FORMERLY ACTA MATHEMATICA ACADEMIAE SCIENTIARUM HUNGARICAE

Acta Mathematica Hungarica

VOLUME 41, NUMBERS 1–2, 1983

Akadémiai Kiadó, Budapest

ACTA MATH. HUNG. HU ISSN 0001—5954

ACTA MATHEMATICA
HUNGARICA

Acta Mathematica publishes papers on mathematics in English, German, French and Russian.

Acta Mathematica is published in two volumes of four issues a year by

AKADÉMIAI KIADÓ
Publishing House of the Hungarian Academy of Sciences
H-1054 Budapest, Alkotmány u. 21.

Manuscripts and editorial correspondence should be addressed to

Acta Mathematica, H-1395 Budapest, P.O.B. 428

Orders may be placed with

KULTURA Foreign Trading Company
H-1389 Budapest P.O. Box 149

or with its representatives abroad.

Acta Mathematica is indexed in Current Contents

ACTA MATHEMATICA HUNGARICA

VOLUME 41

AKADÉMIAI KIADÓ, BUDAPEST

1983

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VOLUME 41

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NOTE OF THE EDITOR

The name of our periodical Acta Mathematica Academiae Scientiarum Hungaricae has been abbreviated to Acta Mathematica Hungarica according to a decision of the Hungarian Academy of Sciences. This modification does not effect the status and the editorial policy of the journal.

Acta Math. Hung.
41 (1—2), (1983), 3—15.

SOME PROPERTIES OF THE CATEGORY OF INTEGRAL DOMAINS

E. FRIED (Budapest)

To the memory of my teacher P. Turán

1. Introduction. A *binding* category is a category C such that there is a full embedding (for the definition see Section 2) $\mathrm{Gra} \to C$ of the category of graphs into C (and therefore there is a full embedding $A \to C$ of any category A of algebras [H $-$ P$_1$], [H $-$ L]). Let Int_0 denote the category of integral domains of characteristic zero with unit elements together with their (unit preserving) homomorphisms. Int_0 has been proved to be binding in [F $-$ S$_2$]. In particular, there are large integral domains with prescribed endomorphism monoids. The integral domains constructed are quite particular, however. For example, they contain no algebraic elements apart from the integers. Therefore, the following natural problem was raised by J. Sichler [S]:

(∗) Given an integral domain I of characteristic zero and a monoid M, does there exist an integral domain J containing I such that $\mathrm{End}\,(J) \cong M$?

This question was answered in [F] for the special case when M is a group. In the present paper we give an affirmative answer to this problem and some stronger versions of it.

Analogous problems (endomorphisms and category of structures containing a given substructure) have been investigated for unary algebras [K$_1$], [K$_2$], lattices [A $-$ S], and graphs [B $-$ N].

The methods employed in this paper are, partly, from earlier papers of Fried and Sichler [F $-$ S$_1$], [F $-$ S$_2$]. In 1976 the author proposed a related problem, not solvable by these methods, at the mathematical competition of the students of Hungarian universities. It has turned out that the proposer's and the solver's methods can be generalized to settle Sichler's mentioned problem.

THEOREM 1.1. *Let I be an integral domain of characteristic zero. Then the category of integral domains containing I is binding.*

Above, as throughout this paper, our integral domains possess unit elements and these are preserved under homomorphisms by definition.

One can ask how these homomorphisms act on I. Of course, in a simple construction proving Theorem 1 it is natural to choose these actions to be trivial (the identity map). For small categories, however, one can essentially prescribe the action of the homomorphisms on I. (Recall that a category is *small* if its objects form a set.) We prove even more.

THEOREM 1.2. *Let $\mathscr{K}$ be a small category. For any functor $\Phi\colon \mathscr{K} \to \mathrm{Int}_0$ there exists a full embedding $\Psi\colon \mathscr{K} \to \mathrm{Int}_0$ such that $\Phi(X) \subseteq \Psi(X)$ for every $X \in \mathrm{Ob}\,(\mathscr{K})$ and for every morphism $\varphi\colon X \to Y$, the homomorphism $\Phi(\varphi)$ is the restriction of $\Psi(X)$ to $\Phi(X)$. In addition, one can require that all objects in the range of Ψ have the same cardinality.*

The following problem has been raised by L. Babai [B]. Let $\mathscr{R}$ be a set of rings with unit element such that each of these rings has the same additive group A. These rings together with their unit-preserving homomorphisms form a (small) category. The question was which categories have such a representation. We prove that every small category can be represented this way even by integral domains.

THEOREM 1.3. *Given a small category $\mathscr{K}$, there exists an Abelian group A and a full embedding $\mathscr{K}\to\mathrm{Int}(A)$ where $\mathrm{Int}(A)$ denotes the category of integral domains having A for their additive group.*

In the study of endomorphisms we exclude characteristic p because of the trouble caused by Frobenius endomorphisms $x\mapsto x^p$. The category Int_p of integral domains of characteristic p is never binding because of these ever present central endomorphisms One can, however, form a factor category $\widehat{\mathrm{Int}_p}$ by setting $\varphi\equiv\psi$ if $\varphi(x)^{p^n}=\varphi(x)^{p^m}$ for suitable positive integers n, m and all x in $\mathrm{dom}\,\varphi=\mathrm{dom}\,\psi$ (φ, ψ any morphisms in Int_p with common domain and range). The author and J. Kollár conjecture the following.

CONJECTURE 1.5. The factor category $\widehat{\mathrm{Int}_p}$ is binding. Moreover, for any integral domain I of characteristic p, the category $\widehat{\mathrm{Int}_p}(I)$ is binding (the objects of $\widehat{\mathrm{Int}_p}(I)$ are those integral domains which contain I and the morphisms are the morphisms out of $\widehat{\mathrm{Int}_p}$).

2. The main result. Before stating the main result we have to quote some definitions and results.

A *concrete category* is a category $\mathscr{K}$ together with an "underlying set" functor $\square\colon \mathscr{K}\to\mathrm{Sets}$. The map $\square\colon \mathrm{Hom}(X, Y)\to\mathrm{Hom}(\square X, \square Y)$ has to be one-to-one for each $X, Y\in\mathrm{Ob}(\mathscr{K})$ ($\square$ is faithful).

A functor $F\colon \mathscr{K}\to\mathscr{L}$ is a *full embedding* if it is an embedding (i.e. one-to-one on objects and morphisms), and it is full (i.e., the map $F\colon \mathrm{Hom}(X, Y)\to\mathrm{Hom}(F(X), F(Y))$ is onto for each $X, Y\in\mathrm{Ob}(\mathscr{K})$).

Let $(\mathscr{K}, \square)$ and $(\mathscr{L}, \square')$ be concrete categories. A full embedding $F\colon \mathscr{K}\to\mathscr{L}$ is a *pseudorealization* if $\square X\subseteq\square' F(X)$ $(X\in\mathrm{Ob}(\mathscr{K}))$ and for any morphism $\varphi\colon X\to Y$ in $\mathscr{K}$, $\square\varphi$ is the restriction of $\square' F(\varphi)$ to $\square X$. A pseudorealization is *exclusive* if the image of the "excess" $\square' F(x)-\square X$ under $\square' F(\varphi)$ is disjoint from $\square Y$, i.e., it is a subset of the "excess" $\square' F(Y)\setminus\square Y$.

Clearly, composition of (exclusive) pseudorealizations is (exclusive) pseudorealization as well.

The following result of Hedrlin, Pultr and Vopenka [V—P—H], [H—P_2] is fundamental for the theory of full embeddings.

THEOREM 2.1. *Every category of general algebras (infinitary operations and relations admitted) has an exclusive pseudorealization in the category of graphs.*

(Graphs are symmetric irreflexive binary relations.)

We need the following two observations on small categories.

PROPOSITION 2.2. *Every small category $\mathscr{K}$ can be endowed with a faithful set functor $\square$.*

Thus, in case of small categories we only have to deal with concrete categories.

PROPOSITION 2.3. *Every small concrete category has an exclusive pseudorealization in the category of graphs.*

For the proofs see, e.g., [H—L].

Under an assumption on set theory much more powerful results have been proved by L. Kučera (see: [P—T] pp. 99—100) and Z. Hedrlin [H], [Ku].

(M) *Assumption.* The class of mesurable cardinals is a set. (In other words, there is a cardinal greater than any measurable cardinal.)

THEOREM 2.4. (Kučera, Hedrlin). *Assuming* (M), *every concrete category has an exclusive pseudorealization in* Gra *(the category of graphs).*

DEFINITION. Let $(\mathscr{K}, \square)$ and $(\mathscr{L}, \square')$ be concrete categories and $\Phi\colon \mathscr{K}\to\mathscr{L}$ be a functor. We denote by $\mathscr{K}\cup\Phi(\mathscr{K})$ the concrete category $(\mathscr{K}, \square'')$ defined by $\square''(\varphi)=\square(\varphi)\dot{\bigcup}\square'\Phi(\varphi)$ $(\varphi\in\mathscr{K})$ (where $\dot{\bigcup}$ stands for disjoint union).

Now, we are able to state the main result of the paper.

THEOREM 2.5. *Let* $(\mathscr{K}, \square)$ *be a concrete category and* $\Phi\colon \mathscr{K}\to\mathrm{Int}_0$ *a functor such that*

(i) $|\Phi(X)|$ *is bounded* $(X\in\mathrm{Ob}\,\mathscr{K})$.

Assume further that

(ii) *there exists an exclusive pseudorealization* $P\colon \mathscr{K}\cup\Phi(\mathscr{K})\to\mathrm{Gra}$.

Then there exists a full embedding $\psi\colon \mathscr{K}\to\mathrm{Int}_0$ *such that* $\Phi(X)$ *is a subdomain of* $\psi(X)$ *and* $\Phi(\varphi)$ *is the restriction of* $\Psi(\varphi)$ *to* $\Phi(X)$ $(\varphi\colon X\to Y\in\mathscr{K})$.

In addition, we may require that for any given small subcategory $\mathscr{K}_0$ *of* $\mathscr{K}$ $\Psi(X)$ *have the same cardinatily when* X *ranges over* $\mathrm{Ob}\,\mathscr{K}_0$.

REMARK 2.6. (i) *and* (ii) *are automatically satisfied for small categories. (For* (ii) *use Proposition* 2.3.*)*

REMARK 2.7. *Under* (M) *condition* (ii) *is always satisfied by the theorem of Kučera and Hedrlin. (Theorem* 2.4.*).*

We would like to get rid of the somewhat sophisticated condition (ii) without assuming (M). We do not know the answer to the following

PROBLEM 2.8. *Let* $\Phi\colon \mathrm{Gra}\to\mathrm{Sets}$ *be a functor such that* $|\Phi(X)|$ *is bounded for all graphs X. Does there exist an (exclusive) pseudorealization* $\mathrm{Gra}\cup\Phi(\mathrm{Gra})\to\mathrm{Gra}$?

(Of course, we may assume that (M) is false.)

A positive answer to this problem would make the assumption (ii) in Theorem 2.5 superfluous.

Next we derive Theorems 1.1 through 1.3 from Theorem 2.5.

PROOF OF THEOREM 1.1. Let us apply Theorem 2.5 with $\mathscr{K}=\mathrm{Gra}$, $\Phi(X)=I$ $(X\in\mathrm{Ob\,Gra})$, $\Phi(\varphi)=\mathrm{id}_I$ $(\varphi\in\mathrm{Mor\,Gra})$ (i) is trivially satisfied. (ii) is an obvious consequence of Theorem 2.1. The conclusion is a full embedding $\Psi\colon \mathrm{Gra}\to\mathrm{Int}_0$ such that $\Psi(X)\supseteq I$ for every X. ∎

PROOF OF THEOREM 1.2. $\mathscr{K}$ can be viewed as a concrete category by Proposition

2.2. (i) and (ii) of Theorem 2.5 are satisfied by Remark 2.6. The conclusion of 2.5 restates that of 1.2 ∎

PROOF OF THEOREM 1.3. We apply Theorem 1.2. Let $I=Q$, the field of rationals. Let us define the functor $\Phi\colon \mathscr{K}\to\mathrm{Int}_0$ by setting $\Phi(X)=I$ for all $X\in\mathrm{Ob}\,\mathscr{K}$. Then $\Phi(\varphi)=\mathrm{id}_Q$ is automatically satisfied for each $\varphi\in\mathscr{K}$. The integral domains $\Psi(X)$ obtained by Theorem 1.2 all have the same cardinality $\varkappa$ and contain Q. Therefore they are $\varkappa$-dimensional vector-spaces over Q hence their additive groups are isomorphic and may be identified. ∎

Let us mention that the construction used in the proof of Theorem 2.5 will give us that A dependes only on the least upper bound of the cardinalities of $\square(X)$ $(X\in\mathrm{Ob}\,\mathscr{K})$.

The method used in the proof of Theorem 2.5 is based on the fact that every homomorphism of integral domains preserves two properties of the elements:

(a) The element is invertible.

(b) The element belongs to the p-component of the multiplicative semigroup of nonzero elements and has infinite height in this component.

In the proof of Theorem 2.5 we shall extend the integral domains of the form $I=\Phi(X)$ in many steps. In the first step we extend it by variables over I. This variables will represent the vertices of the graph $P(X)$ and some auxiliary variables which will be the vertices of a rigid graph. The second extension will make these auxiliary variables infinitely high so they will be "recognizable". In the third step we extend the ring by inverses such that we will be able to recognize the vertices and the edges of the graph $P(X)$. According to this way we shall break the proof into three parts: 1. Which elements are recognizeble and how can we recognize them. 2. How to construct the proper integral domain by means of graphs. 3. The actual proof of Theorem 2.5.

3. Some properties of integral domains. Let I be an integral domain (with unit 1 and char $I=0$) and let X be a set of variables over I. Consider the polynomial-ring $I(X)$. If X' is the least subset of X such that $f\in I[X']$, for some given $f\in I[X]$ then we shall say that the elements of X' *occur* in f and the elements of $X\setminus X'$ *do not occur* in f. If there are disjoint subsets X_1, X_2 of X such that $f\in I[X_1]$ and $g\in I[X_2]$ then we shall call f and g *disjoint polynomials.*

An element a of an integral domain R will be called a *power* (in R) if there exists an element $b\in R$ and an integer $k\geqq 2$ such that $a=b^k$.

PROPOSITION 3.1. *Let $c\neq 0$ be an element of the integral domain I; K the algebraic closure of the quotient field of I; $f\in I[X]$; $x_1, \dots, x_r\in X$ elements not occurring in f; $k_1, \dots, k_r$ non-negative integers with positive sum. Define the polynomial $F(x_1, \dots, x_r)$ by*

$$F(x_1, \dots, x_r)=c\cdot x_1^{k_1}\cdot \ldots \cdot x_r^{k_r}-f.$$

Then we have:

(i) *If $f\neq 0$ then F is not a power in $K[X]$.*

(ii) *If f is not a power in $K[X]$ then F is irreducible in $K[X]$.*

PROOF. (i) For $r=1$, F and its derivative are mutually prime, thus F is not a power in $K[X]$, indeed. In case $r\geqq 2$ the condition $F=G^k$ implies $F(x, \dots, x)=$ $=G(x, \dots, x)^k$ yielding, by the special case, $k=1$. Hence (i) is proven.

(ii) Since K contains all roots of unity, in case $r=1$ the reducibility of F would imply that f is a power in $K[X]$, which is not the case. Suppose $r \geqq 2$, $F=G \cdot H$. Consider $F'=F(x_1, \ldots, x_1)$, $G'=G(x_1, \ldots, x_1)$, $H'=H(x_1, \ldots, x_1)$. Now, we have $F'=G' \cdot H'$ and:

$$\deg F = \deg G + \deg H, \quad \deg F' = \deg G' + \deg H'$$

$$\deg F = \deg F', \quad \deg G \geqq \deg G', \quad \deg H \geqq \deg H'.$$

These yield, using the special case, that one of deg G and deg H equals zero, i.e., F must be irreducible. ∎

NOTATION 3.2. Let X be a set of variables over I and let L be the algebraic closure of (the quotient field of) $I[X]$. For each natural number i and for each $x \in X$ we choose and fix a unique ${}_i x \in L$, satisfying ${}_0 x = x$ and $({}_i x)^2 = {}_{i-1} x$ for $i > 0$ (we could choose any other fixed prime in the exponent instead of 2). Let, further, ${}_i X = \{{}_i x \mid x \in X\}$ and ${}_\omega X = \bigcup {}_i X$ $(i=0, 1, \ldots)$. We shall denote by $I[{}_\omega X]$ the subring of L generated by ${}_\omega X$, i.e., the union of the ascending chain of the polynomial rings $I[{}_i X]$. For disjoint sets X and Y of variables over I, $I[{}_\omega X \cup Y]$ will denote the subring of the algebraic closure of $I[X \cup Y]$ generated by $I[{}_\omega X] \cup I[Y]$.

DEFINITION 3.3. *Let I, X, Y be as in Notation 3.2. A subset H of $I[{}_\omega X \cup Y]$ will be called normal if H does not contain any element of I and the elements of H are not divisible by any element of ${}_\omega X \cup Y$ (in $I[{}_\omega X \cup Y]$).*

NOTATION 3.4. Let I, X, Y be as in Notation 3.2. and let H be a normal subset of $I[{}_\omega X \cup Y]$. We shall denote by $I[{}_\omega X \cup Y/H]$ the subring of the quotient field of $I[{}_\omega X \cup Y]$ generated by $I[{}_\omega X \cup Y] \cup H^{-1}$ (H^{-1} consists of the multiplicative inverses of the elements of H). Further, $I[{}_i X \cup Y/H]$ will denote that subring of the above field which is generated by $I[{}_i X \cup Y] \cup H_i^{-1}$, where H_i stands for $H \cap I[{}_i X \cup Y]$.
We have, clearly:

PROPOSITION 3.5. *For any field K the unique factorization holds in $K[{}_i X \cup Y/H]$.* ∎

In virtue of Proposition 3.5 we shall consider the elements of $K[{}_i X \cup Y/H]$ as quotient of two mutually prime elements of $K[{}_i X \cup Y]$.
Proposition 3.5 yields, also:

PROPOSITION 3.6. *If K is a field then an element of $K[{}_i X \cup Y/H]$ is invertible in $K[{}_i X \cup Y/H]$ iff both the numerator and the denominator are products of an element of K and some irreducible divisors of elements of H.* ∎

DEFINITION 3.7. *An element a of an integral domain I will be called high (in I) if the equations $x^{(2^n)} = a$ have a solution in I, for every natural n.*

We want to find all the high elements of $I[{}_\omega X \cup Y/H]$. Clearly, the elements of ${}_\omega X$ are all high. The converse is, generally, not true but we have:

PROPOSITION 3.8. *Let K be the quotient field of I and let u be a high element of $I[{}_\omega X \cup Y/H]$. Then $u = a \cdot t_1 \cdot \ldots \cdot t_r$ with some high $a \in K$ and with $t_1, \ldots, t_r \in {}_i X$, for suitable i.*

PROOF. We proceed through several claims.

Claim 1. If K is a field and u is high in $K(x)$ then $u \in K$ and u is high in K.

In this case we have mutually prime polynomials f_i and g_i, for $i=0, 1, \ldots$, satisfying $(f_i/g_i)^{2^i}=u$. This yields, by unique factorization, $f_0=(f_i)^{2^i}$ and $g_0=(g_i)^{2^i}$, for each i. This means, however, that both the degrees of f_0 and g_0 are divisible by all powers of two, i.e., $\deg f_0=\deg g_0=0$. Further, this implies $\deg f_i=\deg g_i=0$, hence, $u \in K$ and u is high in K.

Claim 2. If X is a finite set of variables over the field K and $u \in K(X)$ is high, then u is high in K.

This statement follows from Claim 1, using an obvious induction.

Claim 3. Let K be a field, x a variable over K, $L=K(x)$ and $M=L[\{_\omega x\}]$. If u is high in M, then $u=v(_ix)^j$ where v is high in K and j is some integer.

We may suppose, without loss of generality, that $u=x^j \cdot \dfrac{f(x)}{g(x)}$, where x, f and g are mutually prime (j any integer).

If u is high in L we are done, by Claim 1.

Thus, we may suppose that there is a $v \in M$, $v \notin L$ such that $v^{2^k}=u$, for some k. If $v^{2^{k-1}} \in L$, then we start with $v^{2^{k-1}}$ instead of u. Therefore, we may suppose, that u was chosen such that $v^{2^{k-1}} \notin L$. Now, we change v to $v^{2^{k-1}}$. Thus, we have $v \in M$, $v \notin L$, such that $v^2=u$. Now, we choose $y={}_ix$ such that $v \in L(y)$ but $v \notin L(y^2)$ (such an $_ix$ must exists, for $v \notin L$).

Since y (as well as x) is transcendental over K, $y \notin L(y^2)$, i.e., $L(y)$ is an algebraic extension of rank 2 over $L(y^2)$. Thus, v is of the form $v=a+by$ with $a, b \in \in L(y^2)$. Since $v^2=u \in L \subseteq L(y^2)$, we have $a^2+b^2y^2+2aby \in L(y^2)$, yielding $2aby \in \in L(y^2)$. Now, the condition $y \notin L(y^2)$ implies either $a=0$ or $b=0$. The second case is impossible, for $v \notin L(y^2)$, so $v=by$, i.e., $v=y^t \cdot \dfrac{p(y^2)}{q(y^2)}$, with mutually prime p and q, which are not divisible by y and with an odd t. According to our choice, $x=y^{2^i}$, i.e., $y^{2t} \cdot \dfrac{p^2(y^2)}{q^2(y^2)}=v^2=u=x^j \cdot \dfrac{f(x)}{g(x)}=y^{j \cdot 2^i} \cdot \dfrac{f(y^{2^i})}{g(y^{2^i})}$.

Now, we have, by unique factorization

$$y^{2t} = y^{j \cdot 2^i}; \quad p^2(y^2) = f(y^{2^i}); \quad q^2(y^2) = g(y^{2^i}).$$

The first equation gives us $i=1$, $t=j$, for t is odd. This yields $p^2(x)=f(x)$ and $q^2(x)=g(x)$.

Since u is high so is either v or $-v$, and we may suppose v is high. Since $u=v^2$, if v is high in $L(y)$, we are done by Claim 1. Otherwise, similarly as before, there must be a $w \in M$, $w \notin L(y)$ such that $w^{2^k}=v$, for some k. Now, we are going to show that $z=w^{2^{k-1}} \notin L(y)$. Indeed, if z were of the form $z=y^l \cdot \dfrac{r(y^2)}{s(y^2)}$ with mutually prime y, r, s then z^2 could not be of the form $y^t \cdot \dfrac{p(y^2)}{q(y^2)}$ with an odd t. Therefore v has exactly the same properties as u has.

Thus, we can continue our procedure, i.e., we have a sequence of elements

$u={}_0u$, $v={}_1u$, ${}_2u, \ldots$ such that ${}_iu=({}_{i-1}u)^2$ and ${}_iu=({}_ix)^j \cdot \dfrac{f_i({}_ix^2)}{g_i({}_ix^2)}$ $(x, f_i, g_i$ are mutually prim) satisfying $f_i=(f_{i+1})^2$ and $g_i=(g_{i+1})^2$.

These imply, however, that $f_0, g_0 \in K$, i.e., u is of the desired form.

Claim 4. Let X be a finite set of variables over the field K, $L=K(X)$ and $M=L[{}_\omega X]$. If u is high in M, then $u=v \cdot t_1 \cdot \ldots \cdot t_r$ where v is high is K and $t_1, \ldots, t_r \in {}_\omega X$.

The claim follows by an obvious induction from Claim 3.

Claim 5. Proposition 3.8 is true for finite X and Y.

Indeed, applying first Claim 4 to the quotient field of $I(Y)$, then Claim 2 to I we get the desired property.

Claim 6. Proposition 3.8 is true.

Since u depends only on finitely many variables, we may apply Claim 5. ∎

We need one more preliminary result on *monomials*, i.e., on elements of $I[X] \setminus I$ which are products of elements of $I \cup X$.

PROPOSITION 3.9. *Let K be a field containing all roots of unity and let u and v be disjoint monomials in $K[X]$. Then each irreducible divisor of $u-v$ is of the form $w-z$ such that for some natural number k we have $u=w^k$, $v=z^k$. In particular, w and z are disjoint monomials.*

PROOF. We may suppose, by the disjointness, that $u-v$ is of the form $A \cdot x^n - B$ with some positive n, where $x \in X$ and $A, B \in K[X_1]$ for some finite subset X_1 of X not containing x. If $u-v$ is reducible over $K(X_1)$, then it is, also, reducible over $K[X_1]$. Since u and v are disjoint $A \cdot x^n - B = (C \cdot x^n - D) \cdot E$ is impossible with C, D, E in $K[X_1]$, unless E belongs to K. Thus, the reducibility of $u-v$ implies the reducibility of $x^n - B/A$ over $K(X_1)$. Hence, there are polynomials P, Q in $K[X_1]$ and a natural d dividing n such that $(P/Q)^d = B/A$ and P and Q are mutually prime. The unique prime factorization and the disjointness of A and B imply $B=P^d$ and $A=Q^d$. Thus, $u-v$ is the product of all $P \cdot x^{n/d} - \varepsilon \cdot Q$ where ε runs over the d-th roots of unity. The conditions for P and Q give us that they are, also, disjoint monomials. Continuing the procedure the properties of degrees prove the statement. ∎

4. Properties of integral domains constructed by means of graphs. We can formulate the conditions in Theorem 2.5 as follows:

To any object X in the given category $\mathscr{K}$ there is assigned an integral domain I the cardinality of which is under a given bound and a graph whose underlying set contains I, i.e., it is of the form $I \cup Y$. Having an other induced integral domain I' and a graph on the set $I' \cup Y'$ assigned to $x' \in \mathscr{K}$ and any map $\varphi\colon x \to x'$ the exclusive peseudorealization assigns to φ a map $P(\varphi)$ such that the underlying set mapping $\square P(\varphi)$ sends I into I' (and is a "restriction" of a homomorphism) and sends Y into Y'.

Our task is to construct integral domains "containing" $I \cup Y$, $I' \cup Y'$, ... such a way that the extensions of the graph-homomorphisms yield a "one-to-one" correspondence between the graph-homomorphism and the ring homomorphism.

Since we have a bound for the cardinality of all integral domains in question, we have a rigid graph $\mathscr{G}(X, S)$ such that the cardinality of X is greater than the given bound.

Thus, to construct the integral domain we are given an integral domain I, a graph $\mathscr{I}(I \cup Y, T)$ and a rigid graph $\mathscr{G}(X, S)$ with $|I| < |X|$. (The graphs are, for our purpose, directed, connected and loopless. A rigid graph is a graph with no other endomorphism but the identity.)

To manufacture the integral domain, described above, we shall constract a ring of the form $I[_\omega X \cup Y/H]$ with a special H. Our aim is to recognize the elements of X, of $I \cup Y$ and the relations T and S. To this end, we shall use Proposition 3.8. However, in this proposition we had to use the quotient field K of I. This will not do any harm, for the homomorphism will map into $K[_\omega X \cup Y/H]$ and it is enough to recognize the elements in question in this ring. However, this will make the formulazations of the forthcoming propositions a little more complicated.

Firstly, we construct H:

Definition 4.1. *Let I be an integral domain, Y a set of variables over Y, $\mathscr{I}(I \cup Y, T)$ a directed, connected, looples graph and $\mathscr{G}(X, S)$ a directed connected rigid graph.*

We define in $I[_\omega X \cup Y]$

(i) $H_1 = \{x_i - x_j \mid x_i, x_j \in X, \quad x_i \neq x_j\}$

(ii) $H_2 = \{x_i - 2x_j - 1 \mid (x_i, x_j) \in S\}$

(iii) $H_3 = \{x_1 + x_2 + {}_r x_i \mid {}_r x_i \in {}_r X;\ x_1, x_2$ *are two fixed elements of* $X\}$

(iv) $H_4 = \{x_3 + x_4 + u \cdot x_5 + v \mid (u, v) \in T;\ x_3, x_4, x_5$ *are distintc fixed elements of* X *such that* $\{x_1, x_2\} \cap \{x_3, x_4, x_5\} = \emptyset\}$

(v), *We choose and fix an infinite subset* $\{x_6, x_7, \ldots\}$ *of* X *disjoint to* $\{x_1, x_2, x_3, x_4, x_5\}$ *and define* $H_5 = \{x_{3k} + x_{3k+1} + x_{3j+2} + u \mid u \in I \cup Y,\ k > 1$ *integer*$\}$.

Let finally $H = H_1 \cup H_2 \cup H_3 \cup H_4 \cup H_5$.

Now, we are going to show that we can recognize the two graphs using only the algebraic properties of $I[_\omega X \cup Y/H]$.

Proposition 4.2. *The elements of H are not powers and the elements of H_i $(2 \leqq i \leqq 6)$ are irreducible in $K[_\omega X \cup Y]$ (K is the quotient field of I).*

Proof. The conditions of Proposition 3.1 are clearly satisfied, using (i) for H_1 and (i) and (ii) after for the other H_i-s. ∎

Proposition 4.3. *Let u and v be high elements of $I[_\omega X \cup Y]$ not both of them belonging to I. If $u - v$ is invertible in $I[_\omega X \cup Y/H]$ then there exists a natural number k such that both u^{2k} and v^{2k} are products of an element of the quotient field K of I and a power of some element of X. In addition the quotient of the two elements of K is a root of unity.*

Proof. By Proposition 3.8, $u = c \cdot u_1$ and $v = d \cdot v_1$ where c and d are high in K and u_1, v_1 are products of elements of some ${}_n X$. Since $K[_\omega X \cup Y/H]$ is the union of all $K[_i X \cup Y/H]$ we may choose n so that $K[_n X \cup Y/H]$ contains, also, the inverse of $u - v$. The invertibility of $u - v$, the normality of H and Proposition 3.6 imply that u and v are disjoint (monomials). Hence u and v are of the form

$$u = c \cdot y_1^{k_1} \cdot \ldots \cdot y_s^{k_s}, \quad v = d \cdot y_{s+1}^{k_{s+1}} \cdot \ldots \cdot y_r^{k_r},$$

where $y_1, \dots, y_s, y_{s+1}, \dots, y_r$ are distinct elements of ${}_nX$ and $k_1, \dots, k_s, k_{s+1}, \dots, k_r$ are non-negative integers.

Since u and v are disjoint, we may apply Proposition 3.9. Now, consider the irreducible factors of $u-v$ in $L[{}_\omega X \cup Y]$, where L is the algebraic closure of K. These are of the form $w-z$ with disjoint monomials w and z.

By Proposition 3.6, $w-z$ is a divisor of some element of H, i.e., by Proposition 4.2, either an element of $H_2 \cup \dots \cup H_5$ or a divisor of some element of H_1. The first ones are not of the form $w-z$ with disjoint monomials, so $w-z$ must be of the form $\varepsilon \cdot {}_nx_i - \eta \cdot {}_nx_j$ with some 2^n-th roots of the unity ε and η. We may suppose, without loss of generality, that $\varepsilon=1$. By divisibility and disjointness, one of ${}_nx_i$ and ${}_nx_j$ occurs in u, the other in v, say ${}_nx_i = y_1$, ${}_nx_j = y_{s+1}$. The divisibility yields:

$$\varepsilon' \cdot c \cdot (y_{s+1})^{k_1} \cdot \dots \cdot (y_s)^{k_s} = d \cdot (y_{s+1})^{k_s} \cdot \dots \cdot (y_r)^{k_r},$$

where ε' is a root of unity. By unique factorization we must have

$$u = c \cdot y_1^k \quad \text{and} \quad v = d \cdot y_2^k$$

with a positive integer k, and we have $\varepsilon' \cdot c = d$, i.e., d/c is a root of unity. Since $u, v, y_1, y_2 \in K[{}_\omega X \cup Y]$, we have $c, d \in K[{}_\omega X \cup Y]$, i.e., $c, d \in K$. ∎

PROPOSITION 4.4. *Let u and v as in Proposition 4.3, such that $u-2v-1$ is invertible. Then there is an $(x_i, x_j) \in S$ satisfying $u=x_i$ and $v=x_j$.*

PROOF. By Proposition 4.3 $u=c \cdot y_1^k$, $v=d \cdot y_2^k$ $(y_1, y_2 \in {}_nX)$, and d/c is a root of unity. We may, also, suppose that k is odd. Thus, by Proposition 3.1 $u-2v-1$ is irreducible. By Proposition 3.6, the invertibility of $u-2v-1$ implies that it is a product of an element of K and a divisor of elements of some H_i $(i=1, \dots, 5)$. By the irreducibility of $u-2v-1$ we have only one factor. Since in our case $u-2v-1$ depends only on two variables, $i=4$ and $i=5$ is impossible. Since the constant of $u-2v-1$ does not equal 0, we have $i \neq 1$ and $i \neq 3$. Thus, it is of the form $a(x_i - 2x_j - 1)$ with $a \in K$, and $(x_i, x_j) \in S$. This implies immediately $n=0$, $k=1$, $a=1$ and $\{y_1, y_2\} = \{x_i, x_j\}$. If $y_1 = x_j$, $y_2 = x_i$ then $c=-2$, $-2d=1$ and $d/c = 1/4$ is not a root of unity. Hence $y_1 = x_i$, $y_2 = x_j$. ∎

PROPOSITION 4.5. *Let $u \in I[{}_\omega X \cup Y/H]$ such that the elements $x_{3k} + x_{3k+1} + x_{3k+2} + u$ $(k > 1$ integer$)$ are invertible. Then $u \in I \cup Y$.*

PROOF. The above elements are, of course, invertible, also, in $L[{}_\omega X \cup Y/H]$, where L is the algebraic closure of the quotient field K of I. Since $L[{}_\omega X \cup Y/H]$ is the union of all $L[{}_iX \cup Y/H]$, we may suppose that $u=f/g$ with mutually prime $f, g \in L[{}_nX \cup Y]$. Since f and g together depend of finitely many variables only, there exists an integer $k>1$ such that neither of $x_{3k}, x_{3k+1}, x_{3k+2}$ occur in f or g. We may suppose, without loss of generality that $k=2$. We may change n, if necessary, so that $L[{}_nX \cup Y/H]$ contains, also, the inverse of $x_6 + x_7 + x_8 + f/g$. Since H is normal, by Proposition 3.6, we have that both g and $F = g \cdot (x_6 + x_7 + x_8) + f$ are products of elements of L and of irreducible divisors of elements of H.

By Proposition 3.9 and Proposition 4.2 these divisors are either of the form ${}_kx_i - \varepsilon \cdot {}_kx_j$ with some root of unity ε or they belong to $H_2 \cup \dots \cup H_5$.

We take care, firstly, of the elements of H_2 and the elements of the form ${}_kx_i-\varepsilon\cdot{}_kx_j$. Denote $y={}_kx_i$ and $z={}_kx_j$ or $y=x_i$ and $z=x_j$. We are going to prove that $y-ax-b$ $(a, b\in K)$ do not divide F. Otherwise the substitution $y\to ax+b$ would send F to 0. Observe, that x_6, x_7, x_8 do not occur in f and at most two of them occur in $y-ax-b$, therefore the polynomial got after the substitution has positive degree in one of x_6, x_7, x_8, unless the substitution sends g to 0. In that case the image of $f=F-g(x_6+x_7+x_8)$ will be 0 as well, implying that $y-ax-b$ is a common divisor of g and f, in contrary that f and g are mutually prime.

Thus, F is a product of elements of $L\cup H_3\cup H_4\cup H_5$. Since x_6, x_7, x_8 do not occur in f and g, the polynomial F has degree 1 in each of these variables. Therefore, one of the factors contains each of these variables, namely on first degree. Since F does not contain products of x_6, x_7, x_8, these variables must occur in the same irreducible factor. The only elements of $L\cup H_3\cup H_4\cup H_5$ which contain x_6, x_7, x_8 are the elements of H_5, i.e., we get $F=G\cdot(x_6+x_7+x_8+v)$ with some $v\in I\cup Y$.

Comparing the two expressions of F, as polynomial in x_6 we conclude $G=g$ and $f=v\cdot G$. Hence

$$u=\frac{f}{g}=\frac{v\cdot G}{G}=v\in I\cup Y.$$

∎

PROPOSITION 4.6. *Let L be the algebraic closure of the quotient field K of the integral domain I. Suppose, for some $a\in L$ and for some $u, v\in I\cup Y$ the elements $x_1+x_2+a_rx_i$ and $x_3+x_4+ux_5+v$ are invertible in $I[_\omega X\cup Y/H]$. Then, we have $a=1$ and $(u, v)\in T$.*

PROOF. By Proposition 3.1 the given elements are irreducible, thus, by Proposition 3.6 they must belong to H. An obvious comparing gives that only the stated cases are possible. ∎

5. Homomorphisms of integral domains constructed by means of graphs. In Section 4 we have, actually, given the action of the functor ψ mentioned in Theorem 2.5 on the objects. Next, our goal will be to describe the action on morphisms.

DEFINITION 5.1. *Suppose, we are given two integral domains I and I', moreover, two graphs $\mathcal{I}=\mathcal{I}(I\cup Y, T)$ and $\mathcal{I}'=\mathcal{I}(I'\cup Y', T')$.*

A homomorphism $\varphi\colon \mathcal{I}\to\mathcal{I}'$ will be called an $\mathcal{I}$-homomorphism if the restriction of φ to I is a 1-preserving ringhomomorphism into I'.

PROPOSITION 5.2. *Let $\mathcal{I}, \mathcal{I}'\in \mathrm{Int}_0$, $\mathcal{G}=\mathcal{G}(X, S)$ a rigid graph, such that $|X|>\max(|I|, |I'|)$. Besides let each graph-homomorphisms $\varphi\colon \mathcal{I}(I\cup Y, T)\to\mathcal{I}(I'\cup Y', T')$ be an $\mathcal{I}$-homomorphism.*

Then, there is a one-to-one correspondence between the $\mathcal{I}$-homomorphisms

$$\varphi\colon \mathcal{I}(I\cup Y, T)\to\mathcal{I}(I'\cup Y', T')$$

and the 1-preserving ringhomomorphisms

$$\Phi\colon I[_\omega X\cup Y/H]\to I'[_\omega X\cup Y'/H']$$

established by the extension and the restriction, respectively. (H' is constructed by $\mathcal{I}'$ and $\mathcal{G}$ similarly as H is by $\mathcal{I}$ and $\mathcal{G}$.)

PROOF. Suppose, first, that the homomorphism Φ is given. It maps X into the quotient field of $I'[_\omega X\cup Y'/H']$. Since the elements of X are high in $I[_\omega X\cup Y/H]$, so are their images. By Proposition 3.8 these elements are of the form $a\cdot t_1\cdot\ldots\cdot t_r$ ($a\in K'$, $t_i\in{}_nX$, K' is the quotient field of I', n is a natural number). Since the elements of H are invertible, the restriction of Φ to X is an injective mapping. Combining this with the condition $|X|>|I'|$, we get that there exists an $x_i\in X$ satisfying $u=\Phi(x_i)\notin I'$. Since $\mathcal{G}$ is connected, there exists an $x_j\in X$ such that $(x_i, x_j)\in S$. Thus, we can apply Proposition 4.4, which proves $\Phi(x_i)$, $\Phi(x_j)\in X$. Using the connectedness of X and the above argument in finitely many steps, we get that $\Phi(X)\subseteqq X$ and the restriction of Φ to X is a graph-homomorphism and by the rigidity of $\mathcal{G}$ Φ acts on X identically.

Since Φ gives the identity on X, $\Phi({}_rx_i)=\varepsilon\cdot{}_rx_i$ for some root ε of the unity. By Proposition 4.6, we have $\varepsilon=1$, i.e., Φ acts identically on ${}_\omega X$.

Let, now, $u\in I\cup Y$. $x_{3k}+x_{3k+1}+x_{3k+2}+u$ are invertible, so $\Phi(u)$ satisfies the conditions of Proposition 4.5 implying $\Phi(u)\in I'\cup Y'$, i.e., $\Phi(I\cup Y)\subseteqq\Phi(I'\cup Y')$. Using Proposition 4.6, we get $\Phi(T)\subseteqq\Phi(T')$, i.e., the restriction is a graph-homomorphism. By our assumption this is an $\mathcal{I}$-homomorphism.

To prove the converse we start with an $\mathcal{I}$-homomorphism $\varphi\colon\mathcal{I}(I\cup Y, T)\to\mathcal{I}(I'\cup Y', T')$. By the first part of the proof, the only way to extend it to a ring-homomorphism is to define it to act identically on ${}_\omega X$. Since any homomorphism of an integral domain has at most one extension to any subring to its quotient field, there is at most one Φ to which φ extends. We have to show that this is a homomorphism, indeed.

By the definition of an $\mathcal{I}$-homomorphism the restriction of φ gives a 1-preserving ring homomorphism $\Phi\colon I\to I'$. We extend Φ to $I[X\cup Y]$ such that $\Phi(x_i)=x_i$ and $\Phi(y)=\varphi(y)(x_i\in X,\ y\in Y)$. Since $I[X\cup Y]$ is the free ring over I such a homomorphism does exist. Using the properties of algebraic extensions Φ can be extended to $I[_\omega X\cup Y]$ such that $\Phi({}_rx_i)={}_rx_i$ for all ${}_rx_i\in{}_rX$.

A homomorphism of an integral domain can be extended to a quotient ring of it iff no denominator is mapped to 0. However, this is obvious, for φ acts identically on ${}_\omega X$. The only thing we still have to check that Φ maps into $I'[_\omega X\cup Y', H']$. To this end we are going to show that $\Phi(H)\subseteqq H'$. $\Phi(H_i)\subseteqq H_i'$ is clear, for $i=1, 2, 3$, for Φ acts identically on ${}_\omega X$. $\Phi(H_5)\subseteqq H_5'$, for $\varphi(I\cup Y)\subseteqq I'\cup Y'$ and $\Phi(H_4)\subseteqq H_5'$, for φ is a graph homomorphism. ∎

6. The proof of the main theorem. Now, we are going to prove Theorem 2.5.

We start with the concrete category $(\mathcal{K}, \square)$ and let $\varphi\colon A\to B$ be any morphism in $\mathcal{K}$. By condition (i) we have a functor $\Phi\colon\mathcal{K}\to\mathrm{Int}_0$ such that $|\Phi(X)|$ is bounded. We shall use the notation $\Phi(A)=I_A$, $\Phi(B)=I_B$, $\Phi(\varphi)=\varphi_I$. Thus, we have in Int_0: $\varphi_I\colon I_A\to I_B$ and we know that $|I_A|$, $|I_B|$ is less than a universal bound depending only on $\mathcal{K}$.

To deal with condition (ii), we have, first of all, the category $\mathcal{K}\cup\Phi(\mathcal{K})$. This is the same category as $\mathcal{K}$, but the underlying set functor $\square''$ is defined such that

$$\square''A=\square A\,\dot{\cup}\,\square' I_A,\quad \square''B=\square B\,\dot{\cup}\,\square' I_B,\quad \square''\varphi=\square\varphi\,\dot{\cup}\,\square'\varphi_A,$$

where $\square'$ is the underlying set functor in Int_0. Let $\square^*$ denote the underlying set functor of Gra and denote $G_A=P(A)$, $G_B=P(B)$, $\varphi_G=P(\varphi)$. According to (ii) we have:

$$\square''(A)\subseteqq\square^*(G_A),\quad \square''(B)\subseteqq\square^*(G_B),$$

and the restriction of $\square^*\varphi_G$ to $\square^*(G_A)\setminus\square''(A)$ maps into $\square^*(G_B)\setminus\square''(B)$.

Now, define $Y_A=\square^*(G_A)\setminus\square' I_A$ and $Y_B=\square^* G_B\setminus\square' I_B$. Since $\square''\varphi=$ $=\square\varphi\cup\square'\varphi_I$, we have that $\square^*\varphi_G$ sends Y_A into Y_B and I_A into I_B, further this latter restriction is a ring homomorphism.

Thus, we have $G_A=\mathscr{I}(I_A\cup Y_A, T_A)$ and $G_B=\mathscr{I}(I_B\cup Y_B, T_B)$. Since P is full each graph-homomorphism is of the form φ_G, i.e., it is an I-homomorphism.

Now, let X be a set such that $|X|>|I_A|$ and consider the subcategory $\widetilde{\mathrm{Gra}}$ of Gra consisting of all G_A and φ_G $(A\in\mathrm{Ob}\,\mathscr{K},\ \varphi\in\mathrm{Mor}\,\mathscr{K})$. Define $\Sigma\colon\widetilde{\mathrm{Gra}}\to\mathrm{Int}_0$, such that $\Sigma(G_A)=I_A[_\omega X\cup Y_A/H_A]$. Since each graph homomorphism $G_A\to G_B$ is of the form φ_G, we may apply Proposition 5.2 establishing a ring homomorphism $\Phi\colon\Sigma(G_A)\to\Sigma(G_B)$. By Proposition 5.2 the definition $\Sigma(\varphi_G)=\Phi$ gives us a full embedding.

Hence, $\psi=\Sigma\circ P$ is a full embedding as well. $I_A=\Phi(A)$ is, of course, a subdomain of $I_A[_\omega X\cup Y_A/H_A]=\Sigma\big(P(A)\big)$. By Proposition 5.2 the restriction of $\Sigma\big(P(\varphi)\big)$, where $\varphi\colon A\to B$, to the graph G_A is an $\mathscr{I}$-homomorphism, therefore the restriction to I_A is the ring homomorphism φ_I.

To finish the proof, consider any small subcategory $\mathscr{K}_0$ of $\mathscr{K}$. Then the graphs of the form $\{G_A|A\in\mathscr{K}_0\}$ form a set. When choosing X such that $|X|>|I_A|$ for all $A\in\mathscr{K}$ and $|X|>|G_A|$ for all $A\in\mathscr{K}_0$ then we have $|I_A[_\omega X\cup Y_A/H_A]|=|X|$, whenever $A\in\mathscr{K}_0$. ∎

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(Received April 13, 1977; revised January 3, 1982)

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Acta Math. Hung.
41 (1—2), (1983), 17—26.

ON CONGRUENCE n-DISTRIBUTIVITY OF ORDERED ALGEBRAS

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1. Introduction. The triple $(A, F, \leqq)$ is said to be an ordered algebra of type τ if (A, F) is a universal algebra of type τ, $(A, \leqq)$ is a partially ordered set, and all $f \in F$ are monotone with respect to $\leqq$. It is worth mentioning that τ-terms induce monotone term-functions on ordered τ-algebras. For τ-terms g, h the string $g \leqq h$ is called an order-identity or, shortly, identity. (Note that an identity $g=h$ is equivalent to the conjunction of $g \leqq h$ and $h \leqq g$). Let **H**, **S**, **P** be the operators of taking homomorphic images, subalgebras and direct products, respectively. (These concepts are defined in the natural way. I.e., a homomorphism is a *monotone* map preserving the operations, $u \leqq v$ in $\prod_{\gamma \in \Gamma} A_\gamma$ means $(\forall \gamma \in \Gamma)(u(\gamma) \leqq v(\gamma))$ and the original order is restricted in case of subalgebras.) The following result of Bloom [2] shows that the counterpart of the classical Birkhoff Theorem is valid for classes of ordered algebras: **HSP** is a closure operator on classes of similar ordered algebras, and a class of similar ordered algebras is closed under **HSP** iff it can be defined by a set of order-identities.

The concept of n-distributivity was introduced by Huhn [8, 10]. This concept has proved to be a very useful tool in several investigations (cf., e.g., Huhn [8, 9, 10] and Herrmann—Huhn [7]).

A lattice is called *n-distributive* if the n-distributive identity

$$x \wedge \bigvee_{i=0}^{n} y_i \leqq \bigvee_{j=0}^{n} \Big(x \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{n} y_i\Big)$$

holds in it.

A variety of lattices is said to be a congruence variety (Jónsson [13]) if it is generated by the class of congruence lattices of all members of some variety of universal algebras. It is known (cf. Nation [16]) that n-distributive congruence varieties are distributive, and this fact plays an important rôle in the theory of congruence varieties. Our aim is to generalize this result for the case of ordered algebras.

2. Order-congruences. If congruence relations of an ordered algebra $(A, F, \leqq)$ were defined as congruences of (A, F), they would not depend on the ordering. Moreover, there would be no reasonable way to define orders on factor algebras so that factor algebras would be order-homomorphic images under the canonical map. That is why the concept of order-congruences is introduced. Since our motivation will be given only in Proposition 2.1, the following definition might seem astounding at the first sight.

DEFINITION. A binary relation Θ is called an order-congruence of the ordered algebra $(A, F, \leqq)$ if Θ is a congruence of the universal algebra (A, F) and

$$(*)\qquad \begin{cases} \text{whenever}\quad a, b, a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_k \in A \quad \text{such that} \\ a = a_0 \Theta a_1 \leqq a_2 \Theta a_3 \leqq a_4 \Theta a_5 \leqq \ldots a_m = b = b_0 \Theta b_1 \leqq b_2 \Theta b_3 \leqq \\ \leqq b_4 \Theta b_5 \leqq \ldots b_k = a \quad \text{then} \quad a \Theta b. \end{cases}$$

PROPOSITION 2.1. *Assume that Θ is a binary relation on an ordered algebra A. Then Θ is an order-congruence if and only if there exist an ordered algebra B and a homomorphism $\varphi\colon A \to B$ such that $\Theta = \mathrm{Ker}\,\varphi$.*

PROOF. Suppose Θ is an order-congruence of $(A, F, \leqq)$. Set $B = A/\Theta$ and, for $a, b \in A$, define $[a]\Theta \leqq [b]\Theta$ by "there exist $a_0, a_1, \ldots, a_t \in A$ such that $a = a_0 \Theta a_1 \leqq a_2 \Theta a_3 \leqq a_4 \Theta \ldots a_t = b$". The reflexivity of Θ and $\leqq$ (over A) together with $(*)$ yield that $\leqq$ is an ordering of B. If f is an m-ary operation of A and $[a^i]\Theta \leqq \leqq [b^i]\Theta$ $(i = 1, \ldots, m)$ then we have $a^i = a_0^i \Theta a_1^i \leqq a_2^i \Theta \ldots a_t^i = b^i$ where, without loss of generality, we assume that t does not depend on i. Since f preserves both Θ and $\leqq$ we obtain

$$f(a^1, \ldots, a^m) = f(a_0^1, \ldots, a_0^m) \Theta f(a_1^1, \ldots, a_1^m) \leqq f(a_2^1, \ldots, a_2^m) \Theta \ldots f(a_t^1, \ldots, a_t^m) = \\ = f(b^1, \ldots, b^m),$$

which shows that

$$f([a^1]\Theta, \ldots, [a^m]\Theta) = [f(a^1, \ldots, a^m)]\Theta \leqq [f(b^1, \ldots, b^m)]\Theta = f([b^1]\Theta, \ldots, [b^m]\Theta).$$

Hence, equipped with this ordering, B is an ordered algebra. Now the map $\varphi\colon A \to B$, $a\varphi = [a]\Theta$ is a homomorphism and $\Theta = \mathrm{Ker}\,\varphi$.

Conversely, if $\Theta = \mathrm{Ker}\,\varphi$ for some homomorphism φ and $a = a_0 \Theta a_1 \leqq \leqq a_2 \Theta \ldots a_m = b = b_0 \Theta b_1 \leqq b_2 \Theta \ldots b_k = a$ then $a\varphi = a_0\varphi = a_1\varphi \leqq a_2\varphi = \ldots = a_m = b$, implying $a\varphi \leqq b\varphi$. Since $b\varphi \leqq a\varphi$ follows similarly, $a\varphi = b\varphi$, whence $a\Theta b$. Q.e.d.

Let us mention two examples. The additive group $Z = (Z, +, \leqq)$ of integers with the usual ordering has many congruences, but it has only the two trivial order-congruences. (Indeed, its proper factor groups do not admit nontrivial orderings.) In case of lattices equipped with the usual ordering congruences and order-congruences are the same.

For an ordered algebra A let Con (A) denote the set of *order*-congruences of A. Since the meet of arbitrary many order-congruences is an order-congruence again, Con (A) is a complete lattice with respect to the set-theoretic inclusion. The join in Con (A) is described in the following

PROPOSITION 2.2. *Let A be an ordered algebra and let $\Theta_0, \Theta_1, \ldots, \Theta_k$ be order-congruences of A. Set $\Phi = \{(a, b) \in A^2 |$ there exist $a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_t \in A$ such that*

$$a = a_0 \Theta_0 a_1 \Theta_1 a_2 \Theta_2 \ldots a_k \Theta_k a_{k+1} \leqq a_{k+2} \Theta_0 a_{k+3} \Theta_1 \ldots a_{2k+2} \Theta_k a_{2k+3} \leqq \ldots \leqq a_m = b$$

and

$$b = b_0 \Theta_0 b_1 \Theta_1 \ldots b_k \Theta_k b_{k+1} \leqq b_{k+2} \Theta_0 b_{k+3} \Theta_1 \ldots b_{2k+2} \Theta_k b_{2k+3} \leqq \ldots b_t = a\}.$$

Then $\Phi = \bigvee_{i=0}^{n} \Theta_i$ in the lattice Con (A).

PROOF. It is straightforward to check that Φ is an order-congruence. The inclusion $\Theta_i \subseteq \Phi$ is trivial. On the other hand, if $\Psi \in \mathrm{Con}\,(A)$ and $\Theta_i \subseteq \Psi$ for all i then $(*)$ yields $\Phi \subseteq \Psi$. Q.e.d.

3. The main theorem. Now we can formulate

THEOREM 3.1. *For any class $\mathcal{U}$ of ordered algebras which is closed under taking subalgebras and direct products, and for any natural number n the following three conditions are equivalent:*

(i) Con (A) *is n-distributive for all* $A \in \mathcal{U}$;

(ii) Con (A) *is distributive for all* $A \in \mathcal{U}$;

(iii) *There exist a natural number k and ternary terms $t_0(x, y, z), t_1(x, y, z), \dots, t_k(x, y, z)$ (corresponding to the type of $\mathcal{U}$) such that the identities*

$$t_0(x, y, z) = x, \quad t_k(x, y, z) = z, \quad t_i(x, y, x) = x \quad \textit{for} \quad i = 0, 1, \dots, k,$$

$$t_i(x, x, y) = t_{i+1}(x, x, y) \quad \textit{for} \quad i \equiv 0\ (3), 0 \leqq i < k,$$

$$t_i(x, y, y) = t_{i+1}(x, y, y) \quad \textit{for} \quad i \equiv 1\ (3), 0 \leqq i < k,$$

$$t_i(x, y, z) \leqq t_{i+1}(x, y, z) \quad \textit{for} \quad i \equiv 2\ (3), 0 \leqq i < k$$

hold in $\mathcal{U}$.

Before proving this theorem let some consequences and examples be mentioned.

COROLLARY 3.2. (Jónsson [12]). *A variety $\mathcal{V}$ of universal algebras of type τ is congruence distributive iff there exist a natural number k and ternary τ-terms $t_0, t_1, \dots, t_k$ such that the identities*

$$t_0(x, y, z) = x, \quad t_k(x, y, z) = z, \quad t_i(x, y, x) = x \quad \textit{for} \quad 0 \leqq i \leqq k,$$

$$t_i(x, x, y) = t_{i+1}(x, x, y) \quad \textit{for} \quad i \textit{ even}, \ 0 \leqq i < k,$$

$$t_i(x, y, y) = t_{i+1}(x, y, y) \quad \textit{for} \quad i \textit{ odd}, \ 0 \leqq i < k$$

hold in $\mathcal{V}$.

COROLLARY 3.3. (Nation [16]). *If a variety $\mathcal{V}$ of universal algebras is congruence n-distributive then it is congruence distributive.*

Both corollaries follow by the same consideration: Equip the members of $\mathcal{V}$ with the trivial order. Then congruences are the same as order-congruences and an order-identity $t_i(x, y, z) \leqq t_{i+1}(x, y, z)$ is equivalent to $t_i(x, y, z) = t_{i+1}(x, y, z)$.

If we call lattice varieties generated by the class $\{\mathrm{Con}\,(A) | A \in \mathcal{U}\}$ for some variety of ordered algebras $\mathcal{U}$ *order-congruence varieties* and denote by $\mathcal{M}(T)$ the variety of all vector spaces over a field T then we can describe the minimal modular order-congruence varieties:

COROLLARY 3.4. *For any modular but not distributive order-congruence variety $\mathcal{U}$ there exists a prime field T such that the (order-) congruence variety*

$$\mathbf{HSP}\,\{\mathrm{Con}\,(V) | V \in \mathcal{M}(T)\}$$

is a subvariety of $\mathcal{U}$. (Note that $\mathbf{HSP}\,\{\mathrm{Con}\,(V) | V \in \mathcal{M}(T_i)\}$, $i = 1, 2$, *are incomparable provided T_1 and T_2 are non-isomorphic prime fields.)*

For congruence varieties the same result was announced by Freese [4]. Herrmann and Freese [5] gave a very elegant proof for Freese's result. Their proof is based on, among others, Corollary 3.3. Replacing Corollary 3.3 by Theorem 3.1 their argument proves Corollary 3.4. (Since their work [5] had not appeared when the present paper was written, let us mention that their proof can be found in [3, Theorem 3.2], too.)

If a variety $\mathscr{V}$ of algebras is congruence distributive then $\mathscr{V}$, as an **S** and **P** closed class of ordered algebras (with the equality relations as orderings) is order-congruence distributive. (Indeed, Jónsson's condition from Corollary 3.2 is stronger than (iii) of Theorem 3.1) Therefore if we intend to present examples for classes of ordered algebras satisfying the conditions of Theorem 3.1, we can equip any congruence distributive variety of algebras with the trivial orderings. Another example is the class of all lattices with the usual orderings. In order to give a nontrivial example (which is far from lattice orderings) consider the ordered algebra $A=(\{a, b, c\}, f, \leqq)$ where the ordering is $\{(x, x)|x\in A\}\cup\{(a, b), (a, c)\}$, and f is a ternary majority function defined by

$$f(x_1, x_2, x_3) = \begin{cases} a & \text{if} \quad |\{x_1, x_2, x_3\}| = 3 \\ u\in A & \text{if} \quad |\{i|x_i = u\}| \geqq 2. \end{cases}$$

Now the class $\mathbf{SP}\{A\}$ satisfies condition (iii) of Theorem 3.1 since we can put $k=2$ and $t_1(x, y, z)=f(x, y, z)$.

Finally, it is worth mentioning that for a single ordered algebra A the n-distributivity of Con (A) does not imply the distributivity of Con (A). (Indeed, choose a finite ordered A such that Con (A) is not distributive. Then Con (A) is n-distributive for any n greater than $|\text{Con}\,(A)|$.)

4. Proof of the main theorem. Let us define three further conditions besides the conditions of Theorem 3.1:

(iv) The identity

$$\delta\colon\ x\wedge \bigvee_{i=0}^{n} y_i \leqq \Big(x\wedge \bigvee_{i=0}^{n-1} y_i\Big)\vee\Big(x\wedge \bigvee_{i=1}^{n} y_i\Big)$$

holds in Con (A) for any $A\in\mathscr{U}$;

(v) There exist $k\geqq 1$ and $(n+2)$-ary terms $t_0, t_1, \dots, t_k$ such that the identities

$$t_0(x_0, x_1, \dots, x_{n+1}) = x_0, \quad t_k(x_0, x_1, \dots, x_{n+1}) = x_{n+1},$$
$$t_i(x, y_1, y_2, \dots, y_n, x) = x \quad \text{for} \quad 0 \leqq i \leqq n,$$
$$t_i(x, x, \dots, x, \underbrace{y, y \dots, y}_{j+1}) = t_{i+1}(x, x, \dots, x, \underbrace{y, y, \dots, y}_{j+1})$$

where $0\leqq j\leqq n$, $0\leqq i<k$ and $i\equiv j\ (n+2)$,

$$t_i(x_0, x_1, \dots, x_{n+1})\leqq t_{i+1}(x_0, x_1, \dots, x_{n+1})$$

for $i\equiv n+1\ (n+2)$ and $0\leqq i<k$ hold in $\mathscr{U}$;

(vi) $$(x_0, x_{n+1})\in \bigvee_{j=0}^{n} \Big(\Theta_{x_0x_{n+1}}\wedge \bigvee_{\substack{i=0\\ i\neq j}}^{n} \Theta_{x_{n-i}x_{n+1-i}}\Big)$$

where $\Theta_{x_sx_t}$ denotes the smallest order-congruence of $F_{\mathscr{U}}(x_0, x_1, \dots, x_{n+1})$, the free $\mathscr{U}$-algebra over $\{x_0, x_1, \dots, x_{n+1}\}$, under which x_s and x_t collapse.

REMARK. Since $\mathcal{U}$ is closed under **S** and **P**, the free algebra involved in (vi) (and that over an arbitrary generating set) exists. (The definition of free algebras and the proof of this remark are the same as in case of universal algebras, cf. Grätzer [6] or Birkhoff [1].) Note also that (v) is a generalization of Mederly's condition [15].

Via the implications (i)$\to$(vi)$\to$(v)$\to$(iv)$\to$(ii)$\to$(i) we intend to show that all the six conditions, (i), ..., (vi), are equivalent. (No matter that (iii) is not involved above. For $n=1$ (v) and (iii) are the same, and 1-distributivity is the usual distributivity. Thus the equivalence of the other five conditions yields the equivalence of all the six ones.)

Since $(x_0, x_{n+1})\in\Theta_{x_0x_{n+1}}\wedge(\Theta_{x_0x_1}\circ\Theta_{x_1x_2}\circ\ldots\circ\Theta_{x_nx_{n+1}})\subseteq\Theta_{x_0x_{n+1}}\wedge\bigvee_{i=0}^{n}\Theta_{x_{n-i}x_{n+1-i}}$, the implication (i)$\to$(vi) is trivial. Distributive lattices are n-distributive (cf. Huhn [8]), which settles (ii)$\to$(i).

(vi) *implies* (v). Suppose (vi) and let $\bigvee_{\substack{i=0\\ i\neq j}}^{n}\Theta_{x_{n-i}x_{n+1-i}}$ be denoted by Φ_j ($j=0, 1, \ldots, n$). From the assumption $(x_0, x_{n+1})\in\bigvee_{i=0}^{n}(\Theta_{x_0x_{n+1}}\wedge\Phi_j)$ and Proposition 2.2 we obtain that there are elements $t_0(x_0, x_1, \ldots, x_{n+1})$, $t_1(x_0, x_1, \ldots, x_{n+1})$, ..., $t_k(x_0, x_1, \ldots, x_{n+1})$ in $F_{\mathcal{U}}(x_0, \ldots, x_{n+1})$ (where t_i is a term) such that

(1) $$x_0 = t_0(x_0, x_1, \ldots, x_{n+1}),\ t_k(x_0, x_1, \ldots, x_{n+1}) = x_{n+1},$$

(2) $$t_i(x_0, x_1, \ldots, x_{n+1}) \leqq t_{i+1}(x_0, x_1, \ldots, x_{n+1}) \quad\text{for}\quad i\equiv n+1 \quad (n+2),$$

(3) $$t_i(x_0, x_1, \ldots, x_{n+1})\,\Theta_{x_0x_{n+1}}\wedge\Phi_j\ t_{i+1}(x_0, x_1, \ldots, x_{n+1})$$

where $0\leqq j\leqq n$ and $i\equiv j\ (n+2)$.

Since $\Theta_{x_0x_{n+1}}\wedge\Phi_j=\Theta_{x_0x_{n+1}}\cap\Phi_j$, from (3) we obtain

(4) $$t_i(x_0, x_1, \ldots, x_{n+1})\,\Phi_j\,t_{i+1}(x_0, x_1, \ldots, x_{n+1})$$

where $0\leqq j\leqq n$ and $i\equiv j\ (n+2)$. Denoting $(x_0, x_1, \ldots, x_{n+1})$ by $\mathbf{x}$, from (2) and (3) we obtain $x_0=t_0(\mathbf{x})\,\Theta_{x_0x_{n+1}}t_1(\mathbf{x})\,\Theta_{x_0x_{n+1}}\ldots\Theta_{x_0x_{n+1}}t_{n+1}(\mathbf{x})\leqq t_{n+2}(\mathbf{x})\,\Theta_{x_0x_{n+1}}\ldots\leqq\ldots t_i(\mathbf{x})\ldots$ $\ldots\Theta_{x_0x_{n+1}}\ldots\leqq\ldots t_k(\mathbf{x})=x_{n+1}\,\Theta_{x_0x_{n+1}}x_0$, whence $(*)$ yields

(5) $$x_0\,\Theta_{x_0x_{n+1}}t_i(x_0, x_1, \ldots, x_{n+1}) \quad\text{for}\quad i = 0, 1, \ldots, k.$$

Since $F_{\mathcal{U}}(x_0, \ldots, x_{n+1})$ is a *free* ordered algebra in $\mathcal{U}$, (1) and (2) show that all the identities of (v) which contain $n+2$ variables hold in $\mathcal{U}$. For the rest of identities (4) and (5) will be used. Consider indices i, j ($0\leqq j\leqq n$, $0\leqq i\leqq k$ and $i\equiv j\,(n+2)$) and the homomorphism φ: $F_{\mathcal{U}}(x_0, x_1, \ldots, x_{n+1})\to F_{\mathcal{U}}(x, y)$, $x_0\varphi=\ldots=x_{n-j}\varphi=x$, $x_{n+1-j}\varphi=\ldots=x_{n+1}\varphi=y$. Then $\operatorname{Ker}\varphi$ is an order-congruence by Proposition 2.1. Since, for $i\neq j$, $(x_{n-i}, x_{n+1-i})\in\operatorname{Ker}\varphi$, we have $\Phi_j=\bigvee_{i\neq j}\Theta_{x_{n-i}x_{n+1-i}}\subseteq\operatorname{Ker}\varphi$. Thus from (4) we obtain

$$t_i(x, x, \ldots, x, \underbrace{y, y, \ldots, y}_{j+1}) = t_i(x_0\varphi, x_1\varphi, \ldots, x_{n+1}\varphi) = t_i(x_0, \ldots, x_{n+1})\varphi =$$

$$= t_{i+1}(x_0, \ldots, x_{n+1})\varphi = t_{i+1}(x_0\varphi, \ldots, x_{n+1}\varphi) = t_{i+1}(x, \ldots, x, \underbrace{y, \ldots, y}_{j+1}).$$

Hence the identity $t_i(x, \ldots, x, \underbrace{y, \ldots, y}_{j+1})=t_{i+1}(x, \ldots, x, \underbrace{y, \ldots, y}_{j+1})$ holds in $\mathcal{U}$. Similarly, considering the homomorphism $\psi\colon F_{\mathcal{U}}(x_0, \ldots, x_{n+1}) \to F_{\mathcal{U}}(x, y_1, y_2, \ldots, y_n)$, $x_0\psi = x_{n+1}\psi = x$, $x_s\psi = y_s$ for $1 \leqq s \leqq n$, and making use of (5) together with $\Theta_{x_0 x_{n+1}} \subseteq \operatorname{Ker}\psi$, we obtain $t_i(x, y_1, \ldots, y_n, x) = t_i(x_0\psi, x_1\psi, \ldots, x_{n+1}\psi) = t_i(x_0, x_1, \ldots, x_{n+1})\psi = x_0\psi = x$, whence $t_i(x, y_1, \ldots, y_n, x) = x$ holds in $\mathcal{U}$.

(v) *implies* (iv). Suppose (v) and let $A \in \mathcal{U}$, $\alpha, \beta_0, \ldots, \beta_n \in \operatorname{Con}(A)$. Considering a pair (a, b) of elements in $\alpha \wedge \bigvee_{i=0}^{n} \beta_i$ and denoting $\bigvee_{i=0}^{n-1} \beta_i$, $\bigvee_{i=1}^{n} \beta_i$ by γ_n and γ_0, respectively, $(a, b) \in (\alpha \wedge \gamma_n) \vee (\alpha \wedge \gamma_0)$ should be shown. Since the rôle of a and b can be interchanged, by Proposition 2.2 it suffices to find a sequence of elements $a = d_0, d_1, \ldots, d_r = b$ such that for all $i (< r)$ we have either $d_i \, \alpha \wedge \gamma_j \, d_{i+1}$ for some $j \in \{0, n\}$ or $d_i \leqq d_{i+1}$. First of all $(a, b) \in \alpha$ and, by Proposition 2.2, there are elements $c_{ij}, c^{ij}, c_{s+1,0}, c^{s+1,0} \in A$ $(i = 0, 1, \ldots, s, j = 0, 1, \ldots, n+1)$ such that

$$\begin{aligned} a &= c_{00}\beta_0 c_{01}\beta_1 c_{02}\beta_2 c_{03} \ldots \beta_n c_{0,n+1} \leqq c_{10}\beta_0 c_{11}\beta_1 c_{12}\beta_2 c_{13} \ldots \beta_n c_{1,n+1} \leqq \\ &\leqq c_{20}\beta_0 c_{21}\beta_1 c_{22}\beta_2 c_{23} \ldots \beta_n c_{2,n+1} \leqq \ldots \leqq c_{s0}\beta_0 c_{s1}\beta_1 c_{s2}\beta_2 c_{s3} \ldots \beta_n c_{s,n+1} \leqq \\ &\leqq c_{s+1,0} = b \end{aligned}$$

and

$$\begin{aligned} b &= c^{00}\beta_0 c^{01}\beta_1 c^{02} \ldots \beta_n c^{0,n+1} \leqq c^{10}\beta_0 c^{11}\beta_1 c^{12} \ldots \beta_n c^{1,n+1} \leqq \ldots \leqq c^{s0}\beta_0 c^{s1}\beta_1 c^{s2} \ldots \\ &\ldots \beta_n c^{s,n+1} \leqq c^{s+1,0} = a. \end{aligned}$$

Let us compute by the identities of (v) and keeping in mind that all term functions are monotone:

$$\begin{aligned} a &= t_0(a, \ldots, a, a, b) = \\ &= t_1(a, \ldots, a, a, b)\,\gamma_n\, t_1(a, \ldots, a, c_{0n}, b)\,\gamma_0\, t_1(a, \ldots, a, c_{0,n+1}, b) \leqq \\ &\leqq t_1(a, \ldots, a, c_{10}, b)\,\gamma_n\, t_1(a, \ldots, a, c_{1n}, b)\,\gamma_0\, t_1(a, \ldots, a, c_{1,n+1}, b) \leqq \ldots \leqq \\ &\leqq t_1(a, \ldots, a, c_{s0}, b)\,\gamma_n\, t_1(a, \ldots, a, c_{sn}, b)\,\gamma_0\, t_1(a, \ldots, a, c_{s,n+1}, b) \leqq \\ &\leqq t_1(a, \ldots, a, c_{s+1,0}, b) = t_1(a, \ldots, a, b, b) = \\ &= t_2(a, \ldots, a, a, b, b)\,\gamma_n\, t_2(a, \ldots, a, c_{0n}, b, b)\,\gamma_0\, t_2(a, \ldots, a, c_{0,n+1}, b, b) \leqq \\ &\leqq t_2(a, \ldots, a, c_{10}, b, b)\,\gamma_n\, t_2(a, \ldots, a, c_{1n}, b, b)\,\gamma_0\, t_2(a, \ldots a, c_{1,n+1}, b, b) \leqq \ldots \leqq \\ &\leqq t_2(a, \ldots, a, c_{s+1,0}, b, b) = t_2(a, \ldots a, b, b, b) = t_3(a, \ldots, a, b, b, b) = \ldots = \\ &= t_{n+1}(a, b, b, \ldots, b) \leqq t_{n+2}(a, b, \ldots, b) = \\ &= t_{n+2}(a, c^{00}, \ldots, c^{00}, b)\,\gamma_n\, t_{n+2}(a, c^{0n}, \ldots, c^{0n}, b)\,\gamma_0\, t_{n+2}(a, c^{0,n+1}, \ldots, c^{0,n+1}, b) \leqq \\ &\leqq t_{n+2}(a, c^{10}, \ldots, c^{10}, b)\,\gamma_n\, t_{n+2}(a, c^{1n}, \ldots, c^{1n}, b)\,\gamma_0\, t_{n+2}(a, c^{1,n+1}, \ldots, c^{1,n+1}, b) \leqq \ldots \leqq \\ &\leqq t_{n+2}(a, c^{s0}, \ldots, c^{s0}, b)\,\gamma_n\, t_{n+2}(a, c^{sn}, \ldots, c^{sn}, b)\,\gamma_0\, t_{n+2}(a, c^{s,n+1}, \ldots, c^{s,n+1}, b) \leqq \\ &\leqq t_{n+2}(a, c^{s+1,0}, \ldots, c^{s+1,0}, b) = t_{n+2}(a, \ldots, a, a, b) = \\ &= t_{n+3}(a, \ldots, a, a, b)\,\gamma_n\, t_{n+3}(a, \ldots, a, c_{0n}, b)\,\gamma_0\, t_{n+3}(a, \ldots, a, c_{0,n+1}, b) \leqq \\ &\leqq t_{n+3}(a, \ldots, a, c_{10}, b)\,\gamma_n\, t_{n+3}(a, \ldots, a, c_{1n}, b)\,\gamma_0\, t_{n+3}(a, \ldots, a, c_{1,n+1}, b) \leqq \ldots \leqq \ldots = \\ &= t_k(a, \text{ some elements of } A, b) = b. \end{aligned}$$

Now if we replaced γ_0 and γ_n by $\alpha\wedge\gamma_0$ and $\alpha\wedge\gamma_n$, respectively, we would obtain a required sequence $a=d_0, d_1, \dots, d_r=b$. But this is possible since for any $u_1, \dots, u_n\in A$ we have $t_i(a, u_1, \dots, u_n, b)\alpha t_i(a, u_1, \dots, u_n, a)=a$, whence the elements of the above sequence are pairwise congruent modulo α.

(iv) *implies* (ii). Suppose the identity δ holds in a lattice L and let x, y, z be arbitrary elements of L. Let $x\wedge y\wedge z$ be denoted by w. Then

$$x\wedge(y\vee z) = x\wedge(y\vee w\vee w\vee\dots\vee w\vee z) = (x\wedge(y\vee w\vee\dots\vee w))\vee(x\wedge(w\vee\dots\vee w\vee z) =$$

$$= (x\wedge y)\vee(x\wedge z),$$

i.e., L is distributive.

The proof of Theorem 3.1 is complete.

5. Some Mal'cev conditions. Roughly saying, a Mal'cev condition is a condition on classes of algebras (ordered algebras, resp.) of the form "there are certain terms which satisfy certain prescribed identities (order-identities, resp.)." (For a precise definition and classification of Mal'cev conditions cf., e.g., Jónsson [13].) For example, (iii) of Theorem 3.1, Jónsson's condition in Corollary 3.2, and (v) in the previous section are Mal'cev conditions. These conditions are named after A. I. Mal'cev, who has proved in [14] that a variety $\mathcal{U}$ of universal algebras is congruence permutable if and only if there exists a ternary term t, corresponding to the type of $\mathcal{U}$, such that the identities $t(x, z, z)=x$ and $t(x, x, z)=z$ are satisfied in $\mathcal{U}$. An analogous result is true for **SP** closed classes of ordered algebras with the surprising consequence that these classes allow only trivial orderings whenever they are order-congruence permutable. (Therefore the permutability of *order*-congruences seems to have not much importance. However, to claim its unimportance we need the following generalization of Mal'cev's result.)

PROPOSITION 5.1. *For any* **S** *and* **P** *closed class $\mathcal{U}$ of ordered algebras the following three conditions are equivalent:*

(i) *$\mathcal{U}$ is order-congruence permutable, i.e., if Φ and Ψ are order-congruences of any member of $\mathcal{U}$ then $\Phi\circ\Psi=\Psi\circ\Phi$;*

(ii) *$\mathcal{U}$ is congruence permutable (i.e., congruences in the usual sense of its members commute) and its members have trivial (i.e., equality) orderings;*

(iii) *There exists a ternary term t (corresponding to the type of $\mathcal{U}$) such that the (order-) identities $t(x, z, z)=x$, $t(x, x, z)=z$ hold in $\mathcal{U}$.*

PROOF. (i) *implies* (iii). Suppose (i) and consider Θ_{xy}, Θ_{yz}, the order-congruences of the free algebra $F_{\mathcal{U}}(x, y, z)$, generated by (x, y) and (y, z), respectively. Now $(x, z)\in\Theta_{xy}\circ\Theta_{yz}$ implies $(x, z)\in\Theta_{yz}\circ\Theta_{xy}$, whence $(x, t)\in\Theta_{yx}$ and $(t, z)\in\Theta_{xy}$ for some $t=t(x, y, z)\in F_{\mathcal{U}}(x, y, z)$. Defining a homomorphism $\varphi\colon F_{\mathcal{U}}(x, y, z)\to F_{\mathcal{U}}(x, z)$ by $x\mapsto x, y\mapsto z, z\mapsto z$, we have $\Theta_{yz}\subseteq\operatorname{Ker}\varphi$. Thus $x=x\varphi=t(x, y, z)\varphi=t(x\varphi, y\varphi, z\varphi)=t(x, z, z)$, while the satisfaction of the other identity follows similarly.

(iii) *implies* (ii). It suffices to show that the members of $\mathcal{U}$ do not allow nontrivial orderings, because then congruences and order-congruences are the same and Mal'cev's above mentioned theorem applies. (No matter that $\mathcal{U}$ is not necessarily a variety, consider the variety generated (in the usual sense) by it.) Assume that $a, b\in A\in\mathcal{U}$, $a\neq b$ and $a\leqq b$. Then $b=t(a, a, b)\leqq t(a, b, b)=a$ is a contradiction.

Finally, (ii) trivially implies (i).

Now we intend to present an algorithm which associates a strong (i.e., containing a finite number of prescibed formulae) Mal'cev type condition $M(p^{(m)}\leqq q^{(n)})$ with an arbitrary lattice identity $p\leqq q$ and integers $m, n\geqq 2$ such that the following result can be stated. (Note that $M(p^{(m)}\leqq q^{(n)})$ is not a Mal'cev condition in the sense of Jónsson [13].)

THEOREM 5.2. *For any class $\mathcal{U}$ of ordered algebras closed under* **S** *and* **P** *and for any lattice identity $p\leqq q$ the following three conditions are equivalent:*

(i) *The lattice identity $p\leqq q$ holds in the lattice of order-congruences of any member of $\mathcal{U}$;*

(ii) *For any integer $m\geqq 2$ there exists an integer $n\geqq 2$ such that the Mal'cev type condition $M(p^{(m)}\leqq q^{(n)})$ is satisfied in $\mathcal{U}$;*

(iii) *$p\leqq q$ holds in the order-congruence lattices of finitely generated members of $\mathcal{U}$.*

Before defining the Mal'cev type conditions involved in Theorem 5.2 some remarks will be made. This theorem can be considered as a generalization of Wille's one [18]. (Really, if $\mathcal{U}$ happens to consist of trivially ordered algebras then any universal Horn sentence of $M(p^{(m)}\leqq q^{(n)})$ is equivalent to an identity and $M(p^{(m)}\leqq \leqq q^{(n)})$ turns into a strong Mal'cev condition, which only slightly differs from Wille's one.) Even their proofs are similar, the only essential difference is the use of Proposition 2.2 instead of the well-known description of join of congruences. (For the proof of Wille's theorem see, beside [18], Pixley [17], but the proof cited in [11] is also recommended since its form is near to our approach.) Hence the proof of Theorem 5.2 would not be surprising for those who are acquainted with that of Wille's theorem and Theorem 3.1. Thus the proof will be omitted because of its length.

To make our Mal'cev type conditions visible we shall use a pictorial approach. Finally note that if $p\leqq q$ is the distributive law then (ii) of Theorem 5.2 is much less handlable than condition (iii) of Theorem 3.1.

The definition of $M(p^{(m)}\leqq q^{(n)})$ starts with the recursive definition of $G_m(p)$, the graph of the lattice term p of order m. The graph $G_m(p)$ has coloured edges (the colours are the sign $\leqq$ and the variables of p) and two of its vertices, the so-called left and right endpoints, have special rôle. In the figures the left endpoint will be placed on the left-hand side, and dually.

If p is a variable then $G_m(p)$ has only a single edge coloured by p, which connects the two endpoints.

To obtain $G_m(p_1\wedge p_2)$ take disjoint copies of $G_m(p_1)$ and $G_m(p_2)$ and glue their left (right, resp.) endpoints together (Figure 1).

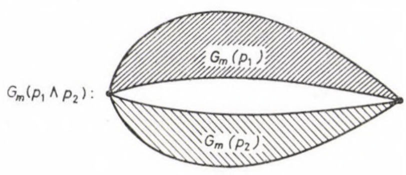

Fig. 1.

To define $G_m(p_1\vee p_2)$ consider $2m$ disjoint graphs $H_1, H_2, \ldots, H_m, H^1, H^2, \ldots, \ldots, H^m$ where H_i and H^i are copies of $G_m(p_j)$ for $i\equiv j$ (3) and $j\in\{1, 2\}$, while for

$i \equiv 0$ (3) let H_i and H^i be copies of the graph consisting of a single oriented edge coloured by $\leqq$:

$$\circ \xrightarrow{\leqq} \circ$$

Now glue together:

the right endpoint of H_i and the left one of H_{i+1} for $i=1, \ldots, n-1$,
the right endpoint of H^i and the left one of H^{i+1} for $i=1, \ldots, n-1$,
the left endpoint of H_1 and the right endpoint of H^m,
the left endpoint of H^1 and the right endpoint of H_m.

The obtained graph is $G_m(p_1 \vee p_2)$, its left (right, resp.) endpoint is the left endpoint of H_1 (H^1, resp.). (Note that, exceptionally, the left endpoint of H^i is placed on the right-hand side on Figure 2, and conversely.)

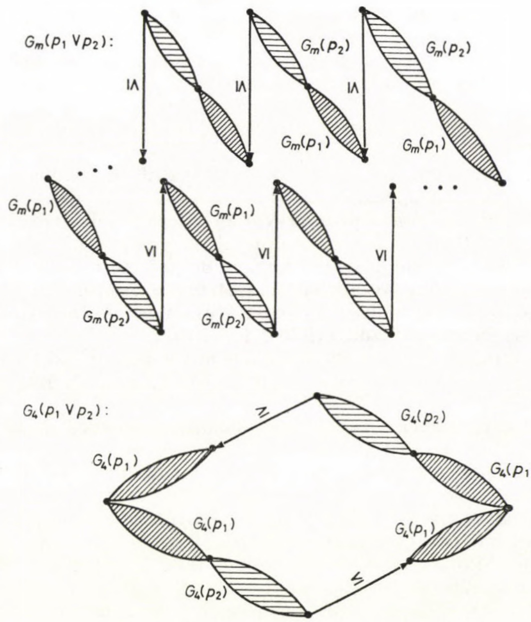

Fig. 2.

The graph $G_n(q)$ is defined in the same way. Let $X=\{x_0, x_1, \ldots, x_k\}$ and $T=\{t_0, t_1, \ldots, t_s\}$ be the vertex set of $G_m(p)$ and $G_n(q)$, respectively, such that x_0 and t_0 are the left endpoints while x_1 and t_1 are the right ones. For each variable α occurring in $p \leqq q$ let Θ_α be the smallest equivalence relation of the set $\{0, 1, \ldots, k\}$ under which i and j collapse whenever x_i and x_j are connected with an α-coloured edge in $G_m(p)$. Now $G(p^{(m)} \leqq q^{(n)})$ is obtained from $G_n(q)$ via replacing the colour α, for all variables α of $p \leqq q$, by Θ_α on each α-coloured edge of $G_n(q)$.

For an equivalence Θ of $\{0, 1, \ldots, k\}$ and $i \in \{0, 1, \ldots, k\}$ let $i\Theta = \min \{j \mid j\Theta i\}$. With a Θ-coloured edge of $G(p^{(m)} \leqq q^{(n)})$ connecting the vertices t_u and t_v we associate the universally quantified Horn sentence "if $x_i \leqq x_j$ for all edges

$x_i \circ\!\xrightarrow{\leqq}\!\circ x_j$ of $G_m(p)$ then $t_u(x_{0\Theta}, x_{1\Theta}, \ldots, x_{k\Theta}) = t_v(x_{0\Theta}, x_{1\Theta}, \ldots, x_{k\Theta})$" while with an edge $t_u \circ\!\xrightarrow{\leqq}\!\circ t_v$ of $G(p^{(m)} \leqq q^{(n)})$ the universal Horn sentence "if $x_i \leqq x_j$ for all edges $x_i \circ\!\xrightarrow{\leqq}\!\circ x_j$ of $G_m(p)$ then $t_u(x_0, x_1, \ldots, x_k) \leqq t_v(x_0, x_1, \ldots, x_k)$" will be associated.

Finally, $M(p^{(m)} \leqq q^{(n)})$ is defined to be the following condition:

"There exist $(k+1)$-ary terms $t_0(x_0, x_1, \ldots, x_k)$, $t_1(x_0, x_1, \ldots, x_k), \ldots, \ldots, t_s(x_0, x_1, \ldots, x_k)$ such that the two endpoint Horn sentences "$x_i \leqq x_j$ for all edges $x_i \circ\!\xrightarrow{\leqq}\!\circ x_j$ of $G_m(p)$ imply $t_l(x_0, x_1, \ldots, x_k) = x_l$" $(l=0, 1)$ and the Horn sentences associated with the edges of $G(p^{(m)} \leqq q^{(n)})$ are satisfied".

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(Received September 25, 1981)

Acta Math. Hung.
41 (1—2), (1983), 27—36.

ENUMERATION OF CONNECTED SPANNING SUBGRAPHS OF A PLANAR GRAPH

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1. Introduction

One of the important solved problems in graph theory is the enumeration of distinct connected spanning *trees* contained in a given connected graph; historically it was first solved by Kirchhoff. Since a tree graph contains *no cycle* by definition, the next question of importance is naturally the problem of enumeration, for a given connected graph, of connected spanning subgraphs each of which contains only a *single cycle* (besides the trees attached to the particular cycle). In general, the problems of interest are the enumeration of the connected spanning subgraphs having a preassigned cyclomatic number, i.e. each containing a fixed number of *cycles* (besides the trees attached), for a given graph.

The enumeration of spanning *trees* for a labelled graph is usually computed by means of the *adjacecy matrix* of the given graph. On the other hand, in contrast to the *adjacency matrix* method (or essentially the "incidence matrix" method), we approached the problem some years ago by a *dual* notion and derived a computational expression using the concept of *cycles* in defining the required matrix entries [1]. The *duality* is, of course, referred to that between the *vertices* and the *cycles*; the matrix entries are indexed by the *vertices* of the given graph in the adjacency matrix method, while they are indexed by the *assigned cycles* in the latter approach. It was already apparent to us, at that time, that the computational effectiveness of these two approaches depends critically on the nature of the given graph. Some examples were given in that paper [1] to point out that the *adjacency matrix* method is clearly not as effective as the "cycle matrix" method if the given graph involves many vertices but very few cycles (and vice versa). However, the formal expressions derived by either approach are of equal simplicity and elegance. In the present investigation, we rely on the concept of *cycles*. However, a direct application of *cycle matrices* [1] does not appear to be very effective. As it turns out, the problems can be handled efficiently by introducing the so-called *cycle-adjacency matrix* for a given connected graph after labelling the cycles considered. In carrying out the dual notion to the usual adjacency-matrix, it is necessary to impose the requirement that each edge of the graph can belong at most to two independent cycles.

Using this matrix, together with some further auxiliary notions, we derive the explicit expressions for the enumeration of the connected subgraphs (of a planar graph) each containing *one* and *two* cycles. These explicit expressions suggest immediately the general expression for n cycles. Though it is natural to try to prove it by a mathematical induction on n, yet the involvement of determinants makes it computationally very complicated. We resolve this by introducing the i-th "annihilation operator" which *deletes* the i-th column and the i-th row in the cycle-adjacency matrix. Together with a *formal procedure*, the operator method provides a straightforward proof of the general expression for any given cycles. As a by-product, this method provides very neat expressions for *sums* of various enumerations; in particular, the case of all spanning subgraphs has the form of an exponentiation of the annihilation operator acting on the cycle-adjacency matrix.

2. The cycle-adjacency matrix

Let G be a finite connected planar graph. Denote by S a set of independent cycles in G over the field $\mathbf{Z}_2$. Let $S\equiv\{C_i\}_{i=1,\dots,n}$ where C_i are cycles in S. Call n the degree of S (write $n\equiv\deg S$). Let N be the cyclomatic number of G, $N\equiv\equiv e-v+1$, where e and v are, respectively, total numbers of edges and vertices in G. For $n=\deg S\leqq N$, the *cycle-adjacency matrix* of G *relative to* S, denote by $\mathbf{E}_S$ (or simply $\mathbf{E}$ when there is no confusion), is defined by

$$\text{(1)}\qquad \mathbf{E}_S=\begin{pmatrix} E_{11} & E_{12} & \dots & E_{1n}\\ E_{21} & E_{22} & \dots & E_{2n}\\ \cdot & \cdot & \ddots & \cdot\\ \cdot & \cdot & & \cdot\\ \cdot & \cdot & & \cdot\\ E_{n1} & E_{n2} & \dots & E_{nn}\end{pmatrix}$$

where $E_{ij}\equiv-$(total number of edges common to C_i and C_j), if $i\neq j$. $\mathbf{E}_{ii}$ is defined as the total number of edges belonging to C_i. We emphasize that, in defining (1), each edge of G can belong at most to *two* independent cycles of S. As an example, consider the graph of Figure 1(a). The cycles considered are indicated by dotted lines. They are ordered, as indicated in the figure. If we consider only *two* of the cycles in this example, for instance, we choose $S'=\{C_1, C_2\}$, then the cycle-adjancency matrix is just

$$\text{(2)}\qquad \mathbf{E}_{S'}=\begin{pmatrix} 4 & -2\\ -2 & 4\end{pmatrix},$$

where $C_1\equiv\{a, b, e, g\}$ and $C_2\equiv\{c, d, e, g\}$ as indicated by dotted circuits in Fig. 1(b.)

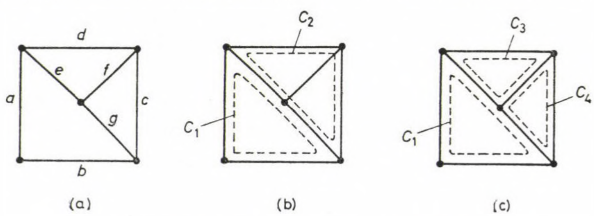

Fig. 1. Examples of cycle-adjacency matrix

If, instead, we consider a set of three independent cycles and choose $S''\equiv\equiv\{C_1, C_3, C_4\}$, then the cycle-adjacency matrix is (see Fig. 1(c))

$$\text{(3)}\qquad \mathbf{E}_{S''}=\begin{pmatrix} 4 & -1 & -1\\ -1 & 3 & -1\\ -1 & -1 & 3\end{pmatrix},$$

where $C_3\equiv\{d, e, f\}$ and $C_4\equiv\{c, f, g\}$.

EXAMPLE. As another example, let us consider Fig. 2 in which the cycles 1, 2 and 3 form an independent set yet the cycles 2 and 3 are *not faces* (in graph-theoretical terms). The corresponding cycle-adjacency matrix is

$$\mathbf{E} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} \tag{4}$$

Fig. 2. Example of cycle-adjacency matrix

For a set S of independent cycles, it is useful to introduce the following equivalence relation: two cycles C_i and C_j are said to be in the same *eqivalent class in S* iff there exists a set of cycles in S (with labels $k_1, \ldots, k_r$ say) such that the product $E_{ik_1}E_{k_1k_2}\ldots E_{k_r j} \neq 0$. Therefore, if there are two equivalent classes in S, then the cycle-adjacency matrix has the form

$$\mathbf{E}_S = \begin{pmatrix} \mathbf{E}'_{S'} & \mathbf{0} \\ \mathbf{0} & \mathbf{E}''_{S''} \end{pmatrix} \tag{5}$$

with $S = S' \cup S''$ and $S' \cap S'' = \emptyset$ where S' and S'' are subcollections in S.

With G, S, C_i, n and $\mathbf{E}_S$ defined as in the beginning of this section, we have the following useful fact.

LEMMA I. *For any given set S of indepedent cycles,* $\det \mathbf{E}_S$ *is equal to all the possible ways of deleting* $n(\equiv \deg S)$ *distinct edges, each of which belongs to a different cycle in S, such that the remaining subgraph is still connected.*

PROOF. First, if all the cycles in S are in different equivalent classes then the lemma is trivial. For two cycles not in the same equivalent class, (5) yields

$$\det \mathbf{E}_S = \det \mathbf{E}'_{S'} \cdot \det \mathbf{E}''_{S''}. \tag{6}$$

Hence it is enough to consider the case where all cycles are in the same equivalent class.

It is convenient to construct a new symmetric matrix $\mathbf{E}^{\#}$ for a given S ($\deg S = n$ is assumed) by

$$E^{\#}_{ij} = E_{ij} \quad \text{for} \quad i, j \leqq n \tag{7}$$

$$E^{\#}_{i,n+1} \equiv -\sum_{j=1}^{n} E^{\#}_{ij}, \quad i = 1, \ldots, n, \tag{8}$$

and

$$E^{\#}_{n+1,n+1} \equiv -\sum_{j=1}^{n} E^{\#}_{n+1,j}. \tag{9}$$

Then this matrix $E^{\#}$ has exactly the form of a Kirchhoff matrix [2], i.e., it is symmetric and the conditions

$$\sum_{j=1}^{n+1} E_{ij}^{\#} = 0, \quad i = 1, \ldots, n+1$$

are satisfied. Hence, from the fact that det**E** is equal to the cofactor of $E_{n+1,n+1}^{\#}$ in $\mathbf{E}^{\#}$, we see that det**E** is the number of ways to delete n distinct edges, each of which belongs to a different cycle in S, such that the remaining subgraph is connected.

THEOREM I. *Let G be a connected planar graph and S be a set of independent cycles in G. If the degree of S is equal to the cyclomatic number N, then the total number of spanning trees σ_0 of G is given by*

$$\sigma_0 = \det \mathbf{E}_S. \tag{10}$$

PROOF. It follows from Lemma I that the deletion of $n(\equiv \deg S)$ distinct edges, each from a different cycle of S, yields a spanning tree of G. This establishes the theorem.

We note that in certain cases it is more convenient to use (10) than the usual adjacency-matrix method. As an example, consider the r-sided polygon. (10) yields the trivial result ($\deg S=1$ here):

$$\sigma_0 = \det \mathbf{E}_S = \det \{\text{a single entry, } r\} = r \tag{11}$$

while the adjacency matrix method is obviously more involved in this case. In this particular example, there is only one choice for S, i.e. the single cycle along the polygon edges.

3. Enumeration of spanning subgraphs containing one and two cycles

To simplify notation, from now on we fix a set S of independent cycles and write $\mathbf{E}$ in place of $\mathbf{E}_S$. We denote by $\mathbf{E}(i)$ the matrix obtained from $\mathbf{E}$ by deleting the i^{th} row and i^{th} column. Similarly, $\mathbf{E}(i,j)$ is the matrix with i^{th} and j^{th} rows as we llas i^{th} and j^{th} columns deleted from $\mathbf{E}$, etc. We next introduce the following conventions:

$$\mathbf{E}(1, \ldots, n) \equiv 1, \quad n \equiv \deg S \tag{12}$$

and

$$\mathbf{E}(i_j, \ldots, i_m) \equiv 0, \quad \text{if some } i_j = i_k \text{ with } j \neq k. \tag{13}$$

It is important to emphasize that in $\mathbf{E}(i)$ (and similarly for $\mathbf{E}(i,j)$, etc.) the deletion implemented in the matrix does not actually correspond to the graphic deletion of the edges associated with the i^{th} cycle. Its precise meaning is rather that the i^{th} cycle does not enter our consideration.

Similar to the definition of σ_0 (i.e., the total number of spanning trees in G), we denote by σ_j the number of ways to delete $(n-j)$ edges, each of which belongs to a different cycle in S ($\deg S \equiv n$), such that the resulting subgraph remains connected. The cyclomatic number of this subgraph is just j in the special case when $n = \deg S = N$, where N is the cyclomatic number of the graph G. *From now on, we shall res-*

trict ourselves to the cases where $n=N$, the cyclomatic number of G, unless otherwise stated.

THEOREM II.

$$\sigma_1 = \sum_{i=1}^{n} \det \mathbf{E}(i) + \frac{1}{2} \sum_{i,j=1}^{n} E_{ij} \det \mathbf{E}(i,j). \tag{14}$$

PROOF. It follows from Lemma I that $\det \mathbf{E}(i)$ is equal to the number of ways to delete $(n-1)$ edges, each of which belongs to a different cycle in $S-C_i$, such that the resulting subgraph remains connected. In the first sum of (14), for any partial sum $\det \mathbf{E}(i)+\det \mathbf{E}(j)$, the deletion of $(n-1)$ edges common to cycles C_i and C_j is effectively carried out *twice*. The number of repetitions amounts to $-E_{ij} \det \mathbf{E}(i,j)$. Hence

$$\det \mathbf{E}(i)+\det \mathbf{E}(j)+E_{ij} \det \mathbf{E}(i,j)$$

is exactly the number of distinct deletions of $(n-1)$ edges either entirely from $S-C_i$ or entirely from $S-C_j$ such that the resulting graph remains connected, without the repetitions of those deletions that involve the removal of any edge in $C_i \cap C_j$. Since the deletion of $(n-1)$ edges, each from a different cycle, yields a spanning subgraph containing only a *single* cycle (with trees atteched), this finishes the proof.

By an argument similar to Theorem II, we find

$$\sigma_2 = \sum_{\substack{i,j \\ (i<j)}}^{n} \det \mathbf{E}(i,j) + \sum_{\substack{i,j,k \\ (i<j)}}^{n} E_{ij} \det \mathbf{E}(i,j,k) + \frac{1}{2} \sum_{\substack{i,j,k,l \\ (k<j;\, k<l)}}^{n} E_{ij} E_{kl} \det \mathbf{E}(i,j,k,l). \tag{15}$$

σ_2 is the number of ways to delete $(n-2)$ edges, each belonging to a different cycle of S such that the resulting spanning subgraph has cyclomatic number 2.

However, the result of σ_0, σ_1 and σ_2, given by (10), (14) and (15), suggests a unified elegant approach by adapting an operator method and a "formal" procedure, to be discussed in Section 4.

4. The general enumeration problem and a formal procedure

Following the discussion of the last section, it is now possible to consider the general enumeration problem of total number of (connected) spanning subgraphs. However, it is much more convenient to use a "formal" procedure. For every $C_i \in S$ we introduce the so-called "annihilation operator" α_i:

$$\alpha_i \mathbf{E} = \mathbf{E}(i), \quad (\alpha_i \alpha_j)\mathbf{E} = \mathbf{E}(i,j), \text{ etc.} \tag{16}$$

Note that $\alpha_i \alpha_i = 0$ and $\alpha_i \alpha_j = \alpha_j \alpha_i$. Hence $\mathbf{E}(i,j)=\mathbf{E}(j,i)$, etc.

To establish the formal procedure to be used in the proofs of theorems to follow, we introduce here a vector-space structure.

