l_1$. l_1 be a Banachs pace with norm $\|x\|=\sum_1^\infty|x_i|$. Let $M=\{x\in l_1|\,\|x\|\leqq 1\}$. We construct a retraction $r: X\to M$ which does not have property $(*)$. In fact, let $g: X\to M$ be defined as follows:

$$g(x)=\begin{cases}x & \|x\|\leqq 1\\ x/\|x\| & \|x\|\geqq 1.\end{cases}$$

By the way, g is a retraction having property $(*)$. For $n=1, 2, 3, \ldots$ let $e^n=$

$=(0, 0, \ldots, 0, 1, 0, \ldots)$. Define a function $f_n: \mathbf{R} \to \mathbf{R}$ by

$$f_n(s) = \begin{cases} -1 & s \leqq -1 \\ s & -1 \leqq s \leqq 1 \\ n+1-ns & 1 \leqq s \leqq 1+\dfrac{1}{n} \\ 0 & 1+\dfrac{1}{n} \leqq s. \end{cases}$$

f_n is continuous. Define a function

$$f: X \to X,\ x \to f(x) = (f_1(x_1), f_2(x_2), f_3(x_3), \ldots).$$

f is continuous and $f|_M = \mathrm{id}_M$. For $n=1, 2, 3, \ldots$ $f((1+1/n)e^n)=0$. Finally, let $r=g \circ f$. r is a retraction of X onto M; and, for $n=1, 2, 3, \ldots$ $r((1+1/n)e^n)=0$. $d((1+1/n)e^n, M)=1/n \to 0$ although the sequence $\{(1+1/n)e^n\}_{n=1}^{\infty}$ is divergent.

It seems to be a rather difficult question whether there is a retraction $r: A(M) \to M$ having property $(*)$ provided M is a retract of $A(M)$ or not.

If M is a closed convex set of a uniformly convex Banach space, then the nearest point map is defined and continuous [10] and, as an immediate consequence, it is a retraction (of the entire space onto M) having property $(*)$.

A simpler version of Theorem 1 was stated in [5] where M and X were restricted to be compact and locally compact, respectively. The proof was based on Urysohn's metrization theorem.

A closed invariant set M is a repeller if it is asymptotically stable with respect to the new dynamical system $S_t: X \to X$, $t \in \mathbf{R}$, defined by $S_t = T_{-t}$. The region of repulsion is denoted by $R(M)$. The concepts of monotone Liapunov functions and of metrics of Liapunov type as well as Theorem 1 can be reformulated for repellers. In particular, a continuous function $V: R(M) \to \mathbf{R}^+$ is called to be a monotone Liapunov function if it satisfies the conditions

(i)′ $V(x)=0$ for $x \in M$ and $V(x)>0$ for $x \in R(M) \setminus M$,

(ii)′ for $x \in R(M)$, the function $k_x: \mathbf{R} \to \mathbf{R}^+$ defined by $k_x(t)=V(T_t x)$ is monotonically increasing and tends to zero as t tends to negative infinity.

The construction used in proving Theorem 1 can be applied in more general situations as well. From now on, assume that $\emptyset \neq M \subset X$ is a closed isolated invariant set [3], i.e. there is an open neighbourhood N of M such that M is not a proper subset of any invariant subset of N. The boundary of N is denoted by ∂N. Let us remark that a continuous function $W: N \to \mathbf{R}^+$ is called to be a hyperbolic Liapunov function if it satisfies the following conditions

(iii) there are upper semicontinuous functions $t_-, t_+: N \to (0, \infty)$ such that for any $x \in N$, $\{T_t x \mid -t_-(x) < t < t_+(x)\} \subset N$ and $T_{-t_-(x)} x \in \partial N$, $T_{t_+(x)} x \in \partial N$. N can be written as the disjoint union of H_1, H_2, H_3 and H_4 where

$$H_1 = \{x \in N \mid t_-(x) = t_+(x) = \infty\}, \quad H_2 = \{x \in N \mid t_-(x) < t_+(x) = \infty\},$$

$$H_3 = \{x \in N \mid t_+(x) < t_-(x) = \infty\}, \quad H_4 = \{x \in N \mid t_+(x) < \infty,\ t_-(x) < \infty\},$$

$W(x)=0$ for $x \in H_1$ and $W(x)>0$ for $x \in N \setminus H_1$.

(iv) for $x\in N$, define the function $l_x\colon (-t_-(x), t_+(x))\to \mathbf{R}^+$ by $l_x(t)=W(T_t x)$. For $x\in H_1$, l_x is identically zero. For $x\in H_2$, l_x tends to zero as $t\to\infty$ and l_x is decreasing. For $x\in H_3$, l_x tends to zero as $t\to-\infty$ and l_x is increasing. For $x\in H_4$, there is a $t^*(x)\in(-t_-(x), t_+(x))$ such that the function l_x is decreasing on $(-t_-(x), t^*(x))$ and increasing on $(t^*(x), t_+(x))$. Recall that $H_1=M$.

On the existence of hyperbolic Liapunov functions, see [12]. Assume that there is a retraction $r\colon N\to M$ having property $(*)$. Assume that W is a hyperbolic Liapunov function on N with respect to M and, for $x\in N$, $W(x)\cong u(d(x, M))$ where $u\colon \mathbf{R}^+\to\mathbf{R}^+$ is a continuous nondecreasing function, $u(0)=0$ and $u(s)>0$ for $s>0$. The method used in proving Theorem 1 gives us the following result.

THEOREM 2. *Under the previous conditions, there is a new metric ϱ on N, topologically equivalent to $d|_N$ such that for any $x\in N$, $W(x)=\varrho(x, M)$.*

All the previous results can be reformulated for discrete dynamical systems as well.

II.

We attempt to globalise the results of Part I. We prove a theorem which is possibly the simplest result in this direction.

First of all, we define two dynamical systems which will play an important role in the proof of Theorem 3.

Let X_1^* be the closed right hemisphere of radius two, i.e.

$$X_1^*=\left\{(2\cos\vartheta\cos\varphi, 2\cos\vartheta\sin\varphi, 2\sin\vartheta)\in\mathbf{R}^3\middle|-\frac{\pi}{2}\leqq\vartheta\leqq\frac{\pi}{2},\ 0\leqq\varphi\leqq\pi\right\}.$$

Define a metric d_1^* on X_1^* as follows: let the distance between two points of X_1^* be the infimum of euclidean length of arcs connecting each other — the spherical distance. Although trajectories do not determine dynamical systems uniquely, we define a dynamical system by its trajectories: any dynamical system having the prescribed trajectories will do. Thus, let T_*^1 be a fixed dynamical system on X_1^* satisfying the following conditions:

(4) The north pole $P=(0, 0, 2)$ is a source, the south pole $Q=(0, 0, -2)$ is a sink. There are two other critical points $R_1=(2, 0, 0)$, $R_2=(-2, 0, 0)$ as well. ∂X_1^*, the boundary of X_1^* consists of these four critical points and of four other trajectories issuing from P, P, R_1, R_2 and tending towards R_1, R_2, Q, Q, respectively. Any other trajectory can be described as $\varphi\equiv\text{constant}\in(0, \pi)$, i.e. it is a longitude issuing from P and tending towards Q.

LEMMA 1. *The metric d_1^* has the following properties:*

(a) *The function $V^Q\colon A(Q)\to\mathbf{R}^+$ defined by $V^Q(x)=d_1^*(x, Q)$ is a monotone Liapunov function with respect to Q.*

(b) *The function $V^P\colon R(P)\to\mathbf{R}^+$ defined by $V^P(x)=d_1^*(x, P)$ is a monotone Liapunov function with respect to P.*

(c) *For $i=1, 2$, there is an open neighbourhood $N_i\subset X_1^*$ of R_i such that the function $W^{R_i}\colon N_i\to\mathbf{R}^+$ defined by $W^{R_i}(x)=d_1^*(x, R_i)$ is a hyperbolic Liapunov function with respect to R_i.*

PROOF. (a) and (b) are obvious.

We carry out the proof of (c) for R_1. Define a function

$$l_\varphi: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbf{R}^+ \quad \text{by} \quad l_\varphi(\vartheta) = d_1^*(S_\varphi(\vartheta), R_1),$$

where $S_\varphi(\vartheta)=(2\cos\vartheta\cos\varphi, 2\cos\vartheta\sin\varphi, 2\sin\vartheta)$. Easy to see that

$$l_\varphi(\vartheta) = 4\arcsin\left(\frac{1-\cos\varphi\cos\vartheta}{2}\right)^{1/2}.$$

For $0<\varphi<\pi/2$, the function l_φ is monotonically decreasing on $(-\pi/2, 0]$, attains its minimum at $\vartheta=0$ — i.e. when the trajectory cuts the equator — and is monotonically increasing on $[0, \pi/2)$. Thus, for arbitrary $\delta\in(0,\pi)$, N_1 can be taken for $\{x\in X_1^* \mid d_1^*(x, R_1)<\delta\}$. It is worth mentioning that (c) follows from purely geometrical considerations as well. Q. e. d.

Let X_2^* be the union of the cylinder

$$E=\{(\cos\varphi, \sin\varphi, z)\in\mathbf{R}^3 \mid -\pi < \varphi \leqq \pi,\ 0 \leqq z \leqq 1\}$$

and of the cone

$$F=\{(r\cos\varphi, r\sin\varphi, 1+m(1-r))\in\mathbf{R}^3 \mid -\pi < \varphi \leqq \pi,\ 0 \leqq r \leqq 1\}$$

where m is a positive constant specified later. Define a metric d_2^* on X_2^* as follows: let the distance between two points of X_2^* be the infimum of euclidean length of arcs connecting each other. Let T_*^2 be a fixed dynamical system on X_2^* satisfying the following conditions:

(5) $P=(1, 0, 0)$ is a source, $Q=(0, 0, m+1)$ is a sink (or conversely, Q is a source and P is a sink). There is an other critical point $R=(-1, 0, 0)$ as well. ∂X_2^*, the boundary of X_2^* consists of R and of the source P and of two other trajectories issuing from P and tending towards R. Any other trajectory issues from P except one and tends to Q and cuts the circle $E\cap F$. We denote this intersection point by $S=(\cos s, \sin s, 1)$. If $s=0$, the trajectory through S is described as $\varphi=0$. If $s=\pi$, the trajectory through S is described as $\varphi=\pi$. If $s\in(0,\pi)$, the trajectory through S is given as

$$\left\{\left(\cos\varphi, \sin\varphi, \operatorname{tg}\frac{\varphi}{2}\Big/\operatorname{tg}\frac{s}{2}\right)\in E \,\middle|\, 0<\varphi\leqq s\right\}\cup$$

$$\cup\{(r\cos s, r\sin s, 1+m(1-r))\in F \mid 0<r\leqq 1\}.$$

If $s\in(-\pi, 0)$, the trajectory through S is given as

$$\left\{\left(\cos\varphi, \sin\varphi, \operatorname{tg}\frac{\varphi}{2}\Big/\operatorname{tg}\frac{s}{2}\right)\in E \,\middle|\, s\leqq\varphi<0\right\}\cup$$

$$\cup\{(r\cos s, r\sin s, 1+m(1-r))\in F \mid 0<r\leqq 1\}.$$

LEMMA 2. *There is an $m_0>0$ such that for $m\geqq m_0$ the metric d_2^* has the following properties:*

(a) *The function* $V^Q: A(Q) \to \mathbf{R}^+$ *defined by* $V^Q(x) = d_2^*(x, Q)$ *is a monotone Liapunov function with respect to* Q.

(b) *The function* $V^P: R(P) \to \mathbf{R}^+$ *defined by* $V^P(x) = d_2^*(x, P)$ *is a monotone Liapunov function with respect to* P.

(c) *There is an open neighbourhood* $N \subset X_2^*$ *of* R *such that the function* $W^R: N \to \mathbf{R}^+$ *defined by* $W^R(x) = d_2^*(x, R)$ *is a hyperbolic Liapunov function with respect to* R.

PROOF. (a) is obvious. We omit the proof of (c) since it is similar to the proof of (c) of Lemma 1. (N can be taken for $\{x \in X_2^* \mid d_2^*(x, R) < \delta\}$ for arbitrary $\delta \in (0, 1)$.) (a) and (c) are valid for any $m > 0$.

(b) is quite clear from geometrical intuition as well. However, for completeness, we carry out the proof of (b) in detail. Pick a point $S = (\cos s, \sin s, 1) \in E \cap F$ and consider the trajectory through S. Using symmetry, we may restrict ourselves to the case $0 \leqq s \leqq \pi$. The case when $s = 0$ is trivial. The case when $s = \pi$ can be excluded since the trajectory through $(\cos \pi, \sin \pi, 1)$ issues from R and not from P. Thus, we may assume that $0 < s < \pi$. Let s be fixed. We have to prove that $d_2^*(P, S_s^E(\varphi))$, where $S_s^E(\varphi) = \left(\cos \varphi, \sin \varphi, \operatorname{tg} \frac{\varphi}{2} \Big/ \operatorname{tg} \frac{s}{2}\right)$ is an increasing function of φ, $0 < \varphi \leqq s$. A direct calculation shows that

$$d_2^*(P, S_s^E(\varphi)) = \left(\varphi^2 + \operatorname{tg}^2 \frac{\varphi}{2} \Big/ \operatorname{tg}^2 \frac{s}{2}\right)^{1/2}$$

an increasing function, indeed.

Further, we have to prove that $H(r) = d_2^*(P, S_s^F(r))$ — where $S_s^F(r) = (r \cos s, r \sin s, 1 + m(1-r))$ — is a decreasing function of r, $0 < r < 1$. Let $Y(x) = (\cos x, \sin x, 1)$. Easy to see that

$$d_2^*(P, S_s^F(r)) = \inf \{d(x, r) \mid 0 \leqq x \leqq s\}$$

where

$$d(x, r) = d_2^*(P, Y(x)) + d_2^*(Y(x), S_s^F(r)).$$

A lengthy but straightforward calculation shows that

$$d(x, r) = (1 + x^2)^{1/2} + M\left(1 + r^2 - 2r \cos \frac{1}{M}(s - x)\right)^{1/2}$$

where $M = (1 + m^2)^{1/2}$. Differentiating with respect to x

$$d_x'(x, r) = \frac{x}{(1 + x^2)^{1/2}} - \frac{r \sin \frac{1}{M}(s - x)}{\left(1 + r^2 - 2r \cos \frac{1}{M}(s - x)\right)^{1/2}}.$$

Since $d_x'(0, r) < 0$ and $d_x'(s, r) > 0$, the infimum is attained at a point $x(r) \in (0, s)$ for which $d_x'(x(r), r) = 0$. We will show that for M being sufficiently large, equation

(6a) $$d_x'(x, r) = 0, \quad 0 < x < s$$

has exactly one solution $x=x(r)$. In fact, (6a) is equivalent to

$$\frac{x^2}{1+x^2}=\frac{r^2\left(1-\cos^2\frac{1}{M}(s-x)\right)}{1+r^2-2r\cos\frac{1}{M}(s-x)},$$

(6b) $$h(x, r)=0, \quad 0<x<s$$

where

$$h(x, r)=x^2-s^2+r^2(1+x^2)\cos^2\frac{1}{M}(s-x)-2rx^2\cos\frac{1}{M}(s-x).$$

A direct computation shows that

$$h'_x(x, r)=\frac{2r}{M}\sin\frac{1}{M}(s-x)\left\{\left(r\cos\frac{1}{M}(s-x)-1\right)(1+x^2)+1\right\}+$$

$$+2x\left(1-r\cos\frac{1}{M}(s-x)\right)^2.$$

For the sake of brevity, let $A=1-r\cos\frac{1}{M}(s-x)$; $\gamma=\frac{2r}{M}\sin\frac{1}{M}(s-x)$ $(0<\gamma\ll 1$, γ can be taken sufficiently small). Let

$$F_{\gamma,x}(A)=\gamma\{-A(1+x^2)+1\}+2xA^2, \quad 0<x<\pi,$$

then

$$F_{\gamma,x}(0)=\gamma>0,$$

$$F_{\gamma,x}(1)=-\gamma x^2+2x=x(2-\gamma x)\geqq x(2-\gamma\pi)>0,$$

$$F_{\gamma,x}(A)=2x\left(A-\gamma\frac{1+x^2}{4x}\right)^2+\gamma\left(1-\gamma(1+x^2)^2/8x\right).$$

The minimum of $F_{\gamma,x}$ is attained at $A=\gamma\frac{1+x^2}{4x}$. If $\gamma\frac{1+x^2}{4x}>1$, then $\min\limits_{0\leqq A\leqq 1}F_{\gamma,x}(A)>$ >0. If $\gamma\frac{1+x^2}{4x}\leqq 1$, then

$$\min_{0\leqq A\leqq 1}F_{\gamma,x}(A)=\gamma\left(1-\gamma(1+x^2)^2/8x\right)=$$

$$=\gamma\left(1-\gamma\frac{1+x^2}{4x}\frac{1+x^2}{2}\right)\geqq\gamma\left(1-(1+x^2)/2\right)=\gamma(1-x^2)/2$$

Therefore, in case of $x<1$, $\min\limits_{0\leqq A\leqq 1}F_{\gamma,x}(A)>0$. If $x\geqq 1$, then

$$F_{\gamma,x}(A)\geqq\gamma\{-A(1+\pi^2)+1\}+2A^2=2\left(A-\gamma\frac{1+\pi^2}{4}\right)^2+\gamma\left(1-\gamma(1+\pi^2)^2/8\right)>0.$$

Thus, for M being sufficiently large,

$$h'_x(x, r)>0, \quad 0<x<s<\pi, \ 0<r<1.$$

Easy to see that $h(0, r)<0$ and $h(s, r)>0$. Therefore, (6a) has exactly one solution $x=x(r)$. By the implicit function theorem, the function

$$x\colon (0, 1) \to (0, s) \subset \mathbf{R}^+, \quad r \to x(r)$$

is differentiable. Thus, $H(r)=d(x(r), r)$ is differentiable as well. We show that $H'(r)<0$.