This vector-space V consists of all formal linear combinations with real coefficients; of the form $a_1 * A_1 + \ldots + a_m * A_m$ where A_i are (symmetric) matrices whose entries are indexed by some subsets of $\mathbf{n} \equiv \{1, \ldots, n\}$, a_i are real numbers. These formal sums form a real vector space under the following rules of addition and scalar multiplication:

$$b(a*\mathbf{A}) = (ba)*\mathbf{A} = (ab)*\mathbf{A} \tag{17}$$

$$(a+b)*\mathbf{A} = a*\mathbf{A}+b*\mathbf{A} \tag{18}$$

$$c(a*\mathbf{A}+b*\mathbf{B}) = (ca)*\mathbf{A}+(cb)*\mathbf{B}. \tag{19}$$

Since the "formal" sum does not allow the matrices to be added in the ordinary sense, we emphasize the following processes are not allowed:

$$a*(b\mathbf{A}) = (ab)*\mathbf{A} \quad (\textit{no-go!})$$

$$a*(\mathbf{A}+\mathbf{B}) = a*\mathbf{A}+a*\mathbf{B} \quad (\textit{no-go!})$$

Hereafter we shall use the notation:

$$\mathbf{E}^* \equiv 1*\mathbf{E}. \tag{20}$$

The annihilation operators are now defined as follows: for any formal sum $\sum a_k * A_j$ and $i \in \mathbf{n}$, define $\alpha_i \sum a_j * A_j = \sum a_k * A_k(i)$ where $A_k(i)$ is obtained from A_k by deleting the i-th row and column if they occur in A_k and is zero otherwise.

$$\alpha_i \mathbf{E}^* = \mathbf{E}(i)^* \tag{21}$$

$$\alpha_j \mathbf{E}(i)^* = \mathbf{E}(i, j)^* \tag{22}$$

$$(\alpha_i \dots \alpha_j)\mathbf{E}^* = \mathbf{E}(i, \dots, j)^* \tag{23}$$

$$(a\alpha_i + b\alpha_j)\mathbf{E}^* = a\mathbf{E}(i)^* + b\mathbf{E}(j)^* \tag{24}$$

$$\alpha_i[a\mathbf{E}(j)^* + b\mathbf{E}(k)^*] = a\mathbf{E}(i, j)^* + b\mathbf{E}(i, k)^*. \tag{25}$$

Further successive operations by operators α_i can be carried out similarly.

The final step in our formal procedure is to define the functional **det** on V:

$$\mathbf{det}\colon V \to \mathbf{R} \tag{26}$$

by

$$\mathbf{det}\left(\sum_i a_i A_i^*\right) = \sum_i a_i \det A_i, \tag{27}$$

where the det on the right-hand side of (27) is just the usual determinant of a square matrix. Denote $\mathbf{det}\,\mathbf{T} = T$ and call $\mathbf{T}$ the *generating formal matrix* of T. As an example of (27), we have (assume $i \neq j$ and $k \neq r$):

$$\mathbf{det}\{a_1 + a_2\mathbf{E}(i)^* + a_3\mathbf{E}(j)^* + a_4\mathbf{E}(k, m)^* + a_5\mathbf{E}(r, s)^*\} = \tag{28}$$

$$= a_1 + a_2 \det \mathbf{E}(i) + a_3 \det \mathbf{E}(j) + a_4 \det \mathbf{E}(k, m) + a_5 \det \mathbf{E}(r, s).$$

Note that $\mathbf{E}(i)$ and $\mathbf{E}(j)$ have the same dimension, but we may not add them in the sense of ordinary matrix addition.

Define the following operator:

$$\alpha \equiv \sum_{i=1}^{n} \alpha_i + \frac{1}{2} \sum_{i,j=1}^{n} E_{ij}\alpha_i\alpha_j. \tag{29}$$

We now complete the "formal" procedure: to each σ_j there corresponds a formal matrix $\boldsymbol{\sigma}_j$ defined inductively by

(30) $$\boldsymbol{\sigma}_0 \equiv E^*,$$

(31) $$\boldsymbol{\sigma}_1 \equiv \sum_i \mathbf{E}(i)^* + \frac{1}{2} \sum_{i,j} E_{ij}\mathbf{E}(i,j)^* = \alpha\mathbf{E}^*$$

(32) $$\boldsymbol{\sigma}_2 \equiv \frac{1}{2!}\alpha^2 \mathbf{E}^*, \text{ etc.}$$

The formal matrices $\boldsymbol{\sigma}_j$ defined here are the "generating" formal matrices of σ_j.

THEOREM III.

(33) $$\boldsymbol{\sigma}_m \equiv \frac{1}{m!}\alpha^m \mathbf{E}^*$$

is a "generating" formal matrix for σ_m, *i.e.* $\det \boldsymbol{\sigma}_m = \sigma_m$.

PROOF. We recall that σ_m is the number of ways of deleting $(n-m)$ edges each belonging to a different cycle in S such that the resulting subgraph remains connected. The proof is carried out by an induction on m. The theorem is trivial for $m=0$ and 1. For $m>1$, let the theorem be true for $m=k>1$. Consider, in particular, when $\mathbf{E}$ is $\mathbf{E}(i)$ or $\mathbf{E}(i,j)$. Then, the term $\frac{1}{k!}\alpha^k\mathbf{E}(i)^*$ corresponds to the ways of deleting $(n-k-1)$ edges each from a different cycle in $S-C_i$ such that the resulting graph remains connected. For $i\neq j$, the sum

(34) $$\frac{1}{k!}\alpha^k\{\mathbf{E}(i)^* + \mathbf{E}(j)^*\}$$

is a generating-matrix term provided the repetition of E_{ij} in the enumeration is removed, i.e.,

(35) $$\frac{1}{k!}\alpha^k\{\mathbf{E}(i)^* + \mathbf{E}(j)^*\} + \frac{1}{k!}E_{ij}\alpha^k\mathbf{E}(i,j)^*,$$

corresponds, in the sense of the "generating" formal matrix, the ways of deleting $(n-k-1)$ edges either entirely from $S-C_i$ or entirely from $S-C_j$ such that the remaining graph remains connected, without the repetitions of those deletions that involve the removal of any edge in $C_i \cap C_j$. We note that the last term in (35) is in fact *negative*, due to the sign of E_{ij} as defined in the beginning. Taking into account all cycles in S, (35) yields the matrix

(36) $$\frac{1}{k!}\alpha^k \sum_{i=1}^{n} \mathbf{E}(i)^* + \frac{1}{k!}\alpha^k \sum_{\substack{i,j \\ (i<j)}} E_{ij}\mathbf{E}(i,j)^*$$

which is just $\frac{1}{k!}\alpha^{k+1}\mathbf{E}^*$. Finally, we must remove the $(k+1)$-fold repetitions appear-ing in $\frac{1}{k!}\alpha^{k+1}\mathbf{E}^*$, i.e.,

$$\sigma_{k+1} = \frac{1}{(k+1)!}\,\alpha^{k+1}\mathbf{E}^* \tag{37}$$

which yields the "generating formal matrix" corresponding to the ways of deleting $(n-k-1)$ edges each from a different cycle in S such that the resulting graph remains connected. This completes the induction.

By definition, it is clear

$$\alpha^m\mathbf{E}^* = 0 \quad \text{for} \quad m > n \equiv \deg S. \tag{38}$$

Hence

$$\sum_{i=0}^{n} \boldsymbol{\sigma}_i = e^\alpha \mathbf{E}^* \tag{39}$$

which yields immediately the following result.

THEOREM IV. *For a planar graph with* $n=N$ (*i.e.*, deg $S=e-v+1$), $e^\alpha\mathbf{E}^*$ *is the "generating" formal matrix that yields the total number of all possible spanning subgraphs* (*with cyclomatic numbers* 0, 1, 2, ..., *etc.*) *of the given graph.*

To get some feeling of Theorem IV, it is instructive to check the terms σ_n and σ_{n-1} by an expansion of $e^\alpha\mathbf{E}^*$. First, by (26)

$$\boldsymbol{\sigma}_n \equiv \frac{1}{n!}\alpha^n\mathbf{E}^* = \frac{1}{n!}\Big(\sum_i \alpha_i + \frac{1}{2}\sum_{i,j} E_{ij}\alpha_i\alpha_j\Big)^n \mathbf{E}^* = \tag{40}$$

$$= \frac{1}{n!}\Big(\sum_i \alpha_i\Big)^n \mathbf{E}^* = \frac{1}{n!}\sum_{\{i\}}\Big(\prod_{r=0}^{n} \alpha_{i_r}\Big)\mathbf{E}^*$$

where $\{i\}=\{i_1, \ldots, i_n\}$ are all possible permutations of $\{1, \ldots, n\}$. But $\mathbf{E}(i_1, \ldots, i_n\}$ is symmetric w.r.t. its arguments, such permutation yields an $n!$ fold repetition hence, by (12),

$$\boldsymbol{\sigma}_n = \frac{1}{n!}\{n!\,\mathbf{E}(1, \ldots, n)^*\} = \mathbf{E}(1, \ldots, n)^* = 1^*. \tag{41}$$

(41) yields immediately

$$\sigma_n = \det \boldsymbol{\sigma}_n = 1. \tag{42}$$

Next, we compute σ_{n-1} from $e^\alpha\mathbf{E}^*$:

$$\boldsymbol{\sigma}_{n-1} = \frac{1}{(n-1)!}\alpha^{n-1}\mathbf{E}^* = \frac{1}{(n-1)!}\Big\{\Big(\sum_i \alpha_i\Big)^{n-1} + \frac{n-1}{2}\sum_{i,j} E_{ij}\Big(\sum_k \alpha_k\Big)^{n-2}\alpha_i\alpha_j\Big\}\mathbf{E}^*. \tag{43}$$

From ($\hat{\imath} \equiv$ deletion of i)

$$\Big(\sum_i \alpha_i\Big)^{n-1}\mathbf{E}^* = \sum_{\{i\}}\prod_{r=1}^{n-1}\alpha_{i_r}\mathbf{E}^* = (n-1)!\sum_{i=1}^{n}\mathbf{E}(1, \ldots, \hat{\imath}, \ldots, n)^* \tag{44}$$

and

$$(45)\quad \left\{\left(\sum_k \alpha_k\right)^{n-2}\alpha_i\alpha_j\right\}\mathbf{E}^* = \left(\sum_k \alpha_k\right)^{n-2}\mathbf{E}(i,j)^* = (n-2)!\mathbf{E}(1,\dots,n)^* = (n-2)!1^*$$

we find

$$(46)\qquad \boldsymbol{\sigma}_{n-1} = \sum_{i=1}^{n}\mathbf{E}(1,\dots,\hat{i},\dots,n)^* + \frac{1}{2}\sum_{\substack{i,j=1\\(i\neq j)}}^{n} E_{ij}1^*.$$

Thus

$$(47)\qquad \sigma_{n-1} = \sum_{i=1}^{n} E_{ii} + \frac{1}{2}\sum_{\substack{i,j=1\\(i\neq j)}}^{n} E_{ij} = -\sum_{\substack{i,j=1\\(i<j)}}^{n} E_{ij}.$$

Finally, let us consider the "generating" formal matrix for σ_m which yields the following result.

THEOREM V.

$$(48)\qquad \sigma_m = \sum_{r=0}^{m}\lambda_r$$

where

$$(49)\quad \lambda_r = \frac{1}{r!}\sum_{i_1<i_2}\sum_{i_3<i_4}\cdots\sum_{i_{2r-1}<i_{2r}}\ \sum_{i_{2r+1}<\dots<i_{m+r}} E_{i_1i_2}E_{i_3i_4}\dots E_{i_{2r-1}i_{2r}}\det\mathbf{E}(i_1,\dots,i_{m+r}),$$

for $r>0$, *and* λ_0 *is given by* $\lambda_0 = \sum_{i_1<\dots<i_m}\det\mathbf{E}(i_1,\dots,i_m)$, *where all* i_k $(k=1,\dots$ $\dots,m+r)$ *are distinct.*

PROOF. By (29) and Theorem IV,

$$(50)\qquad \boldsymbol{\sigma}_m = \frac{1}{m!}\alpha^m\mathbf{E}^* = \frac{1}{m!}\left(\sum_i \alpha_i + \sum_{i<j}E_{ij}\alpha_i\alpha_j\right)^m\mathbf{E}^* =$$

$$= \frac{1}{m!}\sum_{r=0}^{m}\frac{m!}{r!(m-r)!}\left(\sum_{i<j}E_{ij}\alpha_i\alpha_j\right)^r\left(\sum_i\alpha_i\right)^{m-r}\mathbf{E}^*.$$

The expansions of $(\sum_{i<j}E_{ij}\alpha_i\alpha_j)^r$ and $(\sum_i\alpha_i)^{m-r}$ may be written as:

$$(51)\qquad \left(\sum_{i<j}E_{ij}\alpha_i\alpha_j\right)^r = \sum_{i_1<i_2}\cdots\sum_{i_{2r-1}<i_{2r}}E_{i_1i_2}\dots E_{i_{2r-1}i_{2r}}\alpha_{i_1}\dots\alpha_{i_{2r}}$$

and

$$(52)\qquad \left(\sum_i\alpha_i\right)^{m-r} = (m-r)!\sum_{i_{2r+1}<\dots<i_{m+r}}\alpha_{i_{2r+1}}\dots\alpha_{i_{m+r}}$$

where the property $\alpha_i\alpha_i=0$ is used in (52). Hence, one finds

$$(53)\quad \boldsymbol{\sigma}_m = \sum_{r=0}^{m}\frac{1}{r!}\sum_{i_1<i_2}\cdots\sum_{i_{2r-1}<i_{2r}}\ \sum_{i_{2r+1}<\dots<i_{m+r}}E_{i_1i_2}\dots E_{i_{2r-1}i_{2r}}\alpha_{i_1}\dots\alpha_{i_{m+r}}\mathbf{E}^* =$$

$$= \sum_{r=0}^{m}\frac{1}{r!}\sum_{i_1<i_2}\cdots\sum_{i_{2r-1}<i_{2r}}\ \sum_{i_{2r+1}<\dots<i_{m+r}}E_{i_1i_2}\dots E_{i_{2r-1}i_{2r}}\mathbf{E}(i_1,\dots,i_{m+r})^*$$

which establishes the theorem.

5. Discussion

For a 2-connected planar graph, one can choose the cycles to be the faces. From the viewpoint of graphic duality, a spanning subgraph with m independent cycles corresponds to an $(m+1)$-forest in the dual graph. It is thus obvious that the results in Theorems I to V are applicable to the counting of forests in a planar graph. A further generalization of the formal procedure formulated here, via the annihilation operators, has also been carried out. It led to the solution of the enumeration of forests in any *non-planar* graph [3].

We would like to thank the referee for his valuable suggestions which resulted in an improvement of the manuscript.

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(Received July 4, 1980; revised March 5, 1982)

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Acta Math. Hung.
41 (1—2), (1983), 37—46.

MOST DARBOUX BAIRE 1 FUNCTIONS MAP BIG SETS ONTO SMALL SETS

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The space of all bounded Darboux Baire 1 functions on a given real interval is a complete metric space with the sup norm (denoted by $b\mathcal{D}\mathcal{B}_1$). Analogous to the case of the real-valued continuous functions on [0, 1] having the subclass of nowhere differentiable functions as a residual subset, we may also ask what subclasses of "unusual, irregular" functions in $b\mathcal{D}\mathcal{B}_1$ form residual sets. In other words what "unusual, irregular" properties of $b\mathcal{D}\mathcal{B}_1$ functions are typical in the sense of category. In [4] Ceder and Pearson confronted this question, obtaining several residual classes defined in terms of the behaviour of derived numbers and posed many unsolved problems.

In this paper we establish that a typical $b\mathcal{D}\mathcal{B}_1$ function maps the set of its approximate continuity points (resp. continuity points) onto a nowhere dense, null set. Moreover, this result can not be improved by replacing nowhere dense by countable for in this case a 1st category F_σ, dense set is obtained.

In particular, since the class of all constant functions is a nowhere dense subset of $b\mathcal{D}\mathcal{B}_1$, a typical $b\mathcal{D}\mathcal{B}_1$ function maps its set of approximate continuity points which is Borel, residual and has full measure, onto a nowhere dense set of zero measure yet the range of the function is a nondegenerate interval. In other words, a typical $b\mathcal{D}\mathcal{B}_1$ function maps "big" sets into "small" sets, both in the sense of category and measure.

Examples of such typical $b\mathcal{D}\mathcal{B}_1$ functions can be generated using the theorem of Agronsky [1] (see also Theorem 2.4 in [2] p. 13) which asserts that: if A is a bilaterally c-dense in itself F_σ subset of I, then there exists a $b\mathcal{D}\mathcal{B}_1$ function f such that $f(x)\neq 0$ if and only if $x\in A$. If such an A is chosen to have zero measure too, then we obtain one of our typical $b\mathcal{D}\mathcal{B}_1$ functions.

Moreover, our results have analogues in the space $b\mathcal{A}$ of bounded approximately continuous functions (Theorems 9, 10, 11 and Corollary 3).

Notation and terminology

In the sequel I will be any nondegenerate real interval and all the functions in $b\mathcal{D}\mathcal{B}_1$ or $b\mathcal{A}$ are assumed to have domain I. A function will always be identified with its graph. By $C^+(g, x)$, where x is a right limit point of dom g, we mean the right cluster set of g at x. Similarly with $C^-(g, x)$. A function f belongs to $\mathcal{B}_1(P)$ where P is any set if $f|P$, the restriction of f to P, is Baire 1 relative to P. A function f with domain P belongs to $\mathcal{D}(P)$ if $f(x)\in C^+(f, x)$ [resp. $C^-(f, x)$] whenever x is a right [resp. left] limit point of P. If P is an interval then $\mathcal{D}\mathcal{B}_1(P)$ coincides with the $\mathcal{D}\mathcal{B}_1$

functions defined on P. This equivalence does not hold any longer, if $\mathscr{B}_1$ is not supposed, therefore it is emphasized that the symbol $\mathscr{D}$ without denoting a set behind it always means the usual Darboux, that is the intermediate-value property (on an interval). This is important in Theorem 3.

By $|A|$ we mean the Lebesgue measure of A. By a *portion* of a perfect set P we mean a set of the form $(a, b)\cap P$ where a is a right limit point of P and b is a left limit point of P.

By C_f or $C(f)$ we mean the set of continuity points of f. When $f\in\mathscr{B}_1$, C_f is a residual G_δ. By A_f or $A(f)$ we mean the set of approximate continuity points of f. When $f\in\mathscr{B}_1$, A_f is residual, Borel and has full measure. For other information on Darboux Baire 1 and approximate continuity the reader is referred to Bruckner [2].

The proofs of the above mentioned results require some preliminary lemmas and theorems which we now present.

LEMMA 1. *Let P be a perfect set and $f\in\mathscr{B}_1(P)$. Then there exist sequences $\{f_k\}_{k=1}^\infty$ of functions and $\{B_i^k\}_{i,k=1}^\infty$ of sets such that*

(1) *for each k, $\{B_i^k\}_{i=1}^\infty$ is a family of disjoint sets each of which is an F_σ and G_δ subset of P and whose union is P;*

(2) *for each k, $\{B_i^{k+1}\}_{i=1}^\infty$ is a refinement of $\{B_i^k\}_{i=1}^\infty$;*

(3) *for each k and i, $f_k\in\mathscr{B}_1(P)$ and f_k is constant on B_i^k;*

(4) *for each k, $\|f_k-f\|<\frac{1}{2^{k+1}}$ and $\|f_{k+1}-f_k\|<\frac{1}{2^k}$.*

PROOF. See [5] p. 294, § 27. VIII. 3.

The next lemma is a generalization of Lemma 2 of [3].

LEMMA 2. *Let P be a perfect set and C be a first category (relative to P) subset of some portion G of P. Let c be a real number and λ be any positive extended real number. Then there exist $h\in\mathscr{D}\mathscr{B}_1(G)$ and a set $H\subseteqq G$ such that*

(1) *$H\subset P-C$ is a closed (relative to G) set of measure 0 and is of 1^{st} category (relative to P);*

(2) *$\{x\colon h(x)\neq c\}\subseteqq H$, and $|h-c|\leqq\lambda$;*

(3) *If N is any neighbourhood of either a or b then*

$$\operatorname{rng} h|[N\cap(a, b)\cap P]=(c-\lambda, c+\lambda);$$

(4) *$h|\{x\colon h(x)\neq c\}$ is bilaterally c-dense in itself.*

PROOF. First choose $\{I_n\}_{n=1}^\infty$ and $\{J_n\}_{n=1}^\infty$ to be sequences of disjoint portions of G, which converge monotonically to a and b, respectively and such that for each m and n, $I_n\cap J_m=\varnothing$. For each n choose P_n and Q_n to be non-void perfect, first category, null subsets of I_n-C and J_n-C, respectively. Put $H=\bigcup_{n=1}^\infty(P_n\cup Q_n)$.

Choose P_n' and Q_n' to be bilaterally c-dense-in-themselves F_σ subsets of P_n and Q_n, respectively. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence such that $\lambda_n\uparrow\lambda$. According to [1] there exist functions f_n and g_n on I_n and J_n, respectively such that $f_n\in\mathscr{D}\mathscr{B}_1(I_n)$ and $g_n\in\mathscr{D}\mathscr{B}_1(J_n)$ and $\operatorname{rng} f_n=[c, c+\lambda_n]$ or $[c-\lambda_n, c]$ according to whether n is even or odd and such that $\operatorname{rng} g_n=[c, c+\lambda_n]$ or $[c-\lambda_n, c]$ according to whether n is even or odd. Moreover, $f_n(x)\neq c$ if and only if $x\in P_n'$ and $g_n(x)\neq c$ if and only if $x\in Q_n'$. If follows from [1] also that $f_n|P_n'$ and $g_n|Q_n'$ are bilaterally c-dence-in-themselves sets. Finally define

$$h(x)=\begin{cases} f_n(x) & \text{if} \quad x\in I_n \\ g_n(x) & \text{if} \quad x\in J_n \\ c & \text{otherwise.}\end{cases}$$

It is easily checked that h is the desired function.

In Bruckner, Ceder and Keston [3] the following result was proven.

THEOREM 1. *Let $f\in\mathscr{B}_1$ on an interval I and let E be a set of first category. Then there exists $g\in\mathscr{D}\mathscr{B}_1$ such that $\{x\colon f(x)\neq g(x)\}$ is a first category, null subset of $I-E$.*

The next result relativizes this theorem to a perfect set.

THEOREM 2. *Let P be a perfect set and let $f\in\mathscr{B}_1(P)$ and E be a first category (relative to P) subset of P. Then there exists $g\in\mathscr{D}\mathscr{B}_1(P)$ such that $\{x\colon f(x)\neq g(x)\}$ is a first category (relative to P), null subset of $P-E$.*

PROOF. The proof is basically the same as that of Theorem 1, except for the proof that $g\in\mathscr{D}(P)$. Accordingly, assertions whose proofs parallel those of Theorem 1 will not be proven whereas assertions whose proofs are different will be justified. Let f_k $(k=1, 2, \dots)$ and B_k^i $(i, k=1, 2, \dots)$ be specified as in Lemma 1.

For each k put $D_k=P-\bigcup_{i=1}^{\infty}\operatorname{int} B_i^k$ (where interior is taken relative to P). Then $\{D_k\}_{k=1}^{\infty}$ is an ascending chain of closed nowhere dense subsets of P whose union, D, is a first category F_σ subset of P. Let $D_0=\emptyset$.

The proof involves an induction on a double sequence of portions of P, $\{G_m^k\}_{k,m=1}^{\infty}$ such that for a fixed k and m, f_k is constant on G_m^k. In order to describe the induction more clearly let us assume that we have such an open portion G_m^k and apply Lemma 2 where $C=E\cup D$, $G_m^k=G=P\cap(a, b)$, $\{c\}=f_k(G_m^k)$, $\lambda=\frac{3}{2^k}$ when $k>1$ and $+\infty$ when $k=1$. Let h_m^k and H_m^k be functions and sets stipulated by Lemma 2.

For each i put $A_i=\bigcup_{k=1}^{i}\bigcup_{m=1}^{\infty}H_m^k$. Then each A_i is a first category null set disjoint from $E\cup D$ and $A_i\subseteq A_{i+1}$ for each i.

To begin the induction let $\{G_m^1\}_{m=1}^{\infty}$ be an enumeration of the components (i.e. open relative intervals maximal with respect to inclusion) of the sets $\operatorname{int} B_i^1$ $(i=1, 2, \dots)$. (For notational convenience we can assume that there are infinitely many components here and also in later stages of the induction.)

Define g_1 on P as follows

$$g_1(x)=\begin{cases} h_m^1(x) & \text{if} \quad x\in G_m^1 \\ f(x) & \text{if} \quad x\notin \bigcup_{m=1}^{\infty} G_m^1.\end{cases}$$

Let P' denote $P-\{\inf P, \sup P\}$. Then it is easily checked that $g_1\in\mathscr{D}\mathscr{B}_1(P)$ and $A_1\cup D_1$ is closed in P'.

Now assume we have constructed for each i such that $1 \leqq i < k$, a sequence $\{G^i_m\}_{m=1}^\infty$ of disjoint portions of P together with sequences $\{h^i_m\}_{m=1}^\infty$ and $\{H^i_m\}_{m=1}^\infty$ and a function g_i such that

(1) $g_i \in \mathcal{DB}_1(P)$;
(2) $A_i \cup D_i$ is closed in P';
(3) $\|g_i - g_{i-1}\| \leqq \frac{5}{2^i}$ $(i \geqq 2)$;
(4) $h^j_m | H^j_m \subseteq g_i$ whenever $j \leqq i$.

Then $P-(D_k \cup A_{k-1})$ will be open in P', and letting $\{G^k_m\}_{m=1}^\infty$ be an enumeration of all components of the sets of form $S \cap \operatorname{int} B^k_i$ $(i=1, 2, \ldots)$ where S is a component of $P-(D_k \cup A_{k-1})$ we define

$$g_k(x) = \begin{cases} f(x) & \text{if } x \in D_k \\ h^k_m(x) & \text{if } x \in G^k_m \\ g_{k-1}(x) & \text{if } x \in A_{k-1}. \end{cases}$$

Then conditions (1), (2) (3) and (4) follow as in the proof of Theorem 1 with the exception of $g_k \in \mathcal{DB}_1(P)$.

To show that $g_k \in \mathcal{DB}_1(P)$ it will suffice to show that whenever x is a right limit point of P then $(x, g_k(x))$ is a right limit point of g_k. We have three cases to consider.

Case 1. For some m, $x \in G^k_m$. Since $h^k_m \subseteq g_k$ and $h^k_m \in \mathcal{D}(G^k_m)$ it follows that $g_k(x) \in C^+(g_k, x)$.

Case 2. $x \in D_k$ and $g_k(x) = f(x)$. Let i be the integer for which $x \in D_{i+1} - D_i$. Then $i < k$ and $x \in G^i_j$. Let $\{s\} = f_i(G^i_j)$. Moreover $x \notin \bigcup_{m=1}^\infty G^{i+1}_m$. Let $\delta > 0$ and choose $G^{i+1}_n \cap (x, x+\delta) \cap G^i_j \neq \emptyset$ and let $\{t_n\} = f_{i+1}(G^{i+1}_n)$. Then $\operatorname{rng} h^{i+1}_n = \left(t_n - \frac{3}{2^{i+1}},\ t_n + \frac{3}{2^{i+1}}\right)$. Since $h^{i+1}_n | H^{i+1}_n \subseteq g_k$ we also have $\left(t_n - \frac{3}{2^{i+1}},\ t_n + \frac{3}{2^{i+1}}\right) \subseteq \operatorname{rng} g_k | G^{i+1}_n$. Since $\|f_{i+1} - f_i\| < \frac{1}{2^i}$ and $\|f - f_i\| < \frac{1}{2^{i+1}}$ we have $|t_n - s| < \frac{1}{2^i}$ and $|f(x) - s| < \frac{1}{2^{i+1}}$. Consequently $|f(x) - t_n| < \frac{3}{2^{i+1}}$. It now follows that $f(x) \in C^+(g_k, x)$.

Case 3. $x \in A_{k-1}$ and $g_k(x) = g_{k-1}(x)$. Choose m and j so that $x \in H^j_m \subseteq G^j_m$ with $j < k$. Let $\{s\} = f_j(G^j_m)$. If $h^j_m(x) \neq s$, then (4) of Lemma 2 yields $h^j_m(x) \in C^+(g_k, x)$ and hence $g_k(x) \in C^+(g_k, x)$. So we may suppose that $h^j_m(x) = s$. Let $\delta > 0$ and choose n such that $G^{j+1}_n \cap (x, x+\delta) \cap G^j_m \neq \emptyset$. Let $\{t_n\} = f_{j+1}(G^{j+1}_n)$. Then $|t_n - s| < \frac{1}{2^j}$, and $\left(t_n - \frac{3}{2^{j+1}},\ t_n + \frac{3}{2^{j+1}}\right) \subseteq \operatorname{rng} h^{j+1}_n | H^{j+1}_n \subseteq \operatorname{rng} g_k | G^{j+1}_n$. Since $\{A_i\}_{i=1}^\infty$ is ascending we have $g_k(x) = g_{k-1}(x) = \ldots = g_j(x) = h^j_m(x) = s$. It follows that $g_k(x) \in C^+(g_k, x)$.

This completes the induction. Hence, $\{g_k\}_{k=1}^\infty$ is a uniformly Cauchy sequence of functions in $\mathcal{DB}_1(P)$ and therefore converges to some function g. It is easily checked, as in the proof of Theorem 1, that g is the desired function.

COROLLARY 1. *Let P be perfect, $f\in\mathcal{B}_1(P)$,* rng $f\subseteqq[c,d]$ *and E be countable. Then there exists $g\in b\mathcal{D}\mathcal{B}_1(P)$ such that $\{x: g(x)\neq f(x)\}$ is a null subset of $P-E$ and* rng $g=[c,d]$.

PROOF. Truncate the function given by Theorem 2 between c and d.

COROLLARY 2. *Let P be perfect, $a=\inf P$, $b=\sup P$, $f\in\mathcal{B}_1(P)$, E countable, $\delta>0$ and $x\in(a,b)\cap P$. Suppose $y\in P$ implies that $|f(y)-f(x)|<\varepsilon$. Then there exists $g\in b\mathcal{D}\mathcal{B}_1(P)$ such that $\{x: g(x)\neq f(x)\}$ is a null subset of $(P\cap(a,b))-E$ and* rng $g=C^-(g,b)=C^+(g,a)=[f(x)-(\delta+\varepsilon),\ f(x)+(\delta+\varepsilon)]$.

PROOF. Choose monotonic sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ in P such that $a_n\searrow a, b_n\nearrow b$ and the terms are bilateral limit points of P, finally $a_1=b_1$. Let

$$E'=E\cup\{x:\ x=a_n \quad \text{or} \quad x=b_n \quad \text{for some} \quad n\}.$$

We have $\operatorname{rng} f\subseteqq[f(x)-(\delta+\varepsilon), f(x)+\delta+\varepsilon]$. Apply Corollary 1 to E' and each interval $[a_{n+1},a_n]$ and $[b_n,b_{n+1}]$ to obtain $g_n\in b\mathcal{D}\mathcal{B}_1(P\cap[a_{n+1},a_n])$ and $h_n\in b\mathcal{D}\mathcal{B}_1(P\cap[b_n,b_{n+1}])$ such that $\{x: g_n(x)\neq f(x)\}$ and $\{x: h_n(x)\neq f(x)\}$ are null sets of $P\setminus E'$ and rng $g_n=$ rng $h_n=[f(x)-\varepsilon-\delta,\ f(x)+\varepsilon+\delta]$ for each n. Define

$$g=\left(\bigcup_{n=1}^\infty g_n\right)\cup\left(\bigcup_{n=1}^\infty h_n\right)\cup\{(a,f(a)),\quad (b,f(b))\}.$$

Note that $g(x)=f(x)$ for points of E'. It follows that g is the desired function.

THEOREM 3. *Let $f\in b\mathcal{B}_1$, $h\in b\mathcal{D}$ and $\|f-h\|<\varepsilon$ on an interval I. Then there exists $g\in b\mathcal{D}\mathcal{B}_1$ such that $\|g-f\|\leqq 4\varepsilon$ and $g=f$ a.e.*

PROOF. It clearly suffices to prove it for an open interval (a,b).

The proof is carried out by transfinite induction. Let Ω denote the first uncountable ordinal. For every $\alpha<\Omega$ we are going to define a triplet $(g_\alpha, G_\alpha, E_\alpha)$ satisfying

(1) G_α is open, E_α is countable and $E_\alpha\subset G_\alpha$;
(2) $g_\alpha\in\mathcal{D}\mathcal{B}_1(G_\alpha)$ and $g_\alpha=f$ a.e. in G_α;
(3) $\|g_\alpha-f\|\leqq 4\varepsilon$;
(4) for $\alpha<\beta$, $E_\alpha\subset E_\beta$, $G_\alpha\subset G_\beta$ and $g_\beta|_{G_\alpha}=g_\alpha$;
(5) if $G_\alpha\neq(a,b)$ then $G_{\alpha+1}\neq G_\alpha$;
(6) whenever I is a component of G_α and $x\in I\setminus E_\alpha$ then there exist $y_1<x<y_2$, $y_1,y_2\in I$ and $z_1<x<z_2$, $z_1,z_2\in I$ for which $g_\alpha(y_k)>f(x)+2\varepsilon$ $(k=1,2)$ and $g_\alpha(z_k)<f(x)-2\varepsilon$ $(k=1,2)$.

Suppose all these were done. Then the (relative in (a,b)) closed sets $Z_\alpha=(a,b)\setminus G_\alpha$ form a decreasing transnifite sequence, which has to be constant from a countable ordinal α_0. By (5) we have $G_\alpha=(a,b)$ for $\alpha\geqq\alpha_0$ and hence by (2) and (3) g_{α_0} is our desired function.

Let $x_0\in C(f)$ and let the interval $(c_0,d_0)\ni x_0$ be chosen such that for any x, $c_0\leqq x\leqq d_0$ we have $|f(x)-f(x_0)|<\varepsilon$. We apply now Corollary 2 on $P=[c_0,d_0]$ with $E=\emptyset$ and $\delta=2\varepsilon$ to get the function g. We put $(g_0,G_0,E_0)=(g,(c_0,d_0),\emptyset)$.

Now (1) is trivial, (2) is stated by Corollary 2. Since rng $g_0=[f(x_0)-3\varepsilon,\ f(x_0)+3\varepsilon]$, we have for any $y\in(c_0,d_0)$ the estimation $|f(y)-g(y)|\leqq 4\varepsilon$, thus (3) is obvious. Property (6) is again implied by Corollary 2, since $C^+(g_0, c_0)=$ $=C^-(g_0, d_0)=[f(x_0)-3\varepsilon, f(x_0)+3\varepsilon]$. Suppose now that we have defined $(g_\alpha, G_\alpha, E_\alpha)$ for every $\alpha<\beta$, where $\beta<\Omega$ is a given ordinal and $(g_\alpha, G_\alpha, E_\alpha)$ satisfies (1)—(6).

First we put $H_\beta=\bigcup_{\alpha<\beta} G_\alpha$. H_β is open (and if $\beta=\gamma+1$, then of course $H_\beta=G_\gamma$). The function $\bigcup_{\alpha<\beta} g_\alpha$ satisfies all the requirements of our theorem on H_β, therefore we may finish the induction process, if $H_\beta=(a, b)$, by putting $G_\gamma=(a, b)$, $g_\gamma=\bigcup_{\alpha<\beta} g_\alpha$ for every $\gamma\geqq\beta$. Otherwise the closed set $Z_\beta=(a, b)\setminus H_\beta$ is non-empty.

Case 1. Suppose that x_β is an isolated point of Z_β. Then $G_\beta=H_\beta\cup\{x_\beta\}$ is an open set which strictly contains all the former G_α sets ($\alpha<\beta$) and we define

$$g_\beta(x)=\begin{cases}\Big(\bigcup_{\alpha<\beta} g_\alpha\Big)(x), & \text{if } x\in H_\beta\\ f(x), & \text{if } x=x_\beta.\end{cases}$$

Let $E_\beta=\bigcup_{\alpha<\beta} E_\alpha\cup\{x_\beta\}$, then E_β is countable. The only non-trivial assertion to be proved is Darboux property in x_β (even (6) is obvious by $x_\beta\in E_\beta$). Since $h\in\mathcal{D}$ we can find sequences $\{u_n\}_{n=1}^\infty$, $\{v_n\}_{n=1}^\infty$ in $G_\beta\setminus E_\beta$ such that $v_n\to x$ from the left, $u_n\to x$ from the right, and $\lim h(u_n)=\lim h(v_n)=h(x_\beta)$. Since $\|f-h\|<\varepsilon$, we may, without loss of generality, assume that there exist $\lambda_1, \lambda_2\in\big(h(x_\beta)-\varepsilon, h(x_\beta)+\varepsilon\big)$ such that $f(u_n)\to\lambda_1$, $f(v_n)\to\lambda_2$.

Fixing n there exists a component I_α of some G_α where $\alpha<\beta$ such that $u_n\in I_\alpha$. Since $u_n\notin E_\alpha$ and $I_\alpha\subset(x_\beta, +\infty)$, we can apply (6) and thus we obtain y_n, z_n satisfying $x_\beta<y_n, z_n<u_n$ and

$$g_\beta(y_n)=g_\alpha(y_n)>f(u_n)+2\varepsilon>f(u_n)-2\varepsilon>g_\alpha(z_n)=g_\beta(z_n).$$

Hence $\operatorname{rng} g_\beta|I_\alpha\cap(x_\beta, u_n)\supseteqq[f(u_n)-2\varepsilon, f(u_n)+2\varepsilon]$. Letting $n\to+\infty$ we obtain $[\lambda_1-2\varepsilon, \lambda_1+2\varepsilon]\subset C^+(g_\beta, x_\beta)$. On the other hand $|\lambda_1-f(x_\beta)|\leqq|\lambda_1-h(x_\beta)|+|h(x_\beta)-$ $-f(x_\beta)|<2\varepsilon$, thus we have $g_\beta(x_\beta)=f(x_\beta)\in C^+(g_\beta, x_\beta)$. Similarly $g_\beta(x_\beta)\in C^-(g_\beta, x_\beta)$, and property (2) is verified.

Case 2. If the set $Z_\beta\neq\emptyset$ contains no isolated points then it is (relative in (a, b)) perfect, and now we pick a point $x_\beta\in C(f|Z_\beta)$, which is a two-sided limit point in Z_β. Then we can find two-sided limit points $c_\beta, d_\beta\in Z_\beta$, such that $c_\beta<x_\beta<d_\beta$ and $|f(x)-f(x_\beta|<\varepsilon$ for every $x\in Z_\beta\cap[c_\beta, d_\beta]$.

We apply again Corollary 2 on the perfect set $P=Z_\beta\cap[c_\beta, d_\beta]$, and E denotes all those points in P which are end-points of intervals contiguous to P, and $\delta=2\varepsilon$. Denoting by g the function of Corollary 2 we define

$$G_\beta=H_\beta\cup(c_\beta, d_\beta),\quad g_\beta=\Big(\bigcup_{\alpha<\beta} g_\alpha\Big)\cup g,\quad E_\beta=\Big(\bigcup_{\alpha<\beta} E_\alpha\Big)\cup E.$$

Property (1) is obvious for G_β and E_β. It is also clear from the construction, that g_β is Baire 1 on G_β and $g=f$ a.e. on G_β. To check the (local) Darboux property we have to do it at the points of $Z_\beta\cap G_\beta$ only. By Corollary 2, the function $g=g_\beta|Z_\beta\cap G_\beta$ is Darboux on the set $Z_\beta\cap G_\beta$. Therefore g_β is Darboux in the two-sided limit points of $Z_\beta\cap G_\beta$. Suppose, that $x\in Z_\beta\cap G_\beta$ is a left hand side limit point of $Z_\beta\cap G_\beta$, and $Z_\beta\cap(x, x+\eta)=\emptyset$ if η is small enough. Then we can repeat the argument of

Case 1 in the right hand side neighbourhood of x to obtain $g_\beta(x)=f(x)\in C^+(g_\beta, x)$. On the other hand $g_\beta(x)=f(x)=g(x)\in C^-(g, x)\subset C^-(g_\beta, x)$ by Corollary 2, thus g_β is Darboux at x. To verify (3) it is enough to consider again the points of $Z_\beta\cap G_\beta$. For $x\in Z_\beta\cap G_\beta$ we have $f(x_\beta)-\varepsilon<f(x)<f(x_\beta)+\varepsilon$ and

$$f(x_\beta)-3\varepsilon \leqq g_\beta(x) = g(x) \leqq f(x_\beta)+3\varepsilon$$

and hence $|g_\beta(x)-f(x)|\leqq 4\varepsilon$.

Properties (4) and (5) are obvious. Let I be a component of G_β. If $I\neq(c_\beta, d_\beta)$ then (6) holds on I by the induction hypothesis. Let $I=(c_\beta, d_\beta)$ (this is, in fact, a component of G_β, since $c_\beta, d_\beta\notin G_\beta$), and $x\in I\setminus E_\beta$. If $x\notin Z_\beta$, then (6) holds again by the induction hypothesis. If $x\in Z_\beta\cap I$, then we have $f(x_\beta)-\varepsilon<f(x)<f(x_\beta)+\varepsilon$ and by Corollary 2

$$C^+(g, c_\beta) = [f(x_\beta)-3\varepsilon, f(x_\beta)+3\varepsilon], \quad C^-(g, c_\beta) = [f(x_\beta)-3\varepsilon, f(x_\beta)+3\varepsilon],$$

thus we can find points y_1, z_1 (resp. y_2, z_2) in $I\cap Z_\beta$ arbitrarily close to c_β (resp. d_β) such that $g_\beta(y_k)=g(y_k)>f(x)+2\varepsilon$ and $g_\beta(z_k)=g(z_k)<f(x)-2\varepsilon$ $(k=1, 2)$.

This completes the induction construction and the proof of our theorem.

Theorem 3 gives a quick proof of the following surprising result.

THEOREM 4. *The class of all $b\mathcal{D}\mathcal{B}_1$ functions g such that $g(A_g)$ [resp. $g(C_g)$] is finite is dense in $b\mathcal{D}\mathcal{B}_1$.*

PROOF. Let $h\in b\mathcal{D}\mathcal{B}_1$ and $\varepsilon>0$, choose $f\in\mathcal{B}_1$ with $\operatorname{rng} f$ finite and $\|f-h\|<\dfrac{\varepsilon}{5}$. Applying Theorem 3 we can obtain $g\in b\mathcal{D}\mathcal{B}_1$ such that $\|g-h\|<\varepsilon$ and $g=f$ a.e. It is easy to verify that $g(A_g)$ is finite. Since $C_g\subseteqq A_g$, $g(C_g)$ is also finite.

Now we can prove our main result.

THEOREM 5. *The class of all $b\mathcal{D}\mathcal{B}_1$ functions f such that $\overline{f(A_f)}$ [resp. $\overline{f(C_f)}$] has measure zero is a residual G_δ set in $b\mathcal{D}\mathcal{B}_1$*

PROOF. We will carry out the proof for "A_f". The proof for "C_f" is the same. Let $\mathcal{H}$ consist of all f such that $\overline{f(A_f)}$ is null. By Theorem 4 $\mathcal{H}$ is dense so it suffices to show that $\mathcal{H}$ is a G_δ subset of $b\mathcal{D}\mathcal{B}_1$. Let F_n consist of all $f\in b\mathcal{D}\mathcal{B}_1$ for which $|\overline{f(A_f)}|\geqq\dfrac{1}{n}$. Then $\bigcup\limits_{n=1}^{\infty} F_n=b\mathcal{D}\mathcal{B}_1-\mathcal{H}$ so it will suffice to show that each F_n is closed.

For any $\varepsilon>0$ and set A, let $N_\varepsilon(A)$ denote $\{x\colon |x-a|<\varepsilon$ for some $a\in A\}$. For sets A, B let $d(A, B)=\inf\{\varepsilon\colon A\subseteqq N_\varepsilon(B)$ and $B\subseteqq N_\varepsilon(A)\}$. Observe that if $\|f-g\|<\varepsilon$, then $d\big(f(A_f), g(A_g)\big)\leqq 2\varepsilon$. This can be shown as follows: let $x\in A_f$. Then there exists a P with $x\in P$ and $\lambda(P)>0$ and $|f(y)-f(x)|<\varepsilon$ whenever $y\in P$. Then $|g(y)-f(x)|<2\varepsilon$ for all $y\in P$. Pick $z\in P\cap A_g$. Then $|g(z)-f(x)|<2\varepsilon$. Hence, $f(A_f)\subseteqq N_{2\varepsilon}\big(g(A_g)\big)$ from which the result follows.

Now let us show F_n is closed. Let $\|f_k-f\|\to 0$ where $f_k\in F_n$.

Suppose $|\overline{f(A_f)}|<\dfrac{1}{n}$. Then there exists an open set G containing $\overline{f(A_f)}$ such that $|G|<\dfrac{1}{n}$. By compactness we can find $\varepsilon>0$ such that $N_\varepsilon(\overline{f(A_f)})\subseteqq G$. Then choosing k such that $\|f_k-f\|<\frac{\varepsilon}{4}$ we have $d(\overline{f(A_f)}, \overline{f_k(A_{f_k})})\leqq\frac{\varepsilon}{2}$ so that $\overline{f_k(A_{f_k})}\subseteqq G$ and $|\overline{f_k(A_{f_k})}|<\frac{1}{n}$, a contradiction.

COROLLARY 3. *The class of all $b\mathcal{DB}_1$ functions f such that $f(A_f)$ [resp. $f(C_f)$] is nowhere dense and null is a residual subset of $b\mathcal{DB}_1$.*

PROOF. This class contains the residual class of Theorem 5.

It is unknown whether or not the class of all f such that $f(A_f)$ [*resp.* $f(C_f)$] is null is a G_δ set. Hence, we can not improve Corollary 3 to assert that the class is a residual G_δ. However, if we just consider those f such that $f(A_f)$ is nowhere dense, then we do get a residual G_δ as the next result shows.

THEOREM 6. *The class of all $b\mathcal{DB}_1$ functions f such that $f(A_f)$ [resp. $f(C_f)$] is nowhere dense is a residual G_δ set in $b\mathcal{DB}_1$.*

PROOF. By Theorem 4 the class $\mathcal{H}$ of all f such that $f(A_f)$ [resp. $f(C_f)$] is nowhere dense is dense in $b\mathcal{DB}_1$. So it suffices to show $\mathcal{H}$ is a G_δ set. We will carry out the proof for "A_f" since the proof for "C_f" is the same.

If $f\notin\mathcal{H}$ then there exists a rational interval J such that $J\subseteqq\overline{f(A_f)}$. Letting $\{J_n\}_{n=1}^\infty$ be an enumeration of all the rational intervals, let $E_n=\{f\colon J_n\subseteqq\overline{f(A_f)}\}$. Then $b\mathcal{DB}_1-\mathcal{H}=\bigcup_{n=1}^\infty E_n$.

Let us now show that each E_n is closed. Let $f_k\in E_n$ and $\|f_k-f\|\to 0$. It suffices to show that each open subinterval W of J_n hits $f(A_f)$. Suppose $W=(a,b)\subseteqq J_n$ and $\varepsilon=(b-a)/3$. Choose m such that $\|f_m-f\|<\varepsilon$ and put $V=(a+\varepsilon, b-\varepsilon)$. Since $V\cap f_m(A_{f_m})\neq\emptyset$ there exists a set T of positive measure such that $f_m(T)\subseteqq V$. Choose $z\in T\cap A_f$. Then $f_m(z)\in V$ and $f(z)\in W$. Hence, $f(A_f)\cap W\neq\emptyset$ and $\mathcal{H}$ is a G_δ.

We can not improve the above results by requiring $f(A_f)$ [or $f(C_f)$] to be countable instead of nowhere dense and/or null as the next results show.

THEOREM 7. *The class of all $f\in b\mathcal{DB}_1$ such that $f(A_f)$ [resp. $f(C_f)$] is countable is a dense, first category subset of $b\mathcal{DB}_1$.*

PROOF. Let us prove that the class $\mathcal{H}$ of all $b\mathcal{DB}_1$ functions f such that $f(C_f)$ is countable is a dense, first category set in $b\mathcal{DB}_1$. By Theorem 4 $\mathcal{H}$ is dense.

Let $\{J_n\}_{n=1}^\infty$ be an enumeration of the rational intervals. Put $A_n=\{f\colon f$ is constant on $J_n\cap C_f\}$. Let $f_k\in A_n$ and $f_k\to f$. Clearly f is constant on $J_n\cap\left(\bigcap_{k=1}^\infty C_{f_k}\right)$. But $\bigcap_{k=1}^\infty C_{f_k}\cap C_f$ is dense in C_f. Hence, $f\in A_n$ and A_n is closed. Moreover, it is easy to see that each A_n is nowhere dense. Hence, $\bigcup_{n=1}^\infty A_n$ is a first category set. If $f(C_f)$ is countable, then the residual G_δ set C_f is covered by countably many sets of the form $f^{-1}(c)$ where $c\in f(C_f)$. Hence, there exists $c\in f(C_f)$ and J_m such that $f^{-1}(c)$ is dense in J_m, that is, $f\in A_m$. Therefore, $\mathcal{H}\subseteqq\bigcup_{n=1}^\infty A_n$ and $\mathcal{H}$ is of first category.

Let $\mathscr{E}$ be the class of all $f \in b\mathscr{D}\mathscr{B}_1$ such that $f(A_f)$ is countable. Since $\mathscr{E} \subseteqq \mathscr{H}$, $\mathscr{E}$ too is 1st category and dense (by Theorem 4).

THEOREM 8. *The class of all $f \in b\mathscr{D}\mathscr{B}_1$ such that $f(A_f)$ [resp. $f(C_f)$] is finite is a dense, first-category F_σ subset of $b\mathscr{D}\mathscr{B}_1$.*

PROOF. Let $\mathscr{H}$ be the class of all f such that $f(A_f)$ is finite. Then by Theorem 7 $\mathscr{H}$ is dense and first category. For each n define $F_n = \{f\colon \operatorname{card}\left(f(C_f)\right) \leqq n\}$. Then, F_n is easily seen to be closed from the fact that $d\left(f(C_f), g(C_g)\right) \leqq 2\varepsilon$ whenewer $\|f-g\| < \varepsilon$. Since $\mathscr{H} = \bigcup_{n=1}^{\infty} F_n$, $\mathscr{H}$ is an F_σ set. The same proof works for the "$f(A_f)$" case.

THEOREM 9. *Let $f \in b\mathscr{B}_1$ and $\varepsilon > 0$. Then there exists $g \in b\mathscr{A}$ with $|g| < \varepsilon$ such that $h(C_h)$ is finite where $h = f - g$.*

PROOF. Consider a decomposition $[0, 1] = \bigcup_{k=1}^{n} A_k$ such that the sets A_k are pairwise disjoint, each A_k is an ambiguous F_σ-G_δ set and for the oscillation of f the estimation $\omega(f; A_k) < \varepsilon$ holds. Let $M_k \subset A_k$ be a countable set everywhere dense in A_k and pick $m_k \in M_k$. Now we put $\varphi(x) = f(m_k)$ if $x \in A_k$. Then $\varphi \in \mathscr{B}_1$ and $|f - \varphi| < \varepsilon$. According to Theorem 3.2 in [6] (p. 191) there exists $g \in b\mathscr{A}$, $|g| \leqq \varepsilon$ such shat $(f-\varphi)\Big|\bigcup_{k=1}^{n} M_k = g\Big|\bigcup_{k=1}^{n} M_k$. For any $x \in \bigcup_{k=1}^{n} M_k$ we have $f(x) - g(x) = f(x) - [f(x) - \varphi(x)] = \varphi(x)$. Thus the function $h = f - g$ takes only finitely many valus on the everywhere dense set $\bigcup_{k=1}^{n} M_k$. Therefore in a point of continuity it must take one of these values. Hence $h(C_h)$ is finite.

The class of all bounded approximately continuous functions, $b\mathscr{A}$, and the class of all bounded derivatives, $b\Delta$, form closed subspaces of $b\mathscr{D}\mathscr{B}_1$ with $b\mathscr{A} \subseteqq b\Delta$. So it is natural to ask if the analogues of Theorems 5 and 6 are valid in these spaces. The question for $b\Delta$ is open and it seems difficult*. More precisely, for any $f \in \Delta$ we always have $\overline{f(A_f)} = \operatorname{rng} f$. Indeed, referring to a well known theorem of Denjoy, a set $\{x\colon a < f(x) < b\}$ is either empty or has positive measure and then $\overline{f(A_f)} = \operatorname{rng} f$ readily follows. On the other hand, we do not know whether $f(A_f)$ is a null set for a typical derivative*. For the case of $b\mathscr{A}$ or $b\Delta$ we can prove that the analogues of Theorems 5, 6, 7 and 8 and Corollary 3 when applied to "C_f" are all valid.

Note that for $f \in b\mathscr{A}$, A_f is the domain interval I.

First we have the analogue of Theorem 4 which is much simpler because $b\mathscr{A}$ and $b\Delta$ are closed under addition whereas $b\mathscr{D}\mathscr{B}_1$ is not.

THEOREM 10. *The class of all $f \in b\mathscr{A}$ ($f \in b\Delta$) such that $f(C_f)$ is finite is dense in $b\mathscr{A}$ ($b\Delta$).*

PROOF. Let $f \in b\mathscr{A}(b\Delta)$ and $\varepsilon > 0$. Applying Theorem 9 there exists $g \in b\mathscr{A}$ such that $\|g\| < \varepsilon$ and $h(C_h)$ is finite where $h = f - g$. Then $\|f - h\| < \varepsilon$ and $h \in b\mathscr{A}(b\Delta)$.

* This problem has recently been settled by the second named author. It is proved, that for any bounded derivative $f, f(I \cap A_f) = f(I)$ for any subinterval $I \subset [0, 1]$.

An examination of the proofs of Theorems 5, 6, 7 and 8 as well as Corollary 3 shows that for $f(C_f)$ they are all valid when "$b\mathscr{A}$" or "$b\Delta$" is substituted for "$b\mathscr{D}\mathscr{B}_1$" and no other changes are made. Therefore,

THEOREM 11. *The analogues of Theorems* 5, 6, 7 *and* 8 *and Corollary* 3 *stated for* $f(C_f)$ *are valid both in* $b\mathscr{A}$ *and* $b\Delta$.

The authors wish to thank M. Laczkovich for providing useful comments during the preparation of this paper.

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(Received January 9, 1981)

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Acta Math. Hung.
41 (1—2), (1983), 47—52.

SOME RECURRENCE RELATIONS AND EINSTEIN'S CONNECTION IN 2-DIMENSIONAL UNIFIED FIELD THEORY[1]

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I. Introduction

IA. Two dimensional unified field theory (2-g-UFT). In the usual Einstein's 2-g-UFT the generalized 2-dimensional Riemannian space X_2, referred to a real coordinate system x^ν, is endowed with a real nonsymmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$[2]:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}, \tag{1.1a}$$

where

$$\mathfrak{g} \overset{\text{df}}{=} \operatorname{Det}((g_{\lambda\mu})) \neq 0, \quad \mathfrak{h} \overset{\text{df}}{=} \operatorname{Det}((h_{\lambda\mu})) \neq 0, \quad \mathfrak{k} = \operatorname{Det}((k_{\lambda\mu})) = (k_{12})^2 \neq 0. \tag{1.1b}$$

We may define a unique tensor $h^{\lambda\nu}$ by

$$h_{\lambda\mu} h^{\lambda\nu} = \delta_\mu^\nu. \tag{1.2}$$

In 2-g-UFT we use both $h_{\lambda\mu}$ and $h^{\lambda\nu}$ as tensors for raising and/or lowering indices of all tensors defined in X_2 in the usual manner.

The densities defined in (1.1)b are related by

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{k}, \tag{1.3a}$$

so that

$$g = 1 + k, \tag{1.3b}$$

where

$$g \overset{\text{df}}{=} \mathfrak{g}/\mathfrak{h}, \quad k \overset{\text{df}}{=} \mathfrak{k}/\mathfrak{h}. \tag{1.3c}$$

In particular, we note from the last condition of (1.1b) that *there exists only the first class of $k_{\lambda\mu}$ in 2-g-UFT.*

The differential geometric structure is imposed on X_2 by the tensor $g_{\lambda\mu}$ by means of a connection $\Gamma^\nu_{\lambda\mu}$ given by the system of Einstein's equations

$$D_\omega g_{\lambda\mu} = 2 S_{\omega\mu}{}^\alpha g_{\lambda\alpha}, \tag{1.4}$$

where D_ω is the symbolic vector of the covariant derivative with respect to $\Gamma^\nu_{\lambda\mu}$ and

$$S^\nu_{\lambda\mu} \overset{\text{df}}{=} \Gamma^\nu_{[\lambda\mu]}. \tag{1.5}$$

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[1] The results presented in this paper were supported by 1980-Research Grant of Asan Foundation.

[2] Throughout the present paper, all indices take the values 1, 2 and follow the summation convention with the exception of indices x, y, z. Greek indices are used for the holonomic components of a tensor and Roman indices for the nonholonomic components.

It has been shown that ([5], pp. 52) if the system (1.4) admits a solution $\Gamma^{\nu}_{\lambda\mu}$, it must be of the form

$$\Gamma^{\nu}_{\lambda\mu} = \left\{ {\nu \atop \lambda\mu} \right\} + S^{\nu}_{\lambda\mu} + U^{\nu}_{\lambda\mu}, \tag{1.6}$$

where $\left\{ {\nu \atop \lambda\mu} \right\}$ are the Christoffel symbols with respect to the tensor $h_{\lambda\mu}$ and

$$U^{\nu}_{\lambda\mu} = 2h^{\nu\alpha} S_{\alpha(\lambda}{}^{\beta} k_{\mu)\beta}. \tag{1.7}$$

IB. Purpose. The generalized 2-dimensional Riemannian space X_2 has some particular properties, probably due to the simplicity of dimension. In this paper we derive, in the first, several useful recurrence relations in X_2 which do not hold in a higher dimensional space. The purpose of the present paper is to obtain the solution of Einstein's equations in a simple tensorial form in 2-g-UFT, using newly obtained recurrence relations. This solution is the simplest ever obtained.

II. Recurrence relations in X_2.

In this section we derive several powerful recurrence relations, establishing a nonholonomic frame in X_2.

The eigenvalues M and the corresponding eigenvectors a^{ν} in X_2, defined by

$$(Mh_{\lambda\mu} + k_{\lambda\mu}) a^{\mu} = 0, \quad (M\text{: a scalar}) \tag{2.1}$$

are called *basic scalars* and *basic vectors*, respectively. There are exactly two linearly independent basic vectors $\underset{i}{a}{}^{\nu}$ satisfying (2.1), where the corresponding basic scalars $\underset{i}{M}$ are solutions of

$$\mathrm{Det}\,((Mh_{\lambda\mu} + k_{\lambda\mu})) = \mathfrak{h}(M^2 + k) = 0. \tag{2.2}$$

Therefore, the basic scalars $\underset{1}{M}$ and $\underset{2}{M}$ are given by

$$\underset{1}{M} = -\underset{2}{M} = \begin{cases} i\sqrt{k} & \text{for} \quad k > 0; \\ \sqrt{-k} & \text{for} \quad k < 0. \end{cases} \tag{2.3}$$

Since $\underset{1}{a}{}^{\nu}$ and $\underset{2}{a}{}^{\nu}$ are linearly independent, there esixts a unique reciprocal set of two linearly independent covariant vectors $\overset{1}{a}_{\lambda}$ and $\overset{2}{a}_{\lambda}$ such that

$$\underset{i}{a}{}^{\nu} \overset{i}{a}_{\lambda} = \delta^{\nu}_{\lambda}, \quad \underset{j}{a}{}^{\lambda} \overset{i}{a}_{\lambda} = \delta^{i}_{j}. \tag{2.4}$$

Using the vectors $\underset{i}{a}{}^{\nu}$ and $\overset{i}{a}_{\lambda}$ a nonholonomic frame of X_2 will be established in the following way: If $T^{\nu\cdots}_{\lambda\cdots}$ are holonomic components of a tensor, then its nonholonomic components $T^{i\cdots}_{j\cdots}$ are defined by

$$T^{i\cdots}_{j\cdots} \overset{\mathrm{df}}{=} T^{\nu\cdots}_{\lambda\cdots} \overset{i}{a}_{\nu} \underset{j}{a}{}^{\lambda} \cdots, \tag{2.5}$$

which are equivalent to

(2.6) $$T^{\nu\dots}_{\lambda\dots} = T^{i\dots}_{j\dots} \underset{i}{a}^{\nu} \overset{j}{a}_{\lambda} \dots .$$

It has been shown that ([5], pp. 22) the nonholonomic components of ${}^{(p)}k_{\lambda}{}^{\nu}$ are given by

(2.7) $${}^{(p)}k_{x}{}^{i} = \underset{x}{M}^{p}\, \delta^{i}_{x} \quad (p = 0, 1, 2, \dots),$$

where

(2.8) $${}^{(o)}k_{\lambda}{}^{\nu} \overset{\text{df}}{=} \delta^{\nu}_{\lambda}, \quad {}^{(1)}k_{\lambda}{}^{\nu} \overset{\text{df}}{=} k_{\lambda}{}^{\nu}, \quad {}^{(p)}k_{\lambda}{}^{\nu} \overset{\text{df}}{=} {}^{(p-1)}k_{\lambda}{}^{\alpha}\, k_{\alpha}{}^{\nu}.$$

THEOREM (2.1). *(The first set of recurrence relations.) We have*

(2.9) $${}^{(p+2)}k_{\lambda}{}^{\nu} + k\,{}^{(p)}k_{\lambda}{}^{\nu} = 0 \quad (p=0, 1, 2, \dots).$$

PROOF. Since the basic scalars $\underset{x}{M}$ satisfy $\underset{x}{M}^2+k=0$ by virtue of (2.3), the first recurrence relations may be derived in the following way, using (2.6) and (2.7):

$${}^{(p+2)}k_{\lambda}{}^{\nu} + k\,{}^{(p)}k_{\lambda}{}^{\nu} = \sum_{x,y} ({}^{(p+2)}k_{x}{}^{y} + k\,{}^{(p)}k_{x}{}^{y}) \overset{x}{a}_{\lambda} \underset{y}{a}^{\nu} = \sum_{x} \underset{x}{M}^{p} (\underset{x}{M}^2 + k) \overset{x}{a}_{\lambda} \underset{x}{a}^{\nu} = 0.$$

In order to derive further recurrence relations, we use [the following] Mishra's abbreviations ([6]), denoting the tensor $T_{\omega\mu\nu}$ by T:

(2.10a) $$\overset{pqr}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu} \overset{\text{df}}{=} {}^{(p)}k_{\omega}{}^{\alpha}\, {}^{(q)}k_{\mu}{}^{\beta}\, {}^{(r)}k_{\nu}{}^{\gamma}$$

(2.10b) $$\overset{pqr}{T} \overset{\text{df}}{=} \overset{pqr}{T}_{\omega\mu\nu} \overset{\text{df}}{=} \overset{pqr}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu}\, T_{\alpha\beta\gamma}, \quad T \overset{\text{df}}{=} T_{\omega\mu\nu} \overset{\text{df}}{=} \overset{000}{T}.$$

If the tensor $\overset{pqr}{T}_{\omega\mu\nu}$ is skew-symmetric in the first two indices,

(2.11) $$\overset{pqr}{T}_{\omega\mu\nu} = -\overset{qpr}{T}_{\mu\omega\nu}.$$

LEMMA (2.2). *The basic scalars satisfy*

(2.12a) $$\underset{x}{M} + \underset{y}{M} = 0,$$

(2.12b) $$\underset{x}{M}\,\underset{y}{M} - k = 0$$

for all values of x and y when $x \neq y$.

PROOF. The identities (2.12) are direct results of (2.3).

THEOREM (2.3). *(The second set of recurrence relations.) If $T_{\omega\mu\nu}$ is a tensor skew-symmetric in the first two indices, then*

(2.13a) $$\overset{(10)r}{T} = 0$$

$$(r = 0, 1, 2, \dots)$$

(2.13b) $$\overset{11r}{T} = k\, \overset{00r}{T}.$$

PROOF. Using (2.6), (2.7), and (2.10), we have

$$\overset{p(q)r}{T} = \overset{(pq)r}{T}_{\omega\mu\nu} = \sum_{x,y,z} \overset{(pq)r}{T}_{xyz}\, \overset{x}{a}_{\omega}\, \overset{y}{a}_{\mu}\, \overset{z}{a}_{\nu} =$$

$$= \frac{1}{2} \sum_{x,y,z} T_{ijk}\big({}^{(p)}k_x{}^{i}\, {}^{(q)}k_y{}^{j} + {}^{(q)}k_x{}^{i}\, {}^{(p)}k_y{}^{j}\big){}^{(r)}k_z{}^{k}\, \overset{x}{a}_{\omega}\, \overset{y}{a}_{\mu}\, \overset{z}{a}_{\nu} =$$

$$= \frac{1}{2} \sum_{x,y,z} T_{xyz}\big(\underset{x}{M^p}\, \underset{y}{M^q} + \underset{x}{M^q}\, \underset{y}{M^p}\big)\underset{z}{M^r}\, \overset{x}{a}_{\omega}\, \overset{y}{a}_{\mu}\, \overset{z}{a}_{\nu}$$

for $p, q, r=0, 1, 2, 3, \ldots$. Hence the second set of recurrence relations may be derived from the above result in the following way, respectively, using the skew-symmetry of the tensor $T_{\omega\mu\nu}$ when $x=y$ and Lemma (2.2) when $x \neq y$:

$$\overset{(10)r}{T} = \frac{1}{2} \sum_{x,y,z} T_{xyz}\big(\underset{x}{M} + \underset{y}{M}\big)\underset{z}{M^r}\, \overset{x}{a}_{\omega}\, \overset{y}{a}_{\mu}\, \overset{z}{a}_{\nu} = 0,$$

$$\overset{11r}{T} - k\, \overset{00r}{T} = \sum_{x,y,z} T_{xyz}\big(\underset{x}{M}\underset{y}{M} - k\big)\underset{z}{M^r}\, \overset{x}{a}_{\omega}\, \overset{y}{a}_{\mu}\, \overset{z}{a}_{\nu} = 0.$$

The condition (2.13a) implies that the tensor $\overset{10r}{T}_{\omega\mu\nu}$ is symmetric in the first two indices, even though the tensor $T_{\omega\mu\nu}$ is skew-symmetric in the first two indices; that is

$$\overset{10r}{T}_{\omega\mu\nu} = \overset{10r}{T}_{\mu\omega\nu} \quad (r=0, 1, 2, \ldots). \tag{2.14}$$

THEOREM (2.4). *(The third set of recurrence relations.) If $T_{\omega\mu\nu}$ is a tensor skew-symmetric in the first two indices, then*

$$\overset{r(10)}{T}_{\nu[\omega\mu]} = 0 \tag{2.15a}$$

$$(r = 0, 1, 2, \ldots)$$

$$\overset{r11}{T}_{\nu[\omega\mu]} = k\, \overset{r00}{T}_{\nu[\omega\mu]}. \tag{2.15b}$$

PROOF. As in the proof of the previous theorem, we have

$$\overset{r(pq)}{T}_{\nu[\omega\mu]} = \sum_{x,y,z} \overset{r(pq)}{T}_{x[yz]}\, \overset{x}{a}_{\nu}\, \overset{y}{a}_{\omega}\, \overset{z}{a}_{\mu} = \frac{1}{2} \sum_{x,y,z} T_{x[yz]}\, \underset{x}{M^r}\big(\underset{y}{M^p}\, \underset{z}{M^q} + \underset{y}{M^q}\, \underset{z}{M^p}\big)\overset{x}{a}_{\nu}\, \overset{y}{a}_{\omega}\, \overset{z}{a}_{\mu}.$$

Hence the third set of recurrence relations can be obtained in the following way:

$$\overset{r(10)}{T}_{\nu[\omega\mu]} = \frac{1}{2} \sum_{x,y,z} T_{x[yz]}\, \underset{x}{M^r}\big(\underset{y}{M} + \underset{z}{M}\big)\overset{x}{a}_{\nu}\, \overset{y}{a}_{\omega}\, \overset{z}{a}_{\mu} = 0,$$

$$\overset{r11}{T}_{\nu[\omega\mu]} - k\, \overset{r00}{T}_{\nu[\omega\mu]} = \sum_{x,y,z} T_{x[yz]}\, \underset{x}{M^r}\big(\underset{y}{M}\underset{z}{M} - k\big)\overset{x}{a}_{\nu}\, \overset{y}{a}_{\omega}\, \overset{z}{a}_{\mu} = 0.$$

III. Einstein's connection in 2-g-UFT

Using the recurrence relations obtained in the previous section, we find a unique solution of the system of equations (1.4) in this section. The solution obtained in this section is the simplest ever found.

LEMMA (3.5). *In X_2 we have*

$$S_{[\omega\mu\nu]} = 0. \tag{3.1}$$

PROOF. Since all indices take the values 1, 2 only in X_2, (3.1) follows from the skew-symmetry of the tensor $S_{\omega\mu\nu}$ in the first two indices.

THEOREM (3.6). *If the condition*

$$g \neq 0 \tag{3.2}$$

is satisfied, the system (1.4) *admits a unique solution*

$$S_{\omega\mu\nu} = \frac{1}{g} \nabla_\nu k_{\omega\mu}, \tag{3.3}$$

where ∇_ν is the symbolic vector of the covariant derivative with respect to $\{{}^{\nu}_{\omega\mu}\}$.

PROOF. Using the relation

$$2S_{\nu[\omega\mu]} = -S_{\omega\mu\nu} \tag{3.4}$$

equivalent to (3.1), and employing the notations introduced in (2.10), we have from (1.4)

$$D_\nu k_{\omega\mu} = 2S_{\nu[\mu}{}^{\alpha} g_{\omega]\alpha} = 2S_{\nu[\mu\omega]} + 2\overset{001}{S}_{\nu[\mu\omega]} = S_{\omega\mu\nu} + 2\overset{001}{S}_{\nu[\mu\omega]}. \tag{3.5}$$

On the other hand, one finds by virtue of (1.6)

$$\begin{aligned} D_\nu k_{\omega\mu} &= \partial_\nu k_{\omega\mu} - \Gamma^{\alpha}_{\omega\nu} k_{\alpha\mu} - \Gamma^{\alpha}_{\mu\nu} k_{\omega\alpha} = \nabla_\nu k_{\omega\mu} - (S_{\omega\nu}{}^{\alpha} + U^{\alpha}{}_{\omega\nu}) k_{\alpha\mu} - \\ &\quad - (S_{\mu\nu}{}^{\alpha} + U^{\alpha}{}_{\mu\nu}) k_{\omega\alpha} = \nabla_\nu k_{\omega\mu} + 2\overset{001}{S}_{\nu[\mu\omega]} + 2\overset{100}{U}_{[\mu\omega]\nu}. \end{aligned} \tag{3.6a}$$

Substituting

$$U_{\nu\lambda\mu} = 2\overset{001}{S}_{\nu(\lambda\mu)} \tag{3.7}$$

equivalent to (1.7) into (3.6a), and using (3.4), (2.13a), and (2.15b), one obtains from (3.6a) that

$$\begin{aligned} D_\nu k_{\omega\mu} &= \nabla_\nu k_{\omega\mu} + 2\overset{001}{S}_{\nu[\mu\omega]} - 2\overset{(10)1}{S}_{\omega\mu\nu} + 2\overset{011}{S}_{\nu[\omega\mu]} = \nabla_\nu k_{\omega\mu} + 2\overset{001}{S}_{\nu[\mu\omega]} + 2kS_{\nu[\omega\mu]} = \\ &= \nabla_\nu k_{\omega\mu} + 2\overset{001}{S}_{\nu[\mu\omega]} - kS_{\omega\mu\nu}. \end{aligned} \tag{3.6b}$$

Comparing (3.5) and (3.6b), we have (3.3) if the condition (3.2) is satisfied.

THEOREM (3.7). *If the condition* (3.2) *is satisfied, the tensor $U^{\nu}_{\omega\mu}$ is given by*

$$U^{\nu}_{\omega\mu} = -\frac{2}{g} (\nabla_\alpha k_{(\omega}{}^{\nu}) k_{\mu)}{}^{\alpha}. \tag{3.8}$$

PROOF. Substitution of (3.3) into (3.7) gives (3.8) in the following way:

$$U^{\nu}_{\omega\mu} = 2h^{\nu\alpha} \overset{001}{S}_{\alpha(\omega\mu)} = -2h^{\nu\alpha} k_{(\mu}{}^{\beta} S_{\omega)\alpha\beta} = -\frac{2}{g} (\nabla_{\beta} k_{(\omega}{}^{\nu}) k_{\mu)}{}^{\beta}.$$

Now that we have obtained the tensors $S_{\omega\mu}{}^{\nu}$ and $U^{\nu}{}_{\omega\mu}$ in terms of $g_{\lambda\mu}$, it is possible for us to determine the connection $\Gamma^{\nu}_{\omega\mu}$ by only substituting for S and U into (1.6). Formally,

THEOREM (3.8). *If the condition* (3.2) *is satisfied, the Einstein's connection* $\Gamma^{\nu}_{\omega\mu}$ *in* 2-g-UFT *is given by*

$$\Gamma^{\nu}_{\omega\mu} = \begin{Bmatrix} \nu \\ \omega\mu \end{Bmatrix} + \frac{1}{g} h^{\nu\alpha} \nabla_{\alpha} k_{\omega\mu} - \frac{2}{g} (\nabla_{\alpha} k_{(\omega}{}^{\nu}) k_{\mu)}{}^{\alpha}. \tag{3.9}$$

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(Received January 22, 1981)

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Acta Math. Hung.
41 (1—2), (1983), 53—59.

ABSTRACT MEASURE DIFFERENTIAL INEQUALITIES AND APPLICATIONS

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I. Introduction. The problems regarding existence of solutions, uniqueness of solutions as well as the existence of extremal solutions of an abstract measure differential equation (AMDE) have been considered by Sharma [2,3]. In this paper we consider the same AMDE and establish a basic inequality in the form of a monotonicity theorem, using the lattice fixed point theorem due to Tarski [4].

In Section 3, the difference between two approximate solutions of an AMDE is estimated. Uniqueness of a solution and its continuous dependence on initial conditions follow as a consequence of this estimate.

2. Notations and definitions. Let R denote the real line and X a linear space over the field R. For any subset S of X and a σ-algebra M on S, let the symbol $\mathrm{ca}(S, M)$ denote the space of real measures (as a subspace of the set of all complex measures defined on M). It is known that $\mathrm{ca}(S, M)$ is a Banach space [1, p.161] with respect to the norm $\|\cdot\|$ defined by

$$\|p\| = |p|(S), \quad p\in \mathrm{ca}(S, M),$$

where $|p|$ is the total variation measure of p. For any $x\in X$ define the sets S_x and $\bar{S}_x$ as follows: $S_x=\{rx\colon -\infty<r<1\}$ and $\bar{S}_x=\{rx\colon -\infty<r\leqq 1\}$. Let x_0 be a fixed element of X. For any $z\in X$ for which $S_{x_0}\subset S_z$, we shall denote the set $S_z-S_{x_0}$ by $\overline{x_0z}$ and write $z>x_0$. For any $z>x_0$, let M_z denote the smallest σ-algebra on $\bar{S}_z$, containing $\{x_0\}$ and the sets $\bar{S}_x$, $x\in\overline{x_0z}$.

In what follows μ will stand for a finite positive measure and $p\in\mathrm{ca}(S_z, M_z)$. For a given positive number b, let I be the interval $(-b, b)$ and f be a real valued μ-integrable function defined on $S_z\times I$. For any real number $\alpha\in I$ consider the equation

$$\frac{dp}{d\mu} = f(x, p(\bar{S}_x)) \tag{2.1}$$

with the initial condition

$$p(\bar{S}_{x_0}) = \alpha, \tag{2.2}$$

where $\dfrac{dp}{d\mu}$ is the Radon—Nikodym derivative of p with respect to μ.

Definition 2.1. A real measure $\varphi\in\mathrm{ca}(S_z, M_z)$, $z>x_0$, is said to be a solution of (2.1) satisfying (2.2) if

(i) $\varphi(\bar{S}_{x_0}) = \alpha$,
(ii) $\varphi(E)\in I, \quad E\in M_z$,

(iii) $\varphi \ll \mu$ on $\overline{x_0 z}$,
(iv) φ satisfies (2.1) a.e. $[\mu]$ on $\overline{x_0 z}$.

REMARK 2.1. Clearly if φ is a solution of (2.1), it is bounded. However if I is replaced by R, it may be either bounded or unbounded.

REMARK 2.2. From the definition (2.1), it is clear that φ is a solution of (2.1) satisfying (2.2) if and only if $\varphi(\bar{S}_{x_0}) = \alpha$, and

$$\varphi(E) = \int_E f(x, \varphi(\bar{S}_x))\, d\mu, \quad E \subset \overline{x_0 z},\ E \in M_z.$$

A solution φ of (2.1) satisfying (2.2), will be denoted by $\varphi(x_0, \alpha)$.

DEFINITION 2.2. Let g be a real μ-integrable function defined on S_z, $z \geqq x_0$, $p \in \mathrm{ca}(S_z, M_z)$ and $p \ll \mu$, Then by the inequality $\dfrac{dp}{d\mu} \leqq g(x)$ we mean

$$p(E) \leqq \int_E g(x)\, d\mu, \quad E \in M_z,\ E \subset \overline{x_0 z}.$$

DEFINITION 2.3. Let $\varepsilon > 0$. A measure $\varphi \in \mathrm{ca}(S_z, M_z)$ is said to be an $\in$-approximate solution of (2.1) on $\overline{x_0 z}$ if

(i) $\varphi \ll \mu$ on $\overline{x_0 z}$,
(ii) $\left| \dfrac{d\varphi}{d\mu} - f(x, \varphi(\bar{S}_x)) \right| \leqq \varepsilon$, a.e. on $\overline{x_0 z}$.

Note that if $\varepsilon = 0$, φ is a solution of (2.1) on $\overline{x_0 z}$.

REMARK 2.3. The conditions (i) and (ii) in the above definition imply that for an ε-approximate solution φ of (2.1) on $\overline{x_0 z}$, we have

$$\int_E f(x, \varphi(\bar{S}_x))\, d\mu - \varepsilon\mu(E) \leqq \varphi(E) \leqq \int_E f(x, \varphi(\bar{S}_x))\, d\mu + \varepsilon\mu(E),$$

whenever $E \in M_z$ and $E \subset \overline{x_0 z}$.

3. In this section we obtain an estimate for the difference between two approximate solutions of (2.1). We need the following assumptions.

(A_1) $\mu\{x_0\} = 0$.

(A_2) f is a real μ-integrable function defined on $S_z \times R$ for every $z \geqq x_0$, and satisfies the Lipschitz condition

$$|f(x, y_1) - f(x, y_2)| \leqq L|y_1 - y_2|, \quad L > 0.$$

(A_3) There exists a real μ-integrable function w defined on S_z, such that

$$|f(x, y)| \leqq w(x), \quad (x, y) \in S_z \times R.$$

(A_4) For any positive measure $\sigma \in \mathrm{ca}(S_z, M_z)$, and any measurable subset E_x of $\overline{x_0 z}$, there exists a constant $M>0$, such that

$$\text{(i) } LM < 1, \quad \text{(ii) } \int_{E_x} \sigma(E_t)\,d\mu \leqq M\sigma(E_x).$$

We now prove the following theorem.

THEOREM 1. *Let the assumptions* (A_1) *to* (A_4) *hold. Let p_1 and p_2 be ε_1- and ε_2-aproximate solutions of (2.1) on $\overline{x_0 z}$, respectively, satisfying $p_1(\bar{S}_{x_0})=\alpha_1$ and $p_2(\bar{S}_{x_0})=\alpha_2$, where α_1 and α_2 are constants. Let δ and ε be nonnegative real numbers such that $|\alpha_1-\alpha_2|\leqq\delta$ and $\varepsilon_1+\varepsilon_2=\varepsilon$. Then*

$$|p_1(F)-p_2(F)| \leqq \delta + \frac{(\varepsilon+\delta L)\mu(\overline{x_0 z})}{(1-LM)}, \quad F\in M_z.$$

PROOF. Since p_1 and p_2 are ε_1- and ε_2-approximate solutions of (2.1), for any $E\subset\overline{x_0 z}$, we have by Remark 2.3

$$p_1(E)-p_2(E) \leqq \varepsilon\mu(E)+\int_E [f(x, p_1(\bar{S}_x))-f(x, p_2(\bar{S}_x))]\,d\mu. \tag{3.1}$$

Letting $E=\overline{x_0 x}$, $x\in\overline{x_0 z}$ and using (A_2) we obtain from (3.1)

$$|p_1(\overline{x_0 x})-p_2(\overline{x_0 x})| \leqq \varepsilon\mu(\overline{x_0 x})+L\int_{\overline{x_0 x}} |p_1(\bar{S}_x)-p_2(\bar{S}_x)|\,d\mu \leqq$$

$$\leqq \varepsilon\mu(\overline{x_0 z})+L\int_{\overline{x_0 x}} |p_1(\bar{S}_{x_0})-p_2(\bar{S}_{x_0})|\,d\mu+L\int_{\overline{x_0 x}} |p_1(\overline{x_0 x})-p_2(\overline{x_0 x})|\,d\mu.$$

Set $\sigma=p_1-p_2$. Then the above inequality implies that

$$|\sigma(\overline{x_0 x})| \leqq \varepsilon\mu(\overline{x_0 z})+L\delta\mu(\overline{x_0 z})+L\int_{\overline{x_0 x}} |\sigma(\overline{x_0 x})|\,d\mu \leqq$$

$$\leqq (\varepsilon+L\delta)\mu(\overline{x_0 z})+L\int_{\overline{x_0 x}} |\sigma|(\overline{x_0 x})\,d\mu.$$

This, by virtue of definition of $|\sigma|$ implies that

$$|\sigma|(\overline{x_0 x}) \leqq (\varepsilon+L\delta)\mu(\overline{x_0 z})+L\int_{\overline{x_0 x}} |\sigma|(\overline{x_0 x})\,d\mu.$$

Using (A_4) in this inequality we then obtain

$$|\sigma|(\overline{x_0 x}) \leqq \frac{(\varepsilon+L\delta)\mu(\overline{x_0 z})}{(1-LM)}.$$

Since this is true for every $x\in\overline{x_0 z}$, we conclude that

$$|\sigma|(E) \leqq \frac{(\varepsilon+L\delta)\mu(\overline{x_0 z})}{(1-LM)}, \quad E\in M_z, E\subset\overline{x_0 z}.$$

This also implies that

$$|p_1(E)-p_2(E)| \leqq \frac{(\varepsilon+L\delta)\,\mu(\overline{x_0z})}{(1-LM)}, \quad E\subset \overline{x_0z}. \tag{3.2}$$

Now if $F\in M_z$, then F can be written as $F=\bar{S}_{x_0}\cup E$, for some $E\subset\overline{x_0z}$. Hence using (3.2) and the hypothesis $|p_1(\bar{S}_{x_0})-p_2(\bar{S}_{x_0})|\leqq\delta$ we obtain the desired inequality

$$|p_1(F)-p_2(F)| \leqq \delta+\frac{(\varepsilon+L\delta)\,\mu(\overline{x_0z})}{(1-LM)}, \quad F\in M_z. \tag{3.3}$$

REMARK 3.2. It is clear that when $\varepsilon_1=0$, p_1 is the solution of (2.1) satisfying $p_1(\bar{S}_{x_0})=\alpha_1$. In this case the inequality (3.3) reduces to

$$|p_1(F)-p_2(F)| \leqq \delta+\frac{(\varepsilon_2+L\delta)\,\mu(\overline{x_0z})}{(1-LM)}, \quad F\in M_z.$$

Further if ε_2 is also zero, then

$$|p_1(F)-p_2(F)| \leqq \delta\left[1+\frac{L\mu(\overline{x_0z})}{(1-LM)}\right]. \tag{3.4}$$

This gives an estimate for the difference between two solutions p_1, p_2, satisfying $p_1(\bar{S}_{x_0})=\alpha_1$, $p_2(\bar{S}_{x_0})=\alpha_2$.

It is also clear that if $\alpha_1=\alpha_2=\alpha$, we can choose $\delta=0$ and hence (3.4) implies that $p_1(F)=p_2(F)$ for each $F\in M_z$. This means that there is a unique solution of (2.1) satisfying the initial condition (2.2).

REMARK 3.3. The inequality (3.4) also indicates that the solutions of (2.1) depend continuously on the initial conditions in the following sense. If $p_1(x_0, \alpha_1)$ and $p_2(x_0, \alpha_2)$ are two solutions of (2.1) then for a given $\varepsilon>0$, there exists a $\delta=\delta(\varepsilon)>0$ such that $|p_1(F)-p_2(F)|\leqq\varepsilon$, $F\in M_z$ whenever $|\alpha_1-\alpha_2|\leqq\delta$.

4. A monotonicity theorem. We shall now prove a differential inequality which can be viewed as a kind of comparison theorem. The proof is based on the lattice fixed point theorem due to Tarski [4]. The following definition is required.

DEFINITION. A solution $p_M(x_0, \alpha)$ of (2.1) satisfying (2.2) existing on $\overline{x_0z}$, $z\geqq x_0$ is said to be a maximal solution of (2.1) if for any other solution $\varphi(x_0, \alpha)$ of (2.1), the inequality $\varphi(E)\leqq p_M(E)$, $E\in M_z$ holds.

A similar definition can be given for the minimal solution of (2.1). It is clear that, whenever the maximal and minimal solutions exist, they are unique.

We now need the following assumptions.

(B_1) $f(x, y)$ is a real valued monotonically non-decreasing function in y.

(B_2)
$$\int_{\overline{x_0z}} w(x)\,d\mu \leqq W_0 \quad \text{for some} \quad W_0>0,$$

where w is the same function which occurs in (A_3).

(B_3)
$$S=\{p\in\mathrm{ca}(S_z, M_z)\colon\ \|p\|\leqq K\}$$

where $K=|\alpha|+W_0$, α being the same as in (2.2).

If $p_1, p_2 \in S$, then by $p_1 \leqq p_2$, we mean $p_1(E) \leqq p_2(E)$, where $E \in M_z$. Note that S is a complete lattice with respect to this order relation.

We shall first prove the existence of the maximal and minimal solutions of (2.1) and then establish the main inequality.

THEOREM 2. *Under the assumptions* (A_1), (A_3) *and* (B_1) *to* (B_3) *the equation* (2.1) *has maximal and minimal solutions existing on* $\overline{x_0 z}$ *and lying in* S.

PROOF. Define an operator T on S by

$$Tp(\bar{S}_{x_0}) = \alpha, \quad Tp(E) = \int_E f(x, p(\bar{S}_x))\,d\mu, \quad E \subset \overline{x_0 z}. \tag{4.1}$$

Using (A_3), (B_2), (B_3) and the definitions of T, one can show that T maps S into itself. Also the assumption (B_1) together with the positivity of μ it is observed that T is an isotone increasing on S. Since S is a complete lattice, we can apply the lattice fixed point theorem due to Tarski [4] to conclude that the set $\{u \in S\colon Tu = u\}$ is non-empty and is also a complete lattice. Consequently the equation (2.1) possesses maximal and minimal solutions existing on $\overline{x_0 z}$ and lying in S.

THEOREM 3. *Let all the assumptions of Theorem* 2 *hold. If a real measure* $q \in S$ *satisfies the condition* $q(\bar{S}_{x_0}) \leqq \alpha$ *and*

$$q(E) \leqq \int_E f(x, q(\bar{S}_x))\,d\mu, \quad E \subset \overline{x_0 z}, \tag{4.2}$$

then

$$q(E) \leqq p_M(E), \quad E \in M_z, \tag{4.3}$$

where $p_M(x_0, \alpha)$ *is the maximal solution of* (2.1) *existing on* $\overline{x_0 z}$ *and lying in* S.

If the inequality (4.2) *is reversed then*

$$q(E) \geqq p_m(E), \quad E \in M_z, \tag{4.4}$$

where $p_m(x_0, \alpha)$ *is the minimal solution of* (2.1) *existing on* $\overline{x_0 z}$ *and lying in* S.

PROOF. We only prove (4.3) since the proof of (4.4) follows on similar lines. Set $\pi = \sup S$. Clearly π exists, since S is a complete lattice. Consider the lattice interval $[q, \pi]$. This also exists since $q \in S$ and $\pi = \sup S$. Define an operator T by (4.1). Then, as in Theorem 2, T is isotone increasing. We shall now show that T maps $[q, \pi]$ into itself. Let $p \in [q, \pi]$. Then $p \in S$ and $p \geqq q$. Hence by using (B_1) and (4.2) we obtain

$$Tp(E) \geqq \int_E f(x, q(\bar{S}_x))\,d\mu \geqq q(E), \quad E \subset \overline{x_0 z}.$$

This together with the definition of π implies that $Tp \in [q, \pi]$. Now an appropriate application of the lattice fixed point theorem [4] shows that the maximal solution $p_M(x_0, \alpha)$ of (2.1) exists on $\overline{x_0 z}$ and lies in $[q, \pi]$. The desired inequality (4.3) is now an immediate consequence of this.

5. Applications. We shall now give two applications of Theorem 3.

Let α in (2.2) be positive. Let $g(x, y)$ be a nonnegative μ-integrable function defined on $S_z \times R^+$, $z \geqq x_0$, where R^+ is the set of non-negative real numbers. Let r be a positive measure belonging to $\mathrm{ca}(S_z, M_z)$. Consider the differential equation

$$\frac{dr}{d\mu} = g(x, r(\bar{S}_x)) \tag{5.1}$$

with the initial condition

$$r(\bar{S}_{x_0}) = \alpha \tag{5.2}$$

THEOREM 4. *Let the assumptions of Theorem 2 hold, with f replaced by g. Assume further that the function f of* (2.1) *satisfies the condition*

$$|f(x, y)| \leqq g(x, |y|). \tag{5.3}$$

Let $r_M(x_0, \alpha)$ be the maximal solution of (5.1) *existing on $\overline{x_0 z}$. Then for any solution $p(x_0, \alpha)$ of* (2.1) *existing on $\overline{x_0 z}$ we have*

$$|p(E)| \leqq r_M(E), \quad E \in M_z. \tag{5.4}$$

PROOF. Since $p(x_0, \alpha)$ is a solution of (2.1), we have

$$p(E) = \int_E f(x, p(\bar{S}_x)\, d\mu, \quad E \in M_z, \quad E \subset \overline{x_0 z}.$$

This together with (5.3) implies

$$|p(E)| \leqq \int_E g(x, p(\bar{S}_x))\, d\mu, \quad E \subset \overline{x_0 z}.$$

From this inequality it can be deduced that

$$|p|(E) \leqq \int_E g(x, |p|(\bar{S}_x))\, d\mu, \quad E \subset \overline{x_0 z}.$$

Now an application of Theorem 3 yields the desired inequality (5.4).

We now merely state a uniqueness theorem for (2.1). The proof follows by a simple application of Theorem 3.

THEOREM 5. *Let all the assumptions of Theorem 2 hold with f replaced by g. Further let the function f of* (2.1) *satisfy the condition*

$$|f(x, y_1) - f(x, y_2)| \leqq g(x, |y_1 - y_2|).$$

Suppose that the identically zero measure is the only solution of (5.1) *existing on $\overline{x_0 z}$. Then the equation* (2.1) *has at most one solution satisfying* (2.2) *and existing on $\overline{x_0 z}$.*

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(Received May 4, 1981)

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Acta Math. Hung.
41 (1—2), (1983), 61—66.

ON NASH EQUILIBRIUM. II

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In this paper we study Nash equilibria from a view point of topology. Using the methods in [1] and [2] we obtain new results for Nash-Pareto equilibria described in [8].

Let us consider the multiobjective model consisting of two subsystems with

$$\begin{cases} Z_1 = P, \quad Z_2 = Q \\ u_1 = f\colon\ P\times Q \to R^n \\ u_2 = g\colon\ P\times Q \to R^m. \end{cases}$$

Suppose P and Q are smooth manifolds with $\dim P=p$, $\dim Q=q$, f and g are continuously differentiable mappings.

DEFINITION 1 (see [2]). Let W be a smooth manifold with continuously differentiable functions u_i: $W\to R\,(i=1, \dots, m)$, where $m\leqq\dim W$. Let u: $W\to R^m$ be defined by $u=(u_1, \dots, u_m)$, and $\mathrm{Pos}\,(R^m)\subset R^m$ be the set of $(y_1, \dots, y_m)\in R^m$ such that $y_i>0$ for each i. Let $H(x)=\mathrm{Du}\,(x)^{-1}(\mathrm{Pos}\,R^m)$ where $\mathrm{Du}(x)$: $T_x(W)\to R^m$ is the derivative of u at x considered as a linear mapping from the tangent space of W at x to R^m. A point $x\in W$ is called "Pareto Optimal" (or critical Pareto) of u if $H(x)=\emptyset$.

DEFINITION 2. We call $(x,^* y^*)\in P\times Q$ Nash-Pareto equilibrium of model if x^* and y^* are Pareto optimal of the mappings $\bar f(x)=f(x, y^*)$: $P\to R^n$ and $\bar g(y)=g(x^*, y)$: $Q\to R^m$, respectively.

Let θ be the set of Nash-Pareto equilibria (in the sense ofD efinition 2). It is clear that the classical Nash-Pareto equilibria set (see [8]) belongs to θ, but, in general, it does not coincide with θ. The following simple example shows this.

Take $P=Q=R$ and let f, g: $P\times Q\to R^2$ be defined by

$$f(x, y) = \left(-\frac{x^4}{4}+\frac{x^3}{3}+x^2, y\right), \quad g(x, y) = \left(x, -\frac{y^4}{4}+\frac{y^3}{3}+y^2\right).$$

In this model, Jacobians of f and g at the points $(-1, -1)$, $(0, 0)$, $(2, 2)$ are of the following form:

$$A(f) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A(g) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence these points belong to θ, but only $(2, 2)$ is a Nash-Pareto equilibrium in the sense of Definition 2 in [8].

Let us recall some notions from differential topology. Let P be a smooth (C^∞) manifold and Y be a closed subset of P. By a stratification of Y we will mean the closed subsets Y^{p_i}: $Y=Y^{p_1}\supset Y^{p_2}\supset\ldots\supset Y^{p_\lambda}\supset\ldots$, where $p_1>p_2>\ldots>p_\lambda>\ldots$ are such that

i) $Y^{p_\lambda}\setminus Y^{p_{\lambda-1}}$ is a smooth submanifold of dimension p_λ. Its connected components are called strata.

ii) For any stratum A, its boundary $\partial A=\bar{A}\setminus A$ is a union of strata of dimension smaller than dim A.

Let U and V be two strata of Y such that $V\subset\partial U$, $x\in V$. The Whitney's conditions are the following:

(a) if $\{x_i\}$ is a sequence of points in U, $x_i\to x$, and TU_{x_i} converges (in the Grassmannian of (dim U)-planes in TW) and $\tau=\lim TU_{x_i}$ then $TV_x\subset\tau$.

(b) if for the sequences of points $\{x_i\}$ and $\{y_i\}$ in U and V, resp. such that $x_i\to x$, $y_i\to x$, $x_i\neq y_i$, $\overline{x_iy_i}$ converges (in the projective space P^{n-1}) and TU_{x_i} converges (in the Grassmannian of (dim U)-planes in R^n), then $l\in\tau$ where $l=\lim\overline{x_iy_i}$ and $\tau=\lim TU_{x_i}$.

We will say that a stratification satisfying (b) with any triple (U, V, x) where $x\in V$, $V\supset\partial U$, U and V are strata, is Whitney's stratification (see [5]).

1. Generic property

Let now P and Q be smooth compact manifolds without boundary, and let f and g be of class C^r, $r\geqq 3$.

Let $C^r(P\times Q, R^n)$ be a space of mappings of class C^r from $P\times Q$ into R^n with the natural structure of Banach spaces, and let $J^r(P\times Q, R^n)$ be the space of all r-jets $P\times Q\to R^n$ with topology included by that of $C^\infty(P\times Q, R^n)$. Denote

$$\hat{J}_0 = \{j^1\varphi\in J^1(P\times Q, R^n)|D\bar{\varphi}(x, y)^{-1}(\mathrm{Pos}\, R^n)=\emptyset\}$$

$$\hat{\hat{J}}_0 = \{j^1\psi\in J^1(P\times Q, R^m)|D\bar{\bar{\psi}}(x, y)^{-1}(\mathrm{Pos}\, R^m)=\emptyset\},$$

$$M_0=\{(x, y)\in P\times Q|J^1f(x, y)\in\hat{J}_0\},\quad N_0=\{(x, y)\in P\times Q|J^1g(x, y)\in\hat{\hat{J}}_0\},$$

where $\bar{\varphi}(x, y)$ is the mapping $\varphi(x, y)$: $P\to R^n$ when y is fixed, and $\bar{\bar{\psi}}(x, y)$ is the mapping $\psi(x, y)$: $Q\to R^m$ when x is fixed.

From definitions 1 and 2 we conclude at once

LEMMA 1. *$\hat{J}_0(\hat{\hat{J}}_0)$ is a set with Whitney's stratification in $J^1(P\times Q, R^n)$ (in $J^1(P\times Q, R^m)$).*

PROOF. Let Y denote the set $R^n\setminus[(\mathrm{Pos}\, R^n)\cup(-\mathrm{Pos}\, R^n)]$. Taking $Y^n=Y$ and Y^k as the set of all coordinate k-planes of R^n ($k=1, \ldots, n-1$) and $Y^0=\{0\}$ then $Y^n\supset Y^{n-1}\supset\ldots\supset Y^1\supset Y^0$ form a Whitney's stratification of Y. Suppose M is a smooth manifold. We will prove that

$$J_0=\{J^1\varphi\in J^1(M, R^n)|D\varphi(x)^{-1}(\mathrm{Pos}\, R^n)=\emptyset\}$$

is a set with Whitney's stratification. Setting

$$J_0^{N_n} = \{j^1\varphi \in J^1(M, R^n) | \operatorname{Im} D\varphi(x) \subset Y^k\}$$

we have

(1) $$J_0^{N_n} = J_0 \supset J_0^{N_{n-1}} \supset \ldots \supset J_0^{N_1} \supset J_0^{N_0} = J_0^0.$$

This is a Whitney's stratification of J_0. Indeed, by definition

(2) $$J_0^{N_k} \setminus J_0^{N_{k-1}} = \{j^1\varphi \in J^1(M, R^n) | \operatorname{Im} D\varphi(x) \subset Y^k\} \setminus \{j\varphi_\varphi^1 \in J^1(M, R^n) | \operatorname{Im} D\varphi(x) \subset$$
$$\subset Y^{k-1}\} = \{j^1\varphi \in J^1(M, R^n) | \operatorname{Im} D\varphi(x) \subset Y^k \quad \text{and} \quad \operatorname{Im} D\varphi(x) \cap (Y^k \setminus Y^{k-1}) \neq \varnothing\}.$$

For $k=n$, $Y^n \setminus Y^{n-1}$ is open in R^n, therefore $J_0^{N_n} \setminus J_0^{N_{n-1}}$ is open in $J^1(M, R^n)$. Consequently it is a smooth submanifold of $J^1(M, R^n)$.

For $k<n$, $Y^k = \bigcup_i Y_i^k$ is the union of all coordinate k-planes Y^k in R^n. From linearity of $D\varphi(x)$ we have $J_0^{N_k} = \bigcup_i J_{0i}^{N_k}$ where $J_{0_i}^{N_k} = \{j^1\varphi \in J^1(M, R^n) | \operatorname{Im} D\varphi(x) \subset Y_i^k\}$. But Y_i^k is a subspace of R^n, hence $J_{0i}^{N_k}$ is a bundle subspace of $J^1(M, R^n)$.