$$H'(r) = d'_x(x(r), r)\cdot x'(r)+d'_r(x(r), r) = d'_r(x(r), r),$$

$$H'(r) = \frac{M\left(r-\cos\frac{1}{M}(s-x(r))\right)}{\left(1+r^2-2r\cos\frac{1}{M}(s-x(r))\right)^{1/2}}.$$

For any fixed $r\in(0, 1)$, consider the functions

$$f\colon [0, s] \to \mathbf{R},\ x \to \left(r-\cos\frac{1}{M}(s-x)\right)$$

$$g\colon [0, s] \to \mathbf{R},\ x \to h(x, r).$$

g is strictly increasing and $g(0)<0, g(s)>0$. f is strictly decreasing and $f(s)<0$. If $f(0)<0$, then $H'(r)<0$. Therefore, we may assume that $f(0)\geqq 0$. The unique solution of $f(x)=0$ is denoted by $x_0(r)$. Thus,

$$r = \cos\frac{1}{M}(s-x_0(r)), \quad 0 \leqq x_0(r) < s.$$

Substituting $x_0(r)$ into g we obtain

$$g(x_0(r)) = (r^2-1)\left(r^2-x_0^2(r)\,(1-r^2)\right).$$

For M being sufficiently large, the condition $f(0)\geqq 0$ implies that $r>99/100$. Thus, for M being sufficiently large, $g(x_0(r))<0$. The monotonicity of g implies $x_0(r)<x(r)$, which, in turn — using the monotonicity of f — implies $f(x(r))<0$. Therefore, $H'(r)<0$, $0<r<1$, which completes the proof of the Lemma. Q. e. d.

Let (S_2, d) be the two-dimensional unit sphere

$$\{(x_1, x_2, x_3)\in\mathbf{R}^3 \mid x_1^2+x_2^2+x_3^2 = 1\}$$

with the usual spherical metric. Let T be a Morse—Smale dynamical system on S_2. Recall that a Morse—Smale dynamical system can be characterised by the following properties: finitely many critical points, each hyperbolic, finitely many periodic orbits, each hyperbolic, there is no junction between saddle points [11].

Assume that T has no periodic orbits.

THEOREM 3. *Under the previous conditions, there is a new metric ϱ on S_2 topologically equivalent to d with the following properties:*

(a) *If Q is a sink of T, the function $V^Q\colon A(Q)\to\mathbf{R}^+$ defined by $V^Q(x)=\varrho(x, Q)$ is a monotone Liapunov function with respect to Q.*

(b) *If P is a source of T, the function $V^P\colon R(P)\to\mathbf{R}^+$ defined by $V^P(x)=\varrho(x, P)$ is a monotone Liapunov function with respect to P.*

(c) *If R is a saddle point, then there is a neighbourhood $N \subset S_2$ of R such that the function $W^R: N \to \mathbf{R}^+$ defined by $W^R(x) = \varrho(x, R)$ is a hyperbolic Liapunov function with respect to R.*

PROOF. If there are no saddle points, one has exactly one source and one sink, and the Theorem can be proved directly.

Step 1. Thus, we may assume that there is at least one saddle point. The set of separatrices is denoted by Σ. Σ consists of critical points and trajectories connecting saddle points with critical points of other types. The closure of the components of $S_2 \setminus \Sigma$ are called to be elementary cells. We denote them by $C_1, C_2, \ldots, C_N$. Define the set

$$G = \{x \in S_2 | x \in \partial C_i \quad \text{for some} \quad i \in \{1, \ldots, N\}\}$$

(∂C_i denotes the boundary of C_i). Using graph terminology, G is a connected spherical graph. Its vertices are critical points, $V(G) = \{P_1, P_2, \ldots, P_K\}$ its edges are trajectories connecting vertices, $E(G) = \{e_1, \ldots, e_L\}$. Adding both endpoints to e_j, we obtain simple rectifiable arcs $\tilde{e}_j$ on S_2, $j = 1, 2, \ldots, L$. $\tilde{e}_j$ has (euclidean) arc length α_j. For $x, y \in \tilde{e}_j$ we define $\Delta(x, y) = \pi \alpha_j^{-1} l(x, y)$ where $l(x, y)$ is the (euclidean) arc length of $\tilde{e}_j$ between x and y and $\pi = 3.14\ldots$. Especially, if P_i and P_k are connected by an edge, then $\Delta(P_i, P_k) = \pi$. Δ is a metric on each $\tilde{e}_j$, $j = 1, \ldots, L$. We extend Δ onto G as follows. For $x, y \in G$ let $\Delta(x, y) = \min \{\Delta(p_{x,y}) | p_{x,y}$ is a path joining x and y in $G\}$ where $\Delta(p_{x,y})$ is the length of the path $p_{x,y}$. Since the metric Δ is already defined on the edges of G, $\Delta(p_{x,y})$ is simply defined by addition. It follows directly from the above considerations, that Δ is a metric on G, topologically equivalent to $d|_G$.

Step 2. It is well-known [4] that there are only two types of elementary cells which are usually represented on the plane. However, in proving our Theorem, the usual planar representations of the elementary cells can not be used directly. Therefore, we had to slightly modify the usual representations. This was which we have done in the Lemmas.

For $i = 1, 2, \ldots, N$ there exists [4] an orbit preserving homeomorphism

$$h_i: C_i \to X_1^* \quad \text{or} \quad h_i: C_i \to X_2^*.$$

For $x, y \in C_i$, $i = 1, \ldots, N$ let

$$\varrho_i(x, y) = \begin{cases} d_1^*(h_i(x), h_i(y)) & \text{if} \quad h_i: C_i \to X_1^* \\ d_2^*(h_i(x), h_i(y)) & \text{if} \quad h_i: C_i \to X_2^*. \end{cases}$$

ϱ_i is a metric on C_i, topologically equivalent to $d|_{C_i}$.

We may define h_i on ∂C_i at first, then extend it to the entire C_i [8].

The circle of radius r is denoted by γ_r. The distance between any two points of γ_r is defined by arc length. From now on, γ_r is considered with this particular metric. It is worth to note that for $i = 1, \ldots, N$, ∂C_i with the metric Δ restricted to it is isometric to γ_1 or γ_2. This is a basic property of the metric Δ constructed before. Further, we remark that $(\partial X_1^*, d_1^*)$ is isometric to γ_2 and $(\partial X_2^*, d_2^*)$ is isometric to γ_1. If $\partial X_1^*, \partial X_2^*$ and ∂C_i, $i = 1, \ldots, N$ are considered with metrics d_1^*, d_2^* and Δ, respectively, the distance between any two neighbouring critical

points on $\partial X_1^*, \partial X_2^*, \partial C_i$, $i=1, \ldots, N$ is π. Therefore, h_i can be taken to have the additional property: for $x, y \in \partial C_i$,

$$\Delta(x, y) = \begin{cases} d_1^*(h_i(x), h_i(y)) & \text{if } h_i: C_i \to X_1^* \\ d_2^*(h_i(x), h_i(y)) & \text{if } h_i: C_i \to X_2^*. \end{cases}$$

Step 3. We have already defined metrics Δ and ϱ_i on G and C_i, respectively. It is clear that for $x, y \in G \cap C_i$, $\Delta(x, y) = \varrho_i(x, y)$. Since $C_i \cap C_j \subset G$, it follows that for $x, y \in C_i \cap C_j$, $\varrho_i(x, y) = \varrho_j(x, y)$.

Now we are in a position to define ϱ. We use an extension method of [2]. For $x, y \in C_i$, $i=1, \ldots, N$ let

$$\varrho(x, y) = \varrho_i(x, y).$$

For $x \in C_i$, $y \in C_j$, $i \neq j$ let

$$\varrho(x, y) = \inf \{\varrho_i(x, a) + \Delta(a, b) + \varrho_j(b, y) | a \in G \cap C_i, \quad b \in G \cap C_j\}.$$

It is rather lengthy but straightforward task to prove that ϱ is a metric on S_2. It is clear that ϱ is topologically equivalent to d.

If m is to be taken greater than m_0, then, by the Lemmas, ϱ has properties (a), (b) and (c) on any elementary cell C_i. Since C_i, $i=1, \ldots, N$ consists of entire trajectories, this proves that ϱ has properties (a), (b) and (c) on S_2, too. Q. e. d.

REMARK. Let X be a compact two-dimensional oriented manifold with Riemannian metric d. Assume that $\partial X = \varnothing$. Let T be a Morse—Smale dynamical system on X. Assume that T has no periodic orbits.

The conditions on T imply that T has critical points. If $X \neq S_2$, then, as a corollary of the index theorem [6], there must be saddle points among critical points.

Further, the elementary cells are of the same type as in the case $X = S_2$ [9]. Thus, the proof of Theorem 3 can be repeated (no alterations are needed).

We arrived at the conclusion that Theorem 3 is valid for closed oriented surfaces, too.

Acknowledgement

The author wishes to acknowledge valuable discussions with Prof. M. Farkas and A. P. Bosznay in preparing the paper.