Let $A_i^{N_k} = \{j^1\varphi \in J^1(M, R^n) | \operatorname{Im} D\varphi(x) \subset Y_i^k$ and $\operatorname{Im} D\varphi(x) \cap (Y_i^k \setminus Y^{k-1}) \neq \varnothing\}$ then $J_0^{N_k} \setminus J_0^{N_{k-1}} = \bigcup_i A_i^{N_k}$.

We see that $A_i^{N_k} \cap A_j^{N_k} = \varnothing$ for $i \neq j$; $i, j = 1, \ldots, N(k)$ ($N(k)$ is the number of coordinate k-planes of R^n). In fact, if there is a mapping $\varphi\colon M \to R^n$ such that $\operatorname{Im} D\varphi(x) \cap (Y_i^k \setminus Y_i^{k-1}) \neq \varnothing$ and $\operatorname{Im} D\varphi(x) \cap \{[Y_i^k, Y_j^k] \setminus (Y_i^k \cup Y_j^k)\} \neq \varnothing$, then $\operatorname{Im} D\varphi(x)$ does not belong either to Y_i^k or to Y_j^k, i.e. $j^1\varphi \notin A_i^{N_k}$ and $j^1\varphi \notin A_j^{N_k}$. (Here $[Y_i^k, Y_j^k]$ is a linear hull of Y_i^k with Y_j^k.) This implies $A_i^{N_k} \cap A_j^{N_k} = \varnothing$ for $i \neq j$. Moreover $A_i^{N_k}$ is a smooth submanifold of $J^1(M, R^n)$.

Further, (2) yields

$$\partial(J_0^{N_k} \setminus J_0^{N_{k-1}}) = \overline{J_0^{N_k} \setminus J_0^{N_{k-1}}} \setminus (J_0^{N_k} \setminus J_0^{N_{k-1}}) =$$
$$= \{j^1\varphi \in J^1(M, R^n) | \operatorname{Im} D\varphi(x) \subset Y^k \quad \text{and} \quad \operatorname{Im} D\varphi(x) \cap (Y^k \setminus Y^{k-1}) = \varnothing\} =$$
$$= \{j^1\varphi \in J^1(M, R^n) | \operatorname{Im} D\varphi(x) \subset Y^{k-1}\}.$$

This is a union of some strata of smaller dimension.

In this way, (1) is a stratification of J_0. In order to prove that this stratification satisfies Whitney's condition (b), suppose U is a stratum of $J_0^{N_k} \setminus J_0^{N_{k-1}}$ and V is another stratum in ∂U. Let $\{j^1\varphi_i\} = \{(x_i, y_i, p_i)\}$ and $\{j^1\varphi_i'\} = \{(x_i', y_i', p_i')\}$ be sequences of points in U and in V converging to $j^1\varphi = (x, y, p)$ and $j^1\varphi$, respectively. Let $e = \lim \overline{j^1\varphi_i\, j^1\varphi_i'}$ and $\tau = \lim TU_{j^1\varphi_i}$. Does e belong to τ? We have

$$TU_{j^1\varphi_i} = (TM_{x_i}, TR_{y_i}^n, T(J^1(m, k))_{p_i})$$

where $m = \dim M$, k is a dimension of the coordinate k-plane Y_j^k with $\operatorname{Im} D\varphi_i(x_i) \subset Y_j^k$ and

$$\operatorname{Im} D\varphi_i(x_i) \cap (Y_j^k \setminus Y^{k-1}) \neq \varnothing$$

(as $j^1\varphi_i \in U$ for every i, hence Y_j^k is the same for any $j^1\varphi_i$).

$$TU_{j^1\varphi_i} = (TM_{x_i}, R^n, J^1(m, k)).$$

Therefore $\tau=(TM_x, R^n, J^1(m, k))$. On the other hand,

$$\overline{j^1\varphi_i j^1\varphi_i'} = (\overline{x_i x_i'}, \overline{y_i y_i'}, \overline{p_i p_i'})$$

implies $\lim \overline{x_i x_i'} \in TM_x$, $\lim \overline{y_i y_i'} \in R^n$ (as M is a smooth manifold and R^n is a finite-dimensional space.) Since $j^1\varphi_i' \in V \subset \partial U$, Im $D\varphi_i'(x_i')$ belongs to Y_j^k, in other words $p_i' \in J^1(m, k)$. Consequently, $\lim \overline{p_i p_i'} \in J^1(m, k)$. (Note that $J^1(m, k)$ is a finite dimensional space.) Then we conclude $e \in \tau$, as required.

Hypothesis (T). $j^1 f \pitchfork \hat{J}_0$ and $j^1 g \pitchfork \hat{\hat{J}}_0$, $M_0 \pitchfork N_0$ in $P \times Q$.

We explain this hypothesis. By Lemma 3, $\hat{J}_0$ and $\hat{\hat{J}}_0$ are sets with Whitney's stratification. Say, $J^1 f$ is transverse with respect to this stratification if it is transverse to each of the strata. If the first statement of the hypothesis is satisfied then by Lemma 2 below, M_0 and N_0 are sets with stratification. Say, M_0 is transverse to N_0 in $P \times Q$ if any stratum of stratification of M_0 is transverse to any one of the stratification of N_0 in $P \times Q$.

Lemma 2. *Under Hypothesis* (T), M_0 *and* N_0 *are sets with Whitney's stratifications.*

Proof. From differentiability of f, it follows that $J^1 f$: $P \times Q \to J^1(P \times Q, R^n)$ is of class C^2.

By $J^1 f \pitchfork \hat{J}_0$ we can apply Corollary 8.8 in [5], which asserts that in this condition the inverse image of Whitney's stratification is Whitney's stratification. Hence $M_0 = (j^1\varphi)^{-1}(\hat{J}_0)$ is a set with Whitney's stratification. For N_0 the proof is analogous.

Proposition 1. *Under Hypothesis* (T), θ *is a set with stratification.*

Indeed, by Lemma 2, M_0 and N_0 are sets with Whitney's stratification and by Hypothesis (T) $M_0 \pitchfork N_0$, therefore $M_0 \cap N_0 = \theta$ is a set with stratification. (Note, that if M_0 is not transverse to N_0, then in general θ is not a set with stratification since the intersection of two manifolds in general is not a manifold when they are not transverse.)

Proposition 2. *The set of pairs of the mappings* (f, g) *such that Hypothesis* (T) *is satisfied, is open and dense in* $C^3(P \times Q, R^n) \times C^3(P \times Q, R^m)$.

Proof. It is easy to show that condition (b) implies condition (a). Hence, Proposition 2 is an immediate corollary of Lemma 1 and Thom's transversality theorem on sets with stratification. Thom's theorem asserts the following: let Y be a closed set of $J^r(N, P)$ with stratification of finite number of submanifolds, satisfying Whitney's condition (a), and let N be a compact manifold. Then the set

$$\Omega = \{f \in C^\infty(N, P) \mid j^r f \pitchfork Y\}$$

is open and dense for large enough r. (In [4] and in other references, we usually have Thom's transversality theorem and the simple transversality theorem on the set with stratification. However, Thom's transversality theorem is still valid on a set with stratification. The proof is analogous to that of the above mentioned two theorems.)

Remark. In the case $n=m=1$, it is clear that M_0 and N_0 are submanifolds in $P \times Q$ (see [1]) and in this, by Hypothesis (T), $M_0 \cap N_0 = \theta$ is a submanifold. In general, if n or $m \neq 1$, θ is not a submanifold.

2. Necessary and sufficient conditions for Nash-Pareto equilibria

First of all, recall the notion of an admissible curve introduced by Smale in order to examine Pareto Optimal sets (see [2]). Let W be a smooth manifold, $u_1, \ldots, u_m$ be smooth functions defined on W. Let $u\colon W \to R^m$ be defined by $u=(u_1, \ldots, u_m)$. A curve $\varphi\colon (-\varepsilon, \varepsilon) \to W$ defined on an interval is admissible if $\frac{d}{dt} u_i(\varphi(t)) > 0$ for each $t \in (-\varepsilon, \varepsilon)$ and each $i=1, \ldots, m$. Observe that the point $x \in W$ is a Pareto optimum iff there is no admissible path for x.

Here we consider the smooth manifolds P and Q with functions $f\colon P \times Q \to R^n$ and $g\colon P \times Q \to R^m$ of the class $C^k (k \geqq 2)$. A C^1-curve $\varphi\colon (-\varepsilon, \varepsilon) \to P \times Q$ defined on an interval is an admissible curve of (f, g) if $D\bar{f}(\varphi(t))(\varphi_1'(t)) \geqq 0$ and $D\bar{\bar{g}}(\varphi(t))(\varphi_2'(t)) \geqq 0$ for each $t \in (-\varepsilon, \varepsilon)$, and at least one of these inequalities is strict, where φ_1 and φ_2 are components of φ, in detail, $\varphi_1\colon (-\varepsilon, \varepsilon) \to P$ and $\varphi_2\colon (-\varepsilon, \varepsilon) \to Q$.

Proposition 3. *If the Euler—Poincaré characteristic of P and Q is not* 0 *(in the case oriented) or is odd (in the case non-oriented) then* $\theta \neq \emptyset$.

Proof. Consider the vector field $T_x\bar{f} + T_y\bar{\bar{g}}$ on $P \times Q$. By our hypothesis on the Euler—Poincaré characteristic it follows that there is a point $(x, y) \in P \times Q$ at which this vector field vanishes, i.e. $T_x f = 0$ and $T_y g = 0$, consequently x and y are Pareto optima of f and g, respectively (see [3]), or $(x, y) \in \theta$.

Proposition 4. *A point $(x, y) \in P \times Q$ is Nash-Pareto equilibrium if and only if there is no admissible path for (x, y).*

Proof. Suppose (x, y) is a Nash-Pareto equilibrium, i.e. x and y are Pareto optima of $f(\cdot, y)$ and $g(x, \cdot)$, respectively. If there is an admissible path $\varphi = (\varphi_1, \varphi_2)\colon (-\varepsilon, \varepsilon) \to P \times Q$ through (x, y), for example $\varphi(0) = (x, y)$, then$D\bar{f}(\varphi_1(0))(\varphi_1'(0)) > 0$, and $D\bar{\bar{g}}(\varphi_2(0)(\varphi_2'(0)) \geqq 0$.

We can consider $D\bar{f}(\varphi_1(t))(\varphi_1'(t)) > 0$ for each $t \in (-\varepsilon,' \varepsilon')$ for some $\varepsilon'\colon 0 < \varepsilon' < \varepsilon$ (observe that f is of class C^2 and $\bar{f}(\varphi_1(t)) = f(\varphi_1(t), y)$ with y fixed). Thus φ_1 is an admissible curve of $f(\cdot, y)$ which contradicts x being Pareto optimum of $f(\cdot, y)$.

Conversely, if $(x, y) \in P \times Q$ is not a Nash-Pareto equilibrium, i.e. either x or y is not a Pareto optimum. For example, x is not a Pareto optimum of $f(\cdot, y)$. Then there is an admissible path $\varphi_1\colon (-\varepsilon, \varepsilon) \to P$ such that $\varphi_1(0) = x$ and $D\bar{f}(\varphi_1(t))(\varphi_1'(t)) > 0$ for each $t \in (-\varepsilon, \varepsilon)$. φ_2 is defined on $(-\varepsilon, \varepsilon)$ with values in Q by $\varphi_1(t) = y$. The curve $\varphi = (\varphi_1, \varphi_2)$ will be an admissible path of (f, g) at (x, y). This completes our proof.

From Proposition A in [2] and Proposition 1, we have the following:

Proposition 5. *A point $(x, y) \in P \times Q$ is a Nash-Pareto equilibrium if and only if one of following equivalent conditions is satisfied:*

a) *$D\bar{f}_i(x)$ $(i=1, \ldots, n)$ and $D\bar{\bar{g}}_j(y)$ $(j=1, \ldots, m)$ do not lie in the same open half space of* $T^*_{(x,y)}(P \times Q) = T^*_x(P) \oplus T^*_y(Q)$.

b) *There exist* $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m \in R_+$ *with not all* λ_i *zero, not all* μ_j *zero such that*

$$\sum_{i=1}^{n} \lambda_i Df_i(x) = 0 \quad \textit{and} \quad \sum_{j=1}^{m} \mu_j Dg_j(x) = 0.$$

Thanks to the notion of admissible paths of (f, g), one can define the set of stable Nash-Pareto equilibria and obtain results analogous with the stable Pareto set (see [2] and [3]).

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(Received May 29, 1981)

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Acta Math. Hung.
41 (1—2), (1983), 67—76.

DEGREE OF APPROXIMATION OF FUNCTIONS IN THE HÖLDER METRIC

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1. Definitions and notations. Let f be a periodic function of period 2π and integrable in the Lebesgue sense over $[-\pi, \pi]$. Let the Fourier series of f be given by

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\equiv\sum_{n=0}^{\infty}A_n(x). \tag{1.1}$$

Let $C_{2\pi}$ be the Banach space of all 2π-periodic continuous functions defined on $[-\pi, \pi]$ under sup norm. For $0<\alpha\leqq 1$ and some positive constant K, the function space H_α is given by the following:

$$H_\alpha=\{f\in C_{2\pi}\colon\ |f(x)-f(y)|\leqq K|x-y|^\alpha\}.$$

The space H_α is a Banach space ([7]) with the norm $\|\cdot\|_\alpha$ defined by

$$\|f\|_\alpha=\|f\|_c+\sup_{x,y}\{\Delta^\alpha f(x,y)\}. \tag{1.2}$$

where

$$\|f\|_c=\sup_{-\pi\leqq x\leqq\pi}|f(x)|$$

and

$$\Delta^\alpha[f(x,y)]=|f(x)-f(y)|/|x-y|^\alpha\quad (x\neq y). \tag{1.3}$$

We shall use the convention that $\Delta^0 f(x,y)=0$. The metric induced by the norm (1.2) on H_α is called the Hölder metric.

The set of functions f with $\|f\|_\alpha\leqq K$ is a compact subset of $C[0, 1]$.

It can be seen that $\|f\|_\beta\leqq(2\pi)^{\alpha-\beta}\|f\|_\alpha$ for $0\leqq\beta<\alpha\leqq 1$. Thus $\{(H_\alpha, \|\cdot\|_\alpha)\}$ is a family of Banach spaces which decreases as α increases.

Let $A=(a_{nk})$ $(k, n=0, 1, \dots)$ be an infinite matrix of real numbers. We denote by $T_n(f)$ the A-transform of the Fourier series of f given by

$$T_n(f;x)=\sum_{n=0}^{\infty}a_{nk}s_k(x)\quad(n=0, 1, \dots), \tag{1.4}$$

where $s_n(x)$ is the n-th partial sum of the series (1.1). If $A=(a_{n,k})$ is lower-triangular i.e. $a_{nk}=0$ for $k>n$, we write

$$t_n(f;x)=\sum_{k=0}^{n}a_{nk}s_k(x)\quad(n=0, 1, \dots). \tag{1.5}$$

Throughout the paper, we shall let K denote an absolute positive constant which may be different at different occurrences. We shall also use the following notations:

$$\varphi_x(t) = f(x+t)+f(x-t)-2f(x). \tag{1.6}$$

$$\bar{a}_{nk} = \sum_{r=0}^{k} a_{nr}, \tag{1.7}$$

$$\bar{a}_n(k) = \bar{a}_{n,k}, \tag{1.8}$$

$$a'_{nk} = \sum_{r=k}^{n} a_{nr}, \tag{1.9}$$

$$\tau = \left(\frac{\pi}{t}\right) \quad (0 < t \leqq \pi). \tag{1.10}$$

2. Introduction. Alexits [1, p. 301] studied the degree of approximation of functions of H_α by the Cesáro mean of their Fourier series in the sup norm. Since $C_{2\pi} \supseteq \supseteq H_\beta \supseteq H_\alpha$ for $0 \leqq \beta \leqq \alpha \leqq 1$, Prössdorf [7] obtained an estimate for $\|\sigma_n(f)-f\|_\beta$ for $f \in H_\alpha$ where $\sigma_n(f)$ is the Fejér means of the Fourier series of f. Precisely, he proved the following theorem:

THEOREM A [7, Theorem 2]. *Let* $f \in H_\alpha$ $(0<\alpha \leqq 1)$ *and* $0 \leqq \beta < \alpha$. *Then*

$$\|\sigma_n(f)-f\|_\beta = \begin{cases} O(n^{\beta-\alpha}) \quad (0<\alpha<1), \\ O\,[n^{\beta-1}(1+\log n)^{1-\beta}] \quad (\alpha = 1). \end{cases} \tag{2.1}$$

The case $\beta = 0$ of the above result is that of Alexits referred to earlier.

With a view to generalizing Theorem A, Chandra [3, Theorems 1 and 2] obtained estimates analogous to (2.1) by replacing $\sigma_n(f)$ with $(\bar{N}, p_n)$ and Nörlund means of the Fourier series of f.

The object of this paper is to generalize the results of Chandra [3] by obtaining estimates using the A-transform of the Fourier series of $f \in H_\alpha$. In §4, we specialise the matrix A and obtain corollaries some of which cannot be obtained from the results hither-to-known (see Corollary 2).

We shall prove the following:

THEOREM 1. *Let* $A=(a_{nk})$ *be a lower triangular infinite matrix satisfying the following:*

$$a_{nk} \geqq 0 \,(n, k = 0, 1, 2, \ldots) \quad \textit{and} \quad \sum_{k=0}^{n} a_{nk} = 1, \tag{2.1}$$

$$a_{nk} \leqq a_{n,k+1} \quad (k = 0, 1, \ldots, n-1,\ n = 0, 1, \ldots). \tag{2.2}$$

Then, for $0 \leqq \beta < \alpha \leqq 1$ *and* $f \in H_\alpha$

$$\|t_n(f)-f\|_\beta = \begin{cases} O\left(a_{nn}^{\alpha-\beta} \log^{\beta/\alpha}(na_{nn}+1)\right) \quad (0<\alpha<1), \\ O\left(a_{nn}^{1-\beta}\left\{\log^\beta(1+na_{nn})+\log^{1-\beta}\left(\dfrac{1}{a_{nn}}\right)\right\}\right) \quad (\alpha = 1). \end{cases} \tag{2.3}$$

THEOREM 2. *Let the lower triangular matrix* $A=(a_{nk})$ *satisfy* (2.1) *and* (2.2). *Then for* $f\in H_\alpha$, *and* $0\leqq\beta<\alpha\leqq 1$,

$$\|t_n(f)-f\|_\beta=\begin{cases}O(n^{\beta-\alpha})+O(n^{1-\alpha+\beta}a_{nn}) & (\text{for } 0<\alpha<1)\\ O(n^{\beta-1})+O\{a_{nn}n^\beta(\log n)^{1-\beta}\} & (\text{for } \alpha=1).\end{cases}\tag{2.4}$$

THEOREM 3. *Let* $A=(a_{nk})$ *be an infinite matrix satisfying the following:*

$$a_{nk}\geqq 0\quad (n,k=0,1,\ldots),\quad \sum_{k=0}^{\infty}a_{nk}=1\quad (n=0,1,\ldots),\tag{2.5}$$

$$\sum_{k=n+1}^{\infty}(k+1)a_{nk}=O((n+1))\quad (n=1,2,\ldots),\tag{2.6}$$

$$a_{nk}\geqq a_{n,k+1}\quad (k=0,1,\ldots)\ \text{for}\ \ n=0,1,\ldots.\tag{2.7}$$

Then, for $0\leqq\beta<\alpha\leqq 1$ *and* $f\in H_\alpha$,

(2.8)

$$\|T_n(f)-f\|_\beta=O(n^{\beta-\alpha})+O\left[\left\{\sum_{k=1}^{n}k^{-1}\bar a_n((k+1))\right\}^{\beta/\alpha}\left\{\sum_{k=1}^{n}k^{-1-\alpha}\bar a_n((k+1))\right\}^{1-\beta/\alpha}\right].$$

THEOREM 4. *Let the lower-triangular matrix* $A=(a_{nk})$ *satisfy* (2.5) *and* (2.2). *Then for* $f\in H_\alpha$ *and* $0\leqq\beta<\alpha\leqq 1$,

$$\|t_n(f)-f\|_\beta=O(n^{\beta-1})+O\left[\left\{\sum_{k=1}^{n}k^{-1}a'_{n,n-k}\right\}^{\beta/\alpha}\left\{\sum_{k=1}^{n}k^{-1-\alpha}a'_{n,n-k}\right\}^{1-\beta/\alpha}\right].\tag{2.9}$$

3. We shall need the following lemmas:

LEMMA 1. *Let the lower-triangular matrix* $A=(a_{nk})$ *satisfy* $a_{nk}\geqq 0\,(k=0,1,\ldots,n;$ $n=0,1,\ldots)$ *and* (2.2). *Then*

$$\sum_{k=0}^{n}a_{nk}\sin\left(n+\frac{1}{2}\right)t=O\{t^{-1}a_{nn}\},$$

uniformly in $0<t\leqq\pi$. *This is easily proved by Abel's lemma.*

LEMMA 2. *Let* $A=(a_{nk})$ *satisfy* (2.7) *and*

$$a_{nk}\geqq 0\quad (n,k=0,1,\ldots,).\tag{3.1}$$

Then

$$\sum_{k=0}^{n}a_{nk}\sin\left(n+\frac{1}{2}\right)t=O\{\bar a_n(\tau)\}.$$

This lemma can be proved by using arguments similar to that of McFadden [5, p. 182].

LEMMA 3. *Let the lower-triangular matrix* $A=(a_{nk})$ *satisfy* (2.2) *and* $a_{nk}\geqq 0$ $(k=0,1,\ldots,n;\ n=0,1,\ldots)$. *Then*

$$\sum_{k=0}^{n}a_{nk}\sin\left(k+\frac{1}{2}\right)t=O\{a'_{n,n-\tau+1}\}.$$

PROOF. We write

$$\sum_{k=0}^{n} a_{nk} \sin\left(k+\frac{1}{2}\right) t = \left(\sum_{k=0}^{n-\tau} + \sum_{n-\tau-1}^{n}\right) a_{nk} \sin\left(k+\frac{1}{2}\right) t.$$

Clearly, the second sum on the right does not exceed $O(a'_{n,n-\tau+1})$ and since $\{a_{nk}\}_{k=0}^{n}$ is monotonically increasing, the first sum does not exceed

$$O(t^{-1}a_{n,n-\tau+1}) = O\left(\sum_{k=n-\tau+1}^{n} a_{nk}\right) = O(a'_{n,n-\tau+1}).$$

On collecting the results, the lemma follows.

4. In this section, we shall prove the theorems mentioned in § 2.

We observe that for $f \in H_\alpha$, $0<\alpha\leqq 1$,

$$|\varphi_x(t)-\varphi_y(t)| \leqq 4K|x-y|^\alpha \tag{4.1}$$

and also

$$|\varphi_x(t)-\varphi_y(t)| \leqq |f(x+t)-f(x)|+|f(x)-f(x-t)|+|f(y+t)-f(y)|+ \\ +|f(y)-f(y-t)| \leqq 4K|t|^\alpha. \tag{4.2}$$

PROOF OF THEOREM 1. Let $l_n(x)=t_n(f;\, x)-f(x)$, where

$$t_n(f;x)-f(x) = \frac{1}{\pi}\int_0^\pi \frac{\varphi_x(t)}{2\sin\frac{1}{2}t}\left(\sum_{k=0}^{n} a_{nk}\sin\left(k+\frac{1}{2}\right)t\right)dt.$$

Then

$$|l_n(x)-l_n(y)| \leqq \frac{1}{\pi}\int_0^\pi \frac{|\varphi_x(t)-\varphi_y(t)|}{2\sin\frac{1}{2}t}\left|\sum_{k=0}^{n} a_{nk}\sin\left(k+\frac{1}{2}\right)t\right|dt =$$

$$= \frac{1}{\pi}\left(\int_0^{a_{nn}} + \int_{a_{nn}}^{\pi}\right) = I_1+I_2, \text{ say.}$$

However, by (4.2) and (2.1),

$$I_1 = O(1)\int_0^{a_{nn}} t^{-1}|\varphi_x(t)-\varphi_y(t)|\,dt = O(a_{nn}^\alpha), \tag{4.3}$$

and by (4.2) and Lemma 1,

$$I_2 = O(a_{nn})\int_{a_{nn}}^{\pi} t^{-2+\alpha}dt = \begin{cases} O(a_{nn}^\alpha) & \text{for } 0<\alpha<1 \\ O[a_{nn}\log(1/a_{nn})] & \text{for } \alpha=1. \end{cases} \tag{4.4}$$

Since, by (2.1) and (2.2), it follows that $(n+1)a_{nn} \geqq 1$ therefore by (4.1) and (2.1), we have

$$I_1 = O(|x-y|^\alpha) \int_0^{a_{nn}} t^{-1} \left\{ \sum_{k=0}^{n} a_{nk} \left| \sin\left(k+\frac{1}{2}\right) t \right| \right\} dt = \tag{4.5}$$

$$= O(|x-y|^\alpha) \left\{ \int_0^{\frac{1}{n+1}} \left(\sum_{k=0}^{n} (k+1) a_{nk} \right) dt + \int_{\frac{1}{n+1}}^{a_{nn}} t^{-1} \left(\sum_{k=0}^{n} a_{nk} \right) dt \right\} =$$

$$= O\big(|x-y|^\alpha \log\,(na_{nn}+1)\big)$$

and, by Lemma 1 and (4.1), we obtain that

$$I_2 = O\left\{ |x-y|^\alpha a_{nn} \int_{a_{nn}}^{\pi} t^{-2}\, dt \right\} = O\{|x-y|^\alpha\}. \tag{4.6}$$

Now, for $k=1, 2$, we observe that $I_k = I_k^{1-\beta/\alpha} I_k^{\beta/\alpha}$. Thus, by substituting the estimate for I_1 from (4.5) in $I_1^{\beta/\alpha}$ and the estimate for I_1 from (4.3) in $I_1^{1-\beta/\alpha}$, we get

$$I_1 = O\{|x-y|^\beta a_{nn}^{\alpha-\beta} \log^{\beta/\alpha}(na_{nn}+1)\} \tag{4.7}$$

and, by (4.4) and (4.6)

$$I_2 = \begin{cases} O\left[|x-y|^\beta a_{nn}^{\alpha-\beta}\right] & \text{for } 0<\alpha<1, \\ O\left[|x-y|^\beta \{a_{nn} \log\,(1/a_{nn})\}^{1-\beta}\right] & \text{for } \alpha = 1. \end{cases} \tag{4.8}$$

Hence we observe that

$$\sup_{x,y} |\Delta^\beta l_n(x, y)| = \sup_{x \neq y} \frac{|l_n(x) - l_n(y)|}{|x-y|^\beta} =$$

$$= \begin{cases} O\left[a_{nn}^{\alpha-\beta} \log^{\beta/\alpha}(na_{nn}+1)\right] & \text{for } \ 0<\alpha<1, \\ O\left[a_{nn}^{1-\beta} \{\log^\beta(na_{nn}+1) + \log^{1-\beta}(1/a_{nn})\}\right] & \text{for } \alpha = 1. \end{cases} \tag{4.9}$$

Now, proceeding as above, we obtain

$$\|l_n\|_c = \sup_{0<x<2\pi} |t_n(x) - f(x)| = \begin{cases} O(a_{nn}^\alpha) & \text{for } 0<\alpha<1, \\ O\left[a_{nn} \log\,(1/a_{nn})\right] & \text{for } \alpha = 1. \end{cases} \tag{4.10}$$

Since $a_{nn} \leqq 1$ and $\log\,(1/a_{nn}) \leqq \dfrac{1}{a_{nn}}$, we have on collecting results from (4.9) and (4.10),

$$\|t_n(f) - f\|_\beta = \begin{cases} O\left[a_{nn}^{\alpha-\beta} \log^{\beta/\alpha}(na_{nn}+1)\right] & \text{for } 0<\alpha<1, \\ O\left[a_{nn}^{1-\beta} \{\log^\beta(na_{nn}+1) + \log^{1-\beta}) 1/a_{nn})\}\right] & \text{for } \alpha = 1. \end{cases}$$

This completes the proof of Theorem 1.

PROOF OF THEOREM 2. We have, as in Theorem 1,

$$|l_n(x)-l_n(y)| \leqq \frac{1}{\pi}\int_0^{\pi} \frac{|\varphi_x(t)-\varphi_y(t)|}{2\sin\frac{1}{2}t}\left|\sum_{k=0}^{n} a_{nk}\sin\left(k+\frac{1}{2}\right)t\right|dt =$$

$$=\frac{1}{\pi}\left(\int_0^{\pi/n}+\int_{\pi/n}^{0}\right)=J_1+J_2, \text{ say.}$$

Proceeding as in the proof of (4.3) and (4.4), we have $J_1=O(n^{-\alpha})$ $(0<\alpha\leqq 1)$ and

$$J_2=\begin{cases} O(n^{1-\alpha}a_{nn}) & 0<\alpha<1,\\ O(a_{nn}\log n) & \alpha=1.\end{cases}$$

Similarly, letting $a_{nn}=\pi/n$ in (4.5), we obain $J_1=O\{|x-y|^{\alpha}\}$. Also, by (4.1) and Lemma 1, $J_2=O\{|x-y|^{\alpha}na_{nn}\}$. Now $J_k=J_k^{1-\beta/\alpha}J_k^{\beta/\alpha}$ $(k=1,2)$, therefore $J_1= =O\{|x-y|^{\beta}n^{\beta-\alpha}\}$
and

$$J_2=\begin{cases} O\{|x-y|^{\beta}n^{\beta-\alpha+1}a_{nn} & \text{for } 0<\alpha<1,\\ O\{|x-y|^{\beta}n^{\beta/\alpha}a_{nn}\log^{1-\beta/\alpha}n\} & \text{for } \alpha=1.\end{cases}$$

Hence

$$\sup_{x,y}|\varDelta^{\beta}l_n(x,y)|=\begin{cases} O(n^{\beta-\alpha})+O(n^{\beta-\alpha+1}a_{nn}) & \text{for } 0<\alpha<1,\\ O(n^{\beta-1})+O(n^{\beta}a_{nn}\log^{1-\beta}n) & \text{for } \alpha=1.\end{cases}$$

Now, proceeding as above

$$\|l_n\|_c=\begin{cases} O(n^{-\alpha})+O(n^{1-\alpha}a_{nn}) & 0<\alpha<1,\\ O(n^{-1})+O(a_{nn}\log n) & \alpha=1.\end{cases}$$

Now, combining the results, we obtain

$$\|t_n-f\|_{\beta}=\begin{cases} O(n^{\beta-\alpha})+O(n^{\beta-\alpha+1}a_{nn}) & \text{for } 0<\alpha<1,\\ O(n^{\beta-1})+O(a_{nn}n^{\beta}\log^{1-\beta}n) & \text{for } \alpha=1.\end{cases}$$

This completes the proof of Theorem 2.

PROOF OF THEOREM 3. From (1.4) and (2.5), we get

$$T_n(f;x)-f(x)=\frac{1}{2\pi}\sum_{n=0}^{\infty}a_{nk}\int_0^{\pi}\frac{\varphi_x(t)\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,dt=$$

$$=\frac{1}{2\pi}\int_0^{\pi}\frac{\varphi_x(t)}{\sin\frac{1}{2}t}\left\{\sum_{k=0}^{\infty}a_{nk}\sin\left(k+\frac{1}{2}\right)t\right\}dt,$$

the change of order of summation being permissible by the facts that $\Phi_x(t)=O(t^\alpha)$ and $\Sigma_k|a_{nk}|<\infty$.

Writing $l_n(x)=T_n(f;\,x)-f(x)$, we have

$$|l_n(x)-l_n(y)| \leqq \frac{1}{2\pi}\int_0^\pi \frac{|\varphi_x(t)-\varphi_y(t)|}{\sin\frac{1}{2}t}\left|\sum_{k=0}^\infty a_{nk}\sin\left(k+\frac{1}{2}\right)t\right|dt =$$

$$= (2\pi)^{-1}\left(\int_0^{\pi/n}+\int_{\pi/n}^{\pi}\right) = I_1+I_2,\text{ say.}$$

By (4.2) and (2.5),

$$I_1 = O\left(\int_0^{\pi/n} t^{\alpha-1}dt\right) = O(n^{-\alpha}).$$

And by Lemma 2 and (4.2)

$$I_2 = O(1)\int_{\pi/n}^{\pi} t^{\alpha-1}\bar{a}_n([\pi/t])\,dt =$$

$$= O(1)\sum_{k=1}^{n-1}\int_{\pi/(k+1)}^{\pi/k} t^{\alpha-1}\bar{a}_n([\pi/t])\,dt = O(1)\sum_{k=1}^{n} k^{-1-\alpha}\bar{a}_n(k+1).$$

Now using (4.1) in place of (4.2) we estimate I_1 and I_2.

$$I_1 = \int_0^{\pi/n}\frac{|\varphi_x(t)-\varphi_y(t)|}{|\sin(1/2)t|}\left|\left(\sum_{k=0}^{n}+\sum_{k=n+1}^{\infty}\right)a_{nk}\sin\left(k+\frac{1}{2}\right)t\right|dt = I_{11}+I_{12}.$$

Clearly

$$I_{11} = O\left(|x-y|^\alpha\int_0^{\pi/n}\frac{1}{t}\sum_{k=0}^{n}a_{nk}\left(k+\frac{1}{2}\right)t\,dt\right) = O(|x-y|^\alpha)$$

by (2.5). In view of (4.1) and (2.6)

$$I_{12} = O\left(\int_0^{\pi/n}\frac{|x-y|^\alpha}{t}\left|\sum_{k=n+1}^{\infty}a_{nk}(k+1)t\right|dt\right) = O\left(|x-y|^\alpha(n+1)\int_0^{\pi/n}dt\right) =$$

$$= O(|x-y|^\alpha).$$

On collecting the above results we thus get $I_1=O(|x-y|^\alpha)$.

From the definition of I_2, (4.1) and Lemma 2,

$$I_2 = O\left(|x-y|^\alpha\int_{\pi/n}^{\pi}\frac{\bar{a}_n(\tau)}{t}\,dt\right) = O\left(|x-y|^\alpha\sum_{k=1}^{n-1}\int_{\pi/(k+1)}^{\pi/k}\frac{\bar{a}_n\left(\left[\frac{\pi}{t}\right]\right)}{t}\,dt\right) =$$

$$= O\left(|x-y|^\alpha\sum_{k=1}^{n}k^{-1}\bar{a}_n(k+1)\right),$$

since $\bar{a}_n(t)$ is a non-decreasing function of t.

For $r=1,2$, we write $I_r = I_r^{1-\beta/\alpha} I_r^{\beta/\alpha}$. Then $I_1 = O\{|x-y|^\beta n^{\beta-\alpha}\}$ and

$$I_2 = O\left\{|x-y|^\beta \left(\sum_{k=1}^{n} k^{-1}\bar{a}_n(k+1)\right)^{\beta/\alpha} \left(\sum_{k=1}^{n} k^{-1-\alpha}\bar{a}_n(k+1)\right)^{1-\beta/\alpha}\right. .$$

Hence

$$\sup_{x,y} |\Delta^\beta l_n(x,y)| = O(n^{\beta-\alpha}) + O\left\{\left(\sum_{k=1}^{n} k^{-1}\bar{a}_n(k+1)\right)^{\beta/\alpha} \left(\sum_{k=1}^{n} k^{-1-\alpha}\bar{a}_n(k+1)\right)^{1-\beta/\alpha}\right\}.$$

Also, it is easy to observe that

$$\|l_n\|_c = \sup_{0<x<2\pi} |T_n(f;x) - f(x)| = O(n^{-\alpha}) + O\left\{\sum_{k=1}^{n} k^{-1-\alpha}\bar{a}_n(k+1)\right\}.$$

Thus collecting the results obtained for $\|l_n\|_c$ and $\sup_{x,y} |\Delta^\beta l_n(x,y)|$, the proof of the theorem may be completed.

Proof of theorem 4. Observe that by (4.2) and Lemma 3

$$\int_{\pi/n}^{\pi} \frac{|\varphi_x(t)-\varphi_y(t)|}{\sin\frac{1}{2}t} \left|\sum_{k=0}^{n} a_{nk} \sin\left(k+\frac{1}{2}\right)t\right| dt = O(1) \int_{\pi/n}^{\pi} t^{\alpha-1} a'_{n,n-\tau+1}\, dt = \tag{4.11}$$

$$= O(1) \sum_{k=1}^{n-1} \int_{\pi/(k+1)}^{\pi/k} t^{\alpha-1} a'_{n,n-\tau+1}\, dt = O(1) \sum_{k=1}^{n} k^{-1-\alpha} a'_{n,n-k},$$

since $0<\alpha<1$ and $a'_{n,n-\tau}$ is a non-increasing function of t.

By using (4.1) in place of (4.2) and proceeding as above, we see that the integral on the left of (4.11) is

$$O\left\{|x-y|^\alpha \sum_{k=1}^{n} k^{-1} a'_{n,n-k}\right\}.$$

The theorem can be proved by modifying the proof of Theorem 3.

5. In this section, we specialize the matrix A to obtain interesting corollaries.

Let $\{p_n\}$ be a sequence of non-negative constants such that $p_0>0$ and $P_n = p_0+p_1+\ldots+p_n$.

Then the transformations

$$\bar{N}_n(f;x) = (P_n)^{-1} \sum_{k=0}^{n} p_k s_k(x), \tag{5.1}$$

and

$$N_n(f;x) = (P_n)^{-1} \sum_{k=0}^{n} p_{n-k} s_k(x), \tag{5.2}$$

are the $(\bar{N}, p_n)$ and (N, p_n) transformations of the Fourier series $\sum_{n=0}^{\infty} A_n(x)$, respectively (see [4] for definitions of $(\bar{N}, p_n)$ an (N, p_n) methods). Let E_n^α be given by

$$\sum_{n=0}^{\infty} E_n^r x^n = (1-x)^{-r-1} \quad (|x|<1). \tag{5.3}$$

We write for $N_n(f;\,x)$, $\sigma_n^r(f;\,x)$ or $H_n(f;\,x)$ according as $p_n=E_n^{r-1}$ $(r>-1)$ or $p_n=1/(n+1)$ $(n=0,1,\ldots)$ in (5.2), respectively. We also write for $\overline{N}_n(f;\,x)$, $L_n(f;\,x)$ when $p_n=1/(n+1)$, $n=0,1,\ldots$ in (5.1).

COROLLARY 1 (see [3]; Theorem 1). *Let* $f\in H_\alpha$ *and* $0\leqq\beta<\alpha\leqq 1$. *Then*

$$(5.4)\quad \|\overline{N}_n(f)-f\|_\beta = \begin{cases} O\{(p_n/P_n)^{\alpha/\beta}(\log(np_n/P_n))\} & (0<\alpha<1),\\ O\{(p_n/P_n)^{1-\beta}\}\{\log^\beta(np_n/P_n)+\log^{1-\beta}(P_n/p_n)\} & (\alpha=1),\end{cases}$$

where $\{p_n\}$ *is a positive non-decreasing sequence of* n.

PROOF. Under the hypotheses on $\{p_n\}$, $P_n\leqq(n+1)p_n$. Hence the corollary follows from Theorem 1, (2.3), on setting $a_{nn}=p_n/P_n$.

REMARK. On setting $\beta=0$ we obtain a result of Chandra [2].

From Theorem 3, we obtain

COROLLARY 2. *Let* $f\in H_\alpha$ *and* $0\leqq\beta<\alpha/2$, $0<\alpha<1$. *Then*

$$(5.5)\qquad \|L_n(f)-f\|_\beta = O\big((\log n)^{2(\beta/\alpha)-1}\big).$$

REMARK. The case $\beta=0$ of Corollary 2 is considered in [6].

COROLLARY 3 (see [3], Theorem 2). *Let* $\{p_n\}$ *be a positive and non-increasing sequence and* $f\in H_\alpha$, $0\leqq\beta<\alpha\leqq 1$. *Then*

$$(5.6)\quad \|N_n(f)-f\|_\beta = O\left[(P_n)^{-1}\left\{\sum_{k=1}^{n}P_k k^{-1-\alpha}\right\}^{1-\beta/\alpha}\left\{\sum_{k=1}^{n}(P_k/k)\right\}^{\beta/\alpha}\right]+O(n^{\beta-\alpha}).$$

This follows from Theorem 4 by setting $a_{nn}=1/P_n$ and $a'_{n,n-k}=P_k/P_n$.

On setting $p_n=E_n^{\gamma-1}$ $(0<\gamma<1)$ in Corollary 3 we obtain

COROLLARY 4. *Let* $f\in H_\alpha$ *and* $0\leqq\beta<\alpha<\gamma$. *Then*

$$(5.7)\qquad \|\sigma_n^\gamma(f)-f\|_\beta = \begin{cases} O(n^{\beta-\alpha}) & (0<\alpha<1;\ 0<\gamma<1),\\ O\big(n^{\beta-1}(\log n)^{1-\beta}\big) & (\alpha=1=\gamma).\end{cases}$$

REMARK. The case $\beta=0$ of Corollary 4 is due to Alexits [1, p. 301].

COROLLARY 5. *Let* $f\in H_\alpha$ *and* $0<\alpha<1$, $0\leqq\beta<\alpha/2$. *Then*

$$(5.8)\qquad \|H_n(f)-f\|_\beta = O\big((\log n)^{-1+2\beta/\alpha}\big).$$

On putting $\beta=0$ we can obtain estimates in the sup norm from our theorems or corollaries.

The authors owe their thanks to the referee for his comments which led to the improvement of Theorem 3.

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(Received June 9, 1981; revised December 21, 1981)

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Acta Math. Hung.
41 (1—2), (1983), 77—88.

ON THE STRICT TOPOLOGY IN THE NON-LOCALLY CONVEX SETTING. II

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1. Introduction

The strict topology was for the first time defined by Buck [1] on the space of all bounded continuous functions on a locally compact space X. Sentilles [18] and Fremlin—Garling—Haydon [7] have considered the case in which X is completely regular. Several other authors continued the investigation of the strict topology β_0 on the space $C^b(X, E)$ of all bounded continuous E-valued functions on X where E is either the scalar field or an arbitrary locally convex space. The case of an arbitrary topological vector space E was considered by Khan [14] and by the author [13]. In this paper we continue the study of the strict topology assuming that E is a topological vector space. We show that the dual space of $(C^b(X)\otimes E, \beta_0)$ is a certain space $M_t(\mathrm{Bo}(X), E')$ on E'-valued measures defined on the σ-algebra Bo (X) of the Borel subsets of X. In case $C^b(X)\otimes E$ is β_0-dense in $C^b(X, E)$, we have that $(C^b(X, E), \beta_0)' = M_t(\mathrm{Bo}\,(X), E')$. By [13] and [14], this in particular happens in each of the following cases:

1) X has finite covering dimension.
2) Every compact subset of X has finite covering dimension.
3) E has the approximation property.
4) E is a complete metrizable space with a basis.
5) X is a P-space.

It is shown that β_0 is finer than the topology of uniform convergence on the tight subsets of $M_t(\mathrm{Bo}\,(X), E')$. If E is locally bounded and X a P-space, then β_0 is finer than the Mackey topology for the pair $\langle F, F'\rangle$ where $F=(C^b(X, E), \beta_0)$. We also look at the problem of the separability in the strict topology as well as at some other properties of this topology.

2. Preliminaries

Throughout this paper, X will denote a completely regular Hausdorff space, E a real Hausdorff topological vector space, $C^b(X, E)$ the space of all bounded E-valued functions on X and $C^{tb}(X, E)$ the subspace of all totally bounded members of $C^b(X, E)$. We will denote by $C^b(X)$ the space $C^b(X, \mathbf{R})$ ($\mathbf{R}$ is the space of real numbers). The algebraic tensor product $C^b(X)\otimes E$ is isomorphic to the subspace of $C^b(X, E)$ spanned by the functions $f\otimes s$, $f\in C^b(X)$ and $s\in E$, where $f\otimes s$ is defined on X by $(f\otimes s)(x)=f(x)s$. The uniform topology u on $C^b(X, E)$ is the linear topology which has as a base at zero the family of all sets of the form $\{f\in C^b(X, E)\colon f(X)\subset W\}$ where W is a neighborhood of zero in E. The strict topology is the linear topology β_0 which has as a base at zero all sets of the form $\{f\in C^b(X,E)\colon (gf)(X)\subset W\}$ where

W is a neighborhood of zero in E and g a bounded real function on X vanishing at infinity. If p is a seminorm on E, then, for every subset A of X and every E-valued function f on X, we define $\|f\|_{A,p}$ and $\|f\|_p$ by

$$\|f\|_{A,p} = \sup\{p(f(x)) \colon x\in A\}, \quad \|f\|_p = \|f\|_{X,p}.$$

Finally, we will let $M(X)$ denote the space of all bounded real-valued finitely-additive regular (with respect to the family of zero sets) measures on the algebra $B(X)$ of subsets of X generated by the zero sets (see Varadarajan [21]). By $M_\tau(X)$ and $M_t(X)$ we will denote, respectively, the subspaces of all τ-additive and all tight members of $M(X)$.

3. The space $M(B(X), E')$

Let W be a neighborhood of zero in E. We denote by $M_W(B(X), E')$ the space of all $m\colon B(X)\to E'$ with the following two properties:

a) For each $s\in E$, the set function $ms\colon B(X)\to\mathbf{R}$, $(ms)(A)=m(A)s$, belongs to $M(X)$.

b) There exists a member μ of $M(X)$ such that $|ms|\leqq\mu$ for all $s\in W$ ($|ms|$ denotes the total variation of ms).

If $m\in M_W(B(X), E')$, then (since $M(X)$ is an order complete lattice) the supremum $m_W=\sup\{|ms|\colon s\in W\}$ exists.If W is symmetric, then $m_W=\sup\{ms\colon s\in W\}$. We will refer to m_W as the W-variation of m. If $\mathbf{B}$ is a base at zero in E, then we will denote by $M(B(X), E')$ the space $\cup\{M_W(B(X), E')\colon W\in\mathbf{B}\}$.

We omit the proof of the following easily established

PROPOSITION 3.1. *Let W be a symmetric neighborhood of zero in E and let m be an E'-valued function on $B(X)$. Then, $m\in M_W(B(X), E')$ iff $ms\in M(X)$ for all $s\in E$ and $\sup\sum_{i=1}^{n}|m(A_i)s_i|<\infty$ where the supremum is taken over the family of all finite $B(X)$-partitions $(A_i)_{i=1}^n$ of X and all choices of $s_1, \dots, s_n$ in W. Moreover, for each $A\in B(X)$, we have*

$$m_W(A) = \sup\left\{\sum_{i=1}^{n} m(A_i)s_i \colon s_i\in W,\ (A_i)_{i=1}^n \text{ a } B(X)\text{-partition of } A\right\}.$$

For p a continuous seminorm on E, we define $M_p(B(X), E')$ to be the set $M_W(B(X), E')$ where $W=W_p=\{s\in E\colon p(s)\leqq 1\}$. For an $m\in M_p(B(X), E')$, we define m_p to be the measure m_W.

Let now W be a symmetric neighborhood of zero in E and let W^{00} be the bipolar of W with respect to the pair $\langle E, E'\rangle$. Let $q=q_W$ be the Minkowski functional of the absolutely convex set W^{00}. We have the following

LEMMA 3.2. *A set function $m\colon B(X)\to E'$ belongs to $M_W(B(X), E')$ iff $m\in M_q(B(X), E')$. Moreover, we have $m_q=m_W$.*

PROOF. Since $W\subset\{s\in E\colon q(s)\leqq 1\}$, it follows that $m\in M_W(B(X), E')$, whenever $m\in M_q(B(X), E')$, and that $m_W\leqq m_q$. On the other hand, let $m\in M_W(B(X), E')$

and let $A\in B(X)$. For each $x'\in E'$, we define $|x'|_W=\sup_{s\in W}|x'(s)|$. Then $W^0=\{x'\in E'\colon |x'|_W\leqq 1\}$. It is easy to see that $|x'|_W=\sup\{|x'(s)|\colon q(s)\leqq 1\}$. From Proposition 3.1, we get that

$$m_W(A)=\sup\left\{\sum_{i=1}^{n}|m(A_i)|_W\colon (A_i)_{i=1}^n \text{ a } B(X)\text{-partition of } A\right\}.$$

It follows now easily that $m_q(A)\leqq m_W(A)$ and this completes the proof.

Next we define integration of E-valued functions on X with respect to members of $M(B(X), E')$. The definition is similar to the one in the locally convex case (see [7,12]). Let A be a non-empty member of $B(X)$ and consider the collection Ω_A of all $\alpha=\{A_1, \dots, A_n;\ x_1, \dots, x_n\}$ where $\{A_1, \dots, A_n\}$ is a finite $B(X)$-partition of A and $x_i\in A_i$. For $\alpha_1, \alpha_2\in\Omega_A$, we define $\alpha_1\geqq\alpha_2$ iff the partition of A in α_1 is a refinement of the partition in α_2. In this way, Ω_A becomes a directed set. For f and E-valued function on X and $\alpha=\{A_1, \dots, A_n;\ x_1, \dots, x_n\}\in\Omega_A$, we define $s_\alpha(f)=\sum_{i=1}^{n} m(A_i)f(x_i)$. We say that f is m-integrable over A if the limit $\lim_{\alpha\in\Omega_A} s_\alpha(f)$ exists. The value of this limit is called the integral of f over A and we will denote it by $\int_A fdm$. For $A=\emptyset$, we define $\int_A fdm=0$. If f is integrable over each $A\in B(X)$, then we will say that f is integrable. We will write simply $\int fdm$ for the integral $\int_X fdm$.

For W a neighborhood of zero in E, f a bounded E-valued function on X and $A\subset X$, we will denote by $W_A(f)$ the number $\inf\{\lambda>0\colon f(A)\subset\lambda W\}$. Modifying slightly the argument used in the locally convex case (see [7, 12] we get the following.

PROPOSITION 3.3. *Let* $m\in M_W(B(X), E')$ *and let* $f\in C^{tb}(X, E)$. *Then* f *is* m*-integrable and for each* A *in* $B(X)$ *we have* $\left|\int_A fdm\right|\leqq m_W(A)W_A(f)$. *Moreover, for* $g\in C^b(X)$ *and* $s\in E$, *we have*

$$\int_A g\otimes s\,dm=\int_A g\,d(ms).$$

From the preceding proposition we get easily the following.

PROPOISTION 3.4. *If* $m\in M(B(X), E')$, *then the mapping* $f\mapsto m(f)=\int fdm$, $f\in C^{tb}(X, E)$, *is an element of the dual space of* $(C^{tb}(X, E), u)$.

We have also the following result whose proof is analogous to the proof of Theorem 2.2 in [10].

PROPOSITION 3.5. *Let* $m\in M_W(B(X), E')$. *If* W *is balanced, then for each cozero set* A *in* X *we have*

$$m_W(A)=\sup\{|m(f)|\colon f\in C^{tb}(X, E),\ f(A)\subset W,\ f=0 \text{ on } X-A\}=$$
$$=\sup\{|m(f)|\colon f\in C^b(X)\otimes E,\ f(A)\subset W,\ f=0 \text{ on } X-A\}.$$

COROLLARY 3.6. *If* $m_1, m_2 \in M(B(X), E')$ *are such that* $m_1(f)=m_2(f)$ *for each* $f \in C^b(X)\otimes E$, *then* $m_1=m_2$.

PROPOSITION 3.7. *Let* $\varphi \in (C^b(X)\otimes E, u)'$. *Then, there exists a unique* $m \in M(B(X), E')$ *such that* $\varphi(f)=\int f dm$ *for each* $f \in C^b(X)\otimes E$.

PROOF. There exists a balanced neighborhood W of zero in E such that

$$\{f \in C^b(X)\otimes E: f(X)\subset W\}\subset\{f: |\varphi(f)|\leqq 1\}.$$

For each $s \in E$, the mapping T_s: $C^b(X)\to\mathbf{R}$, $T_s(f)=\varphi(f\otimes s)$, is continuous with respect to the uniform topology on $C^b(X)$. Hence (see [21]) there exists a unique $\mu_s \in M(X)$ such that $\varphi(f\otimes s)=\int f d\mu_s$ for each $f \in C^b(X)$. For $A \in B(X)$, we define $m(A)$: $E\to\mathbf{R}$, $m(A)s=\mu_s(A)$. It is shown now easily (the proof is similar to the corresponding one in the locally convex case) that $m \in M(B(X), E')$ and that $\varphi(f)=m(f)$ for each $f \in C^b(X)\otimes E$. The uniqueness of m follows from the preceding corollary.

4. Some properties of the topology β_0

Suppose that F is a real locally convex (not necessarily Hausdorff) space. For p a continuous seminorm on F, we let β_{0p} denote the locally convex topology on $C^b(X, F)$ generated by the family of seminorms p_g, $p_g(f)=\|gf\|_p$, where g ranges over the family of all bounded real functions on X which vanish at infinity.

We have the following easily established

LEMMA 4.1. *The topology* β_{0p} *has as a base at zero the sets of the form*

$$\bigcap_{n=1}^{\infty}\{f \in C^b(X, F): \|f\|_{K_n, p}\leqq a_n\}$$

where (K_n) *is a sequence of compact subsets of* X *and* $0<a_n\to\infty$.

Let now u_p denote the locally convex topology on $C^b(X, F)$ generated by the seminorm $f\mapsto\|f\|_p$ and let $\tau_{c,p}$ be the locally convex topology generated by the seminorms $f\mapsto\|f\|_{K,p}$, where K ranges over the family of all compact subsets of X. If $C_{rc}(X, F)$ is the space of all $f \in C^b(X, F)$ which have relatively-compact range, then it is shown in [8] that on $C_{rc}(X, F)$ the topology β_{0p} coincides with the finest locally convex topology which agrees with $\tau_{c,p}$ on u_p-bounded sets. As the next Theorem shows, the same happens on the space $C^b(X, F)$.

THEOREM 4.2. a) β_{0p} *is the finest locally convex topology on* $C^b(X, F)$ *which agrees with* $\tau_{c,p}$ *on* u_p*-bounded sets.*

b) β_{0p} *is the finest locally convex topology on* $C^b(X, F)$ *which agrees with* β_{0p} *on* u_p*-bounded sets.*

PROOF. a) Using the preceding Lemma it follows easily that β_{0p} agrees with $\tau_{c,p}$ on u_p-bounded sets. Conversely, let τ be a locally convex topology on $C^b(X, F)$ which agrees with $\tau_{c,p}$ on u_p-bounded sets. Assume first that p is a norm on F. For each $f \in C^b(X, F)$, we have $\|f\|_p=\sup\|f\|_{K,p}$ where K ranges over the family of compact subsets of X. Also, let $K_1, \ldots, K_n$ be compact subsets of X, $\varepsilon>0$, $K=\bigcup_{i=1}^{n} K_i$ and $d=\|f\|_{K,p}+\varepsilon$. The open set $V=\{x\in X: p(f(x))<d\}$ contains K and hence there exists a $g\in C^b(X)$, $0\leqq g\leqq 1$, with $g=1$ on K and $g=0$ on $X-V$. If $f_1=gf$ and $f_2=(1-g)f$, then $f=f_1+f_2$, $\|f_2\|_{K,p}=0$ and $\|f_1\|_p\leqq d$. This (by Wiweger [23], Lemma 1) proves that β_{0p} coincides with the mixed topology $\gamma_p=\gamma[u_p, \tau_{c,p}]$. Since every u_p-neighborhood of zero is u_p-bounded and since u_p nad $\tau_{c,p}$ are both Hausdorff, γ_p is the finest locally convex topology on $C^b(X, F)$ which agress with $\tau_{c,p}$ on u_p-bounded sets. Assume next that p is an arbitrary continuous seminorm on F. Set $M_p=\{s\in E: p(s)=0\}$ and $F_p=F/M_p$. On F_p we consider the norm $\bar{p}(s+M_p)==p(s)$. Let W be an absolutely convex τ-closed neighborhood of zero. For each positive integer n, there exist a compact set K_n and $0<\delta_n<1$ such that

$$W_n = \{f\in C^b(X, F)\colon \|f\|_{K_n,p} \leqq \delta_n,\ \|f\|_p \leqq n\}\subset W.$$

Set

$$V_n = \{f\in C^b(X, F_p)\colon \|f\|_{K_n,\bar{p}} \leqq \delta_n,\ \|f\|_{\bar{p}} \leqq n\}.$$

The absolutely convex hull V_0 of the set $\bigcup_{n=1}^{\infty} V_n$ is a $\beta_{0\bar{p}}$ neighborhood of zero since $\bar{p}$ is a norm and V_0 is a neighborhood of zero for the finest locally convex topology on $C^b(X, F_p)$ which agrees with $\tau_{c,\bar{p}}$ on $u_{\bar{p}}$-bounded sets. Hence, there exists a bounded real function on X vanishing at infinity such that

$$V = \{f\in C^b(X, F_p)\colon \|gf\|_{\bar{p}} \leqq 1\}\subset V_0.$$

We will show that the set

$$O = \{f\in C^b(X, F)\colon \|gf\|_p \leqq 1\}$$

is contained in W. In fact, let $f\in W$. Since $\lim_{\delta\to 1} \delta f=f$ in the topology τ, it suffices to show that $\delta f\in W$ for each $0<\delta<1$. So let $0<\delta<1$. For each $h\in C^b(X, F)$, let $\bar{h}=\pi\circ h$ where $\pi\colon F\to F_p$ be the quotient mapping. The set $A_p=\{\bar{h}\colon h\in C^b(X, F)\}$ is $\tau_{c,\bar{p}}$-dense in $C^b(X, F_p)$ because $C^b(X)\otimes F_p\subset A_p$. Since $\bar{f}\in V\subset V_0$, there are $f_i\in V_{n_i}$, $i=1, \ldots, N$, and $\lambda_i\in\mathbf{R}$ with $\sum_{i=1}^{N} |\lambda_i|\leqq 1$ and $\bar{f}=\sum_{i=1}^{N} \lambda_i f_i$. Let $\varepsilon>0$ be such that $\varepsilon+\delta\cdot\delta_{n_i}<\delta_{n_i}$ and $\varepsilon+\delta n_i<n_i$ for $i=1, \ldots, N$. Let K be an arbitrary compact subset of X and set $G_i=K\cup K_{n_i}$, $i=1, \ldots, N$. Let $h_i'\in C^b(X, F)$ be such that $\|\delta f_i-\bar{h}_i'\|_{G_i,\bar{p}}<\varepsilon$, $i=1, \ldots, N$. Then $\|h_i'\|_{G_i,p}=\|\bar{h}_i'\|_{G_i,\bar{p}}\leqq\varepsilon+\delta n_i<n_i$. Hence the open set $Z_i=\{x\in X: p(h_i'(x))<n_i\}$ contains the compact set G_i. Choose $\varphi_i\in C^b(X)$, $0\leqq\varphi_i\leqq 1$, $\varphi_i=1$ on G_i, $\varphi_i=0$ on $X-Z_i$ and set $h_i=\varphi_i h_i'$. Then $\|h_i\|_p\leqq n_i$, $\|\delta f_i-\bar{h}_i\|_{K,\bar{p}}<\varepsilon$ and $\|h_i\|K_{n_i,p}\leqq\varepsilon+\delta\|f_i\|K_{n_i,\bar{p}}\leqq\varepsilon+\delta\cdot\delta_{n_i}<\delta_{n_i}$. Thus $h_i\in W_{n_i}\subset W$ and so $h_0=\sum_{i=1}^{N} \lambda_i h_i\in W$. Let $M>0$ be such that $\|f\|_p<M$ and $n_i<M$ for $i=1, \ldots, N$. The set $A=\{h\in C^b(X, F)\colon \|h\|_p \leqq M\}$ is u_p-bounded and $\delta f\in A$. Also

$$\|h_0-\delta f\|_{K,p} \leqq \sum_{i=1}^{N} |\lambda_i|\, \|\bar{h}_i-\delta f_i\| \leqq \varepsilon$$

and $h_0 \in A$. This proves that δf belongs to the $\tau_{c,p}$-closure of W in A. Since τ agrees with $\tau_{c,p}$ on A and since W is τ-closed we have $\delta f \in W$. This proves that $0 \subset W$ and so W is a β_{0p}-neighborhood of zero.

b) It follows easily from a).

COROLLARY 4.3. *An absoluetly convex subset W of $C^b(X, F)$ is a β_{0p}-neighborhood of zero iff for each $M>0$ there exist a compact subset K of X and $\delta>0$ such that*

$$\{f\in C^b(X, F)\colon \|f\|_{K,p} \leqq \delta,\ \|f\|_p \leqq M\} \subset W.$$

PROPOSITION 4.4. $\beta_{0p}=\gamma[u_p, \tau_{c,p}]$.

PROOF. Let W be an absolutely convex neighborhood of zero in $C^b(X, F)$ for the topology $\gamma_p=\gamma[u_p, \tau_{c,p}]$. By the definition of γ_p (see [23]), there exist a sequence (K_n) of compact subsets of X, a sequence (δ_n) of positive numbers and $\delta>0$ such that

$$\bigcup_{n=1}^{\infty} (V_1 \cap V + V_2 \cap 2V + \ldots + V_n \cap nV) \subset W$$

where $V_i=\{f\in C^b(X, F)\colon \|f\|_{K_i,p} \leqq \delta_i\}$ and $V=\{f\colon \|f\|_p \leqq \delta\}$. Let $M>0$. If n is such that $n\delta>M$, then

$$\{f\in C^b(X, F)\colon \|f\|_p \leqq M,\ \|f\|_{K_n,p} \leqq \delta_n\} \subset V_n \cap nV \subset W.$$

Thus W is β_{0p}-neighborhood of zero by the preceding corollary. This proves that $\gamma_p \leqq \beta_{0,p}$. For the inverse inequality, we observe first that the result holds when p is a norm in F as we have seen in the proof of the Theorem 4.2. In the general case, let M_p, F_p and π be as in the proof of Theorem 4.2 and let W be a β_{0p}-neighborhood of zero. By Lemma 4.1, there exist a sequence (K_n) of compact subsets of X and a sequence (a_n) of real numbers, with $0<a_n\to\infty$, such that

$$W_1 = \bigcap_{n=1}^{\infty} \{f\in C^b(X, F)\colon \|f\|_{K_n,p} \leqq a_n\} \subset W.$$

Since $\bar{p}$ is a norm, we have $\beta_{0,\bar{p}}=\gamma[u_{\bar{p}}, \tau_{c,\bar{p}}]=\gamma_{\bar{p}}$ and hence the set

$$W_2 = \bigcap_{n=1}^{\infty} \{f\in C^b(X, F_p)\colon \|f\|_{K_n,\bar{p}} \leqq a_n\}$$

is a $\gamma_{\bar{p}}$-neighborhood of zero. Hence, there exist a sequence (G_n) of compact subsets of X, a sequence (δ_n) of positive numbers and $\delta>0$ such that

$$O = \bigcup_{n=1}^{\infty} (V_1 \cap V + V_2 \cap 2V + \ldots + V_n \cap nV) \subset W_2$$

where

$$V_n = \{f\in C^b(X, F_p)\colon \|f\|_{G_n,\bar{p}} \leqq \delta_n\},\quad V = \{f\in C^b(X, F_p)\colon \|f\|_{\bar{p}} \leqq \delta\}.$$

Setting

$$Z_n = \{f\in C^b(X, F)\colon \|f\|_{G_n,p} < \delta_n\},\quad Z = \{f\in C^b(X, F)\colon \|f\|_p < \delta\},$$

we have that

$$\bigcup_{n=1}^{\infty} \{Z_1 \cap Z + Z_2 \cap 2Z + \ldots + Z_n \cap nZ\} \subset W$$

and so W is a γ_p-neighborhood of zero. This completes the proof.

As in the locally convex case (see [7]), we will call an element m of $M_W(B(X), E')$ τ-additive iff $ms \in M_\tau(X)$ for each $s \in E$. An argument similar to the one used in the locally convex case shows that an $m \in M_W(B(X), E')$ is τ-additive iff there exists an extension of m to a set function $\mu = m_\tau$: $\mathrm{Bo}(X) \to E'$ such that:

(1) μs is a τ-additive regular Borel measure for each $s \in E$.

(2) $\mu_W(X) < \infty$, where, for $A \in \mathrm{Bo}(X)$, $\mu_W(A)$ denotes the supremum of all $\sum_{i=1}^{n} |\mu(A_i)s_i|$ for all finite $\mathrm{Bo}(X)$-partitions $(A_i)_{i=1}^n$ of A and all choices of $s_i \in W$.

It is also shown that μ is unique, μ is a τ-additive regular Borel measure on X and m_W coincides with the restriction of μ_W to $B(X)$. Integrals, with respect to μ, of E-valued functions on X are defined as in the case of m with the only difference that we consider partitions into Borel sets. We will denote by $M_{W,\tau}(\mathrm{Bo}(X), E')$ the space of all μ: $\mathrm{Bo}(X) \to E'$ with properties (1) and (2) and by $M_\tau(\mathrm{Bo}(X), E')$ the union of all $M_{W,\tau}(\mathrm{B0}(X), E')$ for all neighborhoods W of zero in E.

We omit the proof of the following easily established.

PROPOSITION 4.5. *Let* $\mathbf{B}$ *be a base at zero in* E *and, for each* $W \in \mathbf{B}$, *let* q_W *denote the Minkowski functional of the bipolar* W^{00} *of* W. *Then, the locally convex topology on* E *generated by the family of seminorms* q_W, $W \in \mathbf{B}$, *coincides with the finest locally convex topology on* E *coarser than the given topology of* E.

We will denote by E_c the vector space E equipped with the finest locally convex topology on E coarser than the given topology. It is easy to see that $E' = E'_c$.

DEFINITION 4.6. 1) A set Φ of linear functionals on $C^b(X, E)$ is called tight iff there exists a neighborhood W of zero in E satisfying the following two conditions:

a) There exists $\lambda > 0$ such that the set $\{f \in C^b(X, E)\colon f(X) \subset \lambda W\}$ is contained in the polar Φ^0 of Φ in $C^b(X, E)$.

b) For every $\varepsilon > 0$ there exists a compact subset K of X such that $|\varphi(f)| \leqq \varepsilon$ for each $\varphi \in \Phi$ and each $f \in C^b(X, E)$ with $f(X) \subset W$ and $f = 0$ on K.

2) A subset H of $M(X)$ is called tight iff H is norm bounded and for each $\varepsilon > 0$ there exists a compact subset K of X such that $|m|(V) \leqq \varepsilon$ for each $m \in H$ and each $V \in B(X)$ contained in $X - K$.

3) A subset H of $M(B(X), E')$ is called tight iff there exists a neighborhood W of zero in E with $H \subset M_W(B(X), E')$ and such that the set $\{m_W\colon m \in H\}$ is a tight subset of $M(X)$.

4) A $m \in M(B(X), E')$ is called tight iff the singleton $\{m\}$ is tight.

It is easy to see that every tight member m of $M(B(X), E')$ is τ-additive and hence it has a unique extension m_τ to a member of $M_\tau(\mathrm{Bo}(X), E')$. It is also easy to see that if $m \in M_W(B(X), E')$ is tight, then given $\varepsilon > 0$ there exists a compact subset K of X such that $(m_\tau)_W(X - K) \leqq \varepsilon$. We will denote by $M_t(\mathrm{Bo}(X), E')$ the subspace of all tight members of $M_\tau(\mathrm{Bo}(X), E')$. We can verify easily that if $m \in M_t \cdot$ $\cdot(\mathrm{Bo}(X), E')$, then every $f \in C^b(X, E)$ is m-integrable.

PROPOSITION 4.7. *Let $m \in M_t(\mathrm{Bo}(X), E')$. Then, the mapping $f \mapsto m(f) = \int f\,dm$ is an element of the dual space of $(C^b(X, E), \beta_0)$.*

PROOF. Let W be a neighborhood of zero in E for which $m_W(X) < \infty$ and p denote the Minkowski functional of W^{00}. Then $m_W = m_p$. Let now $r > 0$ and choose a compact subset K of X such that $m_p(X-K) < \frac{1}{2r}$. Let $d > 0$ be such that $dm_p(X) < \frac{1}{2}$. If now $f \in C^b(X, E_c)$ with $\|f\|_{K,p} \leqq d$ and $\|f\|_p \leqq r$, then

$$\left|\int f\,dm\right| \leqq \left|\int_K f\,dm\right| + \left|\int_{X-K} f\,dm\right| \leqq dm_p(X) + rm_p(X-K) \leqq 1.$$

This (by Corollary 4.3) shows that the set $V = \{f \in C^b(X, E_c) \colon \left|\int f\,dm\right| \leqq 1\}$ is a β_{0p}-neighborhood of zero in $C^b(X, E_c)$. Thus there exists a bounded real function g on X vanishing at infinity and such that

$$V_1 = \{f \in C^b(X, E_c) \colon \|gf\|_p \leqq 1\} \subset V.$$

If now $f \in C^b(X, E)$ with $(gf)(X) \subset W$, then $f \in V_1$ and so $\left|\int f\,dm\right| \leqq 1$. It follows that the mapping $f \mapsto m(f)$ is β_0-continuous in $C^b(X, E)$.

THEOREM 4.8. $(C^b(X) \otimes E, \beta_0)' = M_t(\mathrm{Bo}(X), E')$.

PROOF. Every $m \in M_t(\mathrm{Bo}(X), E')$ defines a β_0-continuous linear functional on $C^b(X) \otimes E$ by the preceding proposition. Conversely, let $\varphi \in (C^b(X) \otimes E, \beta_0)'$. Since $\beta_0 \leqq u$, there exists (by Proposition 3.7) a unique $m \in M(B(X), E')$ such that $\varphi(f) = \int f\,dm$ for each $f \in C^b(X) \otimes E$. We will show that m is tight. In fact, let $\varepsilon > 0$ and let $V = \{f \in C^b(X) \otimes E \colon |\varphi(f)| \leqq 1\}$. There exist a balanced neighborhood W of zero in E, a sequence (K_n) of compact sets in X and a sequence (a_n) of reals, with $0 > a_n \to \infty$, such that

$$\bigcap_{n=1}^{\infty} \{f \in C^b(X) \otimes E \colon f(K_n) \subset a_n W\} \subset V$$

(by [13], Theorem 3.1). Let n_0 be such that $a_n \geqq 1/\varepsilon$, if $n > n_0$. If $K = \bigcup_{i=1}^{n_0} K_i$ and $\delta = \min\{\varepsilon a_1, \ldots, \varepsilon a_{n_0}\}$, then

$$\{f \in C^b(X) \otimes E \colon f(X) \subset W, f(K) \subset \delta W\} \subset \varepsilon V.$$

Using Proposition 3.5, we get that $m_W(A) \leqq \varepsilon$ for each cozero set A contained in $X-K$. For every zero set $Z \subset X-K$ there exists a cozero set A with $Z \subset A \subset X-K$ and so $m_W(Z) < m_W(A) \leqq \varepsilon$. By regularity, $m_{W(A)} \leqq \varepsilon$ for each $A \in B(X)$ disjoint from K. This proves that m is tight. If $m_\tau \in M_t(\mathrm{Bo}(X), E')$ is the unique extension of m, then $\varphi(f) = \int f\,dm = \int f\,dm_\tau$ for each $f \in C^b(X) \otimes E$ and the result follows.

COROLLARY 4.9. *If $C^b(X) \otimes E$ is β_0-dense in $C^b(X, E)$, then $(C^b(X, E), \beta_0)' =$ $= M_t(\mathrm{Bo}(X), E')$.*

It is clear that every tight linear functional φ on $C^b(X, E)$ is continuous with respect to the uniform topology. Thus there exists a unique $m_\varphi \in M(\mathrm{B}(X), E')$ such that $\varphi(f) = \int f dm$ for each $f \in C^b(X) \otimes E$. The following Theorem gives the relationship between tight subsets of $M(B(X), E')$ and tight sets of linear functions on $C^b(X, E)$.

THEOREM 4.10. I. *If Φ is a tight set of linear functionals on $C^b(X, E)$, then the set $H_\Phi = \{m_\varphi: \varphi \in \Phi\}$ is a tight subset of $M(B(X), E')$.*

II. *Let $H \subset M_t(B(X), E')$ and let $\Phi_H = \{\varphi_m: m \in H\}$, where $\varphi_m(f) = \int f dm$ for all $f \in C^b(X, E)$. Then H is tight iff Φ_H is tight.*

III. *If $C^b(X) \otimes E$ is β_0-dense in $C^b(X, E)$, then a subset Φ of $(C^b(X, E), \beta_0)'$ is tight iff the set H_Φ is tight.*

PROOF. I. Let W be a balanced neighborhood of zero in E and $\lambda > 0$ be such that a) and b) of 1) in Definition 4.6 are satisfied. By Proposition 3.5, we have $m_W(X) \leqq 1/\lambda$ for all $m = m_\varphi (\varphi \in \Phi)$. Let $\varepsilon > 0$ and let K be a compact subset of X such that $|\varphi(f)| \leqq \varepsilon$ for all $f \in C^b(X, E)$ with $f = 0$ on K and $f(X) \subset W$. If V is a cozero set contained in $X - K$, then (by Proposition 3.5) we have $(m_\varphi)_W(V) \leqq \varepsilon$ for all $\varphi \in \Phi$. Using the regularity of m_φ, we get that $(m_\varphi)_W(A) \leqq \varepsilon$ for each $A \in B(X)$ disjoint from K and each $\varphi \in \Phi$. Hence H_Φ is tight.

II. If Φ is tihgt, then H_H is tight by I. Conversely, suppose that H is tight and let W be a balanced neighborhood of zero in E such that the set $\{m_W: m \in H\}$ is a tight subset of $M(X)$. If p is the Minkowski functional of W^{00}, then $m_p = m_W$ for each $m \in H$. Let $d > 0$ be such that $m_p(X) \leqq d$ for all $m \in H$. If $\varepsilon > 0$, then, using the Corollary 4.3, we get that the set

$$A = \left\{f \in C^b(X, E_c): \left|\int f dm\right| \leqq \varepsilon \text{ for all } m \in H\right\}$$

is a β_0-neighborhood of zero in $C^b(X, E_c)$ and hence there exists a bounded real function g on X vanishing at infinity such that

$$\{f \in C^b(X, E_c): \|gf\|_p \leqq 1\} \subset A.$$

Let K be a compact set in X such that $|g(x)| \leqq 1$ if $x \notin K$. If now $f \in C^b(X, E)$ vanishes on K and $f(X) \subset W$, then $g(x)f(x) \in W$ for each $x \in X$ and so $\|gf\|_p \leqq 1$ which implies that $\left|\int f dm\right| \leqq \varepsilon$ for each $m \in H$. Also if $f(X) \subset \frac{1}{d} W$, then $|\varphi_m(f)| = \left|\int f dm\right| \leqq 1$ for all $m \in H$. This proves that Φ_H is tight.

III. It follows easily from I and II and from Corollary 4.9 since, for $\varphi \in \Phi$ and $m = m_\varphi$, we have $\int f dm = \varphi(f)$ for all $f \in C^b(X, E)$.

PROPOSITION 4.11. *If X is not empty, then β_0 is locally convex iff E is locally convex.*

PROOF. Clearly β_0 is locally convex when E is such a space. Conversely, suppose that β_0 is locally convex and let W we a zero-neighborhood in E. Let $x_0 \in X$ and let h denote the characteristic function of the singleton set $\{x_0\}$. Then the set

$$V = \{f \in C^b(X, E): (hf)(X) \subset W\}$$

is a β_0-neighborhood of zero. By hypothesis, there exists a convex β_0-neighborhood V_1 of zero contained in V. Let W_1 be a balanced neighborhood of zero in E and g a real function on X vanishing at infinity with $\|g\| \leqq 1$ and such that

$$V_2 = \{f \in C^b(X, E) \colon (gf)(X) \subset W_1\} \subset V_1.$$

Since V_1 is convex, it follows easily that the convex hull W_2 of W_1 is contained in W. This proves that E is locally convex.

PROPOSITION 4.12. *On $C^b(X, E)$ the topology of uniform convergence on the tight subsets of $M_t(\mathrm{Bo}(X), E')$ is coarser than β_0. If X is not empty and E not locally convex, then the two topologies are not equal.*

PROOF. Let $H \subset M_t(\mathrm{Bo}(X), E')$ be tight. As we have shown in the proof of Theorem 4.19, there exists a continuous seminorm p on E and a bounded real function on X vanishing at infinity such that

$$\{f \in C^b(X, E) \colon \|gf\|_p \leqq 1\} \subset H^0$$

where H^0 denotes the polar of H in $C^b(X, E)$. This proves the first part of the proposition. The second part follows from the preceding proposition.

Suppose now that E is locally bounded and let $\|\cdot\|$ be a p-norm $(0<p\leqq 1)$ giving the topology of E. If $W = |s \in E \colon \|s\| \leqq 1\}$, then $m_W(X) < \infty$ for each $m \in M(B(X), E')$ and the mapping $m \mapsto m_W(X)$ is a norm on $M(B(X), E')$. Thus $M(B(X), E')$ becomes a normed space. The same happens on the space $M_t \cdot \cdot (\mathrm{Bo}(X), E')$.

THEOREM 4.13. *Let E be locally bounded and let $F = C^b(X, E)$ and $G = M_t(\mathrm{Bo}(X), E')$. Then every absollutely convex $\sigma(G, F)$-compact subset A of G is norm-bounded.*

PROOF. Let $\|\cdot\|$ be a p-norm on E giving its topology and let $W = \{s \in E \colon \|s\| \leqq 1\}$. If q is the Minkowski functional of the bipolar W^{00} of W, then $m_q(X) = m_W(X) < \infty$ for each $m \in M_t(\mathrm{Bo}(X), E')$. If $f \in C^b(X, E)$, then $\left|\int f\,dm\right| \leqq \|f\|_q \cdot m_q(X)$ for each $m \in G$. Hence the set $B = \{f \in F \colon \|f\|_q \leqq 1\}$ is $\sigma(F, G)$-bounded. Also, the polar A^0 of A in F is a Mackey neighborhood of zero in F for the pair $\langle F, G \rangle$. Hence there exists $d>0$ such that $B \subset dA^0$ and so $A \subset A^{00} \subset dB^0$. So, for $m \in A$, we have

$$m_W(X) = m_q(X) = \sup\left\{\left|\int f\,dm\right| \colon f \in B\right\} \leqq d$$

which shows that A is norm-bounded.

In case X is a P-space, every compact subset of X is finite. Thus, by [13], Theorem 3.3, $C^b(X) \otimes E$ is β_0-dense in $C^b(X, E)$ and so the dual of the space $(C^b(X, E), \beta_0)$ coincides with the space $M_t(\mathrm{Bo}(X), E')$. Modifying now the argument used by Kurana-Choo [17] for the case of a normed space E, we have the following analogous result.

THEOREM 4.14. *If E is locally bounded and X a P-space, then for every countably $\sigma(M_t(B_0(X), E'), C^b(X, E))$-compact subset A of $M_t(\mathrm{Bo}(X), E')$ and every $\varepsilon > 0$ there exists a finite subset K of X such that $m_W(X-K) < \varepsilon$ for each $m \in A$.*

THEOREM 4.15. *If E is locally bounded and X a P-space, then the strict topology on $C^b(X, E)$ is finer than the Mackey topology for the pair $\langle F, F'\rangle$, where $F=C^b(X, E)$.*

PROOF. Every absolutely convex $\sigma(F', F)$-compact subset A of $F' = M_t(\mathrm{Bo}(X), E')$ is tight (Theorems 4.13, 4.14). The result now follows from Proposition 4.12.

Next, we will look at the question of when is the space $(C^b(X, E), \beta_0)$ separable. We will need the following

LEMMA 4.16. *The mapping*

$$T\colon (C^b(X), \beta_0)\times E \to (C^b(X, E), \beta_0), \quad T(f, s) = f\otimes s,$$

is continuous.

PROOF. Let $f_\alpha\to f$ in $(C^b(X), \beta_0)$ and $s_\alpha\to s$ in E. We will show that $f_\alpha\otimes s_\alpha\to f\otimes s$ in $(C^b(X, E), \beta_0)$. In fact, let W be a neighborhood of zero in E and h a bounded real function on X vanishing at infinity. Set $V=\{g\in C^b(X, E)\colon (hg)(X)\subset W\}$. Choose a balanced neighborhood W_1 of zero in E with $W_1+W_1+W_1\subset W$. There exists $d\geqq 1$ such that $\|hf\|\leqq d$ and $s\in dW_1$. Let α_0 be such that $s_\alpha-s\in d^{-1}W_1$ and $\|h(f_\alpha-f)\|\leqq d^{-1}$ if $\alpha\geqq\alpha_0$. If now $\alpha\geqq\alpha_0$, then, for each $x\in X$, we have

$$h(x)[f_\alpha(x)s_\alpha-f(x)s] = h(x)[f_\alpha(x)-f(x)](s_\alpha-s)+h(x)[f_\alpha(x)-f(x)]s+$$
$$+h(x)f(x)(s_\alpha-s)\in W_1+W_1+W_1\subset W$$

and so $f_\alpha\otimes s_\alpha-f\otimes s\in V$. This completes the proof.

Recall that X is called separably-submetrizable if it can be mapped by a one-to-one continuous function onto some separable metric space.

THEOREM 4.17. *Suppose that X is not empty and that the dual space E' of E is not trivial. Then:*

1) *If $(C^b(X, E), \beta_0)$ is separable, then E is separable and X is separably-submetrizable.*

2) *$(C^b(X)\otimes E, \beta_0)$ is separable iff E is separable and X is separably-submetrizable.*

3) *If $C^b(X)\otimes E$ is β_0-dense in $C^b(X, E)$, then $(C^b(X, E), \beta_0)$ is separable iff E is separable and X is separably-submetrizable.*

PROOF. 1) Let (f_n) be a sequence in $C^b(X, E)$ which is β_0-dense and let $x\in X$. It is easy to see that the sequence $(f_n(x))$ is dense in E and thus E is separable. Also, choose $\varphi\in E'$, $\varphi\neq 0$, and define $T\colon (C^b(X, E), \beta_0)\to(C^b(X), \beta_0)$, $T(f)=\varphi\circ f$. It is easy to see that T is continuous and onto. Thus $(C^b(X), \beta_0)$ is separable and so X is separably-submetrizable ([20], p. 509).

2) If $(C^b(X)\otimes E, \beta_0)$ is separable, then an argument similar to that used in 1) shows that E is separable and X is separably-submetrizable. Conversely, let E be separable and X separably-submetrizable. Then $(C^b(X), \beta_0)$ is separable ([20], p. 509) and so $F=(C^b(X), \beta_0)\times E$ with the product topology is separable. Let

$$S\colon F\to (C^b(X)\otimes E, \beta_0), \quad S(f) = f\otimes s.$$

By Lemma 4.16, S is continuous and thus $S(F)$, with the topology induced by β_0, is separable. Since $C^b(X)\otimes E$ is the linear span of $S(F)$, the result follows.

3) It follows easily from 1) and 2).

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(Received July 2, 1981)

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Acta Math. Hung.
41 (1—2), (1983), 89—91.

UN THÉORÈME SUR LA MESURABILITÉ DES FONCTIONS DE DEUX VARIABLES

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Soient R l'espace des nombres réels et $R^2=R\times R$. On sait que la mesurabilité (au sens de Lebesgue) [la propriété de Baire] de toutes les sections $f_x(t)=f(x,t)$ d'une fonction $f\colon R^2\to R$ et à la fois la continuité approximative et la continuité presque partout (relativement à la mesure de Lebesgue) de toutes les sections $f^y(t)=f(t,y)$ impliquent la mesurabilité (également au sens de Lebesgue) [la propriété de Baire] de la fonction f comme la fonction de deux variables (voir [6]). D'autre part, l'hypothèse du continu implique l'existence d'une fonction $f\colon R^2\to R$ non-mesurable et n'ayant pas de la propriété de Baire et telle que toutes ses sections f_x sont boréliennes et toutes ses sections f^y sont approximativement continues (voir [2]).

Dans cet article nous introduisons une condition (plus faible que la continuité approximative et la continuité presque partout à la fois) relative aux sections f^y qui implique la mesurabilité [la propriété de Baire] de la fonction f ayant ses sections f_x mesurables [avec la propriété de Baire].

DÉFINITION. On dit qu'une fonction $g\colon R\to R$ a la propriété (H) lorsqu'il existe pour tout point $x\in R$ deux ensembles ouverts $U_1(x)$ et $U_2(x)$ ayant ses densités supérieures positives au point x et tels que la fonction partielle $g/[U_1(x)\cup\{x\}]$ est semi-continue supérieurement au point x et la fonction partielle $g/[U_2(x)\cup\{x\}]$ est semi-continue inférieurement au point x.

THÉORÈME 1. *Si toutes les sections f^y d'une fonction $f\colon R^2\to R$ ont la propriété* (H) *et toutes les sections f_x sont mesurables, la fonction f est également mesurable.*

Dans la démonstration de ce théorème nous profitons des lemmes suivants:

LEMME 1 (voir [1] et [3]). *Soit (X, M, μ) un espace dont la mesure σ-finie est μ. Supposons qu'une fonction $f\colon X\to R$ soit telle que, quel que soit le nombre $\varepsilon>0$, la classe d'ensembles*

$$D_\varepsilon=\{D\in M\colon \operatorname*{osc}_D f\leqq\varepsilon\}$$

satisfasse à la condition suivante:

(E) *il existe pour tout ensemble $A\in M$ de mesure μ positive un ensemble $D\in D_\varepsilon$ tel que $D\subset A$ et $\mu(D)>0$. Alors la fonction f est $\bar\mu$-mesurable, où $\bar\mu$ désigne le complété de la mesure μ.*

LEMME 2 (voir [4]). *Soit $A\subset R^2$ un ensemble mesurable. Il existe un ensemble $B\subset A$ du type F_σ et tel que $m_2(A-B)=0$ (m_2 désigne, comme d'habitude, la mesure de Lebesgue dans R^2) et $B\subset +B$; c'est-à-dire: quel que soit le point $(x,y)\in B$, x est* *un point de densité de l'ensemble* $B^y=\{t\in R\colon (t,y)\in B\}$ *et y est un point de densité de l'ensemble* $B_x=\{t\in R\colon (x,t)\in B\}$.

Démonstration du théorème 1. On peut supposer que la fonction f soit bornée, comme dans le cas contraire on peut considérer la fonction $\operatorname{arctg} f$.

Démontrons que la fonction f satisfait à la condition (E) du lemme 1. Soit $A\subset R^2$ un ensemble mesurable tel que $m_2(A)>0$. Fixons le nombre $\varepsilon>0$. En appliquant le lemme 2 à l'ensemble A, on peut écrire qu'il existe un ensemble $B\subset A$ du type F_σ et tel que $B\subset +B$ et $m_2(A-B)=0$. Désignons par a l'infimum essentiel infess $f(x)$.

L'ensemble $B_1=\{(x,y)\in B\colon a\leqq f(x,y)<a+\varepsilon/8\}$ est de mesure extérieure positive. Les sections f^y ayant la propriété (H), il existe pour tout point $(x,y)\in B_1$ un intervalle ouvert d'extrémités rationnelles $U(x,y)$ tel que $B^y\cap U(x,y)\neq\emptyset$ et $f(t,y)<a+\varepsilon/8+\varepsilon/8=a+\varepsilon/4$ pour tout point $t\in U(x,y)$.

La famille des intervalles ouverts d'extrémités rationnelles étant dénombrable et l'ensemble B_1 étant de mesure extérieure positive, il existe un intervalle de la famille $\{U(x,y)\}_{(x,y)\in B_1}$, que nous désignons par U_0 tel que l'ensemble $C=$ $=\{(x,y)\in B_1;$ au point (x,y) correspond l'intervalle $U_0\}$ est de mesure extérieure positive. Soit $D_0=\{y\in R;$ il existe x tel que $(x,y)\in C\}$. L'ensemble D_0 est de mesure extérieure positive. Désignons par D l'ensemble de tous les points de densité extérieure de l'ensemble D_0 et par F l'ensemble $(U_0\times D)\cap B$. L'ensemble F est mesurable et de mesure positive, puisque $m(F^y)>0$ pour presque tous les points $y\in D_0$ et $F\subset B\subset A$.

Démontrons encore que $f(x,y)\leqq a+\varepsilon$ pour presque tous les points $(x,y)\in F$.

En effet, supposons, au contraire, que l'ensemble $G=\{(x,y)\in F\colon f(x,y)>a+\varepsilon\}$ soit de mesure extérieure positive. Les sections f^y ayant la propriété (H), il existe pour tout point $(x,y)\in G$ un intervalle ouvert d'extrémités rationnelles $V(x,y)\subset U_0$ tel que $V(x,y)\cap B^y\neq\emptyset$ et. $f(t,y)>a+\varepsilon$ pour tout $t\in V(x,y)$. De nouveau, la famille de tous les intervalles d'extrémités rationnelles étant dénombrable, il existe un intervalle de la famille $\{V(x,y)\}_{(x,y)\in G}$, que nous désignons par V_0 tel que l'ensemble

$K=\{y\in R\colon$ il existe un point $x\in R$ tel que $(x,y)\in G$ et au point (x,y) correspond l'intervalle $V_0\}$ est de mesure extérieure positive.

En désignant par N l'ensemble de tous les points de densité extérieure de l'ensemble K, remarquons que $N\subset D$. Fixons un point $(x_1,y_1)\in(V_0\times N)\cap B$. On a, d'une part, $f(x_1,y)>a+\varepsilon$ pour tout $y\in K$ et d'autre part, pour tout $y\in D_0$,

$$f(x_1,y)<a+\varepsilon/8+\varepsilon/8=a+\varepsilon/4,$$

ce qui contredit la mesurabilité de la section f_{x_1}. Par conséquent, l'ensemble

$$L=\{(x,y)\in F\colon a\leqq f(x,y)\leqq a+\varepsilon\}$$

est mesurable, de mesure positive et $\operatorname*{osc}_L f\leqq\varepsilon$. Comme, de plus, $L\subset F\subset B\subset A$, l'hypothèse du lemme 1 est donc satisfaite et notre démonstration est achevée.

Remarque 1. L'hypothèse du continu implique qu'il existe une fonction $f\colon R^2\to R$ non-mesurable et telle que toutes ses sections f_x sont approximativement continues et toutes ses sections f^y sont telles que, quel que soit le point $x\in R$, il existe un ensemble ouvert $U(x)$ tel que $m(U(x) \cap V) > 0$ pour tout entrourage ouvert V du point x et la fonction partielle $f/[U(x) \cup \{x\}]$ est continue au point x (voir [5]).

THÉORÈME 2. *Soit* $f\colon R^2 \to R$ *une fonction. Si toutes les sections* f^y *ont la propriété* (H) *et toutes les sections* f_x *ont la propriété de Baire, la fonction* f *a également la propriété de Baire.*

On peut démontrer ce théorème d'une façon analogue que le théorème 1. Pourtant nous montrons une autre démonstration.

Dans ce but remarquons que:

REMARQUE 2. Si la fonction $g\colon R \to R$ a la propriété (H), elle est ponctuellement discontinue.

REMARQUE 3. Soit $S \subset R$ un ensemble dense et dénombrable. Si la fonction $g\colon R \to R$ a la propriété (H), on a

$$\liminf_{\substack{t \to x \\ t \in S}} g(t) \leqq g(x) \leqq \limsup_{\substack{t \to x \\ t \in S}} g(t)$$

pour tout $x \in R$.

D'après le Théorème 3 de l'article [6] le Théorème 2 résulte immédiatement des Remarques 2 et 3.

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(Reçu le 8, juillet 1981.)

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Acta Math. Hung.
41 (1—2), (1983), 93—100.

DIFFERENTIAL LIPSCHITZIANNESS TESTS ON ABSTRACT QUASI-METRIC SPACES

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0. Introduction

An important problem concerning a wide class of mappings between (quasi-) metric spaces is that of finding sufficient conditions in order that a "local" Lipschitz property should imply a "global" one (the words "local" and "global" being precised by the context of the problem considered). As fundamental results in this direction one must consider Dieudonné's mean value theorem [14, ch. VIII, § 5] on one hand, as well as Brezis—Browder's semigroup invariance theorem [7, Theorem 2] on the other hand, proved — under a continuity hypothesis — by a supremum and, respectively, ordering technique. It is the main aim of the present note to demonstrate that these results may be viewed as particular cases of a general differential lipschitzianness test for a class of (not necessarily continuous) closed mappings depending upon a "spatial" parameter, the basic tool in proving such a common extension being a maximality principle on (quasi-) ordered quasi-metric spaces that may be considered as an abstract version of the well-known Ekeland-Brøndsted's one [16, 8]. A number of further applications of our main result, especially to flow-invariance problems as well as to projective completeness criteria will be discussed elsewhere.

1. A maximality principle

Let X be an abstract nonempty set and let $\leqq$ be a *quasi-ordering* on X (i.e., a reflexive and transitive relation on X). For any nonempty subset Y of X and any $x\in Y$, $Y(x, \leqq)$ will denote the subset of all $y\in Y$ with $x\leqq y$. A sequence $(x_n$: $n\in N)$ in X will be called *monotone* iff $x_i\leqq x_j$ whenever $i\leqq j$, $i, j\in N$, and *bounded above* iff $x_n\leqq y$, all $n\in N$, for some $y\in X$ (in which case, y will be termed an *upper bound* of this sequence). Furthermore, if we introduce a *quasi-metric* d on X (that is, a mapping d: $X^2\to R_+$ satisfying all the requirements of a metric except sufficiency) a sequence $(x_n;\ n\in N)$ in X will be called *quasi-asymptotic* iff for any $\varepsilon>0$ there exists $n=n(\varepsilon)\in N$ with $d(x_n, x_{n+1})<\varepsilon$, and an element $z\in X$ is said to be *d-maximal* iff $y\in X$ and $z\leqq y$ imply $d(z, y)=0$. A satisfactory motivation for introducing these notions will be offered later; for the moment, we are only interested to state and prove a useful Zorn maximality principle on this class of quasi-ordered quasi-metric structures, a result that may be formulated as follows.

THEOREM 1. *Suppose the quasi-metric space* (X, d) *and the quasi-ordering* $\leqq$ *on* X *are such that*

(i) *any monotone sequence in* X *is a quasi-asymptotic one,*
(ii) *any monotone Cauchy sequence in* X *has an upper bound.*

Then, for every $x \in X$, there is a d-maximal element $z \in X$ with the property $x \leqq z$.

PROOF. Firstly, we claim any $x \in X$ has the property

(1) *for every $\varepsilon > 0$ there exists $y \geqq x$ such that $d(y, z) > \varepsilon$, for all $z \geqq y$.*

Indeed, suppose (1) is not valid; then there must be a number $\varepsilon > 0$ such that for every $y \geqq x$ there exists $z \geqq y$ with $d(y, z) \geqq \varepsilon$. It immediately follows that a monotone sequence $(y_n;\ n \in N)$ in $X(x, \leqq)$ may be found with $d(y_n, y_{n+1}) \geqq \varepsilon$, all $n \in N$, contradicting (i) and proving our claim. In this case, given $x \in X$, a monotone sequence $(x_n;\ n \in N)$ in $X(x, \leqq)$ may be constructed with

(2) $$n \in N,\ y \in X \ \text{ and } \ x_n \leqq y \ \text{ imply } \ d(x_n, y) < (1/2)^n.$$

By (ii), $x_n \leqq z$, all $n \in N$, for some $z \in X$; so combining with (2), $x_n \to z$ as $n \to \infty$. Evidently $x \leqq z$. Now suppose $y \in X$ is such that $z \leqq y$; then $x_n \leqq y$, all $n \in N$, so that, again by (2), $x_n \to y$ as $n \to \infty$ and this gives $d(z, y) = 0$. Consequently, z satisfies all the requirements of the theorem and the proof is complete. Q.E.D.

As an important particular case, let $(X, \leqq)$ be an abstract quasi-ordered set satisfying

(iii) *any monotone sequence in X is bounded above*

and let φ be a function from X into R, *decreasing* ($x \leqq y$ implies $\varphi(x) \geqq \varphi(y)$) and *bounded from below* ($\varphi(x) \geqq b$, all $x \in X$, for some $b \in R$). Then, defining a quasimetric d on X by the convention

(3) $$d(x, y) = |\varphi(x) - \varphi(y)|, \quad x, y \in X,$$

conditions (i)+(ii) are automatically satisfied and the above result reduces to the well-known Brezis—Browder's ordering principle [7] (see also I. Ekeland [17]). Moreover, it was shown in Brezis—Browder's paper that their fundamental contribution may be regarded as a considerable extension of the Bishop-Phelps' maximality result [4] (see also J. P. Aubin and J. Siegel [1], A. Brøndsted [10] as well as I. Ekeland [15]) or, equivalently, — after a pattern discovered by N. Bourbaki [5] and refined by A. Brøndsted [9] — of a fixed point Caristi—Kirk's result [12, 19] (see also F. E. Browder [11] as well as C. S. Wong [31]) and therefore the above theorem may be also considered as extending all these results.

A close analysis of the conditions involved in Theorem 1 shows the boundedness property imposed in (ii) is, in fact, not intimately related to the notion of Cauchy sequence so, it seems to be natural to replace it by a more appropriate property such as convergence. To do this, we need a number of new notational conventions. Let X, $\leqq$, and d be as before. A subset Y of X will be termed *order-closed* iff for any monotone sequence $(x_n;\ n \in N)$ in Y and any $x \in X$ with $x_n \to x$ as $n \to \infty$ we have $x \in Y$; in this context, the considered quasi-ordering $\leqq$ on X is said to be *self-closed* iff $X(x, \leqq)$ is order-closed for all $x \in X$. Also, the underlying quasimetric space (X, d) will be termed *order-complete* iff any monotone Cauchy sequence in X is a convergent one. Now, as a useful variant of the above result, we have

THEOREM 2. *Suppose the elements* X, $\leqq$ *and* d *are such that condition* (i) *as well as*

(iv) $\leqq$ *is a self-closed quasi-ordering,*

(v) (X, d) *is order-complete*

hold. Then, for every $x \in X$, *there is a d-maximal element* $z \in X$ *with the property* $x \leqq z$.

PROOF. Let $(x_n;\ n \in N)$ be a monotone Cauchy sequence in X. By (v), $x_n \to x$ as $n \to \infty$ for some $x \in X$ which gives, by (iv), $x_n \leqq x$, all $n \in N$, proving x is an upper bound of this sequence. Consequently, Theorem 1 applies and the proof is complete. Q.E.D.

Let us call a sequence $(x_n;\ n \in N)$ in X *asymptotic* (M. Turinici [28]) iff $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$, and the underlying quasi-ordering $\leqq$ on X *semi-closed* (L. Nachbin [22, p. 100]) iff $X(x, \leqq)$ is closed for all $x \in X$. In such a situation, if we suppose condition (i) is replaced by the stronger one

(i)′ *any monotone sequence in* X *is an asymptotic one*

and/or condition (iv) by

(iv)′ $\leqq$ *is a semi-closed quasi-ordering*

then, the corresponding variant of Theorem 2 appears as a quasi-order as well as quasi-metric extension of a similar author's result [27]. Moreover, if the underlying quasi-ordering $\leqq$ on X is taken of the form $x \leqq y$ iff $d(x,y) \leqq f(x) - f(y)$, f being a function from X into R satisfying

(vi) f *is lsc and bounded from below*

the above theorem reduces to Ekeland-Brøndsted's maximality result [16, 8] (see also M. Turinici [26] and J. D. Weston [30]) or, equivalently — by the same Bourbaki—Brøndsted's pattern — to the Caristi—Kirk's fixed point theorem quoted before (see also S. Kasahara [18], L. Pasicki [23], as well as J. Siegel [25], for a number of interesting new viewpoints in this direction) so, Theorem 2 appears also as a common extension of all these contributions.