References

[1] N. P. Bhatia, G. P. Szegő, *Dynamical Systems: Stability Theory and Applications*. Springer (Berlin—Heidelberg—New York, 1967).
[2] R. H. Bing, Extending a metric, *Duke Math. J.*, **14** (1947), 511—519.
[3] C. Conley, R. Euston, Isolated invariant sets and isolating blocks, *Trans. Amer. Math. Soc.*, **157** (1971), 1—27.
[4] H. F. DeBaggis, Dynamical systems with stable structure, in *Contributions to the Theory of Nonlinear Oscillations*, vol. 2, Princeton Univ. Press (Princeton, N. J., 1952), 37—59.
[5] B. M. Garay, Metrics and level surfaces of Liapunov functions, *ZAMM*, **61** (1981), 238—240.
[6] M. W. Hirsch, Differential topology, in *Graduate Texts in Mathematics* **33**, Springer (New York—Heidelberg—Berlin, 1976).

[7] J. L. Kelley, *General topology*. Van Nostrand (Princeton, 1955).
[8] L. Markus, Global structure of ordinary differential equations in the plane, *Trans. Amer. Math. Soc.*, **76** (1954), 127—148.
[9] M. M. Peixoto, Structural stability on two-dimensional manifolds, *Topology*, **1** (1962), 101—120.
[10] I. Singer, Best approximation in normed linear spaces by elements of linear subspaces, in *Die Grundlehren der Math. Wiss.* B. 171 Springer (Berlin—Heidelberg—New York, 1970).
[11] S. Smale, Differentiable dynamical systems, *Bull. Amer. Math. Soc.*, **73** (1967), 747—817,
[12] F. W. Wilson, J. A. Yorke, Liapunov functions and isolating blocks, *J. of Diff. Eq.*, **13** (1973), 106—123.

(Received April 5, 1982)

TECHNICAL UNIVERSITY
DEPARTMENT OF MATHEMATICS
H-1111 BUDAPEST,
STOCZEK U. 2—4

Acta Math. Hung.
42 (3—4) (1983), 349—354.

A REDUCED FREE PRODUCT OF DISTRIBUTIVE LATTICES. I

A. P. HUHN (Szeged)

This paper was inspired by the problem whether every distributive algebraic lattice can be represented as the congruence lattice of a lattice. However, here we do not attack this problem directly. Our "conjecture" is that the problem is closely connected with the theory of free distributive lattices and free distributive products. In the present paper we start a systematic study of those questions of the theory of free distributive products that we think relevant in this context. Applications to the representation problem are given in [2].

Let D_1 and D_2 be finite distributive lattices such that D_1 is isomorphic with a 0—$\vee$-subsemilattice of D_2. This isomorphism will be denoted by $\delta: a(\in D_1) \mapsto a^+$.

THEOREM 1. *There is a distributive lattice D containing D_1 as a 0-sublattice and D_2 as a 0—1-sublattice such that, for any element a in D_1, we have $a \leqq a^+$ in D. Furthermore, D can be chosen in such a way that, for any elements $a \leqq b \leqq c$ in D_1, b has a relative complement in $[a, c]$ in D only if it has a relative complement in $[a, c]$ in D_1.*

To prepare for the proof, define $D_1 \nearrow D_2$ as follows.

(i) There are a 0-homomorphism $\chi_1: D_1 \to D_1 \nearrow D_2$ anda 0—1-homomorphism $\chi_2: D_2 \to D_1 \nearrow D_2$ such that, for all a in D_1, $a\chi_1 \leqq a^+\chi_2$.

(ii) For any distributive lattice E, for any 0-homomorphism $\chi_1': D_1 \to E$ and 0—1-homomorphism $\chi_2': D_2 \to E$ (E bounded) satisfying $a\chi_1' \leqq a^+\chi_2'$ for all a in D_1, there exists a 0—1-homomorphism $\chi: D_1 \nearrow D_2 \to E$ such that $\chi_i\chi = \chi_i'$, $i = 1, 2$.

$D_1 \nearrow D_2$ obviously exists (the one element lattice satisfies (i)) and is unique, but we have no guarantee so far that D_1 and D_2 are mapped one-to-one into $D_1 \nearrow D_2$. To prove this and especially the second part of Theorem 1, we have to solve the word problem of $D_1 \nearrow D_2$. Now $D_1 \nearrow D_2$ is a homomorphic image of the 0—1-free product $D_1^b * D_2$, where D_1^b denotes the lattice D_1 with a new 1 adjoined. To distinguish the supremum of D_1 from this new bound, $\sup D_1$ will be denoted by ι. We also define $1^+ = 1$. Solving the word problem mentioned is equivalent with describing the congruence associated with the homomorphism of $D_1^b * D_2$ to $D_1 \nearrow D_2$ mapping D_i to $D_i\chi_i$, $i = 1, 2$. Let this homomorphism and the associated congruence be denoted by φ and θ, respectively. We think D_1, D_2 embedded in $D_1^b * D_2$, so the notations a, a^+ apply in $D_1^b * D_2$, too. First we describe a generating set of θ.

LEMMA 1. *θ is generated by the pairs $(j^+, j \vee j^+)$ where j runs over all the join-irreducibles of D_1.*

PROOF. It suffices to show that, for all a in D, $(a^+, a \vee a^+)$ is in the congruence generated by the $(j^+, j \vee j^+)$, and this is clear by

$$(a^+, a \vee a^+) = \left(\bigvee_{j \leqq a} j^+, \bigvee_{j \leqq a} j \vee \bigvee_{j \leqq a} j^+ \right) = \bigvee_{j \leqq a} (j^+, j \vee j^+)$$

where we utilized that $^+$ preserves joins.

To go further we need a description of $D_1^b * D_2$. Let P_2 denote the set of join irreducibles of D_2. Then $D_1^b * D_2$ consists of all the antitone mappings of P_2 to D_1^b, ordered by the component-wise ordering, that is $f \leqq g$ iff $f(k) \leqq g(k)$ in D_1 for all k in P_2. Then D_1^b is represented as the set of all constant mappings with values in D_1^b while D_2 is represented as the set of all characteristic functions, that is, functions with values 0, 1.

We are ready to solve the word problem of $D_1 \nearrow D_2$. Let P_i, $i = 1, 2$ and P_1^b denote the set of join irreducible elements of D_i, $i = 1, 2$ and D_1^b. For any k in P_2, define a filter $\mathscr{F}(k)$ in P_1 by

$$j \in \mathscr{F}(k) \Leftrightarrow k \leqq j^+.$$

Denote by $I(k)$ the set complement of $\mathscr{F}(k)$ in P_1. Then $I(k)$ is an ideal.

With any ideal I of P_1 let us associate the congruence $\theta(I)$ of D_1 generated by all pairs $(0, j)$, $j \in I$, or in other words by the pair $(0, \bigvee_{j \in I} j)$.

THEOREM 2. *For arbitrary elements f, g in $D_1^b * D_2$, $f \equiv g \pmod{\theta}$ iff for each k in P_2, $f(k) \equiv g(k) \pmod{\theta(I(k))}$ in D_1.*

PROOF. First we show that, for all j in P_1, $j^+ \equiv j \vee j^+$ modulo the congruence described in the theorem. We remind that P_1 is a subset of D_1 and D_1 is embedded in $D_1^b * D_2$. Our previous description of $D_1^b * D_2$ represents j as the mapping f_j with $f_j(k) = j$ for all $k \in P_2$. j^+ is represented by the mapping f_{j^+} where

$$f_{j^+}(k) = \begin{cases} 1 & \text{if } k \leqq j^+, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $j \vee j^+$ is represented by

$$f_{j \vee j^+}(k) = \begin{cases} 1 & \text{if } k \leqq j^+, \\ j & \text{otherwise.} \end{cases}$$

Obviously $f_{j^+}(k) \equiv f_{j \vee j^+}(k)$ provided $k \leqq j^+$. Assume $k \not\leqq j^+$. By the definition of $I(k)$, we have $j \in I(k)$. Therefore,

$$f_{j \vee j^+}(k) = j \equiv 0 = f_{j^+}(k) \pmod{\theta(I(k))}.$$

The relation "$f(k) \equiv g(k) \pmod{\theta(I(k))}$ for all k" is obviously a congruence, therefore it is now enough to show that θ contains this relation. Let, indeed,

$$f(k) \equiv g(k) \pmod{\theta(I(k))}$$

for all k in P_2. We may assume $f \leqq g$. It follows that

$$g(k) \leqq f(k) \vee \bigvee_{k \not\leqq j^+} j$$

for all $k \in P_2$. Indeed, $\bigvee_{k \not\leqq j^+} j$ is the greatest element of the ideal in D_1 generating $\theta(I(k))$, thus $f(k) \vee \bigvee_{k \not\leqq j^+} j$ is the greatest element of the $\theta(I(k))$-class of $g(k)$ in D_1. Therefore it is sufficient to prove that f is congruent with $f \vee h$ modulo θ where h is the function mapping $k \in P_2$ to $\bigvee_{k \leqq j^+} j$. (The problem that h is not an element of $D_1^b * D_2$ will be settled below.)