2. The main results

Let (V, d) be a complete quasi-metric space. For any $v \in V$, $r \geqq 0$ and any nonempty subset W of V, let $d(v, W)$ denote the usual distance between v and W (the infimum of all $d(v, w)$, $w \in W$) and, in case $v \in W$, let $W(v, r)$ denote the W-closed sphere with center v and radius r (the subset of all $w \in W$ with $d(v, w) \leqq r$). Let J be a given interval of the real axis and F a (nonempty) closed subset of V. By a *(J, F)-closed process* on V we mean a mapping $(t, v) \mapsto S(t, v) = S(t)v$ from $J \times V$ into V satisfying the closedness condition

(vii) *for any decreasing sequence* $(t_n;\ n \in N)$ *in* J *and any sequence* $(v_n;\ n \in N)$ *in* F *with* $t_n \to t$, $v_n \to v$ *and* $S(t_n)v_n \to w$ *as* $n \to \infty$ *for some* $t \in J$, $v \in F$ *and* $w \in V$ *respectively, we have* $S(t)v = w$.

As a notational convention, for any $t\in J$ and any subset W of V, $S(t)W$ will denote the subset of all $S(t)w$, $w\in W$. Now, under these preparatory facts, the main differential lipschitzianness test of the present note may be stated as follows.

THEOREM 3. *Let J, F and S be as above and suppose there exist a couple of functions K and H from J into R with K increasing, H strictly increasing, a number $\lambda\geqq 0$, and a denumerable subset A of J such that*

(viii) *for any $s\in A$, not identical with the left end-point of J, we have*

$$\liminf_{t\to +0}\,(1/(H(s)-H(s-t)))\,d\big(S(s)v, S(s-t)F(v, K(s)-K(s-t))\big)\leqq\lambda, \tag{4}$$

for all elements $v\in F$,

(ix) *for any $s\in J\cap A$, not identical with the left end-point of J we have*

$$\liminf_{t\to 0+} d\big(S(s)v, S(s-t)F(v, K(s)-K(s-t))\big)\leqq\lambda\big(H(s)-H(s-0)\big), \tag{5}$$

for all elements $v\in F$.

Then, necessarily, the following kind of Lipschitz property holds:

(6) $d\big(S(b)u, S(a)F(u, K(b)-K(a))\big)\leqq\lambda\big(H(b)-H(a)\big)$, *all $u\in F$, all $a, b\in J$, $a<b$.*

PROOF. Let $n\mapsto a_n$ be a bijection of N onto A and let $g\colon J\to R$ be defined, for every $t\in J$ by

$$g(t)=\begin{cases}\Sigma\{(1/2)^n;\ a_n\leqq t\}, & \text{if } \{n\in N;\ a_n\leqq t\}\neq\emptyset,\\ 0, & \text{if } \{n\in N;\ a_n\leqq t\}=\emptyset.\end{cases}$$

Evidently, g is monotone increasing on J and,

$$g(a_n)-g(t)\geqq(1/2)^n,\quad \text{all}\quad t\in J,\ t<a_n,\quad \text{all}\quad n\in N. \tag{7}$$

Define a new function $f\colon J\to R$ by the convention $f(t)=\gamma H(t)+\varepsilon g(t)$, $t\in J$, $\gamma>\lambda$ and $\varepsilon>0$ being arbitrary fixed; clearly, f is also monotone increasing on J. Now, $a, b\in J$, $a<b$ being fixed, let I denote the (compact) interval $[a, b]$ and X the cartesian product $I\times F$ quasi-metrized by the usual "product" quasi-metric

$$e\big((t, u), (s, v)\big)=|t-s|+d(u, v),\quad (t, u), (s, v)\in X$$

and quasi-ordered by the convention

(8) $(t, u)\leqq(s, v)$ iff $t\geqq s$, $d(u, v)\leqq K(t)-K(s)$ and $d\big(S(t)u, S(s)v\big)\leqq f(t)-f(s)$.

Firstly, X is complete (hence order-complete) and, moreover (by a reasoning similar to that exposed in author's paper [28]) order-asymptotic. Secondly, we claim that — in addition to these properties — the quasi-ordering $\leqq$ is also a self-closed one. Indeed, let the element $(s, v)\in X$ and the sequence $((t_n, u_n);\ n\in N)$ in X be such that

$$(s, v)\leqq(t_n, u_n),\ n\in N,\ (t_n, u_n)\leqq(t_m, u_m),\ n\leqq m,\ \text{and}\ (t_n, u_n)\to(t, u)$$

as $n \to \infty$ for some $(t, u) \in X$, that is,

$$(9) \qquad s \geqq t_n,\ d(v, u_n) \leqq K(s) - K(t_n),\ d\big(S(s)v,\ S(t_n)u_n\big) \leqq f(s) - f(t_n),\ n \in N,$$

$$(10)\ t_n \geqq t_m,\ d(u_n, u_m) \leqq K(t_n) - K(t_m),\ d\big(S(t_n)u_n,\ S(t_m)u_m\big) \leqq f(t_n) - f(t_m),\ n \leqq m,$$

$$(11) \qquad t_n \to t \text{ and } u_n \to u \text{ as } n \to \infty.$$

From the first part of (9) and (10) it immediately follows (by (11) and the monotonicity of K) $s \geqq t$ and $d(v, u) \leqq K(s) - K(t)$. Moreover, from the second part of (10), $(S(t_n)u_n;\ n \in N)$ is a Cauchy sequence in V hence, by completeness, $S(t_n)u_n \to w$ as $n \to \infty$ for some $w \in V$ so that, again by (11) combined with our closedness hypothesis (vii), $S(t)u = w$ which gives (taking the limit as $n \to \infty$ in the second part of (9) and remembering f is monotone increasing) $d(S(s)v,\ S(t)u) \leqq f(s) - f(t)$, and this proves our claim. In this case, Theorem 2 can be applied, so, for the prescribed element $(b, u) \in X$ an e-maximal element $(s, v) \in X$ (with respect to the underlying quasi-ordering $\leqq$) may be found with $(b, u) \leqq (s, v)$. We claim $s = a$. Indeed, suppose by contradiction $s > a$. For every $r \in I$, $r < s$, and every $w \in F(v, K(s) - K(r))$, the relation $(s, v) \leqq (r, w)$ does not hold and therefore (since $s \geqq r$ and $d(v, w) \leqq K(s) - K(r)$) we must have (by the convention (8) combined with the definition of the function f)

$$d\big(S(s)v,\ S(r)w\big) > \gamma\big(H(s) - H(r)\big) + \varepsilon\big(g(s) - g(r)\big),$$

so, taking infimum with respect to w and denoting for simplicity $t = s - r$,

$$(12) \qquad d\big(S(s)v,\ S(s-t)F(v,\ K(s) - K(s-t))\big) \geqq \gamma\big(H(s) - H(s-t)\big) + \\ + \varepsilon\big(g(s) - g(s-t)\big), \quad 0 < t \leqq s - a.$$

Now, two cases are open before us: either $s \in J \setminus A$ or $s \in A$. The analysis of these two cases may be performed as follows.

Case 1. $s \in J \setminus A$. In this situation, as an immediate consequence of (12) we have

$$d\big(S(s)v,\ S(s-t)F(v, K(s) - K(s-t))\big) \geqq \gamma\big(H(s) - H(s-t)\big),\ 0 < t \leqq s - a,$$

so, dividing by $H(s) - H(s-t)$ and taking $\liminf$ as $t \to 0+$ we get a contradiction with respect to (4).

Case 2. $s \in A$, that is $s = a_n$ for some $n \in N$. Then, by (12) and the evaluation (7), we get

$$d(S(s)v,\ S(s-t)F(v,\ K(s) - K(s-t))) \geqq \gamma\big(H(s) - H(s-0)\big) + \varepsilon(1/2)^n,\ 0 < t \leqq s - a,$$

so, passing to $\liminf$ as $t \to 0+$, (5) will be contradicted. Therefore, in any case we reached an impossible situation and this shows $s = a$, as claimed. In this case, again by (8) coupled with the definition of f, the relation $(b, u) \leqq (a, v)$ becomes

$$b \geqq a,\ d(u, v) \leqq K(b) - K(a),\ d\big(S(b)u,\ S(a)v\big) \leqq \gamma\big(H(b) - H(a)\big) + \varepsilon\big(g(b) - g(a)\big)$$

and this immediately implies

$$d\big(S(b)u,\ S(a)F(u,\ K(b) - K(a))\big) \leqq \gamma\big(H(b) - H(a)\big) + \varepsilon\big(g(b) - g(a)\big),$$

a relation equivalent in fact to (6), because $\gamma>\lambda$ and $\varepsilon>0$ were arbitrarily chosen. Q.E.D.

Let V be an abstract nonempty set and let D be a family of quasi-metrics on V, with (V, d) complete for any $d\in D$. Also, let J be a given interval of the real axis and F a subset of V, closed in (V, d) for any $d\in D$. By a *(J, F, D)-closed process* on V we mean a mapping $(t, v)\mapsto S(t, v)=S(t)v$ from $J\times V$ into V satisfying the closedness assumption (vii) with respect to every quasimetric d of D. In such a case, as an immediate extension of the main result, we have

THEOREM 4. *Let the elements D, J, F and S be as before and suppose the couple of functions K and H from J into R satisfies the general assumptions of the main result, as well as*

(x) *for every $d\in D$, a number $\lambda(d)\geqq 0$ and a denumerable subset $A(d)$ of J may be found such that* (viii) *and* (ix) *hold with λ replaced by $\lambda(d)$ and A by $A(d)$.*

Then, necessarily, the evaluation (6) *will be valid* $\big($*with* $\lambda=\lambda(d)\big)$ *for any quasi-metric d of D.*

3. Some particularizations

Let (V, d) be a complete metric space and let F be a closed subset of V. By an *F-closed process* on V we mean a mapping $(t, v)\mapsto S(t, v)=S(t)v$ from $R_+\times V$ into V satisfying the closedness hypothesis (vii) with $J=R_+$, as well as

(xi) $S(0)v=v$, *all* $v\in V$.

Of course, any F-closed process is identical with an (R_+, F)-closed process (in the sense of the preceding section) satisfying (xi). In this case, as an important particularization of the main result, we have

THEOREM 5. *Let F and S be as before and suppose there exist a couple of functions K and H from R_+ into itself with K increasing, H strictly increasing and $K(0)=H(0)=0$, a number $\lambda\geqq 0$ and a denumerable subset A of R_+ such that* (viii) *and* (ix) *hold with J replaced by R_+. Then, necessarily,*

$$d\big(S(t)u,\ F\big(u, K(t)\big)\big)\leqq \lambda H(t),\quad \textit{all}\quad t\in R_+,\ u\in F. \tag{13}$$

Concerning this result, it must be observed that in case $\lambda=0$, the evaluation (13) becomes

$$S(t)u\in F\big(u,\ K(t)\big),\ \textit{all}\ \ t\in R_+,\ u\in F, \tag{13$'$}$$

so, it may be interpreted as an *invariance* result with respect to the flows $t\mapsto S(t)u$ issuing from $u\in F$; for such a reason, Theorem 5 above is usually termed a "flow-invariance" result with respect to the F-closed process S on V, in which case, it may be compared with a classical Brezis—Browder's result [7] (see also I. Ekeland [17]) as well as a number of concrete "non-semigroup" invariance conditions used by N. Pavel [24] and R. H. Martin Jr. [21] in case of a Banach space.

Again let (V, d) be a complete metric space and let J be a given interval of R. A mapping $T\colon J\to V$ will be termed *order-closed* provided that

(xii) *for any decreasing sequence* $(t_n;\ n\in N)$ *in* J *with* $t_n\to t$ *and* $Tt_n\to w$ *as* $n\to\infty$ *for some* $t\in J$ *and* $w\in V$ *respectively we have* $Tt=w$.

Evidently, any order-closed mapping T from J into V may be identified with a (J, V)-closed process S on V, constant with respect to its "spatial" variable (i.e., $S(t)v=Tt$, all $t\in J, v\in V$). In this case, as another important particularization of the main result, we have

THEOREM 6. *Let* J *and* T *be as above and suppose there is a strictly increasing function* H *from* J *into* R, *a number* $\lambda\geqq 0$, *and a denumerable subset* A *of* J, *such that*

(xiii) *for any* $s\in J\setminus A$, *not identical with the left end-point of* J, *we have*

$$\liminf_{t\to 0+} (1/(H(s)-H(s-t)))\, d(Ts, T(s-t)) \leqq \lambda, \tag{14}$$

(xiv) *for any* $s\in A$, *not identical with the left end-point of* J, *we have*

$$\liminf_{t\to 0+} d(Ts, T(s-t)) \leqq \lambda(H(s)-H(s-0)). \tag{15}$$

Then, necessarily,

$$d(Tb, Ta) \leqq \lambda(H(b)-H(a)), \quad \text{all} \quad a, b\in J, \quad a<b. \tag{16}$$

Evidently, the above result appears as a *mean value theorem* on abstract metric spaces, extending, from this viewpoint, a classical Dieudonné's one [14, ch. VIII, § 5] (see also A. K. Aziz and J. B. Diaz [2] as well as N. Bourbaki [6, ch. I, § 3]). On the other hand, in case A is empty, the same theorem extends a similar Clarke's result [13] (see also W. A. Kirk and W. O. Ray [20], as well as M. Turinici [29]). Finally, it should be noted that another way of investigating these problems is that offered by the *differential inequalities theory* (see, as a reference, J. Bebernes and G. H. Meisters [3]); some of these aspects will be treated in a forthcoming paper.

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(Received July 13, 1981)

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Acta Math. Hung.
41 (1—2), (1983), 101—105.

ON STRONG SUMMABILITY OF ORTHOGONAL SERIES BY EULER METHODS

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1. Let $\{\varphi_i(x)\}$ be an orthonormal system in [0, 1]. We consider real orthogonal series $\sum_{i=0}^{\infty} c_i\varphi_i(x)$ with corresponding partial sums $s_k(x)$ and the Euler means of order q $\big((E, q)$*-means*; $0<q<1\big)$

$$t_n(x) = \sum_{k=0}^{n} a_{nk}\, s_k(x) = \sum_{i=0}^{n} c_i\varphi_i(x) \sum_{k=i}^{n} a_{nk},$$

where

$$a_{nk} = \begin{cases} \binom{n}{k} q^k (1-q)^{n-k} & (n \geqq k) \\ 0 & (n<k). \end{cases} \tag{1}$$

We will show

THEOREM. *If the orthogonal series* $\sum_{i=0}^{\infty} c_i\varphi_i(x)$, $\sum_{i=0}^{\infty} c_i^2<\infty$, *is* $(E,\ q)$*-summable* $(0<q<1)$ *to* $f(x)$ *in* [0, 1], *then for any* $\gamma>0$

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_{nk}|s_k(x)-f(x)|^{\gamma} = 0. \tag{2}$$

(Summability and convergence have the meaning of summability a.e. and convergence a.e.)

If (2) is satisfied, we call the underlying series *strong* (E, q)*-summable with order* γ $\big([(E, q)]^{\gamma}$*-summable*$\big)$. In the classical case $\gamma=2$ similar results were established for different summability-methods, e.g. for (C, 1) by A. Zygmund [5]. For arbitrary γ, G. Sunouchi [3] was the first to prove an analogous theorem for $[(C, \alpha)]^{\gamma}$-summability $(\alpha>0)$ of orthogonal series.

2. Some lemmas. We require some lemmas:

LEMMA 1. *If* $\sum_{i=0}^{\infty} c_i^2 < \infty$, *then for* $\alpha > 0$, $\gamma > 0$

$$\int_0^1 \left\{ \sup_{1 \le n < \infty} \left(\frac{1}{A_n^\alpha} \sum_{k=0}^n A_{n-k}^{(\alpha-1)} |s_k(x) - \sigma_k(x)|^\gamma \right)^{\frac{1}{\gamma}} \right\}^2 dx \le A(\alpha, \gamma) \sum_{i=0}^{\infty} c_i^2$$

where $\sigma_k(x) = \frac{1}{k+1} \sum_{j=0}^k s_j(x)$. (G. Sunouchi [3], Lemma 2).

LEMMA 2. *The series* $\sum_{i=0}^{\infty} c_i \varphi_i(x)$, $\sum_{i=0}^{\infty} c_i^2 < \infty$, *is* (E, q)*-summable* $(0 < q < 1)$ *if and only if* $\{s_{m_i}(x)\}$ *is convergent, where*

$$\alpha \sqrt{m_i} < m_{i+1} - m_i < \beta \sqrt{m_i} \quad (0 < \alpha < \beta < \infty). \tag{3}$$

PROOF. With $m_i = i^2$ this was proved by O. A. Ziza [4]; c. f. also H. Schwinn [2].

LEMMA 3. *Let* $\{m_i\}$ *satisfy* (3). *If* $\sum_{i=0}^{\infty} c_i^2 < \infty$, *then the means*

$$\begin{cases} \sigma_n^*(x) = 0 \quad \text{if} \quad 0 \le n \le m_1 \\ \sigma_n^*(x) = \dfrac{1}{n - m_{i-1}} \displaystyle\sum_{k=m_{i-1}+1}^{n} (s_k(x) - s_{m_{i-1}}(x)) \end{cases} \tag{4}$$

$$= \sum_{k=m_{i-1}+1}^{n} \left(1 - \frac{k - m_{i-1} - 1}{n - m_{i-1}}\right) c_k \varphi_k(x) \text{ if } m_i < n \le m_{i+1},\ i \ge 1,$$

are convergent to 0.

PROOF. At first we can see that $\sigma_{m_i}^*(x) \to 0$ by

$$\sum_{i=2}^{\infty} \int_0^1 (\sigma_{m_i}^*(x))^2 \, dx \le \sum_{i=2}^{\infty} \sum_{k=m_{i-1}+1}^{m_{i+1}} c_k^2 < \infty.$$

Now let

$$\delta_i(x) := \max_{m_i < n < m_{i+1}} |\sigma_{m_{i+1}}^*(x) - \sigma_n^*(x)|^2 \le \left(\sum_{n=m_i+1}^{m_{i+1}-1} |\sigma_{n+1}^*(x) - \sigma_n^*(x)| \right)^2$$

$$\le (m_{i+1} - m_i) \sum_{n=m_i+1}^{m_{i+1}-1} (\sigma_{n+1}^*(x) - \sigma_n^*(x))^2.$$

We get with (3)

$$\sum_{i=2}^{\infty} \int_0^1 \delta_i(x) \, dx \le \sum_{i=2}^{\infty} \beta \sqrt{m_i} \sum_{n=m_i+1}^{m_{i+1}-1} \frac{1}{(n - m_{i-1})^4} \sum_{k=m_{i-1}+1}^{n+1} (k - m_{i-1})^2 c_k^2 \le$$

$$\le \beta \sum_{i=2}^{\infty} \sqrt{m_i} \sum_{k=m_{i-1}+1}^{m_{i+1}} c_k^2 (k - m_{i-1})^2 \sum_{n=\max(k-1,\, m_i+1)}^{m_{i+1}-1} \frac{1}{(n - m_{i-1})^4} \le$$

$$\le C \sum_{i=2}^{\infty} \frac{\sqrt{m_i}}{m_{i+1} - m_i} \sum_{k=m_{i-1}+1}^{m_{i+1}} c_k^2 \le C^* \sum_{k=0}^{\infty} c_k^2 < \infty,$$

i.e. $\delta_i(x) \to 0$ $(i \to \infty)$ and so $\sigma_k^*(x) \to 0$ $(k \to \infty)$.

LEMMA 4. *For the elements* (1) *of* (E, q) $(0<q<1)$

$$\left\{\sum_{k=1}^{n} a_{nk}^{r}\right\}^{\frac{1}{r}} \leqq C(r) n^{-\frac{1}{2}\left(1-\frac{1}{r}\right)} \quad (r>1)$$

holds.

PROOF. Depending on the order q we define $\bar{n}$ as the smallest natural number ν with $n \leqq q\nu$. Putting

$$n_0 = 1, \quad n_{i+1} = n_i + [\sqrt{\bar{n}_i}]^1 \quad (i = 0, 1, \ldots)$$

each sequence $\{n_i\}$ and $\{\bar{n}_i\}$ satisfies a gap-condition (3). The natural numbers can be devided by $\{n_i\}$ into intervals

$$J_i := \langle n_i, n_{i+1}-1\rangle = \{n \in \mathbf{N} \mid n_i \leqq n < n_{i+1}\}.$$

We will use the sets $J_{i-K_i}, J_{i-K_i+1}, \ldots, J_i, \ldots, J_{i+L_i}$, whereby K_i and L_i are determined by the relations $[(q-\varepsilon)\bar{n}_i] \in J_{i-K_i}$ resp. $[(q+\varepsilon)\bar{n}_i] \in J_{i+L_i}$ (ε may be fixed with $0<q-\varepsilon<q+\varepsilon<1$). We notice that

$$\left|\frac{\nu}{\bar{n}_i} - q\right| > \bar{n}_i^{-\frac{1}{3}} \quad \left(\nu \notin \bigcup_{l=-K_i}^{L_i} J_{i+l}\right).$$

With the estimations

$$\sum_{k:\left|\frac{k}{n}-q\right|>n^{-\frac{1}{3}}} a_{nk} \leqq \frac{C^*(q)}{n^3}$$

(L. KANTOROWITSCH [1], Lemma 1) and

$$\max\{a_{nk} \mid \bar{n}_i \leqq n < \bar{n}_{i+1}; k \in J_{i+l}, -K_i \leqq l < 0\} \leqq C_1 \frac{\gamma^{(l+1)^2}}{\sqrt{\bar{n}_i}},$$

$$\max\{a_{nk} \mid \bar{n}_i \leqq n < \bar{n}_{i+1}; k \in J_{i+l}, 0 < l \leqq L_i\} \leqq C_1 \frac{\gamma^{(l-1)^2}}{\sqrt{\bar{n}_i}},$$

$$\max\{a_{nk} \mid \bar{n}_i \leqq n < \bar{n}_{i+1}; k \in J_i\} \leqq \frac{C_2}{\sqrt{\bar{n}_i}}$$

for $0 < \gamma = \gamma(q) < 1$ (cf. [2]) and with

$$|J_{i+l}| := n_{i+l+1} - n_{i+l} = [\sqrt{\bar{n}_{i+l}}] \leqq \sqrt{\bar{n}_{i+L_i}} \leqq C_3 \sqrt{\bar{n}_i} \quad (-K_i \leqq l \leqq L_i)$$

we get finally for $\bar{n}_i \leqq n < \bar{n}_{i+1}$

$$\sum_{k=0}^{n} a_{nk}^r \leqq \sum_{k:\left|\frac{k}{n}-q\right|>n^{-\frac{1}{3}}} a_{nk}^r + \sum_{l=-K_i}^{L_i} \sum_{k \in J_{i+l}} a_{nk}^r \leqq \left(\frac{C^*(q)}{n^3}\right)^r n + C_4 \sum_{l=-K_i}^{L_i} \sqrt{\bar{n}_{i+l}} \left(\frac{\gamma^{|l|}}{\sqrt{\bar{n}_i}}\right)^r \leqq$$

$$\leqq C_5 \bar{n}_i^{\frac{1}{2}(1-r)} \leqq C_6 n^{\frac{1}{2}(1-r)}$$

as asserted.

---

[1] $[\alpha]$ denotes the integral part of α.

3. Proof of the theorem. (I) *Case* $0<\gamma\leqq 2$: In [2] it was shown that the assertion is true if $\gamma=2$. If $\gamma<2$ we get by Hölder's inequality

$$\sum_{k=0}^{n} a_{nk}|s_k(x)-f(x)|^\gamma \leqq \left\{\sum_{k=0}^{n} a_{nk}\right\}^{1-\frac{\gamma}{2}} \left\{\sum_{k=0}^{n} a_{nk}|s_k(x)-f(x)|^2\right\}^{\frac{\gamma}{2}} \to 0.$$

(II) *Case* $\gamma>2$: With an arbitrary index sequence $\{m_i\}$ $(m_0=0)$ satisfying (3) and defining $\lambda(k)$ by $\lambda(0)=0$, $m_{\lambda(k)}<k\leqq m_{\lambda(k)+1}$ $(k\geqq 1)$ we consider now (cf. (4); $m_{-1}:=0$)

$$\tau_n(x):=\sum_{k=0}^{n} a_{nk}|s_k(x)-f(x)|^\gamma \leqq K_1\left\{\sum_{k=0}^{n} a_{nk}|s_k(x)-s_{m_{\lambda(k)-1}}(x)-\sigma_k^*(x)|^\gamma+\right.$$

$$\left.+\sum_{k=0}^{n} a_{nk}|s_{m_{\lambda(k)-1}}(x)-f(x)|^\gamma+\sum_{k=0}^{n} a_{nk}|\sigma_k^*(x)|^\gamma\right\} = K_1\{\tau_n^{(1)}(x)+\tau_n^{(2)}(x)+\tau_n^{(3)}(x)\}.$$

By the regularity of (E, q) and with the aid of Lemmas 2 and 3 $\tau_n^{(2)}(x)\to 0$ and $\tau_n^{(3)}(x)\to 0$ are true. For the remaining part we get for an arbitrary $r>1\left(r'=\dfrac{r}{r-1}\right)$ by Hölder inequality and Lemma 4

$$\tau_n^{(1)}(x) \leqq \left\{\sum_{k=0}^{n} a_{nk}^r\right\}^{\frac{1}{r}} \left\{\sum_{k=1}^{n} |s_k(x)-s_{m_{\lambda(k)-1}}(x)-\sigma_k^*(x)|^{\gamma r'}\right\}^{\frac{1}{r'}} \leqq K_2\left\{\frac{1}{\sqrt{n}}\sum_{k=1}^{n}|s_k(x)-\right.$$

$$\left.-s_{m_{\lambda(k)-1}}(x)-\sigma_k^*(x)|^{\gamma r'}\right\}^{\frac{1}{r'}} \leqq K_2\left\{\frac{1}{\sqrt{m_{\lambda(n)}}}\sum_{i=0}^{\lambda(n)}\sum_{k=m_i+1}^{\min(m_{i+1},n)}|s_k(x)-s_{m_{i-1}}(x)-\sigma_k^*(x)|^{\gamma r'}\right\}^{\frac{1}{r'}}.$$

The assertion is proved by Kronecker's lemma if we show that

$$\tau(x):=\sum_{i=1}^{\infty}\frac{1}{\sqrt{m_i}}\sum_{k=m_i+1}^{m_{i+1}}|s_k(x)-s_{m_{i-1}}(x)-\sigma_k^*(x)|^{\gamma r'}<\infty. \tag{5}$$

To this end we consider $(\gamma r'>2)$

$$\int_0^1 (\tau(x))^{\frac{2}{\gamma r'}}\,dx \leqq \int_0^1 \sum_{i=1}^{\infty}\left\{\frac{1}{\sqrt{m_i}}\sum_{k=m_i+1}^{m_{i+1}}|s_k(x)-s_{m_{i-1}}(x)-\sigma_k^*(x)|^{\gamma r'}\right\}^{\frac{2}{\gamma r'}}dx.$$

With the aid of Lemma 1, putting for any k with $m_i<k\leqq m_{i+1}$ (i fixed) $s_k(x)-s_{m_{i-1}}(x)=:S'_\nu(x)$ $(\nu:=k-m_{i-1};\ k>m_{i-1})$ and $\sigma'_\nu(x):=\dfrac{1}{\nu}\sum_{l=1}^{\nu}S'_l(x)$ $(=\sigma_k^*(x)$ if $m_i<k\leqq m_{i+1})$, we get with $m_{i+1}-m_{i-1}\leqq 2\beta\sqrt{m_i}$ (cf. (3))

$$\int_0^1\left\{\frac{1}{\sqrt{m_i}}\sum_{k=m_i+1}^{m_{i+1}}|s_k(x)-s_{m_{i-1}}(x)-\sigma_k^*(x)|^{\gamma r'}\right\}^{\frac{2}{\gamma r'}}dx \leqq$$

$$\leqq K_3\int_0^1\left\{\frac{1}{(m_{i+1}-m_{i-1})}\sum_{\nu=1}^{m_{i+1}-m_{i-1}}|S'_\nu(x)-\sigma'_\nu(x)|^{\gamma r'}\right\}^{\frac{2}{\gamma r'}}dx \leqq K_4\sum_{k=m_{i-1}+1}^{m_{i+1}}c_k^2$$

and thus

$$\int_0^1 (\tau(x))^{\frac{2}{\gamma r'}}\, dx \leqq K_4 \sum_{i=1}^{\infty} \sum_{k=m_{i-1}+1}^{m_{i+1}} c_k^2 < \infty.$$

B. Levi's theorem finally gives (5) which completes the proof.

I would like to express my sincere gratitude to Professor Dr. L. Leindler for his interest and remarks.

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(Received July 22, 1981)

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and thus

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By [illegible] theorem finally gives (7) which completes the proof.

I would like to express my sincere gratitude to Professor Dr. L. Leindler for his interest and remarks.

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Acta Math. Hung.
41 (1—2), (1983), 107—110.

ON VERY STRONG SUMMABILITY OF ORTHOGONAL SERIES BY EULER METHOD

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1. Let $\{\varphi_n(x)\}$ be an orthonormal system on the interval [0, 1]. We shall consider orthogonal series

$$(1.1)\qquad \sum_{n=0}^{\infty} c_n\varphi_n(x)$$

with real coefficients satisfying

$$(1.2)\qquad \sum_{n=0}^{\infty} c_n^2 < \infty.$$

By the Riesz—Fischer theorem, the series (1.1) converges in the mean to a square integrable function $f(x)$. By $s_n(x)$ and $t_n(x)$ we denote the nth partial sums and the nth Euler means of order q $((E, q)$-means; $0<q<1)$ of (1.1), i.e.

$$s_n(x) = \sum_{i=0}^{n} c_i\varphi_i(x) \quad\text{and}\quad t_n(x) = \sum_{k=0}^{n} a_{nk}s_k(x),$$

where $0<q<1$ and

$$a_{nk} = \begin{cases} \binom{n}{k} q^k(1-q)^{n-k} & \text{if } n \geqq k,\\ 0 & \text{if } n<k.\end{cases}$$

Very recently H. Schwinn [6] proved the following theorem in connection with the strong Euler summability:

THEOREM A. *If the orthogonal series* (1.1) *is* (E, q)*-summable to* $f(x)$ *in* [0, 1] *almost everywhere, then for any positive* p

$$(1.3)\qquad \lim_{n\to\infty} \sum_{k=0}^{n} a_{nk}|s_k(x)-f(x)|^p = 0$$

holds in [0, 1] *almost everywhere, i.e.* (1.1) *is strong* (E, q)*-summable with order* p.

The aim of the present paper is to give a sufficient condition in order that (1.1) should be *very strong Euler-summable,* i.e. that for any increasing sequence $\{\nu_k\}$ of natural numbers

$$(1.4)\qquad \lim_{n\to\infty} \sum_{k=0}^{n} a_{nk}|s_{\nu_k}(x)-f(x)|^p = 0$$

should hold in [0, 1] almost everywhere.

For the classical arithmetical means or for the $(C, \alpha>0)$-means problems of this type have been discussed in great detail (see e.g. the book of G. Alexits [1], p. 107, or the papers of K. Tandori [8], G. Sunouchi [7] and L. Leindler [2]), as regards the Riesz-means we refer to J. Meder [4] and L. Leindler [3].

To prove (1.4) we shall use Theorem A and the method of proof given by us in [3]. First we prove

THEOREM 1. *Let $\{\nu_k\}$ be an arbitrary increasing sequence of natural numbers. If* (1.1) *is almost everywhere (E, q)-summable to $f(x)$ in* [0, 1] *and*

$$\sum_{m=0}^{\infty}\left(\sum_{k=m^2+1}^{(m+1)^2} c_k^2\right)\log^2(2+h_m)<\infty, \tag{1.5}$$

where h_m denotes the number of indices ν_{k^2} lying between m^2 and $(m+1)^2$, then

$$\lim_{n\to\infty}\sum_{k=0}^{n} a_{nk}|s_{\nu_k}(x)-f(x)|^p=0 \tag{1.6}$$

also holds for any positive p in [0, 1] *almost everywhere.*

Using Theorem A and Theorem 1 we prove

THEOREM 2. *If* (1.1) *is almost everywhere (E, q)-summable to $f(x)$ in* [0, 1] *and there exists a positive sequence $\{a_n\}$ with the following properties: $\sum_{n=0}^{\infty} a_n^2<\infty$, $c_n^2=O(a_n^2)$ and*

$$A_n^2:=\sum_{k=n^2+1}^{(n+1)^2} a_k^2 \geqq A_{n+1}^2,\text{[1]} \tag{1.7}$$

then (1.1) *is also very strong Euler-summable in* [0, 1] *almost everywhere, i.e.* (1.4) *holds for any increasing sequence $\{\nu_k\}$ in* [0, 1] *almost everywhere.*

From Theorem 2 we get immediately

COROLLARY. *If* (1.1) *is (E, q)-summable and*

$$\left(\sqrt{n}-\frac{1}{2}\right)c_n^2 \geqq \left(\sqrt{n}+\frac{1}{2}\right)c_{n+1}^2$$

then (1.1) *is also very strong Euler-summable.*

2. We require the following lemmas. For the sake of brevity, from now on, convergence and summability have the meaning of convergence and summability almost everywhere in [0, 1].

[1] We mention that condition (1.7) can be weakened, namely instead of (1.7) it is sufficient to claim that there exists a constant K and a natural number μ such that for any positive l and m

$$K\sum_{s=0}^{\mu} A_{l+s}^2 \geqq A_{l+m}^2$$

holds.

LEMMA 1. (1.1) *is* (E, q)*-summable if and only if the partial sums* $s_{n^2}(x)$ *are convergent.*

This lemma was proved by O. A. Ziza [9]; cf. also H. Schwinn [5].

LEMMA 2 ([3]). *Let* $\{p_m\}$ *and* $\{q_k\}$ *be increasing sequences of natural numbers. If* r_m *denotes the number of the terms* q_k *lying between* p_m *and* p_{m+1}, *then the condition*

$$\sum_{m=1}^{\infty}\left(\sum_{n=p_m+1}^{p_{m+1}} c_n^2\right)\log^2(2+r_m)<\infty$$

implies that

$$\lim_{m\to\infty}\left(s_{p_m}(x)-s_{q_k}(x)\right)=0$$

holds for any q_k *with* $p_m<q_k<p_{m+1}$.

3. PROOF OF THEOREM 1. By Lemma 1 the (E, q)-summability of (1.1) implies that the partial sums $s_{m^2}(x)$ converge to $f(x)$. Thus, using Lemma 2 with $p_m=m^2$, $q_k=\nu_{k^2}$ and $r_m=h_m$, on account of (1.5), we obtain that the partial sums $s_{\nu_{k^2}}(x)$ also converge to $f(x)$.

Now we construct a new orthonormal system and a new-sequence of coefficients by means of the given sequence $\{\nu_k\}$. Putting $\nu_{-1}=-1$ we define

$$C_n^2:=\sum_{i=\nu_{n-1}+1}^{\nu_n} c_i^2 \quad\text{for}\quad n\geqq 0,$$

and

$$\Phi_n(x):=\begin{cases} C_n^{-1}\displaystyle\sum_{i=\nu_{n-1}+1}^{\nu_n} c_i\varphi_i(x) & \text{if}\quad C_n\neq 0,\\ (\nu_n-\nu_{n-1})^{-1/2}\displaystyle\sum_{i=\nu+_{n-1}1}^{\nu_n}\varphi_i(x) & \text{if}\quad C_n=0.\end{cases}$$

It is clear that

$$S_k(x):=\sum_{j=0}^{k} C_j\Phi_j(x)=\sum_{i=0}^{\nu_k} c_i\varphi_i(x)=s_{\nu_k}(x), \tag{3.1}$$

thus the convergence of $s_{\nu_{k^2}}(x)$ implies that of $S_{k^2}(x)$, whence by Lemma 1 we obtain that the orthogonal series $\sum_{n=0}^{\infty} C_n\Phi_n(x)$ is (E, q)-summable.

Thus Theorem A gives

$$\lim_{n\to\infty}\sum_{k=0}^{n} a_{nk}|S_k(x)-f(x)|^p=0, \tag{3.2}$$

and by (3.1) statement (3.2) means exactly the same as (1.6) does.

The proof is complete.

PROOF OF THEOREM 2. On account of Theorem 1, it will suffice to show that under the assumptions of Theorem 2, (1.5) holds for any increasing sequence $\{\nu_k\}$.

Indeed it is obvious that[2]

$$(3.3)\qquad \sum_{m=0}^{\infty}\left(\sum_{k=m^2+1}^{(m+1)^2} c_k^2\right)\log^2(2+h_m) = K_1\sum_{m=0}^{\infty} A_m^2\log^2(2+h_m) \leqq$$

$$\leqq K_2\sum_{m=0}^{\infty} A_m^2(2+h_m) = K_3+K_2\sum_{m=0}^{\infty} A_m^2 h_m.$$

To prove the finiteness of $\sum_{m=0}^{\infty} A_m^2 h_m$ we mention that by the definition of h_u, the inequality

$$(3.4)\qquad \sum_{n=0}^{N} h_n \leqq N+1,$$

follows for any N.

Hence, by (1.7), we obtain that

$$(3.5)\qquad \sum_{n=0}^{N} A_n^2 h_n = \sum_{n=0}^{\infty} A_n^2.$$

Namely, if e.g. $h_m > 1$ then by (3.4) there exist at least $h_m - 1$ indices $n_i < m$ such that $h_{n_i} = 0$, and if we replace the sum A_m^2 by $A_{n_i}^2$ $(h_m - 1)$-times, then

$$A_m^2 h_m = \sum_{i=1}^{h_m-1} A_{n_i}^2 + A_m^2$$

holds obviously; and this "replacing-procedure" conveys statement (3.5).

The estimations (3.3) and (3.5) verify (1.5) for any $\{v_k\}$ under the assumptions of Theorem 2, and this ends the proof.

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(Received August 19, 1981)

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[2] $K_1, K_2, \ldots$ denote positive constants.

Acta Math. Hung.
41 (1—2), (1983), 111—117.

BI-SELECTIVE DERIVATIVES ARE OF HONORARY BAIRE CLASS TWO

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I. Introduction

In this paper, we answer a question posed by M. Laczkovicz in [4]. In one sense this answer is a by-product of research into selective derivatives, in particular [8]. The author feels that the reader could benefit from a brief restatement of the history and basis of the problem.

In [6], the present author introduced and developed the idea of selective differentiation theory. In that paper it was shown that selective derivatives need not be Baire class one but in some cases could be seen to be Baire class two. M. Laczkovich, in [4], was able to show that selective derivatives were always Baire class two. In establishing this result, he introduced the concept of $1-r$ differentiation which includes the aforementioned selective process. He also proved that a $1-r$ derivative was of Baire class two. Then he asked whether all $1-r$ derivatives are of honorary Baire class two. (In [1], Bagemihl and Piranian defined a function g as honorary Baire class two if there exists a Baire class one function h such that $\{x\colon h(x)\neq g(x)\}$ is at most countable. See also [3], [9].) Initially, using a technique developed in [8], we answered this question for selective derivatives. It was a corollary of a simple but very useful lemma about selections. However, we did not publish this lemma or corollary. Later we realized that the process of $1-r$ differentiation could be equivalently redefined in a way that made its connection with the selective process clearer. Further, the lemma could be restated in this framework. The connecting term is something we label a bi-selection. We have presumed to state our results in this context.

II. Definitions, notation and background results

Throughout the paper $[x, y]$ will denote the interval having endpoints x and y regardless of whether $x<y$ or $y<x$.

DEFINITION. A selection is an interval function s defined on the set of all nondegenerate closed subintervals of $[0, 1]$ satisfying $x<s[x, y]<y$ for every $[x, y]$, $x<y$.

Next, let $f\colon[0, 1]\to R$ be fixed. The various analogues of the classical derivatives of f at x with respect to the selection s are typified by the definition and notation for the selective derivative.

DEFINITION. A finite function $g\colon[0, 1]\to R$ is called the selective derivative of f, denoted sf', if for all x in $[0, 1]$

$$\lim_{h\to 0}\frac{f(s[x, x+h])-f(x)}{s[x, x+h]-x} = g(x) = sf'(x).$$

* Research supported in part by N. S. F. Grant # MCS 8102494.

For the same function f, let l and r be two interval functions, defined on all nondegenerate closed subintervals of $[0, 1]$, satisfying

$$\min(l[x, y],\ r[x, y]) < \frac{f(y)-f(x)}{y-x} \leqq \max(l[x, y],\ r[x, y]).$$

We say that l and r are admissible for f when they satisfy the above inequality.

Again, the concept of $\mathrm{l}-\mathrm{r}$ differentiation can be illustrated by the definition and notation associated with the $\mathrm{l}-\mathrm{r}$ derivative of f.

Definition. A finite function $g\colon [0,1]\to R$ called the $\mathrm{l}-\mathrm{r}$ derivative of f, denoted ${}_{\mathrm{l}}^{\mathrm{r}}f'$, if for each x in $[0, 1]$,

$$\lim_{y\to z-} l[y, x] = g(x) = {}_{\mathrm{l}}^{\mathrm{r}}f'(x) = \lim_{y\to x+} r[x, y].$$

M. Laczkovich pointed out, [4], that all selective derivatives are $\mathrm{l}-\mathrm{r}$ derivatives. We will have need of several results of that paper. Included is the following about pairs of admissible interval functions and the relations between the resulting extreme derivates.

L-Theorem 2 [page 102, 4]. *If both* $\mathrm{l}_1[x, y]$, $\mathrm{r}_1[x, y]$ *and* $\mathrm{l}_2[x, y]$, $\mathrm{r}_2[x, y]$ *are admissible for* f *then* $\{x\colon {}_{\mathrm{l}_1}^{\mathrm{r}_1}\bar{f}(x) < {}_{\mathrm{l}_2}^{\mathrm{r}_2}f(x)\}$ *is countable.*

(Here, for example, ${}_{\mathrm{l}_1}^{\mathrm{r}_1}\bar{f}$ is the notation for the upper extreme derivate of f relative to $\mathrm{l}_1-\mathrm{r}_2$.)

We introduce now the notion of a bi-selection.

Definition. A bi-selection, b, consists of an ordered pair of interval functions u and v defined on all closed nondegenerate subintervals of $[0, 1]$, satisfying the condition that u is a selection. An alternate equivalent statement may help clarify matters. Namely: A selection can be thought of as picking a point out of the interior of the interval $[a, b]$. A bi-selection can be thought of as picking a point out of the interior of the infinite strip $\{(x, y)\colon a \leqq x \leqq b\}$.

Next, the notion of bi-selective differentiation for our function f should seem very natural.

Definition. A finite function $g\colon [0, 1]$ is called the bi-selective derivative of f, denoted bf', if for each x in $[0, 1]$,

$$\lim_{h\to 0} \frac{v[x, x+h]-f(x)}{u[x, x+h]-x} = g(x) = bf'(x).$$

It is clear that each selective derivative is a bi-selective derivative.

III. New results

First we establish the equivalence between $\mathrm{l}-\mathrm{r}$ differentiation theory and bi-selective differentiation theory. This will be done only for the corresponding derivatives. Yet the proof indicates how we may switch between admissible pairs and bi-selections.

THEOREM 1. *Let* f: $[0, 1]\to R$ *be fixed. Then* g: $[0, 1]\to R$ *is a bi selective derivative of f if and only if it is a* $l-r$ *derivative of f.*

PROOF. ($\Rightarrow$). Suppose $u[x, y]$, $v[x, y]$ form the bi-selection. We merely define

$$l[x, y] = \frac{f(y)-v[x, y]}{y-u[x, y]} \quad \text{and} \quad r[x, y] = \frac{v[x, y]-f(x)}{u[x, y]-x}.$$

Then it follows that l and r are admissible for f and that ${}_l^r f'=g$.

($\Leftarrow$) This requires more computation. First, let l and r be the admissible pair generating $g(x)$. Let us consider the secant chord, C, connecting $(x, f(x))$ and $(y, f(y))$ for a fixed pair $0\leqq x<y\leqq 1$. Over $[x, y]$, consider the segment, L_1, through $(x, f(x))$ with slope $r[x, y]$ and also the segment L_2, through $(y, f(y))$ with slope $l[x, y]$. The fact that l and r are admissible for f forces one of the following cases to occur.

Case 1. $C=L_1=L_2$.

Case 2. $C=L_1\neq L_2$.

Case 3. $C=L_2\neq L_1$.

Case 4. $C\neq L_1\neq L_2\neq C$, but $L_1\cap L_2=(x_0, y_0)$ with $x<x_0<y$.

We begin to define the bi-selection.

In Case 1: let $u[x, y]=\dfrac{x+y}{2}$, $v[x, y]=\dfrac{f(x)+f(y)}{2}$

In Case 4: let $u[x, y]=x_0$, $v[x, y]=y_0$.

The definition in Cases 2 and 3 is slightly more complicated. But because Case 3 follows the same reasoning as Case 2 we will only present that situation: One of the two line segments, over $[x, y]$, through $(x, f(x))$ with slopes $r[x, y]\pm(y-x)$ intersects L_2 at a point (x_0, y_0) with $x<x_0<y$. In Case 2, we define $u[x, y]=x_0$, $v[x, y]=y_0$.

It is not hard to see that relative to this bi-selection b, f has $bf'=g$. In fact for fixed x and $h>0$ we have both

$$\left|\frac{v(x-h, x]-f(x)}{u[x-h, x]-x}-l(x-h, x)\right| \leqq h \quad\text{and}\quad \left|\frac{v[x, x+h]-f(x)}{u[x, x+h]-x}-r(x, x+h)\right| \leqq h.$$

From this point, we will state our results in the parlance of bi-selections.

We are now ready to state our basic lemma about bi-selections. As above, we will not state the lemma as it applies to all the various bi-selective derivatives but only enough to indicate the scope.

LEMMA 1. *Let* f: $[0, 1]\to R$ *and* b, *a bi-selection, be fixed. Let* P *be any closed subset of* $[0, 1]$. *Then there is a new bi-selection* t *such that:*

i) *for nearly all points* x *of* P

$$\liminf_{\substack{y\to x\\ y\in P}} \frac{f(y)-f(x)}{y-x} \leqq {}_t f'(x),$$

ii) *for all x in* $[0, 1]\setminus P$

$$_t f'(x) = {}_b f'(x).$$

Here $_t f'$ *is the notation for the lower extreme derivate of f relative to t.*

PROOF. The proof is extremely simple. Let $[x, y]$ be fixed, $0 \leqq x < y \leqq 1$. If P intersects the interior of $[x, y]$ let x_0 be a point of that intersection. We define the selection part, u, of t as $u[x, y] = x_0$. The other part v, of t, we define as $v[x, y] = f(u[x, y]) = f(x_0)$. If P does not intersect the interior of $[x, y]$ let $u[x, y]$ and $v[x, y]$ be the values given by the original bi-selection b. For every bilateral limit point x of P i) holds and for all x in $[0, 1]\setminus P$, ii) is valid.

Though simple, this lemma, in conjunction with L-Theorem 2, becomes very useful when f is assumed to have a bi-selective derivative over P.

We restate L-Theorem 2 in terms of bi-selections.

L-THEOREM 2. *For a given function* $f\colon [0, 1] \to R$ *and two bi-selections,* b_1 *and* b *the set* $\{x\colon 0 \leqq x \leqq 1$ *and* ${}_{b_1}f'(x) > {}^{b_2}f'(x)\}$ *is countable.*

PROPOSITION 1. *Let* $f\colon [0, 1] \to R$ *and P a closed set be fixed. Suppose f is bi-selectively differentiable, with respect to b, for every x in P. Let* $D = \{x\colon f$ *has a derivative, relative to P, at* $x\}$ *i.e.* $D = \left\{x\colon \lim\limits_{\substack{y \to x \\ y \in P}} \dfrac{f(y)-f(x)}{y-x} \text{ exists}\right\}$. *Then for nearly all x in D,* $bf'(x) = \lim\limits_{\substack{y \to x \\ y \in P}} \dfrac{f(y)-f(x)}{y-x}$.

PROOF. Let t be the modification of b mentioned in the proof of the lemma. Then for nearly all x in D we have

$$bf'(x) = {}_b f'(x) \leqq {}^t f'(x) \leqq \limsup_{\substack{y \to x \\ y \in P}} \frac{f(y)-f(x)}{y-x} = \liminf_{\substack{y \to x \\ y \in P}} \frac{f(y)-f(x)}{y-x} \leqq$$

$$\leqq {}_t f'(x) \leqq {}^b f'(x) = bf'(x).$$

PROPOSITION 2. *Let* $f\colon [0, 1] \to R$, *P and b be as in Proposition* 1. *Suppose in addition,* $\liminf\limits_{\substack{y \to x \\ y \in P}} \dfrac{f(y)-f(x)}{y-x} \geqq a$ *for a fixed. Then for nearly all x in P* $bf'(x) \geqq a$.

PROOF. Obvious.

Perhaps the relevance of these propositions to the present discussion should be explained. In [6, page 88] an example was constructed to show that a selective derivative need not be of Baire class 1. It was pointed out by M. Laczkovich that this selective derivative was of honorary Baire class 2. The characteristics of the example were as follows:

1) $[0, 1] = U \cup P$ where U is a dense open set, $U \cap P = \emptyset$, and P is perfect.
2) sf' is Baire 1 on each component of vU.
3) f was differentiable, relative to P, at every point of P.
4) sf' is not Baire 1 over P.

The above results indicate that any such example would be of honorary Baire class 2. The function h defined as:

$$h(x) = \begin{cases} sf'(x) & (x \in U) \\ \lim\limits_{\substack{y \to x \\ y \in P}} \dfrac{f(y)-f(x)}{y-x} & (x \in P), \end{cases}$$

is Baire class 1 and equals sf' nearly everywhere.

We now proceed to show that all bi-selective derivatives are of honorary Baire class 2. We will need an additional lemma. Its nature and proof are similar to Lemma 2 [4, page 103] and Lemma 3 [6, page 86].

LEMMA 2. *Let f: $[0, 1] \to R$ be bi-selectively differentiable, relative to b. Let $c<d$ be fixed and $\varepsilon>0$ be given.*

$$A = \left\{x\colon\ c < \frac{v(x, y)-f(x)}{u(x, y)-x} < d \ \text{whenever}\ 0 < |x-y| < \varepsilon\right\}$$

and let A^* denote the closure of A.

Then if x_1 and x_2 belong to A^* and $0<|x_1-x_2|<\varepsilon$, $c \leqq \dfrac{f(x_2)-f(x_1)}{x_2-x_1} \leqq d$.

PROOF. The conclusion is valid if x_1 and x_2 both belong to A. Next, let x_0 be the limit of a sequence of points x_k from A. Assume without loss of generality that $x_0<x_k<x_0+\varepsilon$. Let $u_k=u[x_0, x_k]$ and $v_k=v[x_0, x_k]$. Since $bf'(x_0)$ is finite and u_k converge to x_0 it follows that v_k converge to $f(x_0)$. In addition, we have $v_k+c(x_k-u_k)<f(x_k)<v_k+d(x_k-u_k)$, so $f(x_k)$ converge to $f(x_0)$. This suffices to show that f is continuous relative to A^*. In turn this is sufficient to yield the entire conclusion.

PROPOSITION 3. *Let f: $[0, 1] \to R$ be bi-selectively differentiable with respect to b. Then bf' is honorary Baire class 2.*

PROOF. There are several ways of establishing a function as honorary Baire class 2 see e.g. [2], [3], [9]. We have chosen to use one based on [2]. Namely: a function h is of honorary Baire class 2 if and only if each of its associated sets differs from an F_σ by a countable set. Let a be given and consider the sets $U=\{x\colon bf'(x)>a\}$ and $B=\{x\colon bf'(x)<a\}$. It will suffice to show that U has the desired structure.

$$U = \bigcup_{n=1}^{\infty} \left\{x\colon\ \frac{1}{n}+a < \frac{v(x, y)-f(x)}{u(x, y)-x} < n,\ 0 < |x-y| < \frac{1}{n}\right\} = \bigcup_{n=1}^{\infty} U_n.$$

Lemma 2 gives that for each n and for x in U_n^*

$$a < a+\frac{1}{n} \leqq \liminf_{\substack{y \to x \\ y \in U_n}} \frac{f(y)-f(x)}{y-x}.$$

Proposition 2 yields that for nearly all x in U_n^* $bf'(x) \geqq a+\dfrac{1}{n}>a$. Therefore $U=$ $=\bigcup\limits_{n=1}^{\infty} U_n^* \setminus C$ where C is at most countable.

We will make use of the machinery of Lemmas 1 and 2 to prove a final result.

PROPOSITION 4. *Let f: [0, 1] be bi-selectively differentiable with respect to b. Let P be any perfect set. Then there is a point x_0 in P at which f is differentiable relative to P, and its relative derivative is $bf'(x_0)$.*

PROOF. We wish to show the existence of a point x_0 in P such that

$$\lim_{\substack{y\to x_0 \\ y\in P}} \frac{f(y)-f(x_0)}{y-x_0} = bf'(x_0).$$

We show first that there is a dense G_δ subset of P at which the relative derivative exists. Then by Proposition 1 we are finished. Let $\varepsilon>0$ be given. We may define a double sequence of set $A_{n,m}$ such that

i) $A_{n,m} = \left\{x\colon c_n < \dfrac{v(x,y)-f(x)}{u(x,y)-x} < d_n \text{ whenever } 0<|x-y| \leqq \dfrac{1}{m}\right\}$

ii) $0 < d_n - c_n < \varepsilon$

iii) $\bigcup_{n=1}^{\infty} (c_n, d_n) = R$

iv) $\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_{n,m} = [0, 1]$.

We know the behavior of f over the closure of $A_{n,m}=A^*_{n,m}$ by Lemma 2. By applying the Baire category theorem to $A^*_{n,m}\cap P$, we get a set U_ε dense and open, relative to P, such that for each x in U_ε we have a c_n, d_n and m such that $0<d_n-c_n<\varepsilon$ and $c_n<\dfrac{f(y)-f(x)}{y-x}<d_n$ whenever $0<|y-x|<\dfrac{1}{m}$ and $y\in P$. This will show the existence of the desired dense G_δ.

It is interesting to compare this result to one obtained in [7]. Suppose f is approximately differentiable. Then in the conclusion of Proposition 4 rather than a point we can find an entire portion where the relative derivative exists and equals the approximate derivative.

We end the paper with a question. In what way are all selective or bi-selective derivatives characterized by properties exhibited by the example in [6, page 88]? More specifically, if g is a selective derivative is there a dense open set on which it is Baire 1?

Acknowledgement. The author would like to thank Prof. Bagemihl for his ready assistance in obtaining information about honorary Baire class 2 functions.

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(Received July 31, 1981)

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Acta Math. Hung.
41 (1—2), (1983), 119—126.

A UNIFICATION OF GENERALIZATIONS OF THE LAPLACE TRANSFORM AND GENERALIZED* FUNCTIONS

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1. Introduction. Some well known generalizations of the classical Laplace transform exist in many forms whose kernels involve various kinds of special functions. To cite some of them, Meijer [6] has given in the form

$$F(p) = \int_0^\infty (pt)^{-k-\frac{1}{2}} e^{-\frac{1}{2}pt} W_{k+\frac{1}{2},m}(pt) f(t)\, dt. \tag{1.1}$$

Meijer's another generalization [5] is

$$F(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty (pt)^{\frac{1}{2}} K_\nu(pt) f(t)\, dt. \tag{1.2}$$

Varma's generalizations [14], [15] are

$$F(p) = \int_0^\infty (2pt)^{-\frac{1}{4}} W_{k,m}(2pt) f(t)\, dt \tag{1.3}$$

and

$$F(p) = \int_0^\infty (pt)^{m-\frac{1}{2}} e^{-\frac{1}{2}pt} W_{k,m}(pt) f(t)\, dt. \tag{1.4}$$

In 1950, Erdélyi [3] gave an important generalization on which the author published some considerable work in the distributional sense. In 1968, Srivastava [12] gave a unification of all these generalizations in an elegant form as follows:

$$S_{q,k,m}^{\varrho,\sigma}[f(t); p] = F(p) = \int_0^\infty (pt)^{\sigma-\frac{1}{2}} e^{-\frac{1}{2}qpt} W_{k,m}(\varrho pt) f(t)\, dt \tag{1.5}$$

where

$$f(t) = \begin{cases} O(t^\delta e^{\varepsilon t}) & \text{for large } t > 0 \\ O(t^{\delta'}) & \text{for small } t \end{cases}$$

and $\mathrm{Re}[(q+\varrho)p - 2\varepsilon] > 0$ and $\mathrm{Re}(\sigma+\delta' \pm m+1) > 0$. The reducibility of (1.1) to

* A condensed version of this paper was presented by the author at the International Conference on Generalized Functions and their applications for Mathematical Physics in Moscow in November, 1980.

(1.5) to the classical Laplace transform (see also [13]) is based on the known formulas

$$K_{\pm\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}, W_{\frac{1}{2}+m,\, \pm m}(z) = z^{m+\frac{1}{2}} e^{-\frac{1}{2}z}$$

and the relation

$$W_{0,\nu}(2z) = \sqrt{\frac{2z}{\pi}}\, K_\nu(z).$$

For example, for $\varrho=1$, $\sigma=\frac{1}{2}-m$, $q=1$ (1.5) renduces to classical Laplace transform. Saksena [9] also gave another generalization of the Laplace transform in the form

$$F(p) = p^a \int_0^\infty (qpt)^{c-\frac{1}{2}} e^{-\left(p-\frac{q}{2}\right)pt} W_{l,m}(qpt)\varphi(t)\,dt$$

which, for different values of the parameters, reduces to the various generalizations.

L. Schwartz [10] was the pioneer in the work of extension of the Laplace transform to distributions. Later many authors including H. G. Garnir, V. S. Vladimirov, A. H. Zemanian, J. L. B. Cooper, John Benedetto and T. Ishihira discussed the Laplace transform of several classes of generalized functions and deduced some important properties.

2. Notation, terminology and some definitions. The notation and terminology used in this paper follow that in Zemanian [17] and [18] and the author [7], [8]. The definitions of testing function spaces $D(I)$, $L_a(I)$, S, $L_{a,b}$ and their duals are given in [18], [7] and [8]. However the definition of $L_a(I)$ is given as follows:

Let I denote the open interval $(0, \infty)$. Let a be a real number. We define $L_a(I)$ as the space of all smooth functions $\varphi(y)$ on I such that

$$\gamma_{a,n}(\varphi) = \sup_{0<t<\infty} |e^{at} D^n \varphi(t)| < \infty \quad (n = 0, 1, 2, \ldots) \tag{2.1}$$

and its topology is generated by the semi-norms $\{\gamma_{a,n}\}_{n=1}^\infty$. $L'_a(I)$ is the space of continuous linear functionals on I. We also note that $D(I)\subset L_a(I)$ and the convergence in $L'_a(I)$ implies convergence in $D'(I)$. In a similar way we can define the test function space $L_{a,b}$ as in Zemanian [18, p. 48] in order to enable us to discuss the two-sided W-transform of generalized functions in Section 4.

3. Some preliminary results. Let us put, for convenience,

$$T(pt) = (pt)^{m-\frac{1}{2}} e^{\frac{-pqt}{2}} W_{k,m}(\varrho pt).$$

$$\text{(a)} \qquad T(pt)\in L_a(I) \quad \text{if} \quad a < \frac{p}{2}(\varrho+q).$$

PROOF. By using the seminorms (2.1), the asymptotic relations of $W_{k,m}$ function [11, p. 61] namely $W_{k,m}(x)=O\left(x^k e^{-\frac{x}{2}}\right)$, x large and the differential property

[11, p. 25] namely

$$D_x^n\left\{e^{-\frac{1}{2}x}x^{m-\frac{1}{2}}W_{k,m}(x)\right\} = (-1)^n e^{-\frac{x}{2}}x^{m-\frac{1}{2}-\frac{n}{2}}W_{k+\frac{n}{2},\, m-\frac{1}{2}n}(x)$$

we have with $x=\varrho pt$

$$D_t^n T(pt) = \left[(\varrho p)^n e^{\frac{1}{2}-\sigma}\right] D_x^n\left[\{x^{m-\frac{1}{2}}e^{-\frac{x}{2}}W_{k,m}(x)\}x^{\sigma-m}e^{\frac{x}{2}\left(1-\frac{q}{e}\right)}\right] =$$

$$= (\varrho p)^n e^{\frac{1}{2}-\sigma}\left[\sum_{j=0}^{n}\binom{j}{n}(-1)^j e^{-x/2}x^{m-\frac{1}{2}-\frac{j}{2}}W_{k+\frac{j}{2},\, m-\frac{j}{2}}(x)\times\right.$$

$$\left.\times\sum_{l=0}^{n-j}\binom{1}{n-j}\times A_{\sigma,m\,j,l}\,x^{\sigma-m-l}e^{x/2\left(1-\frac{q}{e}\right)}\times B_{q,\varrho,n-j-l}\right]$$

where $A_{\sigma.m,j,l}$ and $B_{q,\varrho,n-j-l}$ are some polynomial expressions without p or x. Consider now

$$|e^{at}D_t^n T(pt)|. \tag{3.1}$$

A typical term of this expression is found to be asymptotic to

$$e^{-t\left(\frac{\varrho p}{2}+\frac{pq}{2}-a\right)}(\varrho pt)^{-\frac{1}{2}+\varrho-l+k}$$

which $\to 0$ as $t\to\infty$ if $a<\frac{p}{2}(\varrho+q)$. Similarly we can prove that $T(pt)\in L_{a,b}$ [18, p. 48] if $a<\frac{p}{2}(\varrho+q)<b$. Let now $t\to 0$. We have

$$W_{k+\frac{j}{2},\, m-\frac{j}{2}}(\varrho pt)\sim(\varrho pt)^{\pm\left(m-\frac{j}{2}\right)+\frac{1}{2}}$$

whence, as before, a typical term of the expression (3.1) is asymptotic to $e^{t\left(a-\frac{pq}{2}\right)}\times$ $\times t^{\sigma-l\pm m-j}$ which $\to 0$ as $t\to 0$ if $\sigma\pm m>l+j$ $(l=0, 1, \dots, n-j)$, $(j=0, 1, \dots, n)$. Hence $T(pt)\in L_a(I)$ if $a<\frac{p}{2}(\varrho+q)$. Furthermore $T(pt)\in L_{a,b}$ if $a<\frac{p}{2}(\varrho+q)<b$ and $\varrho\pm m>l+j$.

(b) The Lemma 3.2 of [7] can also be proved in one dimensional case. It can be stated as follows. Let $a, b, \sigma\in R^1$ with $a<\operatorname{Re} p<b$. If $\theta\in S$, then $\theta[T(pt)]\in L_{a,b}$. If $\{\theta_\nu\}_{\nu=1}^{\infty}$ converges to 0 in S, then $\{[T(pt)]\theta_\nu\}_{\nu=1}^{\infty}$ also converges in $L_{a,b}$ to 0.

4. (a) **The *n*-dimensional *W*-transform.** We follow the notation in a paper of the author [7]. For example

$$t=\{t_1, t_2, \dots, t_n\}\in R^n,\ [pt]^* = p_1t_1\cdot p_2t_2\cdot\ldots\cdot p_nt_n,\quad D_x^n = \frac{\partial|n|}{\partial_{x_1}^{n_1}\partial_{x_2}^{n_2}\ldots\partial_{x_n}^{n_n}}$$

where $|n|=n_1+n_2+\ldots+n_n$. $a<b$ means $a_j<b_j$ $(j=1, \dots, n)$.

THEOREM 4.1. *Let* $a, p, \varrho, q, \sigma\in R^n$. *Then* $[T(pt)]^*\in L_{a,b}$ *if* $a<p\left(\frac{\varrho+q}{2}\right)<b$.

To prove this, it is enough if we prove

$$\sup_{0<t<\infty} \left|[e^{at}]^* D_t^n [T(pt)]^*\right| < \infty.$$

By considering the j^{th} component, we can prove this as we did in Section 3. The following results are also easy to prove as in [7] and [17].

THEOREM 4.2. *$[T(pt)]^* f \in S'$ if and only if $\frac{f(t)}{K_{a,b}(t)} \in S'$, provided $a < \frac{p}{2} \times \times (\varrho + q) < b$.*

THEOREM 4.3. *If $f \in L'_{a,b}$ then $[T(pt)]^* f \in S'$.*

(b) **Boundedness property for generalized functions in $L'_{a,b}$.** This can be stated and proved as in [7] and [17] and hence omitted.

(c) **The two-sided W-transform of generalized functions:** A generalized function f is W-transformable if there exist two points $a, b \in R^n$ $(a<b)$ such that $f \in L'_{a,b}$. A point $p \in C^n$ is said to be in Γf if there exist two points $a, b \in R^n$ $(a<b)$ such that $f \in L'_{a,b}$. The W-transform of a W-transformable generalized function f is defined as (Fp) from the subset Γf of C^n into C^1 given by

$$F(p) = \langle f(t), [T(pt)]^* \rangle \tag{4.1}$$

where $p \in \Gamma f$. The rihgt-hand side of (4.1) has a sense as the application of $f \in L_{a,b}$ to $[T(pt)]^* \in L_{a,b}$ for every fixed value of p in Γf.

5. Analyticity. In this section it is assumed that $a, b, s, t, \varrho, s, q, \sigma \in R^1$, p being replaced by s.

THEOREM 5.1. *Let $F(s)$ be given by (4.1) p being replaced by s and in one dimension. Then $F(s)$ is an analytic function of s in Γf and*

$$D_s F(s) = \left\langle f(t), \frac{\partial}{\partial s} T(st) \right\rangle.$$

PROOF. Let s be an arbitrary but fixed point in Γf and let $r>0$ be such that $a < \operatorname{Re} s - r < \operatorname{Re} s + r < b$. With s as centre we construct a circle C of radius r_1 in Γf such that $r < r_1$. Let $|\Delta s|$ be a non-zero complex increment in s such that $|\Delta s| < r$. Consider

$$\frac{F(s+\Delta s) - F(s)}{\Delta s} - \left\langle f(t), \frac{\partial}{\partial s} T(st) \right\rangle = \left\langle f(t), \psi_{\Delta s}(t) \right\rangle \tag{5.1}$$

where

$$\psi_{\Delta s}(t) = \frac{t^{\sigma - \frac{1}{2}}}{\Delta s} \left[(s+\Delta s)^{\sigma - \frac{1}{2}} e^{-\frac{q}{2}(s+\Delta s)t} W_{k,m}(\varrho(s+\Delta s)t) - s^{\sigma - \frac{1}{2}} e^{-\frac{1}{2}qst} W_{k,m}(\varrho st) \right].$$

Our theorem would be proved if we show that (5.1) converges to zero as $|\Delta s| \to 0$. This can be done by showing that $\psi_{\Delta s}(t)$ converges in $L_{a,b}$ to zero as $|\Delta s| \to 0$ since

$f(t)\in L'_{a,b}$. After some simplification $\psi_{\Delta s}(t)$ can be seen to be equal to

$$(st)^{m-\frac{1}{2}} e^{\frac{-qst}{2}} \left[\sum_0^n (-1)^n t^n N_n(\varrho, q, \Delta s, m) W_{k+\frac{n}{2}, m+\frac{n}{2}}(\varrho st)\right.$$

by the help of [11, p. 29, equation 2.6.10], where N_n is a polynomial in $\varrho, p, q, \Delta s, m$ of degree n. We have again

$$D_t^\nu \psi_{\Delta s}(t) = D_t^\nu \left[\frac{(\varrho st)^{m-\frac{1}{2}}}{\varrho^{m-\frac{1}{2}}} e^{-\varrho\frac{st}{2}} e^{\frac{st}{2}(\varrho-q)} W_{k,m}(\varrho st) N_0(\varrho, p, q, \Delta s, m)\right].$$

Now by Cauchy's integral formula [2] we have, after some simplification, by using the definition of $K_{a,b}(t)$ [18, Section 3.11] and the asymptotic properties of $W_{k,m}$ function for all $z\in C$ and $-\infty<t<\infty$ and since $|z-s|=r_1$ and $|z-s-\Delta s|>r_1-r$,

$$|K_{a,b}(t) D_t^\nu \psi_{\Delta s}(t)| \leqq \frac{|\Delta s| \sum KQ}{2\pi} \int_c \frac{|dz|}{r_1^2(r_1-r)} \leqq \frac{|\Delta s| \sum K\varphi}{r_1(r_1-r)}$$

where K_Q is some constant independent of z and t for $q=0, 1, \ldots, \nu$. Thus as $|\Delta s|\to 0$, the right hand side converges to zero and $\psi_{\Delta s}(t)\to 0$ in $L_{a,b}$ as $|\Delta s|\to 0$. Hence (5.1) varishes as $|\Delta s|\to 0$ and the theorem is proved.

The above theorem can be extended to n dimensions by Hartog's theorem [1, p. 140]. The following result is also true and can be proved as in [18, p. 59]:

$$D_s^n F(s) = \left\langle f(t), \frac{\partial n}{\partial s^n}[T(st)]^*\right\rangle (s\in \Gamma f) \quad \text{for} \quad n = 1, 2 \ldots.$$

6. Inversion. The classical inversion theorem for W-transform proved by Srivastava [12] is now extended to a class of generalized functions in the following theorem.

THEOREM 6.1. *Let* $F(x)$ *be given by* $F(x)=\langle f(y), T(xy)\rangle$ *as in* (4.1) *where*

$$T(xy) = (xy)^{\sigma-\frac{1}{2}} e - \frac{xyq}{2} W_{k,m}(\varrho xy).$$

Let $f(y)\in L'_a(I)$ *and*

$$\Phi(s) = \int_0^\infty x^{-s} F(x)\, dx.$$

Then

$$\left\langle \frac{1}{2\pi i} \operatorname*{limit}_{\tau\to\infty} \int_{c-i\tau}^{c+i\tau} \Phi(s) y^{-s} M ds,\ \varphi(y)\right\rangle = \left\langle f(y), \varphi(y)\right\rangle \tag{6.1}$$

in the sense of convergence in $D'(I)$ *where*

$$M^{-1}=\frac{\varrho m+\frac{1}{2}\Gamma(\sigma\pm m+1-s)}{\left(\frac{q+\varrho}{2}\right)^{\sigma+m+1-s\Gamma}\left(\sigma-k+\frac{3}{2}-s\right)}\times$$

$$\times 2^{F_1}\left[m-k+1-s,\ m-k+\frac{1}{2};\ \sigma-k+\frac{3}{2}-s;\ \frac{q-\varrho}{q+\varrho}\right],$$

provided $\sigma-k+\frac{3}{2}-c>0$ *and* $\sigma\pm m+1-c>0$ *where* $s=c+iw$.

PROOF. Since the integral on s on the left-hand side of (6.1) is a continuous function of y, the right-hand side of (6.1) without limit notation can be rewritten as

$$\left\langle(2\pi i)^{-1}\int_{c-i\tau}^{c+i\tau} M\Phi(s)\,y^{-s}\,ds,\ \varphi(y)\right\rangle.$$

Let $\varphi(y)\in D(I)$. Then the above expression is equal to

$$\left\langle(2\pi i)^{-1}\int_{c-i\tau}^{c+i\tau} M\int_0^\infty x^{-s}\langle f(y), T(xy)\rangle dx\, y^{-s}ds,\ \varphi(y)\right\rangle$$

$$=\left\langle(2\pi)^{-1}\int_{-\tau}^{\tau} M\int_0^\infty x^{-s}\langle f(y), T(xy)\rangle dx\, y^{-s}dw,\ \varphi(y)\right\rangle \tag{6.2}$$

$$s=c+iw$$

$$=\left\langle(2\pi)^{-1}\int_{-\tau}^{\tau}\int_0^\infty \langle f(y), T(xy)x^{-s}M\rangle dx\, y^{-s}dw,\ \varphi(y)\right\rangle \tag{6.3}$$

$$=\left\langle(2\pi)^{-1}\int_{-\tau}^{\tau}\left\langle f(y),\int_0^\infty T(xy)x^{-s}M dx\right\rangle y^{-s}dw,\ \varphi(y)\right\rangle. \tag{6.4}$$

That (6.2) is equal to (6.3) is obvious by the ordinary properties of generalized functions. (6.3) and (6.4) are equal by a result [16, Corollary 5.3—2b]. By another result [4, p. 337]

$$\int_0^\infty (xy)^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}xyq}\,W_{k,m}(\varrho xy)x^{-s}M dx=$$

$$=\frac{\varrho^{m+\frac{1}{2}}M\Gamma(\sigma-s\pm m+1)\,y^{s-1}}{\left(\frac{q+\varrho}{2}\right)^{\sigma-s+m+1}\Gamma\left(\sigma-k+\frac{3}{2}-s\right)}\,{}_2F_1\times$$

$$\times\left[m-s+\sigma+1;\ m-k+\frac{1}{2},\ \sigma+\frac{3}{2}-k-s;\ \frac{q-\varrho}{q+\varrho}\right]=y^{s-1}.$$

Hence (6.4) is equal to

(6.5)

$$\left\langle (2\pi)^{-1} \int_{-\tau}^{\tau} y^{-s}\langle f(y), y^{s-1}\rangle \, dw, \varphi(y) \right\rangle = (2\pi)^{-1} \int_{-\tau}^{\tau} \langle y^{-s}\langle f(y), y^{s-1}\rangle, \varphi(y)\rangle dw =$$

$$= (2\pi)^{-1} \int_{-\tau}^{\tau} \langle f(y), y^{s-1}\rangle \langle y^{-s}, \varphi(y)\rangle dw = (2\pi)^{-1} \int_{-\tau}^{\tau} \langle f(y), \langle y^{-s}y^{s-1}, \varphi(y)\rangle\rangle dw =$$

$$= \left\langle f(y), (2\pi)^{-1} \int_{-\tau}^{\tau} \langle y^{-s}y^{s-1}, \varphi(y)\rangle dw \right\rangle =$$

$$(6.6) = \left\langle f(y), (2\pi)^{-1} \int_{-\tau}^{\tau} y^{-s} \int_{0}^{\infty} y^{s-1}\varphi(y)\, dy\, dw \right\rangle = \left\langle f(y), (2\pi)^{-1} \int_{-\tau}^{\tau} y^{-s}\tilde{\varphi}(s)\, dw \right\rangle$$

where $\tilde{\varphi}(s)$ is the Millin transform of $\varphi(y)$. The equality of steps from (6.5) to (6.6) can be established by using the standard results on the integration odf distributions and testing functions with respect to parameters [16, Sec 2.8]. As $\tau \to \infty$ $\int_{-\tau}^{\tau} y^{-s}\tilde{\varphi}(s) \to 2\pi\varphi(y)$ uniformly with respect to y [18, Theorem 4.3.3], it is therefore proved that

$$\left\langle (2\pi)^{-1} \lim_{\tau\to\infty}\text{it} \int_{c-i\tau}^{c+i\tau} M\Phi(s) y^{-s}\, ds, \varphi(y) \right\rangle = (2\pi)^{-1}\langle f(y), 2\pi\varphi(y)\rangle = \langle f(y), \varphi(y)\rangle$$

which proves (6.1)

REMARK. Some more results pertaining to the W-transform can be proved just like in the author's published papers.

Acknowledgements. Hearty thanks are extended to U.S.S.R. Academy of Sciences for their kind invitation and hospitality in Moscow. Financial help from IMUISA and U.G.C. is gratefully acknowledged. Thanks to Professor H. M. Srivastava for helpful suggestions.

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(Received July 31, 1981)

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Acta Math. Hung.
41 (1—2), (1983), 127—136.

QUASI-CENTERS, QUASI-COMMUTATORS, AND RING COMMUTATIVITY

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For an arbitrary associative ring R, define the *left quasi-center* $C_l=C_l(R)$ to be the set of all $y\in R$ with the property that for each $x\in R$ there exists an integer $n=n(x,y)$ for which $yx-nxy=0$; and define the *right quasi-center* $C_r=C_r(R)$ analogously. The first major result of this paper (Theorem 2) asserts that for rings with 1, both quasi-centers coincide with the center. This theorem suggests that quasi-commutators — that is, elements of form $yx-nxy$ for n an integer — may be used in place of ordinary commutators in studying commutativity; and in Section 3 we present some theorems showing that this is in fact true.

The motivation for attempting to prove Theorem 2 was an application to commutativity theorems, which will be reported in a separate paper [3]. The proofs of Theorems 5 and 7 contain other applications.

Various authors have recently done closely-related work [4, 8, 9, 10], some of it in the context of non-associative rings; in particular, a result of Chung and Luh [4] contains the special case of Theorem 2 in which C_l is assumed equal to R, and Theorem 4.2 of [4] may be regarded as a variant of Theorem 2 under a more restrictive definition of quasi-center. However, their proofs do not seem to yield Theorem 2.

Throughout the paper, C or $C(R)$ will denote the center of R, $\mathscr{C}(R)$ the commutator ideal of R, and R^+ the additive group of R. For $S\subseteq R$, $A_l(S)$, $A_r(S)$ and $A(S)$ will denote the left, right, and two-sided annihilators of S. For $x, y\in R$, the usual symbol $[x, y]$ will represent the commutator $xy-yx$. The symbol Z will be reserved for the integers, considered as a set or a ring according to context.

1. The left and right quasi-centers

The sets C_l and C_r are in some sense center-like, and they obviously contain C; however they may be quite different from C, as the following examples show.

EXAMPLE 1. Let p be a prime greater than 3, and let a and b be nonzero elements of $GF(p)$ with $a^2\neq b^2$. Denote by A the algebra over $GF(p)$ with basis $\{\alpha_1, \alpha_2, \alpha_3\}$ and multiplication given by $\alpha_1\alpha_2=a\alpha_3$, $\alpha_2\alpha_1=b\alpha_3$, and $\alpha_i\alpha_j=0$ for all other choices of i, j. It is readily verified that α_1 and α_2 are in $C_l(A)$. Moreover, if we choose k and j so that $ak+bj=1$ and $bk+aj=0$, as the fact that $a^2\neq b^2$ permits us to do, we have $(\alpha_1+j\alpha_2)(\alpha_1+k\alpha_2)=\alpha_3$ and $(\alpha_1+k\alpha_2)(\alpha_1+j\alpha_2)=0$, hence $\alpha_1+j\alpha_2\notin C_l(A)$. Thus, in general, $C_l(R)$ need not be a subring of R.

* Supported by the Natural Sciences and Engineering Research Council of Canada, Grant No. A 3961.

EXAMPLE 2 [10, p. 378]. For any field F of characteristic $\neq 2$, let A be the algebra over F with basis $\{\alpha_1, \alpha_2, \alpha_3\}$ and multiplication given by $\alpha_1\alpha_2=\alpha_3$, $\alpha_2\alpha_1=-\alpha_3$ and $\alpha_i\alpha_j=0$ for all other choices of i, j. Then A satisfies the identity $xy=-yx$, hence $C_l(A)=A$. Thus, even if $C_l(A)$ is a subring, it need not be commutative. In fact, it need not be anti-commutative either, as is pointed out in [8].

EXAMPLE 3. Let Z_6 denote the ring of integers mod 6 and let R be the ring with additive group $Z_6 \oplus Z_6$ and multiplication given by $(a, b)(c, d)=(ac, ad+3bc)$. It is easily verified that $C_l(A)=\{(k, b)|k=0 \text{ or } 3\}$, hence $C_l(A)$ is a commutative subring of A. Since (0,1) is clearly not central, we see that even when C_l is a commutative subring, it may be strictly larger than the center. As may be readily verified, this example also shows that in general $C_l(R)\neq C_r(R)$.

It is not clear what properties of R force $C_l(R)$ to be a subring, or in the event that $C_l(R)$ is a subring, what further properties imply that R is commutative. In the case of rings with torsion-free additive group, however, the situation appears to be slightly better, as the following easy theorem shows.

THEOREM 1. *Let R be a ring with R^+ torsion-free. If $C_l(R)$ is a subring, then $C_l(R)$ is either commutative or anti-commutative. If, moreover, $A_l(R)=\{0\}$, then $C_l(R)$ must be commutative.*

PROOF. We use repeatedly the following easy result, which we denote by EG — if H_1 and H_2 are subgroups of the group G with $H_1 \cup H_2=G$, then $H_1=G$ or $H_2=G$.

Let $y, z\in C_l(R)$; then there are $m=m(y, z)$ and $n=n(y, z)\in Z$ such that $yz=nzy$ and $zy=myz$, hence $(mn-1)yz=(mn-1)zy=0$. Either $yz=zy=0$, in which case we obviously have $[z, y]=0$, or $m=n=\pm 1$. For fixed y, let $S_1(y)=\{z\in C_l(R)|yz=zy\}$ and $S_2(y)=\{z\in C_l(R)|yz=-zy\}$. Clearly $S_1(y)$ and $S_2(y)$ are subgroups of $C_l(R)^+$ with union equal to the entire group. Thus, by EG, each $y\in C_l(R)$ satisfies (i) $[y, z]=0$ for all $z\in C_l(R)$, or (ii) $yz=-zy$ for all $z\in C_l(R)$. Letting S_1 and S_2 be the sets of all y satisfying (i) and (ii) respectively, and applying EG again, shows that $C_l(R)$ is either commutative or anti-commutative.

Assume henceforth that $A_l(R)$ is trivial, and that $C_l(R)$ is anticommutative. To show that $C_l(R)$ is commutative, we need only show that $[y, z]x=0$ for all $x\in R$ and $y, z\in C_l(R)$.

Consider any pair $y, z\in C_l(R)$ with $yz\neq 0$ and let $x\in R$ be arbitrary. Choose $n, m, k\in Z$ for which

$$yx = nxy, \quad zx = mxz, \quad (y+z)x = kx(y+z). \tag{1}$$

Denoting $(n-k)y-(k-m)z$ by w, we see from (1) that $xw=rxw=0$ for any $r\in R$. Since $w\in C_l(R)$, we now have $wx=wrx=0$, hence w belongs to the left annihilator A of the left ideal generated by x. Now $\bar{R}=R/A$ also has torsion-free additive group; and

$$(n-k)\bar{y} = (k-m)\bar{z}, \tag{2}$$

where $\bar{y}, \bar{z}$ denote the canonical images of y and z in $\bar{R}$. Of course, if either $\bar{y}$ or $\bar{z}$ is $\bar{0}$, then $[y, z]x=0$; otherwise, unless $n-k=k-m=0$, (2) yields $[\bar{y}, \bar{z}]=\bar{0}$, which forces $[y, z]x=0$.

Now assume $n=m=k$, and choose $j\in Z$ such that $(y+yz)x=jx(y+yz)$; it follows, by appeal to (1), that

$$(n-j)xy = jxyz-yzx. \tag{3}$$

The anti-commutativity of $C_l(R)$ forces $z^2=0$, and the fact that each $u\in R$ yields $t\in Z$ for which $zu=tuz$ now implies $zuz=0$ for all $u\in R$. Thus, right-multiplying (3) by z gives $(n-j)xyz=0$; and if $n\neq j$, we get $xyz=0=yzx=zyx$, so $[y,z]x=0$. The only remaining possibility is that $n=j$, in which case (3) yields $yzx=nxyz$. But (1) and the fact that $m=n$ give $yzx=nyxz=n^2xyz$, hence $(n^2-n)xyz=0$. Of course, if $xyz=0$ we get $[y,z]x=0$; otherwise $n^2-n=0$, so that $n=0$ or 1. Since $n=0$ means that $yx=0=zyx=yzx=[y,z]x$, we conclude that if $[y,z]x\neq 0$, we must have $n=m=1$.

Summarizing our results, we have that for fixed $y,z\in C_l(R)$ and $x\in R$, either (a) $[y,z]x=0$ or (b) $[x,y]=[x,z]=0$. Keep y,z fixed, and let T_1 and T_2 be the sets of all $x\in R$ satisfying (a) and (b) respectively. Another application of EG shows that $T_1=R$ or $T_2=R$. If $T_1=R$, then our hypothesis on the left-annihilator of R gives $[y,z]=0$; if $T_2=R$, then $y,z\in C$. Thus, $[y,z]=0$ for all $y,z\in C_l(R)$.

For rings R with multiplicative identity, the results on quasi-centers are the best we could hope for, as the following theorem shows.

THEOREM 2. *If R has* 1, *then* $C_l=C_r=C$.

PROOF. Since $C\subseteqq C_r$, we need only show $C_r\subseteqq C$. Accordingly, suppose that $y\in C_r\setminus C$, and consider first the case where y has infinite order in R^+. Choose x such that $[x,y]\neq 0$; then there exist integers $n,k\neq 1$ for which

$$xy = nyx, \quad (x+1)y = ky(x+1), \tag{4}$$

and hence $(k-1)y=(n-k)yx$. Letting $f=k-1$ and $g=n-k$, we have

$$fy = gyx; \tag{5}$$

and multiplying this equation by n yields

$$gxy = nfy. \tag{6}$$

Since $y\in C_r$, there exists $t\in Z$ for which $(-f+gx)y=ty(-f+gx)$; and substituting (5) and (6) into this equality gives

$$f(n-1)y = 0, \tag{7}$$

which is impossible since $f(n-1)\neq 0$. Therefore $y\in C$.

It remains to consider the case of torsion elements y, which we may assume to be of additive order p^s for some prime p. We proceed by induction on s. If $s=1$, then in (4) neither $n-1$ nor $k-1$ can be congruent to $0 \pmod p$, so we obtain a contradiction by simply repeating the above argument modulo p. Moreover, for any s, if there exists an x_0 such that $[x_0,y]\neq 0$ and $x_0y=jyx_0$ with $j\not\equiv 1 \pmod p$, then in equation (4) we may assume that $k\not\equiv 1 \pmod p$ and in equation (5) that $f\not\equiv 0 \pmod p$, so that (7) yields the contradiction $n\equiv 1 \pmod{p^s}$.

Assume as inductive hypothesis that for all $h<s$, elements of C_r of order p^h are central; and suppose $y\in C_r\setminus C$, $p^sy=0\neq p^{s-1}y$ and $xy=nyx$ with $n\equiv 1 \pmod p$.

Note that by the inductive hypothesis $py \in C$, so that $p[x, y]=0$. If $r=r(x)$ is the largest integer for which $n\equiv 1 \pmod{p^r}$, then

$$xy=(bp^r+1)yx \text{ and } [x, y]=bp^r yx \tag{8}$$

for some b prime to p, hence

$$0=p[x, y]=p^{r+1}yx. \tag{9}$$

Choose x such that $r(x)$ is the minimal member of $\{r(u) \mid [u, y]\neq 0\}$; and let $r=s-s_1$, $s_1\geqq 1$. Then using the minimality of $r(x)$, we obtain for each $i\in Z$ an integer $m\not\equiv 0 \pmod{p^{s_1}}$ such that

$$(x+ip^{s_1-1})y=(mp^r+1)y(x+ip^{s_1-1}). \tag{10}$$

Since $p^{r+1}yx=p^s y=0$, we immediately get $[x, y]=0$ if $m\equiv 0 \pmod p$; furthermore, if $m\equiv b \pmod p$ for some choice of $i\not\equiv 0 \pmod p$, then subtracting the first equation of (8) from (10) yields the contradiction $p^{s-1}y=0$. Therefore for each $i=1, 2, \ldots, p-1$, in (10) we have $m\not\equiv 0, b \pmod p$, so there exist distinct $i_1, i_2\in\{1, \ldots, p-1\}$ whose corresponding m_1 and m_2 in (10) are congruent $(\mathrm{mod}\, p)$. Subtracting the two versions of (10) yields

$$(m_2-m_1)p^r yx+(m_2 i_2-m_1 i_1)p^{s-1}y=0; \tag{11}$$

and letting $m_2=vp+m_1$ and recalling (9) gives $0=m_1(i_2-i_1)p^{s-1}y=p^{s-1}y$, a contradiction. This completes the induction, and hence the proof.

Semi-prime rings have a comparable theory of quasi-centers; note the following result, which appears in [4].

THEOREM 3. *If R is semi-prime, then* $C_l=C_r=C$.

2. Some applications of quasi-centers to commutativity theorems

Theorem 3 yields a slight improvement of a class of commutativity theorems. Let $n\geqq 1$, and let $Z[n]=Z[X_1, \ldots, X_n]$ denote the ring of polynomials with integer coefficients in n non-commuting indeterminates; let $\mathcal{Q}$ be a subset of $Z[n]$ containing no polynomials with non-zero constant term. A ring R will be called *$\mathcal{Q}$-central (left $\mathcal{Q}$-quasi-central)* if for each ordered n-tuple $(x_1, x_2, \ldots, x_n)$ of ring elements, there exists a polynomial $f\in\mathcal{Q}$ such that $f(x_1, \ldots, x_n)\in C(\in C_l)$. A number of known theorems assert that for certain $\mathcal{Q}$, any $\mathcal{Q}$-central ring must have nil commutator ideal $\mathcal{C}(R)$ — for example, if $\mathcal{Q}_0$ is the set of all positive integral powers of a single indeterminate X, a $\mathcal{Q}_0$-central ring is radical over its center, and the assertion that $\mathcal{Q}_0$-central rings have $\mathcal{C}(R)$ nil is a classic result of Herstein [5].

THEOREM 4. *Suppose $\mathcal{Q}$ is a set of polynomials with the property that every $\mathcal{Q}$-central ring has nil commutator ideal. Then every left $\mathcal{Q}$-quasi-central ring has nil commutator ideal.*

PROOF. Let R be left $\mathcal{Q}$-quasi-central, and factor out the maximal nil ideal N. Then, by Theorem 3, $\bar{R}=R/N$ is $\mathcal{Q}$-central, hence commutative.

A number of standard commutativity theorems may be formulated as statements that for certain sets $\mathcal{Q}$, all $\mathcal{Q}$-central rings are commutative — this is the case if we use $\mathcal{Q}_1=\{X-X^n|n=2, 3, \ldots\}$. While it is natural to ask whether the hypotheses can be weakened to require only that R be left $\mathcal{Q}$-quasi-central, the answer is in general negative. For example, the ring of Example 3 is left $\mathcal{Q}_1$-quasi-central but non-commutative. However, we do have the following extension of Herstein's result that periodic rings with all nilpotent elements in the center must be commutative [6].

THEOREM 5. *Let R be a periodic ring in which all nilpotent elements are in $C_l \cap C_r$. Then R is a subdirect product of a family $\{R_\alpha\}$, each member of which is either a nil ring, or a commutative ring in which every non-nilpotent element is invertible.*

PROOF. We require two basic facts about periodic rings, namely that some power of each element is idempotent and that nilpotent elements of homomorphic images of R may be lifted to R [2, Lemma 1]. The latter guarantees that the property of nilpotent elements lying in $C_l \cap C_r$ persists under taking homomorphic images.

Let R be any ring satisfying our hypotheses, and let e be any idempotent of R. Then for each $x \in R$, $ex-exe$ and $xe-exe$ are both in $C_l \cap C_r$, so that $ex-exe= =xe-exe=0$, hence idempotents are central. Now consider any subdirectly irreducible image R_α of R. Idempotents being central, any non-zero idempotent must be a multiplicative identity; hence either R_α is nil, or it has 1 and every non-nilpotent element has a power equal to 1. In the latter case, Theorem 2 guarantees that nilpotent elements are central, and R_α is commutative by Herstein's original result.

It is to be noted that $C_l \cap C_r$ cannot be replaced by C_r or C_l in the hypotheses of Theorem 5; indeed, the ring of Example 3 has the property that nilpotent elements are in C_l, but it cannot satisfy the conclusion of Theorem 5 because it has a non-central idempotent.

3. Applications of quasi-commutators to commutativity theorems

The theorems in this section indicate that quasi-commutators may play a role in the study of commutativity theorems analogous to that normally played by ordinary commutators. Theorem 6 is a generalization of the well-known result that $\mathscr{C}(R)$ is nil if R satisfies the identity $[[x, y], [z, w]]=0$, while Theorem 7, apparently a rather deep result, extends a well-known theorem of Herstein [7]. We shall make frequent use of the following lemma.

LEMMA 1 [1, Theorem 1]. *Let R be a ring satisfying an identity $q(X)=0$, where $q(X)$ is a polynomial in a finite number of non-commuting indeterminates, its coefficients being integers with highest common factor 1. If there exists no prime p for which the ring of 2×2 matrices over $GF(p)$ satisfies $q(X)=0$, then R has nil commutator ideal and the nilpotent elements of R form an ideal.*

THEOREM 6. *Let R have the property that for every quasicommutator $c=xy-$ $-jyx$ in R, and every pair z, w of elements of R, there exists a quasi-commutator $d=zw-kwz$ for which $[c, d]=0$. Then $\mathscr{C}(R)$ is nil.*

PROOF. By standard structure-theory arguments, it suffices to establish commutativity of R under the additional hypotheses that R is prime with no non-zero nil ideals. Suppose first that R has characteristic 0. Then beginning with $x, y \in R$ and $c=xy-2yx$, we are guaranteed $k \in Z$ such that $[xy-2yx, yx-kxy]=0$, a condition which reduces to $(1-2k)[xy, yx]=0$; hence R satisfies the identity

(12) $$[xy, yx] = 0.$$

On the other hand, if R has characteristic $p>0$, our hypothesis implies that R satisfies any identity of the form

(13) $$\Pi[xy-iyx, zw-jwz] = 0$$

where i, j range over the set $\{1, 2, ..., p\}$ and the factors are multiplied in arbitrary but fixed order. But, as is easily verified, there exists no prime q for which the ring $M_2(GF(q))$ satisfies either (12) or (13); thus R is commutative by Lemma 1.

I suspect that the hypotheses of Theorem 6 can be weakened to the following: given $x, y, z, w \in R$, there exist $j, k \in Z$ such that $[xy-jyx, zw-kwz]=0$.

THEOREM 7. *Let R be a ring with* 1, *and suppose that for each* $x, y \in R$, *there exist an integer* $j=j(x, y)$ *and an integer* $n=n(x, y)>1$, *such that*

(†) $$(xy-jyx)^n = xy-jyx.$$

Then R is commutative.

PROOF. We begin as is traditional, with the case of R a division ring. If for each $x, y \in R$, there exists $k=k(x, y)$ for which $xy-kyx \in C$, then R satisfies the identity $[xy, yx]=0$ and is necessarily commutative by Lemma 1. Therefore we assume there exist $x, y \in R$ and $k, n \in Z$ with $n>1$, such that $a=xy-kyx \notin C$ and $a^n=1$. Noting that $C(a)$ is a normal extension of C, and letting φ be a non-trivial automorphism of $C(a)$ over C, we apply the Skolem-Noether theorem to obtain $b \in R$ for which $\varphi(a)=bab^{-1}=a^i \neq a$. Thus, we have $b \in R$ such that

(14) $$ba = a^i b, \quad a^i \neq a.$$

Now choose $j \in Z$ and $m>1$ such that

(15) $$(ab-jba)^m = ab-jba;$$

and substitute (14) into (15), obtaining

(16) $$((a-ja^i)b)^m = (a-ja^i)b.$$

Now it follows from (14) that $ba^s=a^{is}b$ for each positive integer s; hence for any polynomial $p(X) \in XZ[X]$, there exists a polynomial $q(X)$ in $XZ[X]$ for which $bp(a)=q(a)b$. Thus from (16) we obtain a polynomial $f(X) \in XZ[X]$ such that $f(a)b^m=(a-ja^i)b$; and *provided that* $a-ja^i \neq 0$, we must have $f(a) \neq 0$ and hence $b^{m-1}=(a-ja^i)(f(a))^{-1}$. In particular, letting P be the prime subfield of R and noting that $P(a)$ is a finite extension of P, we see that b is algebraic over P.

We also wish to show b is algebraic over P if $a-ja^i=0$, in which case (14) yields

(17) $$ab = ja^i b = jba.$$

Choose $f, m \in Z$, $m>1$, such that $y^m=y$, where y denotes $(a+1)b-fb(a+1)$; then

(18) $$((j-f)ba+(1-f)b)^m = (j-f)ba+(1-f)b.$$

If $y=0$ and $(j-f)1=0$, then $(1-f)1=0$ and we get the contradiction $[a, b]=0$; on the other hand, if $y=0$ and $(j-f)1\neq 0$, we have $(j-f)a+(1-f)1=0$, which yields the contradiction $a\in C$. Therefore, we conclude $y\neq 0$. If it happens that $(j-f)1=0$, (18) then yields $((1-f)b)^m=(1-f)b$ and b is therefore algebraic over P; hence we may suppose $(b(sa+t))^m=b(sa+t)$, where $s, t\in Z$ and $s\neq 0$. Now $(sa+t)b=sab+tb=sjba+tb=b(sja+t)$; and we use this result to obtain a polynomial $p(X)\in Z[X]$, of degree $h>1$, for which $b^m p(a)=b(sa+t)$. Since $y=$ $=b(sa+t)\neq 0$, $p(a)\neq 0$ and $b^{m-1}=(sa+t)(p(a))^{-1}\in P(a)$, hence again b is algebraic over P, which is what we set out to prove. Thus, in view of (14) and the fact that $a^n=1$, in all cases the algebra $A(a, b)$ generated by a and b over P is a finite extension of P.

If P is finite, then $A(a, b)$ is commutative by Wedderburn's theorem, contradicting our assumption that $[a, b]\neq 0$; consequently we may assume P is the rational field Q. Note that in a finite extension of Q, there must be a fixed $N\in Z$ such that if u is any root of unity, then $u^N=1$.

Assume that $A(a, b)$ is non-commutative, and choose two of its elements x, y for which $xy\neq jyx$ for all $j\in Z$. Letting N be as above, consider integers $j_1, j_2, \dots$ $\dots, j_N, j_{N+1}$ such that

$$(19) \qquad (xiy-j_i\, iyx)^N = 1,\ i = 1, \dots, N+1;$$

and note that the j_i must be distinct, since $j_s=j_t$ for $s\neq t$ implies

$$(s^N-t^N)(xy-j_s yx)^N = 0,$$

contrary to the choice of x, y and the assumption that s and t are distinct positive integers. For $i=0, 1, \dots, N$, denote by $p_i(x, y)$ the sum of all terms in the expansion of $(xy-yx)^N$ having i factors of yx; then (19) reduces to the statement that the $p_i(x, y)$ satisfy the system

$$\sum_{i=0}^{N} j_k^i\, p_i(x, y) = 1/(k^N), \quad k = 1, \dots, N+1.$$

Since this system has as coefficient matrix a non-singular Vandermonde matrix, we see that $p_0(x, y)=(xy)^N$ is a rational multiple of 1, and hence

$$(20) \qquad [(xy)^N, yx] = 0.$$

The argument above applies to any $x, y\in A(a, b)$ such that $xy\neq jyx$ for all $j\in Z$, and it is obvious that (20) also holds for all other $x, y\in A(a, b)$. Thus (20) is a polynomial identity satisfied by $A(a, b)$, and a straightforward application of Lemma 1 yields commutativity of $A(a, b)$, contradicting our original property that $[a, b]\neq 0$. This completes the division ring case of Theorem 7.

As usual, to treat the semi-simple case, we need only consider the primitive case; and verifying that 2×2 total matrix rings do not satisfy our hypotheses shows that primitive rings satisfying these hypotheses must be division rings. Thus, we have R commutative if R is semi-simple; and we proceed now to the general case, *assuming without loss that R is subdirectly irreducible*. The subdirect irreducibility guarantees that 1 is the unique non-trivial central idempotent, and that R^+ has p-torsion for at most one prime p. A consequence of (†) is that whenever $ab=0$, $ba=0$ as well;

this property, in the presence of subdirect irreducibility, implies that the set D of zero divisors is an ideal (see [1, Lemma 2]).