θ is the join of the congruences θ_j, by Lemma 1, where θ_j is the congruence generated by the pair $(f_{j^+}, f_{j \vee j^+})$ or by $(f_j, f_{j \wedge j^+})$. Now we can either use the description of the congruence generated by an interval or we can work in a bigger lattice, namely in the lattice $D_1^b * B_2$, where B_2 is the (unique) Boolean extension of D_2. The class of distributive lattices has the congruence extension property, thus we can return to $D_1^b * D_2$. This latter treatment is more convenient, therefore we shall take this approach. $D_1^b * B_2$ consists of all (not necessarily antitone) mappings of P_2 to D_1^b. Their ordering is also the componentwise ordering.

$$f_j(k) = \begin{cases} j & \text{if } k \leqq j^+, \\ j & \text{if } k \not\leqq j^+. \end{cases}$$

$$f_{j \wedge j^+}(k) = \begin{cases} j & \text{if } k \leqq j^+, \\ 0 & \text{if } k \not\leqq j^+. \end{cases}$$

Let

$$k_j(k) = \begin{cases} 0 & \text{if } k \leqq j^+, \\ j & \text{if } k \not\leqq j^+. \end{cases}$$

$h_j(k) \in D_1^b * B_2$. Forming the meet of h_j with f_j and with $f_{j \wedge j^+}$ we obtain that h_j is congruent with the zero mapping modulo θ_j. Now let

$$h = \bigvee_{j \in P_1} h_j.$$

Then h is congruent with the zero mapping modulo θ. An expression of h is

$$h(k) = \bigvee_{k \leqq j^+} j.$$

Hence $f \equiv f \vee h \pmod{\theta}$, as claimed.

PROOF OF THEOREM 1. The proof of the first statement of Theorem 1 still does not need the solution of the word problem, it is even easier directly. It is well known that D_1 and D_2 are embedded into $D_1^b * D_2$. Besides $D_1^b * D_2$ is embedded into the free Boolean product $B_1^b * B_2$ where B_2 is as before and B_1^b is the unique Boolean extension of D_1^b. Let $d, d' \in D_2$. We have to show that $d \neq d'$ implies $f_d \neq f_{d'}(\theta)$. We may assume $d \leqq d'$. For an element x of a Boolean algebra let $\bar{x}$ denote its complement. Suppose, indirectly, that for some $d \neq d'$ in D_2,

$f_d \equiv f_{d'} \pmod{\theta}$. This means

$$f_{d'} \wedge \bar{f}_d \leqq \bigvee_{j \in P_1} (f_j \wedge \bar{f}_j^+)$$

as the extension of θ to $B_1^b * B_2$ is generated by the right hand side. We shall deduce $d = d'$ from this last inequality. Now

$$f_{d'}(k) = \begin{cases} 1 & \text{if } k \leqq d', \\ 0 & \text{otherwise.} \end{cases}$$

$$f_d(k) = \begin{cases} 1 & \text{if } k \leqq d, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$(f_{d'} \wedge \bar{f}_d)(k) = \begin{cases} 1 & \text{if } k \leqq d', \quad k \nleqq d. \\ 0 & \text{otherwise.} \end{cases}$$

Assume $f_{d'} \wedge \bar{f}_d$ is not the zero mapping. Then there is a mapping c such that c has the value 1 at exactly one place k, $c \leqq f_{d'} \wedge \bar{f}_d$ and $c(k') = 0$ for all $k' \neq k$ (c is not necessarily antitone). Now let P_1^c consist of all $j \in P_1$ with $c \wedge f_j \wedge \bar{f}_{j+} \neq 0$, that is, $(c \wedge f_j \wedge \bar{f}_{j+})(k) \neq 0$. Obviously,

$$c \leqq f_{d'} \wedge \bar{f}_d \leqq \bigvee_{j \in P_1} (f_j \wedge \bar{f}_{j+}).$$

Hence

$$c \leqq \bigvee_{j \in P_1^c} (f_j \wedge \bar{f}_{j+}) \leqq \bigvee_{j \in P_1^c} f_j.$$

$\bigvee_{j \in P_1^c} f_j$ is a constant function and one of the values of c is 1, thus $\bigvee_{j \in P_1^c} f_j$ is identically 1. In other words, $\bigvee_{j \in P_1^c} j \geqq 1$ in D_1. Hence, utilizing that $^+$ preserves joins.

$$\bigvee_{j \in P_1^c} j^+ = \Big(\bigvee_{j \in P_1^c} j \Big)^+ \geqq 1^+$$

holds in D_2, that is $\bigvee_{j \in P_1^c} f_{j+}$ is the greatest element of D_2, whence $\bigwedge_{j \in P_1^c} f_{j+}$ is the zero mapping. Returning to c, $c \wedge f_j \wedge \bar{f}_{j+}$ is not the zero mapping, and neither is $c \wedge \bar{f}_{j+}$. Both c and $\bar{f}_{j+}$ being characteristic functions, and c being a minimal one, we have $c \leqq \bar{f}_{j+}$ for all j in P_1^c. Thus

$$c \leqq \bigwedge_{j \in P_1^c} \bar{f}_{j+} = 0,$$

a contradiction.

To show the analogous statement for D_1, let $d \leqq d'$, $d \neq d'$, $d, d' \in D_1$. Now the indirect hypothesis is

$$f_{d'} \wedge \bar{f}_d \leqq \bigvee_{j \in P_1} (f_j \wedge \bar{f}_{j+}).$$

Assume $f_{d'} \wedge \bar{f}_d \neq 0$, and choose a mapping c', such that $c' \leqq f_{d'} \wedge \bar{f}_d$, c' is a nonzero constant mapping and c' is minimal relative to these assumptions. Then

$$c' \leqq \bigvee_{j \in P_1^{c'}} (f_j \wedge \bar{f}_{j+})$$

where, in $P_1^{c'}$, we kept only those irreducibles $j \in P_1$ satisfying $c' \wedge f_j \wedge \bar{f}_{j^+} \neq 0$. Now

$$c' \leqq \bigvee_{j \in P_1^{c'}} \bar{f}_{j^+}.$$

c' is a constant, $c' \neq 0$, the $\bar{f}_{j^+}$ are characteristic functions, hence $\bigvee_{j \in P_1^{c'}} \bar{f}_{j^+}$ is identically 1. Therefore,

$$\bigwedge_{j \in P_1^{c'}} f_{j^+} = \overline{\bigvee_{j \in P_1^{c'}} \bar{f}_{j^+}}$$

which is the zero mapping. Hence, in the lattice D_2, we have $\bigwedge_{j \in P_1^{c'}} j^+ = 0$. Consider $\bigwedge_{j \in P_1^{c'}} j$ in D_1

$$\bigwedge_{j \in P_1^{c'}} j \leqq i$$

for all $i \in P_1^{c'}$. Therefore

$$(\bigwedge_{j \in P_1^{c'}} j)^+ \leqq i^+.$$

Hence

$$(\bigwedge_{j \in P_1^{c'}} j)^+ \leqq \bigwedge_{j \in P_1^{c'}} j^+ = 0.$$

Therefore,

$$\bigwedge_{j \in P_1^{c'}} j = 0.$$

Returning to c', we have $c' \wedge f_j \wedge \bar{f}_{j^+} \neq 0$, whence $c' \wedge f_j \neq 0$ for $j \in P_1^{c'}$. c' is a minimal constant mapping, f_j is constant, thus $c' \leqq f_j$. Summarizing,

$$c' \leqq \bigwedge_{j \in P_1^{c'}} f_j = 0,$$

a contradiction. This proves the first statement of the theorem.

Now we apply the solution of the word problem to the proof of the second assertion of Theorem 1. Let $a, b, c \in D_1$, $a \leqq b \leqq c$ and assume b has no relative complement in D_1 in $[a, c]$. In the language of join irreducibles this means the following. Let I_a, I_b, I_c denote the set of join-irreducibles below a, b, c, respectively. Then the relative set complement of I_b in $[I_a, I_c]$, that is, the set $(C(I_b) \cap I_c) \cup I_a$ ($C(I_b)$ denotes set complement relative to P_1) is not an ideal. Let $C'(I_b)$ be the ideal of P_1 generated by $C(I_b)$. $I_a \subseteq C'(I_b) \subseteq I_c$. $C(I_b)$ being a proper subset of $C'(I_b)$, there must exist an element $j \in C'(I_b)$ with $j \in I_b$. Assume j is maximal relative to this property. Let a be the join of all join-irreducibles i with $i \not\leqq j$. Then, in D_1, $j \not\geqq a$. Hence, in D_2, $j^+ \not\geqq a^+$. (Otherwise we would have $a^+ = j^+ \vee a^+ = (j \vee a)^+$ and $a \neq j \vee a$, a contradiction.) Thus, there is a join-irreducible $k \in P_2$ with $k \leqq j^+$, $k \not\leqq a$ for $j \not\leqq a$, and $k \leqq (\bigvee_{i \in I_c} i)^+$. Let $\theta(I(k))$ be also denoted by $\theta^{(k)}$. In $D_1/\theta^{(k)}$, $\theta^{(k)}(j)$ is the only minimal element, $\theta^{(k)}(a) = 0$ and $\theta^{(k)}(b) \geqq \theta^{(k)}(j)$. These imply that $\theta^{(k)}(b)$ cannot have any relative complement in any interval whose bounds are both different from $\theta^{(k)}(b)$. $\theta^{(k)}(c) > \theta^{(k)}(b)$, thus

$\theta^{(k)}(b)$ has no relative complement in $[\theta^{(k)}(a), \theta^{(k)}(c)]$. $\theta^{(k)}(a)=\theta(f_a)$, $\theta^{(k)}(b)=$ $=\theta(f_b)$, $\theta^{(k)}(c)=\theta(f_c)$ in D_1 where f_a, f_b, f_c denote the elements of $D_1^b * D_2$ representing a, b, c, respectively. Hence b does not have a complement in $[a, c]$ either.