Let $J(R)$ denote the Jacobson radical, and let $a \in J(R)$ be arbitrary. For $x \in R$, choose $j \in Z$ and $n > 1$ for which $(ax - jxa)^n = ax - jxa$. Now $(ax - jxa)^{n-1}$ is an idempotent belonging to $J(R)$, hence must be trivial; hence $ax = jxa$ and $J(R) \subseteq C$ by Theorem 2. Now $R/J(R)$ is commutative by our previous work, so $\mathscr{C}(R) \subseteq \subseteq J(R) \subseteq C$; and it follows easily that R satisfies the identity $[x, y][v, w] = 0$, hence that $(\mathscr{C}(R))^2 = 0$. The fact that $\mathscr{C}(R) \subseteq C$ also shows that for each $x \in R$ and each idempotent e, $[e, ex] = ex - exe$ and $[xe, e] = xe - exe$ are both central, hence idempotents of R are central and 1 is the unique non-zero idempotent.

Suppose now that R is non-commutative and choose $x \notin C$. Then by Theorem 2, there must exist $y \in R$ such that $xy \neq kyx$ for every $k \in Z$; and choosing $j \in Z$ and $n > 1$ such that $(xy - jyx)^n = xy - jyx$, we see that $(xy - jyx)^{n-2}$ is a non-zero idempotent, necessarily 1. Now

$$xy - jyx = (1-j)yx + [x, y], \tag{21}$$

hence computing modulo $\mathscr{C}(R)$, which we have shown to be a nilpotent ideal, we have a nilpotent element u such that

$$((1-j)yx)^{n-1} = 1 + u. \tag{22}$$

But $1 + u$ is invertible, hence so is x; therefore all non-invertible elements are central — in particular, $D \subseteq C$.

Continue with x, y, j and n as in the preceding paragraph, and let q be any prime not dividing $1-j$, such that R^+ is q-torsion-free. If we had $qxy = kqyx$ for some $k \in Z$, we would have $q(xy - kyx) = 0$, contradicting our choice of x and y. Thus there exist $f \in Z$ and $m > 1$ such that $(qxy - qfyx)^m = qxy - qfyx \neq 0$ and hence $(qxy - qfyx)^{m-1} = 1$. Of course, we may assume that $n = m$ — that is

$$(xy - jyx)^{n-1} = (qx - qfyx)^{n-1} = 1. \tag{23}$$

Consider $\bar{R} = R/D$, and for arbitrary $w \in R$, let $\bar{w}$ be the canonical image of w in $\bar{R}$. Since $\mathscr{C}(R) \subseteq D$, (23) yields

$$((1-j)^{n-1} - q^{n-1}(1-f)^{n-1})(\bar{x}\bar{y})^{n-1} = \bar{0};$$

hence, since $j \neq 1$ and q does not divide $1-j$, $\bar{R}$ must have finite characteristic, which we denote by p. It follows that $px \in D \subseteq C$ for all $x \in R$; and from (21) and the fact that $xy - jyx$ is invertible, we have $1-j$ not divisible by p. It now follows from (22) that in $\bar{R}$, $\bar{x}\bar{y} = k\bar{1}$ for some positive integers m and k, hence $\bar{x}\bar{y}$ generates a finite subring, necessarily a subfield, of $\bar{R}$. Thus, for some positive integer n, $\bar{x}\bar{y}^{p^r} = \bar{x}\bar{y}$, so that $(xy)^{p^r} - xy \in D \subseteq C$. But since $\mathscr{C}(R) \subseteq C$ and $pw \in C$ for all $w \in R$, we get $[(xy)^p, w] = p(xy)^{p-1}[xy, w] = 0$ for all $w \in R$; hence $(xy)^{p^r} \in C$ and therefore $xy \in C$, contradicting the fact that $[x, y] \neq 0$. This completes the proof of Theorem 7.

In the statement of Theorem 7, we cannot dispense with the assumption that R has 1; Example 2 makes that clear. However, we do have some measure of commutativity even if R fails to have 1.

THEOREM 8. *If R is an associative ring and if for each $x, y \in R$ there are integers j and $n, n>1$, for which $(xy-jyx)^n=xy-jyx$, then $\mathscr{C}(R)$ is nil.*

PROOF. We need only establish commutativity under the additional hypothesis that R is prime with no non-trivial nil ideals. If R is semi-simple, then R is commutative by the proof of Theorem 7; in particular, if $J(R)$ denotes the Jacobson radical, $[x, y] \in J(R)$ for all $x, y \in R$. Now $J(R)$ is itself a prime ring, and it contains no non-trivial idempotents; hence for each $x, y \in J(R)$, there exists $j \in Z$ such that $xy=jyx$ — that is $C_l(J(R))=J(R)$. Therefore, by Theorem 3, $J(R)$ is commutative and R satisfies the identity $[[x, y], [z, w]]=0$. A routine application of Lemma 1 now yields commutativity of R.

In view of the work in [9], it is natural to ask whether a ring R with 1 is necessarily commutative if for each $x, y \in R$, there exist relatively prime $j, k \in Z$ and an integer $n>1$ for which $(jxy-kyx)^n=jxy-kyx$. The following example demolishes that possibility.

EXAMPLE 4. Consider the ring R of 2×2 upper triangular matrices over $GF(2)$, which is a non-commutative ring with 1. For arbitrary $x=\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ and $y=\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$ in R, we have

$$xy = \begin{bmatrix} ad & ae+bf \\ 0 & cf \end{bmatrix} \text{ and } yx = \begin{bmatrix} ad & db+ec \\ 0 & cf \end{bmatrix}.$$

If $ad=cf=1$, then $xy=xy-2yx$ is invertible and there exists n for which $(xy-2yx)^n=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. If $ad=1$ and $cf=0$, then $xy-2yx$, being of form $\begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}$, is idempotent; and a similar argument applies if $ad=0$ and $cf=1$. If $ad=$ $=cf=0$, then either $xy=yx$ or one of xy and yx is trivial with twice the other also trivial. Thus, in all cases, there exist relatively prime $j, k \in Z$ with $(jxy-kyx)^n=$ $=jxy-kyx$.

Our final theorem shows that this difficulty can occur only if j, k are permitted to vary with x and y.

THEOREM 9. *Let R be a ring with 1; and suppose there exists a fixed pair j, k of relatively prime integers such that for each $x, y \in R$ there exists $n=(x, y)>1$ with $(jxy-kyx)^n=jxy-kyx$. Then R is commutative.*

PROOF. Making the substitutions $x=1, y=j$ and $x=k, y=1$ yields $s, t>1$ such that $(j^2-kj)^s=j^2-kj$ and $(jk-k^2)^t=jk-k^2$. Rewriting these conditions as $j^s(j-k)^s=j(j-k)$ and $(j-k)^t k^t=(j-k)k$, and invoking unique factorization in Z, we see that either $j=k=\pm 1$, or one of j, k is 0 and the other is ± 1. In both cases, R satisfies the hypotheses of Theorem 7, hence is commutative.

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(Received August 19, 1981)

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Acta Math. Hung.
41 (1—2), (1983), 137—150.

THE DISTRIBUTION OF THE CHARACTER DEGREES OF THE SYMMETRIC *p*-GROUPS

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1. Introduction. For *fixed* prime p, let P_n denote the Sylow p-subgroup of the symmetric group S_{p^n} on p^n letters. We call these groups P_n $(n=1, 2, \ldots)$ "symmetric p-groups". We have obviously

$$|P_n| = p^{1+p+\ldots+p^{n-1}}. \tag{1.1}$$

The thorough study of the symmetric p-groups was initiated by L. A. Kaloujnine [6], [7].

In this paper we shall investigate the degrees of the irreducible characters (i.e., the dimensions of the pairwise non-equivalent irreducible representations (over the complex field)) of P_n from the point of view of the statistical (or probabilistic) group theory. Since the character degrees divide the order of the group, in our case each degree is a power of p, therefore we rather consider $\log_p \chi(1)$. The probability measure will simply be the proportion of a subset of irreducible characters. So the number h_n of conjugacy classes of P_n will also play an important role. *Our main purpose is to prove that* $\log_p \chi(1)$ *shows an asymptotically normal distribution.*

In a previous paper (P. P. Pálfy and M. Szalay [10]) we investigated the distribution of the orders of the elements of P_n. This work was inspired by the celebrated series of papers [4] by P. Erdős and P. Turán on the statistical group theory dealing with the distribution of the orders of the elements of S_n and related problems. (For a simpler proof of their main distribution theorem, see J. D. Bovey [2]. J. Dénes, P. Erdős and P. Turán [3] proved an analogous distribution theorem for the alternating group A_n.)

For the dimensions of the complex irreducible representations of S_n, one has the trivial *upper* bound

$$\sqrt{n!} = \exp\left\{\frac{1}{2} n \log n - \frac{1}{2} n + O(\log n)\right\}.$$

Dealing with the value distribution of the complex irreducible characters of S_n, P. Turán [13] (see also [14]) remarked that the *maximal* dimension is

$$\exp\left\{\frac{1}{2} n \log n - \frac{1}{2} n + O(\sqrt{n})\right\} \tag{1.2}$$

owing to the relatively small class number of S_n which is $p(n)$, the number of partitions of n. M. P. Schützenberger called afterwards his attention to the interest of the question what can be said on the distribution of the dimensions. According to the first result (M. Szalay [11]), the dimensions of *almost all* complex irreducible representations of S_n are of the form

$$\exp\left\{\frac{1}{2}\, n \log n - O(n \log\log n)\right\}. \tag{1.3}$$

By means of the statistical investigation of partitions (M. Szalay and P. Turán [12]), this was improved to

$$\exp\left\{\frac{1}{2}\, n \log n - \left(\frac{1}{2} + A\right) n + O(n^{7/8} \log^4 n)\right\} \tag{1.4}$$

with a positive constant A. As P. Erdős remarked, this *cannot* be improved to

$$\exp\{g(n) + O(n^{1/2} \log^{-1} n)\}.$$

Another natural probability measure for the irreducible representations of a group G is the Plancherel measure, where the probability of the character χ is $\chi^2(1)/|G|$. Results concerning the Plancherel measure of S_n have been obtained by A. M. Veršik and S. V. Kerov [15].

In what follows we are going to investigate the *distribution* of the dimensions for the symmetric p-group P_n. As to the *maximal* dimension, we shall show that, e.g., for $p>2$ and $n>1$, it is

$$p^{1+p+\ldots+p^{n-2}}$$

(see Proposition 1). Our main purpose is to prove the following distribution theorem by the moment method. (Our random field consists of all possible choices of complex irreducible characters of P_n with equal probabilities.)

THEOREM. *There exist positive constants $\bar{\alpha}=\bar{\alpha}(p)$, $\bar{\beta}=\bar{\beta}(p)$ with*

$$1 + \frac{1}{p-1} - \frac{1}{p^{p-3}} < \bar{\alpha} < 1 + \frac{1}{p-1} \tag{1.5}$$

and

$$1 - \frac{1}{p^{p-3}} < \bar{\beta} < 1 - \frac{1}{p^{p-3}} + \frac{2}{p^{p-2}} \tag{1.6}$$

such that, for a randomly chosen complex irreducible character χ of P_n, we have

$$\lim_{n\to\infty} \operatorname{Prob}\left(\frac{\log_p \chi(1) - \bar{\alpha} p^{n-2}}{\bar{\beta} p^{(n-p+1)/2}} < x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt. \tag{1.7}$$

Another result worth mentioning here is the following formula for the class number h_n of P_n (see (3.4)—(3.5)):

$$h_n = \left[p^{\frac{\bar{\gamma}+1}{p-1}} \cdot |P_n|^{\bar{\gamma}}\right] - \begin{cases} 0 & \text{if } p>2 \\ 1 & \text{if } p=2 \end{cases}$$

where $\bar{\gamma}=\bar{\gamma}(p)$ is a constant with $0<\bar{\gamma}<1$ (cf. (3.6)).

The constants $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ are given by infinite series containing the class numbers h_n $(n=1, 2, \ldots)$ (cf. (4.6) and (4.11); (4.9) and (4.11); (3.3) and (3.5)). Fortunately h_n grows rapidly, hence the convergence is fast. Here we give the values of $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ for the smallest primes.

p	$\bar{\alpha}$	$\bar{\beta}$	$\bar{\gamma}$
2	0.904 366	0.465 601	0.428 430
3	0.960 487	0.521 875	0.463 819
5	1.211 479	0.962 249	0.603 746
7	1.166 250	0.999 588	0.714 311

For $p \geqq 11$, the approximate values are

$$\bar{\alpha} \approx 1+\frac{1}{p-1}, \quad \bar{\beta} \approx 1, \quad \bar{\gamma} \approx 1-\frac{2}{p}$$

within the accuracy of $p^{-(p-3)}$ $(<10^{-8})$.

2. The recurrence formula. Let ξ_n denote a random variable which assigns the logarithm of the degree of χ (to the base p) to a randomly chosen complex irreducible character χ of P_n, i.e., $\xi_n(\chi)=\log_p \chi(1)$, ξ_n is integer-valued. The distribution of ξ_n is reflected by the generating function

$$A_n(z) = \sum_{k=0}^{\infty} a_{nk} z^k, \tag{2.1}$$

where a_{nk} denotes the number of complex irreducible characters of P_n with $\log_p \chi(1)=k$. The sum in (2.1) is clearly finite, so $A_n(z)$ is a polynomial. Substituting $z=1$, we obtain the number of complex irreducible characters of P_n, i.e., the class number h_n of P_n,

$$A_n(1) = h_n. \tag{2.2}$$

Similarly, substituting $z=p^2$, we get the square-sum of the degrees of complex irreducible characters, i.e., the order of P_n,

$$A_n(p^2) = |P_n| = p^{\frac{p^n-1}{p-1}}. \tag{2.3}$$

We define P_0 as the trivial group and $A_0(z)=1$, $h_0=1$. Obviously P_1 is a cyclic group of order p and $A_1(z)=p$, $h_1=p$. We remind the reader of the recurrence relation

$$P_{n+1} = P_n \wr P_1 \quad (n \geqq 1) \tag{2.4}$$

where $\wr$ denotes the wreath product (see, e.g., [8], 2.30). The characters of wreath products are completely described by A. Kerber (see [8], 5.20). In our case P_1 is of prime order and this fact considerably simplifies the construction. P_{n+1} contains a normal subgroup of index p which is isomorphic to the p^{th} direct power of P_n. The complex irreducible representations of this normal subgroup are the outer tensor products of those of P_n, the corresponding characters are the direct products of complex irreducible characters of P_n. The characters that are stabilized by P_{n+1}, namely the direct *powers* of complex irreducible characters of P_n, have p distinct extensions to P_{n+1}. The *remaining* characters induce irreducible characters of P_{n+1} and such characters induce the same if they are conjugate in P_{n+1}, so always p of them induce the same character of P_{n+1}. In these two ways we obtain every complex irreducible character of P_{n+1} (cf. these considerations also with [5], p. 86, and [1], Theorem 4). Since the degree of direct product of characters is the product of the degrees and the induction from a subgroup of index p multiplies the degree by p, we obtain our basic recurrence formula

$$A_{n+1}(z) = pA_n(z^p)+\frac{z}{p}\left(A_n^p(z)-A_n(z^p)\right) = \frac{z}{p}A_n^p(z)+\left(p-\frac{z}{p}\right)A_n(z^p) \tag{2.5}$$

for $n\geqq 0$. We have $A_0(z)=1$ and $A_1(z)=p$. Hence e.g., $A_2(z)=p^2+(p^{p-1}-1)z$.

As a first and easiest application of (2.5) we determine the maximal degree of irreducible characters of P_n.

Proposition 1. *The maximal degree of complex irreducible characters of P_n is*

$$p^{\frac{p^{n-1}-1}{p-1}} \quad \textit{for } p>2,\ n\geqq 1 \textit{ and } p=2,\ n=1,2; \tag{2.6}$$

and

$$2^{3\cdot 2^{n-3}-1} \quad \textit{for } p=2,\ n\geqq 3. \tag{2.6'}$$

Proof. By definition, this maximal degree is $p^{\deg A_n(z)}$. For $\deg A_n(z)=s$, $A_n(z)=\sum_{k=0}^{s} a_{nk}z^k$, the formula (2.5) implies that $a_{n+1,k}=0$ for $k>ps+1$ and $a_{n+1,ps+1}=p^{-1}(a_{ns}^p-a_{ns})$.

For $p>2$ and $a_{ns}\geqq 2$, we get

$$a_{n+1,ps+1} = \frac{a_{ns}(a_{ns}^{p-1}-1)}{p} \geqq \frac{2(2^{p-1}-1)}{p} \geqq 2$$

(since $p/2^{p-1}-1$).

For $p=2$ and $a_{ns}\geqq 3$, we get

$$a_{n+1,2s+1} = \frac{a_{ns}(a_{ns}-1)}{2} \geqq 3.$$

In the above cases, for $N\geqq n$, we have

$$\deg A_{N+1}(z) = 1+p\deg A_N(z).$$

This gives the result by induction beginning with $A_1(z)=p$ for $p>2$ and with $A_3(z)=8+6z+6z^2$ for $p=2$.

3. The class number. Substituting $z=1$ in (2.5), we obtain a recurrence formula for the number of complex irreducible characters of P_n, i.e., for the number h_n of conjugacy classes of P_n, namely,

$$h_{n+1} = \frac{1}{p}h_n^p+\left(p-\frac{1}{p}\right)h_n. \tag{3.1}$$

We have $h_0=1$, $h_1=p$. Now, (3.1) implies that $h_2=p^{p-1}+p^2-1$ and $h_{n+1}\geqq\frac{1}{p}h_n^p$. By induction, we get $h_n\geqq p^{\frac{p-2}{p(p-1)}p^n+\frac{1}{p-1}}$ for $n\geqq 1$. To get a nontrivial lower bound for $p=2$ too, we begin with $n=2$ and obtain $h_n \geqq 2\left(\frac{5}{2}\right)^{2n-2}$ for $p=2$, $n\geqq 2$. This rapid growing of h_n (compare with $|P_n|=p^{(p^n-1)/(p-1)}$), will enable us to derive good estimations in the forthcoming calculations.

PROPOSITION 2. *For* $n\geqq 1$, *we have*

$$h_n = \left[p^{\gamma p^n+\frac{1}{p-1}}\right] \text{ if } p>2 \tag{3.2}$$

and

$$h_n = \left[p^{\gamma p^n+\frac{1}{p-1}}\right]-1 \text{ if } p=2, \tag{3.2'}$$

where $\gamma=\gamma(p)$ *is an (implicit) constant defined by*

$$\gamma = \frac{p-2}{p(p-1)}+\frac{1}{\log p}\sum_{k=2}^{\infty}\frac{1}{p^k}\log\left(1+\frac{p^2-1}{h_{k-1}^{p-1}}\right). \tag{3.3}$$

PROOF. The series defining γ is obviously convergent. Rewriting (3.1), for $k\geqq 1$ we get

$$\frac{\log h_k}{p^k} = \frac{\log h_{k-1}}{p^{k-1}}-\frac{\log p}{p^k}+\frac{1}{p^k}\log\left(1+\frac{p^2-1}{h_{k-1}^{p-1}}\right).$$

Since $h_0=1$, we have

$$\frac{\log h_n}{p^n} = \sum_{k=1}^{n}\left(-\frac{\log p}{p^k}+\frac{1}{p^k}\log\left(1+\frac{p^2-1}{h_{k-1}^{p-1}}\right)\right) =$$

$$= -\frac{\log p}{p-1}\left(1-\frac{1}{p^n}\right)+\frac{2\log p}{p}+\sum_{k=2}^{\infty}\frac{1}{p^k}\log\left(1+\frac{p^2-1}{h_{k-1}^{p-1}}\right)-$$

$$-\sum_{k=n+1}^{\infty}\frac{1}{p^k}\log\left(1+\frac{p^2-1}{h_{k-1}^{p-1}}\right) = \gamma\log p+\frac{\log p}{p-1}\cdot\frac{1}{p^n}-\sum_{k=n+1}^{\infty}\frac{1}{p^k}\log\left(1+\frac{p^2-1}{h_{k-1}^{p-1}}\right)$$

for $n\geqq 1$. Hence,

$$\log h_n = \gamma p^n\log p+\frac{\log p}{p-1}-\frac{1}{p}\log\left(1+\frac{p^2-1}{h_n^{p-1}}\right)-\sum_{k=n+2}^{\infty}\frac{1}{p^{k-n}}\log\left(1+\frac{p^2-1}{h_{k-1}^{p-1}}\right),$$

i.e.,

$$\log(ph_{n+1})^{1/p} = \gamma p^n\log p+\frac{\log p}{p-1}-\sum_{k=n+2}^{\infty}\frac{1}{p^{k-n}}\log\left(1+\frac{p^2-1}{h_{k-1}^{p-1}}\right).$$

Here,

$$0<\sum_{k=n+2}^{\infty}\frac{1}{p^{k-n}}\log\left(1+\frac{p^2-1}{h_{k-1}^{p-1}}\right)<\frac{1}{p(p-1)}\log\left(1+\frac{p^2-1}{h_{n+1}^{p-1}}\right).$$

Consequently,

$$1<\frac{p^{\gamma p^n+\frac{1}{p-1}}}{(ph_{n+1})^{1/p}}<\left(1+\frac{p^2-1}{h_{n+1}^{p-1}}\right)^{\frac{1}{p(p-1)}}\leqq\left(1+\frac{p+1}{h_{n+1}^{p-1}}\right)^{1/p}.$$

Therefore,

$$(ph_{n+1})^{1/p} < p^{\gamma p^n + \frac{1}{p-1}} < \left(ph_{n+1} + \frac{p(p+1)}{h_{n+1}^{p-2}}\right)^{1/p} \leqq \left(h_n^p + (p^2-1)h_n + \frac{p(p+1)}{h_2^{p-2}}\right)^{1/p}$$

for $n \geqq 1$.

For $p > 2$ and $n \geqq 1$, we get

$$h_n < (ph_{n+1})^{1/p} < p^{\gamma p^n + \frac{1}{p-1}} < \left(h_n^p + (p^2-1)h_n + \frac{p(p+1)}{h_2}\right)^{1/p} \leqq$$

$$\leqq (h_n^p + p^2 h_n + 1)^{1/p} \leqq (h_n^p + p h_n^{p-1} + 1)^{1/p} < h_n + 1.$$

For $p=2$ and $n \geqq 1$, we get

$$h_n + 1 < (h_n^2 + 3h_n)^{1/2} = (2h_{n+1})^{1/2} < 2^{\gamma 2^n + 1} < (h_n^2 + 3h_n + 6)^{1/2} \leqq h_n + 2.$$

COROLLARY 1. *For* $n \geqq 1$, *we have*

$$h_n = \left[p^{\gamma + \frac{1}{p-1}} \cdot |P_n|^{(p-1)\gamma}\right] - \begin{cases} 0 & \text{if } p > 2 \\ 1 & \text{if } p = 2 \end{cases} \tag{3.4}$$

where $\gamma = \gamma(p)$ *is given by* (3.3).

As to the magnitude of

$$\bar{\gamma} = (p-1)\gamma, \tag{3.5}$$

we infer from (3.3) that

$$1 - \frac{2}{p} < \bar{\gamma} < 1 - \frac{2}{p} + \frac{p-1}{\log p} \cdot \frac{1}{p(p-1)} \log\left(1 + \frac{p^2-1}{h_1^{p-1}}\right) < \tag{3.6}$$

$$< 1 - \frac{2}{p} + \frac{1}{p^{p-2}\log p}.$$

4. The expectation and the variance. As usual, the expectation $M(\xi_n)$ and the variance $D^2(\xi_n)$ can be expressed in terms of the derivatives of the generating function, namely,

$$M(\xi_n) = \frac{A_n'(1)}{h_n}, \tag{4.1}$$

$$D^2(\xi_n) = \frac{A_n''(1)}{h_n} + \frac{A_n'(1)}{h_n} - \left(\frac{A_n'(1)}{h_n}\right)^2. \tag{4.2}$$

Now our aim is to derive recurrence formulae for the expectations and the variances. Differentiating (2.5) and substituting $z=1$ (notice that $A_n(1) = h_n$), the application of (3.1) yields

$$A_{n+1}'(1) = (h_n^{p-1} + p^2 - 1)A_n'(1) + \frac{1}{p}(h_n^p - h_n) = \frac{ph_{n+1}}{h_n} A_n'(1) + (h_{n+1} - ph_n)$$

and

$$A''_{n+1}(1) = (h_n^{p-1}+p^3-p)A''_n(1)+(p-1)h_n^{p-2}(A'_n(1))^2+$$
$$+\big(2(h_n^{p-1}-1)+p^3-p^2-p+1\big)A'_n(1) =$$
$$= \left(\frac{ph_{n+1}}{h_n}+p^3-p^2-p+1\right)A''_n(1)+(p-1)h_n^p\left(\frac{A'_n(1)}{h_n}\right)^2+$$
$$+\big(2(h_n^{p-1}-1)+p^3-p^2-p+1\big)A'_n(1).$$

Now, owing to (4.1),

$$M(\xi_{n+1}) = \frac{A'_{n+1}(1)}{h_{n+1}} = p\,\frac{A'_n(1)}{h_n}+\left(1-\frac{ph_n}{h_{n+1}}\right) = pM(\xi_n)+\left(1-\frac{ph_n}{h_{n+1}}\right)$$

and, by (4.2),

$$D^2(\xi_{n+1}) = \frac{A''_{n+1}(1)}{h_{n+1}}+\frac{A'_{n+1}(1)}{h_{n+1}}-\left(\frac{A'_{n+1}(1)}{h_{n+1}}\right)^2 =$$
$$= \left(p+(p^2-1)(p-1)\frac{h_n}{h_{n+1}}\right)\frac{A''_n(1)}{h_n}+\frac{(p-1)h_n^p}{h_{n+1}}\left(\frac{A'_n(1)}{h_n}\right)^2+$$
$$+\left(\frac{2(h_n^p-h_n)}{h_{n+1}}+(p^2-1)(p-1)\frac{h_n}{h_{n+1}}\right)\frac{A'_n(1)}{h_n}+p\,\frac{A'_n(1)}{h_n}-$$
$$-p^2\left(\frac{A'_n(1)}{h_n}\right)^2-2p\left(1-\frac{ph_n}{h_{n+1}}\right)\frac{A'_n(1)}{h_n}+\left(1-\frac{ph_n}{h_{n+1}}\right)\frac{ph_n}{h_{n+1}} =$$
$$= \left(p+(p^2-1)(p-1)\frac{h_n}{h_{n+1}}\right)D^2(\xi_n)+\left(1-\frac{ph_n}{h_{n+1}}\right)\frac{ph_n}{h_{n+1}}.$$

Thus we have the recurrence formulae

$$M(\xi_{n+1}) = pM(\xi_n)+\left(1-\frac{ph_n}{h_{n+1}}\right) \tag{4.3}$$

and

$$D^2(\xi_{n+1}) = \left(p+(p^2-1)(p-1)\frac{h_n}{h_{n+1}}\right)D^2(\xi_n)+\left(1-\frac{ph_n}{h_{n+1}}\right)\frac{ph_n}{h_{n+1}} \tag{4.4}$$

for $n\geqq 1$. We notice that $M(\xi_1)=0$ and $D^2(\xi_1)=0$.

PROPOSITION 3. *For* $n\geqq 1$, *we have*

$$\alpha p^n-\frac{1}{p-1} < M(\xi_n) < \alpha p^n-\frac{1}{p-1}+\frac{2p}{h_n^{p-1}}, \tag{4.5}$$

where $\alpha=\alpha(p)$ *is a constant defined by*

$$\alpha = \frac{1}{p(p-1)}-\sum_{k=2}^{\infty}\frac{h_{k-1}}{p^{k-1}h_k}. \tag{4.6}$$

PROOF. The series defining α is obviously convergent. Rewriting (4.3), we get

$$\frac{M(\xi_k)}{p^k}=\frac{M(\xi_{k-1})}{p^{k-1}}+\frac{1}{p^k}-\frac{h_{k-1}}{p^{k-1}h_k}$$

for $k\geqq 2$. Since $M(\xi_1)=0$, we have

$$\frac{M(\xi_n)}{p^n}=\sum_{k=2}^{n}\left(\frac{1}{p^k}-\frac{h_{k-1}}{p^{k-1}h_k}\right)=\frac{1}{p-1}\left(\frac{1}{p}-\frac{1}{p^n}\right)-\sum_{k=2}^{\infty}\frac{h_{k-1}}{p^{k-1}h_k}+\sum_{k=n+1}^{\infty}\frac{h_{k-1}}{p^{k-1}h_k}$$

for $n\geqq 2$ and, trivially, for $n=1$ too. Hence,

$$M(\xi_n)=\alpha p^n-\frac{1}{p-1}+\sum_{k=n+1}^{\infty}\frac{h_{k-1}}{p^{k-1-n}h_k}$$

and here

$$0<\sum_{k=n+1}^{\infty}\frac{h_{k-1}}{p^{k-1-n}h_k}<\sum_{k=n+1}^{\infty}\frac{p}{p^{k-1-n}h_{k-1}^{p-1}}<\frac{2p}{h_n^{p-1}}.$$

COROLLARY 2. *For $n\to\infty$, we have*

$$M(\xi_n)=\alpha p^n-\frac{1}{p-1}+o(1), \tag{4.7}$$

where $\alpha=\alpha(p)$ is given by (4.6).

PROPOSITION 4. *For $n\geqq 1$, we have*

$$\beta^2 p^n-\frac{2\beta^2 p^{n+3}+2p}{h_n^{p-1}}<D^2(\xi_n)<\beta^2 p^n, \tag{4.8}$$

where $\beta^2=\beta^2(p)$ is a constant defined by

$$\beta^2=\sum_{k=2}^{\infty}\frac{h_{k-1}}{p^{k-1}h_k}\left(1-\frac{ph_{k-1}}{h_k}\right)\prod_{l=k+1}^{\infty}\left(1+\frac{(p^2-1)(p-1)h_{l-1}}{ph_l}\right). \tag{4.9}$$

PROOF. Owing to the rapid growing of h_n (cf. (3.1)—(3.3)), the infinite products in the definition of β^2 are convergent. Rewriting (4.4), we get

$$\frac{D^2(\xi_k)}{p^k}=\left(1+\frac{(p^2-1)(p-1)h_{k-1}}{ph_k}\right)\frac{D^2(\xi_{k-1})}{p^{k-1}}+\frac{1}{p^k}\left(1-\frac{ph_{k-1}}{h_k}\right)\frac{ph_{k-1}}{h_k}$$

for $k\geqq 2$. Since $D^2(\xi_1)=0$, we have

$$\frac{D^2(\xi_n)}{p^n}\prod_{l=n+1}^{\infty}\left(1+\frac{(p^2-1)(p-1)h_{l-1}}{ph_l}\right)=$$

$$=\sum_{k=2}^{n}\frac{h_{k-1}}{p^{k-1}h_k}\left(1-\frac{ph_{k-1}}{h_k}\right)\prod_{l=k+1}^{\infty}\left(1+\frac{(p^2-1)(p-1)h_{l-1}}{ph_l}\right)=$$

$$=\beta^2-\sum_{k=n+1}^{\infty}\frac{h_{k-1}}{p^{k-1}h_k}\left(1-\frac{ph_{k-1}}{h_k}\right)\prod_{l=k+1}^{\infty}\left(1+\frac{(p^2-1)(p-1)h_{l-1}}{ph_l}\right)$$

for $n \geqq 2$ and, trivially, for $n=1$ too. Since

$$h_k = \frac{1}{p} h_{k-1}^p + \left(p - \frac{1}{p}\right) h_{k-1} \geqq p h_{k-1},$$

we get

$$\beta^2 p^n > D^2(\xi_n) = \beta^2 p^n \prod_{l=n+1}^{\infty} \left(1 + \frac{(p^2-1)(p-1)h_{l-1}}{p h_l}\right)^{-1} -$$

$$- \sum_{k=n+1}^{\infty} \frac{h_{k-1}}{p^{k-1-n} h_k} \left(1 - \frac{p h_{k-1}}{h_k}\right) \prod_{l=n+1}^{k} \left(1 + \frac{(p^2-1)(p-1)h_{l-1}}{p h_l}\right)^{-1} >$$

$$> \beta^2 p^n \exp\left\{-p^3 \sum_{l=n+1}^{\infty} h_{l-1}^{-(p-1)}\right\} - \sum_{k=n+1}^{\infty} \frac{h_{k-1}}{p^{k-1-n} h_k} >$$

$$> \beta^2 p^n \exp\left\{-p^3 \sum_{l=n+1}^{\infty} (p^{l-1-n} h_n)^{-(p-1)}\right\} - 2p h_n^{-(p-1)} \geqq$$

$$\geqq \beta^2 p^n \exp\{-2p^3 h_n^{-(p-1)}\} - 2p h_n^{-(p-1)} > \beta^2 p^n - \frac{2\beta^2 p^{n+3} + 2p}{h_n^{p-1}}.$$

Taking into account (3.2) and (3.2′) we obtain

COROLLARY 3. *For* $n \to \infty$, *we have*

$$D^2(\xi_n) = \beta^2 p^n - o(1), \tag{4.10}$$

where $\beta^2 = \beta^2(p)$ *is given by* (4.9).

In order to get a better imagination we multiply α and β by suitable powers of p (because $\alpha, \beta \to 0$ if $p \to \infty$). Let

$$\bar{\alpha} = p^2 \alpha, \quad \bar{\beta} = p^{\frac{p-1}{2}} \beta. \tag{4.11}$$

PROPOSITION 5. *With notation* (4.11), *we have*

$$1 + \frac{1}{p-1} - \frac{1}{p^{p-3}} < \bar{\alpha} < 1 + \frac{1}{p-1} \tag{4.12}$$

and

$$1 - \frac{1}{p^{p-3}} < \bar{\beta} < 1 - \frac{1}{p^{p-3}} + \frac{2}{p^{p-2}}. \tag{4.13}$$

PROOF. Multiplying (4.6) by p^2 we get

$$\bar{\alpha} = 1 + \frac{1}{p-1} - \sum_{k=2}^{\infty} \frac{h_{k-1}}{p^{k-3} h_k}.$$

Here,

$$0 < \sum_{k=2}^{\infty} \frac{h_{k-1}}{p^{k-3} h_k} < \frac{p^2}{h_2} + \sum_{k=3}^{\infty} \frac{p}{p^{k-3} h_{k-1}^{p-1}} <$$

$$< \frac{p^2}{h_2} + \frac{p^2}{(p-1) h_2^{p-1}} \leqq \frac{p^2}{h_2} \left(1 + \frac{1}{h_2^{p-2}}\right).$$

For $p=2$,

$$\frac{2^2}{h_2}(1+1)=\frac{8}{5}<\frac{1}{2^{2-3}}.$$

For $p\geqq 3$,

$$\frac{p^2}{h_2}\left(1+\frac{1}{h_2^{p-2}}\right)\leqq\frac{p^2}{h_2}\left(1+\frac{1}{h_2}\right)\leqq\frac{p^2}{p^{p-1}+p^2-1}\left(1+\frac{1}{p^{p-1}}\right)<\frac{1}{p^{p-3}}.$$

These yield the estimates in (4.12).

Multiplying (4.9) by p^{p-1} we get

$$\bar{\beta}^2=p^{p-1}\sum_{k=2}^{\infty}\frac{h_{k-1}}{p^{k-1}h_k}\left(1-\frac{ph_{k-1}}{h_k}\right)\prod_{l=k+1}^{\infty}\left(1+\frac{(p^2-1)(p-1)h_{l-1}}{ph_l}\right).$$

For $p=2$, $\bar{\beta}>0>1-\frac{1}{2^{2-3}}$. For $p\geqq 3$, we get

$$\bar{\beta}^2>p^{p-1}\frac{h_1}{ph_2}\left(1-\frac{ph_1}{h_2}\right)=\frac{p^{p-1}(p^{p-1}-1)}{h_2^2}>$$

$$>\left(\frac{(p^{p-1}+p^2-1)(1-p^{-p+3})}{h_2}\right)^2=\left(1-\frac{1}{p^{p-3}}\right)^2.$$

These yield the lower estimate in (4.13). We turn to the upper estimation of $\bar{\beta}$. For $k\geqq 3$,

$$\frac{h_k}{h_{k+1}}\left(\frac{h_{k-1}}{h_k}\right)^{-1}=\frac{h_k^2}{h_{k+1}h_{k-1}}<\frac{h_k^2}{\frac{1}{p}h_k^p h_{k-1}}\leqq\frac{p}{h_{k-1}}\leqq\frac{p}{h_2}<\frac{1}{p}.$$

Thus, for $j\geqq 3$, $\frac{h_{j-1}}{h_j}\leqq\frac{h_2}{p^{j-3}h_3}$. Therefore,

$$\bar{\beta}^2<p^{p-1}\left\{\prod_{l=3}^{\infty}\exp\left(\frac{(p^2-1)(p-1)h_{l-1}}{ph_l}\right)\right\}\left\{\frac{h_1}{ph_2}\left(1-\frac{ph_1}{h_2}\right)+\sum_{k=3}^{\infty}\frac{h_{k-1}}{p^{k-1}h_k}\right\}\leqq$$

$$\leqq p^{p-1}\exp\left(\frac{(p^2-1)(p-1)}{p}\sum_{l=3}^{\infty}\frac{h_2}{p^{l-3}h_3}\right)\left\{\frac{h_2-ph_1}{h_2^2}+\sum_{k=3}^{\infty}\frac{h_2}{p^{2k-4}h_3}\right\}=$$

$$=p^{p-1}\exp\left((p^2-1)\frac{h_2}{h_3}\right)\left\{\frac{p^{p-1}-1}{h_2^2}+\frac{h_2}{(p^2-1)h_3}\right\}.$$

For $p=2$, this is

$$2\exp\left(3\cdot\frac{5}{20}\right)\left\{\frac{1}{5^2}+\frac{5}{3\cdot 20}\right\}<1=\left(1-\frac{1}{2^{2-3}}+\frac{2}{2^{2-2}}\right)^2.$$

For $p=3$,

$$3^2\exp\left(8\cdot\frac{17}{17\cdot 99}\right)\left\{\frac{8}{17^2}+\frac{17}{8\cdot 17\cdot 99}\right\}<\frac{4}{9}=\left(1-\frac{1}{3^{3-3}}+\frac{2}{3^{3-2}}\right)^2.$$

For $p \geqq 5$,

$$\frac{h_3}{h_3} < \frac{p}{h_2^{p-1}} < \frac{1}{p^7 h_2^2},$$

$$\exp\left((p^2-1)\frac{h_2}{h_3}\right) < \exp\left(\frac{1}{p^5 h_2^2}\right) < 1+\frac{1}{p^5 h_2^2}\exp\left(\frac{1}{p^5 h_2^2}\right) < 1+\frac{1}{p^p},$$

$$\bar{\beta}^2 < p^{p-1}\left(1+\frac{1}{p^p}\right)\left\{\frac{p^{p-1}-1}{h_2^2}+\frac{1}{p^8 h_2^2}\right\} <$$

$$< \frac{p^{p-1}(p^{p-1}-1)}{h_2^2}+\frac{p^{p-1}}{p^8 h_2^2}+\frac{(p^{p-1})^2}{p^p h_2^2} < \left(\frac{p^{p-1}}{h_2}\right)^2,$$

$$\bar{\beta} < \frac{p^{p-1}}{h_2} = 1-\frac{p^2-1}{p^{p-1}}+\left(\frac{p^2-1}{p^{p-1}}\right)^2 \frac{1}{1+\dfrac{p^2-1}{p^{p-1}}} <$$

$$< 1-\frac{1}{p^{p-3}}+\frac{1}{p^{p-1}}+\frac{p^4}{p^{2p-2}} < 1-\frac{1}{p^{p-3}}+\frac{2}{p^{p-2}}.$$

5. The higher moments. Now we are going to investigate the random variables

$$\eta_n = \frac{\xi_n - \alpha p^n + \dfrac{1}{p-1}}{\beta p^{n/2}} \quad (n = 1, 2, \ldots). \tag{5.1}$$

(η_n is, within a small error, the standardized form of ξ_n.) Observe that η_n takes on only finitely many values, hence every moment $M(\eta_n^k)$ exists and is finite.

PROPOSITION 6. *For every* $k=1, 2, \ldots,$ *we have*

$$\lim_{n\to\infty} M(\eta_n^k) = \begin{cases} 1\cdot 3\cdot \ldots \cdot (k-1) & \textit{if } k \textit{ even,} \\ 0 & \textit{if } k \textit{ odd;} \end{cases} \tag{5.2}$$

i.e., the k-th moment of the standard normal distribution.

PROOF. For $k=1, 2$ the assertion follows from Corollaries 2 and 3. For $k \geqq 3$ we proceed by induction on k. The moments of η_n appear as coefficients in the characteristic function $\psi_n(t)$ of η_n, namely, for real t,

$$\psi_n(t) = M(e^{\eta_n it}) = \sum_{k=0}^{\infty} \frac{M(\eta_n^k)}{k!}(it)^k.$$

The definition (5.1) of η_n yields that

$$\psi_n(t) = \exp\left(\frac{-\alpha p^n + \dfrac{1}{p-1}}{\beta p^{n/2}}\, it\right)\frac{1}{h_n} A_n(e^{it/\beta p^{n/2}}).$$

Now the recurrence formula (2.5) implies that

$$\psi_{n+1}(t)=\exp\left(\frac{-\alpha p^{n+1}+\dfrac{1}{p-1}}{\beta p^{(n+1)/2}}\,it\right)\frac{1}{h_{n+1}}\left\{\frac{1}{p}\,e^{it/\beta p^{(n+1)/2}}A_n^p\big(e^{it/\beta p^{(n+1)/2}}\big)+\right.$$

$$\left.+\left(p-\frac{1}{p}\,e^{it/\beta p^{(n+1)/2}}\right)A_n\big(e^{pit/\beta p^{(n+1)/2}}\big)\right\}=$$

$$=\frac{1}{h_{n+1}}\exp\left(\frac{-\alpha p^{n+1}+\dfrac{1}{p-1}+1}{\beta p^{(n+1)/2}}\,it\right)\left\{\frac{h_n^p}{p}\exp\left(\frac{\alpha p^n-\dfrac{1}{p-1}}{\beta p^{n/2}}\cdot\frac{pit}{\sqrt{p}}\right)\psi_n^p\left(\frac{t}{\sqrt{p}}\right)+\right.$$

$$\left.+\left(pe^{-it/\beta p^{(n+1)/2}}-\frac{1}{p}\right)h_n\exp\left(\frac{\alpha p^n-\dfrac{1}{p-1}}{\beta p^{n/2}}\,i\sqrt{p}\,t\right)\psi_n(\sqrt{p}\,t)\right\}=$$

$$=\frac{1}{h_{n+1}}\left\{\frac{h_n^p}{p}\,\psi_n^p\left(\frac{t}{\sqrt{p}}\right)+\left(pe^{-it/\beta p^{(n+1)/2}}-\frac{1}{p}\right)h_n\psi_n(\sqrt{p}\,t)\right\}.$$

Finally,

$$\text{(5.3)}\quad \psi_{n+1}(t)=\left(1-\frac{h_n}{h_{n+1}}\left(p-\frac{1}{p}\right)\right)\psi_n^p\left(\frac{t}{\sqrt{p}}\right)+\frac{h_n}{h_{n+1}}\left(pe^{-it/\beta p^{(n+1)/2}}-\frac{1}{p}\right)\psi_n(\sqrt{p}\,t).$$

This formula involves a recurrence for the moments of η_{n+1}. The coefficient of $(it)^k/k!$ in $\psi_n^p(t/\sqrt{p})$ is a polynomial in $M(\eta_n^0),\ \dots,\ M(\eta_n^k)$, more exactly, it is

$$F_k\big(M(\eta_n^0),\ \dots,\ M(\eta_n^{k-1})\big)+p\,\frac{M(\eta_n^k)}{(\sqrt{p})^k},$$

where F_k is a polynomial. Denoting the moments of the standard normal distribution by m_j $(j=0,1,\dots)$, the equation

$$\big(\exp(-(t/\sqrt{p})^2/2)\big)^p=\exp(-t^2/2)$$

(the characteristic function of the standard normal distribution) implies that F_k satisfies the equation

$$\text{(5.4)}\qquad F_k(m_0,\ \dots,\ m_{k-1})+p\,\frac{m_k}{(\sqrt{p})^k}=m_k.$$

Now, we infer from (5.3) that

$$M(\eta_{n+1}^k)=\left(1-\frac{h_n}{h_{n+1}}\left(p-\frac{1}{p}\right)\right)\big(F_k(M(\eta_n^0),\ \dots,\ M(\eta_n^{k-1}))+p^{1-k/2}M(\eta_n^k)\big)+$$

$$+\frac{h_n}{h_{n+1}}\left(\sum_{j=0}^{k}\binom{k}{j}\frac{p^{1+j/2}M(\eta_n^j)}{(-\beta p^{(n+1)/2})^{k-j}}-p^{-1+k/2}M(\eta_n^k)\right).$$

Consequently,

$$M(\eta_{n+1}^k)-m_k = \left(1-\frac{h_n}{h_{n+1}}\left(p-\frac{1}{p}\right)\right)\left(F_k(M(\eta_n^0), \ldots, M(\eta_n^{k-1}))-F_k(m_0, \ldots, m_{k-1})\right)+$$

$$+\left(1-\frac{h_n}{h_{n+1}}\left(p-\frac{1}{p}\right)\right)p^{1-k/2}\,(M(\eta_n^k)-m_k)+\frac{h_n}{h_{n+1}}\left(p-\frac{1}{p}\right)p^{k/2}(M(\eta_n^k)-m_k)+$$

$$+\frac{h_n}{h_{n+1}}\left(p-\frac{1}{p}\right)(-1+p^{k/2})m_k+\frac{h_n}{h_{n+1}}\sum_{j=0}^{k-1}\binom{k}{j}\frac{p^{1+j/2}M(\eta_n^j)}{(-\beta p^{(n+1)/2})^{k-j}}.$$

By the induction hypothesis, $\lim_{n\to\infty} M(\eta_n^j) = m_j$ for $j = 0, \ldots, k-1$. Thus the moments $M(\eta_n^0), \ldots, M(\eta_n^{k-1})$ are bounded and

$$\lim_{n\to\infty} F_k(M(\eta_n^0), \ldots, M(\eta_n^{k-1})) = F_k(m_0, \ldots, m_{k-1}).$$

Using also the relation

$$\frac{h_n}{h_{n+1}} = o(1) \tag{5.5}$$

we get

$$M(\eta_{n+1}^k)-m_k = \left\{\left(1-\frac{h_n}{h_{n+1}}\left(p-\frac{1}{p}\right)\right)p^{1-k/2}+\frac{h_n}{h_{n+1}}\left(p-\frac{1}{p}\right)p^{k/2}\right\}(M(\eta_n^k)-m_k)+o(1).$$

Hence, for every $\varepsilon>0$, there exists an $n_1=n_1(\varepsilon, k)$ such that the relation

$$|M(\eta_{n+1}^k)-m_k| \leqq \left(p^{1-k/2}+\frac{h_n}{h_{n+1}}\,p^{1+k/2}\right)|M(\eta_n^k)-m_k|+\varepsilon$$

holds for $n\geqq n_1$. Let $\sigma=\sqrt{\frac{3}{2p}}$. Since $k\geqq 3$ and $p\geqq 2$, we have $p^{1-k/2}<\sigma<1$. Using again (5.5) we get $|M(\eta_{n+1}^k)-m_k|\leqq\sigma|M(\eta_n^k)-m_k|+\varepsilon$ for $n\geqq n_2(\geqq n_1)$. Consequently,

$$\limsup_{n\to\infty} |M(\eta_n^k)-m_k| \leqq \frac{\varepsilon}{1-\sigma},$$

and this holds for every $\varepsilon>0$, therefore $\lim_{n\to\infty} M(\eta_n^k)=m_k$.

6. The limit distribution. In the simplest special case the celebrated Moment Convergence Theorem (see, e.g., [9], p. 185) states that if the k^{th} moments (for every $k=1, 2, \ldots$) of a sequence η_n of random variables converge to the corresponding moment of a normal distribution, then the distribution functions of η_n converge to the distribution function of the given normal distribution. Hence in our case Proposition 6 implies that the distribution functions of η_n converge to

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt.$$

Translating it for ξ_n we obtain

$$\lim_{n\to\infty} \operatorname{Prob}\left(\xi_n < \alpha p^n - \frac{1}{p-1} + x\beta p^{n/2}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt.$$

Obviously, the term $-(p-1)^{-1}$ does not play a significant role, so we can get rid of it without altering the limit relation. What we obtain is exactly the statement of our Theorem.

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(Received August 31, 1981)

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Acta Math. Hung.
41 (1—2), (1983), 151—160.

ÜBER DIE CESÀROSCHE SUMMIERBARKEIT DER FUNKTIONENREIHEN

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1. Es sei $\lambda=\{\lambda_k\}_0^\infty$ eine nichtabnehmende, von unten konkave Folge von positiven Zahlen mit $\lambda_k\to\infty$ $(k\to\infty)$. Ohne Beschränkung der Allgemeinheit können wir $\lambda_1=1$ voraussetzen. Für jede nicht-negative ganze Zahl s sei Z_s die Menge der natürlichen Zahlen k, für die $2^s<\lambda_k\leqq 2^{s+1}$ erfüllt ist. Auf Grund der Voraussetzungen gilt $Z_s\neq\emptyset$ $(s=0, 1, \ldots)$. Es sei $Z_s=\{\nu(s)+1, \ldots, \nu(s+1)\}$. Aus der Konkavität folgt

$$\frac{\nu(s+1)}{\nu(s)}\geqq\frac{\lambda\nu(s+1)}{\lambda\nu(s)}\geqq\frac{2^{s+1}-1}{2^s}=2-\frac{1}{2^s}\geqq q(>1)\quad (s=1,2,\ldots). \tag{1}$$

Für eine reelle Zahlenfolge $a=\{a_k\}_0^\infty$ setzen wir

$$A_s^2=\sum_{k\in Z_s}a_k^2\lambda_k\quad (s=0,1,\ldots).$$

Es seien (X, A, μ) ein Maßraum mit $\mu(X)<\infty$ und $\varphi=\{\varphi_k(x)\}_0^\infty$ ein System der Funktionen $\varphi_k(x)\in L(X, A, \mu)$ $(k=0, 1, \ldots)$, weiterhin sei

$$L_n^1(\varphi;\, x)=\int_X\left|\sum_{k=0}^n\left(1-\frac{k}{n+1}\right)\varphi_k(x)\varphi_k(t)\right|d\mu(t)\quad (n=0,1,\ldots).$$

Wir werden erstens den folgenden Satz beweisen.

SATZ I. *Gelten*

$$L_n^1(\varphi;\, x)=O(\lambda_n)\quad (x\in X;\ n=0,1,\ldots)$$

und

$$\sum_{s=0}^\infty A_s<\infty,$$

so ist die Reihe

$$\sum_{k=0}^\infty a_k\varphi_k(x)$$

in X fast überall $(C, 1)$*-nummierbar.*

Einen ähnlichen Satz hat der Verfasser vorher für die Konvergenz bewiesen (*Acta Sci. Math. Szeged,* **42** (1980), 175—182).

Ähnliche Sätze können auch für andere Summationsmethoden, z. B. für die Rieszsche Methode bewiesen werden.

2. Zum Beweis des Satzes I benötigen wir gewisse Hilfssätze.

HILFSSATZ I. *Unter der Bedingung* $L_n^1(\varphi; x) = O(\lambda_n)$ $(x \in X;\ n = 0, 1, \dots)$ *gilt*

$$\int_X \left| \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) b_k \varphi_k(x) \right| d\mu = O\left(\left\{ \sum_{k=0}^{n} b_k^2 \right\}^{1/2} \lambda_{\nu(s)+1}^{\frac{1}{2}} \right)$$

für beliebige Folge $\{b_k\}_0^n$ *mit* $n \in Z_s$ $(s = 0, 1, \dots)$.

BEWEIS DES HILFSSATZES I. Es sein $n \in Z_s$ und

$$E_n^+ \left\{ x \in X : \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) b_k \varphi_k(x) > 0 \right\},\ E_n^- = \left\{ x \in X : \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) b_k \varphi_k(x) < 0 \right\}.$$

Mit den Rademacherschen Funktionen $r_n(t) = \operatorname{sign} \sin 2^n \pi t$ gilt

$$\int_{E_n^+} \left(\sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) b_k \varphi_k(x) \right) d\mu = \tag{2}$$

$$= \int_{E_n^+} \left(\int_0^1 \left(\sum_{k=0}^{n} b_k r_k(t) \right) \left(\sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) r_k(t) \varphi_k(x) \right) dt \right) d\mu(x) =$$

$$= \int_0^1 \left(\sum_{k=0}^{n} b_k r_k(t) \right) \left(\int_{E_n^+} \left(\sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) r_k(t) \varphi_k(x) \right) d\mu(x) \right) dt \leqq$$

$$\leqq \left\{ \int_0^1 \left(\sum_{k=0}^{n} b_k r_k(t) \right)^2 dt \right\}^{1/2} \left\{ \int_0^1 \left(\int_{E_n^+} \left(\sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) r_k(t) \varphi_k(x) \right) d\mu(x) \right)^2 dt \right\}^{1/2} =$$

$$= \left\{ \sum_{k=0}^{n} b_k^2 \right\}^{1/2} \left\{ \int_0^1 \left(\int_{E_n^+} \left(\sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) r_k(t) \varphi_k(x) \right) d\mu(x) \right) \times \right.$$

$$\left. \times \left(\int_{E_n^+} \left(\sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) r_k(t) \varphi_k(y) \right) d\mu(y) \right) dt \right\}^{1/2} = \left\{ \sum_{k=0}^{n} b_k^2 \right\}^{1/2} \times$$

$$\times \left\{ \int_{E_n^+} \int_{E_n^+} \left(\int_0^1 \left(\sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) r_k(t) \varphi_k(x) \right) \left(\sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) r_k(t) \varphi_k(y) \right) dt \right) d\mu(x) d\mu(y) \right\}^{1/2} =$$

$$= \left\{ \sum_{k=0}^{n} b_k^2 \right\}^{1/2} \left\{ \int_{E_n^+} \int_{E_n^+} \left(\sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right)^2 \varphi_k(x) \varphi_k(y) \right) d\mu(x) d\mu(y) \right\}^{1/2} \leqq$$

$$\leqq \left\{ \sum_{k=0}^{n} b_k^2 \right\}^{1/2} \left\{ \int_X \left(\int_X \left| \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right)^2 \varphi_k(x) \varphi_k(y) \right| d\mu(x) \right) d\mu(y) \right\}^{1/2}.$$

Durch zweimahlige Abelsche Umformung erhalten wir

$$\sum_{k=0}^{n}\left(1-\frac{k}{n+1}\right)^2 \varphi_k(x)\varphi_k(y) = \frac{2}{(n+1)^2}\sum_{k=0}^{n}(k+1)K_k^1(x,y)+\frac{1}{n+1}K_n^1(x,y),$$

worbei

$$K_k^1(x,y) = \sum_{l=0}^{k}\left(1-\frac{l}{k+1}\right)\varphi_l(x)\varphi_l(y)$$

bezeichnet. Daraus, folgt auf Grund der Voraussetzung

$$\int_X \left|\sum_{k=0}^{n}\left(1-\frac{k}{n+1}\right)^2 \varphi_k(x)\varphi_k(y)\right| d\mu(x) = O(\lambda_n) = O(\lambda_{\nu(s)+1}) \quad (y\in X)$$

in Falle $n\in Z_s$. Daraus und aus (2) erhalten wir

$$\int_{E_n^+}\left|\sum_{k=0}^{n}\left(1-\frac{k}{n+1}\right)b_k\varphi_k(x)\right|d\mu = O\left(\left\{\sum_{k=0}^{n} b_k^2\right\}^{1/2}\lambda_{\nu(s)+1}^{1/2}\right). \tag{3}$$

Durch Anwendung dieser Abschätzung mit dem System $\{-\varphi_k(x)\}_0^\infty$ folgt

$$\int_{E_n^-}\left|\sum_{k=0}^{n}\left(1-\frac{k}{n+1}\right)b_k\varphi_k(x)\right|d\mu = O\left(\left\{\sum_{k=0}^{n} b_k^2\right\}^{1/2}\lambda_{\nu(s)+1}^{1/2}\right).$$

Daraus and aus (3) ergibt sich die Behauptung des Hilfssatzes I.

HILFSSATZ II. *Unter der Bedingung* $L_n^1(\varphi;\, x)=O(\lambda_n)$ $(x\in X;\ n=0, 1, \dots)$ *gilt*

$$\int_X\left|\sum_{k=0}^{n} b_k\varphi_k(x)\right|d\mu = O\left(\left\{\sum_{k=0}^{n} b_k^2\right\}^{1/2}\lambda_{\nu(s)+1}^{1/2}\right)$$

für jede Folge $\{b_k\}_0^n$ *mit* $n\in Z_s$ $(s=1, 2, \dots)$.

BEWEIS DES HILFSSATZES II. Es sei $n\in Z_s$ $(s\geqq 1)$. Wir betrachten die Folge

$$c_k = \begin{cases} b_k, & k = 0, \dots, n,\\ 0, & k = n+1, \dots, \nu(s+2).\end{cases}$$

Da

$$\sum_{k=0}^{\nu(s+2)}\left(1-\frac{k}{\nu(s+2)+1}\right)c_k\varphi_k(x) = \sum_{k=0}^{n}\left(1-\frac{k}{\nu(s+2)+1}\right)b_k\varphi_k(x) =$$

$$=\frac{1}{\nu(s+2)+1}\sum_{k=0}^{n-1}\left(\sum_{l=0}^{k} b_l\varphi_l(x)\right)+\left(1-\frac{n}{\nu(s+2)+1}\right)\sum_{k=0}^{n} b_k\varphi_k(x)$$

gilt, besteht die Gleichung

$$\text{(4)}\qquad \left(1-\frac{n}{\nu(s+2)+1}\right)\sum_{k=0}^{n} b_k\varphi_k(x) = \sum_{k=0}^{\nu(s+2)}\left(1-\frac{k}{\nu(s+2)+1}\right)c_k\varphi_k(x)-$$

$$-\frac{1}{\nu(s+2)+1}\sum_{k=0}^{n-1}\left(\sum_{l=0}^{k} b_l\varphi_l(x)\right).$$

Durch Anwendung des Hilfssatzes I auf die Folge $\{c_k\}_0^{\nu(s+2)}$ erhalten wir

$$\text{(5)}\qquad \int_X\left|\sum_{k=0}^{\nu(s+2)}\left(1-\frac{k}{\nu(s+2)+1}\right)c_k\varphi_k(x)\right|d\mu = O\left(\left\{\sum_{k=0}^{\nu(s+2)} c_k^2\right\}^{1/2}\lambda_{\nu(s+1)+1}^{1/2}\right) =$$

$$= O\left(\left\{\sum_{k=0}^{n} b_k^2\right\}^{1/2}\lambda_{\nu(s)+1}^{1/2}\right).$$

Weiterhin erhalten wir durch Anwendung des Hilfssatzes I auf die Folge $\{b_k\}_0^{n-1}$

$$\text{(6)}\qquad \int_X\left|\frac{1}{n}\sum_{k=0}^{n-1}\left(\sum_{l=0}^{k} b_l\varphi_l(x)\right)\right|d\mu = O\left(\left\{\sum_{k=0}^{n-1} b_k^2\right\}^{1/2}\lambda_{\nu(s)+1}^{1/2}\right) =$$

$$= O\left(\left\{\sum_{k=0}^{n} b_k^2\right\}^{1/2}\lambda_{\nu(s)+1}^{1/2}\right).$$

Aus (4), (5) und (6) folgt

$$\text{(7)}\qquad \int_X\left(1-\frac{n}{\nu(s+2)+1}\right)\left|\sum_{k=0}^{n} b_k\varphi_k(x)\right|d\mu = O\left(\left\{\sum_{k=0}^{n} b_k^2\right\}^{1/2}\lambda_{\nu(s)+1}^{1/2}\right).$$

Im Falle $n\in Z_s$ gilt auf Grund von (1)

$$1-\frac{n}{\nu(s+2)+1} \geqq 1-\frac{\nu(s+1)}{\nu(s+2)} \geqq 1-\frac{1}{q}\;(>0).$$

Daraus und aus (7) ergibt sich die Behauptung des Hilfssatzes II.

Hilfssatz III. *Unter der Bedingung* $L_n^1(\varphi;\, x)=O(\lambda_n)$ $(x\in X;\, n=0, 1, \dots)$ *gilt*

$$\int_X\left|\sum_{k=\nu(s)+1}^{n(x)}\left(1-\frac{k}{n(x)+1}\right)a_k\varphi_k(x)\right|d\mu = O(A_s)$$

für beliebige meßbare Funktion $n(x)$ *mit ganzzahligen Werten,* $\nu(s)<n(x)\leqq\nu(s+1)$ $(s=0, 1, \dots)$.

Beweis des Hilfssatzes III. Es seien

$$E_s^+ = \left\{x\in X\colon \sum_{k=\nu(s)+1}^{n(x)}\left(1-\frac{k}{n(x)+1}\right)a_k\varphi_k(x) > 0\right\},$$

$$E_s^- = \left\{x\in X\colon \sum_{k=\nu(s)+1}^{n(x)}\left(1-\frac{k}{n(x)+1}\right)a_k\varphi_k(x) < 0\right\}.$$

Mit der im Beweis des Hilfssatzes I angewandten Methode ergibt sich

$$\int_{E_s^+}\left(\sum_{k=0}^{n(x)}\left(1-\frac{k}{n(x)+1}\right)a_k\varphi_k(x)\right)d\mu = \tag{8}$$

$$=\int_{E_s^+}\left(\int_0^1\left(\sum_{k=\nu(s)+1}^{\nu(s+1)}a_kr_k(t)\right)\left(\sum_{k=0}^{n(x)}\left(1-\frac{k}{n(x)+1}\right)r_k(t)\varphi_k(x)\right)dt\right)d\mu(x)=$$

$$=\int_0^1\left(\sum_{k=\nu(s)+1}^{\nu(s+1)}a_kr_k(t)\right)\left(\int_{E_s^+}\left(\sum_{k=0}^{n(x)}\left(1-\frac{k}{n(x)+1}\right)r_k(t)\varphi_k(x)\right)d\mu(x)\right)dt \leqq$$

$$\leqq\left\{\int_0^1\left(\sum_{k=\nu(s)+1}^{\nu(s+1)}a_kr_k(t)\right)^2dt\right\}^{1/2}\left\{\int_0^1\left(\int_{E_s^+}\left(\sum_{k=0}^{n(x)}\left(1-\frac{k}{n(x)+1}\right)r_k(t)\varphi_k(x)\right)d\mu(x)\right)^2dt\right\}^{1/2}=$$

$$=\left\{\sum_{k=\nu(s)+1}^{\nu(s)+1}a_k^2\right\}^{1/2}\left\{\left(\int_{E_s^+}\left(\sum_{k=0}^{n(x)}\left(1-\frac{k}{n(x)+1}\right)r_k(t)\varphi_k(x)\right)d\mu(x)\right)\times\right.$$

$$\left.\times\left(\int_{E_s^+}\left(\sum_{k=0}^{n(y)}\left(1-\frac{k}{n(y)+1}\right)r_k(t)\varphi_k(y)\right)d\mu(y)\right)dt\right\}^{1/2}=\left\{\sum_{k=\nu(s)+1}^{\nu(s)+1}a_k^2\right\}^{1/2}\times$$

$$\times\left\{\int_{E_s^+}\int_{E_s^+}\left(\int_0^1\left(\sum_{k=0}^{n(x)}\sum_{l=0}^{n(y)}\left(1-\frac{k}{n(x)+1}\right)\left(1-\frac{l}{n(y)+1}\right)r_k(t)\varphi_k(x)r_l(t)\varphi_l(y)\right)dt\right)\times\right.$$

$$\left.\times\,d\mu(x)d\mu(y)\right\}^{1/2}=\left\{\sum_{k=\nu(s)+1}^{\nu(s+1)}a_k^2\right\}^{1/2}\left\{\int_{E_s^+}\int_{E_s^+}\left(\sum_{k=0}^{\min(n(x),n(y))}\left(1-\frac{k}{n(x)+1}\right)\times\right.\right.$$

$$\left.\left.\times\left(1-\frac{k}{n(y)+1}\right)\varphi_k(x)\varphi_k(y)\right)d\mu(x)d\mu(y)\right\}^{1/2}\leqq\left\{\sum_{k=\nu(s)+1}^{\nu(s+1)}a_k^2\right\}^{1/2}\times$$

$$\times\left\{\int_X\left(\int_X\left|\sum_{k=0}^{\min(n(x),n(y))}\left(1-\frac{k}{n(x)+1}\right)\left(1-\frac{k}{n(y)+1}\right)\varphi_k(x)\varphi_k(y)\right|d\mu(x)\right)d\mu(y)\right\}^{1/2}.$$

Durch zweimahlige Abelsche Umformung erhalten wir

$$\sum_{k=0}^{\min(n(x),n(y))}\left(1-\frac{k}{n(x)+1}\right)\left(1-\frac{k}{n(y)+1}\right)\varphi_k(x)\varphi_k(y)=$$

$$=\frac{2}{(n(x)+1)(n(y)+1)}\sum_{k=0}^{\min(n(x),n(y))}(k+1)K_k^1(x,y)-$$

$$-\frac{\min(n(x),n(y))}{(n(x)+1)(n(y)+1)}K^1_{\min(n(x),n(y))}(x,y).$$

Daraus folgt auf Grund der Voraussetzung

$$\int_X \left| \sum_{k=0}^{\min(n(x),n(y))} \left(1-\frac{k}{n(x)+1}\right)\left(1-\frac{k}{n(y)+1}\right)\varphi_k(x)\varphi_k(y)\right| d\mu(x) = O(\lambda_{\nu(s)+1}) \quad (y\in X).$$

Hieraus und aus (8) ergibt sich

$$\int_{E_s^+} \left(\sum_{k=\nu(s)+1}^{n(x)} \left(1-\frac{k}{n(x)+1}\right) a_k\varphi_k(x)\right) d\mu = O\left(\left\{\sum_{k=\nu(s)+1}^{\nu(s+1)} a_k^2\right\}^{1/2} \lambda_{\nu(s)+1}^{1/2}\right) = O(A_s). \tag{9}$$

Durch Anwendung dieser Abschätzung auf das System $\{-\varphi_k(x)\}_0^\infty$ erhalten wir

$$\int_{E_s^-} \left| \sum_{k=\nu(s)+1}^{n(x)} \left(1-\frac{k}{n+1}\right) a_k\varphi_k(x)\right| d\mu = O(A_s).$$

Daraus und aus (9) folgt aber die Behauptung des Hilfssatzes III.

Hilfssatz IV. *Unter der Bedingung* $L_n^1(\varphi; x)=O(\lambda_n)$ $(x\in X;\ n=0, 1, \ldots)$ *gilt*

$$\int_X \left| \sum_{k=\nu(\sigma)+1}^{\nu(\sigma+1)} \left(1-\frac{k}{n(x)+1}\right) a_k\varphi_k(x) - \sum_{k=\nu(\sigma)+1}^{\nu(\sigma)+1} \left(1-\frac{k}{\nu(s)+1}\right) a_k\varphi_k(x)\right| d\mu = O\left(\frac{\nu(\sigma)}{\nu(s)} A_\sigma\right)$$

für jede meßbare Funktion $n(x)$ mit ganzzahligen Werten, $\nu(s)<n(x)\leqq\nu(s+1)$ und für alle Indizes $\sigma=1, \ldots, s-1$.

Beweis des Hilfssatzes IV. Nach der Umformung

$$\sum_{k=\nu(\sigma)+1}^{\nu(\sigma+1)} \left(1-\frac{k}{n(x)+1}\right) a_k\varphi_k(x) - \sum_{k=\nu(\sigma)+1}^{\nu(\sigma+1)} \left(1-\frac{k}{\nu(s)+1}\right) a_k\varphi_k(x) =$$

$$= \left(\frac{1}{\nu(s)+1} - \frac{1}{n(x)+1}\right) \sum_{k=\nu(\sigma)+1}^{\nu(\sigma+1)} k a_k\varphi_k(x)$$

und nach Anwendung des Hilfssatzes II erhalten wir

$$\int_X \left| \sum_{k=\nu(\sigma)+1}^{\nu(\sigma+1)} \left(1-\frac{k}{n(x)+1}\right) a_k\varphi_k(x) - \sum_{k=\nu(\sigma)+1}^{\nu(\sigma+1)} \left(1-\frac{k}{\nu(s)+1}\right) a_k\varphi_k(x)\right| d\mu \leqq$$

$$\leqq \frac{1}{\nu(s)} \int_X \left| \sum_{k=\nu(\sigma)+1}^{\nu(\sigma+1)} k a_k\varphi_k(x) d\mu = O\left(\frac{1}{\nu(s)} \left\{\sum_{k=\nu(\sigma)+1}^{\nu(\sigma+1)} k^2 a_k^2\right\}^{1/2} \lambda_{\nu(\sigma)+1}^{1/2}\right),\right.$$

woraus sich die Behauptung des Hilfssatzes IV ergibt.

3. Beweis des Satzes I. Ohne Beschränkung der Allgemeinheit können wir $a_k=0$ $(k=0, \ldots, \nu(1))$ voraussetzen. Es sei

$$\delta_s(x) = \max_{\nu(s)<n\leqq\nu(s+1)} |\sigma_n(x)-\sigma_{\nu(s)}(x)| \quad (s = 1, 2, \ldots),$$

wobei

$$\sigma_n(x) = \sum_{k=0}^{n} \left(1-\frac{k}{n+1}\right) a_k\varphi_k(x)$$

ist. Für jede natürliche Zahl $s(\geqq 1)$ sei $n_s(x)$ die kleinste ganze Zahl, für welche $\nu(s)<n_s(x)\leqq\nu(s+1)$ und

$$\delta_s(x)=|\sigma_{n_s(x)}(x)-\sigma_{\nu(s)}(x)|\quad (x\in X)$$

bestehen. Dann gilt

$$\delta_s(x)\leqq\sum_{\sigma=1}^{s-1}\left|\sum_{k=\nu(\sigma)+1}^{\nu(\sigma+1)}\left(1-\frac{k}{n_s(x)+1}\right)a_k\varphi_k(x)-\sum_{k=\nu(\sigma)+1}^{\nu(\sigma+1)}\left(1-\frac{k}{\nu(s)+1}\right)a_k\varphi_k(x)\right|+$$

$$+\left|\sum_{k=\nu(s)+1}^{n_s(x)}\left(1-\frac{k}{n_s(x)+1}\right)a_k\varphi_k(x)\right|.$$

Durch Anwendung der Hilfssätze III und IV bekommen wir

$$\int_X\delta_s(x)d\mu=O\left(\sum_{\sigma=1}^{s-1}\frac{\nu(\sigma)}{\nu(s)}A_\sigma+A_s\right)\quad (s=1,2,\ldots),$$

und so gilt auf Grund von (1)

$$\sum_{s=1}^{\infty}\int_X\delta_s(x)d\mu=O\left(\sum_{s=1}^{\infty}\left(\frac{1}{\nu(s)}\sum_{\sigma=1}^{s-1}\nu(\sigma)A_\sigma+A_s\right)\right)=O\left(\sum_{\sigma=1}^{\infty}A_\sigma\left(1+\nu(\sigma)\sum_{s=\sigma+1}^{\infty}\frac{1}{\nu(s)}\right)\right)=$$

$$=O\left(\sum_{\sigma=1}^{\infty}A_\sigma\right)<\infty.$$

Daraus folgt, daß

$$\sum_{s=1}^{\infty}\delta_s(x)<\infty \tag{10}$$

fast überall besteht. Da

$$|\sigma_{\nu(s+1)}(x)-\sigma_{\nu(s)}(x)|\leqq\delta_s(x)\quad (x\in X;\ s=1,2,\ldots)$$

gilt, ergibt sich, aus (10) daß

$$\sum_{s=1}^{\infty}|\sigma_{\nu(s+1)}(x)-\sigma_{\nu(s)}(x)|<\infty$$

auch fast überall gilt, also konvergiert die Reihe

$$\sum_{s=1}^{\infty}\left(\sigma_{\nu(s+1)}(x)-\sigma_{\nu(s)}(x)\right)$$

fast überall. Daraus folgt, daß $\lim_{s\to\infty}\sigma_{\nu(s)}(x)$ fast überall existiert. Es sei n beliebige natürliche Zahl $(n>\nu(1))$ und sei $\nu(s)<n\leqq\nu(s+1)$. Da $|\sigma_n(x)-\sigma_{\nu(s)}(x)|\leqq\leqq\delta_s(x)\to 0$ $(s\to\infty)$ fast überall besteht, folgt, daß die Folge $\{\sigma_n(x)\}$ fast überall konvergiert.

Damit haben wir Satz I bewiesen.

4. Man kann auch den folgenden Satz zeigen.

SATZ II. *Unter den Bedingungen des Satzes* I *konvergiert die Folge* $\left\{\sum_{k=0}^{\nu(s)} a_k\varphi_k(x)\right\}_0^\infty$ *fast überall.*

BEWEIS DES SATZES II. Aus dem Hilfssatz II folgt

$$\int_X \left|\sum_{k=\nu(s)+1}^{\nu(s+1)} a_k\varphi_k(x)\right| d\mu = O\left(\left\{\sum_{k=\nu(s)+1}^{\nu(s+1)} a_k^2\right\}^{1/2} \lambda_{\nu(s)+1}^{1/2}\right) = O(A_s) \quad (s = 1, 2, \dots),$$

woraus sich

$$\sum_{s=1}^{\infty} \int_X \left|\sum_{k=\nu(s)+1}^{\nu(s+1)} a_k\varphi_k(x)\right| d\mu = O\left(\sum_{s=1}^{\infty} A_s\right) < \infty,$$

und daher die Behauptung des Satzes II ergibt.

5. Wir erwähnen die folgenden Behauptung.

SATZ III. *Unter der Bedingung* $L_n^1(\varphi; (x) = O(\lambda_n)$ $(x\in X;\ n=0, 1, \dots)$ *gilt*

$$\int_X |\varphi_n(x)| d\mu = O(\lambda_n^{1/2}).$$

BEWEIS DES SATZES III. Im Falle $n\in Z_s$ folgt aus dem Hilfssatz II

$$\int_X |\varphi_n(x)| d\mu = O(\lambda_{\nu(s)+1}^{1/2}) = O(\lambda_n^{1/2})$$

auf Grund der Definition der Folge $\{\nu(s)\}$.

6. Endlich beweisen wir, daß ohne der Bedingung $\sum_{s=0}^{\infty} A_s < \infty$ die Behauptung des Satzes I im allgemeinen nicht zutrifft.

SATZ IV. *Gilt für die Folge a* $\sum_{s=0}^{\infty} A_s = \infty$, *so gibt es ein System* $\varphi = \{\varphi_k(x)\}_0^\infty$ *der Funktionen* $\varphi_k(x)\in L(0, 1)$ $(k=0, 1, \dots)$ *mit*

$$L_n^1(\varphi;\ x) = \int_0^1 \left|\sum_{k=0}^{n}\left(1-\frac{k}{n+1}\right)\varphi_k(x)\varphi_k(t)\right| dt = O(\lambda_n) \quad (x\in(0, 1);\ n = 0, 1, \dots)$$

derart, daß die Reihe

$$\sum_{k=0}^{n} a_k\varphi_k(x) \tag{11}$$

in $(0, 1)$ *überall nicht* $(C, 1)$*-summierbar ist.*

BEWEIS DES SATZES IV. Es sei $\varphi_k(x)\equiv 0$ $\big(x\in(0, 1);\ k=0, \dots, \nu(0)\big)$. Es sei $s(\geqq 0)$ eine ganze Zahl. Ist $A_s=0$, so seien $\varphi_k(x)\equiv 0$ $\big(x\in(0, 1);\ n=\nu(s)+1, \dots$ $\dots, \nu(s+1)\big)$. Ist aber $A_s>0$, so seien $I_k(s)$ $\big(k=\nu(s)+1, \dots, \nu(s+1)\big)$ paarweise disjunkte Intervalle mit

$$\bigcup_{k=\nu(s)+1}^{\nu(s+1)} I_k(s) = (0,1), \quad \operatorname{mes} I_k(s) = \frac{a_k^2}{\sum_{l=\nu(s)+1}^{\nu(s)+1} a_l^2} \quad (k = \nu(s)+1, \ldots, \nu(s+1)),$$

und wir setzen

$$\varphi_k(x) = \begin{cases} A_s/a_k, & x \in I_k(s) \\ 0 & \text{sonst} \end{cases} \quad (k = \nu(s)+1, \ldots, \nu(s+1)).$$

(Im Falle $a_k=0$ sei $\varphi_k(x) \equiv 0$ $(x \in (0, 1))$).

Es sei $x \in (0, 1)$. Dann gibt es für jede natürliche Zahl $s(\geqq 1)$ eine Folge $\{n_s(x)\}_1^\infty$ mit folgenden Eigenschaften: es gelten

$$\nu(s) < n_s(x) \leqq \nu(s+1) \quad (s = 1, 2, \ldots,),$$

$$\varphi_{n_s}(x) = A_s/a_{n_s} \quad (s = 1, 2, \ldots),$$

$$\varphi_k(x) = 0 \quad (\nu(s) < k \leqq \nu(s+1),\ k \neq n_s(x);\ s = 1, 2, \ldots).$$

So gilt für jedes $x \in (0, 1)$ und für jedes $s(\geqq 1)$, auf Grund von (1):

$$\sum_{k=0}^{\nu(s+1)} \left(1 - \frac{k}{\nu(s+1)+1}\right) a_k \varphi_k(x) = \sum_{\sigma=1}^{s+1} \left(1 - \frac{n_\sigma(x)}{\nu(s+1)+1}\right) A_\sigma \geqq$$

$$\geqq \sum_{\sigma=1}^{s} \left(1 - \frac{n_\sigma(x)}{\nu(s+1)}\right) A_\sigma \geqq \left(1 - \frac{1}{q}\right) \sum_{\sigma=1}^{s} A_\sigma,$$

Daraus folgt, daß die Reihe (11) überall in (0, 1) nicht $(C, 1)$-summierbar ist.

Es sei $x \in (0, 1)$. Dann gilt

$$L_n^1(\varphi; x) \equiv 0 = O(\lambda_n) \quad (n = 0, \ldots, \nu(0)). \tag{12}$$

Es sei $n > \nu(0)$. Dann gibt es eine natürliche Zahl s mit $\nu(s) < n \leqq \nu(s+1)$. Ist $n < n_s(x)$, so gilt

$$L_n^1(\varphi; x) \leqq \int_0^1 \sum_{\sigma=1}^{s-1} |\varphi_{n_\sigma(x)}(x)\varphi_{n_\sigma(x)}(t)|\, dt = \sum_{\sigma=1}^{s-1} \int_{I_{n_\sigma}(x)} \varphi_{n_\sigma(x)}(x)\varphi_{n_\sigma(x)}(t)\, dt = \tag{13}$$

$$= \sum_{\sigma=1}^{s-1} \frac{A_\sigma^2}{a_{n_\sigma(x)}^2} a_{n_\sigma(x)}^2 \Big/ \sum_{k=\nu(\sigma)+1}^{\nu(\sigma+1)} a_k^2 = \sum_{\sigma=1}^{s-1} A_\sigma^2 \Big/ \sum_{k=\nu(\sigma)+1}^{\nu(\sigma+1)} a_k^2 = O(1) \sum_{\sigma=1}^{s-1} \lambda_{\nu(\sigma+1)} =$$

$$= O(\lambda_{\nu(s)}) = O(\lambda_n),$$

auf Grund des Definition der Folge $\{\nu(s)\}$.

Ist aber $n_s(x) \leqq n$, so gilt ähnlicherweise

$$L_n^1(\varphi; x) = O\left(\sum_{\sigma=1}^{s} \lambda_{\nu(\sigma+1)}\right) = O(\lambda_{\nu(s+1)}) = O(\lambda_{\nu(s)}) = O(\lambda_n). \tag{14}$$

Aus (12), (13) und (14) folgt $L_n^1(\varphi; x) = O(\lambda_n)$ $(x \in (0, 1);\ n = 0, 1, \ldots)$.

Damit haben wir den Satz IV bewiesen.

BEMERKUNG. Für das Funktionensystem φ im Beweis des Satzes IV ist auch die strengere Forderung

$$\int_0^1 \sum_{k=0}^{n} |\varphi_k(x)\varphi_k(t)|\,dt = O(\lambda_n)$$

$$(x\in(0, 1);\ n = 0, 1, \ldots)$$

erfüllt.

(Eingegangen am 9. September 1981)

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Acta Math. Hung.
41 (1—2), (1983), 161—168.

SOME REMARKS ON THE NOTION OF REGULAR CONVERGENCE OF MULTIPLE SERIES

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1. Consider a d-multiple series

$$\sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} a_{k_1, k_2, \ldots, k_d}, \tag{1}$$

whose terms are real (or complex) numbers and d is a fixed positive integer. The rectangular sums defined by

$$s(m_1, \ldots, m_d; n_1, \ldots, n_d) = \sum_{k_1=m_1}^{n_1} \cdots \sum_{k_d=m_d}^{n_d} a_{k_1, \ldots, k_d},$$

where $1 \leqq m_j \leqq n_j$ for each j, play a decisive role in the study of convergence behaviour of the series (1). In particular, if $m_j=1$ for each j,

$$s_{n_1, \ldots, n_d} = s(1, \ldots, 1; n_1, \ldots, n_d) = \sum_{k_1=1}^{n_1} \cdots \sum_{k_d=1}^{n_d} a_{k_1, \ldots, k_d}$$

are called the rectangular partial sums of (1).

We remind that (1) *converges in Pringsheim's sense* to a finite number s if for every $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that

$$|s_{n_1, \ldots, n_d} - s| < \varepsilon \quad \text{whenever} \quad \min(n_1, \ldots, n_d) \geqq N. \tag{2}$$

Following Hardy [3], the *d-multiple series* (1) is said to be *regularly convergent* if

(i) it converges in Pringsheim's sense, and

(ii) for each choice of the indices $1 \leqq j_1 < \ldots < j_e \leqq d$ with $1 \leqq e < d$, and, denoting the remaining indices of $\{1, 2, \ldots, d\}$ by $1 \leqq l_{e+1} < \ldots < l_d \leqq d$, for all fixed values of $k_{l_{e+1}}, \ldots, k_{l_d}$ $(=1, 2, \ldots)$, the e-multiple series

$$\sum_{k_{j_1}=1}^{\infty} \cdots \sum_{k_{j_e}=1}^{\infty} a_{k_1, \ldots, k_d}$$

also converge in Pringsheim's sense.

This definition is actually given in [3] only for double series, while for arbitrary multiple series it can be found e.g. in [4, p. 34]. We mention that this kind of convergence of multiple series is essentially contained already in [2] in a special case.

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[1] This research was conducted while the author was on leave from Szeged University and a Visiting Professor at Ulm University, West Germany.

The notion of regular convergence was rediscovered by the present author in [5], where it was defined by an equivalent condition and called convergence "in the restricted sense". More precisely, our definition is the following: the *d-multiple series* (1) is said to be *regularly convergent* if for every $\varepsilon>0$ there exists an $N=N(\varepsilon)$ such that

$$|s(m_1, \ldots, m_d; n_1, \ldots, n_d)| < \varepsilon \tag{3}$$

whenever

$$\max(m_1, \ldots, m_d) \geqq N \quad \text{and} \quad 1 \leqq m_j \leqq n_j \quad \text{for each } j. \tag{4}$$

The equivalence of the two definitions can be verified with the aid of [5, Theorem 1′] which, however, is stated there in a wrong form. The corrected statement reads as follows.

THEOREM 1. *The series* (1) *is regularly convergent if and only if*

(i) *it converges in Pringsheim's sense, and*

(ii) *for each choice of the index* $j(=1, \ldots, d)$ *and for all fixed values of* $k_j(=1, 2, \ldots)$, *the* $(d-1)$*-multiple series*

$$\sum_{k_1=1}^{\infty} \cdots \sum_{k_{j-1}=1}^{\infty} \sum_{k_{j+1}=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} a_{k_1, \ldots, k_d}$$

are regularly convergent.

The requirement expressed by conditions (i) and (ii) in Theorem 1 is a reformulation of the definition due to Hardy. On the other hand, using our definition, conditions (i) and (ii) together are equivalent to the following two ones: for every $\varepsilon>0$ there exists an $N=N(\varepsilon)$ such that

(i′) inequality (3) is satisfied whenever

$$\min(m_1, \ldots, m_d) \geqq N \quad \text{and} \quad 1 \leqq m_j \leqq n_j \quad \text{for each } j; \text{ and}$$

(ii′) inequality (3) is satisfied whenever $m_j=n_j=1, 2, \ldots, N-1$ for some j, but

$$\max(m_1, \ldots, m_{j-1}, m_{j+1}, \ldots, m_d) \geqq N \quad \text{and} \quad 1 \leqq m_j \leqq n_j \quad \text{for each } j.$$

Now on the basis of (i′) and (ii′) one can apply an induction argument on d for the proof of equivalence of the two definitions of regular convergence.

EXAMPLE 1. Consider the triple series (1), whose members are defined as follows: for $k=1, 2, \ldots$ set

$$a_{k,k,1} = -a_{k,k,2} = 1,$$

$$a_{2k-1,2k,1} = a_{2k,2k-1,1} = -a_{2k-1,2k,2} = -a_{2k,2k-1,2} = 1;$$

otherwise $a_{k_1,k_2,k_3}=0$.

This triple series converges to 0 in Pringsheim's sense and all its single series (i.e. fixing two indices of k_1, k_2, k_3 arbitrarily and letting the third index run over $1, 2, \ldots$) are also convergent, but it fails to be regularly convergent.

REMARK 1. A *d-multiple sequence* $\{s_{k_1, \ldots, k_d}$: $k_j=1, 2, \ldots$ for each $j\}$ is said to be *regularly convergent* if

(i) it converges in Pringsheim's sense, i.e. for every $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ such that (2) is satisfied;

(ii) for each choice of the indices $1 \leqq j_1 < \ldots < j_e \leqq d$ with $1 \leqq e < d$, and, denoting the remaining indices of $\{1, 2, \ldots, d\}$ by $1 \leqq l_{e+1} < \ldots < l_d \leqq d$, for all fixed values of $k_{l_{e+1}}, \ldots, k_{l_d}$ $(=1, 2, \ldots)$, the e-multiple sequences $\{s_{k_1, \ldots, k_d}\colon k_{j_1} = 1, 2, \ldots; \ldots; k_{j_e} = 1, 2, \ldots\}$ converge in Pringsheim's sense.

This definition for double sequences is given also by Hardy [3].

In accordance with the connection between the members of a multpile series and its rectangular partial sums, this definition can be reformulated as follows. The *d-multiple sequence* $\{s_{k_1, \ldots, k_d}\}$ is *regularly convergent* if for every $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ such that

$$\left| \sum_{\varepsilon_1=0}^{1} \cdots \sum_{\varepsilon_d=0}^{1} (-1)^{\varepsilon_1 + \cdots + \varepsilon_d} s_{\varepsilon_1 m_1 + (1-\varepsilon_1) n_1, \ldots, \varepsilon_d m_d + (1-\varepsilon_d) n_d} \right| < \varepsilon$$

whenever (4) is satisfied.

2. Now we are going to present the generalization of a few well-known convergence tests from single series to multiple series, while using the notion of regular convergence. It is also of some interest to point out that these generalizations become, in general, false if one uses the convergence notion in Pringsheim's sense. That will be illustrated by counterexamples.

We begin with the extension of the so-called *Leibniz rule* to multiple series. To this effect, set

$$\Delta_{\varepsilon_1, \ldots, \varepsilon_d} a_{k_1, \ldots, k_d} = \delta_{j_1}(\ldots(\delta_{j_e} a_{k_1, \ldots, k_d})\ldots),$$

where $\varepsilon_j = 1$ for $j = j_1, \ldots, j_e$ with $1 \leqq e \leqq d$ and $\varepsilon_j = 0$ for the other j from $\{1, 2, \ldots \ldots, d\}$, and

$$\delta_j a_{k_1, \ldots, k_d} = a_{k_1, \ldots, k_{j-1}, k_j, k_{j+1}, \ldots, k_d} - a_{k_1, \ldots, k_{j-1}, k_j+1, k_{j+1}, \ldots, k_d}.$$

The following theorem is known for $d = 1$ as the Leibniz rule.

THEOREM 2. *Let $\{a_{k_1, \ldots, k_d}\colon k_j = 1, 2, \ldots$ for each $j\}$ be a d-multiple sequence of positive numbers with the following poperties: for all fixed values of $\varepsilon_1, \ldots, \varepsilon_d$, where $\varepsilon_j = 0$ or 1 for each j and $\varepsilon_j = 1$ for at least one j, denoting by e the number of those ε_j for which $\varepsilon_j = 1$, $1 \leqq e \leqq d$, we have*

(5) $\Delta_{\varepsilon_1, \ldots, \varepsilon_d} a_{k_1, \ldots, k_d}$ *is* $\begin{cases} \text{nonnegatvie in the cases } e = 1; \\ \text{of constant sign (depending perhaps on } \varepsilon_1, \ldots, \varepsilon_d) \text{ for all} \\ \text{values of } k_1, \ldots, k_d \ (=1, 2, \ldots) \text{ in the cases } e \geqq 2; \end{cases}$

and

(6) $$a_{k_1, \ldots, k_d} \to 0 \quad as \quad \max(k_1, \ldots, k_d) \to \infty.$$

Then the series

(7) $$\sum_{k_1=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} (-1)^{k_1 + \cdots + k_d} a_{k_1, \ldots, k_d}$$

is regularly convergent.

The proof of Theorem 2 is based on the following elementary

LEMMA. *Under conditions* (5) *we have*

$$\left|\sum_{k_1=m_1}^{n_1} \dots \sum_{k_d=m_d}^{n_d} (-1)^{k_1+\dots+k_d} a_{k_1,\dots,k_d}\right| \leqq C_d a_{m_1,\dots,m_d}, \tag{8}$$

where C_d is a positive constant depending only on d.

Now from (8) and (4), the definition (3) of regular convergence follows immediately.

PROOF OF THE LEMMA. We apply an induction with respect to d. Inequality (8) is obvious for $d=1$. Assume that (8) has already been proved for $1, 2, \dots, d-1$ and prove it for d itself.

If $m_j=n_j$ for some j, then (8) is a consequence of the induction hypothesis. Thus we may assume that $m_j<n_j$ for each j.

It is enough to verify (8) with a constant $C_d^{(e)}$ in the special case when n_j-m_j+1 is even for each j. Indeed, in the general case

$$s(m_1, \dots, m_d; n_1, \dots, n_d) = \sum_{k_1=m_1}^{n_1} \dots \sum_{k_d=m_d}^{n_d} (-1)^{k_1+\dots+k_d} a_{k_1,\dots,k_d},$$

can be represented as a sum of a "smaller" rectangular sum $s(m_1, \dots, m_d; n_1', \dots, n_d')$, where $n_j'=n_j$ or n_j-1 according as n_j-m_j+1 is even or odd, and of certain $(d-1)$-multiple "remainder" rectangular sums whose number is at most d. Applying the induction hypothesis, we obtain (8) in the general case with $C_d=C_d^{(e)}+dC_{d-1}$.

From now on suppose that n_j-m_j+1 is even for each j. Groupping the terms of $s(m_1, \dots, m_d; n_1, \dots, n_d)$ into $2\times2\times\dots\times2$ blocks in the way as it is suggested by the definition of $\Delta_{1,\dots,1}a_{k_1,\dots,k_d}$, we can see that $s(m_1, \dots, m_d; n_1, \dots, n_d)\geqq 0$ or $\leqq 0$ according as $\Delta_{1,\dots,1}a_{k_1,\dots,k_d}$ and $(-1)^{m_1+\dots+m_d}$ are of the same sign or not, respectively.

On the other hand, consider the decomposition

$$\begin{aligned} s(m_1, \dots, m_d; n_1, \dots, n_d) &= s(m_1, m_2, \dots, m_d; m_1, n_2, \dots, n_d)+ \\ &+s(m_1+1, m_2, \dots, m_d; n_1-1, n_2, \dots, n_d)+s(n_1, m_2, \dots, m_d; n_1, n_2, \dots, n_d), \end{aligned} \tag{9}$$

where the middle term on the right is 0 if $n_1=m_1+1$. By the above observation concerning groupping the terms, $s(m_1+1, m_2, \dots, m_d; n_1-1, n_2, \dots, n_d)$ and $s(m_1, \dots, m_d; n_1, \dots, n_d)$ have opposite signs. Consequently, applying the induction hypothesis to the first and third terms on the right in (9), we get $2C_{d-1}a_{m_1,\dots,m_d}$ as an upper bound for $|s(m_1, \dots, m_d; n_1, \dots, n_d)|$, i.e. $C_d^{(e)}=2C_{d-1}$.

Remembering that $C_d=C_d^{(e)}+dC_{d-1}$, we can write $C_d=(d+2)C_{d-1}$. The proof of our lemma is complete.

It seems very likely that (8) is true even with $C_d=1$, but we are unable to prove this stronger statement.

REMARK 2. Both conditions (5) and (6) are essential for the conclusion in Theorem 2, as it is shown by the following examples.

EXAMPLE 2. Set

$$a_{1,2n-1} = \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2}$$

and

$$a_{1,2n} = \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} - \frac{1}{(n+1)^3} \quad (n = 1, 2, \ldots),$$

further, let $a_{ik}=a_{1,i+k}$ $(i, k=1, 2, \ldots)$.

It is clear that $\Delta_{1,0}a_{ik} \geqq 0$ and $\Delta_{0,1}a_{ik} \geqq 0$ (but now $\Delta_{1,1}a_{ik}$ is not of constant sign). Since

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6},$$

condition (4) is also satisfied. Nevertheless, the double series

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} (-1)^{i+k} a_{ik} \tag{10}$$

diverges even in Pringsheim's sense. Indeed, we have

$$-\sum_{i=1}^{2n} (-1)^i a_{i,2l} = \sum_{i=1}^{n} \left(\frac{1}{(i+l)^2} - \frac{1}{(i+l)^3} \right) \geqq \frac{n}{l(l+n)} - \frac{1}{2l^2} \geqq \frac{1}{3l} - \frac{1}{2l^2}$$

for $l=n, n+1, \ldots, 2n-1$, while

$$\sum_{i=1}^{2n} (-1)^{i+1} a_{i,2l+1} = \sum_{i=1}^{n} \frac{1}{(i+l+1)^3} \leqq \frac{1}{2(l+1)^2} \quad \text{for} \quad l = n, n+1, \ldots.$$

Hence, for $n=2, 3, \ldots$

$$s_{2n-1,2n} - s_{4n-2,2n} = \sum_{l=n}^{2n-1} \sum_{i=1}^{n} \left(\frac{1}{(i+l)^2} - \frac{1}{(i+l)^3} \right) - \sum_{l=n}^{2n-2} \sum_{i=1}^{n} \frac{1}{(i+l+1)^3} \geqq$$

$$\geqq \sum_{l=n}^{2n-1} \left(\frac{1}{3l} - \frac{1}{2l^2} \right) - \sum_{l=n}^{2n-2} \frac{1}{2(l+1)^2} \geqq \frac{1}{6} - \frac{1}{2(n-1)} - \frac{1}{2n}.$$

EXAMPLE 3. Setting $a_{ik}=1/\min(i, k)$ $(i, k=1, 2, \ldots)$, condition (5) is satisfied with $\Delta_{1,1}a_{ik} \leqq 0$. But instead of (6) we have only

$$a_{ik} \to 0 \quad \text{as} \quad \min(i, k) \to \infty. \tag{11}$$

Since

$$s_{nn} - s_{n-1,n} = \sum_{k=1}^{n} (-1)^{n+k} \frac{1}{k} \not\to 0 \quad \text{as} \quad n \to \infty,$$

the series (10) does not converge in Pringsheim's sense.

EXAMPLE 4. Setting $a_{ik}=(i+1)/ik$ $(i, k=1, 2, \ldots)$, conditions (5) (now $\Delta_{1,1}a_{ik} \geqq 0$) and (11) are satisfied. But again

$$s_{nn} - s_{n-1,n} = \frac{n+1}{n} \sum_{k=1}^{n} (-1)^{n+k} \frac{1}{k} \not\to 0 \quad \text{as} \quad n \to \infty.$$

3. The following Theorem A of Hardy [3] and Theorem 3 (the latter is also stated in [3] but in a wrong form) demonstrate the power of the notion of regular convergence. First we recall a definition (cf. [3] for $d=2$). A d-multiple sequence $\{\lambda_{k_1,\dots,k_d}$: $k_j=1,2,\dots$ for each $j\}$ is said to be *of bounded variation* if

$$\sum_{k_1=0}^{\infty}\cdots\sum_{k_d=0}^{\infty}|\underbrace{\Delta_{1,\dots,1}}_{d\,\text{times}}\lambda_{k_1,\dots,k_d}|<\infty,$$

where we agree to set $\lambda_{k_1,\dots,k_d}=0$ if $k_j=0$ for at least one j.

Hence it follows, in particular, that for each choice of the indices $1\leqq j_1<\dots$ $\dots<j_e\leqq d$ with $1\leqq e<d$, and, denoting the remaining indices of $\{1,2,\dots,d\}$ by $1\leqq l_{e+1}<\dots<l_d\leqq d$, for all fixed values of $k_{l_{e+1}},\dots,k_{l_d}$ $(=1,2,\dots)$, the e-multiple sequences $\{\lambda_{k_1,\dots,k_d}$: $k_{j_1}=1,2,\dots;\ \dots;\ k_{j_e}=1,2,\dots\}$ are also of bounded variation.

THEOREM A [3, Theorem 13]. *The necessary and sufficient condition that the series*

$$\sum_{k_1=1}^{\infty}\cdots\sum_{k_d=1}^{\infty}\lambda_{k_1,\dots,k_d}a_{k_1,\dots,k_d} \tag{12}$$

be regularly convergent whenever the series (1) *is regularly convergent is that the d-multiple sequence* $\{\lambda_{k_1,\dots,k_d}\}$ *be of bounded variation.*

THEOREM 3. *If* $\{\lambda_{k_1,\dots,k_d}\}$ *is of bounded variation,*

$$\lambda_{k_1,\dots,k_d}\to 0 \quad as \quad \max(k_1,\dots,k_d)\to\infty, \tag{13}$$

and the rectangular partial sums of the series (1) *are bounded, then the series* (12) *is regularly convergent.*

Theorem A is proved in [3] for $d=2$. Both the proof of Theorem A in the general case and the proof of Theorem 3 can be carried out analogously with the aid of a d-multiple Abel transformation (for a d-multiple forward Abel transformation see [1] and also [6], while for a d-multiple backward one see [6]).

REMARK 3. Theorem 3 is stated in [3, cf. Theorem 11] in a wrong form, requiring, instead of (13), only the weaker condition that $\{\lambda_{k_1,\dots,k_d}\}$ tends regularly to zero. But this condition (together with the condition that $\{\lambda_{k_1,\dots,k_d}\}$ is of bounded variation) is not enough to imply the conclusion of Theorem 3. This is shown by the following

EXAMPLE 5. Let

$$a_{ik}=\begin{cases}(-1)^{i+k} & \text{for } i=1,2 \text{ and } k=1,2,\dots;\\ 0 & \text{otherwise;}\end{cases}$$

and

$$\lambda_{ik}=\begin{cases}1 & \text{for } i=1 \text{ and } k=1,2,\dots;\\ 0 & \text{otherwise.}\end{cases}$$

Then the double series $\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}a_{ik}$ converges in Pringsheim's sense and, in addition, its rectangular partial sums are bounded. Further, $\{\lambda_{ik}\}$ is of bounded variation and tends regularly to zero. Nevertheless, the series $\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\lambda_{ik}a_{ik}$ does not converge even in Pringsheim's sense.

REMARK 4. Example 5 also shows that Theorem A is no longer true if in it the term "regular convergence" is replaced by the term "convergence in Pringheim's sense".

On the other hand, for certain particular choices of $\{\lambda_{k_1,\dots,k_d}\}$ one can still assert an analogous theorem to Theorem A, using the convergence notion in Pringsheim's sense both in the condition and in the conclusion.

THEOREM 4. *If*

$$\lambda_{k_1,\dots,k_d}=\mu_{\max(k_1,\dots,k_d)}\quad (k_j=1,2,\dots \text{ for each } j),$$

where $\{\mu_i\colon i=1,2,\dots\}$ *is of bounded variation, and the series* (1) *converges in Pringsheim's sense, then the series* (12) *also converges in Pringsheim's sense.*

PROOF. We only sketch it for $d=2$. Given an $\varepsilon>0$, by (2) there exists an $N=N(\varepsilon)$ such that

$$|s(p,1;\,m,N)|<\varepsilon,\quad |s(1,q;\,N,n)|<\varepsilon \tag{14}$$

and

$$|s(p,q;\,m,n)|<\varepsilon \text{ whenever both } m\geqq p\geqq N \text{ and } n\geqq q\geqq N, \tag{15}$$

where, in accordance with the notation used so far, we write

$$s(p,q;\,m,n)=\sum_{i=p}^{m}\sum_{k=q}^{n}a_{ik}.$$

Setting also $S(p,q;\,m,n)=\sum_{i=p}^{m}\sum_{k=q}^{n}\lambda_{ik}a_{ik}$ and $S_{mn}=S(1,1;\,m,n)$, one can decompose

$$S_{mn}-S_{NN}=S(N+1,1;\,m,N)+S(1,n+1;\,N,n)+S(N+1,N+1;\,m,n).$$

A single backward Abel transformation and (14) provide that

$$|S(N+1,1;\,m,N)|\leqq\varepsilon\sum_{i=N+2}^{m}|\mu_i-\mu_{i-1}|+\varepsilon|\mu_{N+1}|\leqq\varepsilon|\sum_{i=1}^{m}|\mu_i-\mu_{i-1}|,$$

where $\mu_0=0$. Similarly,

$$|S(1,N+1;\,N,n|)\leqq\varepsilon\sum_{i=1}^{n}|\mu_i-\mu_{i-1}|.$$

Finally, using a double backward Abel transformation (see, e.g. [6]):

$$S(N+1,N+1;\,m,n)=\sum_{i=N+2}^{m}\sum_{k=N+2}^{n}s(i,k;\,m,n)(\lambda_{ik}-\lambda_{i-1,k}-\lambda_{i,k-1}+\lambda_{i-1,k-1})+$$

$$+\sum_{k=N+2}^{m}s(i,N+1;\,m,n)(\lambda_{i,N+1}-\lambda_{i-1,N+1})+\sum_{k=N+2}^{n}s(N+1,k;\,m,n)(\lambda_{N+1,k}-$$

$$-\lambda_{N+1,k-1})+s(N+1,N+1;\,m,n)\lambda_{N+1,N+1},$$

whence, by (15), we obtain

$$|S(N+1, N+1; m, n)| \leqq 3\varepsilon \sum_{i=1}^{\max(m, n)} |\mu_i - \mu_{i-1}|.$$

Since $\{\mu_i\}$ is of bounded variation, the proof is complete.

Acknowledgement. I express my sincere gratitude to Professor W. Kratz for raising the problem of finding convergence tests concerning regular convergence.

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(Received September 11, 1981)

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Acta Math. Hung.
41(1—2), (1983), 169—176.

ARITHMETICAL PROPERTIES OF PERMUTATIONS OF INTEGERS

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For the finite case let $a_1, a_2, \ldots, a_n$ be a permutation of the integers $1, 2, \ldots, n$ and for the infinite case let $a_1, a_2, \ldots, a_i, \ldots$ be a permutation of all positive integers.

Some problems and results concerning such permutations and related questions can be found in [2] (see in particular p. 94). In [3] the density of the sums a_i+a_{i+1} is estimated from several points of view.

In the present paper we shall investigate the least common multiple and the greatest common divisor of two subsequent elements. First we deal with the least common multiple. For the identical permutation we have $[a_i, a_{i+1}]=i(i+1)$. We show that for suitable other permutations this value becomes considerably smaller.

First we consider the finite case

THEOREM 1. *We have*

$$\min \max_{l \leq i \leq n-1} [a_i, a_{i+1}] = (1+o(1)) \frac{n^2}{4 \log n} \tag{1}$$

where the minimum is to be taken for all permutations $a_1, a_2, \ldots, a_n$.

One might think that the main reason for not being able to get a smaller value lies in the presence of the large primes (see also the proof). Theorem 2 shows that this is only partly true.

THEOREM 2. *Omit arbitrarily* $g(n)=o(n)$ *numbers from* $1, 2, \ldots, n$ *and form a permutation of the remaining ones.*

Then for any fix $\varepsilon>0$, *and n large enough we have*

$$\min \max_{l \leq i \leq n-g(n)-1} [a_i, a_{i+1}] > n^{2-\varepsilon}. \tag{2}$$

On the other hand, for any $\varepsilon(n)\to 0$ *we have with a suitable* $g(n)=o(n)$

$$\min \max_{l \leq i \leq n-g(n)-1} [a_i, a_{i+1}] < n^{2-\varepsilon(n)}. \tag{3}$$

An equivalent form of Theorem 2 is: $\dfrac{\log \{\min \max_{l \leq i \leq n-g(n)-1} [a_i, a_{i+1}]\}}{\log n}$ must tend to 2 for any $g(n)=o(n)$, but it can do this from below arbitrarily slowly for suitable $g(n)=o(n)$.

In the infinite case we obtain a much smaller upper bound:

THEOREM 3. *We can construct an infinite permutation satisfying*

$$[a_i, a_{i+1}] < ie^{c\sqrt{\log i \log\log i}} \tag{4}$$

for all i.

In the opposite direction we can prove only a very poor result:

THEOREM 4. *For any permutation*

$$\limsup_i \frac{[a_i, a_{i+1}]}{i} \geqq \frac{1}{1-\log 2} \sim 3{,}26. \tag{5}$$

Very probably this lim sup must be infinite, and one can expect an even sharper rate of growth.

Concerning the greatest common divisor only the infinite case is interesting.

THEOREM 5. *We can construct an infinite permutation satisfying*

$$(a_i, a_{i+1}) > \frac{1}{2} i \tag{6}$$

for all i.

On the other hand, for any permutation

$$\liminf_i \frac{(a_i, a_{i+1})}{i} \leqq \frac{61}{90}. \tag{7}$$

The right value is probably $\frac{1}{2}$, but we could not yet prove this.

Proofs

PROOF OF THEOREM 1. First we show that any permutation must contain an a_i for which

$$[a_i, a_{i+1}] \geqq (1+o(1)) \frac{n^2}{4\log n}.$$

Consider the primes between $\frac{n}{2}$ and n, the number of these is about $\frac{n}{2\log n}$. Hence at least one of them has a left neighbour $\geqq (1+o(1)) \frac{n}{2\log n}$, and thus the least common multiple here is $\geqq (1+o(1)) \frac{n}{2\log n} \cdot \frac{n}{2}$.

Now we construct a permutation satisfying

$$[a_i, a_{i+1}] \leqq \{1+o(1)\} \frac{n^2}{4\log n} \tag{8}$$

for all $i \leqq n-1$.

The idea is to take the multiples of a prime p as a block, and to separate the blocks by "small" numbers. Then the l.c.m. will not be too large at the border of the blocks. And inside a block

$$[a_i, a_{i+1}] \leqq \frac{n^2}{p} \tag{9}$$

which is good if p is not too small. Finally we have to arrange the numbers having only small prime factors.

Let us see the details. For the primes p up to n let k_p be the minimal exponent for which $q_p = p^{k_p} \geqq 4 \log n$ (i.e. $q_p = p$ if $p \geqq 4 \log n$).

We now define the set S of the "small" separator numbers: take $\Pi(n) - \Pi(\sqrt{n})$ numbers from 1 to some L just leaving out the values q_p and $2q_p$. Obviously $L = (1+o(1))\frac{n}{\log n}$.

We start the permutation by writing down alternately the primes between n and $\frac{n}{2}$ in decreasing order and the first elements of S in increasing order. (Here a block consists of p alone.) To show (8) we observe that when we arrive to $p \sim cn$, then we have used up $\Pi(n) - \Pi(cn) \sim (1-c)\frac{n}{\log n}$ small numbers, i.e. the l.c.m. of p and its neighbour is

$$c(1-c)\frac{n^2}{\log n} \leqq 4 \log n. \tag{10}$$

For the primes between $\frac{n}{2}$ and $\sqrt{n}$ we slightly improve the construction. We take the largest prime, insert all its multiples (up to n) after it, leaving its double to the end. Now we choose the next even number of S as separator, start the next block with the double of the next prime, put in all the multiples and terminate it by the prime itself. Then we insert the next odd number of S as separator and repeat the alogirthm. (9) and (10) show that (8) is satisfied. We note that for $p < \frac{n}{8}$ we do not have to be so careful about the parity of the separator number, and for $p < \frac{n}{4\sqrt{\log n}}$ we do not really need separators at all.

Next we proceed similarly with the q_p values between $\sqrt{n}$ and $4 \log n$, but here of course we take only those multiples of q_p which have not yet been used up (either in the blocks, or as separators). q_p and $2q_p$ lie at the two ends of a block (they were excluded from S to be now at disposal), hence we can either omit the separators, or put in arbitrarily large numbers as separators. We shall insert as separators the numbers still left, i.e. which have all their prime power factors less than $4 \log n$ (and which were not in S). There are at most

$$2^{\Pi^*(4 \log n)} \sim n^{\frac{c}{\log \log n}}$$

Such numbers, where $\Pi^*(x)$ denotes the number of prime-powers up to x since there are $\Pi(\sqrt{n}) > n^{1/2-\varepsilon}$ blocks for $4 \log n \leqq q_p \leqq \sqrt{n}$, we can consume as separators all the numbers left. We have obviously

$$[a_i, a_{i+1}] \leqq \begin{cases} 2q_p n \leqq 2n^{3/2} & \text{at the border of the blocks} \\ \dfrac{n^2}{q_p} \leqq \dfrac{n^2}{4\log n} & \text{inside.} \end{cases}$$

PROOF OF THEOREM 2. To prove (2) we observe the well-known fact that there are cn numbers up to n which have a prime factor greater than $n^{1-\varepsilon/2}$ $\left(c=(1+o(1))\log\dfrac{1}{1-\varepsilon/2}\right)$, hence we must keep nearly all of them. When we jump from a multiple of a large prime to a multiple of another large prime, then we either jump directly, but then the l.c.m. is at least $(n^{1-\varepsilon/2})^2$, or we insert a small number as separator, but then we need at least $(1+o(1))\dfrac{n}{\log n}$ separators, and so we obtain a l.c.m. greater than $n^{1-\varepsilon/2}\dfrac{n}{\log n}$.

To prove (3) we keep only those numbers whose largest prime factor lies between $n^{\varepsilon(n)}$ and $n^{1-\varepsilon(n)}$. It is well known that we omitted just $o(n)$ numbers (see e.g. [1]). We start the permutation by the largest prime left and its multiples, then we put the next prime followed by its multiples, etc. Here

$$[a_i, a_{i+1}] \leqq \begin{cases} \dfrac{n^2}{p} \leqq n^{2-\varepsilon(n)}, & \text{for two multiples of the same } p \\ n\cdot n^{1-\varepsilon(n)}, & \text{when jumping to a next prime.} \end{cases}$$

PROOF OF THEOREM 3. First we note that it is enough to construct a permutation $a_1, a_2, \ldots,$ of a *subsequence* of the natural numbers which satisfies (4), since we can insert the remaining elements afterwards arbitrarily rarely into this permutation.

We shall use the (probably well-known and nearly trivial) statement of the following lemma:

LEMMA. *Let H be a finite set, $|H|=h$ and $t\leqq h$. Then we can order the subsets having exactly t elements so that $|H_i\cap H_{i+1}|=t-1$ holds for all i.*

PROOF OF THE LEMMA. We prove by induction on h. The initial step is obvious. Now assume that the assertion is true for $h-1$ and for all t. Consider now h and any t. We fix an element x_0, take first all subsets containing x_0 and then take the other ones. Both parts can be ordered suitably by the induction hypothesis for $h-1$, $t-1$, and for $h-1$, t, resp. We have no difficulty either at joining the two parts, since if a "good" order exists, then a simple bijection of H can transform it into another "good" order with a prescribed first (or last) subset.

The construction of the permutation runs by an iterative process. Assume that for some n and $k=n^{\log n}$ we have $a_1, a_2, \ldots, a_k$ ready and no one of them has a prime factor greater than $\dfrac{n}{2}$. We take now all primes between $\dfrac{n}{2}$ and n, and form all the products consisting of v such (distinct) primes where $v=\log n+4\log\log n$. By the lemma we can arrange these products so that any two subsequent terms should differ only in one prime factor. This arrangement will be the next segment of the permutation from a_{k+2}. For a transition element a_{k+1} we can take e.g. any prime between $\frac{n}{2}$ and n. For $i \geqq k+1$ clearly

$$[a_i, {}_{i+1}] \leqq n^{v+1} \sim ke^{2\sqrt{\log k}\log\log k} \leqq ie^{2\sqrt{\log i}\log\log i}.$$

We have formed about

$$(11) \qquad r = \binom{\dfrac{n}{2\log n}}{\log n + 4\log\log n}$$

new terms of the permutation thus we arrived at least to a_r.

The algorithm will work if

$$(12) \qquad r > (2n)^{\log 2n}$$

holds. Estimating the binomial coefficient in (11) as a power of the smallest factor in the numerator and the greatest factor in the denominator we obtain

$$r \geqq \left(\frac{\dfrac{n}{2\log n} - \log n - 4\log\log n}{\log n + 4\log\log n}\right)^{\log n + 4\log\log n} \geqq \left(\frac{n}{\log^3 n}\right)^{\log n + 4\log\log n}$$

and (12) follows by an easy calculation.

Proof of Theorem 4. First we give a very simple proof of weaker form of (5) with $\frac{3}{2}$ instead of $\frac{1}{1-\log 2}$, i.e. that no permutation can satisfy

$$(13) \qquad [a_i, a_{i+1}] < \left(\frac{3}{2} - \varepsilon\right) i \quad \text{with a fix } \varepsilon \text{ for } i \geqq i_0.$$

We use the inequality

$$(14) \qquad \frac{1}{[a_i, a_{i+1}]} \leqq \frac{1}{3}\left\{\frac{1}{a_i} + \frac{1}{a_{i+1}}\right\}$$

which is equivalent to

$$(15) \qquad 3 \leqq \frac{a_i}{(a_i, a_{i+1})} + \frac{a_{i+1}}{(a_i, a_{i+1})}$$

and hence it is obvious, since the minimal value of the two terms on the right-hand side of (15) is 1 and 2.

Assuming (13) we obtain

$$\sum_{i=1}^{n} \frac{1}{[a_i, a_{i+1}]} \geqq \sum_{i=i_0}^{n} \frac{1}{[a_i, a_{i+1}]} > \left(\frac{2}{3} + \varepsilon'\right) \sum_{i=i_0}^{n} \frac{1}{i} \geqq \left(\frac{2}{3} + \varepsilon'\right) \log n - K.$$

On the other hand, using (14) we have

$$\sum_{i=1}^{n} \frac{1}{[a_i, a_{i+1}]} \leqq \frac{1}{3}\left\{\sum_{i=1}^{n} \frac{1}{a_i} + \frac{1}{a_{i+1}}\right\} < \frac{2}{3}\sum_{i=1}^{n} \frac{1}{i} \leqq \frac{2}{3}\log n + K'$$

which is a contradiction if n is large enough.

Now we turn to the proof of (5). Assume indirectly that for some permutation, $\varepsilon>0$ and i_0 we have

$$[a_i, a_{i+1}] < i\frac{1}{1-\log 2+\varepsilon} \quad \text{if} \quad i \geqq i_0. \tag{16}$$

This clearly implies also

$$a_i < i\frac{1}{1-\log 2+\varepsilon} \quad \text{for} \quad i \geqq i_0$$

hence $a_1, a_2, \ldots, a_n$ are all smaller than

$$N = n\frac{1}{1-\log 2-\varepsilon} \tag{17}$$

if n is large enough. From now on we shall consider only the a_i-s with $i \leqq n$.

Let us call the primes greater than $\sqrt{N}$ and smaller then N "large primes". If a_i and a_{i+1} have different large prime factors, then $[a_i, a_{i+1}] \geqq N$ in contradiction to (16). Hence we must insert "separators" between a_i-s containing different large prime factors (the separators cannot have large prime factors, of course). If a_i is the greatest separator element and a_{i+1} has a large prime factor then $[a_i, a_{i+1}] \geqq a_i\sqrt{N}$. Hence we again arrive at a contradiction by showing that there are at least $\sqrt{N}$ separators, or equivalently, there are at least $\sqrt{N}$ large primes which occur as factors of a_i-s.

We know that there are $(1+o(1))N\log 2$ numbers up to N having a large prime factor and we have $(1-\log 2+\varepsilon)N$ a_i-s [see (17)], hence at least εN a_i-s have a large prime factor. All of these a_i-s cannot be multiples of less than $\sqrt{N}$ large primes: indeed, the number of multiples up to N of $\sqrt{N}$ large primes is

$$\sum_{\text{,,these } p\text{''}} \left[\frac{N}{p}\right] < \sum_{\sqrt{N}<p\leqq N^{1/2+\varepsilon/2}} \left[\frac{N}{p}\right] = (1+o(1))N\log\frac{\frac{1}{2}+\frac{\varepsilon}{2}}{1/2} < \varepsilon N.$$

PROOF OF THEOREM 5. The permutation 1, 2, 6, 3, 12, 4, 20, 5, 35, 7, … clearly satisfies (6), i.e., if we have already constructed a_{2n} and k is the smallest number which was not yet used, then a_{2n+1} should be a common multiple of a_{2n} and k (e.g. the smallest one still available) anf put $a_{2n+2}=k$.

To prove (7) we observe first the following facts:

Let $b_1, b_2, \ldots$ be arbitrary different natural numbers not greater than n. Then:

$$(b_1, b_2) \leqq \frac{n}{2}, \tag{18}$$

$$(b_1, b_2) \leqq \frac{n}{3} \quad \text{or} \quad (b_2, b_3) \leqq \frac{n}{3}, \tag{19}$$

$$\min_{l\leqq i\leqq 4} (b_i, b_{i+1}) \leqq \frac{n}{4}. \tag{20}$$

(18) is obvious. To show (19) assume indirectly that e.g. $\frac{n}{3} < d = (b_1, b_2) \geqq (b_2, b_3)$. Then either $b_1=d$ and $b_2=2d$ or $b_1=2d$ and $b_2=d$, but in both cases b_3 must be

at least $3d$ which is a contradiction. We can show (20) by similar methods. Put

$$d = \min_{1 \leqq i \leqq 4} (b_i, b_{i+1}) = (b_k, b_{k+1}).$$

Arguing again indirectly we have $\{b_k, b_{k+1}\}$ $\{d, 2d, 3d\}$, and taking step by step the neighbouring numbers, and having in mind that the greatest common divisor must be at least d, we obtain that

$$\{b_1, \ldots, b_5\} \subseteq \left\{d, 2d, 3d, \frac{3}{2} d\right\}$$

which is a contradiction.

Now we are ready to prove (7). Assume that $(a_i, a_{i+1}) > \frac{1}{c} i$ if i is large enough. Then $\{a_1, a_2, \ldots, a_{cn}\} \supseteqq \{1, 2, \ldots, n\}$ if n is large enough. On the other hand, taking $a_{cn/2}, \ldots, a_{cn-1}, a_{cn}$, at mos tevery second number can be less than or equal to n, since

$$(a_i, a_{i+1}) > \frac{1}{c} i \geqq \frac{1}{c} \cdot \frac{cn}{2} = \frac{n}{2}$$

which is impossible by (18) if both a_i and a_{i+1} are less than or equal to n.

Similarly, using (19) we obtain that at most the two third part of $a_{cn/3}, \ldots, a_{cn/2}$ is not greater than n, and finally using (20) we conclude that at most the 4/5 part of $a_{cn/4}, \ldots, a_{cn/3}$ is smaller than n. Hence

$$n \leqq \left(cn - \frac{cn}{2}\right) \cdot \frac{1}{2} + \left(\frac{cn}{2} - \frac{cn}{3}\right) \cdot \frac{2}{3} + \left(\frac{cn}{3} - \frac{cn}{4}\right) \cdot \frac{4}{5} + \frac{cn}{4}$$

i.e. $\frac{1}{c} \leqq \frac{61}{90}$, as asserted.

Remarks. 1. We can improve (7) somewhat, if we use further inequalities of the type (18), (19) and (20). But this does not seem to give a serious reduction, and also the discovery of the proper inequalities is not too easy. E.g. $\min_{1 \leqq i \leqq 12} (b_i, b_{i+1}) \leqq \frac{n}{5}$, and here 12 cannot be replaced by 11, as shown by the numbers

$$81j,\ 162j,\ 108j,\ 54j,\ 216j,\ 72j,\ 144j,\ 48j,\ 96j,\ 192j,\ 128j,\ 64j$$

$$\left(n = 216j,\ d = 48j = \frac{2}{9} n\right).$$

2. We mention the following related problem, where we can determine the extremum exactly:

Theorem 6.

$$\liminf_i \frac{\min \{a_i, |a_{i+1} - a_i|\}}{i} \leqq \frac{3}{4} \tag{21}$$

is true for any permutation, and we can construct a permutation where equality holds.

PROOF OF THEOREM 6. The following permutation shows the possibility of equality:

$$1, 2, 3, 6, 4, 8, 5, 10, \ldots,$$

i.e. we always take the smallest number still available followed by its double.

To prove (21) we assume indirectly that there is a permutation satisfying (22)

$$a_i > \left(\frac{3}{4}+\varepsilon\right) i \quad \text{and} \quad |a_{i+1}-a_i| > \left(\frac{3}{4}+\varepsilon\right) i \quad \text{for} \quad i \geqq i_0 \quad \text{with a fix} \quad \varepsilon > 0.$$

Then all the numbers up to $\left(\frac{3}{4}+\varepsilon\right)N$ must occur among $a_1, \ldots, a_N$, if N is large enough. This also means that at least $\left(\frac{1}{4}+\varepsilon\right)N$ numbers smaller than $\left(\frac{3}{4}+\varepsilon\right)N$ must appear among $a_{N/2+1}, \ldots, a_N$. Thus we obtain an $i > \frac{N}{2}$, for which both a_i and a_{i+1} are smaller than $\left(\frac{3}{4}+\varepsilon\right)N$. Say $a_{i+1} > a_i$, then

$$\left(\frac{3}{4}+\varepsilon\right) N > a_{i+1} = a_i + (a_{i+1}-a_i) > 2\left(\frac{3}{4}+\varepsilon\right) i > 2\left(\frac{3}{4}+\varepsilon\right)\frac{N}{2}$$

which is a contradiction.

We note that the proof gives slightly more, since we did not make really use of the ε in (22).

Acknowledgement. The authors express their gratitude to Prof. M. Simonovits for this remarks concerning Theorem 1.

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(Received September 22, 1981)

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Acta Math. Hung.
41 (1—2), (1983), 177—185.

ON SUMS OF SUBSEQUENT TERMS OF PERMUTATIONS

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For the finite case let $a_1, a_2, \ldots, a_n$ be a permutation of the integers $1, 2, \ldots, n$ and for the infinite case let $a_1, a_2, \ldots, a_i, \ldots$ be a permutation of all positive integers.

Some problems and results concerning such permutations and related questions can be found in [2] (see in particular p. 94). In [1] the least common multiple and the greatest common divisor of two subsequent elements is estimated.

In the present paper we shall investigate the density of the sums of subsequent elements. As a special case we shall also obtain an answer to a question of N. Hegyvári, which asked for the smallest possible value of the maximum of the sums a_i+a_{i+1} $(1\leqq i\leqq n-1)$ concerning all such permutations of $1, 2, \ldots, n$, where the sums a_i+a_{i+1} are distinct (see the results on $f(n)$ in Theorem 3).

Let us consider first the infinite case. For an infinite permutation put $T=\{t|t=a_i+a_{i+1}$ is solvable for some $i\}$, and

$$R=\limsup_{x\to\infty}\frac{T(x)}{x},\quad r=\liminf_{x\to\infty}\frac{T(x)}{x} \tag{1}$$

where $T(x)$ denotes the number of elements of T up to x.

For the identical permutation T consists of the odd numbers greater than 1, and $R=r=\frac{1}{2}$.

We shall show that the identical permutation is, in some sense, best possible (see 1.2 in Theorem 1), but this is not the case from several other points of view (see 1.1 in Theorem 1, and Theorems 2 and 3).

THEOREM 1. 1.1. *The largest possible value of R is $2/3$. More precisely, to any function $H(x)$ with $\limsup_{x\to\infty} H(x)=\infty$ we can construct a permutation satisfying*

$$T(n)>\frac{2}{3}n-H(n) \tag{2}$$

for infinitely many values of n, and this is best possible, since for any permutation

$$T(n)-\frac{2}{3}n\to-\infty. \tag{3}$$

Moreover $R=\frac{2}{3}$ implies $r=0$.

1.2. *The largest possible value of r is* $\frac{1}{2}$. *More precisely, for any permutation*

$$T(n) < \left[\frac{n}{2}\right] \tag{4}$$

must hold for infinitely many values of n, and this is best possible, since e.g. for the identical permutation

$$T(n) = \left[\frac{n-1}{2}\right] \tag{5}$$

for all n. *Moreover* $r=\frac{1}{2}$ *implies* $R=\frac{1}{2}$.

1.3. *In general*

$$2R^2-1+\frac{(1-R)^2}{1-r} \leqq 0 \tag{6}$$

or equivalently (since by (6) *clearly* $1-2R^2 \geqq 0$*):*

$$r \leqq \frac{R(2-3R)}{1-2R^2} \tag{7}$$

REMARKS. 1. (6) [or (7)] contain some part of 1.1 and 1.2 as special case, namely: $R \leqq \frac{2}{3}$, $R=\frac{2}{3}$ implies $r=0$, $r \leqq \frac{1}{2}$ and $r=\frac{1}{2}$ implies $R=\frac{1}{2}$.

2. Concerning 1.2, there are other "extremal" permutations too which satisfy (5), e.g.: $j, j-1, j-2, \ldots, 1, 2j, 2j-1, \ldots, j+1, 3j, \ldots$

3. The results in 1.1 and 1.2 might suggest that one or more of the following inequalities hold:

$$3R+r \leqq 2, \tag{i}$$

$$R+r \leqq 1, \tag{ii}$$

$$Rr \leqq \frac{1}{4}. \tag{iii}$$

It is easy to check that (i) implies (ii) and (ii) implies (iii), and they clearly hold for $R \leqq \frac{1}{2}$, and also for $R=\frac{2}{3}$.

For the identical permutation we have equality in all three cases, thus the truth of (i)—(iii) would show another extremal property of the identical permutation. These inequalities are, however, not always valid:

THEOREM 2. *There are permutations for which* (i), (ii) *and* (iii) *are false. Namely we can construct permutations with*

$$r = \frac{1}{8}\cdot\frac{9-14R}{2-3R}\left(R \leqq \frac{9}{14}\right) \quad \textit{or} \quad R = \frac{9-16r}{14-24r}, \tag{8}$$

and then optimizing the left hand sides of (i)—(iii) *we obtain*

$$(9)\qquad \begin{cases} 3R+r = 2+\dfrac{7-2\sqrt{6}}{12} \sim 2{,}175 & \left(\textit{for}\quad R=\dfrac{2}{3}-\dfrac{1}{6\sqrt{6}}\right) \\ R+r = 1+\dfrac{3-2\sqrt{2}}{12} \sim 1{,}014 & \left(\textit{for}\quad R=\dfrac{2}{3}-\dfrac{1}{6\sqrt{2}}\right) \\ R\cdot r = \dfrac{1}{4}+\dfrac{11-4\sqrt{7}}{72} \sim 0{,}256 & \left(\textit{for}\quad R=\dfrac{2}{3}-\dfrac{1}{3\sqrt{7}}\right). \end{cases}$$

REMARKS. 1. On the other hand, using (7) we can easily check that (ii) holds for $R \geqq 0{,}619$ and (iii) holds for $R \geqq 0{,}608$, and for all permutations we have

$$(10)\qquad \begin{cases} 3R+r \leqq 2{,}236 \\ R+r \leqq 1{,}042 \\ Rr \leqq 0{,}270. \end{cases}$$

2. (8) and consequently (9) can be improved, e.g. we can construct permutations also with

$$(11)\qquad R = \frac{12r^2-7r}{12r^2-6r-1}$$

which is greater than the value of R given in (8) for $\frac{3}{8}<r<\frac{1}{2}$. We shall sketch this construction at the end of the proof of Theorem 2. Even this construction can be improved, but we think that the main interest lies in the falsity of (i)—(iii), and therefore we did not elaborate the technical details of the further improvement. Anyway, we mention that using (11) optimization yields $3R+r \sim 2{,}184$, $R+r \sim 1{,}018$ and $Rr \sim 0{,}257$.

Now we turn to the finite case, where we have two different though similar formulations of the problem.

For the first one consider permutations of $1, 2, \ldots, n$ and put

$$f(n) = \max \frac{\text{the number of different values of } a_i+a_{i+1}}{\max\limits_{1\leqq i\leqq n-1} \{a_i+a_{i+1}\}}$$

where the maximum is to be taken for all permutations.

For the second formulation take permutations of $1, 2, \ldots, m$ for any m, and put

$$g(n) = \frac{\max\{\text{the number of different values of } a_i+a_{i+1}, \text{ where } a_i+a_{i+1}\leqq n\}}{n}$$

and the maximum is to be taken for all m and for all permutations.

THEOREM 3. *Both* $f(n)$ *and* $g(n)$ *are* $\frac{2}{3}+O\left(\frac{1}{n}\right)$. *More precisely*

$$(12)\qquad f(2j) = \frac{2j-1}{3j},\quad f(2j-1) = \frac{2j-2}{3j-2}$$

and

$$g(3j)=\frac{2j-1}{3j},\quad g(3j-1)=\frac{2j-2}{3j-1},\quad g(3j-2)=\frac{2j-2}{3j-2}. \tag{13}$$

Finally, we mention that we can obtain analogous results for the sums of more than two subsequent terms of permutations. We just state here some results for the infinite case. Let

$$R_k=\limsup_{x\to\infty}\frac{T_k(x)}{x}\quad\text{and}\quad r_k=\liminf_{x\to\infty}\frac{T_k(x)}{x},$$

where $T_k(x)$ is the number of different values of $a_i+a_{i+1}+\ldots+a_{i+k-1}$ up to x. ($T_2(x)=T(x)$, $R_2=R$, $r_2=r$.)

THEOREM 4.

$$kR_k^2-1+\frac{(1-R_k)^2}{1-r_k}\leqq 0$$

or equivalently

$$r_k\leqq\frac{R_k[2-(k+1)R_k]}{1-kR_k^2}.$$

In particular

$$R_k\leqq\frac{2}{k+1},\quad r_k\leqq\frac{1}{k}.$$

Equality can hold, but then necessarily $r_k=0$ *and* $R_k=\frac{1}{k}$, *resp.*

PROOF OF THEOREM 3. Concerning $f(n)$ consider the permutation

$$2j,\ j,\ 1,\ j+1,\ 2,\ j+2,\ldots,j-1,\ 2j-1 \tag{14}$$

for $n=2j$ and omit the first term for $n=2j-1$. This shows that $f(2j)\geqq\frac{2j-1}{3j}$ and $f(2j-1)\geqq\frac{2j-2}{3j-2}$. To prove the converse inequalities in (12), take an arbitrary permutation $a_1,a_2,\ldots,a_n$, and assume that there are $n-1-v$ different values of the sums a_i+a_{i+1} while $\max\limits_{1\leqq i\leqq n-1}\{a_i+a_{i+1}\}=M$. Then the quotient in question is $\frac{n-1-v}{M}$. Clearly

$$\frac{n-1-v}{M}\leqq\frac{n-1}{M+v} \tag{15}$$

and

$$S=\sum_{i=1}^{n-1}(a_i+a_{i+1})\leqq(M+v)+(M+v-1)+\ldots+(M+v-n+2)= \tag{16}$$

$$=(M+v)(n-1)-\frac{(n-1)(n-2)}{2}.$$

By (15) it is enough to show that

$$M+v\geqq 3j\quad\text{for}\quad n=2j \tag{17a}$$

and

(17b) $$M+v \geqq 3j-2 \quad \text{for} \quad n=2j-1.$$

We carry through the calculations for (17a), the case of odd n runs similarly. Assume indirectly that $M+v \leqq 3j-1$ $(n=2j)$, then applying (16) we obtain

(18) $$S \leqq 2j(2j-1).$$

On the other hand, we get a lower estimation for S by observing

$$S = 2(1+2+\ldots+n)-(a_1+a_n) \geqq (n+1)n-2n+1 = 2j(2j-1)+1,$$

which is a contradiction to (18).

Concerning $g(n)$, consider again the permutation (14) for $n=3j$, and omit the first term for $n=3j-1$ and $n=3j-2$. This shows that

$$g(3j) \geqq \frac{2j-1}{3j}, \quad g(3j-1) \geqq \frac{2j-2}{3j-1} \quad \text{and} \quad g(3j-2) \geqq \frac{2j-2}{3j-2}.$$

To prove the converse inequalities in (13) assume that for some permutation there are w different values of a_i+a_{i+1} up to n. If we take the sum S' of these a_i+a_{i+1}, then

(19) $$S' \leqq n+(n-1)+\ldots+(n-w+1) = nw-\frac{w(w-1)}{2}$$

and on the other hand

(20) $$S' \geqq 2\{1+2+\ldots+(w-1)\}+w+(w+1) = w(w+1)+1,$$

since any number can be a term in at most two sums a_i+a_{i+1} (once as a_i, once as a_{i+1}) and also not all numbers can occur twice. Hence the best case is to take the first $(w-1)$ numbers twice plus w and $w+1$ once. Confronting the two estimations we obtain

$$w(w+1)+1 \leqq nw-\frac{w(w-1)}{2}$$

or

$$w \leqq \frac{2n-1}{3}-\frac{2}{3w}$$

i.e.

$$w \leqq \begin{cases} \left[\frac{2n-1}{3}\right] & \text{if} \quad 3 \nmid 2n-1 \\ \left[\frac{2n-1}{3}\right]-1 & \text{if} \quad 3 \mid 2n-1 \end{cases}$$

which give exactly the right hand sides of (13).

Proof of Theorem 1. 1.1. The result we obtained for $g(n)$ in Theorem 3 immediately yields $R \leqq \frac{2}{3}$, and the proof also shows that $R=\frac{2}{3}$ implies $r=0$: if for a sequence n_k we have

$$T(n_k) = \frac{2}{3} n_k + o(n_k)$$

then

$$T\left(\frac{n_k}{3}\right) = o\left(\frac{n_k}{3}\right)$$

must also hold.

To prove (2) we use an iterative translated version of (14). Let $c_1 < c_2 < \ldots < c_k < \ldots$ be arbitrary positive itegers, and consider the following permutation:

$$\begin{cases} 1,\ c_1+1,\ 2,\ c_1+2, \ldots, c_1,\ 2c_1,\ 2c_1+1,\ c_1+c_2+1,\ 2c_1+2, \\ c_1+c_2+2, \ldots, c_1+c_2,\ 2c_2,\ 2c_2+1,\ c_2+c_3+1,\ 2c_2+2,\ c_2+c_3+2, \ldots \end{cases} \tag{21}$$

Thus while the a_i-s run through the interval $[2c_k+1, 2c_{k+1}]$ similarly to construction (14), the sums a_i+a_{i+1} cover the interval $[3c_k+c_{k+1}+2, c_k+3c_{k+1}]$ (and the sums of the border points of the subsequent intervals give the values $4c_k+1$).

It we choose c_{k+1} to be very large compared with c_k, then we can satisfy (2) for $n = c_k + 3c_{k+1}$.

To show (3) we use estimation (20) and a modified version of (19). Take a large but fixed K, and let n_0 be a number for which $T(n_0) \geqq K$. Denote the sum of the (distinct) values of a_i+a_{i+1} up to n_0 by U. Consider now an n much larger than n_0, put $w = T(n)$, and estimate the sum S' of the different values of a_i+a_{i+1} up to n. On the one hand

$$S' \leqq U + n + (n-1) + \ldots + (n-w+1+K) = U + n(w-K) - \frac{(w-K)(w-K-1)}{2},$$

and on the other hand (20) is valid. Hence

$$w(w+1)+1 \leqq n(w-K) - \frac{(w-K)(w-K-1)}{2} + U$$

or

$$n > \frac{w(w+1)}{w-K} + \frac{w-K-1}{2} - \frac{U}{w-K} = w + \frac{(K+1)w}{w-K} + \frac{w}{2} - \frac{K+1}{2} - \frac{U}{w-K} \geqq$$

$$\geqq \frac{3}{2} w + (K+1) - \frac{K+1}{2} - \varepsilon$$

i.e.

$$w \leqq \frac{2}{3} n - \frac{K+1}{3} + \varepsilon$$

which proves (3).

1.2. We have to show only (4) (the implication $r = \frac{1}{2} \Rightarrow R = \frac{1}{2}$ can be derived — as we have already mentioned — from (6)).

Assume indirectly that for some permutation $T(n) \geqq \left[\frac{n}{2}\right]$ for $n \geqq n_0$.

Denote the distinct elements of T in increasing order by $t_1, t_2, \ldots$. Let $2j_1$ be the maximal even number for which $T(2j_1) < j_1$ (possibly $j_1 = 1$). Then obviously

$T(2j_1)=j_1-1$, and

$$t_{j_1}=2j_1+1,\ t_{j_1+1}=2j_1+2,\ t_{j_1+2}\leqq 2j_1+4,\ \dots,\ t_j\leqq 2j,\ \dots .$$

Hence

$$S'=\sum_{k=1}^{w} t_k \leqq L+2+4+\dots+(2w-2)+2w = L+w(w+1) \tag{22}$$

where

$$L=\sum_{k=1}^{j_1}(t_k-2_k).$$

Estimating S' from below by (20) we do not arrive to the desired contradiction; we need a slightly sharpened form. Since a_1 can be a term in at most one t_k, we have

$$S'\geqq 2(1+2+\dots+w)-a_1+(w+1) = w(w+1)+(w+1)-a_1. \tag{23}$$

Comparing (22) and (23) we obtain $(w+1)-a_1\leqq L$ which is a contradiction if w is large enough.

1.3. Take an n with $w=T(n)\sim Rn$, and consider S'. For an upper estimation — roughly speaking — the "worst" case is, if the t_k elements follow each other as rarely as r permits this, and at the end we take each integer that $T(n)$ should really grow up for the value Rn. Thus till an y we take about every $1/r$-th integer (and do not take anything if $r=0$), and between y and n we take all integers. Then $ry+(n-y)\sim Rn$ i.e.

$$y\sim n\frac{1-R}{1-r},$$

and we have

$$S'\leqq\ \sim\frac{1}{r}(1+2+\dots+yr)^{*}+(y+1)+(y+2)+\dots+n\sim$$

$$\sim\frac{y^2r}{2}+\frac{n^2-y^2}{2}\sim\frac{1}{2}n^2\left\{1+\frac{(1-R)^2}{r-1}\right\}.$$

On the other hand by (20) $S'\geqq\sim(Rn)^2$ i.e.

$$R^2n^2\leqq\frac{1}{2}n^2\left\{1+\frac{(1-R)^2}{r-1}\right\}$$

and dividing by n^2 we obtain (6). Clearly, the proof can also be purely formalized.

Proof of Theorem 2. First we note that the falsity of (i) can be shown already by construction (21). To calculate R and r we have to estimate $T(n)/n$ for $n=c_k+3c_{k+1}$ and for $n=3c_k+c_{k+1}$, resp. and we obtain

$$\frac{2c_{k+1}}{3c_{k+1}+c_k}=\frac{2}{3+\dfrac{c_k}{c_{k+1}}}\quad\text{and}\quad\frac{2c_k}{3c_k+c_{k+1}}=\frac{2}{3+\dfrac{c_{k+1}}{c_k}},\ \text{resp.}$$

Thus denoting $\liminf_{k\to\infty}\frac{c_k}{c_{k+1}}$ by d, we have $R=\frac{2}{3+d}$ and $r=\frac{2}{3+1/d}$.

* For $r=0$ we simply omit this term.

Hence

$$3R+r = 2+4\frac{d-d^2}{3+10d+3d^2}$$

which gives the maximal value $2+4\cdot\frac{2-\sqrt{3}}{8}\sim 2{,}134$ $\left(\text{for } d=\frac{-3+\sqrt{3}}{13}\right)$. Thus (i) is false, but it is easy to see that (ii) [and (iii)] cannot be disproved using construction (21).

Now we construct permutations satisfying (8), by an iterative process.

Step 1. We assume that for some x we have already constructed the segment of the permutation using the first Rx numbers as a_i-s and obtained about Rx distinct sums a_i+a_{i+1} up to x $\left(\text{i.e. } \frac{T(x)}{x}\sim R\right)$.

Step 2. Between x and

$$\frac{R}{r}x = 2Sx \tag{24}$$

no number will occur as a_i+a_{i+1}, hence for $n=2Sx$, $\frac{T(n)}{n}\sim r$.

Step 3. We continue the permutation by taking each integer between $Sx+1$ and Tx (T will be suitably chosen later) and hence we obtain the odd numbers between $2Sx+2$ and $2Tx$ as the sums a_i+a_{i+1}. $\frac{T(n)}{n}$ did not fall below r, moreover it is increasing, since $r\leqq\frac{1}{2}$.

Step 4. The next segment of the permutation should be

$$Rx+1,\ (2T-R)x+1,\ Rx+2,\ (2T-R)x+2,\ \dots,\ Sx,\ (2T+S-2R)x,$$

i.e. we take the numbers from Rx to Sx and from $(2T-R)x$ to $(2T+S-2R)x$ alternately. Thus we obtain all numbers between $2Tx+2$ and $(2T+2S-2R)x$ as the sums a_i+a_{i+1}. $T(n)/n$ clearly keeps increasing.

Step 5. The next segment will be again a (14)-type construction for the numbers from Tx to $(2T-R)x$:

$$Tx+1,\ \frac{3T-R}{2}x+1,\ Tx+2,\ \frac{3T-R}{2}x+2,\dots,\frac{3T-R}{2}x,\ (2T-R)x.$$

The corresponding sums a_i+a_{i+1} will be all numbers from $\frac{5T-R}{2}x+2$ to $\frac{7T-3R}{2}x$. We want that these values should join to the values obtained in Step 4, i.e.

$$2T+2S-2R = \frac{5T-R}{2}$$

or

$$T = 4S - 3R \tag{25}$$

and we choose this value for T.

Finally we want that we should arrive after Step 5 to a similar situation as in Step 1, with $\frac{7T-3R}{2}x$ instead of x, and this requires only the equality

$$R\frac{7T-3R}{2} = 2T+S-2R. \tag{26}$$

Inserting first (25) and then (24) into (26) we obtain (8).

Now we sketch the construction of permutations satisfying (11). Steps 1, 2 and 3are the same as before.

Step 4. Between $2Tx$ and

$$\frac{T-S+R}{r}x = Ux \tag{27}$$

again no number will occur as a_i+a_{i+1}, hence also for $n=Ux$ we have $\frac{T(n)}{n}\sim r$.

Step 5. We take the numbers from Rx to Sx and from $(U-R)x$ to $(U+S-2R)x$ alternately (cf. Step 4 in the previous proof), and so the sums a_i+a_{i+1} cover the interval $[Ux+2,\ (U+2S-2R)x]$.

Step 6. We take the numbers from Tx to $\frac{U-R+T}{2}x$ and from $\frac{U-R+T}{2}x$ to $(U-R)x$ alternately, and so the sums a_i+a_{i+1} cover the interval $\left[\frac{U-R+3T}{2}x+2,\ \frac{3U-3R+T}{2}x\right]$. We want this to join to the pervious interval i.e.

$$U+2S-2R = \frac{U-R+3T}{2}. \tag{28}$$

And finally, after Step 6 we want to arrive to a similar situation as described in Step 1, i.e.

$$R\frac{3U-3R+T}{2} = U+S-2R. \tag{29}$$

(24), (27), (28) and (29) imply (11) by an easy calculation.

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(Received September 22, 1981)

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[1] G. Szegő, *Orthogonal polynomials,* AMS Coll, Publ. Vol. XXIII (Providence, 1939).
[2] A. Zygmund, Smooth functions, *Duke Math. J.,* **12** (1945), 47—76.

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FORMERLY ACTA MATHEMATICA ACADEMIAE SCIENTIARUM HUNGARICAE

Acta Mathematica Hungarica

VOLUME 41, NUMBERS 3–4, 1983

Akadémiai Kiadó, Budapest

ACTA MATH. HUNG. HU ISSN 0001—5954

ACTA MATHEMATICA
HUNGARICA

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Acta Math. Hung.
41 (3—4) (1983), 187—194.

A GENERALIZATION OF THE FREUD—SHARMA OPERATORS

N. MISRA (Lucknow)

1. Introduction

In 1964, G. Freud [1] constructed an almost interpolatory operator of degree at most $4n-3$, which led to an independent proof of Jackson's theorem in the closed interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. This result gave rise to an extensive literature in this direction. In 1974, G. Freud and A. Sharma [5] considered the similar problem and obtained a direct proof of A. F. Timan's theorem [4] for a continuous function in $[-1, 1]$, through the construction of some good sequences of polynomials of degree $\leqq n(1+C)$; $C>0$. They further improved the operator to obtain Teljakowski [2], Gopengauz [3] type estimate in [6].

In this paper we generalize Freud—Sharma (shortly F—S) operators $J_n^{(\alpha,\beta)}(f;x)$ and $A_n^{(\alpha,\beta)}(f;x)$ respectively based on the zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and $(1-x^2)P_n^{(\alpha,\beta)}(x)$ with $\alpha,\beta>-1$, defined in [5] without affecting their degree. The generalized operator $J_{n,p}^{(\alpha,\beta)}(f;x)$ (see Section 2, 2.6) is non interpolatory while $A_{n,p}^{(\alpha,\beta)}(f;x)$ (see Section 2, 2.10) is interpolatory in each closed sub-interval of $(-1,1)$ and both of them produce Timan's estimate for $f^{(p)}\in C[-1,1]$. Our main aim for generalizing the F—S operators is to achieve theorems on simultaneous approximations for a differentiable function in the special case of $\alpha=\beta=-\frac{1}{2}$. We prove

THEOREM 1. *Let $f^{(p)}\in C[-1,1]$ and denote $\omega_{f^{(p)}}(\delta)$ its modulus of continuity. Then for $0\leqq t\leqq p$*

$$|f^{(t)}(x)-J_{n,p}^{(-1/2,-1/2)(t)}(f;x)| \leqq C_p\left[\frac{(1-x^2)^{1/2}}{n}+\frac{|x|}{n^2}\right]^{p-t}\left[\omega_{f^{(p)}}\left(\frac{(1-x^2)^{1/2}}{n}\right)+\right.$$
$$\left.+\omega_{f^{(p)}}\left(\frac{1}{n^2}\right)\right],$$

where C_p is a constant depending on p and independent of n and x.

THEOREM 2. *Let $f^{(p)}\in C[-1,1]$. Then for $0\leqq t\leqq p$*

$$|f^{(t)}(x)-A_{n,p}^{(-1/2,-1/2)(t)}(f;x)| \leqq C_p\left[\frac{(1-x^2)^{1/2}}{n}+\frac{|x|}{n^2}\right]^{p-t}\left[\omega_{f^{(p)}}\left(\frac{(1-x^2)^{1/2}}{n}\right)+\right.$$
$$\left.+\omega_{f^{(p)}}\left(\frac{1}{n^2}\right)\right],$$

where C_p is a constant depending on p and independent of n and x.

2. The operators $J_{n,p}^{(\alpha,\beta)}(f, x)$ and $A_{n,p}^{(\alpha,\beta)}(f; x)$

Let $\{x_{k,n}\}_{k=1}^{n}$ be the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$; $\alpha, \beta > -1$ and denote $l_{k,n}(x)$ the fundamental polynomial of Lagrange interpolation based on these nodes. We shall denote $x_{k,n}$ by x_k; $l_{k,n}$ by l_k for the sake of convenience.

Let $m = [n\varrho]$, for some ϱ, $0 < \varrho < \dfrac{1}{(r+p)2^{(p+1)}}$, $p \geqq 0$ where $2r > \max(4, \alpha + 5/2, \beta + 5/2)$. We set

$$\varphi_m(x, y) = \frac{1}{m} \frac{T_{m+1}(x)T_m(y) - T_{m+1}(y)T_m(x)}{(x-y)}$$

where $T_m(x) = \cos m\theta$, $x = \cos\theta$ so that ([5], p. 238)

$$\varphi_m(x, x) = \frac{1}{m}\left[m + \frac{1}{2} + \frac{1}{2}\,\frac{\sin(2m+1)\theta}{\sin\theta}\right]. \tag{2.1}$$

Now we introduce the polynomials $\psi_p(x, y)$ of degree $\leqq 2^p m$ defined as follows:

$$\Psi_p(x, y) = \begin{cases} \varphi_m(x, y) & \text{if} \quad p = 0 \\ \Psi_{p-1}(x, y)\overline{\Psi}_{p-1}(x, y), & \text{if} \quad p \geqq 1, \end{cases} \tag{2.2}$$

where $\overline{\Psi}_{p-1}(x, y) = 2 - \Psi_{p-1}(x, y)$. Simplifying (2.2) we obtain

$$\Psi_p(x, y) = \varphi_m(x, y) \prod_{i=1}^{p} \left[1 + \{1 - \varphi_m(x,y)\}^{2^{i-1}}\right], \ p \geqq 1 \tag{2.3}$$

and

$$\Psi_p(x, x) = 1 - \left[\frac{1}{2m}\left(1 + \frac{\sin(2m+1)\theta}{\sin\theta}\right)\right]^{2^p}, (p \geqq 1). \tag{2.4}$$

Let

$$\lambda_p(x) = \frac{1}{2^{2p+1}}\left[\sum_{\nu=0}^{p}\binom{2p+1}{\nu}\right]\left[(1+x)^{2p+1-\nu}(1-x)^{\nu} \sum_{i=0}^{p} \frac{(x-1)^i}{i!} f^{(i)}_{(1)} + \right. \tag{2.5}$$

$$\left. + (1-x)^{2p+1-\nu}(1+x)^{\nu} \sum_{i=0}^{p} \frac{(x+1)^i}{i!} f^{(i)}(-1)\right].$$

Now we define

$$J_{n,p}^{(\alpha,\beta)}(f; x) = \lambda_p(x) + \sum_{k=1}^{n}\left[\sum_{i=0}^{p} \frac{(x-x_k)^i}{i!} f^{(i)}(x_k) - \lambda_p(x)\right]\Psi_p^{2r+2p}(x_k, x) l_k(x). \tag{2.6}$$

The operator $J_{n,p}^{(\alpha,\beta)}(f; x)$ is non-interpolatory and of degree $n + 3p + m(r+p)2^{p+1} \leqq \leqq n(1+C)$, $C > 0$ being fixed. Since $2^{p+1}(r+p)m \leqq n-1$

$$\Psi_p^{2r+2p}(x, x) = \sum_{k=1}^{n} \Psi_p^{2r+2p}(x_k, x) l_k(x). \tag{2.7}$$

From (2.6) and (2.7) we obtain

$$(2.8)\quad f(x)-J_{n,p}^{(\alpha,\beta)}(f;x)=\sum_{k=1}^{n}\left[f(x)-\sum_{i=0}^{p}\frac{(x-x_k)^i}{i!}f^{(i)}(x_k)\right]\cdot\Psi_p^{2r+2p}(x_k,x)l_k(x)+$$
$$+[f(x)-\lambda_p(x)][1-\Psi_p^{2r+2p}(x,x)].$$

Let $f^p\in C[-1,1]$, then using (2, 3), Lemma 3 (F—S [5]) and adopting the method of proof of Theorem 1 of Freud—Sharma [5], we obtain from (2.8) the Timan type estimate

$$(2.9)\quad |f(x)-J_{n,p}^{(\alpha,\beta)}(f;x)|\leqq C_p\left[\frac{(1-x^2)^{1/2}}{n}+\frac{|x|}{n^2}\right]^p\left[\omega_f(p)\left(\frac{(1-x^2)^{1/2}}{n}\right)+\omega_f(p)\left(\frac{1}{n^2}\right)\right].$$

The operator $J_{n,p}^{(\alpha,\beta)}(f;x)$ cannot be made interpolating like the F—S operator $J_n^{(\alpha,\beta)}(f;x)$, because the expression $\psi_p(x_k,x_k)$, $(p\geqq 1)$ vanishes when $\varphi_m(x_k,x_k)=2$. However, if we set for $\delta>0$ (δ being fixed) and $m\geqq 2$,

$$(2.10)\qquad A_{n,p}^{(\alpha,\beta)}(f;x)=J_{n,p}^{(\alpha,\beta)}(f;x)+$$
$$+\sum_{x_k\in[-1+\delta,1-\delta]}\left[\sum_{i=0}^{p}\frac{(x-x_k)^i}{i!}f^{(i)}(x_k)-\lambda_p(x)\right]\frac{\Psi_p^{2r+2p}(x_k,x)1_k(x)[1-\Psi_p^{2r+2p}(x_k,x)]}{\Psi_p^{2r+2p}(x_k,x_k)}$$

then $A_{n,p}^{(\alpha,\beta)}(f;x_k)=f(x_k)$ for $x_k\in[-1+\delta,1-\delta]$.

Again using the facts that for $x_k\in[-1+\delta,\ 1-\delta]$

$$(2.11)\qquad \Psi_p(x_k,x_k)\geqq 1-\left(\frac{1}{m}\right)^{2p}\geqq 1-\frac{1}{2^{2p}}\quad\text{for}\quad m\geqq 2$$

$$(2.12)\qquad |1-\Psi_p(x_k,x_k)|\leqq\frac{1}{m^{2p}},$$

we obtain as in Theorem 2 (F—S [5])

$$(2.13)\qquad |f(x)-A_{n,p}^{(\alpha,\beta)}(f;x)|\leqq$$
$$\leqq C_p\left[\frac{(1-x^2)^{1/2}}{n}+\frac{|x|}{n^2}\right]^p\left[\omega_f(p)\left(\frac{(1-x^2)^{1/2}}{n}\right)+\omega_f(p)\left(\frac{1}{n^2}\right)\right].$$

3. Some lemmas

In this section we establish lemmas needed for our purpose. Let

$$x_k=\cos\frac{(2k-1)\pi}{2n};\ k=1,\ldots,n$$

be the zeros of the Tchebycheff polynomial $T_n(x)=\cos n\theta$; $\cos\theta=x$ and let $l_k(x)$ be the fundamental polynomials of the Lagrange interpolation based on these nodes. Then we have the following:

LEMMA 1.

(i) $\sin\left(\frac{\theta-\theta_k}{2}\right) D^q\varphi_m(x_k, x) \leqq C_1 m^{2q-1}$,

(ii) $\sin^q\theta \sin\left(\frac{\theta-\theta_k}{2}\right) D^q\varphi_m(x_k, x) \leqq C_2 m^{q-1}$,

(iii) $\sin\left(\frac{\theta-\theta_k}{2}\right) D^q l_k(x) \leqq C_3 n^{2q-1}$,

(iv) $\sin^q\theta \sin\left(\frac{\theta-\theta_k}{2}\right) D^q l_k(x) \leqq C_4 n^{q-1}$,

where $D^q \equiv \frac{d^q}{dx^q}$.

LEMMA 2.

(i) $(x-x_k)D^q l_k(x) \leqq C_5\left[\left|\sin^{2q-1}\left(\frac{\theta-\theta_k}{2}\right)\right| n^{2q-1} + \left|\sin^{q-1}\left(\frac{\theta-\theta_k}{2}\right)\right| n^{q-1}\right]$

(ii) $(x-x_k)^q D^q[\Psi_p^{2r+2p}(x_k, x)] \leqq C_6 \varphi_m^{2r+2p-2q}(x_k, x)\ (q \leqq p)$.

PROOF OF LEMMA 1. We shall first show that

(a) $|T_{m+1}^{(q)}(x) - T_m^{(q)}(x)| \leqq \bar{C}_1[m^{2q}\sin\theta/2 + m^{2q-1}]$

(b) $\sin^q(\theta)|T_{m+1}^{(q)} - T_m^{(q)}(x)| \leqq \bar{C}_2[m^q \sin\theta/2 + m^{q-1}]$.

We shall obtain the estimates (a) and (b) by induction. Let $q=1$, so

$$T'_{m+1}(x) - T'_m(x) = m\,\frac{\sin(m\theta)}{\sin\theta} - (m+1)\,\frac{\sin(m+1)\theta}{\sin\theta}.$$

Therefore $T'_{m+1}(1) - T'_m(1) = -(2m+1)$; and

$$|T'_{m+1}(x) - T'_m(x)| \leqq m\left|\frac{\sin m\theta}{\sin\theta} - \frac{\sin(m+1)\theta}{\sin\theta}\right| + \frac{|\sin(m+1)\theta|}{\sin\theta} \leqq$$

$$\leqq \frac{2m\sin\theta/2}{\sin\theta} + \frac{|\sin(m+1)\theta|}{\sin\theta}.$$

So that (a) and (b) are true for $q=1$.

Let us assume that the result is true for integers $\leqq q-1$. From the differential equation satisfied by $T_m(x)$; $(1-x^2)T''_m(x) - xT'_m(x) + m^2 T_m(x) = 0$ we get

$$(1-x^2)T_m^{(q)}(x) - (2q-3)xT_m^{(q-1)}(x) + (m^2-(q-2)^2)T_m^{(q-2)}(x) = 0.$$

Hence

$$(1-x^2)|T^{(q)}_{m+1}(x)-T^{(q)}_{m}(x)| \leqq (2q-3)|x|\|[T^{(q-1)}_{m+1}(x)-T^{(q-1)}_{m}(x)]|+$$
$$+|[m^2-(q-2)^2]|\,|[T^{(q-2)}_{m+1}(x)-T^{(q-2)}_{m}(x)]|+2m+1)|T^{(q-2)}_{m+1}(x)| \leqq$$
$$\leqq C[m^{2(q-1)}\sin\theta/2+m^{2q-3}].$$

Therefore estimates (a) and (b) follow.

Now we prove Lemma 1(i) and (ii). Since

$$\varphi_m(x_k, x) = \frac{1}{m}\left[\frac{T_{m+1}(x)T_m(x_k)-T_{m+1}(x_k)T_m(x)}{(x-x_k)}\right],$$

therefore

(3.2) $$(x-x_k)D^q\varphi_m(x_k, x)+qD^{q-1}\varphi_m(x_k, x) =$$

$$= \frac{1}{m}[(T^{(q)}_{m+1}(x)-T^{(q)}_{m}(x))T_m(x_k)-(T_{m+1}(x_k)-T_m(x_k))T^{q}_{m}(x)].$$

If $q=1$, then using Lemma 3(i) (iv) (F—S [5]) and estimate (a), we get

$$\left|\sin\left(\frac{\theta-\theta_k}{2}\right)\varphi_m'(x_k, x)\right| \leqq \frac{\varphi_m(x_k, x)}{2\sin\left(\frac{\theta+\theta_k}{2}\right)}+C\frac{\left[m\left(\sin\frac{\theta}{2}+\sin\frac{\theta_k}{2}\right)+1\right]}{2\sin\left(\frac{\theta+\theta_k}{2}\right)} \leqq m$$

and

$$\left|\sin\theta\sin\left(\frac{\theta-\theta_k}{2}\right)\varphi_m'(x_k, x)\right| \leqq C.$$

Thus Lemma 1(i) and (ii) is true for $q=1$. If we assume that the lemma is true for $q-1$, then using estimates (a) and (b) and the fact that $T^{(q)}_m(x) \leqq m^{2q}$, Lemma 1(i) and (ii) follow from (3.2).

We omit the proof of Lemma 1(iii) and (iv) as it is similar to the proof of Lemma 1(i) and (ii).

Proof of Lemma 2. Owing to the inequality $\left(\frac{a+b}{2}\right)^m \leqq \frac{a^m+b^m}{2}$ and Lemma 1(iii) and (iv) we get

$$(x-x_k)^q D^q l_k(x) \leqq C\left[\sin\theta\sin\left|\left(\frac{\theta-\theta_k}{2}\right)\right|+\sin^2\left(\frac{\theta-\theta_k}{2}\right)\right]^q l_k^{(q)}(x) \leqq$$

$$\leqq C\left[\sin^{2q}\left(\frac{\theta-\theta_k}{2}\right)+\sin^q\theta\left|\sin^q\left(\frac{\theta-\theta_k}{2}\right)\right|\right] l_k^{(q)}(x) \leqq$$

$$\leqq C\left[\left|\sin^{2q-1}\left(\frac{\theta-\theta_k}{2}\right)\right| n^{2q-1}+\left|\sin^{q-1}\left(\frac{\theta-\theta_k}{2}\right)\right| n^{q-1}\right].$$

Thus lemma 2(i) is proved.

We establish Lemma 2(ii) by the method of induction. Now owing to (2.3), Lemma 3(i)(iv) (F.S [5]) and Lemma 1(i) and (ii) we have

$$|(x-x_k)D[\Psi_p^{2r+2p}(x_k,x)]| \leqq C\left[\sin\theta\left|\sin\left(\frac{\theta-\theta_k}{2}\right)\right|+\sin^2\left(\frac{\theta-\theta_k}{2}\right)\right]\cdot$$

$$\cdot|\varphi_m^{2r+2p-1}(x_k,x)|\,|\varphi_m'(x_k,x)| \leqq C\left[1+m\sin\frac{|\theta-\theta_k|}{2}\right]|\varphi_m^{2r+2p-1}(x_k,x)| \leqq$$

$$\leqq C\varphi_m^{2r+2p-2}(x_k,x).$$

Thus the lemma is true for $q=1$.

Let us assume that the lemma is true for $q=t$. Then

$$|(x-x_k)^t[\Psi_p^{2r+2p}(x_k,x)]^{(t)} \leqq C\varphi_m^{2r+2p-2t}(x_k,x). \tag{3.3}$$

Differentiating (3.3) and using Lemmas 1(i), (ii), Lemma 3 (F—S [5]) we get

$$|(x-x_k)^{t+1}\Psi_p^{2r+2p}(x_k,x)]^{t+1}| \leqq C\varphi_m^{2r+2p-2(t+1)}(x_k,x).$$

Hence Lemma 2(ii) is true for $q\leqq p$.

4. Proof of Theorem 1

In the sequel C_p denotes constant not necessarily the same depending only on p. From (2.8) we have

$$f^{(t)}(x)-J_{n,p}^{(-1/2,-1/2)(t)}(f;x) = \sum_{k=1}^{n}\sum_{s=0}^{t}\binom{t}{s}D^{t-s}\Bigg[f(x)- \tag{4.1}$$

$$-\sum_{i=1}^{p}\frac{(x-x_k)^i}{i!}f^{(i)}(x_k)\Bigg]D^s[\Psi_p^{2r+2p}(x_k,x)l_k(x)]+\sum_{s=0}^{t}\binom{t}{s}D^{t-s}[f(x)-$$

$$-\lambda_p(x)]D^s[1-\Psi_p^{2r+2p}(x,x)] = A_1^{(t)}(x)+A_2^{(t)}(x).$$

Using (2.3), Lemma 2, Lemma 3 (i), IV (F—S [5]), we have

$$|A_1^{(t)}(x)| \leqq C_p\sum_{k=1}^{n}|x-x_k|^{p-t}\omega_{f^{(p)}}|[x-x_k]|+ \tag{4.2}$$

$$+\sum_{s=0}^{t}\sum_{q=0}^{s}|x-x_k|^s\,|D^{s-q}\Psi_p^{2r+2p}(x_k,x)D^q l_k l(x)| \leqq$$

$$\leqq C_p\left[\frac{(1-x^2)^{1/2}}{n}+\frac{|x|}{n^2}\right]^{p-t}\sum_{k=1}^{n}\omega_{f^{(p)}}(|x-x_k|)\,[\varphi_m^{2r}(x_k,x)|l_k(x)|+|\varphi_m^{2r+1}(x_k,x)|] \leqq$$

$$\leqq C_p\left[\frac{(1-x^2)^{1/2}}{n}+\frac{|x|}{n^2}\right]^{p-t}\left[\omega_{f^{(p)}}\left(\frac{(1-x^2)^{1/2}}{n}\right)+\omega_{f^{(p)}}\left(\frac{1}{n^2}\right)\right].$$

From (2.4) and the identity

$$\frac{1}{2^{2p+1}}\sum_{\nu=0}^{p}\binom{2p+1}{\nu}[(1+x)^{2p+1-\nu}(1-x)^{\nu}+(1-x)^{2p+1-\nu}(1+x)^{\nu}]=1$$

we have

$$(4.3)\quad |D^{t-s}[f(x)-\lambda_p(x)]| \leqq C_p(1-x^2)^{p-t+s}[(1+x)\omega_{f^{(p)}}(1-x)+(1-x)\omega_{f^{(p)}}(1+x)]$$

and

$$(4.4)\qquad |(1-x^2)^s D^s[1-\Psi_p^{2r+2p}(x,x)]| \leqq \frac{1}{(2m)^{2^p-s}}[1+|U_{2m}(x)|]^{2^p-s},$$

where $U_{2m}(x)=\dfrac{\sin(2m+1)\theta}{\sin\theta}$. Thus from (4.3) and (4.4) we obtain

$$(4.5)\qquad |A_2^{(t)}(x)| \leqq C_p\left[\frac{(1-x^2)^{1/2}}{m}\right]^{p-t}[(1+x)\omega_{f^{(p)}}(1-x)+(1-x)\omega_{f^{(p)}}(1+x)]\cdot$$

$$\cdot\frac{1}{m^{2^p-p}}[1+U_{2m}(x)]|^{2^p-p} \leqq C_p\left[\frac{(1-x^2)^{1/2}}{n}+\frac{|x|}{n^2}\right]^{p-t}\left[\omega_{f^{(p)}}\left(\frac{(1-x^2)^{1/2}}{n}\right)+\omega_{f^{(p)}}\left(\frac{1}{n^2}\right)\right].$$

Thus from (4.1), (4.2) and (4.5) we have Theorem 1.

5. Proof of Theorem 2

The proof of the theorem is based on Theorem 1. From (2.10) we have

$$(5.1)\quad f(x)-A_{n,p}^{(-1/2,-1/2)}(f;x) = f(x)-J_{n,p}^{(-1/2,-1/2)}(f;x)+\sum_{x_k\varepsilon[-1+\delta,\,1-\delta]}\Bigg[f(x)-$$

$$-\sum_{i=0}^{p}\frac{(x-x_k)^i}{i!}f^{(i)}(x_k)\Bigg]\frac{\Psi_p^{2r+2p}(x_k,x)1_k(x)[1-\Psi_p^{2r+2p}(x_k,x_k)]}{\Psi_p^{2r+2p}(x_k,x_k)}-$$

$$-\sum_{x_k\varepsilon[-1+\delta,1-\delta]}[f(x)-\lambda_p(x)]\frac{\Psi_p^{2r+2p}(x_k,x)l_k(x)[1-\Psi_p^{2r+2p}(x_k,x_k)]}{\Psi_p^{2r+2p}(x_k,x_k)}=$$

$$=f(x)-J_{n,p}^{(-1/2,-1/2)}(f;x)+B_1(x)+B_2(x).$$

Using (2.11) and (2.12) and (4.2) we have

$$(5.2)\qquad |B_1^{(t)}(x)| \leqq \left(1-\frac{1}{2^{2^p}}\right)^{-1}\frac{1}{m^{2^p}}|A_1^{(t)}(x)| \leqq$$

$$\leqq C_p\left[\frac{(1-x^2)^{1/2}}{n}+\frac{|x|}{n^2}\right]^{p-t}\left[\omega_{f^{(p)}}\left(\frac{1-x^2)^{1/2}}{n}+\omega_{f^{(p)}}\left(\frac{1}{n^2}\right)\right].$$

Similarly we have

$$(5.3)\quad |B_2^{(t)}(x)| \leqq C_p\left[\frac{(1-x^2)^{1/2}}{n}+\frac{|x|}{n^2}\right]^{p-t}\left[\omega_{f^{(p)}}\left(\frac{(1-x^2)^{1/2}}{n}\right)+\omega_{f^{(p)}}\left(\frac{1}{n^2}\right)\right].$$

Thus from (5.1), (5.2), (5.3) and Theorem 1, we have

$$|f^{(t)}(x)-A_{n,p}^{(-1/2,-1/2)(t)}(f;x)| \leqq$$

$$\leqq C_p\left[\frac{(1-x^2)^{1/2}}{n}+\frac{|x|}{n^2}\right]^{p-t}\left[\omega_{f^{(p)}}\left(\frac{(1-x^2)^{1/2}}{n}\right)+\omega_{f^{(p)}}\left(\frac{1}{n^2}\right)\right].$$

This completes the proof of Theorem 2.

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(Received October 10, 1980; revised May 25, 1982)

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Acta Math. Hung.
41 (3—4) (1983), 195—200.

SETS OF MULTIPLICITY ON THE DYADIC GROUP

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1. Introduction

Let

$$\mu \sim \sum_{k=0}^{\infty} \hat{\mu}(k) w_k(x)$$

be a *Walsh—Fourier series of a Radon measure* m_μ on the dyadic group [1]. A dyadic interval of rank n, $\left[\frac{p}{2^n}, \frac{(p+1)^-}{2^n}\right]$, is the set of all $x=(x_1, x_2, \ldots)$ such that $\sum_{k=1}^{n} x_k/2^k = p/2^n$. Let $I_n(x)$ be the dyadic interval of rank n containing x. For convenience, $\left(\sum_{k=1}^{\infty} x_k/2^k\right)^-$ denotes $x=(x_1, x_2, \ldots)$ if $\lim_{k\to\infty} x_k=1$ and $\left(\sum_{k=1}^{\infty} x_k/2^k\right)$ denotes x otherwise.

A set E is said to be *a set of multiplicity for the class of Walsh—Fourier series of Radon measures satisfying a condition* (A) (simply: E is *a set of multiplicity under the condition* (A)), if there exists a Walsh—Fourier series of Radon measure m_μ which satisfies

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) = 0 \tag{1}$$

except on E and (A), but $\hat{\mu}(k) \neq 0$ for some k. When a set is not a set of multiplicity, it is called *a set of uniqueness.*

In this paper we shall prove the following three theorems.

THEOREM 1. *A perfect set of Haar measure zero is a set of multiplicity under the condition*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\hat{\mu}(k)|^2 = 0. \tag{2}$$

A perfect set of Haar measure zero needs not to be a set of multiplicity in the classical sense.

THEOREM 2. *There exists a perfect set of Haar measure zero which is a set of multiplicity under the condition*

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{p=0}^{2^n-1} \left| \sum_{k=2^n}^{2^{n+1}-1} \hat{\mu}(k) w_k(p/2^n) \right| = 0. \tag{3}$$

It is easy to see that the condition (3) implies

$$\lim_{n\to\infty} \hat{\mu}(k) = 0. \tag{4}$$

THEOREM 3. *If* $\{\delta_n\}_{n=1}^{\infty}$ *is a sequence of positive numbers such that* $\sum_{n=1}^{\infty}|\delta_n|^2=\infty$ *and* $\lim_{n\to\infty}\delta_n=0$, *then there exists a Walsh—Fourier series of positive Radon measure* m_μ *satisfying* (1) *except on some dense set of Haar measure zero and* $\max_{2^n\le k<2^{n+1}}|\hat{\mu}(k)|\le\delta_n$ *for all n.*

2. Proof of Theorem 1.

Set

$$E_n=\bigcup_{p\in W(n)}\left[\frac{p}{2^n},\ \frac{(p+1)^-}{2^n}\right]$$

for $n=0,1,\ldots$ where

$$W(n)=\left\{p:\left[\frac{p}{2^n},\ \frac{(p+1)^-}{2^n}\right]\cap E\ne\varnothing.\right.$$

It is clear that $E=\bigcap_{n=1}^{\infty}E_n$. For $n=0,1,\ldots$ and $p=0,1,\ldots,2^n-1$, let $N_n(p)$ be the number of k satisfying

$$\left[\frac{k}{2^{n+1}},\ \frac{(k+1)^-}{2^{n+1}}\right]\cap\left[\frac{p}{2^n},\ \frac{(p+1)^-}{2^n}\right]\cap E\ne\varnothing.$$

Set $m_\mu[0,1^-]=1$ and

$$m_\mu\left[\frac{s}{2^{n+1}},\ \frac{(s+1)^-}{2^{n+1}}\right]=\begin{cases}1/N_n(p)\cdot m_\mu\left[\frac{p}{2^n},\ \frac{(p+1)^-}{2^n}\right] & \text{if}\quad N_n(p)\ne 0\\ & \text{and}\quad \left[\frac{s}{2^{n+1}},\ \frac{(s+1)^-}{2^{n+1}}\right]\cap E\ne\varnothing\\ 0\quad\text{otherwise}\end{cases}$$

for $n=1,2,\ldots$ and $s=2p,\ 2p+1$. It is easy to see that m_μ is a positive Radon measure. Since E is a perfect set, if $x\in E$, $\{I_n(x)\setminus I_{n+1}(x)\}\cap E\ne\varnothing$ for infinitely many n. Hence we have

$$2m_\mu(I_{n+1}(x))=m_\mu(I_n(x))$$

for finitely many n. For $x\in E$, we have $\lim_{n\to\infty}m_\mu(I_n(x))=0$. If $x\notin E$, then $m_\mu(I_n(x))=$ $=0$ for sufficiently large n and

$$2^n m_\mu(I_n(x))=\sum_{k=0}^{2^n-1}\hat{\mu}(k)w_k(x).$$

Therefore (1) holds except on E. It is obvious that

$$\lim_{n\to\infty}m_\mu(I_n(x))=\lim_{n\to\infty}1/2^n\sum_{k=0}^{2^n-1}\hat{\mu}(x)w_k(x)=0$$

everywhere. From Theorem 6 of [2], (2) follows.

3. Proof of Theorem 2. Let $\{i_n\}_{n=1}^{\infty}$ be a sequence of integers tending to infinity and set

$$\begin{cases} E_0 = [0,1^-]\diagdown\left[\dfrac{j_0(0)}{2^{N_1}},\ \dfrac{(j_0(0)+1)^-}{2^{N_1}}\right] \\ E_n = \sum\limits_{p=0}^{2^{N_n}-1}\left\{\left[\dfrac{p}{2^{N_n}},\ \dfrac{(p+1)^-}{2^{N_n}}\right]\diagdown\left[\dfrac{j_p(n)}{2^{N_{n+1}}},\ \dfrac{(j_p(n)+1)^-}{2^{N_{n+1}}}\right]\right\} \end{cases}$$

for $n=1,2,\ldots$ where $N_n=i_1+i_2+\ldots+i_n$ and $2^{i_{n+1}}p\leqq j_p(n)<2^{i_{n+1}}(p+1)$. Set $E=\bigcap\limits_{n=0}^{\infty}E_n$.

Let m_μ be a positive Radon measure satisfying the following conditions:

$$\begin{cases} m_\mu[0,1^-]=1 \\ m_\mu\left[\dfrac{k}{2^{N_n}},\ \dfrac{(k+1)^-}{2^{N_n}}\right]=\prod\limits_{j=1}^{n}(1/(2^{i}j-1)) \ \text{ if } \left[\dfrac{k}{2^{N_n}},\ \dfrac{(k+1)^-}{2^{N_n}}\right]\cap E\neq\varnothing \\ m_\mu\left[\dfrac{k}{2^{s}},\ \dfrac{(k+1)^-}{2^{s}}\right]=0 \ \text{ if } \left[\dfrac{k}{2^{s}},\ \dfrac{(k+1)^-}{2^{s}}\right]\cap E=\varnothing. \end{cases}$$

For $N_{n-1}\leqq N<N_n$ there exists only one ν such that

$$\begin{cases} \left|\Delta m_\mu\left[\dfrac{\nu}{2^{N}},\ \dfrac{(\nu+1)^-}{2^{N}}\right]\right|=\prod\limits_{j=1}^{n}(1/2^{i}j-1)) \\ \Delta m_\mu\left[\dfrac{p}{2^{N}},\ \dfrac{(p+1)^-}{2^{N}}\right]=0 \ \text{ if } \ p\neq\nu \end{cases}$$

where

$$\Delta m_\mu\left[\frac{q}{2^N},\ \frac{(q+1)^-}{2^N}\right]=m_\mu\left[\frac{2q}{2^{N+1}},\ \frac{(2q+1)^-}{2^{N+1}}\right]-m_\mu\left[\frac{(2q+1)}{2^{N+1}},\ \frac{(2q+2)^-}{2^{N+1}}\right].$$

Therefore we have

$$\sum_{p=0}^{2^N-1}\left|\Delta m_\mu\left[\frac{p}{2^N},\ \frac{(p+1)^-}{2^N}\right]\right|=\prod_{k=1}^{n-1}(2^{i_k}-1)\prod_{k=1}^{n}(1/(2^{i_k}-1))=1/(2^{i_n}-1).$$

Since

$$\Delta m_\mu\left[\frac{p}{2^N},\ \frac{(p+1)^-}{2^N}\right]=1/2^N\sum_{k=2^N}^{2^{N+1}-1}\hat{\mu}(k)w_k(p/2^N),$$

m_μ satisfies (3). Obviously (1) holds except on E.

COROLLARY 1. *The perfect set of Haar measure zero in Theorem 2 is a set of multiplicity in the classical sense.*

Corollary follows immediately from the following two lemmas:

LEMMA 1. *If* (1) *holds everywhere, then* $\hat{\mu}(k)=0$ *for all* k.

PROOF. Set

$$m_\mu\left[\frac{p}{2^n},\ \frac{(p+1)^-}{2^n}\right] = \lim_{N\to\infty} \int\limits_{\frac{p}{2^N}}^{\frac{(p+1)^-}{2^N}} \sum_{k=0}^{N} \hat\mu(k) w_k(x)\,dx$$

for $n=0, 1, \ldots$ and $p=0, 1, \ldots, 2^n-1$. Then m_μ satisfies

$$m_\mu\left[\frac{p}{2^n},\ \frac{(p+1)^-}{2^n}\right] = m_\mu\left[\frac{2p}{2^{n+1}},\ \frac{(2p+1)^-}{2^{n+1}}\right] + m_\mu\left[\frac{(2p+1)}{2^{n+1}},\ \frac{(2p+2)^-}{2^{n+1}}\right].$$

There exists a sequence of dyadic intervals

$$\left\{\left[\frac{p_n}{2^n},\ \frac{(p_n+1)^-}{2^n}\right]\right\}_{n=1}^{\infty}$$

such that

$$\begin{cases} |m_\mu[0, 1^-]| \leqq 2\left|m_\mu\left[\frac{p_1}{2},\ \frac{(p_1+1)^-}{2}\right]\right| \leqq \ldots \leqq 2^n \left|m_\mu\left[\frac{p_n}{2^n},\ \frac{(p_n+1)^-}{2^n}\right]\right| \leqq \ldots \\ [0, 1^-] \supseteqq \ldots \supseteqq \left[\frac{p_n}{2^n},\ \frac{(p_n+1)^-}{2^n}\right] \supseteqq \ldots . \end{cases}$$

Set $\bigcap\limits_{n=1}^{\infty}\left[\frac{p_n}{2^n},\ \frac{(p_n+1)^-}{2^n}\right] = \{x_0\}$. Then

$$|m_\mu[0, 1^-]| \leqq 2^n \left|m_\mu\big(I_n(x_0)\big)\right| \equiv \left|\sum_{k=0}^{2^n-1} \hat\mu(k) w_k(x_0)\right|.$$

From the hypothesis the right hand side of preceding formula converges to zero as $n\to\infty$. Hence $m_\mu[0, 1^-]=0$. Similarly $m_\mu(I)=0$ for all dyadic intervals I. This proves that $\hat\mu(k)=0$ for all k.

LEMMA 2. *When E is a closed set of Haar measure zero, then*

$$\sum_{k=0}^{\infty} \hat\mu(k) w_k(x) = 0 \quad \textit{except on} \quad E \tag{5}$$

if and only if (1) *holds except on E and* (4) *holds.*

PROOF. Obviously the sufficient condition holds. We shall prove the necessary condition. If I is a dyadic interval adjacent to E, then from Lemma 1, (1) holds in I. For $x\in I\equiv\left[\frac{p}{2^s},\ \frac{(p+1)^-}{2^s}\right]$ set $x = p/2^s \dotplus t/2^s$. We have

$$\sum_{k=0}^{2^n-1} \hat\mu(k) w_k(x) = \sum_{j=0}^{2^{n-s}-1} \left\{\sum_{k=j2^s}^{(j+1)2^s-1} \hat\mu(k) w_k(x)\right\} =$$

$$= \sum_{j=0}^{2^{n-s}-1} \left\{\sum_{k=j2^s}^{(j+1)2^s-1} \hat\mu(k) w_k(p/2^s \dotplus t/2^s)\right\} = \sum_{j=0}^{2^{n-s}-1} \left\{\sum_{k=j2^s}^{(j+1)2^s-1} \hat\mu(k) w_k(p/2_s)\right\} w_j(t).$$

Hence for every t the above expression converges to zero as $n\to\infty$. From Lemma 1, for $j=0, 1, \ldots$

$$\sum_{k=j2^s}^{(j+1)2^s-1} \hat{\mu}(k) w_k(x) = 0 \quad \text{in} \quad I.$$

For $x\in I$ and $j2^s \leqq n < (j+1)2^s$, we have

$$\left|\sum_{k=0}^{n} \hat{\mu}(k) w_k(x)\right| = \left|\sum_{k=j2^s}^{n} \hat{\mu}(k) w_k(x)\right| \leqq 2^s \max_{j2^s \leqq k < (j+1)2^s} |\hat{\mu}(k)|.$$

This proves (5).

4. Proof of Theorem 3. Set

$$\begin{cases} m_\mu[0, 1^-] = 1 \\ m_\mu\left[\frac{2p+i}{2^{n+1}}, \frac{((2p+i)+1)^-}{2^{n+1}}\right] = 1/2\big(1+(-1)^i\delta_n\big) m_\mu\left[\frac{p}{2^n}, \frac{(p+1)^-}{2^n}\right]. \end{cases}$$

Since $1+t \leqq e^{t-\frac{t^2}{4}}$ for $-1\leqq t\leqq 1$ and by the law of large numbers

$$\sum_{k=0}^{n-1} \delta_k w_{2^k}(x) = o_x\left(\sum_{k=0}^{n-1} \delta_k^2\right)^{1/2+\varepsilon} \quad (\varepsilon>0)$$

a.e. if $n\to\infty$, we get

$$2^n m_\mu\big(I_m(x)\big) \leqq \exp\left(\sum_{k=0}^{n-1} \delta_k w_{2^k}(x) - \frac{1}{n}\sum_{k=0}^{n-1} \delta_k^2\right) \to 0$$

a.e. if $n\to\infty$ for $i=0, 1, n=0, 1, \ldots$ and $p=0, 1, \ldots, 2^n-1$. It is obvious that m_μ is a positive Radon measure and $|\hat{\mu}(k)|\leqq 1$ for all k. For $k=0, 1, \ldots, 2^n-1$, we have

$$\hat{\mu}(2^n+k) = \sum_{p=0}^{2^n-1} \Delta m_\mu\left[\frac{p}{2^n}, \frac{(p+1)^-}{2^n}\right] w_k(p/2^n) =$$

$$= \delta_n \sum_{p=0}^{2^n-1} m_\mu\left[\frac{p}{2^n}, \frac{(p+1)^-}{2^n}\right] w_k(p/2^n) = \delta_n \hat{\mu}(k).$$

Then from the hypothesis (4) holds. Moreover since $\Delta m_\mu\left[\frac{p}{2^n}, \frac{(p+1)^-}{2^n}\right] \geqq 0$ and $w_0(p/2^n)=1$ we have

$$\hat{\mu}(2^n) = \sum_{p=0}^{2^n-1} \left|\Delta m_\mu\left[\frac{p}{2^n}, \frac{(p+1)^-}{2^n}\right]\right| \leqq \delta_n.$$

On the other hand for $x=(x_1, x_2, \ldots)$ we have

$$2^n m_\mu\big(I_n(x)\big) = \prod_{k=0}^{n-1} \big(1+w_{2^k}(x)\delta_k\big).$$

COROLLARY 2. *Under the condition of Theorem 3, there exists a dense set of Haar measure zero which is a set of multiplicity under the condition*

$$\sum_{p=0}^{2^n-1}\left|\Delta m\left[\frac{p}{2^n},\ \frac{(p+1)^-}{2^n}\right]\right| \equiv \frac{1}{2^n}\sum_{p=0}^{2^n-1}\left|\sum_{k=2^n}^{2^{n+1}-1}\hat{\mu}(k)w_k(p/2^n)\right| \leqq \delta_n$$

for all n.

Set especially $\delta_1=1$ and $\delta_k=\log k/\sqrt{k}$ for $k\geqq 2$, then $\{\delta_k\}_{k=1}^{\infty}$ satisfies the hypothesis of Theorem 3. It is easy to see that

$$\sum_{k=2^n}^{2^{n+1}-1}|\hat{\mu}(k)|^2 = 2^n\sum_{p=0}^{2^n-1}\left|\Delta m_\mu\left[\frac{p}{2^n},\ \frac{(p+1)^-}{2^n}\right]\right|^2 \sim \prod_{k=1}^{n}(1+\delta_k)^2\delta_{n+1}^2$$

as $n\to\infty$. Since the last formula tends to zero as $n\to\infty$, we have

$$\lim_{n\to\infty}\sum_{k=2^n}^{2^{n+1}-1}|\hat{\mu}(k)|^2 = 0. \tag{8}$$

COROLLARY 3. *There exists a dense set of Haar measure zero which is a set of multiplicity under the condition* (8).

When $\delta_k=\delta$ for all k where $0\leqq|\delta|<1$, set

$$\begin{cases} m_\mu(I_n(x)) = 1/2^n(1+\delta)^{N_n(x)}(1-\delta)^{n-N_n(x)} \\ S = \{x\colon \lim\limits_{n\to\infty} N_n(x)/n = 1/2\} \end{cases}$$

where $N_n(x)$ is the number of elements of $\{1\leqq k\leqq n\colon x_k=0\}$. Then m_μ is a positive Radon measure and (1) holds on S. Obviously mes $S=1$.

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(Received April 21, 1981; revised October 8, 1981)

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Acta Math. Hung.
41 (3—4) (1983), 201—211.

THE SEMIGROUP OF PARTIAL *l*-ISOMORPHISMS OF AN ABELIAN *l*-GROUP

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1. Introduction

If G is an abelian l-group then $\mathscr{I}(G)$ will denote the semigroup of all l-isomorphisms whose domains and ranges are l-subgroups of G. The operation on $\mathscr{I}(G)$ is partial composition of functions. An element b in a semigroup S has an inverse a if $a=aba$ and $b=bab$. If every element in S has a unique inverse then S is called an inverse semigroup and the inverse of b is denoted by b^{-1}. If $f\in\mathscr{I}(G)$ and $f\colon H\to K$ then the inverse mapping $f^{-1}\colon K\to H$ is the unique inverse for f and so $\mathscr{I}(G)$ is an inverse semigroup. In this paper we will investigate the structure of $\mathscr{I}(G)$, particularly its lattice of ideals, and use this to gain information about the l-group G.

The technique of characterizing a mathematical object by an inverse semigroup of one-to-one partial morphisms has often been employed. For example, if G is a group and $T(G)$ the inverse semigroup of all isomorphisms between subgroups of G then $T(G)$ is a complete invariant for finite abelian groups; that is, for finite abelian groups G and G', $T(G)$ is isomorphic to $T(G')$ if and only if G is isomorphic to G' ([9]). $T(G)$ is also a complete invariant for finitely generated abelian groups ([7]). For a different type of example let X be a topological T_1 space and let $T(X)$ be the inverse semigroup of all homeomorphisms between closed subsets of X. Then $T(X)$ is a complete invariant for X ([1]). If you replace closed subsets with open subsets the result still holds.

In general, for an l-group G, the inverse semigroup $\mathscr{I}(G)$ is not a complete invariant. For example, if p is a prime number, let $Q_p=\left\{\frac{a}{n}\colon\ a, n \text{ integers with } (a,n)=1\right\}$ with the inherited order of the reals $\mathbf{R}$. If q is a prime number different from p then Q_p and Q_q are not isomorphic even though $\mathscr{I}(Q_p)$ and $\mathscr{I}(Q_q)$ are isomorphic. The isomorphism of the inverse semigroups stems from the fact that the only subgroups of Q_p are trivial, cyclic, or isomorphic to Q_p. Thus $\mathscr{I}(Q_p)$ has a very sparse structure. Even though the semigroup $\mathscr{I}(G)$ is not in general a complete invariant, it does give us information about G.

Idempotents in $\mathscr{I}(G)$ (maps f such that $ff=f$) are precisely the identity maps on l-subgroups H and will be denoted by i_H. The identity on the trivial subgroup will be denoted by 0. We investigate Green's relations for $\mathscr{I}(G)$. Two maps f and g are $\mathscr{L}$-equivalent if they generate the same principal left ideal; i.e., $\mathscr{I}(G)f=\mathscr{I}(G)g$. For $f,g\in\mathscr{I}(G)$ $f\mathscr{L}g$ if and only if domain of $f=$domain of g. The definition of $\mathscr{R}$-equivalence is analogous and we obtain: $f\mathscr{R}g$ if and only if range of $f=$range of g. We say that $f\mathscr{H}g$ if both $f\mathscr{R}g$ and $f\mathscr{L}g$. Thus for an idempotent i_H, its $\mathscr{H}$-class is the group of all l-automorphisms from H onto H (denoted $\mathscr{A}(H)$). Since $\mathscr{L}\circ\mathscr{R}=\mathscr{R}\circ\mathscr{L}$ we define $\mathscr{D}=\mathscr{L}\circ\mathscr{R}$. Two maps $f, g\in\mathscr{I}(G)$ are $\mathscr{D}$-related if the domain of f is l-isomorphic to the domain of g. Elements f and g are $\mathcal{J}$-equivalent if they generate the same principal two-sided ideal. Idempotents i_H and i_K will be $\mathcal{J}$-related if and only if H contains an l-subgroup l-isomorphic to K and conversely. For a general discussion of inverse semigroups and Green's relations we refer the reader to ([5]).

We will denote the lattice of ideals of $\mathcal{I}(G)$ by $I(G)$. If G is nontrivial then $I(G)$ always has a smallest nontrivial ideal, which is generated by i_H where H is a cyclic totally ordered group. We denote this ideal by I_1. Our principal results concern the ideals which cover I_1. We enumerate all possible covers and examine the implications for G in each of these cases (these different possibilities can be distinguished within the semigroup structure of $\mathcal{I}(G)$). The ideal I_1 may not have a cover and this situation is also explored.

The last section of the paper deals with the ideal structure of the isomorphism semigroup of the l-group of bounded integer-valued functions, and closes by posing several questions.

2. Covers for I_1

An l-group is a group with an underlying lattice so that the group operation distributes over the lattice operations. We assume throughout the paper that all l-groups G are abelian. **Q** will denote the rationals and **Z** the integers; we assume that subgroups of the reals **R** have the total order inherited from **R**. If $\{a_\lambda\colon \lambda\in\Lambda\}$ is a subset of an l-group G then (a_λ) will denote the l-subgroup generated by the a_λ's. If $a\in G$ $a>0$ then (a) is an o-subgroup of G. If $f\in G$ then $\langle f\rangle$ will denote the principal two-sided ideal of $\mathcal{I}(G)$ generated by f. Every principal two-sided ideal can be generated by an idempotent i_H and the ideal $\langle i_H\rangle=\{f\in\mathcal{I}(G)\colon$ domain and range of f are l-isomorphic to l-subgroups of $H\}$. An ideal I *covers* an ideal J if $J<I$ and there exists no ideals strictly between J and I.

We present a few results and notation from the theory of l-groups (for reference see [2]). Let I be a well ordered set and for each $i\in I$ suppose G_i is a totally ordered group. If G is the direct product of the G_i then for $g\in G$ we define $g>0$ if the first nonzero component g_k of g is positive (in its o-group G_k). This makes G a totally ordered group, called the *direct lexicographic* product of the G_i and we denote it by $\overrightarrow{\prod\limits_{i\in I}} G_i$. If I is finite we write $G_1\vec{x}\ldots\vec{x}G_{n-1}\vec{x}G_n$. If G is the direct sum of the G_i we can form an l-group by defining $g\in G$ to be positive if each g_i is positive. Then we denote G by $\underset{i\in I}{\boxed{+}} G_i$; if I is finite we write $G_1\boxed{+}G_2\boxed{+}\ldots\boxed{+}G_n$. This will be called the pointwise order on G.

An l-group G is called *archimedean* if for each pair of positive elements a and b in G there is a positive integer n such that $na\nless b$. By Hölder's theorem ([4]) an archimedean o-group is o-isomorphic to a subgroup of **R**. In addition, if G and G' are subgroups of **R** then each o-isomorphism from G into G' is a multiplication by some positive real number ([4]).

An l-subgroup M of an l-group G is *convex* if whenever $a>0$ and $a\in M$ and $0\leqq b\leqq a$ then $b\in M$. Note that any l-automorphism of G induces an automorphism of the lattice of convex l-subgroups of G. An l-group G with no convex l-subgroups must be archimedean and totally ordered and hence is a subgroup of **R** ([4]). Convex l-subgroups M are called *values* if they are maximal with respect to not containing an element $x \in G$; M is called *a value of* x. Every value M will have a unique convex l-subgroup (denoted M^*) which covers it. Since the factor M^*/M contains no convex l-subgroups it must be an o-subgroup of the reals.

We will also need a few results from group theory about the notion of types of rank one torsion-free abelian groups (for reference see [6]). These groups can be thought of as subgroups of the rationals Q. For p a prime and a an element of G the *p-height* of a is the largest integer k such that p^k divides a; if no such largest integer exists then the p-height of a is ∞. Thus to each element a in G can be associated a sequence $(t_2, t_3, t_5, \dots)$ of p-heights. Two such sequences $(t_2, t_3, t_5, \dots)$ and $(l_2, l_3, l_5, \dots)$ are called equivalent if $\sum_p |t_p - 1_p|$ is finite (where $\infty - \infty = 0$). We call an equivalence class a *type*. Now if G is a subgroup of Q then all elements of G have the same type and so we can associate with G a type $(t_2, t_3, \dots)$. Two torsion free groups of rank one are isomorphic if and only if they have the same type; every type is realized by a rational group ([6]). Note that the group Q_p mentioned in the introduction has type $(0, \dots, 0, \infty, 0, \dots)$ where ∞ occurs in the pth position. Thus if q is a prime different from p we have that Q_p and Q_q are not isomorphic.

Recall that I_1 is the ideal generated by $i_{(a)}$ where a is any positive element in G; I_1 is the smallest nontrivial ideal in $I(G)$. We consider whether or not I_1 has covers; and if so which covers can occur. We call I_1 *chain accessible* if there exists an infinite chain of ideals $J_1 > J_2 > \dots$ such that $\bigcap J_n = I_1$.

PROPOSITION 1. *Suppose G is a noncyclic rank one group of type $(t_2, t_3, \dots)$ where $t_i \neq \infty$ for all i. Then I_1 has no covers, is chain accessible, and contains an infinite number of distinct $\mathcal{J}$-classes.*

PROOF. First suppose, on the contrary, that I_1 has a cover I. Since $I_1 < I$ there exists an isomorphism $f \in I$ such that the domain H of f is not a cyclic group. Then $I_1 < \langle i_H \rangle \leqq I$ and so $\langle i_H \rangle = I$. Now H is not cyclic, so H is not of type $(0, 0, \dots)$. Since $H \leqq G$ this means H is of type $(l_2, l_3, \dots)$ where $l_i \neq \infty$ for all i and infinitely many $l_i \neq 0$. Without loss of generality suppose $l_i \neq 0$ for all i. Then H contains a subgroup K of type $(l_2, 0, l_5, 0, \dots)$. Thus $I_1 < \langle i_K \rangle < I$, which is a contradiction since I covers I_1. Thus I_1 has no covers.

Now, by using the same idea we can partition the type $(t_2, t_3, \dots)$ of G (again assume, without loss of generality, that $t_i \neq 0$ for all i) to get infinite descending chains of types:

$$\begin{tikzcd}[column sep=tiny]
 & & (t_2, t_3, \dots) \arrow[dl, dash] \arrow[dr, dash] & & \\
 & (t_2, 0, t_5, \dots) \arrow[dl, dash] \arrow[dr, dash] & & (0, t_3, 0, t_7, \dots) \arrow[dl, dash] \arrow[dr, dash] & \\
(t_2, 0, 0, 0, t_1, 0, \dots) & & \vdots & & \\
 & & (0, 0, 0, \dots) & &
\end{tikzcd}$$

Each of these types will give rise to an o-subgroup K of that type which will in turn generate an ideal $\langle i_K \rangle$; different types will correspond to distinct ideals and hence G contains an infinite number of distinct $\mathcal{J}$-classes.

Now, if I_1 does have a cover we enumerate the possibilities:

THEOREM 2. *If G is an abelian l-group then I is a cover for I_1 if and only $I=\langle i_H\rangle$ for some l-subgroup H of G where H is l-isomoprhic to one of the following l-groups:*

1) $\mathbf{Z}\boxed{+}\mathbf{Z}$
2) $\mathbf{Z}\vec{\times}\mathbf{Z}$
3) *the o-subgroup* $(1; r)$ *of* $\mathbf{R}$, *where r is an irrational number*
4) Q_p *for some prime p.*

PROOF. Suppose that I_1 has a cover I. As shown in the proof of Proposition 1, I_1 must be a principal ideal $\langle i_H\rangle$. If H contains two disjoint positive elements then it contains an l-subgroup l-isomorphic to $\mathbf{Z}\boxed{+}\mathbf{Z}$, which implies that i_H and $i_{\mathbf{Z}\boxed{+}\mathbf{Z}}$ are $\mathscr{J}$-equivalent since $\langle i_H\rangle$ covers I_1. But the only l-subgroups of $\mathbf{Z}\boxed{+}\mathbf{Z}$ are trivial, cyclic or l-isomorphic to $\mathbf{Z}\boxed{+}\mathbf{Z}$. Thus H is itself l-isomorphic to $\mathbf{Z}\boxed{+}\mathbf{Z}$. Now if H has no pairwise disjoint elements it must be totally ordered. Suppose there exist $0<a$, $b\in H$ such that $na<b$ for all positive integers n. Then (a, b) is o-isomorphic to $\mathbf{Z}\vec{\times}\mathbf{Z}$ and once again we have that H must be o-isomoprhic to $\mathbf{Z}\vec{\times}\mathbf{Z}$.