References

[1] G. Grätzer, *Lattice Theory: First Concepts and Distributive Lattices*, Freeman and Co. (San Francisco, 1971).
[2] A. P. Huhn, On the representation of distributive algebraic lattices. I, to appear.
[3] P. Pudlák, On the congruence lattices of lattices. Preprint.
[4] E. T. Schmidt, The ideal lattice of a distributive lattice with 0 is the congruence lattice of a lattice, *Acta Sci. Math. Szeged*, **43** (1981), 153—168.

(Received April 26, 1982)

JÓZSEF ATTILA UNIVERSITY
BOLYAI INSTITUTE
SZEGED. ARADI VÉRTANÚK TERE 1.
H-6720

Acta Math. Hung.
42 (3—4) (1983), 355—360.

DERIVATIVES TAKE EVERY VALUE ON THE SET OF APPROXIMATE CONTINUITY POINTS

G. PETRUSKA (Budapest)

1. Introduction

The typical behaviour of Baire 1 functions and some subclasses has recently been studied in [1] and [2]. It is a general phenomenon that, roughly speaking these functions map large sets onto small ones. A typical "typical" result is, for instance,

> "the set of functions $f\in\mathscr{F}$ such that $\operatorname{rng} f|_{C_f}$ is a nowhere dense null set is a residual subset in $\mathscr{F}$".

Here $\mathscr{F}$ may denote the Banach space of bounded functions consisting of Baire 1 functions, Darboux Baire 1 functions, derivatives or approximately continuous functions. In the first two classes the same result holds also for $\operatorname{rng} f|_{A_f}$. The behaviour of bounded derivatives on A_f remained open. I mentioned the problem whether $\operatorname{rng} f|_{A_f}=\operatorname{rng} f$ holds for typical $f\in b\Delta$ (bounded derivatives) on the Lodz summer school on real analysis (1981), and it was answered by D. Preiss affirmatively (communicated in a private letter). In this paper we are going to prove that this result is more than typical, it holds for *every* $f\in b\Delta$, moreover for every derivative.

2. Notations, definitions and some simple remarks

— For any measurable subset $H\subset[0, 1]$ its Lebesgue measure is denoted by $|H|$.

— For any function f defined on $[0, 1]$ we denote the set of continuity and approximate continuity points by C_f and A_f, respectively.

— Let $\{I_n\}$ $(n=0, 1, 2, \ldots)$ be a sequence of intervals. A point x is said to be the strong limit point of $\{I_n\}$, if $\operatorname{dist}(x, I_n)\to 0$ $(n\to\infty)$ and $\frac{1}{|I_n|}\operatorname{dist}(x, I_n)\leqq M$, where M is independent of n.

— Let P be a perfect set, and let $\mathscr{M}$ denote a family of intervals. $\mathscr{M}$ is called everywhere dense in P, if for any open interval J, $P\cap J\neq\emptyset$ implies $I\subset J$ for some $I\in\mathscr{M}$. Observe, that if P is a nowhere dense perfect set, then the open intervals contiguous to P form a family everywhere dense in P.

$\mathscr{M}$ is said to be nowhere dense in P, if for every open interval J with $J\cap P\neq\emptyset$ there exists a subinterval $J'\subset J$ with $J'\cap P\neq\emptyset$ and $J'\cap I=\emptyset$ for any $I\in\mathscr{M}$.

Observe that, if $\mathscr{M}$ is *not* nowhere dense in $P\neq\emptyset$, and $I\cap P=\emptyset$ for $I\in\mathscr{M}$, then $\mathscr{M}$ is everywhere dense in a portion of P, that is in a perfect set $Q=J\cap P\neq\emptyset$, where J is an open interval with endpoints out of P.

The simple assertions below are well-known. For the reader's convenience we provide the proofs.

PROPOSITION 1. *Let x be the strong limit point of $\{I_n\}=\{(a_n, b_n)\}$. If F is differentiable in x, then*

$$\lim_{n\to\infty}\frac{F(b_n)-F(a_n)}{b_n-a_n}=F'(x).$$

PROOF. For a given n we have

$$F(b_n)=F(x)+F'(x)(b_n-x)+\varepsilon(b_n)(b_n-x),$$

$$F(a_n)=F(x)+F'(x)(a_n-x)+\varepsilon(a_n)(a_n-x).$$

By subtraction

$$\left|\frac{F(b_n)-F(a_n)}{b_n-a_n}-F'(x)\right|\leqq|\varepsilon(b_n)|\frac{|b_n-x|}{b_n-a_n}+|\varepsilon(a_n)|\frac{|a_n-x|}{b_n-a_n}\leqq$$

$$\leqq(|\varepsilon(a_n)|+|\varepsilon(b_n)|)(1+M),$$

since both $|b_n-x|$ and $|a_n-x|\leqq|I_n|+\operatorname{dist}(x, I_n)$. Hence the assertion follows.

PROPOSITION 2. *Let $f\geqq 0$ be measurable on $[a, b]$, let $\varepsilon>0$, $\eta>0$ be given numbers, and denote*

$$L_\eta=\{x\in[a, b]: f(x)\geqq\eta\}.$$

If $\int_a^b f(x)\,dx<\varepsilon\eta(b-a)$, then $|L_\eta|<\varepsilon(b-a)$. In particular, if F is differentiable in $[a, b]$ and $f(x)=F'(x)\geqq 0$ then $F(b)-F(a)<\varepsilon\eta(b-a)$ implies $|L_\eta|<\varepsilon(b-a)$.

PROOF. $\varepsilon\eta(b-a)>\int_a^b f(x)\,dx\geqq\int_{L_\eta}f(x)\,dx\geqq\eta|L_\eta|$ and hence we get our assertion.

3. Lemmas

LEMMA 1. *Let $P\neq\varnothing$ be a nowhere dense perfect set and let $H\subset P$ be a G_δ subset everywhere dense in P. Let $\mathcal{M}$ denote a family of intervals contiguous to P. If $\mathcal{M}$ is everywhere dense in P then there exists $x\in H$ and a sequence $I_n\in\mathcal{M}$ such that x is a strong limit point of $\{I_n\}$.*

PROOF. Let $H=\bigcap_{n=1}^{\infty}H_n$, where H_n is open and $H_n\cap P$ is everywhere dense in P. Let $I_1=(a_1, b_1)\in\mathcal{M}$ be an arbitrary element of $\mathcal{M}$, and denote $I_1'=(b_1, 2b_1-a_1)=I_1+(b_1-a_1)$. We certainly have $I_1'\cap P\neq\varnothing$ and hence we can choose an open interval $J_1\subset H_1$ such that $J_1\subset I_1'$ and $J_1\cap P\neq\varnothing$. Thus there exists $I_2\in\mathcal{M}$ with $\operatorname{cl} I_2\subset J_1$. Suppose that the intervals $I_1, I_2, \ldots, I_n; J_1, J_2, \ldots, J_{n-1}$ have been chosen such that

(i) $I_k\in\mathcal{M}$ and $(k=1, \ldots, n)$;
(ii) $J_k\subset H_k$ $(k=1, \ldots, n-1)$ and $\operatorname{cl} J_k\subset J_{k-1}$ $(k=2, \ldots, n-1)$;
(iii) $\operatorname{cl} I_k\subset J_{k-1}\subset I_{k-1}'$ $(k=2, \ldots, n)$ where $I_{k-1}'=I_{k-1}+|I_{k-1}|$.

Now we translate I_n by its length to the right and get $I_n'=I_n+|I_n|$. Since $I_n' \cap P \neq \emptyset$, we can find an open interval $J_n \subset H_n$, $\operatorname{cl} J_n \subset J_{n-1}$ such that $J_n \subset I_n'$ and $J_n \cap P \neq \emptyset$. Thus we can select $I_{n+1} \in \mathcal{M}$ with $\operatorname{cl} I_{n+1} \subset J_n$. By induction we obtain the sequences $I_1, \ldots; J_1, \ldots$ satisfying (i)—(ii)—(iii) for every index k. Since $\{\operatorname{cl} J_k\}$ is a nested sequence of closed intervals, we can pick $x \in \bigcap_{k=1}^{\infty} \operatorname{cl} J_k$ (actually, this intersection contains just one point x, because I_k runs through intervals contiguous to P and hence $|I_k|$ tends to zero, and by (iii) the same holds for $|J_k|$). Now $x \in \operatorname{cl} J_k \subset J_{k-1} \subset H_{k-1}$, thus $x \in \bigcap H_k$, that is $x \in H$. By (iii) we have $x \in I_k'$ for every $k=1, 2, \ldots$, and hence $\operatorname{dist}(x, I_k) \leqq |I_k'| = |I_k|$, $\frac{1}{|I_k|} \operatorname{dist}(x, I_k) \leqq 1$. Thus x is a strong limit point of $\{I_n\}$.