If neither of the above two cases occurs we may suppose that H is an archimedean o-group, and hence that H is a subgroup of $\mathbf{R}$. If the rank of H exceeds two then H contains a subgroup K of rank two. But then $I_1<\langle i_K\rangle<I$, which is a contradiction. Thus H is of rank one or two. If H is of rank one then suppose H has type $(t_2, t_3, \ldots)$. If $t_i\neq\infty$ for all i then infinitely many $t_i\neq 0$ (otherwise H is cyclic). But then by Proposition 1 I_1 has no cover. Thus there is some t_{i_0} such that $t_{i_0}=\infty$. But then H contains an l-subgroup K of type $(0, 0, \ldots, 0, \infty, 0, \ldots)$ where $t_{i_0}=\infty$ and so $I_1<\langle i_K\rangle\leqq$ $\leqq\langle i_K\rangle$. Since $\langle i_H\rangle$ covers I_1 the maps i_K and i_H must be $\mathscr{J}$-equivalent. Thus the type of K equals the type of H and hence 4) holds.

The only remaining case is where H is a rank two subgroup of the reals. We may, without loss of generality, assume $1\in H$. Then H is generated by 1 and an irrational number r (rank of H is two) and hence 3) holds.

For the converse, note that in cases 1), 2) and 4) the only l-subgroups of H are trivial, cyclic, or l-isomorphic to H. Thus $\langle i_H\rangle$ covers I_1. We will show that this is also true for case 3). So suppose H is of the form $(1, r)$ where r is irrational and K is a nontrivial, noncyclic l-subgroup of H. Then K is of the form $n_0\mathbf{Z}+rm_0\mathbf{Z}$ where $n_0, m_0\neq 0$. Consider the l-isomorphism $\alpha\colon H\to n_0\mathbf{Z}+rm_0\mathbf{Z}$ where α is multiplication by n_0m_0. Then $\alpha(H)\leqq n_0\mathbf{Z}+rm_0\mathbf{Z}$ and so K contains a copy of H. Thus $i_H\in\langle i_K\rangle$ and hence i_H and i_K are $\mathscr{J}$-equivalent. So we have that $\langle i_H\rangle$ covers I_1.

DEFINITION. Following the order of the scheme of Theorem 2, if $\langle i_H\rangle$ covers I_1 then we will call H and $\langle i_H\rangle$ of class i (for $i=1, \ldots, 4$).

Proposition 1 and Theorem 2 immediately yield:

COROLLARY 3. *Let G be a noncyclic abelian l-group. Then I_1 has no covers if and only if G is o-isomorphic to a rank one subgroup of Q of type $(t_2, t_3, \ldots)$ where $t_i\neq\infty$ for all i and $t_i\neq 0$ for infinitely many i.*

PROOF. One direction is the content of Proposition 1. For the other direction suppose that I_1 has no covers. Then note that, as shown in the proof of Theorem 2, G must be rank one archimedean o-group. If G has type $(t_2, t_3, \ldots)$ where $t_i=\infty$ for some i then I_1 has a cover of class 4. Thus $t_i\neq\infty$ for all i and since G is not cyclic G cannot have type $(0, 0, \ldots)$ and so $t_i\neq 0$ for infinitely many i.

One of the aims in studying $\mathscr{I}(G)$ for an l-group G is to use the inverse semigroup to gain information about the l-group. Recall that the group of l-automorphisms of an l-subgroup H $(\mathscr{A}(H))$ is just the $\mathscr{H}$-class of the idempotent i_H and hence can be recognized within $\mathscr{I}(G)$. Now l-subgroups K of H satisfy the equation $i_K i_H = i_K$. Therefore, for each l-subgroup H of G its lattice of l-subgroups is also recognizable in $\mathscr{I}(G)$. If I_1 has no covers then Corollary 3 gives us information about G. On the other hand, if I_1 has covers then we would like to be able, within the semigroup $\mathscr{I}(G)$, to distinguish between different classes of covers. The next propositions assure us of this. Before stating the first of these let us agree to call a real number r *quadratic* if it satisfies an equation $ax^2+bx+c=0$ with $a, b, c \in \mathbf{Z}$, $a \neq 0$.

PROPOSITION 4. *Suppose* $I=\langle i_H \rangle$ *is a cover of* I_1. *Then*
1) $\mathscr{A}(H) \cong \mathbf{Z}_2$ *if* $i=1$,
2) $\mathscr{A}(H) \cong \{0\}$ *if* $i=3$ *and r is not quadratic,*
3) $\mathscr{A}(H) \cong \mathbf{Z}$ *if* $i=2$, *if* $i=4$, *or if* $i=3$ *and r is quadratic.*

PROOF. Observe that if $i=1$ then the only automorphisms of $\mathbf{Z} \boxed{+} \mathbf{Z}$ are the identity and the map that sends (a, b) to (b, a).

Now suppose that $i=3$; hence we may assume $H=(1, r)$ with r irrational. If r is not quadratic then any automorphism of $(1, r)$ is multiplication by a real number and so is of the form $n_0+m_0 r$. But then $(n+mr)(n_0+m_0 r)=1$ for some $n, m \in \mathbf{Z}$ and hence (since r is not quadratic) $m_0=0$ and $n_0=1$. If r is quadratic it is known ([3]) that $\mathscr{A}(H)$ is cyclic.

If $i=2$ then assume $H=\mathbf{Z} \overrightarrow{\times} \mathbf{Z}$ and thus contains only one nontrivial proper convex subgroup, $M=\{(0, m) : m \in \mathbf{Z}\}$, and each $\varphi \in \mathscr{A}(H)$ must leave this subgroup fixed and so induces the identity on it. Moreover, φ induces the identity on H/M. Thus φ acts as follows: $\varphi(m, n)=\varphi(m, km+n)$ where k depends on φ. Indeed, the assignment $\varphi \to k$ sets up on isomorphism of $\mathscr{A}(\mathbf{Z} \overrightarrow{\times} \mathbf{Z})$ onto $\mathbf{Z}$.

If $i=4$ then $H \cong Q_p$ for some prime p. The o-automorphism of H are the powers of p; hence $\mathscr{A}(H) \cong \mathbf{Z}$.

Note that two l-subgroups H and K of G are l-isomorphic if and onyl if i_H and i_K are $\mathscr{D}$-related; distinct $\mathscr{D}$-classes can be distinguished within $\mathscr{I}(G)$. This idea will help to differentiate the cases where $\mathscr{A}(H) \cong \mathbf{Z}$.

PROPOSITION 5. *Suppose* $I=\langle i_H \rangle$ *is a cover of* I_1 *and* $\mathscr{A}(H) \cong \mathbf{Z}$.
1) *If* $i=3$ *then* $\mathscr{I}(H)$ *has infinitely many* $\mathscr{D}$*-classes.*
2) *If* $i=2$ *or* 4 *then* $\mathscr{I}(H)$ *has exactly three* $\mathscr{D}$*-classes and in fact* $\mathscr{D}=\mathscr{J}$ *for* $\mathscr{I}(H)$.
3) *If* $i=2$ *or* 3 *then there exist two nontrivial idempotents* $i_K, i_J \in I$ *such that* $i_K i_J=0$. *If* $i=4$ *then no such idempotents exist in* $\mathscr{I}(H)$.

PROOF. 1) By Proposition 4, 2) r must be quadratic if $i=3$. If r is quadratic we may assume without loss of generality that $r=\sqrt{n}$, with n a positive integer. Then for each prime number p such that $(p, n)=1$ the subgroups $(p, \sqrt{n})$ give an infinite class of nonisomorphic groups.

2) Note that if $H=\mathbf{Z} \overrightarrow{\times} \mathbf{Z}$ then all o-subgroups of H are either trivial, cyclic, or of the form $n\mathbf{Z} \overrightarrow{\times} m\mathbf{Z}$ and the map $\varphi(a, b)=(na, mb)$ from $\mathbf{Z} \overrightarrow{\times} \mathbf{Z}$ onto $n\mathbf{Z} \overrightarrow{\times} m\mathbf{Z}$ is an o-isomorphism. This means $\mathscr{I}(H)$ has exactly three distinct $\mathscr{D}$-classes. If $H=Q_p$ (i.e., $i=4$) then again, all subgroups are either trivial, cyclic, or siomorphic to Q_p.

3) If $i=2$ or 3 then H has rank two and therefore has two subgroups with trivial intersection. From this we easily get the desired idempontents. If $i=4$ then no such subgroups exist, since the rank of H is one.

REMARK. If H and K are both l-subgroups of class 1 or both of class 2 then i_H and i_K are $\mathcal{D}$-equivalent. Hence they are $\mathcal{J}$-equivalent and so I_1 can admit at most one cover of class 1 and at most one of class 2. However, I_1 may admit infinitely many covers of class 3. For instance, if $\{r_\lambda\colon \lambda\in\Lambda\}$ is an infinite family of algebraically independent real numbers then the ideals generated by the $i_{(1,r_\lambda)}$ are distinct. Note also that if p and q are distinct primes then i_{Q_p} and i_{Q_q} are not $\mathcal{J}$-equivalent. So I_1 may admit infinitely many covers of class 4.

Let us now turn to a result that describes how the presence in $I(G)$ of certain covers of I_1 (or else the absence of these covers) reflects on the structure of G.

THEOREM 6. *Suppose G is a non-cyclic abelian l-group. Then*

1) *I_1 admits no covers of class* 1 *if and only if G is an o-group.*

2) *I_1 admits no covers of class* 2 *if and only if G is archimedean.*

3) *I_1 admits no covers of class* 1 *or* 3 *if and only if G is an o-group and for each value M of G, M^*/M has rank one.*

4) *I_1 admits no covers of class* 2 *or* 3 *if and only if G is archimedean and every o-subgroups has rank one.*

5) *I_1 admits no covers of class* 1, 2 *or* 3 *if and only if G is a rank one subgroup of* $\mathbf{R}$, *(i.e. a subgroup of* $\mathbf{Q}$*).*

6) *I_1 is chain inaccessible and only admits covers of class* 2 *if and only if G is an o-group and for each value M, M^*/M is cyclic.*

7) *I_1 is chain inaccessible and only admits covers of class* 1 *if and only if G is archimedean and every o-subroup is cyclic.*

PROOF. 1) If G is not totally ordered then there exist $a, b\in G$ such that a is incomparable to b. Let $x=a-(a\wedge b)$ and $y=b-(a\wedge b)$; then $x, y>0$ and $x\wedge y=0$. The l-subgroup generated by x and y is l-isomorphic to $\mathbf{Z}\boxed{+}\mathbf{Z}$ which means I_1 admits a cover of class 1. Since $\mathbf{Z}\boxed{+}\mathbf{Z}$ is not an o-group the converse is obvious.

2) Clear.

3) Suppose that I_1 admits no covers of class 1 or 3. Then G is an o-group by 1). Let M be a value of G and consider M^*/M. If M^*/M has rank greater than 1, there exist $a, b\in M^*$ such that $(a+M,\ b+M)$ has rank 2; i.e., $na+mb\notin M$ for all $n, m\in Z$. Hence $(a, b)\cap M=0$. Thus

$$(a, b)\cong\frac{(a, b)+M}{M}\cong\frac{(a+M,\ b+M)}{M}.$$

But then (a, b) has rank 2, which means that I_1 admits a cover of class 3, this is a contradiction. Hence M^*/M has rank 1.

Conversely, observe that by 1) I_1 admits no covers of class 1. I_1 will admit no covers of class 3 if and only if G contains no archimedean o-subgroup of rank 2. Suppose H is such an o-subgroup. Let M be the largest convex subgroup of G such that $H\cap M=\{0\}$. Since H is of rank 2 there exist $0<a, b\in H$ such that if $na=mb$ for some $n, m\in Z$ then $n=m=0$. Let K be the convex subgroup generated by M

and a. Then since H is archimedean, $b \in K$. Then M is a value of a and b, and $M^* = K$. Observe that $H \cap M^* \neq \{0\}$, and since H contains no proper convex subgroups this means $H \cap M^* = H$. Now

$$H \cong \frac{(H, M)}{M} \leqq \frac{M^*}{M},$$

and so M^*/M contains a rank 2 subgroup, which is a contradiction.

4) and 5) are obvious.

6) Suppose I_1 is chain inaccessible and only admits covers of class 2. Then by 3) G is an o-group and each factor M^*/M is of rank one. Now suppose M^*/M is not cyclic for some value M. Then there exist $0 < a, b \in M^*$ such that $(a+m, b+m)$ is not cyclic; hence (a, b) is not cyclic. Furthermore, the o-subgroup (a, b) is archimedean since a and b have the same value M, and M^*/M is archimedean. Hence if (a, b) is of rank 2 or more, I_1 admits a cover of class 3. If (a, b) is of rank 1 but not cyclic then either I_1 admits a cover of class 4, or else I_1 is chain accessible by Proposition 1. Either way we have a contradiction, and so M^*/M is cyclic.

For the converse, note that by 3) I_1 admits no covers of class 1 or 3. Suppose that G contains an o-subgroup H of rank 1. By using the same technique of the proof of 3), there is a value M such that H is o-isomorphic to a subgroup of M^*/M. Hence H is cyclic. This means that I_1 admits no covers of class 4.

Now suppose we have a chain of ideals $\{J_n\}$ such that $I_1 = \bigcap J_n$. If each J_n contains an idempotent i_{H_n} such that H_n is not archimedean, then each J_n contains an idempotent i_{K_n} where K_n is o-isomorphic to $\mathbf{Z} \overrightarrow{\times} \mathbf{Z}$. Thus each J_n-and, consequently also I_1-contains all such idempotents, which is a contradiction. Hence we have an n_0 such that for each $m \geqq n_0$ all the idempotents of J_m are of the form i_H with H archimedean. As the J_n form a chain we may then without loss of generality assume that all J_n have the above property.

Suppose H is an o-subgroup and $i_H \in J_n$. If the rank of H exceeds one, then H contains a subgroup of rank 2, and therefore I_1 admits a cover of class 3, which is a contradiction. Thus H has rank one, and by the previous part of the proof H must be cyclic. Hence every idempotent of J_n belongs to I_1, which says that $J_n = I_1$. This proves I_1 is chain inaccessible.

7) Suppose that I_1 is chain inaccessible and only admits covers of class 1. Then by 2) is archimedean. If H is an o-subgroup of G then H can be considered a subgroup of $\mathbf{R}$. If the rank of H is greater than one then H contains a copy of $(1, r)$ with r irrational. This gives a cover of class 3 for I_1; therefore H must have rank one. If the type of H is $(t_2, t_3, \ldots)$ with $t_p = \infty$ for some p then H contains a copy of Q_p, and so I_1 admits a cover of class 4. Thus $t_p < \infty$ for every prime p. Since I_1 is chain inaccessible only finitely many $t_p > 0$; this suffices to conclude that H is cyclic.

If, on the other hand, G is an archimedean l-group and each o-subgroup is cyclic, then clearly I_1 admits no covers of class 2), 3) or 4). As in 6) suppose $I_1 = \bigcap J_n$ where $\{J_n\}$ is a chain of ideals. If $J_n \neq I_1$ for each n, then there is an idempotent $i_{H_n} \in J_n$ such that H_n is not totally ordered; (otherwise $H_4 \cong Z$). But then H_n contains a copy of $\mathbf{Z} \boxplus \mathbf{Z}$ and hence $I_1 = \bigcap J_n$ contains an idempotent i_H with $H \cong \mathbf{Z} \boxplus \mathbf{Z}$. This is a contradiction, and hence I_1 is chain inaccessible.

We have completed the proof of Theorem 6.

REMARK. If G is l-isomorphic to an l-subgroup of $\mathbf{Q}^X$ (where the rational-valued functions are ordered pointwise) then every o-subgroup of G has rank one, and by 4) of Theorem 6, I_1 admits no covers of class 2 or 3. It is unknown whether every l-group in which all o-subgroups have rank one is embeddable as an l-group of rational-valued functions as specified above.

Similarly, if G is l-isomorphic to an l-subgroup of $\mathbf{Z}^X$, then every o-subgroup of G is cyclic, and hence I_1 is chain inaccessible and only admits a cover of class 1. As above it is unknown whether the converse is true.

3. Examples and covers

In this section we first restrict our attention to l-groups which can be embedded in the l-group $B(X, \mathbf{Z})$ of bounded, integer-valued functions on the the set X. Such l-groups G are hyperarchimedean (every l-homomorphism is archimedean) and indeed have the property that each value M is maximal and G/M is cyclic ([8]). In particular, $I(G)$ only admits a cover for I_1 of class 1.

Suppose G is an l-subgroup of $B(X, \mathbf{Z})$ and G has an infinite pairwise disjoint set $\{x_1, x_2, \dots\}$. Let H_∞ be the l-subgroup generated by this set and let H_n be the l-subgroup generated by $\{x_1, \dots, x_n\}$. Then H_n is l-isomorphic to $\mathbf{Z} \boxplus \mathbf{Z} \boxplus \dots \boxplus Z$($n$-fold). We will denote the corresponding principal ideals generated by i_{H_∞} and i_{H_n} by I_∞ and I_n. Note that $\bigcup_{n+1}^{\infty} I_n$ is also an ideal. We examine $I(G)$:

PROPOSITION 1. *Let G be an l-subgroup of $B(X, \mathbf{Z})$; suppose G contains an infinite pairwise disjoint set and suppose J is an ideal of $\mathscr{I}(G)$. Then*

1) $\{0\} < I_1 < I_2 < \dots < \bigcup_{n=1}^{\infty} I_n < I_\infty$

2) *Either J is one of these ideals of* 1) *or else* $I_\infty < J$.

PROOF. 1) It is clear that i_{H_n} and i_{H_m} are $\mathscr{I}$-equivalent if and only if $n=m$ and that $\bigcup_{n=1}^{\infty} I_n$ is an ideal properly containing each I_n. Note that I_∞ properly contains $\bigcup_{n=1}^{\infty} I_n$ since I_∞ is a principal ideal whereas $\bigcup_{n=1}^{\infty} I_n$ is not.

2) Let J be an ideal of $\mathscr{I}(G)$. If there exists an idempotent $i_H \in J$ such that H contains an infinite pairwise disjoint set, then $I_\infty \leqq J$. In an archimedean l-group K, if n is the maximum number of pairwise disjoint elements then K is l-isomorphic to a direct sum of subgroups of $\mathbf{R}$ [(2)]. Thus if n is the maximum number of pairwise disjoint elements in any l-subgroup K, where $i_K \in J$, then $J = I_n$. If no such bound exists but all such sets are finite then $J = \bigcup_{n=1}^{\infty} I_n$.

We now consider the question of whether or not $I(G)$ is a chain. The answer is no in general.

PROPOSITION 2. *Let X be an uncountable set; $H = \{f \in B(X, \mathbf{Z})\colon f$ is eventually constant$\}$, $K = \{f \in B(X, \mathbf{Z})\colon f$ has countably infinite support$\}$. Then $\langle i_H \rangle$ and $\langle i_K \rangle$ are incomparable ideals and hence $I(B(X, \mathbf{Z}))$ is not a chain.*

PROOF. The l-subgroup K has the following property: for each $0<f\in K$ there exist $f_1, f_2>0$ in K so that $f_1\wedge f_2=0$ and $f=f_1\vee f_2$. This property is not true for the elements of H and hence H contains no l-isomorphic copy of K. Thus $i_K\notin\langle i_H\rangle$. On the other hand, H contains an element (the constant function 1) with uncountably many values but in K each nonzero element has only countably many values. Thus $i_H\notin\langle i_K\rangle$. Hence the ideals are incomparable.

We would still like to know when $I(G)$ is a chain (G again consists of bounded integer-valued functions), so let us restrict ourselves to the l-group of bounded integer-valued sequences; we denote this l-group by B. Our first result lengthens the chain of ideals found in Proposition 1.

PROPOSITION 3. *Let $I_{\infty,1}$ be the principal ideal of $\mathcal{I}(B)$ generated by i_H where H is the l-group of all eventually constant sequences. Then*

1) $I_\infty<I_{\infty,1}$

2) *If J is an ideal of $I(B)$ then either $J\leqq I_\infty$ or $I_{\infty,1}\leqq J$.*

PROOF. 1) H contains elements with infinitely many values; this in conjunction with Proposition 1 gives us that $I_\infty<I_{\infty,1}$.

2) Suppose J is an ideal of $I(B)$ such that $J\nleqq I_\infty$. Then J property contains I_∞. Then there is an l-subgroup K of B such that $i_K\in J-I_\infty$; K must then contain an element $x>0$ with infinitely many values. We wish to show that K contains a copy of H. First suppose that K' is the convex l-subgroup of K generated by x. If K' has no infinite pairwise disjoint sets then, as before, K' is l-isomorphism to a finite cardinal sum of copies of $\mathbf{Z}$. This is impossible since x has infinitely many values. Thus, suppose that $\{x_1, x_2, \ldots\}$ is an infinite pairwise disjoint subset of K'. Since K' is generated (as a convex l-subgroup) by x, $x\wedge x_n>0$ for each $n=1, 2, \ldots$. We may therefore replace x_n by $x\wedge x_n$ and assume that $x\geqq x_n$ for each n.

Furthermore, since K is hyper-archimedean ([2]), there is, for each integer n, a positive integer k_n such that $x\wedge k_nx_n=x\wedge(k_n+1)x_n$. Replacing x_n by $k_nx_n\wedge x$ we may assume that x agrees with x_n (for each n) at every point of the support of x. From this it is clear that $x=\sup\{x_1, x_2, \ldots\}$. Now in H, for each n, define e_n by $e_n(m)=1$ if $n=m$ and 0 otherwise and let e be the constant function 1. The elements e_n and e generate H in the obvious way. Define $\varphi(e_n)=x_n$ and $\varphi(e)=x$, and extend φ to all of H. Then φ will be an l-isomorphism from H into K. Thus $i_H\in\in\langle i_K\rangle\leqq J$ and hence $I_{\infty,1}\leqq J$.

Above $I_{\infty,1}$ things are a bit more obscure. We shall next discuss the role of various l-groups of periodic sequences and conclude this discussion with several open questions.

Let P denote the l-subgroup of B of periodic sequences. Since P is a countable set and B is not, the ideal $\langle i_p\rangle$ is properly contained in $\mathcal{I}(B)$. For each positive number q let P_q denote the l-subgroup of P generated by all periodic sequences whose period divides q^k, where k is any non-negative integer. Note that the l-subgroup generated by P_q and P_t will be P_m where m is the least common multiple of q and t. Although we do not know if $\langle i_P\rangle=\langle i_{P_q}\rangle$ we do have the following

PROPOSITION 4. *For any positive numbers q and t, $\langle i_{P_q}\rangle=\langle i_{q_t}\rangle$.*

PROOF. We will show that P_q can be l-embedded in P_t for any numbers q and t; this will prove the proposition. To do this it will be sufficient to define an embedding

φ on elements of P_q whose periods are powers of q (these elements generate P_q). So, let r be the smallest positive integer such that $q<t^r$. We define φ inductively on n for elements of P_q of period q^n.

Suppose $s=(s_1s_2s_3\ldots)\in P_q$ and period of s is q. Define φs of period t^r as follows:

$$(\varphi s)_i = \begin{cases} s_i & 1 \leqq i \leqq q-1 \\ s_q & q \leqq i \leqq t^r \\ (\varphi s)_j & j \equiv i \bmod t^r \quad \text{with} \quad 1 \leqq j \leqq t^r. \end{cases}$$

Now assume that φs is defined for elements in P_q of period less than or equal to q^n in such a manner that if the period of s is q^k then the period of φs is t^{rk} and φ is an l-embedding. Suppose $s\in P_q$ has period q^{n+1} and let $s=S_1S_2\ldots S_qS_1S_2\ldots S_q\ldots$, where $S_i=s_{(i-1)q^n+2}\ldots s_{(i-1)q^n+q^n}$. We define φs of period $t^{(n+1)r}$ as follows:

$$\varphi s = T_1T_2 \ldots T_{q-1}\underbrace{T_qT_q \ldots T_q}_{t^r-(g-1)}T_1T_2 \ldots$$

where T_i is the block of length t^{nr} that defines $\varphi(S_iS_iS_i\ldots)$. We leave the details of verifying that φs an l-embedding to the reader.

EXAMPLES. 1) $q=3,\ t=2,\ r=2,$
$\varphi(\underline{123456789}\ldots)=(\underline{1233456678997899}\ldots)$
$\varphi(\underline{123112233111222333112133123}\ldots)=$
$=(12331122233323333111122223333333311222333123312331122233312331233\ldots)$
2) $q=3,\ t=5,\ r=1.$
$\varphi(\underline{123456789}\ldots)=(\underline{1233345666789997899978999}\ldots).$

We close the article with a list of open questions and several remarks pertaining to them. Let us start by mentioning some specific questions relating to B, the l-group of bounded, integer-valued sequences.

I. Are i_P, and i_{P_q} $\mathscr{J}$-related? Is $\langle i_P\rangle$ a maximal proper ideal? Is $I(B)$ a chain?

II. What are the implications on an l-group G of having a finite lattice of ideals $I(G)$? We can mention a few (the arguments are straightforward). G must then have finite rank and thus the root system of values is also finite. In addition, if M is a value of G and M^*/M has rank 1, its type $(t_2, t_3, \ldots)$ must satisfy $t_i=0$ for all but finitely many indices. We have not been able to decide whether these conditions on G and its values are sufficient to make $I(G)$ finite.

If we make restrictions on $\mathscr{I}(G)$ we obtain more information. Suppose $\mathscr{I}(G)$ has a finite number of $\mathscr{D}$-classes. Since an archimedean o-group of the form $(1, r)$ (with r irrational) has infinitely many non-isomorphic subgroups, I_1 cannot admit class 3 covers. Now $\mathscr{D}\subseteqq\mathscr{J}$ and so we have that $I(G)$ is finite and G therefore has the properties mentioned in the above paragraph. Now an abelian l-group having a finite root-system of values must be a Hahn-group $V(\Lambda, R_\lambda)$ ([2]) where Λ is a finite root system and R_λ is a subgroup of $\mathbf{R}$. Since I_1 admits no class 3 covers each R_λ is a group of rank 1 whose type has at most finitely many nonzero entries. Conversely, suppose that $G=V(\Lambda, R_\lambda)$ where Λ is finite and each R_λ is a rank 1 group with the above property. It is then clear that G has (up to l-isomorphisms) only finitely many l-subgroups; that is, $\mathscr{I}(G)$ has finitely many $\mathscr{D}$-classes. We summarize:

THEOREM 5. *For an l-group G the following are equivalent:*

1) *$\mathcal{I}(G)$ has finitely many $\mathcal{D}$-classes.*

2) *$G=V(\Lambda, R_\lambda)$ where Λ is a finite root-system and each R_λ is a rank 1 group whose type $(t_2, t_3, t_5, \ldots)$ has at most finitely many nonzero entries.*

In this case I_1 admits no class 3 covers.

III. Suppose $\mathcal{I}(G)$ has finitely many $\mathcal{D}$-classes (and therefore satisfies Theorem 5 above) and also I_1 admits no class 4 covers. Then G is a Hahn-group $V(\Lambda, Z_\lambda)$ where Λ is finite and each Z_λ is cyclic. Is $\mathcal{I}(G)$ a complete invariant under these circumstances? This leads to a more general question: what conditions on G or $\mathcal{I}(G)$ are necessary to make $\mathcal{I}(G)$ a complete invariant?

IV. We know that if $\mathcal{D}=\mathcal{J}$ in $\mathcal{I}(G)$ then I_1 admits no class 3 covers. The converse is not true but what conditions must hold to have $\mathcal{D}=\mathcal{J}$? Or what are the implications for G if $\mathcal{D}=\mathcal{J}$?

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(Received April 30, 1981)

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Acta Math. Hung.
41 (3—4) (1983), 213—218.

α-CONTINUOUS AND α-OPEN MAPPINGS

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Introduction

Let X, Y and Z be topological spaces on which no separation axioms are assumed unless explicitly stated. Let S be a subset of X. The closure (resp. interior) of S will be denoted by $\bar{S}$ (resp. S^0). A subset S of a space X is called α-set [6] (resp. semi-open set [4], preopen set [5]) if $S \subset S^{0-0}$ (resp. $S \subset S^{0-}$, $S \subset S^{-0}$), the complement of an α-set (resp. semi-open set, preopen set) is called α-closed set (resp. semi-closed set, preclosed set). The familly of all α-sets (resp. semi-open sets, preopen sets) in X will be denoted by $\alpha(X)$ (resp. $SO(X)$, $PO(X)$). It is clear that each α-set (resp. α-closed set) is semi-open and preopen (resp. semi-closed, preclosed), but the converses are not true. A mapping $f\colon X \to Y$ is called almost continuous [8] (resp. θ-continuous [1], weakly continuous [3]), briefly a.c. (resp. θ-cont., w. c.) if for each $x \in X$ and each open neighbourhood V of $f(x)$ there exists an open neighbourhood U of x such that $f(U) \subset V^{-0}$ (resp. $f(\bar{U}) \subset \bar{V}$, $f(U) \subset \bar{V}$) and it is called semi-continuous [4] (resp. precontinuous [5]), briefly s.c. (resp. p.c.) if the inverse image of each open set is semi-open (resp. preopen). A mapping $f\colon X \to Y$ is called semi-open [4] (resp. preopen [5]) if the image of each open set in X is semi-open (resp. preopen) and it is called semi-closed [7] (resp. preclosed) if the image of each closed set in X is semi-closed (resp. preclosed).

In the present note we introduce and study the concepts of α-continuity and α-open mappings. Also we strengthen some results in [5] by using this type of non-continuous mapping.

1. α-continuity

DEFINITION 1.1. A mapping $f\colon X \to Y$ is called α-continuous (briefly, α-cont.) if the inverse image of each open set of Y is an α-set.

Theorem 1.1 is an easy consequence of Definition 1.1, and the proof is thus omitted.

THEOREM 1.1. *Let $f\colon X \to Y$ be a mapping, then the following statements are equivalent.*

(i) *f is α-cont.*

(ii) *For each $x \in X$ and each open set $V \subset Y$ containing $f(x)$, there exists $W \in \alpha(X)$ such that $x \in W$, $f(W) \subset V$.*

(iii) *The inverse image of each closed set in Y is α-closed.*

(iv) $f(A^{-0-}) \subset (f(A))^-$ *for each* $A \subset X$.

(v) $(f^{-1}(M))^{-0-} \subset f^{-1}(M^-)$ *for each* $M \subset Y$.

COROLLARY 1.1. *Let $f: X \to Y$ be α-cont., then*

(i) $f(A^-) \subset (f(A))^-$ *for each* $A \in PO(X)$.
(ii) $(f^{-1}(M))^- \subset f^{-1}(M^-)$ *for each* $M \in PO(Y)$.

PROOF. Since for each $A \in PO(X)$, $A^- = A^{-0-}$, therefore the proof follows directly from statements (iv), (v) of Theorem 1.1.

THEOREM 1.2. *Every α-continuous mapping $f: X \to Y$ is θ-continuous.*

PROOF. Let $x \in X$ and $V \subset Y$ be an open set containing $f(x)$. By statement (v) of Theorem 1.1, $(f^{-1}(V))^{-0-} \subset f^{-1}(V^-)$. Since f is α-cont., we have, $f^{-1}(V) \subset \subset (f^{-1}(V))^{0-0} \subset ((f^{-1}(V))^{0-0})^- \subset (f^{-1}(V))^{-0-} \subset f^{-1}(V^-)$. Put $(f^{-1}(V))^{0-0} = = U$, so U is a neighbourhood of x such that $U^- \subset f^{-1}(V^-)$, namely $f(U^-) \subset V^-$. Therefore, f is θ-cont.

It is clear that the class of α-continuity contains the class of continuity but it is contained in the class of each of θ-continuity [1], precontinuity [5] (=almost continuity in the sense of Husain [2]) and semi-continuity [4]. The concepts of α-continuity and almost continuity in the sense of Singal and Singal [8] are independent of each other.

The following diagram summarizes the above discussion.

$$\begin{array}{ccccc} & & \text{semi-continuity} & & \\ & & \Uparrow & & \\ \text{continuity} & \Longrightarrow & \alpha\text{-continuity} & \Longrightarrow & \text{precontinuity} \\ \Downarrow & & \Downarrow & & \\ \text{almost continuity} & \Longrightarrow & \theta\text{-continuity} & \Longrightarrow & \text{weak continuity} \end{array}$$

The examples given below show that the converses of these implications are not true in general.

EXAMPLE 1.1. An injective mapping $f: X \to Y$ form an indiscrete space into a discrete space is p.c., but not α-cont.

EXAMPLE 1.2. Take $X = \{a, b, c\}$, $T_1 = \{X, \varphi, \{a\}, \{c\}, \{a, c\}\}$ and T_2=Discrete topology. Let $f: (X, T_1) \to (X, T_2)$ such that $f(a) = a$, $f(c) = b$, $f(b) = b$. So, f is s.c., but not α-cont.

EXAMPLE 1.3. An injective mapping $f: X \to Y$ from an excluding topological space X with excluding point p into a particular topological space Y with a particular point $f(p)$ is θ-cont., but not α-cont.

EXAMPLE 1.4. Let $X = Y = \{a, b, c\}$ and $T_X = T_Y = \{X, \varphi, \{a\}\}$. Then, a mapping $f: X \to Y$ which is defined by $f(a) = f(b) = a$ and $f(c) = c$ is α-cont., but not continuous.

REMARK. According to the above diagram and the fact, "w.c. mapping is continuous if the range is a regular space [3]", we have that for a mapping f from a space into a regular space, the following are equivalent.

(1) f is w.c.
(2) f is θ-cont.
(3) f is a.c.
(4) f is α-cont.
(5) f is continuous.

LEMMA 1.1. *Let $A\in PO(X)$ and $B\in\alpha(X)$, then $A\cap B\in\alpha(A)$.*

PROOF. Since $A\cap B\subset A^{-0}\cap B^{0-0}=(A^{-0}\cap B^{0-})^{0}\subset(A\cap B^{0})^{-0}$, so $A\cap B\subset$ $\subset(A\cap B^{0})^{-0}\cap A=\mathrm{Int}_A\left((A\cap B^{0})^{-0}\cap A\right)\subset\mathrm{Int}_A\left((A\cap B^{0})^{-}\cap A\right)=\mathrm{Int}_A\left(\mathrm{Cl}_A(A\cap B^{0})\right)=$ $=\mathrm{Int}_A\left(\mathrm{Cl}_A\left(\mathrm{Int}_A(A\cap B^{0})\right)\right)\subset\mathrm{Int}_A\left(\mathrm{Cl}_A\left(\mathrm{Int}_A(A\cap B)\right)\right)$, where $\mathrm{Int}_A(\dots)$ and $\mathrm{Cl}_A(\dots)$ denote the interior and the closure with respect to the subspace A. This implies that $A\cap B\in\alpha(A)$.

THEOREM 1.3. *If $f\colon X\to Y$ is α-cont. and $U\in PO(X)$, then $f|U$ is α-cont.*

PROOF. Let $V\subset Y$ be an open set, then $f^{-1}(V)\in\alpha(X)$. Since $U\in PO(X)$, by Lemma 1.1, we have $U\cap f^{-1}(V)=(f|U)^{-1}(V)\in\alpha(U)$. Therefore $f|U$ is α-cont.

LEMMA 1.2. *Let $A\subset Y\subset X$, $Y\in\alpha(X)$ and $A\in\alpha(Y)$, then $A\in\alpha(X)$.*

PROOF. Since $\mathrm{Int}_Y\left(\mathrm{Cl}_Y\left(\mathrm{Int}_Y(A)\right)\right)$ is open in Y, there exists an open set $U\subset X$ such that $\mathrm{Int}_Y\left(\mathrm{Cl}_Y\left(\mathrm{Int}_Y(A)\right)\right)=Y\cap U$. Since $Y\in\alpha(X)$, $A\subset U\cap Y^{0-0}\subset(U\cap Y)^{0-0}=$ $=\left(\mathrm{Int}_Y\left(\mathrm{Cl}_Y\left(\mathrm{Int}_Y(A)\right)\right)\right)^{0-0}\Big|\left(\left(\mathrm{Int}_Y(A)\right)^{-}\right)^{0-0}=\left(\mathrm{Int}_Y(A)\right)^{-0}$. Since $\mathrm{Int}_Y(A)$ is open in Y, there exists an open set $V\subset X$ such that $\mathrm{Int}_Y(A)=V\cap Y$, so $A\subset(V\cap Y^{0-0})^{-0}\subset$ $\subset(V\cap Y)^{0-0}=\left(\mathrm{Int}_Y(A)\right)^{0-0}\subset A^{0-0}$. Hence, $A\in\alpha(X)$.

THEOREM 1.4. *Let $f\colon X\to Y$ be a mapping and $\{U_i\colon i\in I\}$ be a cover of X such that $U_i\in\alpha(X)$ for each $i\in I$. Then, f is α-cont. iff $f|U_i$ is α-cont. for each $i\in I$.*

PROOF. Let $V\subset Y$ be an open set, then $(f|U_i)^{-1}(V)\in\alpha(U_i)$. Since $U_i\in\alpha(X)$, by Lemma 1.2, $(f|U_i)^{-1}(V)\in\alpha(X)$ for each $i\in I$. But, $f^{-1}(V)=\bigcup_{i\in I}\left((f|U_i)^{-1}(V)\right)$, then $f^{-1}(V)\in\alpha(X)$ because the union of α-sets is an α-set. This implies that f is α-cont.

More characterizations of α-cont. mappings $f\colon X\to Y$ are given in the following.

(i) If X is a connected space, then $f(X)$ is connected.
(ii) If f is surjective, then Y is almost compact if X is almost compact.

Also one may deduce that:

(1) Let $f\colon X\to Y$ be a mapping and let $g\colon X\to X\times Y$, given by $g(x)=\left(x, f(x)\right)$ be its graph mapping. Then f is α-cont. if and only if g is α-cont.

(2) Let $f_i\colon X_i\to Y_i$ be a mapping for each $i\in I$ and $f\colon \pi X_i\to\pi Y_i$ be a mapping defined by $f(\{x_i\})=\{f_i(x_i)\}$ for each $\{x_i\}\in\pi X_i$. Then, f is α-cont. if and only if f_i is α-cont. for each $i\in I$.

2. α-open and α-closed mappings

Now, we introduce new classes of mappings called α-open and α-closed mappings.

DEFINITION 2.1. A mapping f: $X \to Y$ is called α-open (resp. α-closed) if the image of each open (resp. closed) set in X is an α-set (resp. α-closed).

From the above definition one may have the following diagram.

$$\begin{array}{ccc} & & \text{preopen (resp. preclosed)} \\ & & \Uparrow \\ \text{open (resp. closed) mapping} & \Rightarrow & \alpha\text{-open (resp. } \alpha\text{-closed)} \\ & & \Downarrow \\ & & \text{semi-open (resp. semi-closed)} \end{array}$$

The converses of these implications are not true as the following examples illustrate.

EXAMPLE 2.1. An injective mapping from a discrete space into an excluding topological space is semi-open and semi-closed, but neither α-open nor α-closed.

EXAMPLE 2.2. An injective mapping from a discrete space into an indiscrete space is preopen and preclosed, but neither α-open nor α-closed.

EXAMPLE 2.3. Let $X=Y=\{x, y, z\}$ and $T_X=T_Y=\{X, \varphi, \{x, y\}, \{x\}\}$. Then, a mapping f: $X \to Y$ which is defined by $f(x)=x$, $f(y)=z$ and $f(xz)=y$ is α-open and α-closed but neither open nor closed.

THEOREM 2.1. *A mapping f: $X \to Y$ is α-open if and only if for each $x \in X$ and each open set U of X containing x, there exists an α-set $W \subset Y$ containing $f(x)$ such that $W \subset f(U)$.*

PROOF. Follows immediately from Definition 2.1.

DEFINITION 2.2. The intersection of all α-closed sets containing a subset $A \subset X$ is called α-closure of A and is denoted by $Cl_\alpha(A)$.

THEOREM 2.2. *A mapping f: $X \to Y$ is α-closed if and only if* $\mathrm{Cl}_\alpha(f(A)) \subset f(A^-)$ *for each $A \subset X$.*

PROOF. Follows directly from Definition 2.2.

THEOREM 2.3. *Let f: $X \to Y$ be α-open (resp. α-closed). If $W \subset Y$ and $F \subset X$ is a closed (resp. open) set containing $f^{-1}(W)$, then there exists an α-closed set (resp. an α-set) $H \subset Y$ containing W such that $f^{-1}(H) \subset F$.*

PROOF. Let $H=Y-f(X-F)$. Since $f^{-1}(W) \subset F$, we have $f(X-F) \subset Y-W$. Since f is α-open (resp. α-closed), then H is α-closed (resp. an α-set) and $f^{-1}(H)= =X-f^{-1}(f(X-F)) \subset (X-F)=F$.

COROLLARY 2.1. *If f: $X \to Y$ is α-open, then*
(i) $f^{-1}(B^{-0-}) \subset (f^{-1}(B))^-$ *for each set* $B \subset Y$.
(ii) $f^{-1}(A^-) \subset (f^{-}(A))^-$ *for each* $A \in PO(Y)$.

PROOF. (i) $(f^{-1}(B))^-$ is closed in X containing $f^{-1}(B)$ for a set $B\subset Y$. By Theorem 2.3, there exists an α-closed set $H\subset Y$, $H\supset B$ such that $f^{-1}(H)\subset \subset(f^{-1}(B))^-$. Thus, $f^{-1}(B^{-0-})\subset f^{-1}(H^{-0-})\subset f^{-1}(H)\subset(f^{-1}(B))^-$.

(ii) Follows directly from (i).

3. Comparison

The following lemma is very useful in the sequel.

LEMMA 3.1. [5]. *If f: $X\to Y$ is preopen, then $f^{-1}(V^{0-})\subset(f^{-1}(V))^-$ for each $V\subset Y$.*

THEOREM 3.1. *If f: $X\to Y$ is p.c. and s.c., then f is α-cont.*

PROOF. Let $V\subset Y$ be an open set, so $f^{-1}(V)\in PO(X)$ and $f^{-1}(V)\in SO(X)$. Then, $f^{-1}(V)\subset(f^{-1}(V))^{0-}$, $f^{-1}(V)\subset((f^{-1}(V))^{0-})^{-0}=(f^{-1}(V))^{0-0}$. Therefore, f is α-cont.

THEOREM 3.2. *If f: $X\to Y$ is α-cont., preopen, then the inverse image of each α-set in Y is an α-set.*

PROOF. Let $V\subset Y$ be an α-set, so $f^{-1}(V)\subset f^{-1}(V^{0-0})\subset(f^{-1}(V^{0-0}))^{0-0}\subset$

$\subset(f^{-1}(V^{0-}))^{0-0}$. By Lemma 3.1, we have $f^{-1}(V)\subset(f^{-1}(V^0))^{-0}$. Since f is α-cont., by Theorem 1.1, $f^{-1}(V)\subset(f^{-}(V))^{0-0}$. This implies that $f^{-1}(V)\in\alpha(X)$.

COROLLARY 3.1. *If f: $X\to Y$ is α-cont. and preopen, we have:*

(1) *The inverse image of each α-closed set is α-closed.*
(2) $f(\mathrm{Cl}_\alpha(A))\subset\mathrm{Cl}_\alpha f(A)$ *for each set* $A\subset X$.

PROOF. This follows immediately from Theorem 3.2.

THEOREM 3.3. *Let f: $X\to Y$ and g: $Y\to Z$ be two mappings. If f is preopen, α-continuous and g is α-continuous, then $g\circ f$ is α-continuous.*

PROOF. Follows immediately from Theorem 3.2.

A. S. Mashhour, et al. [5] have shown that for every p.c., open mapping the inverse image of each preopen set is preopen. The following theorem is a slight improvement of this fact.

THEOROM 3.4. *Let f: $X\to Y$ be p.c., and α-open. Then the inverse image of each preopen set is preopen.*

PROOF. Let $V\in PO(Y)$, so $f^{-1}(V)\subset f^{-1}(V^{-0})\subset(f^{-1}(V^{-0}))^{-0}\subset(f^{-1}(V^-))^{-0}$. Since f is α-open, by Corollary 2.1, we have, $f^{-1}(V)\subset(f^{-1}(V^-))^{-0}\subset((f^1(V)^-)^{-0}=$ $=(f^{-1}(V))^{-0}$. Therefore, $f^{-1}(V)\in PO(X)$.

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(Received June 8, 1981)

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Acta Math. Hung.
41 (3—4) (1983), 219—224.

ON THE SHADOW LINES OF A CONVEX SURFACE

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0. Introduction

Shadow lines are called the curves on a convex surface which are defined as separating lines between illuminated and shadowed sides of the surface, when the latter is illuminated by a point light-source (see Fig. 1) or through parallel light in a given direction e (Fig. 2). Several properties of the shadow lines relating to the geometry of

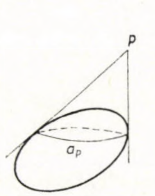

Fig. 1

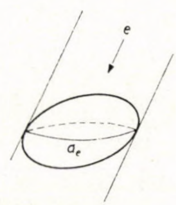

Fig. 2

the convex surface are known (see [1], [2], [3], [4]). The problem we are concerned with here, is the calculation of variation of the function $f_1(p)$ (resp. $f_2(e)$) which gives the length of the shadow line a_p as a function of the point p (resp. the length of a_e as a function of e). This is done by means of a naturaly defined submersion whose fibres are isometric to the shadow lines. We use also a simple general lemma on variations of integrals over the compact fibres of a submersion. The variation formulae thus obtained (see (23), (24)) show that if for some point p (resp. direction e) the corresponding shadow line a_p (resp. a_e) coincides with a geodesic of the surface, then p (resp. e) is a stationary point of f_1 (resp. f_2).

Every function, vector field, etc. apearing in our discussion is supposed to be C^∞-differentiable. An alternative discussion on functions similar to our f_1, f_2 but using approximations of the convex surface by convex polyedra can be found in [5]. Note that our method can easily be adapted in order to compute variations of any order of the functions f_1, f_2.

1. Submersions whose fibres are isometric to the shadow lines

Let M be a C^∞-differentiable convex surface of the Euclidean space E^3. Let ξ be the unit normal vector field on M pointing "outside" M. We think of the convexity of M as equivalent with the condition

$$(1) \qquad \langle Aw, w\rangle \neq 0, \quad \text{for every tangent vector } w \neq 0 \text{ of } M.$$

Here $\langle ..., ... \rangle$ denotes the induced metric of M (from E^3), A denotes the second fundamental form with respect to ξ (see [6] for general definitions and formulae).

A. First we construct a submersion whose fibres are the a_p's. Let P denote the open unbounded part of E^3 consisting of points lying "outside" M. Let $M\times P$ have the Riemannian product structure and F be the C^∞-function

$$(2)\qquad \begin{cases} F\colon\ M\times P\to R \\ F(m, p) = \langle \xi_m, m-p\rangle, \text{ for every } m\in M \text{ and } p\in P. \end{cases}$$

$m-p\in E^3$ is identified with a tangent vector of E^3 at m, according to the standard identification

$$(3)\qquad a\left.\frac{\partial}{\partial x_1}\right|_m + b\left.\frac{\partial}{\partial x_2}\right|_m + c\left.\frac{\partial}{\partial x_3}\right|_m \leftrightarrow (a, b, c).$$

The same identification will be used throughout our discussion without other explicit mention. Let now $\overline{M}$ be defined as

$$(4)\qquad \overline{M} = F^{-1}(\{0\}) = \{(m, p)\in M\times P | \langle \xi_m, m-p\rangle = 0\}.$$

LEMMA 1.1. *$\overline{M}$ is a four dimensional submanifold of $M\times P$.*

PROOF. In fact, if $(m, p)\in\overline{M}$ and $(\alpha_1(t), \alpha_2(t))$ is a curve lying in $\overline{M}$ with $\alpha_1(0)=m$, $\alpha_2(0)=p$, $\dot\alpha_1(0)=v_1\in T_mM$, $\dot\alpha_2(0)=v_2\in T_pP$ then we have

$$\left.\frac{d}{dt}\right|_0 F(\alpha_1(t), \alpha_2(t)) = \left.\frac{d}{dt}\right|_0 \langle \xi_{\alpha_1(t)}, \alpha_1(t)-\alpha_2(t)\rangle =$$
$$= \langle -A_m v_1, m-p\rangle + \langle \xi_m, v_1-v_2\rangle = \langle -A_m(m-p), v_1\rangle + \langle \xi_m, -v_2\rangle,$$

because $\langle m-p, \xi_m\rangle = 0$ and $\langle v_1, \xi_m\rangle = 0$. Hence the gradient of F at points $(m, p)\in\overline{M}$ is

$$(5)\qquad -(A_m(m-p)\oplus\xi_m)\neq 0,$$

and the lemma follows from the implicit function theorem.

PROPOSITION 1.2. *The restriction $\pi=\pi'|\overline{M}$ of the natural projection π': $M\times P\to P$ on the second factor is a submersion of $\overline{M}$ whose fibres $\pi^{-1}(p)$ are isometric to the shadow lines a_p.*

PROOF. The last assertion becomes clear if we observe that

$$(6)\qquad \pi^{-1}(p) = \{(m, p) | \langle \xi_m, m-p\rangle = 0\}\subset M\times\{p\},$$

which, for fixed p, is precisely the differential geometric description of the shadow line a_p, as a set of points of $M\times\{p\}=M$ (Fig. 3). Let now $(m, p)\in\overline{M}$. From (5) and the definition of $\overline{M}$ we have

$$(7)\qquad T_{(m,p)}\overline{M} = \{w_1\oplus w_2\in T_{(m,p)}M\times P | \langle A_m(m-p), w_1\rangle + \langle \xi_m, w_2\rangle = 0\}.$$

Hence to prove that π is a submersion, we have to show that for every $w_2\in T_pP$, there is a $w_1\in T_mM$ such that

$$\langle A_m(m-p), w_1\rangle = -\langle \xi_m, w_2\rangle.$$

From (1) we have $A_m(m-p)\neq 0$, hence this is always possible.

B. The construction of a submersion whose fibres are the a_e's follows the same pattern:

We first consider the product $M \times S$ of M with the unit sphere S of E^3, whose points are identified with the "directions" of E^3. We define F by

$$\begin{cases} F\colon M \times S \to R \\ F(m, e) = \langle \xi_m, e \rangle, \text{ for every } m \in M \text{ and } e \in S. \end{cases} \tag{8}$$

Let $\overline{M}$ be the subset of $M \times S$ defined by

$$\overline{M} = F^{-1}(\{0\}) = \{(m, e) \in M \times S \mid \langle \xi_m, e \rangle = 0\}. \tag{9}$$

At $(m, e) \in \overline{M}$ the gradient of F is easily computed to be

$$(-A_m(e)) \oplus \xi_m \neq 0. \tag{10}$$

Hence we have the following:

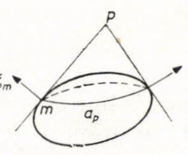

Fig. 3

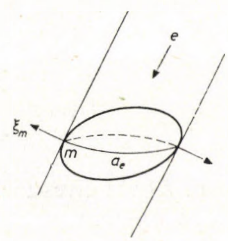

Fig. 4

LEMMA 1.3. *$\overline{M}$ is a three dimensional submanifold of $M \times S$.*

PROPOSITION 1.4. *The restriction $\pi = \pi' | \overline{M}$ of the natural projection π': $M \times S \to S$ on the second factor is a submersion of $\overline{M}$ whose fibres $\pi^{-1}(e)$ are isometric to the shadow lines a_e.*

PROOF. The last assertion becomes clear if we look at

$$\pi^{-1}(e) = \{(m, e) \mid \langle \xi_m, e \rangle = 0\} \subset M \times \{e\}, \tag{11}$$

which for fixed e is precisely the differential geometric description (Fig. 4) of the shadow line a_e as a set of points of $M \times \{e\} = M$. The metric of $\overline{M}$ is the induced one from the product $M \times S$.

Let now $(m, e) \in \overline{M}$. From (10) and the definition of $\overline{M}$ we have

$$T_{(m,e)}\overline{M} = \{w_1 \oplus w_2 \in T_{(m,e)} M \times S \mid \langle A_m(e), w_1 \rangle = \langle \xi_m, w_2 \rangle\}. \tag{12}$$

Since M is convex, given $w_2 \in T_e S$ we can, always find a $w_1 \in T_m M$ such that $\langle A_m(e), w_1 \rangle = \langle \xi_m, w_2 \rangle$. This proves that $\pi_*|_{(m,e)}$ is surjective, hence π is a submersion.

2. Formulation of the problem

Let J_m denote the positive $\pi/2$-rotation of the tangent plane $T_m M$. From (5) (resp. (10)) we know that

$$X_{(m,p)} = J_m A_m(m-p) \oplus 0_p, \tag{13}$$

(resp.

$$X_{(m,e)} = J_m A_m(e) \oplus 0_e) \tag{14}$$

is a tangent vector field on $\overline{M}$ which spans at every point $(m, p) \in \overline{M}$ the tangent space of the shadow line a_p (resp. a_e) at that point. Thus the dual ω of $E = X/\|X\|$ with respect to the metric of $\overline{M}$ is a one form on $\overline{M}$ which, when restricted to the shadow line, coincides with its volume form. By means of this form ω let f_1 (resp. f_2) be defined as

$$f_1(p) = \int_{\pi^{-1}(p)} \omega, \text{ for every } p \in P. \tag{15}$$

(resp.

$$f_2(e) = \int_{\pi^{-1}(e)} \omega, \text{ for every } e \in S.) \tag{16}$$

Clearly $f_1(p)$ (resp. $f_2(p)$) gives the length of the shadow line a_p (resp. a_e). Thus our problem is reduced on that of computation of the variation of integrals of a differential form ω over the fibres of a submersion.

3. Variation of integrals over the fibres of a submersion

Although it is a simple one, I do not know whether the following formula is well known or not (see however [7]). Anyway, for the sake of completeness I give a short proof of it. Recall first that if $\pi: \overline{M} \to N$ is a submersion between Riemannian manifolds $(\overline{M}, g_1)$, (N, g_2), then the restriction of the tangent maps π_{*_p} to the orthogonal complements of the tangent spaces of the fibres is an isomorphism.[1] By means of this isomorphism every vector field X on N can be "lifted" to a vector field $\overline{X}$ on $\overline{M}$ such that

(i) $\overline{X}$ is orthogonal to the fibres
(ii) $\overline{X}$ is π-related to X: $\pi_* \overline{X} = X \circ \pi$.

LEMMA 3.1. *With the preceding notation and the additional condition that the fibres $\pi_p = \pi^{-1}(p)$ for $p \in N$ are compact, let ω be a differential form on $\overline{M}$ of degree $r = \dim \overline{M} = \dim N$. Define $f(p) = \int_{\pi_p} \omega$ for every $p \in N$. Then for every vector field X on N we have*

$$Xf = \int_{\pi_p} L_{\overline{X}} \omega.$$

[1] Not necessarily an isometry.

PROOF. Let $\bar{\varphi}(q,t)$, $\varphi(p,t)$ denote the flows of $\bar{X}$ resp. X. From (ii) we have

$$\pi(\bar{\varphi}(q,t)) = \varphi(\pi(q),t). \tag{17}$$

$$Xf_{|p} = \lim_{h\to 0}\frac{1}{h}\big(f(\varphi(p,h))-f(p)\big) = \lim_{h\to 0}\frac{1}{h}\Big(f\big(\pi(\bar{\varphi}(q,h))\big)-f(p)\Big), \text{ where } \pi(q)=p.$$

$$(*) \qquad = \lim_{h\to 0}\frac{1}{h}\left(\int\limits_{\pi(\pi(\bar{\varphi}(q,h)))}\omega-\int\limits_{h_p}\omega\right)$$

From (17) we have

$$(**) \qquad \pi_{(\pi(\bar{\varphi}(q,h)))} = \bar{\varphi}_h(\pi_p)$$

hence from $(*)$ and $(**)$ it follows that

$$Xf_p = \lim_{h\to 0}\frac{1}{h}\left(\int\limits_{\bar{\varphi}_h(\pi_p)}\omega-\int\limits_{\pi_p}\omega\right) = \int\limits_{\pi_p}\lim_{h\to 0}\frac{1}{h}(\bar{\varphi}_h^*\omega-\omega) = \int\limits_{\pi_p}L_{\bar{X}}\bar{\omega}.$$

4. First variation of f_1 and f_2

Let P, S, π be the same as in Section 1 and ω as in Section 2. We apply Lemma 3.1. For this we need the lift $\bar{X}$ of a vector field X on P (resp. S) which is easily computed as follows.

A. If $\pi\colon \bar{M}\to P$ then $T_{(m,p)}\bar{M}$ is given by

$$T_{(m,p)}\bar{M} = \{(w_1,w_2)\in T_{(m,p)}M\times P \mid \langle A_m(m-p), w_1\rangle+\langle \xi_m, w_2\rangle = 0\}. \tag{18}$$

Condition (i) of 3.1 is equivalent to

$$\langle J_m A(m-p), w_1\rangle = 0. \tag{19}$$

These relations together with $\langle\xi_m, w_1\rangle=0$ imply that $\bar{X}$ has the form

$$\bar{X}|_{(m,p)} = -\frac{\langle\xi_m, X\rangle}{\mu_1}\big(A_m(m-p)\oplus(-\mu_1\xi_m)\big)+0_m\oplus(X-\langle\xi_m, X\rangle\xi_m), \tag{20}$$

where $\mu_1=\langle A_m(m-p), A_m(m-p)\rangle$.

B. Analogously when $\pi\colon M\to S$ we find that

$$\bar{X}_{(m,e)} = \frac{\langle\xi_m, X\rangle}{\mu_2}\big(A_m(e)\oplus\mu_2\xi_m\big)+0_m\oplus(X-\langle X, \xi_m\rangle\xi_m), \tag{21}$$

where we put again $\mu_2=\langle A_m(e), A_m(e)\rangle$. For the evaluation of the integral of $L_{\bar{X}}\omega$ we note that

(22)

$$L_{\bar{X}}\omega(E) = \bar{X}\omega(E)-\omega([\bar{X},E]) = -\omega(\nabla_{\bar{X}}E)+\omega(\nabla_E\bar{X}) = \omega(\nabla_E\bar{X}) = -(\nabla_E E, \bar{X}),$$

where $\langle ..., ... \rangle$ denotes the product metric of $M \times P$ and ∇ the corresponding Levi-Civita connection. Because of (20) the last equation is equivalent to

$$L_X \omega(E) = -\langle \xi_m, X \rangle \langle kA_m(m-p), A_m(m-p) \rangle (\mu_1)^{-3/2} = -\langle \xi_m, X \rangle k(\mu_1)^{-1/2}$$

where k is the geodesic curvature of the shadow line, hence we have

$$Xf_1|_p = -\int_0^{f_1(p)} \langle \xi, X_p \rangle k(\mu_1)^{-1/2} ds. \tag{23}$$

By a similar reasoning we obtain, using (21),

$$Xf_2|_e = \int_0^{f_2(e)} \langle \xi, X_e \rangle k(\mu_2)^{-1/2} ds. \tag{24}$$

From these formulae it imediately follows that:

PROPOSITION 4.1. *If for some point p (resp. direction e) the shadow line a_p (resp. a_e) coincides with a geodesic of the surface M, then p (resp. e) is a stationary point of f_1 (resp. f_2).*

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(Received June 23, 1981)

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Acta Math. Hung.
41 (3—4) (1983), 225—239.

ZAPPA PRODUCTS*

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In this paper we study a general type of product that was introduced for groups by G. Zappa in 1940 [31]. Sometimes this product is referred to as a bilateral semidirect product. About 1950—1960 applications in *group theory* were investigated by Szép, Rédei, Huppert, Itô, Wielandt, and others. In spite of some inherent difficulties, important results eventually were proved (cf. Section 4). Moreover, several useful constructions in *semigroup theory* can be reduced to the Zappa product, but originally they were obtained by other means (Section 5). In fact, the Zappa product cannot be understood very well unless considered as a construction at the crossroads of *automata theory* and algebra (Section 7). Many questions concerning semigroup decompositions (Section 6) and semigroup automata remain open. Nevertheless, in both cases the concept of Zappa products turns out to be meaningful. Furthermore the intimate relationship between automata theory and classical algebra — without sliding to classicism — that is made explicit by the notion of Zappa product is remarkable.

1. Definition and basic properties

DEFINITION (1.1). A *Zappa product* of two semigroups S, T is a semigroup isomorphic to $S\times T, \circ$ where

$$(s_1, t_1)\circ(s_2, t_2) = (s_1\cdot\langle t_1, s_2\rangle, [t_1, s_2]\cdot t_2)$$

for some functions $\langle\,,\rangle\colon T\times S\to S$ and $[\,,]\colon T\times S\to T$. For a Zappa product of monoids we require $(1, 1)$ to be the identity element of $S\times T, \circ$. Similarly, a Zappa product with zero is a semigroup isomorphic to $S\times T\cup\{(0, 0)\}, \circ$ where $(0, 0)$ is a zero element and $\circ$ is defined as above, but $\langle\,,\rangle$, $[\,,]$ are functions with range $S\cup\{0\}$, resp. $T\cup\{0\}$, satisfying

LP0. $$[t, s] = 0 \Leftrightarrow \langle t, s\rangle = 0,$$
$$[0, s] = [t, 0] = 0 = \langle 0, s\rangle = \langle t, 0\rangle.$$

For any set X denote by $\mathcal{T}(X)$ the full transformation semigroup on X. Obviously, we can rewrite the functions $\langle\,,\rangle$, $[\,,]$ in terms of transformations by the follow-

* This research was supported by Grant A7877 of the Natural Sciences and Engineering Research Council of Canada.

ing mappings δ, τ:

$$\delta\colon T \to \mathscr{T}(S \cup \{0\}),\ t \mapsto (\delta_t\colon s \mapsto \delta_t(s) = \langle t, s\rangle),$$

$$\tau\colon S \to \mathscr{T}(T \cup \{0\}),\ s \mapsto (\tau(s)\colon t \mapsto t^s = [t, s]).$$

Since both the symmetric notation and the asymmetric one will be used later on, we state the following conditions in both notations for the convenience of the reader.

LP1.

(a) $t^1 = t \qquad [t, 1] = t$

(b) $1^s = 1 \qquad [1, s] = 1$

(c) $\delta_t(1) = 1 \qquad \langle t, 1\rangle = 1$

(d) $\delta_1(s) = s \qquad \langle 1, s\rangle = s.$

Of course, this condition is applicable only if S and T are monoids. On the other hand, in most cases identity elements due to condition LP1 may be adjoined to S and T, if desired. The essential conditions are

LP2. $t^{(s_1s_2)} = (t^{s_1})^{s_2} \qquad [t, s_2s_2] = [[t, s_1], s_2]$

LP3. $(t_1t_2)^s = t_1^{\delta_{t_2}(s)} \cdot t_2^s \qquad [t_1t_2, s] = [t_1, \langle t_2, s\rangle] \cdot [t_2, s]$

LP4. $\delta_{t_1t_2}(s) = \delta_{t_1}(\delta_{t_2}(s)) \qquad \langle t_1t_2, s\rangle = \langle t_1, \langle t_2, s\rangle\rangle$

LP5. $\delta_t(s_1s_2) = \delta_t(s_1) \cdot \delta_{t^{s_1}}(s_2) \qquad \langle t, s_1s_2\rangle = \langle t, s_1\rangle \cdot \langle [t, s_1], s_2\rangle.$

LP2 says that S is acting on $T \cup \{0\}$; i.e., τ is a homomorphism. Dually LP4 states that δ is an anti-homomorphism. We do not know an illuminating interpretation of LP3 and LP5 in purely algebraic terms, but those conditions are to be understood easily in automata theory as will be clear in section 7.

PROPOSITION (1.2). *Let S, T be semigroups and $\circ$ a multiplication on $S \times T$ as defined above.*

(a) *If* LP2–LP5 *are valid, then $S \times T, \circ$ is a semigroup.*

(b) *If S, T are monoids and* LP1–LP5 *are satisfied, then $S \times T, \circ$ is a monoid with identity* (1, 1).

(c) [15, 31] *If S, T are groups, then every Zappa product of S, T is a group. Similarly Zappa products with zero are obtained if* LP0—LP5 *are fulfilled.*

PROOF. (a) [Cf. 31] $((s_1, t_1) \circ (s_2, t_2)) \circ (s_3, t_3) = (s_1 \cdot \langle t_1, s_2\rangle, [t_1, s_2]t_2) \circ (s_3, t_3) =$

$$= \Big(s_1\langle t_1, s_2\rangle \underbrace{\langle [t_1, s_2]t_2, s_3\rangle}_{\text{LP4}},\ \underbrace{[[t_1, s_2]t_2, s_3]}_{\text{LP3}}t_3\Big) =$$

$$= \Big(\underbrace{s_1\langle t_1, s_2\rangle\langle [t_1, s_2], \langle t_2, s_3\rangle\rangle}_{\text{LP5}},\ \underbrace{[[t_1, s_2], \langle t_2, s_3\rangle]}_{\text{LP2}}[t_2, s_3]t_3\Big) =$$

$$= (s_1\langle t_1, s_2 \cdot \langle t_2, s_3\rangle\rangle,\ [t_1, s_2 \cdot \langle t_2, s_3\rangle][t_2, s_3]t_3) =$$

$$= (s_1, t_1) \circ (s_2\langle t_2, s_3\rangle, [t_2, s_3]t_3) = (s_1, t_1) \circ ((s_2, t_2) \circ (s_3, t_3))$$

(b) $(s, t)\circ(1, 1) = (s\langle t, 1\rangle, [t, 1]) = (s, t)$ by LP1a, c,

$(1,1)\circ(s, t) = (\langle 1, s\rangle, [1, s]t) = (s, t)$ by LP1b, d.

(c) The inverse element of (s, t) is $(1, t^{-1})\circ(s^{-1}, 1) = (\langle t^{-1}, s^{-1}\rangle, [t^{-1}, s^{-1}])$.

PROPOSITION (1.3). *If $S\times T$, $\circ$ is a Zappa product of monoids S, T [with identity element $(1, 1)$], then* LP1—LP5 *are valid.* [Cf. 15, 31].

PROOF. LP1a, c. $(1, t) = (1, t)\circ(1, 1) = (\langle t, 1\rangle, [t, 1])$.

LP1b, d. $(s, 1) = (1, 1)\circ(s, 1) = (\langle 1, s\rangle, [1, s])$.

In order to exploit the associativity, we compute

$$((1, t_1)\circ(s_2, t_2))\circ(s_3, 1) = (\langle t_1, s_2\rangle, [t_1, s_2]t_2)\circ(s_3, 1) =$$
$$= (\langle t_1, s_2\rangle\langle [t_1, s_2]t_2, s_3\rangle, [[t_1, s_2]t_2, s_3]),$$
$$(1, t_1)\circ((s_2, t_2)\circ(s_3, 1)) = (1, t_1)\circ(s_2\langle t_2, s_3\rangle, [t_2, s_3]) =$$
$$= (\langle t_1, s_2\langle t_2, s_3\rangle\rangle, [t_1, s_2\langle t_2, s_3\rangle][t_2, s_3]).$$

By considering the second components we get

LP23 $$[[t_1, s_2]t_2, s_3] = [t_1, s_2\langle t_2, s_3\rangle][t_2, s_3].$$

For LP2 put $t_2=1$: $[[t_1, s_2], s_3]=[t_1, s_2\langle 1, s_3\rangle][1, s_3]$ and note $\langle 1, s_3\rangle=s_3$, $[1, s_3]=1$.

For LP3 put $s_2=1$: $[[t_1, 1]t_2, s_3]=[t_1, \langle t_2, s_3\rangle][t_2, s_3]$ and note $[t_1, 1]=t_1$.

By considering the first components of the above calculation we get

LP45 $$\langle t_1, s_2\rangle\langle [t_1, s_2]t_2, s_3\rangle = \langle t_1, s_2\langle t_2, s_3\rangle\rangle.$$

For LP4 put $s_2=1$: $\langle t_1, 1\rangle\langle [t_1, 1]t_2, s_3\rangle=\langle t_1, \langle t_2, s_3\rangle\rangle$ and note $\langle t_1, 1\rangle=1$, $[t_1, 1]=t_1$.

For LP5 put $t_2=1$: $\langle t_1, s_2\rangle\langle [t_1, s_2], s_3\rangle=\langle t_1, s_2\langle 1, s_3\rangle\rangle$ and note $\langle 1, s_3\rangle=s_3$.

REMARK (1.4). If $S\times T$, $\circ$ is a Zappa product of monoids, then both $S\to S\times T$, $s\mapsto(s, 1)$ and $T\to S\times T$, $t\mapsto(1, t)$ are monomorphisms.

PROOF. $(s_1, 1)\circ(s_2, 1)=(s_1\langle 1, s_2\rangle, [1, s_2])=(s_1s_2, 1)$ by LP1b, d. $(1, t_1)\circ(1, t_2)=(\langle t_1, 1\rangle, [t_1, 1]t_2)=(1, t_1t_2)$ by LP1a, c.

Green's relationson S, T determine those on $S\times T$, $\circ$ to a certain extent.

LEMMA (1.5). *Let $S\times T$, $\circ$ be a Zappa product of S and T. Then*

$$(s_1, t_1) \equiv_{\mathcal{R}} (s_2, t_2)\Rightarrow s_1 \equiv_{\mathcal{R}} s_2 \text{ in } S,$$
$$(s_1, t_1) \equiv_{\mathcal{L}} (s_2, t_2)\Rightarrow t_1 \equiv_{\mathcal{L}} t_2 \text{ in } T,$$
$$(s_1, t_1) \leqq_{\mathcal{R}} (s_2, t_2)\Rightarrow s_1 \leqq_{\mathcal{R}} s_2 \text{ in } S,$$
$$(s_1, t_1) \leqq_{\mathcal{L}} (s_2, t_2)\Rightarrow t_1 \leqq_{\mathcal{L}} t_2 \text{ in } T.$$

PROOF. Straightforward.

REMARK (1.6). The mappings δ, τ of a Zappa product $S\times T$, $\circ$ satisfying LP2—LP5 have the following properties:

(a) Every $\tau(s)$ preserves the left ideals of T: $x\in Tt_2 \Rightarrow x^s \in Tt_2^s$ by LP3, i.e., $x \leqq_{\mathcal{L}} y \Rightarrow x^s \leqq_{\mathcal{L}} y^s$ for every $x, y\in T$. Similarly, every δ preserves the right ideals of S, by LP5: $x \leqq_{\mathcal{R}} y \Rightarrow \delta_t(x) \leqq_{\mathcal{R}} \delta_t(y)$ for every $x, y\in S$.

(b) $\tau(s)$ is an endomorphism, if $s\in \bigcap_{t\in T} \mathrm{Fix}\ \delta_t$. $S\times T$, $\circ$ is a semidirect product, if every $\delta_t = \mathrm{id}_S$. (In that case LP4, LP5 are fulfilled vacuously.)

(c) $\tau(s)$ is a right translation, if $\delta_t(s)=1$ for every $t\in T$. If every $\delta_t = \mathrm{const}_1 = (s\mapsto 1_S)$, then LP4, LP5 are fulfilled trivially.

(1.6)b and c may indicate already that the range of Zappa products is very comprehensive.

2. Zappa products and semigroup extensions

If M is a Zappa product of monoids S and T, then we know by (1.4) and (1.3) that M contains isomorphic copies $\bar{S}$, $\bar{T}$ of S, T such that every element of $M \cong S\times T$, $\circ$ is uniquely representable as a product of elements of those submonoids: $(s, t)=(s, 1)\circ(1, t)$. Conversely we show

PROPOSITION (2.1). *Let H be a semigroup with subsemigroups S, T such that*
(i) $H = S\cdot T$,
(ii) $s_1 t_1 = s_2 t_2 \Rightarrow s_1 = s_2 \wedge t_1 = t_2$ *for every* s_1, $s_2\in S$ *and* t_1, $t_2 \in T$; *i.e., every element of H is uniquely representable as $s\cdot t$ where $s\in S$ and $t\in T$. Then H is a Zappa product of S and T. Similarly, if $0\in H\setminus ST$ and every element of $H\setminus\{0\}$ is uniquely representable as a product $s\cdot t$, $s\in S$, $t\in T$, then H is a Zappa product of S and T with zero.*

PROOF. Assume that (i), (ii) are valid and define

$$\langle t, s\rangle = \text{the } v\in S \text{ such that } ts=v\cdot u \text{ where } v\in S \text{ and } u\in T,$$

$$[t, s] = \text{the } u\in T \text{ such that } ts=v\cdot u \text{ where } v\in S \text{ and } u\in T.$$

By (i), (ii) these mappings $T\times S\to S$, resp. T, are well defined and $S\times T$, $\circ$ is a groupoid where $\circ$ is given by the formula of (1.1). Consider the mapping $\varphi\colon H\to S\times T$ $x\mapsto(s, t)$ such that $x=s\cdot t$, $s\in S$, $t\in T$. We have to prove that φ is an isomorphism H, $\cdot \to S\times T$, $\circ$. For this purpose take $s_1, s_2\in S$, $t_1, t_2\in T$ and compute

$$(s_1t_1)\varphi\circ(s_2t_2)\varphi = (s_1, t_1)\circ(s_2, t_2) = (s_1\langle t_1, s_2\rangle, [t_1, s_2]t_2) =$$
$$= \big((s_1\langle t_1, s_2\rangle)([t_1, s_2]t_2)\big)\varphi = \big(s_1(\langle t_1, s_2\rangle[t_1, s_2])t_2\big)\varphi = \big(s_1(t_1 s_2)t_2\big)\varphi = \big((s_1t_1)\cdot(s_2t_2)\big)\varphi$$

by definition of $\langle\,,\rangle$, $[\,,]$. Concerning the Zappa product with zero, define $\langle t, s\rangle=[t, s]=0$ in case $ts=0$ and as above otherwise. Furthermore let $0\varphi = (0, 0)$. Clearly $x\varphi\circ y\varphi=(x\cdot y)\varphi$ is trivial, if $x=0$ or $y=0$. But we have to consider the possibility $t_1s_2=0$ in the above calculation:

Case $t_1s_2=0$. Then $\langle t_1, s_2\rangle=[t_1, s_2]=0$, and therefore

$$(s_1t_1)\varphi\circ(s_2t_2)\varphi = (s_1\cdot 0,\ 0\cdot t_2) = (0, 0) = 0\varphi = \big((s_1t_1)\cdot(s_2t_2)\big)\varphi.$$

Case $t_1 s_2 \neq 0$. Then $\langle t_1, s_2\rangle \in S$ and $[t_1, s_2] \in T$ and therefore $s_1\langle t_1, s_2\rangle \in S$ and $[t_1, s_2] t_2 \in T$. Moreover $s_1\langle t_1, s_2\rangle [t_1, s_2] t_2 \neq 0$, because $0 \notin S \cdot T$. Thus the above calculation remains valid in this case.

Let us compare the concept of Zappa products with the ordinary (i.e., Schreier type) extension theory of semigroups. A *semigroup extension* (E, θ) of a semigroup T by a semigroup S is a semigroup E containing a subsemigroup $\bar{T}$ isomorphic to T, together with a congruence θ on E such that $\bar{T}$ is a θ-class and that the factor semigroup E/θ is isomorphic to S. [27]

In contrary to that situation of semigroup extensions, a Zappa product $S \times T, \circ$ of monoids contains isomorphic copies of both T and S as noted above. Nevertheless, we may ask for the intersection of the two theories.

PROPOSITION (2.2). *A Zappa product $S \times T, \circ$ of monoids is a semigroup extension $(S \times T, \theta)$ of T with respect to the isomorphic copy $\{1\} \times T$ of T and the mapping $S \to (S \times T)/\theta$, $s \mapsto [(s, 1)]_\theta$, if and only if it is a semidirect product of T and S with respect to the homomorphism $\tau\colon S \to \mathcal{T}(T)$.*

PROOF. Given a Zappa product $S \times T, \circ$ of monoids, define an equivalence by

$$(s_1, t_1) \sim (s_2, t_2) \Leftrightarrow s_1 = s_2.$$

Clearly $\sim$ is a left congruence on $S \times T, \circ$. In fact, $\sim$ is the unique left congruence θ containing $\{1\} \times T$ as an equivalence class and separating $(s_1, 1)$, $(s_2, 1)$ for $s_1 \neq s_2$:

$$(1, 1)\theta(1, t) \Rightarrow (s, 1)\theta(s, t).$$

Thus $\sim$ is a congruence on $S \times T, \circ$ iff for every $(s, t), (s, t') \in S \times T$

$$(s, t)\circ(s_2, t_2) \sim (s, t')\circ(s_2, t_2),$$

i.e., $s\langle t, s_2\rangle = s\langle t', s_2\rangle$. For $s = 1$ and $t' = 1$ this condition reads $\langle t, s_2\rangle = \langle 1, s_2\rangle = s_2$ by LP1d. Therefore $\sim$ is a congruence, iff all $\delta_t = \mathrm{id}_S$ in the alternative notation. But in that case $S \times T, \circ$ is a semidirect product by (1.6).

3. Congruences

As an immediate consequence of LP2—LP5 the functions $\langle , \rangle, [,]$ of a Zappa product $S \times T, \circ$ are usually determined by their values on a generating system (for exceptions see the general case of LP-machines in Section 7), i.e., by their restrictions to $T' \times S'$ where S' and T' are generating systems of the *semigroups* S and T respectively. (As noted by Rédei, the situation is different for generating systems of groups; cf. (4.7)).

But while the definition of some functions $\langle , \rangle, [,]$ on free generating systems is subject to no conditions whatsoever, verification of axioms LP2—LP5 may be difficult in general. The following proposition establishes an alternative access to Zappa products via factorization.

PROPOSITION (3.1). *Let $S \times T, \circ$ be a Zappa product with respect to $[,], \langle , \rangle$. Given two congruences, both denoted by $\sim$, on S and T respectively, define*

$$(s, t) \sim (s'\ t') \Leftrightarrow s \sim s' \wedge t \sim t'.$$

Then $\sim$ *is a congruence on* $S\times T, \circ$ *if*

$$(*) \qquad s\sim s' \wedge t\sim t' \Rightarrow \begin{cases} \langle t, s\rangle \sim \langle t', s'\rangle \text{ and} \\ [t, s]\sim[t', s']. \end{cases}$$

For Zappa products of monoids the converse is also true.

If $(*)$ *is valid, then we have* $S\times T/\sim\, \cong (S/\sim)\times(T/\sim)$ *for appropriately defined functions* $\langle , \rangle, [,]$ *on* $(T/\sim)\times(S/\sim)$.

PROOF. If $(*)$ is valid and $s_i\sim s_i'$, $t_i\sim t_i'$ for $i=1, 2$, then

$$(s_1, t_1)\circ(s_2, t_2) = (s_1\langle t_1, s_2\rangle,\ [t_1, s_2]t_2),$$
$$(s_1', t_1')\circ(s_2', t_2') = (s_1'\langle t_1', s_2'\rangle,\ [t_1', s_1']t_2')$$

show that $\sim$ is a congruence on $S\times T, \circ$.

Conversely assume that $S\times T$, $\circ$ is a Zappa product of monoids, $s\sim s'$, $t\sim t'$, but $[t, s]\nsim[t', s']$ or $\langle t, s\rangle\nsim\langle t', s'\rangle$. Then $(s, 1)\sim(s', 1)$ and $(1, t)\sim(1, t')$, but

$$(1, t)\circ(s, 1) = (\langle t, s\rangle,\ [t, s])\nsim(\langle t', s'\rangle,\ [t', s']) = (1, t')\circ(s', 1).$$

Of course we cannot expect to describe all congruences of $S\times T$, $\circ$ in such a simple manner.

PROPOSITION (3.2). *Let* $S\times T, \circ$ *be a Zappa product of monoids with respect to* $\langle , \rangle, [,]$ *and* $\sim$ *a congruence relation on* T *such that*

$$(+) \qquad t\sim t' \Rightarrow [t, s]\sim[t', s] \text{ for every } s\in S.$$

Define an equivalence on S *by*

$$(++) \qquad s\approx s' \Leftrightarrow \forall t\in T\colon\ [t, s]\sim[t, s'].$$

If $T/\sim$ *is right cancellative, then* $(s, t)\sim(s', t')\Leftrightarrow s\approx s'\wedge t\sim t'$ *is a congruence on* $S\times T, \circ$.

PROOF. $\approx$ is a right congruence because

$$\begin{aligned} s\approx s' &\Rightarrow [t, s]\sim[t, s'] && \text{by } (++) \\ &\Rightarrow \big[[t, s], s_2\big]\sim\big[[t, s'], s_2\big] && \text{by } (+) \\ &\Rightarrow [t, ss_2]\sim[t, s' s_2] && \text{by LP2.} \end{aligned}$$

Clearly $\approx$ is also a left congruence by LP2. Now let $t\sim t'$ and $s\approx s'$. Since $[t, s]\sim[t', s]$ by $(+)$ and $[t', s]\sim[t', s']$ by $(++)$ we have $[t, s]\sim[t', s']$, i.e., the second part of $(*)$. In order to prove $\langle t, s\rangle\approx\langle t', s'\rangle$, we have to show $[t_1, \langle t, s\rangle]\sim$ $\sim[t_1, \langle t', s'\rangle]$ for every $t_1\in T$. For arbitrary $t_1\in T$ we know $[t_1 t, s]\sim[t_1 t', s']$ by the second part of $(*)$ and therefore $[t_1, \langle t, s\rangle]\cdot[t, s]\sim[t_1, \langle t', s'\rangle]\cdot[t', s']$ by LP3. If φ denotes the canonical homomorphism $T\to T/\sim$, this statement reads as follows:

$$[t_1, \langle t, s\rangle]\varphi\cdot[t, s]\varphi = [t_1, \langle t', s'\rangle]\varphi\cdot[t', s']\varphi.$$

But $[t, s]\varphi=[t', s']\varphi$ as noted above, and $T/\sim$ is right cancellative. Thus $[t_1, \langle t, s\rangle]\sim[t_1, \langle t', s'\rangle]$, and (3.1) yields the desired result.

As an example for a situation dual to (3.2) take $T=\Sigma^*$ and S as the set of states of an accepting automaton $A=(S, \Sigma, \delta, q_0, F)$ for a language $L\subseteq\Sigma^*$. Equip S with right zero multiplication to fulfill the additional assumptions of (3.2) vacuously (cf. 5.1 and section 7), put $[a, s]=1$, $\langle a, s\rangle=\delta_a(s)$ where δ is the state transition function of A, and consider any equivalence $\sim$ on S satisfying: $s_1\sim s_2$ implies $\delta_w(s_1)\in F \Leftrightarrow \delta_w(s_2)\in F$ for every $w\in\Sigma^*$. If $\sim$ is the equality, then $\Sigma^*/\approx$ is the transition monoid of A. If $\sim$ is the coarsest possible equivalence (yielding the *minimal automaton* for L), then $\Sigma^*/\approx$ is the syntactic monoid of L. Thus, (3.1) and (3.2) are related to the reduction of automata, especially in cases where we can understand the Zappa product $S\times T$, $\circ$ itself as an automaton by means developed in Section 7.

4. Zappa products in group theory

In order to give an impression of group theoretic aspects of the Zappa product, let us recall the following theorems.

THEOREM (4.1). *If a finite group G is a Zappa product of cyclic groups, then G is supersolvable.* [Itô, 4].

On the other hand it turned out to be difficult to determine all Zappa products of groups of a given type, even for the case of cyclic groups [1, 16].

THEOREM (4.2). *If an arbitrary group G is a Zappa product of Abelian groups, then* $G''=\{1\}$ *where* G'' *denotes the second commutator group of G.* [8, 5].

THEOREM (4.3). *Every finite group which is a Zappa product of nilpotent groups is solvable.* [30, 11, 5].

Sometimes the possibilities for decompositions of a given group into a Zappa product of subgroups are recognized easily.

REMARK (4.4). If G is a finite solvable group and $|G|$ is not a prime power, then G is a nontrivial Zappa product.

PROOF. Indeed G is a Zappa product of Hall subgroups.

REMARK (4.5). No cyclic p-group (where p is a prime number) is factorizable as any Zappa product in a nontrivial manner.

PROOF. Note that Zappa factors of a finite group have to be subgroups. But cyclic p-groups contain a unique maximal subgroup.

For further results on decompositions see Section 6. The relation between the functions δ and τ of a Zappa product has been studied by Rédei and Szép:

PROPOSITION (4.6). *Let S, T be groups and* $\tau\colon S\to\mathcal{T}(T)$ *a homomorphism such that* $1_T\in\operatorname{Fix}\tau(s)$ *for every* $s\in S$ *(i.e.,* LP1a, b *and* LP2 *are valid). If* $\operatorname{Kern}\tau=\{1\}$, *then there exists at most one function* $\delta\colon T\to\mathcal{T}(S)$ *such that* $a^{\delta_c(s)}=(ac)^s\cdot(c^s)^{-1}$ *for every* $a, c\in T$ *and* $s\in S$. *In this case* δ, τ *give rise to a Zappa product of S and T.* [15]

In contrary to the situation in (4.6) we mention Rédei's

EXAMPLE (4.7). [Cf. 16]. Let T be the infinite cyclic group $\langle a\rangle$ and S the cyclic group $\langle A\rangle$ of order 8. Then the action of S on T defined by $x^A=x^{-1}$ may be combined with the following "state transition" functions δ (cf. section 7):

$$\begin{array}{c|c} \langle\,,\,\rangle_1 & A \\ \hline a & A^3 \\ a^{-1} & A^3 \end{array} \qquad \begin{array}{c|c} \langle\,,\,\rangle_2 & A \\ \hline a & A^3 \\ a^{-1} & A^7 \end{array}$$

$$(A^g, a^i)\circ_1(A^k, a^j) = (A^{g+k+2k(i \bmod 2)},\ a^{i\cdot(-1)^k+j}),$$

$$(A^g, a^i)\circ_2(A^k, a^j) = (A^{g+k+2i(k \bmod 2)},\ a^{i\cdot(-1)^k+j}).$$

5. Applications to semigroups

The following examples will give an impression of the range covered by the concept of Zappa product in semigroup theory. Sometimes a certain semigroup H is easily recognized as not expressable as a Zappa product of two proper subsemigroups. However, such a statement does not imply immediately that the construction of Zappa products is meaningless for H. There are examples where the gap between Zappa products and such a semigroup H is very slight and may be bridged naturally by either a factorization or an embedding.

5.1. Transformation monoids

Let S be a transformation monoid acting on a set T. Consider T as a right zero semigroup and define $[t, s]=ts$ by the action of S on T and $\langle t, s\rangle=1_S$ for every $t\in T$, $s\in S$. Then LP2 is a restatement of the definition of transformation semigroups and LP3—LP5 are fulfilled vacuously. $S\times T, \circ$ is a Zappa product with multiplication $(s_1, t_1)\circ(s_2, t_2)=(s_1, t_2)$ which is rather trivial. If an identity element 1_T is adjoined to the right zero semigroup T such that 1_T is a fixed point of the transformations in S and furthermore LP1d is observed, then the Zappa product $S\times(T\cup\{1\}), \circ$ is reflecting the transformation behaviour of (S, T), i.e., $(s_0, t)\circ\circ(s, 1)=(s_0, ts)$ for every $t\in T$. On the other hand, if we consider T as a left zero semigroup and define $\langle t, s\rangle=s$, $[t, s]=ts$, we obtain a semigroup $S\times T, \circ$ with multiplication $(s_1, t_1)\circ(s_2, t_2)=(s_1s_2, t_1s_2)$ representing both the structure of S and the action of S on T.

5.2. Translational hull

Let T be a semigroup, S a monoid, and $\tau\colon S\to P(T)$ a homomorphism of S into the monoid $P(T)$ of right translations of T. Define $\delta_t=$const for every $t\in T$. Then LP1a, c, LP2, LP3 hold by assumption, and LP4, LP5 are fulfilled trivially as noted already in (1.6). The multiplication in $S\times T, \circ$ is

$$(s_1, t_1)\circ(s_2, t_2) = (s_1, t_1^{\tau(s_2)}\cdot t_2).$$

An interesting *special case* of such a product is $S\subseteq P(T)$ and $\tau=$id. An identity element subject to conditions LP1b, d may be adjoined to T if desired.

5.3. Ideal extensions

Let E be a semigroup with 0, 1 and T an ideal of E. Consider $S=E/T$ and define $\langle t, s\rangle=1$,

$$[t, s] = \begin{cases} ts & \text{if } s\in E\setminus T \\ 0 & \text{if } s = T. \end{cases}$$

By the special case of 5.2, $S\times T, \circ$ is a semigroup with multiplication $(s_1, t_1)\circ\circ(s_2, t_2)=(s_1, t_1 s_2 t_2)$ where $t_1\cdot T$ is defined to be 0. If an identity element is adjoined to T, we have

$$(s_1, t_1)\circ(s_2, t_2) = \begin{cases} (s_1 s_2, t_2) & \text{if } t_1 = 1 \\ (s_1, t_1 s_2 t_2) & \text{otherwise.} \end{cases}$$

Therefore the mapping $\varphi\colon S\times(T\cup\{1\})\to E,\ (s, t)\mapsto st$ (where $T\in S=E/T$ is treated as zero element) is a surjective homomorphism. The corresponding congruence on $S\times(T\cup\{1\})$ is $(s_1, t_1)\sim(s_2, t_2)\Leftrightarrow s_1 t_1=s_2 t_2$.

5.4. Bruck-Reilly extensions

Let H be a monoid and θ an endomorphism of H. The generalized Bruck-Reilly extension $\mathrm{BR}(H, \theta)$ is the set $\mathbf{N}_0\times H\times\mathbf{N}_0$ with multiplication

$$(m, a, n)\cdot(p, b, q) = (m-n+t,\ a\theta^{t-n}\cdot b\theta^{t-p},\ q-p+t)$$

where $t=\max(n, p)$ and $\theta^0=\mathrm{id}_H$ [3, 10]. $\mathrm{BR}(H, \theta)$ is the Zappa product of $\mathbf{N}_0, +$ and the semidirect product $H\underset{\theta}{\times}\mathbf{N}_0$ with operations

$$[(a, n), p] = \big(a\theta^{\max(n,p)-n}, \max(n, p)-p\big), \quad \langle(a, n), p\rangle = \max(n, p)-n.$$

5.5. Rees matrix semigroups

Every Rees matrix semigroup $\mathcal{M}^\circ[G, I, \Lambda, P]$ may be canonically extended to a Rees matrix semigroup $\mathcal{M}^\circ[G, I', \Lambda', P']$ where $I'=I\mathbin{\dot\cup}\{|\}$, $\Lambda'=\Lambda\mathbin{\dot\cup}\{|\}$ and

$$p'_{i\lambda} = \begin{cases} p_{i\lambda} & \text{if } i\in I,\ \lambda\in\Lambda, \\ 1 & \text{otherwise;} \end{cases}$$

i.e.,

$$P' = \begin{array}{cc} \begin{pmatrix} 1 & 1 \cdots 1 \\ \vdots & P \\ 1 & \end{pmatrix} & \begin{array}{l} | \\ \left.\vphantom{\begin{matrix}a\\a\end{matrix}}\right\}\Lambda. \end{array} \\ \begin{array}{cc} | & \underbrace{\qquad\qquad}_{I} \end{array} & \end{array}$$

Then $S=\{(i, 1, |)\,|\,i\in I'\}$ and $T=\{(|, a, \lambda)\,|\,a\in G,\ \lambda\in\Lambda'\}$ are subsemigroups of $\mathcal{M}^\circ[G, I', \Lambda', P']$, because $p_{|j}=1$. Now consider I' as a left zero semigroup and Λ' as a right zero semigroup. Clearly $S\cong I'$ and $T\cong G\times\Lambda'$ (direct product), because

$p_{\lambda|}=1$. Furthermore every element $(i, a, \lambda)\in\mathcal{M}^\circ[G, I', \Lambda', P']\setminus\{0\}$ is uniquely representable as a product $\in S\cdot T$:

$$(i, 1, |)\cdot(|, a, \lambda) = (i, p_{|\ |}a, \lambda) = (i, a, \lambda).$$

Thus $\mathcal{M}^\circ[G, I', \Lambda', P']$ is a Zappa product with zero by (2.1). The function τ involved therein is given by $[(a, \lambda), j]=(ap_{\lambda j}, |)$, while the function δ given by $\langle(a, \lambda), j\rangle=|$ is not interesting because of the left zero multiplication in I'. De facto this construction is an application of a slightly generalized version of 5.2.

5.6. Semigroup extensions of groups

Let T be a group and $S=C_{\mathcal{T}(T)}(1)=\{f\colon T\to T\,|\,1f=1\}$ the stabilizer of the element 1_T within the full transformation semigroup on the set T. Define $[,]$ to be the action of $\mathcal{T}(T)$ on T, and $\langle x, f\rangle\colon T\to T$, $y\mapsto(yx)f\cdot(xf)^{-1}$. Clearly $1_T\in\operatorname{Fix}(\langle x, f\rangle)$, i.e., $\langle x, f\rangle\in S$ for every $x\in T$, $f\in S$.
LP1a. id_T is the identity element of S.
b. by definition of S.
c. $y\langle x, 1\rangle=yx\cdot x^{-1}=y$, i.e., $\langle x, 1\rangle=\mathrm{id}_T$.
d. $y\langle 1, f\rangle=yf\cdot(1f)^{-1}=yf$, i.e., $\langle 1, f\rangle=f$.
LP2. composition of mappings.

LP3. $$[x_1, \langle x_2, f\rangle]\cdot[x_2, f] = x_1\langle x_2, f\rangle\cdot x_2f = (x_1x_2)f\cdot(x_2f)^{-1})\cdot x_2f = $$
$$= (x_1x_2)f = [x_1x_2, f].$$

LP4. $$y\langle x_1x_2, f\rangle = (yx_1x_2)f\cdot((x_1x_2)f)^{-1},\ y\langle x_1, \langle x_2, f\rangle\rangle =$$
$$= (yx_1)\langle x_2, f\rangle\cdot(x_1\langle x_2, f\rangle)^{-1} = (yx_1x_2)f\cdot(x_2f)^{-1}\cdot((x_1x_2)f\cdot(x_2f)^{-1})^{-1} =$$
$$= (yx_1x_2)f\cdot(x_2f)^{-1}\cdot((x_2f)^{-1})^{-1}\cdot((x_1x_2)f)^{-1}.$$

LP5. $$y\langle x, f_1f_2\rangle = (yx)f_1f_2\cdot(x(f_1f_2))^{-1},$$
$$y\langle[x, f_1\rangle\langle[x, f_1], f_2\rangle = (((yx)f_1\cdot(xf_1)^{-1})\cdot(xf_1))f_2\cdot((xf_1)f_2)^{-1} = (yx)f_1f_2\cdot(x(f_1f_2))^{-1}.$$
If f is the mapping $x\mapsto x^{-1}$, then $\langle x, f\rangle$ is the composition of f and the inner automorphism induced by x:

$$y\langle x, f\rangle = (yx)f\cdot(xf)^{-1} = (yx)^{-1}\cdot x = x^{-1}y^{-1}x = x^{-1}(yf)x.$$

Furthermore $f\in S$ is an endomorphism of T iff $\langle x, f\rangle=f$, and $f\in S$ is a right translation of T iff $\langle x, f\rangle=\mathrm{id}_T=1_S$. Replacing S by a subsemigroup S' satisfying $x\in T\wedge\wedge f\in S'\Rightarrow\langle x, f\rangle\in S'$ yields further examples. For $S'=\operatorname{Aut}(T)$ we obtain the *holomorph* of T. In the general case it may be interesting to notice that there are virtually no constraints for multiplications in a semigroup extension of a group insofar as any transformation f may become an inner left translation (induced by $(f, 1)$) with respect to the T-component. But when we drop the assumption that T is a group, the situaton becomes rather different as noted in (1.6)a.

6. Decompositions of semigroups

Much work remains to be done in this area. But because of the inherent interest let us mention some preliminary remarks and results.

If a semigroup H is a representable as a homomorphic image of a subsemigroup of a Zappa product $S\times T, \circ$, then we have a decomposition of H depending on the following data:

— the multiplication in S and T,

— the operations $[,]$ and $\langle,\rangle$,

— a description of the involved subsemigroup and/or congruence.

The usefulness of such a decomposition of a given semigroup corresponds more or less to the accessability of those data. (Compare e.g. decompositions by means of the wreath product.) Especially one may ask for decompositions $H, \cdot \cong S\times T, \circ$ where S and T are $\neq\{1\}$. In that case H is called *factorizable* [15]. Besides (4.4) and (4.5) we have

EXAMPLE (6.1). PSL(2, 13) is not factorizable. [7].

On the other hand there are interesting examples of simple groups which are factorizable.

EXAMPLE (6.2). The alternating group A_5 is factorizable into $S=\langle(123), (234)\rangle\cong$ $\cong A_4$ and a cyclic group $T=\langle(12345)\rangle$ of order 5. [cf. 29].

PROOF. Since $(124)=(234)^2(123)$ and $(134)=(234)(123)^2$, S contains all cycles of length 3 leaving the letter 5 fixed. Therefore S consists of all even permutations with the letter 5 as fixed point [e.g. 2, Lemma 5.4.1], i.e., $S\cong A_4$. If $s_1t_1=s_2t_2$ where $s_1, s_2\in S$ and $t_1, t_2\in T$, then $s_2^{-1}s_1=t_2t_1^{-1}\in S\cap T$. But $S\cap T=\{1\}$, because $|S|=12$ and $|T|=5$. Thus $S\cdot T$ contains $12\cdot 5=60$ different elements of A_5 and hence $S\cdot T=A_5$.

In terms of generators we can represent the Zappa product $S\cdot T$ by the following table (cf. Section 7).

A_5:

	A	B
c	B^2c^2	Ac

where $A^3=B^3=(AB)^2=1$, $c^5=1$.

Concerning decompositions by the wreath product every finite aperiodic semigroup may be obtained from almost nothing, i.e., from the right zero semigroup of order 3. But let us start investigating what kind of semigroups can be obtained from right zero semigroups by the Zappa product.

PROPOSITION (6.3). *For a semigroup H the following properties are equivalent:*

(i) *H is a homomorphic image of a subsemigroup of a Zappa product of two right zero semigroups. (i.e., H divides a Zappa product of two right zero semigroups.)*

(ii) *Every product of two elements of H is a right zero, i.e., $x\cdot y\cdot z=y\cdot z$ for every $x, y, z\in H$.*

PROOF. If S and T are right zero semigroups, the multiplication in any Zappa product $S\times T, \circ$ reads as

$$(s_1 . t_1)\circ(s_2, t_2)=(\langle t_1, s_2\rangle, t_2).$$

Therefore every product $(s_1, t_1)\circ(s_2, t_2)\circ(s_3, t_3)$ equals $(\langle t_2; s_3\rangle, t_3)=(s_2, t_2)\circ\circ(s_3, t_3)$. Since property (ii) is invariant under homomorphisms and passing to subsemigroups, this proves the assertion (i)$\Rightarrow$(ii).