LEMMA 2. *Let $P \neq \emptyset$ be a nowhere dense perfect set and let $H \subset P$ be a G_δ subset everywhere dense in P. Let $\mathcal{M}$ denote a family of intervals and suppose $\mathcal{M} = \bigcup_{k=1}^{\infty} \mathcal{M}_k$, $\mathcal{M}_k$ is nowhere dense in P $(k=1, 2, \ldots)$. Then there exists a point $x \in H$ such that for every n a $\delta > 0$ can be found with $(x-\delta, x+\delta) \cap I = \emptyset$ for any $I \in \mathcal{M}_k$ $(k \leqq n)$.*

PROOF. The proof runs on a similar way as in Lemma 1. Let $H = \bigcap_{n=1}^{\infty} H_n$, where H_n is an open set, $H_n \cap P$ is everywhere dense in P. Let $J_1 \subset H_1$ be an arbitrary open interval with $J_1 \cap P \neq \emptyset$. Then we can find another open interval $I_1 \subset J_1$ such that $I_1 \cap I = \emptyset$ for any $I \in \mathcal{M}_1$, but $I_1 \cap P \neq \emptyset$. Now we can select J_2 such that $J_2 \cap P \neq \emptyset$, $\operatorname{cl} J_2 \subset I_1$, $J_2 \subset H_2$, and a subinterval $I_2 \subset J_2$ with $I_2 \cap P \neq 0$ but $I_2 \cap I = \emptyset$ for any $I \in \mathcal{M}_1 \cup \mathcal{M}_2$ etc. By induction we obtain $I_1, \ldots$; $J_1, \ldots$ such that $H_k \supset J_k \supset I_k \supset \operatorname{cl} J_{k+1}$ $(k=1, 2, \ldots)$ and $I_k \cap I = 0$ for $I \in \mathcal{M}_1 \cup \ldots \cup \ldots \cup \mathcal{M}_k$. Let $x \in \bigcap_{k=1}^{\infty} \operatorname{cl} J_k$. Then $x \in H$ (in particular $x \in P$), and $x \in I_k$ $(k=1, 2, \ldots)$. Hence our assertion is obvious.

4. The result

Now we state and prove our main result.

THEOREM. *Let f be a derivative. Then f takes each of its values on A_f.*

PROOF. The primitive of f is denoted by F. It is obviously enough to prove $Z \cap A_f \neq \emptyset$, if $Z = \{x: f(x)=0\} \neq \emptyset$. Let x_0 be a right isolated point of the set Z. We show that f is approximately continuous from the right in x_0. Our derivative f cannot change its sign in a neighbourhood (x_0, x_0+h) (Darboux property), thus we can suppose $f>0$ in this interval. Suppose that f were not approximately continuous from the right in x_0. Then we could find a number $\delta>0$ and intervals $I_n=[x_0, x_0+h_n]$ such that $|L_\delta \cap I_n| \geqq \delta h_n$, where $L_\delta = \{x: f(x) \geqq \delta\}$. Proposition 2 (applied with $\varepsilon=\eta=\delta$) gives

$$F(x_0+h_n)-F(x_0) \geqq \delta^2 h_n$$

and hence the contradiction

$$0 = f(x_0) = \lim \frac{1}{h_n}[F(x_0+h_n)-F(x_0)] \geqq \delta^2$$

proves our assertion. In particular, if x_0 is an isolated point of Z, then $x_0 \in A_f$, that is $Z \cap A_f \neq \varnothing$ and we can finish the proof of the theorem. Thus we suppose now, that Z has no isolated points. We can also suppose that Z is nowhere dense (otherwise we have the stronger result $Z \cap C_f \neq \varnothing$, both Z and C_f being G_δ and everywhere dense in an open subinterval). Therefore $P = \operatorname{cl} Z$ is a nowhere dense perfect set. Let C denote the continuity points of f relative in P, that is the points where the restricted function $f|_P$ is continuous. C and Z are everywhere dense and G_δ subsets in P, and hence $H = Z \cap C$ is of the same character. By the Darboux property and the definition of Z, f cannot change its sign on any of the intervals (a, b) contiguous to P, thus we can call them positive or negative intervals according to sign f. For a given $\delta > 0$ let $\mathcal{M}_\delta^+$ denote the family of intervals (a, b), which are contiguous to P and

$$F(b)-F(a) \geqq \delta(b-a).$$

We prove that $\mathcal{M}_\delta^+$ is nowhere dense in P. If this were not the case, then $\mathcal{M}_\delta^+$ would be everywhere dense in a portion $Q = I \cap P \neq \varnothing$. Then we apply Lemma 1 and find $x \in Z$, and a sequence $\{(a_n, b_n)\}$ from $\mathcal{M}_\delta^+$ such that x is a strong limit point of $\{(a_n, b_n)\}$. Now by Proposition 1, we have

$$\delta \leqq \left| \frac{F(b_n)-F(a_n)}{b_n-a_n} \right| \to 0,$$

a contradiction. Similar result holds for $\mathcal{M}_\delta^-$ that is for the family of contiguous intervals with $F(b)-F(a) \leqq -\delta(b-a)$. We put $\mathcal{M}_k = \mathcal{M}_{1/k}^+ \cup \mathcal{M}_{1/k}^-$, and apply Lemma 2 for $\mathcal{M} = \bigcup_{k=1}^{\infty} \mathcal{M}_k$; we obtain $x_0 \in H = C \cap Z$ such that for any n there exists a neighbourhood U_n of x_0 with $U_n \cap I = \varnothing$ for any $I \in \mathcal{M}_k$ $(k \leqq n)$. We prove, that $x_0 \in A_f$. Let $\eta > 0$ be fixed, and consider $L_\eta = \{x : f(x) \geqq \eta\}$. We show that L is of density zero in x_0. Let $\varepsilon > 0$ be given. Choose the neighbourhood $U = (x_0-\delta, x_0+\delta)$ such that

(i) $I \notin \mathcal{M}_k \left(k \geqq \frac{2}{\varepsilon\eta}+1\right)$ for any interval I contiguous to P and $I \cap U \neq \varnothing$;

(ii) $|f(x)-f(x_0)| = |f(x)| < \eta$ holds for $x \in U \cap P$ (this can be done by $x_0 \in C$);

(iii) $|F(x)-F(x_0)| < \eta \frac{\varepsilon^2}{4} |x-x_0|$ $(|x-x_0| < \delta)$ (this is possible by $F'(x_0) = f(x_0) = 0$).

Now let $0 < h < \delta$ and consider $[x_0, x_0+h]$.

(a) If $x_0+h \in P$ then by (ii), $|L_\eta \cap [x_0, x_0+h]| = \sum_{k=1}^{\infty} |L_\eta \cap I_k|$ where I_k $(k = 1, 2, \ldots)$ runs through the positive intervals $I_k \subset (x_0, x_0+h)$. Let $I_k = (a_k, b_k)$, then, referring to (i) we have $F(b_k)-F(a_k) < \frac{\varepsilon\eta}{2}(b_k-a_k)$, and hence $|L_\eta \cap I_k| <$

$< \frac{\varepsilon}{2}(b_k - a_k)$ by Proposition 2. Thus $|L_\eta \cap [x_0, x_0+h]| < \frac{\varepsilon}{2} \sum_{k=1}^{\infty} |I_k| \leqq \frac{\varepsilon h}{2}$.

(b) If $a < x_0+h < b$, where (a, b) is a negative interval, then $|L_\eta \cap [x_0, x_0+h]| = |L_\eta \cap [x_0, a]| < \frac{\varepsilon}{2}(a-x_0) < \frac{\varepsilon h}{2}$ by the case (a).

(c) Let $a < x_0+h < b$, where (a, b) is a positive interval. We consider two cases.

(c1) $x_0+h-a < \frac{1}{2}\varepsilon h$. Then, applying (a) we obtain

$$|L_\eta \cap [x_0, x_0+h]| = |L_\eta \cap [x_0, a]| + |L_\eta \cap [a, x_0+h]| <$$

$$< \frac{\varepsilon}{2}(a-x_0) + x_0 + h - a < \frac{\varepsilon}{2}(a-x_0) + \frac{\varepsilon}{2} h < \varepsilon h.$$

(c2) $x_0+h-a \geqq \frac{1}{2}\varepsilon h$. Now we have

$$|F(x_0+h) - F(x_0)| < \eta \frac{\varepsilon^2}{4} h, \quad |F(a) - F(x_0)| < \eta \frac{\varepsilon^2}{4}(a-x_0),$$

by (iii). Thus

$$0 < \frac{F(x_0+h)-F(a)}{x_0+h-a} < \frac{1}{4}\eta\varepsilon^2 \left(\frac{h}{x_0+h-a} + \frac{a-x_0}{x_0+h-a} \right) \leqq \frac{1}{4}\eta\varepsilon^2 \left(\frac{2}{\varepsilon} + \frac{2}{\varepsilon} \right) = \eta\varepsilon.$$

Therefore Proposition 2 gives

$$|L_\eta \cap [a, x_0+h]| < \varepsilon(x_0+h-a).$$

Hence

$$|L_\eta \cap [x_0, x_0+h]| = |L_\eta \cap [x_0, a]| + |L_\eta \cap [a, x_0+h]| <$$

$$< \frac{\varepsilon}{2}(a-x_0) + \varepsilon(x_0+h-a) < \varepsilon h.$$

Thus we obtain $|L_\eta \cap [x_0, x_0+h]| < \varepsilon h$ in all cases, and the same result holds for $L_\eta \cap [x_0-h, x_0]$ as well. This shows that

$$\lim_{h \to +0} \frac{1}{h} |L_\eta \cap [x_0, x_0+h]| = \lim_{h \to +0} \frac{1}{h} |L_\eta \cap [x_0-h, x_0]| = 0.$$

Similar argument applies for the density of $K_\eta = \{x : f(x) \leqq -\eta\}$, which proves $x_0 \in A_f$ and our theorem.

REMARK. It follows from a result of O'Malley and Weil, that our Theorem holds for approximate derivatives as well ([3], Theorem 5.1).

References

[1] J. Ceder and G. Petruska, Most Darboux Baire 1 functions map big set onto small, *Acta Math. Hung.*, **41** (1983), 37—46.

[2] A. M. Bruckner and G. Petruska, Some typical results on bounded Baire 1 functions (to appear).

[3] R. J. O'Malley, Clifford E. Weil, The oscillatory behavior of certain derivatives, *Transactions of AMS*, **234** (1977), 467—481.

(Received April 30, 1982)

EÖTVÖS LORÁND UNIVERSITY
DEPARMENT OF ANALYSIS
BUDAPEST, MÚZEUM KRT. 6—8.
H-1088

PRINTED IN HUNGARY
Szegedi Nyomda, Szeged

Instructions for authors. Manuscripts should be typed on standard size papers (25 rows; 50 characters in each row). Mathematical symbols (except sin, log, etc.) will be set in italics, so *use handwritten letters, or indicate them by simple underlining.* The text of theorems, lemmas, corollaries are to be printed also in italics. The words THEOREM, LEMMA, COROLLARY, PROOF, REMARK, PROBLEM, etc. (or their German and French equivalents) should be typed in capital letters. Sub-headings will be set in boldface lower case; indicate them by red underlining. Other special types of symbols (such as German, Greek, script, etc.) should also be clearly identified. Mark footnotes by consecutive superscript numbers. When listing references, please follow the following pattern:

[1] G. Szegő, *Orthogonal polynomials,* AMS Coll, Publ. Vol. XXIII (Providence, 1939).
[2] A. Zygmund, Smooth functions, *Duke Math. J.,* **12** (1945), 47—76.

For abbreviation of names of journals follow the Mathematical Reviews. After the references give the author's affiliation.

Do not send abstract and second copy of the manuscript. Authors will receive only galley-proofs (one copy). Manuscripts will not be sent back to authors (neither for the purpose of proof-reading nor when rejecting a paper).

Authors obtain 100 reprints free of charge. Additional copies may be ordered from the publisher.

Periodicals of the Hungarian Academy of Sciences are obtainable at the following addresses:

AUSTRALIA
C.B.D. LIBRARY AND SUBSCRIPTION SERVICE
Box 4886, G.P.O., *Sydney N.S.W. 2001*
COSMOS BOOKSHOP, 145 Ackland Street
St. Kilda (Melbourne), Victoria 3182

AUSTRIA
GLOBUS, Höchstädtplatz 3, *1206 Wien XX*

BELGIUM
OFFICE INTERNATIONAL DE LIBRAIRIF
30 Avenue Marnix, *1050 Bruxelles*
LIBRAIRIE DE MONDE ENTIER
162 rue du Midi, *1000 Bruxelles*

BULGARIA
HEMUS, Bulvar Ruszki 6, *Sofia*

CANADA
PANNONIA BOOKS, P.O. Box 1017
Postal Station „B", *Toronto, Ontario M5T 2T8*

CHINA
CNPICOR, Periodical Department, P.O. Box 50 *Peking*

CZECHOSLOVAKIA
MAD'ARSKÁ KULTURA, Národní třida 22
115 66 Praha
PNS DOVOZ TISKU, Vinohradská 46, *Praha 2*
PNS DOVOZ TLAČE, *Bratislava 2*

DENMARK
EJNAR MUNKSGAARD, Norregade 6
1165 Copenhagen K

FEDERAL REPUBLIC OF GERMANY
KUNST UND WISSEN ERICH BIEBER
Postfach 46, *7000 Stuttgart 1*

FINLAND
AKATEEMINEN KIRJAKAUPPA, P.O. Box 128
SF-00101 Helsinki 10

FRANCE
DAWSON-FRANCE S. A., B. P. 40, *91121 Palaiseau*
EUROPÉRIODIQUES S. A., 31 Avenue de Versailles, *78170 La Celle St. Cloud*
OFFICE INTERNATIONAL DE DOCUMENTATION ET LIBRAIRIE, 48 rue Gay-Lussac *75240 Paris Cedex 05*

GERMAN DEMOCRATIC REPUBLIC
HAUS DER UNGARISCHEN KULTUR
Karl Liebknecht-Straße 9, *DDR-102 Berlin*
DEUTSCHE POST ZEITUNGSVERTRIEBSAMT
Straße der Pariser Kommüne 3-4, *DDR-104 Berlin*

GREAT BRITAIN
BLACKWELL'S PERIODICALS DIVISION
Hythe Bridge Street, *Oxford OXI 2ET*
BUMPUS, HALDANE AND MAXWELL LTD.
Cowper Works, *Olney, Bucks MK46 4BN*
COLLET'S HOLDINGS LTD. Denington Estate *Wellingborough, Northants NN8 2QT*
WM. DAWSON AND SONS LTD., Cannon House *Folkstone, Kent CT19 5EE*
H. K. LEWIS AND CO., 136 Gower Street
London WC1E 6BS

GREECE
KOSTARAKIS BROTHERS INTERNATIONAL
BOOKSELLERS, 2 Hippokratous Street, *Athens-143*

HOLLAND
MEULENHOFF-BRUNA B.V, Beulingstraat 2, Amsterdam
MARTINUS NIJHOFF B.V.
Lange Voorhout 9-11, *Den Haag*
SWETS SUBSCRIPTION SERVICE
347b Heereweg, *Lisse*

INDIA
ALLIED PUBLISHING PRIVATE LTD., 13/14
Asaf Ali Road, *New Delhi 110001*
150 B-6 Mount Road, *Madras 600002*
INTERNATIONAL BOOK HOUSE PVT. LTD.
Madame Cama Road, *Bombay 400039*
THE STATE TRADING CORPORATION OF INDIA LTD., Books Import Division, Chandralok
36 Janpath, *New Delhi 110001*

ITALY
INTERSCIENTIA, Via Mazzé 28, *10149 Torino*
LIBRERIA COMMISSIONARIA SANSONI, Via Lamarmora 45, *50121 Firenze*
SANTO VANASIA, Via M. Macchi 58 *20124 Milano*
D. E. A., Via Lima 28, *00198 Roma*

JAPAN
KINOKUNIYA BOOK-STORE CO. LTD.
17-7 Shinjuku 3 chome, Shinjuku-ku, *Tokyo 160-91*
MARUZEN COMPANY LTD., Book Department, P.O. Box 5050 Tokyo International, *Tokyo 100-31*
NAUKA LTD. IMPORT DEPARTMENT
2-30-19 Minami Ikebukuro, Toshima-ku, *Tokyo 171*

KOREA
CHULPANMUL, *Phenjan*

NORWAY
TANUM-TIDSKRIFT-SENTRALEN A.S.,
Karl Johansgatan 41–43, *1000 Oslo*

POLAND
WĘGIERSKI INSTYTUT KULTURY,
Marszalkowska 80, *00-517 Warszawa*
CKP IW, ul. Towarowa 28, *00-958 Warszawa*

ROUMANIA
D. E. P., *Bucureşti*
ILEXIM, Calea Grivitei 64–66, *Bucureşti*

SOVIET UNION
SOJUZPECHAT — IMPORT, *Moscow*
and the post offices in each town
MEZHDUNARODNAYA KNIGA, *Moscow G-200*

SPAIN
DIAZ DE SANTOS, Lagasca 95, *Madrid 6*

SWEDEN
GUMPERTS UNIVERSITETSBOKHANDEL AB
Box 346, *401 25 Göteborg 1*

SWITZERLAND
KARGER LIBRI AG., Petersgraben 31, *4011 Basel*

USA
EBSCO SUBSCRIPTION SERVICES
P.O. Box 1943, *Birmingham, Alabama 35201*
F.W. FAXON COMPANY, INC.
15 Southwest Park, *Westwood Mass. 02090*
READ-MORE PUBLICATIONS, INC.
140 Cedar Street, *New York, N.Y. 10006*

YUGOSLAVIA
JUGOSLAVENSKA KNJIGA, Terazije 27, *Beograd*
FORUM, Vojvode Mišića 1, *21000 Novi Sad*

Index: 26.014