For the converse let H be a semigroup satisfying the equation $x\cdot y\cdot z=y\cdot z$ and take the sets S and T equal to H, but equipped with right zero multiplication. Then the definitions $\langle x, y\rangle = x\cdot y$ in H, $[x, y]=x$ give rise to a Zappa product, because LP2—LP5 are fulfilled:

LP4. $$\langle t_1t_2, s\rangle = \langle t_2, s\rangle = t_2\cdot s = t_1\cdot t_2\cdot s = \langle t_1, \langle t_2, s\rangle\rangle,$$

LP5. $$\langle t, s_1\, s_2\rangle = \langle t, s_2\rangle = \langle [t, s_1], s_2\rangle = \langle t, s_1\rangle\langle [t, s_1]\, s_2\rangle,$$

and LP2, LP3 are valid trivially. Now consider the subsemigroup of $S\times T, \circ$ generated by $D=\{(x, x)|x\in S=T=H\}$. Every element of $\langle D\rangle$ is either in D or is of the form $(x\cdot y, y)$ where $x, y\in H$:

$$(x, x)\circ(y, y) = (x\cdot y, y),\ (z, z)\circ(x\cdot y, y) = (z\cdot x\cdot y\cdot, y) = (x\cdot y, y).$$

We shall complete the proof by showing that the mapping $\varphi\colon S\times T\to H,\ (x, y)\mapsto x$ restricted to $\langle D\rangle, \circ$ is a homomorphism.

Case 1. $(x, x)\varphi\cdot(z, u)\varphi=x\cdot z=(x\cdot z, u)\varphi=\big((x, x)\circ(z, u)\big)\varphi$.

Case 2. $(x\cdot y, y)\varphi\cdot(z, u)\varphi=(x\cdot y)\cdot z=y\cdot z=(y\cdot z, u)\varphi=\big((x\cdot y, y)\circ(z, u)\big)\varphi$.

Thus $H, \cdot$ is a homomorphic image of the subsemigroup $\langle D\rangle$ of $S\times T, \circ$.

7. Semigroup automata

In this section we present some ideas of formal language theory and interpret them from the algebraic point of view. Concerning the Zappa product, this section provides a natural motivation and surveys several useful applications related to automata theory. On the other hand little will be done to develop an algebraic theory of semigroup automata in its own right.

In the algebraic setting, a formal language (Σ^*, L) is a structure consisting of the free monoid Σ^* generated by a finite alphabet Σ together with a unary relation, i.e., a subset L. A central question is that of asking for effective representations of L. It belongs to the folklore of theoretical computer science to understand the definition of a language L by a generating grammar as an embedding $(\Sigma^*, L)\subseteq(V^*, \{w\in V^*|S\overset{*}{\Rightarrow}w\})$ such that L is the intersection of Σ^* and the standard type language $\{w\in V^*|S\overset{*}{\Rightarrow}w\}$ for some $S\in V$, given by the compatible pre-order relation $\overset{*}{\Rightarrow}$ induced by a grammar. But in a similar manner it is possible to understand the concept of recognition devices for languages as a special case of the algebraic concept of embedding $(\Sigma^*, L)\subseteq(M, P)$: in this case the multiplication in M is to be defined effectively by some type of algorithm while the standard type language P may be taken as $\{x\in M|x\cdot q_0=1\}$, i.e., the set of left inverses of a fixed element $q_0\in M$. A suitable multiplication algorithm for M is that of derivations according to a formal grammar.

DEFINITION (7.1). A *semigroup automaton* $(\bar{S}, \bar{T}, \delta, \tau)$ consists of two generated semigroups $\bar{S}=(S, S')$, $\bar{T}=(T, T')$ and wto mappings $\delta\colon T\to\mathcal{T}(S\cup\{0\})$, $\tau\colon S\to\mathcal{T}(T\cup\{0\})$ such that the conditions LP0—LP5 are satisfied.

A *generated semigroup* $\bar{S}=(S, S')$ is a semigroup S together with a distinguished set S' of generators of S. The structure $(\bar{S}, \bar{T}, \delta, \tau)$ is to be understood as an automaton by means of the following *dictionary:*

S	set of states
S'	set of operators
$\cdot$ in S	sequential processing
T	input/output
$\cdot$ in T	concatenation
δ	state transition function
τ	output function
0	operation undefined
1_S	no-operation state
1_T	empty input/output
LP3	*sequential processing rule*
LP5	*serial composition rule*

Axioms LP2—LP4 say that S is acting like a sequential transducer. In this framework LP3 and LP5 appear not only as fundamental properties, but also as the most natural ones to impose. A standard type of semigroup automata is provided by the *linear parallel processing machines* (LP-machines) where operators $A, B, C, \ldots$ are acting from one side on an input sequence $a_1 \ldots a_k \in \Sigma^*$ by processing rules $aA \to B_1 \ldots$ $\ldots B_n b_1 \ldots b_m$ in such a way that is governed by the formalism of grammars. ("linear" stands for 1-dimensional, and "parallel" indicates that operators may act simultaneously as usually in derivations.)

DEFINITION (7.2). A *deterministic* LP-*machine* is a formal grammar $(\Sigma, \Sigma \cup \Gamma, P, q_0)$ such that $P \subseteqq \Sigma\Gamma \times \Gamma^* \Sigma^*$ and P is a partial function. [12, 13].

PROPOSITION (7.3). *Every deterministic* LP-*machine* $(\Sigma, \Sigma \cup \Gamma, P, q_0)$ *corresponds canonically to a semigroup automaton* $(\Gamma^*, \Sigma^*, \delta, \tau)$ *with initial state* q_0 *where the functions* δ, τ *are given by*

$$aA \to \underbrace{B_1 \ldots B_n}_{\langle a, A\rangle} \underbrace{b_1 \ldots b_m}_{[a, A]} \quad in\ P$$

and $\langle a, A\rangle=[a, A]=0$ *if there is no rule in* P *with left side* aA.

Furthermore the operations $\langle\, ,\, \rangle, [\, ,\,]$ are defined to be 0 in case a derivation of $a_1 \ldots a_k A_1 \ldots A_j$ would run forever rather than yielding a final result. Final results of derivations according to LP-machine grammars are unique whenever existing [12]. LP-machines are conveniently given by a partial multiplication table of the extension semigroup $\Gamma^* \Sigma^* \cup \{0\}$, $\circ$ where

$$\alpha \circ \beta = \begin{cases} \gamma \text{ if } \alpha\beta \overset{*}{\Rightarrow} \gamma \text{ such that } \gamma \in \Gamma^* \Sigma^*, \\ 0 \text{ otherwise,} \end{cases}$$

as in (1.1). By virtue of (7.3) and (1.2) associativity is provided automatically. A small example of such a table is

terminals \ operators	E (initial state)	I	Q
a	IE	Qa	I
$\#$	1	$\#$	/

for an LP-machine that will accept the language $\{\# a^{2^n-1} \mid n \in \mathbf{N}_0\}$. Without going into details let us mention the following

Applications (7.4) *of* LP-*machines* [12]:

— computation of functions: linear-time integer multiplication

— uniform representation of Chomsky hierarchy:

type of rules for LP-machines	corresponding classical automata
$aA \to B$	finite automaton
$aA \to B_1 \dots B_n$	deterministic pushdown automaton with a single state
$aA \to \begin{cases} B_1 \dots B_n \\ b \end{cases}$	deterministic pushdown automaton
$aA \to \begin{cases} B_1 \dots B_n \\ B_1 \dots B_n b \end{cases}$	——— / (subclass of context-sensitive languages)
$aA \to B_1 \dots B_n b_1 \dots b_m$	equivalent to Turing machine

— context-sensitive languages: parsing algorithms for (special) non-context-free generating grammars for a rather comprehensive subclass [13].

Besides the case of LP-machines where the semigroup $\bar{S}$ of states in a semigroup automaton $(\bar{S}, \Sigma^*, \delta, \tau)$ is chosen to be a free monoid Γ^*, it is worth-while to study finite semigroups of states (with non-trivial multiplication, in contrary to the example given in Section 3). This possibility seems to allow an interesting access for investigations in complexity of rational and especially finite languages where many other tools fail to be useful.

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(Received August 31, 1981)

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Acta Math. Hung.
41 (3—4) (1983), 241—250.

CONTINUOUS INCIDENCE RELATIONS OF TOPOLOGICAL PLANES

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0. In topological planes the link between the geometrical and the topological structure is established by the requirement that the partial external operations "joining of points" and "intersecting of lines" be (op-) continuous (for the terminology, see Section 1). It seems that nowhere in the literature one has discussed the question, if and under which conditions the incidence relation (or other relations derived from it) underlying to the notion of a topological plane is (resp. are) "continuous" in some natural sense, or whether, in a more general setting (see Remark 6), the incidence relation should be required to be "continuous". In this paper, it turns out that the incidence relation of each topological plane and its inverse are lower-semicontinuous and they are continuous (in the sense introduced by the author in [3]) in certain classes of topological planes (Theorem 4, Propositions 5 through 8).

As a byproduct in this paper, we characterize (in Theorem 1) the continuity of certain relations between a topological space and a product of topological spaces (which has a simple consequence in Proposition 9) and describe (in Theorem 3) the topology of the set of lines (of a topological plane) in terms of limits (via the notion of the power of a topology).

1. Auxiliary considerations from the general topology

In this section, we collect some facts from the general topology with a minimum of a systematic representation.

For each set M, we denote by ΦM the class of all filtered families $(f, K, \mathfrak{a})$ in M; for each topological space (M, λ), we denote by $\mathfrak{B}_\lambda$, Lim_λ, $\liminf_\lambda$, $\mathfrak{P}[\lambda]$ the neighborhood operator induced by λ, the limit operator induced by λ (extended to ΦM), the limit inferior induced by λ (which is defined on $\Phi\mathfrak{P}M$) and the (first) power of λ (which is a topology of the power set $\mathfrak{P}M$ of M and has been denoted by $\mathfrak{P}\lambda$ in [3]), respectively (see [3]). For the set-theoretical terminology used in this paper, see Monk [8] and [3].

For the whole section, let (E, τ), (F, σ) be topological spaces. Down to Proposition 1, let φ be a mapping *from* $E\times E$ into F, i.e. with $\mathrm{Dmn}\,\varphi \subseteq E\times E$ and $\mathrm{Rng}\,\varphi \subseteq F$ (read Dmn as "domain of definition of" and Rng as "range of"). One may call such a mapping a *(binary) partial external operation in E*. Occasionally, it is useful to

* Support by Deutscher Akademischer Austauschdienst (Bundesrepublik Deutschland) and by the University Paris VI is gratefully acknowledged.

consider a kind of "continuity" of φ which differs in general from the $(\tau\times\tau, \sigma)$-continuity:

We call φ (τ, σ)-*op-continuous* (or: *operation-continuous*) if and only if for each $(x, y)\in\operatorname{Dmn}\varphi$ and each $U\in\mathfrak{B}_\sigma\varphi(x, y)$ there are sets $V\in\mathfrak{B}_\tau x$ and $W\in\mathfrak{B}_\tau y$ such that $V\times W\subseteqq\operatorname{Dmn}\varphi$ and $\varphi(V\times W)\subseteqq U$. (For $E=F$ and $\tau=\sigma$ this definition is contained in the definition of op-continuity in [4], p. 184.)

PROPOSITION 1. *φ is (τ, σ)-op-continuous if and only if* $\operatorname{Dmn}\varphi$ *is $(\tau\times\tau)$-open and φ is $(\tau\times\tau, \sigma)$-continuous.*

PROOF. □

We recall that a relation $R\subseteqq E\times F$ is said to be (τ, σ)-*continuous* if and only if for each $(x, y)\in R$ there is a (τ, σ)-continuous mapping $\varphi\colon \operatorname{Dmn}R\to F$ such that $(x, y)\in\varphi\subseteqq R$, or, equivalently, if and only if there is a set $\mathfrak{B}\subseteqq\mathscr{C}(\operatorname{Dmn}R, \operatorname{Rng}R)$ such that $R=\bigcup\mathfrak{B}$, where $\operatorname{Dmn}R$ and $\operatorname{Rng}R$ are endowed with the corresponding trace topologies (see [3], p. 37). For Theorem 1 and Remark 1, let $(F_d, \sigma_d)_{d\in D}$ be a family of topological spaces, (F, σ) their product and $(R_d)_{d\in D}$ a family of relations $R_d\subseteqq E\times F_d$ (D is a set).

THEOREM 1. *We define a relation $S\subseteqq E\times F$ by letting $y\ S\ f$ if and only if $y\ R_d\ f(d)$ for all $d\in D$ (for all $(y, f)\in E\times F$). Then, S is (τ, σ)-continuous if and — provided that, for some set M,* $\operatorname{Dmn}R_d=M$ *holds for all $d\in D$ — only if R_d is (τ, σ_d)-continuous for all $d\in D$.*

PROOF. 1. Assume that all R_d are (τ, σ_d)-continuous. Let $y\ S\ f$. Then $y\ R_d\ f(d)$ holds for all $d\in D$. Therefore, there exists a family $(\varphi_d)_{d\in D}$ of (τ, σ_d)-continuous mappings $\varphi_d\colon \operatorname{Dmn}R_d\to F_d$ such that $(y, f(d))\in\varphi_d\subseteqq R_d$ (use of the continuity of R_d and of the axiom of choice). By the definition of S, we have $\operatorname{Dmn}S=\bigcap_{d\in D}^{E}\operatorname{Dmn}R_d$ (where the index E indicates the convention $\bigcap_E\varnothing=E$). We define a mapping $\chi\colon\operatorname{Dmn}S\to F$ by letting $\chi(z)=(\varphi_d(z))_{d\in D}$ for all $z\in\operatorname{Dmn}S$. Since all φ_d are (τ, σ_d)-continuous, χ is (τ, σ)-continuous (see Dugundji [2], p. 201, Corollary 2.3). Using the definition of χ and the choice of the mappings φ_d, one obtains $(y, f)\in\chi$; using additionally the definition of S, one gets that $\chi\subseteqq S$. We have showed that S is (τ, σ)-continuous.

2. Assume that for some set M, $\operatorname{Dmn}R_d=M$ holds for all $d\in D$. Let S be (τ, σ)-continuous. Let $c\in D$ and $y\ R_c\ x$. Then $y\in\operatorname{Dmn}R_d=M$ holds for all $d\in D$. By the axiom of choice, there is a mapping $h\in F$ such that $y\ R_d\ h(d)$ holds for all $d\in D$. We define a mapping $f\in F$ by letting $f(d)=x$ for $d=c$ and $f(d)=h(d)$ for all $d\in D\setminus\{c\}$. Then, by the definition of S, one has $y\ S\ f$. Since S is (τ, σ)-continuous, there is a (τ, σ)-continuous mapping $\varphi\colon M\to F$ such that $(y, f)\in\varphi\subseteqq S$ (use of $M=\operatorname{Dmn}S$). We define a mapping ψ by $\psi=\operatorname{pr}_c\circ\varphi$ (where $\operatorname{pr}_c\colon F\to F_c$ denotes the c-th projection). ψ is (τ, σ_c)-continuous, and one has $\operatorname{Dmn}\psi=\operatorname{Dmn}\varphi==M=\operatorname{Dmn}R_c$. Since $(y, f)\in\varphi$, one has $(y, x)=(y, \operatorname{pr}_c(f))\in\psi$. In order to show the continuity of R_c, it is left to show $\psi\subseteqq R_c$. Let $(z, w)\in\psi$. Then, one obtains, by the definition of ψ, $(\varphi(z))(c)=w$, therefore (using that $\varphi\subseteqq S$ implies $(z, \varphi(z))\in S$), in view of the definition of S, $z\ R_c\ w$. We have showed that R_c is (τ, σ_c)-continuous. □

REMARK 1. If, especially, R_d is a mapping on E into F_d for all $d \in D$, then S, defined in Theorem 1, is a mapping on E into F, and we regain a well-known theorem saying that S is continuous if and only if all R_d are continuous (see, e.g., Dugundji [2], p. 101, Corollary 2.3).

We formulate, next, a simple assertion serving as a lemma soon:

PROPOSITION 2. *τ is the finest topology ϱ of E such that for all topological spaces (G, λ) and all mappings $g\colon G \to E$ the statement $(*)$ holds:*

$(*)$ *If g is (λ, τ)-continuous, then g is (λ, ϱ)-continuous.*

PROOF. $\square$

Now, we recall the notion of lower-semicontinuity of a set-valued mapping. Let for the following φ be a mapping *from* E into $\mathfrak{P}F$, i.e. with $\operatorname{Dmn}\varphi \subseteq E$ and $\operatorname{Rng}\varphi \subseteq \mathfrak{P}F$. Then, φ is said to be *(τ, σ)-lower-semicontinuous* if and only if

$$\varphi\big(\operatorname{Lim}_\tau(f, K, \mathfrak{a})\big) \subseteq \mathfrak{P}\big(\liminf_\sigma(\varphi\circ f, K, \mathfrak{a})\big)$$

holds for all $(f, K, \mathfrak{a}) \in \Phi \operatorname{Dmn}\varphi$ (see [3], p. 39, where we chose this definition as a characterization). A relation $R \subseteq E \times F$ is said to be *(τ, σ)-lower-semicontinuous* if and only if the canonical mapping $\hat{R}$ induced by R, which is defined by $\hat{R}(x) = \{y | xRy\}$ for all $x \in \operatorname{Dmn} R$, is (τ, σ)-lower-semicontinuous. Recall (from [3], p. 40) that if R is (τ, σ)-continuous, then R is (τ, σ)-lower-semicontinuous.

We remark that φ is (τ, σ)-lower-semicontinuous if and only if φ is $(\tau, \mathfrak{P}[\sigma])$-continuous (see [3], p. 39, where this characterization of lower-semicontinuity is chosen as its definition); therefore, one obtains by means of Proposition 2 the following description of $\mathfrak{P}[\tau]$:

PROPOSITION 3. *$\mathfrak{P}[\tau]$ is the finest topology ϱ of $\mathfrak{P}E$ such that for all topological spaces (G, λ) and all mappings $g\colon G \to \mathfrak{P}E$ the next statement $(*)$ holds:*

$(*)$ *If g is (λ, τ)-lower-semicontinuous, then g is (λ, ϱ)-continuous.*

Because of Proposition 3, the author inclines to call $\mathfrak{P}[\tau]$ the *topology of lower-semicontinuity of $\mathfrak{P}E$ w.r. to τ* (cf. also Čech [1], p. 623, and Grimeisen [5]).

For later use, we agree to denote the limit operation in a Hausdorff space (E, τ), i.e. the relation $R \subseteq (\Phi E) \times E$ defined by $(f, K, \mathfrak{a})\ R\ x$ if and only if $x \in \operatorname{Lim}_\tau(f, K, \mathfrak{a})$ (for all $((f, K, \mathfrak{a}), x) \in (\Phi E) \times E$) by $\lim_\tau$.

2. The space of the lines as a subspace of the power of the space of the points

A class M is said to be *decomposed into* classes A and B if and only if $M = A \cup B$ and $\varnothing = A \cap B$. If R is a relation, it is decomposed into a function $\operatorname{fun} R := \{(x, y) \in R |$ if $(x, z) \in R$, then $y = z\}$ (called the *functional part of* R) and a relation $\operatorname{rel} R := R \setminus \operatorname{fun} R$ (called the *relational part of* R), which is a function (namely empty) if and only if R is a function.

Let $(P, \mathfrak{G}, I)$ be a triple of sets $P, \mathfrak{G}, I$ such that $I \subseteq P \times \mathfrak{G}$. We define a relation $S(\subseteq (P \times P) \times \mathfrak{G})$ by letting $(p, q)\ S\ G$ if and only if $p\ I\ G$ and $q\ I\ G$ for all $((p, q), G) \in (P \times P) \times \mathfrak{G}$ and a relation $T(\subseteq (\mathfrak{G} \times \mathfrak{G}) \times P)$ by letting $(G, H)\ T\ p$

if and only if $p\ I\ G$ and $p\ I\ H$ for all $((G, H), p)\in(\mathfrak{G}\times\mathfrak{G})\times P$. We denote fun S by $\vee$ and fun T by $\wedge$.

$(P, \mathfrak{G}, I)$ is called an *incidence structure* if and only if $\operatorname{Dmn} I=P$, $\operatorname{Rng} I=\mathfrak{G}$, $\operatorname{rel} S=S\restriction\mathrm{id}_P$ and $\operatorname{rel} T=T\restriction\mathrm{id}_{\mathfrak{G}}$ (where $\mathrm{id}_K:=\{(k, k)|k\in K\}$ for a set K and $\restriction$ stands for "restricted to"). (Realize that, up to the natural bijective mapping between $\mathrm{id}_P\times\mathfrak{G}$ and $P\times\mathfrak{G}$, the relation $S\restriction\mathrm{id}_P$ coincides with I, while in an analogous sense, $T\restriction\mathrm{id}_{\mathfrak{G}}$ coincides with I^{-1}; so, $(P, \mathfrak{G}, I)$ is an incidence structure if and only if $\operatorname{Dmn} I=P$, $\operatorname{Rng} I=\mathfrak{G}$, S is (essentially) decomposed into $\vee$ and I and T is (essentially) decomposed into $\wedge$ and I^{-1}.) In an incidence structure $(P, \mathfrak{G}, I)$, we adapt the common use to call the elements of P "points", those of $\mathfrak{G}$, "lines", I the "incidence relation", $\vee$ the "joining of points" and $\wedge$ the "intersecting of lines".

REMARK 2. The triple $(P, \mathfrak{G}, I)$ is an incidence structure if and only if it is a "reguläre Inzidenzstruktur" as defined by Pickert [9], p. 2, such that for each $p\in P$ there are $G, H\in\mathfrak{G}$ with the properties $G\neq H$, $p\ I\ G$ and $p\ I\ H$. Furthermore, if $(P, \mathfrak{G}, I)$ is an incidence structure, then for each $G\in\mathfrak{G}$ there is a $p\in P$ with non $(p\ I\ G)$ and, dually, for each $p\in P$ there is a $G\in\mathfrak{G}$ with non $(p\ I\ G)$. Of course, $(P, \mathfrak{G}, I)$ is an incidence structure if and only if $(\mathfrak{G}, P, I^{-1})$ is an incidence structure.

If $(P, \mathfrak{G}, I)$ is an incidence structure, then $\operatorname{Dmn}\vee\subseteqq(P\times P)\setminus\mathrm{id}_P$ and $\operatorname{Dmn}\wedge\subseteqq(\mathfrak{G}\times\mathfrak{G})\setminus\mathrm{id}_{\mathfrak{G}}$.

For the remainder of this paper, let $(P, \mathfrak{G}, I)$ be an incidence structure; without loss of generality we assume that $\mathfrak{G}\subseteqq\mathfrak{P}P$ and that I is defined by $p\ I\ G$ if and only if $p\in G$ (for all $(p, G)\in P\times\mathfrak{G}$).

$(P, \mathfrak{G}, I)$ is called a *plane* if and only if $(P\times P)\setminus\mathrm{id}_P\subseteqq\operatorname{Dmn}\vee$. Of course, projective planes, affine planes and absolute planes (for the terminology, see Karzel—Sörensen—Windelberg [6], pp. 12, 29 and 96) are planes. Let τ and σ be Hausdorff topologies of P and $\mathfrak{G}$, respectively. We call (cf. Salzmann [11], p. 4, where the notion of the underlying incidence structure differs slightly from that chosen here) $E:=((P, \tau), (\mathfrak{G}, \sigma), I)$ a *topological incidence structure* (Salzmann, loc. cit., says "topological geometry") if and only if $\vee$ is (τ, σ)-op-continuous and $\wedge$ is (σ, τ)-op-continuous. Of course, if E is a topological incidence structure, then E is called a *topological plane* [Salzmann, loc. cit., says "topological plane geometry"] (*topological projective plane,* etc.) if and only if $(P, \mathfrak{G}, I)$ is a plane (projective plane, etc.).

It is well known that in topological incidence structures τ determines σ and, conversely, σ determines τ in the sense of the following

PROPOSITION 4 (Salzmann [10], p. 490, there for topological projective planes). *Let E be a topological incidence structure. Then, one has* (a) *and* (b):

(a) *For each $G\in\mathfrak{G}$ and all p_1, p_2 with $p_1\neq p_2$ and $p_i\ I\ G$ $(i=1, 2)$, one has*

$$\mathfrak{V}_\sigma G = \mathscr{H}_{\mathfrak{G}}\{V_1\vee V_2|V_i\in\mathfrak{V}_\tau p_i \quad for \quad i = 1, 2\}.$$

(b) *denotes the dualization of* (a) *(one obtains from* (a) *by replacing $G, \mathfrak{G}, p_1, p_2, I, \sigma, \tau$ and $\vee$ by $p, P, G_1, G_2, I^{-1}, \tau, \sigma$ and $\wedge$ in this order).*

PROOF. Since $((\mathfrak{G}, \sigma), (P, \tau), I^{-1})$ is a topological incidence structure, it suffices to prove (a):

1. We show that $\mathfrak{V}_\sigma G\subseteqq\mathscr{H}_{\mathfrak{G}}\{V_1\vee V_2|V_i\in\mathfrak{V}_\tau p_i$ for $i=1, 2\}$. Let $\gamma\in\mathfrak{V}_\sigma G$. Since $p_1\neq p_2$ and $(p_1, p_2)\ S\ G$ hold, we have $G=p_1\vee p_2$ (because of $\operatorname{rel} S=S\restriction\mathrm{id}_P$).

Since $\vee$ is op-continuous, there are neighborhoods $V_i \in \mathfrak{V}_\tau p_i$ $(i=1,2)$ such that $V_1 \times V_2 \subseteq \mathrm{Dmn} \vee$ and $\vee(V_1 \times V_2) \subseteq \gamma$.

2. We show that $\{V_1 \vee V_2 | V_i \in \mathfrak{V}_\tau p_i$ for $i=1,2\} \subseteq \mathfrak{V}_\sigma G$. (From this follows $\mathscr{H}_{\mathfrak{G}}\{V_1 \vee V_2 | V_i \in R_\tau p_i$ for $i=1,2\} \subseteq \mathfrak{V}_\sigma G$.) Let $V_i \in \mathfrak{V}_\tau p_i$ for $i=1,2$. Since τ is Hausdorff, there are neighborhoods $W_i \in \mathfrak{V}_\tau p_i$ $(i=1,2)$ such that $W_1 \cap W_2 = \emptyset$ and $W_i \subseteq V_i$ for $i=1,2$.

Let $i \in \{1,2\}$. Because of $p_i\ I\ G$, there is an $H_i \in \mathfrak{G}$ with $p_i\ I\ H_i$ and $G \neq H_i$ (see Remark 2). Since $(G, H_i) \notin \mathrm{id}_{\mathfrak{G}}$ and $(G, H_i)\ T\ p_i$, we have $p_i = G \wedge H_i$ (use of rel $T = T \restriction \mathrm{id}_{\mathfrak{G}}$). Since $\wedge$ is op-continuous, there are neighborhoods $\gamma_i \in \mathfrak{V}_\sigma G$ and $\delta_i \in \mathfrak{V}_\sigma H_i$ such that $\gamma_i \times \delta_i \subseteq \mathrm{Dmn} \wedge$ and $\wedge(\gamma_i \times \delta_i) \subseteq W_i$; especially, one has $L \wedge H_i \in W_i$ for all $L \in \gamma_i$.

Now, let $L \in \gamma_1 \cap \gamma_2$. Then, one has $q_i := L \wedge H_i \in W_i$ for $i=1,2$. Since $W_1 \cap W_2 = \emptyset$, one obtains $(q_1, q_2) \notin \mathrm{id}_P$; on the other hand, one has $(q_1, q_2)\ S\ L$ thus $L = q_1 \vee q_2$ (use of rel $S = S \restriction \mathrm{id}_P$) and so $L \in W_1 \vee W_2 \subseteq V_1 \vee V_2$. In view of the choice of L, we have proved that $\gamma_1 \cap \gamma_2 \subseteq V_1 \vee V_2$, therefore, because of $\gamma_1 \cap \gamma_2 \in \mathfrak{V}_\sigma G$, $V_1 \vee V_2 \in \mathfrak{V}_\sigma G$. □

In Proposition 4 and later the following conventions are used: If $A, B \subseteq P$ then $A \vee B$ denotes the set $\vee(A \times B) := \vee((A \times B) \cap \mathrm{Dmn} \vee)$; if $A, B \subseteq \mathfrak{G}$, then $A \wedge B$ is defined correspondingly. If C is a set and $\mathfrak{a} \subseteq \mathfrak{P}C$, then $\mathscr{H}_C \mathfrak{a}$ denotes the set $\{D \in \mathfrak{P}C | A \subseteq D$ for some $A \in \mathfrak{a}\}$.

For the remainder of this paper, we assume that E is a topological plane.

As the set $\mathfrak{G}$ "can be reproduced" from the set $\mathfrak{P}P$ (by applying the mapping $\mathrm{id}_{\mathfrak{G}}$ to $\mathfrak{P}P$), the topology σ of $\mathfrak{G}$ can be reproduced from the power $\mathfrak{P}[\tau]$ of the topology τ (see Section 1):

THEOREM 2. $\sigma = (\mathfrak{P}[\tau])_{\mathfrak{G}}$, *i.e., σ is the trace of the topology $\mathfrak{P}[\tau]$ in the set $\mathfrak{G}$.*

PROOF. We prove that $\mathrm{Lim}_\sigma = \mathrm{Lim}_{((\mathfrak{P}[\tau])_{\mathfrak{G}})}$. Let $(f, K, \mathfrak{a}) \in \Phi\mathfrak{G}$.

1. Let $G \in \mathrm{Lim}_\sigma(f, K, \mathfrak{a})$. By the definition of the power $\mathfrak{P}[\tau]$ of τ, one has $\mathrm{Lim}_{((\mathfrak{P}[\tau])_{\mathfrak{G}})}(f, K, \mathfrak{a}) = \mathfrak{G} \cap (\mathfrak{P}(\liminf_\tau (f, K, \mathfrak{a})))$. We show that $G \in \mathfrak{P} \liminf_\tau (f, K, \mathfrak{a})$ and take, for this purpose, $p \in G$. Let $U \in \mathfrak{V}_\tau p$. Then, there is a $q \in G$ with $q \neq p$ (since $(P, \mathfrak{G}, I)$ is an incidence structure); thus $G = p \vee q$ (since $\mathrm{Dmn} \vee = (P \times P) \setminus \mathrm{id}_P$). Since $P \in \mathfrak{V}_\tau q$, one has by Proposition 4 $U \vee P \in \mathfrak{V}_\sigma G$. Thus, there is an $A \in \mathfrak{a}$ such that $fk \in U \vee P$ for all $k \in A$. Let $k \in A$. Then, there is a $z_k \in U$ and a $w_k \in P$ such that $(z_k, w_k) \in \mathrm{Dmn} \vee$ and $fk = z_k \vee w_k$. One has $z_k \in U \cap fk$, thus $U \cap fk \neq \emptyset$. This holds for all $k \in A$, therefore, since $A \in \mathfrak{a}$, one obtains (by the choice of U) $p \in \liminf_\tau (f, K, \mathfrak{a})$, thus (by the choice of p) $G \subseteq \liminf_\tau (f, K, \mathfrak{a})$, therefore, because of $G \in \mathfrak{G}$, $G \in \mathfrak{G} \cap (\mathfrak{P} \liminf_\tau (f, K, \mathfrak{a}))$.

2. Conversely, let $G \in \mathfrak{G} \cap (\mathfrak{P} \liminf_\tau (f, K, \mathfrak{a}))$. We show that then $G \in \mathrm{Lim}_\sigma (f, K, \mathfrak{a})$. Let $U \in \mathfrak{V}_\sigma G$. Since $(P, \mathfrak{G}, I)$ is an incidence structure, there are p, $q \in G$ with $p \neq q$. Since τ is Hausdorff, there are (by Proposition 4) neighborhoods $V \in \mathfrak{V}_\tau p$ and $W \in \mathfrak{V}_\tau q$ such that $V \cap W = \emptyset$, $V \vee W \subseteq U$ and $V \vee W \in \mathfrak{V}_\sigma G$. Because $V \in \mathfrak{V}_\tau p$, $p \in G$ and $G \subseteq \liminf_\tau (f, K, \mathfrak{a})$, there is a set $A \in \mathfrak{a}$ such that $V \cap fk \neq \emptyset$ holds for all $k \in A$. By the axiom of choice, there is a mapping φ: $A \to P$ such that $\varphi(k) \in V \cap fk$ holds for all $k \in A$. By the analogous argument, one obtains the existence of a set $B \in \mathfrak{a}$ and of a mapping ψ: $B \to P$ such that $\psi(k) \in W \cap fk$ holds for all $k \in B$. Let $k \in A \cap B$. Since $V \cap W = \emptyset$, one has $(\varphi(k), \psi(k)) \in \mathrm{Dmn} \vee$ (because

Dmn $\vee=(P\times P)\setminus \mathrm{id}_P$), therefore (because of the choice of the mappings φ and ψ) $fk=\varphi(k)\vee\psi(k)\in V\vee W$ and so $fk\in U$. This holds for all $k\in A\cap B$. In view of the choice of U, one has finally $G\in \mathrm{Lim}_\sigma(f, K, \mathfrak{a})$, since $A\cap B\in\mathfrak{a}$. $\square$

Theorem 2 allows to describe the limits of filtered families of lines in a simple way:

THEOREM 3. *For all* $(f, K, \mathfrak{a})\in\Phi\mathfrak{G}$ *and all* $G\in\mathfrak{G}$, *one has* $G=\lim_\sigma(f, K, \mathfrak{a})$ *if and only if* $G=\liminf_\tau(f, K, \mathfrak{a})$.

PROOF. Let $(f, K, \mathfrak{a})\in\Phi\mathfrak{G}$ and $G\in\mathfrak{G}$.

1. Let $G=\liminf_\tau(f, K, \mathfrak{a})$. Then, by Theorem 2, $G\in\mathrm{Lim}_\sigma(f, K, \mathfrak{a})$. Since σ is Hausdorff, one obtains $G=\lim_\sigma(f, K, \mathfrak{a})$ (for the terminology, see Section 1).

2. Conversely, let $G=\lim_\sigma(f, K, \mathfrak{a})$, thus $G\in\mathrm{Lim}_\sigma(f, K, \mathfrak{a})$. Then, by Theorem 2, one has $G\subseteq\liminf_\tau(f, K, \mathfrak{a})$.

We show that $\liminf_\tau(f, K, \mathfrak{a})\subseteq G$. Let $p\in\liminf_\tau(f, K, \mathfrak{a})$. Since $(P, \mathfrak{G}, I)$ is an incidence structure, there is a $q\in G$ such that $q\neq p$. Since Dmn $\vee=(P\times P)\setminus\mathrm{id}_P$, $p\vee q$ is defined; put $p\vee q=H$. If we can show that $H=G$, we obtain $p\in G$ (since $p\in H$) and $\liminf_\tau(f, K, \mathfrak{a})\subseteq G$ will be proved.

In order to show $H=G$, we take $U\in\mathfrak{B}_\sigma H$. By Proposition 4, there are neighborhoods $V\in\mathfrak{B}_\tau p$ and $W\in\mathfrak{B}_\tau q$ such that $V\vee W\subseteq U$ and (since τ is Hausdorff) $V\cap W=\emptyset$. Since $p, q\in\liminf_\tau(f, K, \mathfrak{a})$ (here, we use $q\in G\subseteq\liminf_\tau(f, K, \mathfrak{a})$), there are sets $A, B\in\mathfrak{a}$ such that $V\cap fk\neq\emptyset$ for all $k\in A$ and $W\cap fk\neq\emptyset$ for all $k\in B$. Therefore, by the axiom of choice, there are mappings $\varphi\colon A\to P$ and $\psi\colon B\to P$ such that $\varphi(k)\in V\cap fk$ and $\psi(k)\in W\cap fk$ for all $k\in A\cap B$. Let $k\in A\cap B$. Since $V\cap W=\emptyset$, one has $(\varphi(k), \psi(k))\in$ Dmn $\vee$, thus (by the choice of the mappings φ and ψ) $fk=\varphi(k)\vee\psi(k)$ and so $fk\in V\vee W$. This holds for all $k\in A\cap B$, thus $f(A\cap B)\subseteq U$. Since $A\cap B\in\mathfrak{a}$, one obtains (in view of the choice of U) that $H\in\mathrm{Lim}_\sigma(f, K, \mathfrak{a})$. Together with $G\in\mathrm{Lim}_\sigma(f, K, \mathfrak{a})$, this implies $H=G$, since σ is Hausdorff.

So, we have proved $G=\liminf_\tau(f, K, \mathfrak{a})$. $\square$

In order to illustrate Theorem 3, we use it to prove a simple (certainly well-known)

COROLLARY. *I is $\tau\times\sigma$-closed.*

PROOF. Let $(f, K, \mathfrak{a})\in\Phi I$, say $f(k)=(g(k), h(k))$ for all $k\in K$, and $(p, G)\in\mathrm{Lim}_\tau(g, K, \mathfrak{a})\times\mathrm{Lim}_\sigma(h, K, \mathfrak{a})$. Let $U\in\mathfrak{B}_\tau p$. Then, there is an $A\in\mathfrak{a}$ such that $g(A)\subseteq U$; thus, since $g(k)\in h(k)$ for all $k\in K$, $U\cap h(k)\neq\emptyset$ holds for all $k\in A$, therefore, in view of the choice of U, $p\in\liminf_\tau(h, K, \mathfrak{a})$. By Theorem 3, one obtains $p\in G$. $\square$

3. Lower-semicontinuity and continuity of incidence relations

For the remainder of this paper, we omit (except for some cases) all prefixes (namely exactly the prefixes (τ, σ)-, (σ, τ)-, $(\tau\times\tau, \sigma)$-, $(\sigma\times\sigma, \tau)$-, $(\tau\times\sigma, \sigma)$- and $(\tau\times\sigma, \sigma\times\sigma)$-) for the adjectives "continuous" and "lower-semicontinuous" of mappings or relations occuring in the text, without causing any confusion.

THEOREM 4. *I and I^{-1} are lower-semicontinuous.*

PROOF. 1. In order to prove that I is lower-semicontinuous, we have to show that

$$\hat{I}(\mathrm{Lim}_\tau(f, K, \mathfrak{a})) \subseteq \mathfrak{P}(\liminf_\sigma(\hat{I}\circ f, K, \mathfrak{a})) \tag{1}$$

holds for all $(f, K, \mathfrak{a})\in \Phi P$. (We have used that $\mathrm{Dmn}\, I=P$; for the terminology, see Section 1.)

Let $(f, K, \mathfrak{a})\in \Phi P$, $p\in \mathrm{Lim}_\tau(f, K, \mathfrak{a})$ and $G\in \hat{I}(p)$. Since $(P, \mathfrak{G}, I)$ is an incidence structure, there is a $q\in G$ such that $p\neq q$. Let $U\in \mathfrak{B}_\sigma G$. By Proposition 4, there are sets $V\in \mathfrak{B}_\tau p$ and $W\in \mathfrak{B}_\tau q$ such that $V\vee W\subseteq U$, $V\vee W\in \mathfrak{B}_\sigma G$ and $V\cap W=\emptyset$ (since τ is Hausdorff). In view of the choice of p, there is an $A\in \mathfrak{a}$ such that $f(A)\subseteq V$. Let $k\in A$. Since $q\in W$, $fk\in V$ and $V\cap W=\emptyset$, one has $(fk, q)\in \mathrm{Dmn} \vee$ (use of $\mathrm{Dmn} \vee =(P\times P)\setminus \mathrm{id}_P$) and $fk\vee q\in V\vee W$; furthermore, one has $fk\vee q\in \hat{I}(fk)$. Therefore, one obtains $U\cap (\hat{I}(fk))\neq\emptyset$, because $V\vee W\subseteq U$. In view of the choice of U, A and k, one gets $G\in\liminf_\sigma(\hat{I}\circ f, K, \mathfrak{a})$, therefore, in view of the choice of G, $\hat{I}(p)\in \mathfrak{P}(\liminf_\sigma(\hat{I}\circ f, K, \mathfrak{a}))$. We have proved (1).

2. In order to show that I^{-1} is lower-semicontinuous, we have to show that

$$\widehat{I^{-1}}(\mathrm{Lim}_\sigma(f, K, \mathfrak{a})) \subseteq \mathfrak{P}\left(\liminf_\tau\left(\widehat{I^{-1}}\circ f, K, \mathfrak{a}\right)\right) \tag{2}$$

holds for all $(f, K, \mathfrak{a})\in \Phi\mathfrak{G}$. (We have used that $\mathrm{Dmn}\, I^{-1}=\mathfrak{G}$.)

Let $(f, K, \mathfrak{a})\in \Phi\mathfrak{G}$. Since $\widehat{I^{-1}}=\mathrm{id}_{\mathfrak{G}}$ (where $\mathrm{id}_{\mathfrak{G}}=\{(G, G)|G\in\mathfrak{G}\}$), the inclusion (2) holds by Theorem 3. □

Let for the moment E be a topological affine plane and denote by par the mapping on $P\times\mathfrak{G}$ into $\mathfrak{G}$ assigning to each $(p, G)\in P\times\mathfrak{G}$ the line H through p and parallel to G. Then, we say that E has *continuous parallelism* (see Salzmann [11], p. 52, where this notion is mentioned but not applied) if and only if par is continuous. In this terminology, we have

PROPOSITION 5. *Let E be a topological affine plane with continuous parallelism. Then I is continuous.*

PROOF. Let $p\ I\ G$. Since E has continuous parallelism, the mapping $\mathrm{par}(., G)$: $P\to\mathfrak{G}$, defined by $\mathrm{par}(., G)(x)=\mathrm{par}(x, G)$ for all $x\in P$, is continuous. Furthermore, one has $\mathrm{par}(., G)(p)=\mathrm{par}(p, G)=G$ and $\mathrm{par}(., G)\subseteq I$. □

REMARK 3. In Proposition 5, it suffices (in view of its proof) to require that for each $G\in\mathfrak{G}$ the mapping $\mathrm{par}(., G)\colon P\to\mathfrak{G}$ be continuous.

Using this Remark, we prove

PROPOSITION 6. *Let E be a topological affine plane such that τ is induced by a metric d of P and $(*)$ holds for all $G, H\in\mathfrak{G}$:*

$(*)$ *If G and H are parallel, then* $\mathrm{dist}_d(x, H)=\mathrm{const}$ *for all* $x\in G$.

Then I is continuous.

REMARK 4. The converse of (*) holds in each metric affine plane.

PROOF OF PROPOSITION 6. By Remark 3, it suffices to show that the mapping $\mathrm{par}(., G)$ is continuous for each $G\in\mathfrak{G}$. Let $G\in\mathfrak{G}$. For abbreviation, we put par

$(., G)=\varphi$. By Theorem 2, it suffices to show that

$$\varphi(\mathrm{Lim}_\tau(f, K, \mathfrak{a})) \subseteq \mathfrak{P}(\liminf_\tau(\varphi\circ f, K, \mathfrak{a})) \tag{3}$$

holds for all $(f, K, \mathfrak{a})\in \Phi P$. Let $(f, K, \mathfrak{a})\in \Phi P$ and $x\in \mathrm{Lim}_\tau(f, K, \mathfrak{a})$. Let $y\in\varphi(x)$ and $U\in \mathfrak{B}_\tau y$. Then, there is a real $\varepsilon>0$ such that $B(y,\varepsilon)\subseteq U$ (where $B(y,\varepsilon)$ denotes the open ball around y with radius ε). By the choice of x, there is an $A\in\mathfrak{a}$ with $f(A)\subseteq B(x,\varepsilon)$. Let $k\in A$. Then,

$$\mathrm{dist}_d(y, \varphi(f(k))) = \inf_{z\in\varphi(f(k))} d(y, z) = \inf_{z\in\varphi(f(k))} d(x, z) \leqq d(x, f(k)) < \varepsilon,$$

where we used (*). Thus, there is a $w\in\varphi(f(k))$ such that $d(y, w)<\varepsilon$, therefore $U\cap\varphi(f(k))\neq\emptyset$. This holds for all $k\in A$. In view of the choice of y, U and A, one has proved that $\varphi(x)\subseteq\liminf_\tau(\varphi\circ f, K, \mathfrak{a})$ and so (3). □

Let for the moment E be a topological absolute plane (for the terminology, see Karzel—Sörensen—Windelberg [6], p. 96) and denote by orth the mapping on $P\times\mathfrak{G}$ into $\mathfrak{G}$ assigning to each $(p, G)\in P\times\mathfrak{G}$ the line H through p and orthogonal to G. Then, we say that E has *continuous orthogonalism* if and only if orth is continuous. In this terminology, we have

PROPOSITION 7. *Let E be a topological absolute plane with continuous orthogonalism. Then I^{-1} is continuous.*

PROOF. Let $(G, p)\in I^{-1}$. We define an auxiliary mapping $\lambda\colon P\times\mathfrak{G}\to\mathfrak{G}\times\mathfrak{G}$ by letting $\lambda(y, X)=(X, \mathrm{orth}(y, X))$ for all $(y, X)\in P\times\mathfrak{G}$ and, by means of λ, the mapping $\psi\colon \mathfrak{G}\to P$ by $\psi(X)=(\wedge\circ\lambda)(p, X)$ for all $X\in\mathfrak{G}$.

The mapping λ is continuous, since the projection $(y, X)\mapsto X$ $((y, X)\in P\times\mathfrak{G})$ and the mapping orth are continuous, furthermore $\wedge$ is continuous, thus, since $\mathrm{Rng}\,\lambda\subseteq\mathrm{Dmn}\wedge$ (see [6], p. 100), the mapping ψ is continuous.

Furthermore, one has $(G, p)\in\psi$ and $\psi\subseteq I^{-1}$. □

Combining the Propositions 5 and 7, one obtains (for the terminology, see Karzel—Sörensen—Windelberg [6], p. 133)

PROPOSITION 8. *Let E be a topological Euclidean plane with continuous orthogonalism. Then I and I^{-1} are continuous.*

PROOF. Since $\mathrm{par}(y, X)=\mathrm{orth}(y, \mathrm{orth}(y, X))$ holds for all $(y, X)\in P\times\mathfrak{G}$ (see [6], p. 133) and, by supposition, orth is continuous, also the mapping par is continuous, i.e. E has continuous parallelism. Thus, we are allowed to apply Proposition 5, and so I is continuous. On the other hand, E is a topological absolute plane, thus Proposition 7 is applicable. □

4. On the relations S, T, S^{-1} and T^{-1}

We refer to the definition of S and T in Section 2 and recall that $\vee$ is the functional part of S and $\wedge$ is the functional part of T.

Theorem 1 implies as a very special case

PROPOSITION 9. *S^{-1} is continuous if and only if I^{-1} is continuous. T^{-1} is continuous if and only if I is continuous.*

REMARK 5. Of course, instead of S and T one may study, for each set D, the relations S_D and T_D defined by letting $f\ S_D\ G$ if and only if $f(d)\ I\ G$ for all $d\in D$ (for all $(f, G)\in P^D\times\mathfrak{G}$) and $f\ T_D\ p$ if and only if $p\ I\ f(d)$ for all $d\in D$ (for all $(f, p)\in\mathfrak{G}^D\times P$). Let now $D\neq\emptyset$.

Then, by Theorem 1, S_D^{-1} *is* (σ, τ^D)*-continuous if and only if* I^{-1} *is continuous;* furthermore, T_D^{-1} *is* (τ, σ^D)*-continuous if and only if* I *is continuous.* For $D=2$, one has $S=S_D$ and $T_D=T$. (Here, τ^D and σ^D denote product topologies.)

REMARK 6. In the proof of Proposition 9, the continuity of $\vee$ and $\wedge$ is not used in any way. This suggests to study incidence structures $(P, \mathfrak{G}, I)$ together with the topologies τ and σ of P and $\mathfrak{G}$ such that the only link between the geometric structure and the topological structure consists in the requirement that I and (or) I^{-1} be continuous.

We recall the remarks immediately after the definition of an incidence structure and add, now: Since the mapping $(p, p)\mapsto p\,(p\in P)$ resp. $(G, G)\mapsto G\,(G\in\mathfrak{G})$ is a homeomorphism between $(\mathrm{id}_P, (\tau\times\tau)_{\mathrm{id}_P})$ and (P, τ) resp. between $(\mathrm{id}_{\mathfrak{G}}, (\sigma\times\sigma)_{\mathrm{id}_{\mathfrak{G}}})$ and $(\mathfrak{G}, \sigma)$,

$$\text{(4)}\qquad \begin{cases} \textit{the relation } S{\restriction}\mathrm{id}_P \textit{ resp. } T{\restriction}\mathrm{id}_{\mathfrak{G}} \textit{ is } (\tau\times\tau, \sigma)\textit{-lower semicontinuous} \\ \textit{resp. } (\sigma\times\sigma, \tau)\textit{-lower-semicontinuous} \end{cases}$$

(by Theorem 4), and

$$\text{(5)}\qquad \begin{cases} S{\restriction}\mathrm{id}_P \textit{ resp. } T{\restriction}\mathrm{id}_{\mathfrak{G}} \textit{ is } (\tau\times\tau, \sigma)\textit{-continuous resp. } (\sigma\times\sigma, \tau)\textit{-continuous} \\ \textit{if and only if } I \textit{ resp. } I^{-1} \textit{ is continuous.} \end{cases}$$

Although in the sense of the preceding paragraph, the relation S resp. T is (essentially) decomposed into the semicontinuous or (possibly) continuous relation I resp. I^{-1} and the continuous relation (mapping) $\vee$ resp. $\wedge$, S resp. T need not be even semicontinuous, as Proposition 10 resp. Proposition 11 below shows:

PROPOSITION 10. *Assume the existence of a* $(p, G)\in I$ *such that* p *is a* τ*-accumulation point of the set* G *(for the terminology, see* [7], p. 41*). Then* S *is not lower-semicontinuous (thus not continuous).*

PROOF. Assume that S is lower-semicontinuous. Since p is a τ-accumulation point of G, there is an $(f, K, \mathfrak{a})\in\Phi(G\setminus\{p\})$ such that $p\in\mathrm{Lim}_\tau(f, K, \mathfrak{a})$. Since S is lower-semicontinuous, one has

$$\text{(6)}\qquad \hat{S}(\mathrm{Lim}_{\tau\times\tau}(g, K, \mathfrak{a})) \subseteq \mathfrak{P}(\liminf_\sigma(\hat{S}\circ g, K, \mathfrak{a})),$$

where the mapping g be defined by $g(k)=(p, f(k))$ for all $k\in K$. Because $p\ I\ G$, $f(k)\ I\ G$, $f(k)\neq p$ and $(P, \mathfrak{G}, I)$ is an incidence structure, $p\vee f(k)$ is defined and $\hat{S}(p, f(k))=\{p\vee f(k)\}=\{G\}$ holds for all $k\in K$. Therefore, the right side of (6) is equal to $\mathfrak{P}(\sigma\{G\})=\mathfrak{P}\{G\}$, since σ is Hausdorff (observe that in this paper (as in [3]) "topology" is a synonym for "closure operator" in the sense of Kuratowski). On the other hand, (since τ is Hausdorff) the left side of (6) equals to $\hat{S}\{(p, p)\}= =\{\{Z\in\mathfrak{G}\,|\,p\ I\ Z\}\}$, therefore, by (6) (since no line is empty), $\hat{S}\{(p, p)\}=\{\{G\}\}$, which contradicts the supposition that $(P, \mathfrak{G}, I)$ is an incidence structure. □

PROPOSITION 11. *Assume* $\emptyset \notin P$ *and the existence of a* $(p, G) \in I$ *such that* G *is a* σ*-accumulation point of the set* $\{Z \in \mathfrak{G} \mid p \; I \; Z\}$. *Then* T *is not lower-semicontinuous (thus not continuous).*

PROOF. Dualize the proof of Proposition 10 and, doing so, use $\emptyset \notin P$ instead of the words "no line is empty". □

REMARK 7. In the proofs of the Propositions 10 and 11, we did not make use of the continuity of $\vee$ or $\wedge$, nor did we use $(P \times P) \setminus \mathrm{id}_P \subseteq \mathrm{Dmn} \vee$. Thus, they remain valid if one replaces the underlying hypothesis "E is a topological plane" by "$(P, \mathfrak{G}, I)$ is an incidence structure together with Hausdorff topologies τ and σ of P and $\mathfrak{G}$, respectively".

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(Received September 9, 1981)

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Acta Math. Hung.
41 (3—4) (1983), 251—254.

ON SUMS OF DISTINCT INTEGERS BELONGING TO CERTAIN SEQUENCES

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Many papers have been devoted to the problem of representing integers as sums of distinct numbers belonging to fixed sequences.

We mention for example the paper of Birch [1], in which he proves the conjecture of Erdös concerning the representability of all large integers as sum of distinct terms of the type $p^a q^b$ where p, q are coprime, the very general results of Cassels [2], which contain Birch's result, and the paper of Erdös [3].

Here we give conditions, different from those of Cassels and Erdös, on a sequence guaranteeing that all large natural numbers are sums of its distinct terms.

We introduce some notation: $\mathcal{A}$ is said to be *of type* δ if, for every large x, we have $(x,(1+\delta)x)\cap\mathcal{A}\neq\emptyset$. Let $\mathcal{A}$ and $\mathcal{B}$ denote sequences of natural numbers; we say that the product $\mathcal{A}\mathcal{B}=\{ab,\ a\in\mathcal{A},\ b\in\mathcal{B}\}$ is *direct* if $ab=a'b'$ implies $a=a'$, $b=b'$.

Further, let $S(\mathcal{A})=\{m,\ m=a_1+\ldots+a_v,\ a_1>a_2>\ldots>a_v,\ a_j\in\mathcal{A}\}$, and, as usual, set $\mathcal{A}(x)=\sum_{\substack{a\in\mathcal{A}\\ a\leq x}}1$.

Our results are the following

THEOREM 1. *Let $\mathcal{A}$ be of type* 1 *such that* $1\in\mathcal{A}$. *Then there exists a number $L=L(\mathcal{A})$ with the following property: if $\mathcal{B}$ is a sequence such that*

(i) *$\mathcal{A}\mathcal{B}$ is direct*

(ii) *$\mathcal{B}(y)>L\log^2 y$ for some $y>10$,*

then $S(\mathcal{A}\mathcal{B})$ contains an arithmetical progression of the form $\{km,\ m\in\mathbf{N}\}$.

Before stating our second theorem, we need another definition. We say that $d=d(\mathcal{A})$ is the g.c.d. of the sequence $\mathcal{A}$ if d is the largest positive integer dividing all sufficiently large elements of $\mathcal{A}$, provided these numbers are bounded.

THEOREM 2. *Let $\mathcal{A}$ and $\mathcal{B}$ be as in Theorem* 1, *with* $1\in\mathcal{B}$. *Assume* $d(\mathcal{A})=1$. *Then $S(\mathcal{A}\mathcal{B})$ contains all large integers.*

REMARK. If the product is not direct and we allow each term with its "multiplicity", then the theorems maintain their validity; the proofs are the same.

A straightforward application of Theorem 2 gives a result of Birch's type, but slightly weaker, namely

COROLLARY. *Let n be a positive integer and $p_1, \ldots, p_5$ be pairwise coprime numbers. Then there exists $L=L(n,p_1,\ldots,p_5)$ such that every large m may be written as*

$$m=r_1^n+\ldots+r_v^n,\ r_1>\ldots>r_v>0 \ \text{and}\ r_j=\prod_{j=1}^{5}p_j^{a_j},\ a_j\geqq 0\ \text{and}\ a_3+a_4+a_5\leqq L.$$

We remark that the proofs are entirely elementary.

Some lemmas

LEMMA 1. *Let M be a finite set. Let $C \subset [M]^T = \{B \subset M, |B| = T\}$ be such that $B_1, \dots, B_s \in C \Rightarrow \exists i, j,\ i \neq j,\ B_i \cap B_j \neq \emptyset$. Then $\exists m \in M$ such that*

$$|\{B \in C, m \in B\}| \geqq \frac{|C| + (T-1)(s-1)}{T(s-1)}.$$

PROOF. Let s_0 be the least integer such that the hypothesis holds with s replaced by s_0. Then there exist $B_1, \dots, B_{s_0-1} \in C$ with $B_i \cap B_j = \emptyset$ if $i \neq j$. Moreover if $B \in C$ then there exists j, $1 \leqq j \leqq s_0 - 1$, such that $B \cap B_j \neq \emptyset$. By Dirichlet's box principle we can find μ, $1 \leqq \mu \leqq s_0 - 1$, such that

$$|\{B \in C, B \cap B_\mu \neq \emptyset\}| \geqq \frac{|C| - s_0 + 1}{s_0 - 1} + 1.$$

But $|B_\mu| = T$, hence, by the box principle again, we obtain the conclusion.

LEMMA 2. *Let $f\colon \bigcup_{1 \leqq j \leqq T} [M]^j \to A$ satisfy the following properties:*

(i) *$B_1, \dots, B_s \in [M]^j \cap f^{-1}(a)$, for some j and $a \in A \Rightarrow \exists \mu, \nu,\ \mu \neq \nu,\ B_\mu \cap B_\nu \neq \emptyset$,*

(ii) *$f(\{m\} \cup B) = f(\{m\} \cup B') \Rightarrow f(B) = f(B')$.*

Then we have $|M| \leqq (T!)^{2/T} s |A|^{1/T} + T$.

PROOF. Let $a_0 \in A$. We apply Lemma 1 setting $C = f^{-1}(a_0) \cup [M]^T$, thus obtaining the existence of $m \in M$ such that

$$|\{B \in [M]^T \cap f^{-1}(a_0), m \in B\}| \geqq \frac{|f^{-1}(a_0) \cap [M]^T|}{Ts}.$$

Using property (ii) we deduce

$$|[M]^{T-1} \cap f^{-1}(a_1)| \geqq \frac{|[M]^T \cap f^{-1}(a_1)|}{Ts}$$

for some $a_1 \in A$.

By a repeated application of this argument we obtain, for every j, the existence of $a_j \in A$ such that

$$|[M]^{T-j-1} \cap f^{-1}(a_{j+1})| \geqq \frac{|[M]^{T-j} \cap f^{-1}(a_j)|}{s(T-j)}. \tag{2}$$

Set $j = T-2$. Then, by (i) we have $|[M]^{T-j-1} \cap f^{-1}(a_{j+1})| \leqq s$, and using this fact together with the inequalities (2) for $j = T-2, T-3, \dots, 0$, we get

$$s \geqq \frac{|[M]^T \cap f^{-1}(a_0)|}{s^{T-1} T!} \quad \text{for every } a_0 \in A. \tag{3}$$

By the box principle there exists $a_0 \in A$ such that $|[M]^T \cap f^{-1}(a_0)| \geqq \dfrac{|[M]^T|}{|A|}$ and now (3) gives the estimate $|[M]^T| \leqq |A| s^T T!$. Recalling that $|[M]^T| = \binom{|M|}{T} \geqq \dfrac{(|M| - T)^T}{T!}$ the lemma follows.

LEMMA 3. *Let $\mathcal{A}$ be of type* 1. *Then there exists $L_1=L_1(\mathcal{A})$ such that every integer m can be written in the form $m=a_1+a_2+\ldots+a_v+c$ where $a_1>a_2>\ldots>a_v>c$, $a_i\in\mathcal{A}$ and $0\leqq c\leqq L_1$, $a_v\geqq 2$.*

PROOF. Let L_1 be such that $(x/2, x)\cap\mathcal{A}\neq\emptyset$ if $x>L_1$. We prove the lemma by induction on m. If $m\leqq L_1$ the assertion is clearly true. Suppose $k>L_1$ and the lemma true for $m=1, \ldots, k-1$. There exists $a\in\mathcal{A}$ such that $k/2<a<k$. Applying the inductive hypothesis to $k-a$ we obtain easily the lemma.

LEMMA 4. *Suppose $\mathcal{B}$ is a sequence such that $\mathcal{B}(y)\geqq 4F\log^2 y$ for some $y>10$. Then there is a number B such that*

$$B=\sum_{j=1}^{j(v)} b_{jv}, \quad v=1,2,\ldots,[F],$$

where the $b_{jv}\in\mathcal{B}$ for every j, v, and are all distinct.

PROOF. Set $M=\{b\in\mathcal{B}, b\leqq y\}$, $T=[\log y]$, $s=[F]$, $f(R)=\sum_{r\in\mathcal{R}} r$, $A=\{1,2,\ldots,\ldots,[Ty]\}$ in Lemma 2.

(ii) is trivially satisfied. If Lemma 4 is not true, then (i) too is satisfied. We then should obtain: $\mathcal{B}(y)\leqq(T!)^{2/T}F(Ty)^{1/T}+T\leqq 2(\log y)^2 F y^{1/T}+T<4F\log^2 y$, for $y>10$, which is inconsistent with our hypothesis.

PROOF OF THEOREM 1. Set $L=4(L_1+1)$ and let B be as in Lemma 4, with $F=L_1+1$. By Lemma 3 we may write $Bm=Ba_1+\ldots+Ba_v+Bc$ with $a_1>\ldots>a_v>c$, $0\leqq c\leqq L_1$, $a_j\in\mathcal{A}$; $a_j\geqq 2$. Using Lemma 4 we obtain:

$$Bm=(b_{11}+\ldots+b_{j(1)1})(a_1+\ldots+a_v)+\sum_{r=2}^{c+1}\sum_{j=1}^{j(r)} b_{jr}=\sum_{j=1}^{j(1)}\sum_{s=1}^{v}=b_{j1}a_s+\sum_{j,r} b_{jr},$$

whence $Bm\in S(\mathcal{A}\mathcal{B})$ since $1\in\mathcal{A}$ and the numbers $b_{j,1}a_s$, $b_{j,r}$ are pairwise distinct.

PROOF OF THEOREM 2. By replacing eventually the constant L with a larger one, we may assume Theorem 1 true with $\mathcal{B}\setminus\{1\}$ in place of $\mathcal{B}$. We thus obtain the existence of B such that $\{Bm, m\in N\}\subseteq S(\mathcal{A}(\mathcal{B}-\{1\}))$. Let d be the least integer with the following property: there exists a finite set $\mathcal{F}_d\subset\mathcal{A}$ such that for every $m>m_d$, $dm=\Sigma a_i b_i+\sum_{f\in\mathcal{F}'} a_f$ where $a_i\in\mathcal{A}$, $b_i\in\mathcal{B}-\{1\}$, $\mathcal{F}'\subseteq\mathcal{F}_d$, $a_i b_i$ distinct.

The above observation implies the existence of such a d. We note that this property implies $\{dm, m>m_0\subset S(\mathcal{A}\mathcal{B})\}$. We shall prove that $d=1$.

Let $\mathfrak{D}=\{\sigma\ (\mathrm{mod}\ d)$, there exist infinitely many $a\in\mathcal{A}$, $a\equiv\sigma\ (\mathrm{mod}\ d)\}$ and pick $\sigma\in\mathfrak{D}$ such that $(\sigma, d)=\text{minimum}=d'$. Set $\sigma=\gamma d'$, $d=\lambda d'$, $(\gamma,\lambda)=1$. Choose $a_1<\ldots<a_\lambda$ such that $a_i\equiv\sigma\ (\mathrm{mod}\ d)$, $a_1>\max_{a\in\mathcal{F}_d} a$. Set $\mathcal{F}_{d'}=\mathcal{F}_d\cup\{a_1,\ldots,a_\lambda\}$, $m_{d'}=\sum_{i=1}^{\lambda} a_i+dm_d$. Let $m>m_{d'}$, $m\equiv\lambda'\ (\mathrm{mod}\ \lambda)$, $1\leqq\lambda'\leqq\lambda$; and find f, $1\leqq f\leqq\lambda$, $f\gamma\equiv\lambda'\ (\mathrm{mod}\ \lambda)$.

We have:

$$d'm-a_1-\ldots-a_f\equiv d'\lambda'-f\sigma\equiv d'\lambda'-f\gamma d'\equiv 0\ (\mathrm{mod}\ d)$$

and $d'm-a_1-\dots-a_f>dm_d$. By our assumption we may write

$$d'm = a_1+\dots+a_f+\sum_{t\in\mathcal{F}'} a_t+\Sigma a_i b_i$$

whence, by the minimality of d, $d'=d$. In view of our choice of d', we obtain $a\equiv 0 \pmod d$ for all large $a\in\mathcal{A}$. But this means $d=1$.

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(Received September 15, 1981)

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Acta Math. Hung.
41 (3—4) (1983), 255—277.

$\mathfrak{M}$-ADDITIVE FUNCTIONS. I

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§ 1. Definition of $\mathfrak{M}$-decomposition

Let $q>1$ be a fixed integer. Let I and I_0 denote the set of natural numbers and of nonnegative integers, respectively. Let $I_0^N=\{0, 1, ..., q^N-1\}$. R and C will denote the field of real and complex numbers, respectively. R_k is the k-dimensional Euclidean-space over R.

Let $\mathfrak{M}_k$ $(k=1, 2, ...)$ be a sequence of disjoint subsets of nonnegative integers having the following properties:

(1) Every element m_k of $\mathfrak{M}_k$ is smaller than q^k. The set $\mathfrak{M}_k$ may be empty.

(2) For every $n\in I_0$ there exist exactly one k and $m_k\in\mathfrak{M}_k$ for which $n\equiv m_k$ (mod q^k).

Let $\mathfrak{M}=\bigcup\mathfrak{M}_k$ and for $m\in\mathfrak{M}$ let $\lambda(m)$ be defined as the unique index k for which $m\in\mathfrak{M}_k$, i.e. $m\in\mathfrak{M}_{\lambda(m)}$.

A set $\mathfrak{M}$ being given, we define the $\mathfrak{M}$-decomposition of $n\in I_0$ as follows:

$$\begin{cases} n = n_0;\ n_j = m_{k_{j+1}}+q^{k_{j+1}}n_{j+1} \quad (j = 0, ..., \mu(n)-1) \\ n_{\mu(n)} = 0, \quad n_h \neq 0 \quad \text{for} \quad h < \mu(n). \end{cases} \tag{1.1}$$

The numbers $m_{k_1}, ..., m_{k_{\mu(m)}}$ are called the digits of n in the $\mathfrak{M}$-decomposition. $\mu(n)=0$ only if $n=0$. For $n=0$ the set of digits is empty.

DEFINITION 1.1. A function f: $I_0\to R_k$ is called *quasi-$\mathfrak{M}$-additive* if it can be represented as

$$f(n) = \sum_{j=1}^{\mu(n)} H(m_{k_j}; j) \tag{1.2}$$

where $H(\cdot\,; j)$ is defined on $\mathfrak{M}$, and has values in R_k for every $j=1, 2, ...$ If, in addition, $H(0; j)=0$ $(j=1, 2, ...)$, then we say that f is an *$\mathfrak{M}$-additive function.*

DEFINITION 1.2. A function g: $I_0\to C$ (or $I_0\to R$) is called *quasi-$\mathfrak{M}$-multiplicative,* if it can be represented as

$$g(n) = \prod_{j=1}^{\mu(n)} K(m_{k_j}; j) \tag{1.3}$$

where $K(\cdot\,; j)$ is defined on $\mathfrak{M}$ and has complex or real values for every $j=1, 2, ...$ If, in addition, $K(0; j)=1$ $(j=1, 2, ...)$, then we say that g is *$\mathfrak{M}$-multiplicative.*

§ 2. q-additive and q-multiplicative functions

If we choose $\mathfrak{M}_1$ to be $\{0, 1, ..., q-1\}$, $\mathfrak{M}=\mathfrak{M}_1$, then we shall come to the q-ary expansion of integers:

$$n = a_0+a_1 q+...+a_r q^r \quad (a_j \in \mathfrak{M}).$$

The notion of q-additivity has been introduced by A. O. Gelfond [1]. $f\colon I_0 \to R$ is q-additive if

$$f(n) = \sum_{j=0}^{r} f(a_j q^j), \quad f(0) = 0.$$

f is q-multiplicative if $f(0)=1$, and

$$f(n) = \prod_{j=0}^{r} f(a_j q^j).$$

The existence of limit distribution of q-additive functions was considered by H. Delange [2].

We say that a real valued function $f(n)$ has a limit distribution with the distribution function $F(x)$ if

$$\lim_{y\to\infty} y^{-1} \# \{n < y | f(n) < x\} = F(x)$$

for every continuity point of $F(x)$.

Delange proved that a q-additive function has a limit distribution if and only if the following series converge:

$$\sum_{j=0}^{\infty} \sum_{a=1}^{q-1} f(aq^j), \tag{2.1}$$

$$\sum_{j=0}^{\infty} \sum_{a=1}^{q=1} f^2(aq^j). \tag{2.2}$$

Furthermore, for q-multiplicative functions he proved the following assertion. For a q-multiplicative function satisfying $|f(n)| \leqq 1$ the mean-value

$$\lim_{x} x^{-1} \sum_{n\leqq x} f(n) = M(f)$$

exists and is nonzero if and only if

$$\sum_{a=0}^{q-1} f(aq^j) \neq 0 \quad (j = 0, 1, 2, ...) \tag{2.3}$$

and

$$\sum_{j=0}^{\infty} \sum_{a=0}^{q-1} f(aq^j) \tag{2.4}$$

converges.

Let $\mathscr{A}$ be an infinite subset of I_0 with counting function $A(x)$, that is

$$A(x) = \# \{n \leqq x | n \in \mathscr{A}\}.$$

We say that a real valued function $f(n)$ (defined at least on $\mathcal{A}$), has a limit distribution $F(y)$ on $\mathcal{A}$, if

$$\lim_x \frac{1}{A(x)} \# \{n \leqq x | n \in \mathcal{A}, f(n) < y\} = F(y)$$

for almost all y.

The following assertion is proved in [3]. Let q be an odd prime, $\mathcal{A}$ be the set of primes, $\pi(x) = A(x)$. Then the q-additive function f has a limit distribution on the set of primes if the series (2.1), (2.2) converge. If $g(n)$ is a q-multiplicative function, $|g(n)| \leqq 1$ and the series

$$\sum_{r=1}^{\infty} \left(q-1-\sum_{a=1}^{q-1} g(aq^r)\right)$$

converges, then

$$\frac{1}{\pi(x)} \sum_{p \leqq x} g(p) \to H(g) \quad (x \to \infty),$$

where

$$H(g) = \frac{1}{\varphi(q)} \sum_{\substack{(a,q)=1 \\ 1 \leqq a < q}} g(a) \prod_{r=1}^{\infty} \frac{1}{q} \left(1 + \prod_{a=1}^{\infty} g(aq^r)\right).$$

The proof is based upon the following result due to Barban, Linnik and Tshudakov [4]. Let q be an odd prime, D run over the powers of q. Then

$$(*) \qquad \pi(x, D, l) = \big(1 + O((\log x)^{-c})\big) \frac{\operatorname{li} x}{\varphi(D)}$$

holds uniformly for $(l, D) = 1$ and $x \geqq D^3$.

By using some recent results of Iwaniec we get $(*)$ for every $q > 1$ and consequently our theorem holds for $q > 1$.

The question whether the convergence of the series is necessary for the existence of the limit distribution is still open and seems to be quite hard.

§ 3. The $\mathfrak{M}$-star-decomposition

Let $\mathfrak{M}_k, \mathfrak{M}$ be as above, N an integer. For every $n \in J_0^N$ we define

$$(3.1) \qquad n = n_0,\ n_j = m_{k_{j+1}} + q^{k_{j+1}} n_{j+1} \quad (j = 0, 1, \ldots, \nu_N(n) - 1),$$

where $\nu_N(n)$ is defined by the inequalities:

$$(3.2) \qquad k_1 + \ldots + k_{\nu_N(n)} \geqq N, \quad k_1 + \ldots + k_{\nu_N(n)-1} < N.$$

This decomposition is almost the same as the $\mathfrak{M}$-decomposition, the only difference is that we put some 0-digits down as many times as it is needed for the fulfillment of (3.2).

Now we define the quasi-$\mathfrak{M}$-star additivity and multiplicativity as follows.

DEFINITION 3.1. A function $f\colon I_0^N \to R_k$ is called a *quasi-$\mathfrak{M}$-star additive function* if

$$f(n) = \sum_{j=1}^{\nu_N(n)} H(m_{k_j}; j), \tag{3.3}$$

where $H(\cdot\,; j)$ is defined on $\mathfrak{M}$ and has values in R_k for every $j=1, 2, \dots$. If, additionally, $H(0; j)=0$ $(j=1, 2, \dots)$, we say that f is an *$\mathfrak{M}$-star additive function.*

DEFINITION 3.2. A function $g\colon I_0^N \to C$ (or $I_0^N \to R$) is called a *quasi-$\mathfrak{M}$-star-multiplicative function,* if it can be represented as

$$g(n) = \prod_{j=1}^{\nu_N(n)} K(m_{k_j}; j)$$

where $K(m, j)$ is defined on $\mathfrak{M}$, and has complex or real values. Moreover if $K(0, j)=1$ $(j=1, 2, \dots)$, then we say that $g(n)$ is an *$\mathfrak{M}$-star-multiplicative function.*

We can see immediately that the notions of quasi-star additivity and of the quasi-additivity are not the same, while the star-additivity and additivity are identical notions.

Let $H(m, j)$ $(m \in \mathfrak{M},\ j=1, 2, \dots)$ be given functions, $f(n)=f_N(n)$ be defined by (3.3). We consider the frequencies

$$q^{-N} \# \left\{ n \in I_0^N \,\middle|\, \frac{f_N(n)-A_N}{B_N} < \underline{x} \right\} = F_N(\underline{x})$$

(A_N, B_N are real numbers).

We shall say that $B_N^{-1}(f_N(n)-A_N)$ $(n \in I_0^N)$ has a limit distribution $F(\underline{x})$ if $F_N(\underline{x}) \to F(\underline{x})$ for almost all $\underline{x} \in R_k$.

We are interested in determining under what conditions there are suitable sequences of A_N and B_N by which limit distribution exists.

§ 4. Change of digits after multiplication

Every $n \in I_0^N$ can be be written uniquely in the form

$$n = \sum_{j=0}^{N-1} \varepsilon_j q^j \quad (\varepsilon_j = 0, 1, \dots, q-1). \tag{4.1}$$

Let $\alpha(n) = \Sigma \varepsilon_j$ and consider the difference $(\Delta_h(n) =)\Delta(n) = \alpha(hn) - \alpha(n)$ where $h \geqq 2$ is an integer.

We are interested in dealing with the statistical behaviour of $\Delta(n)$. We can see that $\Delta_h(n)$ is an $\mathfrak{M}$-additive function for a suitable $\mathfrak{M}$-set.

Let t be an integer satisfying $h < q^t$. Let $\mathfrak{M}_1 = \mathfrak{M}_{t-1} = \emptyset$, and for $k > t$ $\mathfrak{M}_k$ contains those elements m_k for which $1 \leqq m_k < q^{k-t}$ and that do not occur in $\mathfrak{M}_r$ for $r < k$. In other words, putting every $n \in I_0^{k-t}$ in q-ary expansion with k digits: $n = \sum_{j=0}^{k-1} \delta_j q^j$, $\mathfrak{M}_k$ contains exactly those of them for which the sequence $(\delta_0, \delta_1, \dots, \delta_{k-1})$ contains t consecutive zeros exactly at the end of the sequence: $\delta_{k-t} = \dots = \delta_{k-1} = 0$.

Let m_{k_1} be the first $\mathfrak{M}$-component of n, i.e. $n=m_{k_1}+q^{k_1}n_1$. Then $hn=hm_{k_1}+$ $+q^{k_1}(hn_1)$.

Observing that $hm_{k_1}<q^{k_1}$ and that $\alpha(n)=\alpha(m_{k_1})+\alpha(n_1)$, we get

$$\Delta(n) = \Delta(m_{k_1})+\Delta(n_1).$$

Hence we get that

$$\Delta(n) = \sum_{j=1}^{\nu_N(n)} \Delta(m_{k_j}). \tag{4.2}$$

By using this representation, and some ideas usual in probability theory, L. Dringó and I ([5], [6]) proved the following assertion: If $q=2$, h is not a power of 2, and

$$K_N(x) = 2^{-N}\,\#\{\Delta(n) < x\beta_h\sqrt{N},\ n\in I_0^N\},$$

then $K_N(x)\to\Phi(x)$. β_h is a suitable constant.

The case $q\neq 2$ can be considered in the same way. We could get local limit theorems for $\Delta(n)$ as well. They will follow from some more general theorems.

A lot of other questions lead to quasi-$\mathfrak{M}$-additive functions.

1. Let $(1=)h_0<h_1<\ldots<h_K(<q^t)$ and

$$R(n) = \big(\alpha(h_0n), \alpha(h_1n), \ldots, \alpha(h_Kn)\big).$$

Then $R(n)$ is a $(K+1)$-dimensional $\mathfrak{M}$-additive function.

2. For every $n\in I$ we consider the q-ary expansion $(\varepsilon_{N-1}, \ldots, \varepsilon_0)$ defined by (4.1). Completing it with $\varepsilon_N=\ldots=\varepsilon_{N+t-1}=0$, we get

$$n \to (\varepsilon_{N+t-1}, \ldots, \varepsilon_N, \varepsilon_{N-1}, \ldots, 0).$$

For $h<q^t$ and $n\in I_0^N$ we have $hn\in I_0^{N+t}$, and so

$$hn = \sum_{j=0}^{N+t-1} \delta_j q^j.$$

Given u and v $\in\{0, 1, \ldots, q-1\}$, we define

$$f(n, u, v) = \sum_{\substack{i=0\\ \varepsilon_i=u,\,\delta_i=v}}^{N+t-1} 1.$$

It is obvious that $f(n, u, v)$ is a quasi-$\mathfrak{M}$-star-additive function.

Let $A_k=\operatorname{card}(\mathfrak{M}_k)$. It is quite easy to see that

$$\sum_{k=0}^{\infty} A_k/q^k = 1, \tag{4.3}$$

and that

$$\sum_k \frac{A_k k^\alpha}{q^k} < \infty \tag{4.4}$$

for every fixed α.

§ 5. Length of the decomposition

For arbitrary $\varepsilon > 0$ we can construct a set $\mathfrak{M}$ for which

$$\sum \frac{A_k}{q^k} < \varepsilon.$$

Furthermore, for every $\mathfrak{M}$ satisfying the conditions stated in § 1 we have

$$\varrho := \sum_k \frac{A_k}{q^k} \leqq 1. \tag{5.1}$$

Let

$$\alpha_N(H) := q^{-N} \# \{n \in I_0^N | \nu_N(n) = H\}, \tag{5.2}$$

$$\beta_N(H) := q^{-N} \# \{n \in I_0^N | \mu(n) = H\}. \tag{5.3}$$

Let $N \geqq 0$. For $n < q^N$ we have $\nu_N(n) \geqq 2$ if and only if $n = m_k + q^k u$ for $k < N$, $m_k \in \mathfrak{M}$.

Therefore,

$$\sum_{H \geqq 2} \alpha_N(H) = q^{-N} \sum_{k<N} A_k q^{N-k}.$$

Since $\alpha_N(0) = 0$, we have

$$\alpha_N(1) = 1 - \sum_{H \geqq 2} \alpha_N(H) = 1 - \sum_{k<N} \frac{A_k}{q^k} \searrow 1 - \varrho. \tag{5.4}$$

Let $H \geqq 2$. For $n \in I_0^N$ we have $\nu_N(n) = H$ if and only if $n = m_k + q^k u$, $\nu_{N-k}(u) = H - 1$. So we have

$$q^N \alpha_N(H) = \sum_{k<N} A_k q^{N-k} \alpha_{N-k}(H-1)$$

and hence

$$\alpha_N(H) = \sum_{k<N} \frac{A_k}{q^k} \alpha_{N-k}(H-1). \tag{5.5}$$

Let us assume that $\varrho < 1$. We shall prove that

$$\alpha_N(H) \to \varrho^{H-1}(1-\varrho) \quad (H \geqq 1). \tag{5.6}$$

Relation (5.6) holds for $H = 1$. Assuming (5.6) with $H-1$ instead of H, and using (5.5) we get (5.6) for H.

Let us consider now $\beta_N(H)$. Assume that $\varrho < 1$. It is obvious that $\mu(n) = 0$ only if $n = 0$. Furthermore, from $\mu(n) \leqq \nu_N(n)$ we have

$$\beta_N(1) + \beta_N(0) \geqq \nu_N(1),$$

and so by (5.4) we have

$$\liminf \beta_N(1) \geqq 1 - \varrho. \tag{5.7}$$

Let $H \geqq 2$. We have, as before, that

$$q^N \beta_N(H) = \sum_{k<N} A_k q^{N-k} \beta_{N-k}(H-1),$$

and deduce by induction that

$$\liminf \beta_N(H) \geqq (1-\varrho)\varrho^{H-1} \quad (H \geqq 1). \tag{5.8}$$

Since the sum over H of the numbers standing on the right hand side of (5.8) is 1, these inequalities involve that

$$\lim_N \beta_N(H) = (1-\varrho)\varrho^{H-1} \quad (N \to \infty). \tag{5.9}$$

Indeed, assume that $\limsup \beta_N(H^*) = (1-\varrho)\varrho^{H^*-1}+\delta$, $\delta>0$ and let H_1 be so large that

$$\sum_{H>H_1} (1-\varrho)\varrho^{H-1} < \frac{\delta}{2}.$$

Observing that

$$1 \geqq \limsup(\beta_N(1)+\ldots+\beta_N(H_1)) \geqq \limsup \beta_N(H^*)+$$

$$+\sum_{\substack{z=1\\ j\neq H^*}}^{H_1} \liminf \beta_N(j) \geqq \sum_{j\leqq H_1} (1-\varrho)\varrho^{j-1}+\delta > 1+\frac{\delta}{2},$$

we get a contradiction.

We have proved

THEOREM 5.1. *Assume that* $\varrho<1$. *Then for every* $H\geqq 1$

$$\lim \beta_N(H) = \lim \alpha_N(H) = (1-\varrho)\varrho^{H-1}. \tag{5.9}$$

The case $\varrho=1$ is more interesting. In this case $\lim \nu_N(H)=0$ for every H, as it follows by the previous argument.

Under some additional conditions for $\mathfrak{M}$ we can give local limit theorems for $\nu_N(n)$ and $\mu(n)$, that are quite easy consequences of well-known limit-theorems for the sum of independent lattice random variables. It is possible to deduce asymptotic expansion for $\alpha_N(H)$; $\beta_N(H)$ as well.

Let $X_1, X_2, \ldots$ be a sequence of independent random variables distributed identically according to the law

$$P(X_r = k) = \frac{A_k}{q^k}.$$

Let $Y_t = X_1+\ldots+X_t$. It is obvious that

$$\#\{n\in I_0^N | \nu_N(n) = H\} = \sum_{S<N-1} \#\{\lambda(m^1)+\ldots+\lambda(m^{H-1}) = S\} \sum_{\substack{m^H<q^{N-S}\\ \lambda(m^H)>N-S}} 1. \tag{5.10}$$

Here on the right hand side $m^1, \ldots, m^H$ run over all the possible values of $\mathfrak{M}$, as indicated in (5.10).

First we observe that

$$\#\{\lambda(m^1)+\ldots+\lambda(m^{H-1}) = S\} = q^S P(Y_{H-1}=S). \tag{5.11}$$

Let

$$T_R = \sum_{\substack{m<q^R\\ \lambda(m)\geqq R}} 1 \quad (=\alpha_R(1)); \tag{5.12}$$

m runs over the elements of $\mathfrak{M}$. For $k<R$ m_k occurs as first component q^{N-k} times among the integers of I_0^N. By using the properties of $\mathfrak{M}$ we get

$$q^R = \sum_{k<R} A_k q^{R-k} + T_R.$$

Hence, and from $\varrho=1$ we get

$$q^{-R} T_R = \sum_{k\geqq R} \frac{A_k}{q^k} = P(X_1 \geqq R). \tag{5.13}$$

Taking into account (5.10)—(5.13) we get

$$\alpha_N(H) = \sum_{t=1}^{N} P(X_1 \geqq t) P(Y_{H-1} = N-t) \quad (H \geqq 2). \tag{5.14}$$

If we define $P(Y_0=0)=1$, $P(Y_0=k)=0$ for $k>0$, then (5.14) is correct for $H=1$ too.

In a similar way we can deduce the formula

$$q^N \beta_N(H) = \sum_{0\leqq S\leqq N-1} q^s P(Y_{H-1} = S) \sum_{m<q^{N-S}}^{*} 1$$

and thus

$$\beta_N(H) = \sum_{0\leqq S\leqq N-1} P(Y_{H-1} = S) \beta_{N-S}(1) = \sum_{t=1}^{N} \beta_t(1) P(Y_{H-1} = N-t). \tag{5.15}$$

Let

$$a = MX_1 = \sum_k \frac{kA_k}{q}, \tag{5.16}$$

$$\sigma^2 = MX_1^2 - a^2. \tag{5.17}$$

We shall prove that

$$\sum_{t\leqq 1} \beta_t(1) = a. \tag{5.18}$$

Let

$$\gamma_t(1) = q^{-t} \sum_{m<q^t} 1.$$

Then $\beta_t(1) = \gamma_t(1) - 1/q^t$. Using (5.12), (5.13), we have

$$q^t \gamma_t(1) = \sum_{\substack{m<q^t \\ \lambda(m)\geqq t}} 1 + \sum_{\substack{m<q^t \\ \lambda(m)<t}} 1 = q^t P(X_1 \geqq t) + B_t.$$

To prove (5.18) it is enough to show that

$$\sum_{t\geqq 1} \frac{1}{q^t} (B_t - 1) = 0. \tag{5.19}$$

Since

$$\sum \frac{1}{q^t} B_t = \sum_m \sum_{j=1}^{\infty} \frac{1}{q^{\lambda(m)+j}} = \sum_m \frac{1}{q^{\lambda(m)}} \cdot \frac{1}{q-1} = \frac{1}{q-1},$$

therefore (5.19) and so (5.18) hold.

Now we prove

LEMMA 5.1. *If* $MX_1^\alpha<\infty$ *for* $\alpha\geqq 2$, *then*

$$\sum t^{\alpha-1}\beta_t(1)<\infty. \tag{5.20}$$

PROOF.

$$\sum t^{\alpha-1}\beta_t(1)\leqq \sum t^{\alpha-1}\gamma_t(1)=\sum\nolimits_1+\sum\nolimits_2,$$

where

$$\sum\nolimits_1=\sum t^{\alpha-1}P(X_1\geqq t),\ \sum\nolimits_2=\sum t^{\alpha-1}q^{-t}B_t.$$

We have

$$\sum\nolimits_1=\sum_k P(X_1=k)\Big(\sum_{t\leqq k}t^{\alpha-1}\Big)\leqq \sum_k \frac{k^\alpha A_k}{q^k}=MX_1^\alpha<\infty.$$

Furthermore

$$\sum\nolimits_2=\sum_t t^{\alpha-1}q^{-t}\sum_{\substack{m<q^t\\ \lambda(m)<t}}1=\sum_m q^{-\lambda(m)}\sum_j\frac{(\lambda(m)+j)^{\alpha+1}}{q^j}\ll$$

$$\ll \sum_m q^{-\lambda(m)}(\lambda(m))^{\alpha-1}\ll MX_1^{\alpha-1}<\infty.$$

So (5.20) holds.

Let X be a lattice-random variable that takes the values from the arithmetical progression $a+kd$ ($k=0,\pm1,\pm2,\ldots$). We shall say that X is of maximal step d if there does not exist a proper subset of $a+kd$ that is an arithmetical progression which contains all possible values of X.

Now we assume that the maximal step for X_1 is 1. We suppose that $MX_1^\alpha<\infty$. Under these conditions we have

$$\lim_{H\to\infty}\sup_N|\sigma\sqrt{H}\,P(Y_H=N)-\varphi(x_{H,N})|=0, \tag{5.21}$$

where

$$\overline{X}_{H,N}=\frac{N-Ha}{\sigma\sqrt{N}},\quad \varphi(x)=\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{x^2}{2}\Big) \tag{5.22}$$

as it is asserted by a theorem due to Gnedenko (see [7]).

Hence we deduce easily

THEOREM 5.2. *Let* $a=MX_1$, $\sigma^2=MX_1^2-a^2$ *be finite. Assume that* X_1 *is of maximal step* 1. *Then*

$$\alpha_N(H)=\frac{a^{3/2}}{\sigma\sqrt{N}}\varphi\Big(\frac{Ha-N}{\sigma\sqrt{N/a}}\Big)+o(1/\sqrt{N})\quad (N\to\infty) \tag{5.23}$$

$$\beta_N(H)=\frac{a^{3/2}}{\sigma\sqrt{N}}\varphi\Big(\frac{Ha-N}{\sigma\sqrt{N/a}}\Big)+o(1/\sqrt{N})\quad (N\to\infty) \tag{5.24}$$

uniformly in H.

PROOF. We prove only (5.24). The proof of (5.23) is almost the same.

By using (5.15), (5.18), (5.21) we get

$$\sigma\sqrt{H-1}\,\beta_N(H)=\sum_{t=1}^{N}\beta_t(1)\varphi(x_{H-1,N-t})+o(1)=\sum_{t=1}^{T_N}\beta_t(1)\varphi(x_{H-1,N-t})+o(1),$$

where T_N tends to infinity arbitrary slowly. Assume that $N^{-1/2}T_N \to 0$. If $H < N/(2a)$ or $H > 2N/a$, then $|X_{H-1,N-t}| \to \infty$ uniformly in $t \leqq T_N$, consequently the right hand side tends to zero, as well as $a\varphi(x_{H-1,N}) + o(1)$ tends to zero. For $H \in \left[\frac{N}{2a}, \frac{2N}{a}\right]$ we have

$$\varphi(x_{H-1;N-t}) = \varphi(x_{H-1,N}) + o(1),$$

so

$$\sigma\sqrt{(H-1)}\,\beta_N(H) = a\,\varphi(x_{H-1,N}) + o(1) \tag{5.25}$$

uniformly in H.

Let $H_0 = N/a$, $H = H_0 + L$, assume that $|H - H_0| \leqq \varepsilon_N N^{3/4}$, $\varepsilon_N \to 0$ slowly. Then

$$x_{H-1,N} = \frac{N-(H-1)a}{\sigma\sqrt{H-1}} = \frac{a-La}{\sigma\sqrt{H_0+L-1}} = \frac{-La}{\sigma\sqrt{H_0}}\left(1+O\left(\frac{L}{N}\right)\right) + O\left(\frac{1}{\sqrt{N}}\right) =$$

$$= \frac{-La}{\sigma\sqrt{\frac{N}{a}}} + O\left(\frac{L^2}{N^{3/2}}\right) + O(1/\sqrt{N}) = \frac{-La}{\sigma\sqrt{\frac{N}{a}}} + o(1),$$

consequently

$$\varphi(x_{H-1,N}) = \varphi\left(\frac{Ha-N}{\sigma\sqrt{\frac{N}{a}}}\right) + o(1),$$

and (5.24) holds.

If $H \geqq H_0 + \varepsilon_N N^{3/4}$, then $\varphi(x_{H-1,N}) \to 0$. From (5.25) we have $\beta_N(H) = o(1/\sqrt{H})$, consequently $\beta_N(H) = o(1/\sqrt{N})$ which involves (5.24). Repeating this argument we can prove (5.24) in the range

$$H \in \left[\frac{N}{2a},\ H_0 - \varepsilon_N N^{3/4}\right].$$

For $H < N/(2a)$ we consider (5.15), (5.20) and use the Chebyshev-inequality. We have

$$P(Y_{H-1} = N-t) \leqq P(Y_{H-1} \geqq N-t) \ll \frac{H}{N^2} \quad \text{for} \quad t \leqq \frac{N}{4a} (= T),$$

while from (5.20) we have

$$\sum_{t \geqq T} \beta_t(1) \ll T^{-1} = O(N^{-1}).$$

From these we get

$$\beta_N(H) = o(N^{-1/2}) \quad (H \leqq N/2a).$$

Let $P(Y_0 = k) = 1$ or 0 according to $k = 0$ or $k \neq 0$. We define $Q(n)$ as

$$Q(n) = \sum_{j=0}^{\infty} P(Y_j = n), \quad (n \geqq 1);\ Q(0) = 1.$$

We shall use the following

LEMMA 5.2. ([8]) *If* $MX_1^2<\infty$, *then*

$$Q(n)=\frac{1}{a}+o(1/n); \tag{5.26}$$

if $MX_1^4<\infty$, *then*

$$Q(n)=\frac{1}{a}+o\left(\frac{1}{n^3}\right). \tag{5.26$'$}$$

THEOREM 5.3. *Let the maximal step of* X_1 *be* 1. *Assume that* $MX_1^4<\infty$. *Then*

(5.27)

$$\beta_N(H)=\frac{a^{3/2}}{\sigma\sqrt{N}}\varphi\left(-L\frac{a^{3/2}}{\sigma\sqrt{N}}\right)+O\left(\frac{1}{N}\left[\left(\frac{|L|}{\sqrt{N}}\right)^3+1\right]\right)\varphi\left(-L\frac{a^{3/2}}{\sigma\sqrt{N}}\right)+o(N^{-3/2})$$

uniformly in H, *where* $L=H-\dfrac{N}{a}$. *Furthermore*

$$\sum_{H>H_1}\beta_N(H)\ll P(Y_{H_1}\leqq N), \tag{5.28}$$

$$\sum_{H<H_2}\beta_N(H)\ll P\left(Y_{H_2}\geqq N-\frac{R}{2}\right)+O(R^{-3}), \tag{5.29}$$

where $H_2=\dfrac{N}{a}-R,\quad R\geqq 1.$

PROOF. First we prove (5.28) and (5.29). We have

$$\sum_{H>H_1}\beta_N(H)=\sum_{t\leqq N}\beta_t(1)\sum_{H\geqq H_1}P(Y_{H-1}=N-t).$$

Taking into account that

$$\sum_{H>H_1}P(Y_H=S)=\sum_{M\leqq S}P(Y_{H_1}=M)\sum_{H>H_1}P(Y_{H-H_1}=S-M)\leqq$$
$$\leqq\sum_{M\leqq S}P(Y_{H_1}=M)=P(Y_{H_1}\leqq S),$$

we have

$$\sum_{H>H_1}\beta_N(H)\ll\sum_t\beta_t(1)P(Y_{H_1}\leqq N-t)\ll P(Y_{H_1}\leqq N),$$

and so (5.28) holds.

To prove (5.29) we observe that

$$\sum_{t\geqq R/2}\beta_t(1)\ll R^{-3}.$$

So we have

$$\sum_{H<H_2}\beta_N(H)\leqq\sum_{t\leqq R/2}\beta_t(1)\sum_{H<H_2}P(Y_{H-1}=N-t)+O(R^{-3}).$$

Since

$$\sum_{H<H_2}P(Y_{H-1}=N-t)\leqq P\left(Y_{H_2}\geqq N-\frac{R}{2}\right)\quad\text{for}\quad t\leqq R/2$$

we get (5.29) immediately.

Now we prove (5.27). Under the conditions of the theorem

$$\sigma\sqrt{H-1}\,P(Y_{H-1}=r)=s(x_{H-1,r};\,H)+o(H^{-1}), \tag{5.30}$$

where

$$s(x;\,H)=\left(1+\frac{Q_1(x)}{\sqrt{H-1}}+\frac{Q_2(x)}{H-1}\right)\varphi(x), \tag{5.31}$$

$Q_1(x)$, $Q_2(x)$ are suitable polynomials, where $\deg Q_1\leqq 3$, $\deg Q_2\leqq 6$ and

$$x_{u,r}=\frac{r-ua}{\sigma\sqrt{u}}$$

(see [9]).

Taking into account (5.28), (5.29) it is enough to prove (5.27) for $\left|H-\frac{N}{a}\right|\leqq$ $\leqq c\sqrt{N}\log N$. Assume that $H=H_0+L$, $H_0=\frac{N}{a}$; $|L|\leqq c\sqrt{N}\log N$. Let $L^*=$ $=\max(1,|L|)$. From (5.31) we deduce immediately that

$$|s(x,H)-s(x,H_0+1)|\leqq(1+|x|^6)\frac{L^*}{N^{3/2}}\varphi(x). \tag{5.32}$$

Furthermore, by an easy calculation we get

$$|s(x_{H-1,N-t};\,H)-s(x_{H-1,N};\,H)|\ll\frac{t}{\sqrt{N}}(1+|x_{H-1,N}|^6)\varphi(x_{H-1,N}) \tag{5.33}$$

in the interval $1\leqq t\leqq T$, $T=N^{1/3}\log N$. Hence and from (5.30) we get that

(5.34)
$$|P(Y_{H-1}=N-t)-P(Y_{H-1}=N)|\ll\frac{t}{N}(1+|x_{H-1,N}|^6)\varphi(x_{H-1,N})+o(N^{-3/2}).$$

Starting from (5.15)

$$\beta_N(H)=\textstyle\sum_1+\delta(H,N),$$

where

$$\textstyle\sum_1=\sum\limits_{t\leqq T}\beta_t(1)P(Y_{H-1}=N-t),\ \delta(H,N)=\sum\limits_{t>T}\beta_t(1)P(Y_{H-1}=N-t).$$

Observing that $P(Y_H=r)\ll N^{-1/2}$ holds for every r, by (5.20) we get

$$\delta(H,N)\ll N^{-1/2}\sum_{t>T}\beta_t(1)\ll\frac{1}{T^3\sqrt{N}}. \tag{5.35}$$

From (5.34) by (5.18), (5.20) we deduce that

(5.36)
$$\textstyle\sum_1=P(Y_{H-1}=N)a+O(N^{-1/2}T^3)+o(N^{-3/2})+O\left(\frac{1}{N}(1+|x_{H-1,N}|^6)\varphi(x_{H-1,N})\right).$$

Taking into account (5.35), (5.36), (5.30), (5.32) we have

$$\beta_N(H) = a\,\frac{s(x_{H-1,N};\; H_0+1)}{\sigma\sqrt{H-1}} + \tag{5.37}$$

$$+O\left((1+|x_{H-1,N}|^6)\frac{1}{N}\left(1+\frac{L^*}{N}\right)\varphi(x_{H-1,N})\right)+o(N^{-3/2}).$$

Since

$$|(H-1)^{-1/2}-H_0^{-1/2}| \ll L^* N^{-3/2}$$

the same relation holds, if we change $\sqrt{H-1}$ to $\sqrt{H_0}$ in the denominator of the first term on the right hand side.

Furthermore we have

$$x_{H-1,N} = x_{H_0+L-1,N} = \frac{-La}{\sigma\sqrt{\dfrac{N}{a}+L}} = \xi_1+\xi_2,$$

where

$$\xi_1 = \frac{-La^{3/2}}{\sigma\sqrt{N}}, \quad \xi_2 = O(L^{*2}N^{-3/2}).$$

Since $\xi_2=O(1)$, we get

$$|s(\xi_1+\xi_2,\, H_0+1)-s(\xi_1,\, H_0+1)| \ll |\xi_2|\varphi(\xi_1).$$

Taking into account this inequality and that

$$|s(\xi_1,\, H_0+1)-\varphi(\xi_1)| \ll N^{-1/2}(1+|\xi_1|^3)\varphi(\xi_1),$$

we get the desired result.

§ 6. Mean value of $\mathfrak{M}$-quasi-multiplicative functions and some consequences

THEOREM 6.1. *Assume that* $\Sigma q^{-k}A_k=1$,

$$MX_1^4 = \sum q^{-k}A_k k^4 < \infty, \tag{6.1}$$

and that X_1 *is of maximal step* 1.

Let g *be a complex-valued quasi-*$\mathfrak{M}$*-multiplicative function defined by*

$$g(n) = \prod_{j=1}^{\mu(n)} K(m^j;\, j) \tag{6.2}$$

where $m^1, \dots, m^{\mu(n)}$ *are the* $\mathfrak{M}$*-digits of* n. *Assume that*

$$|K(m;\, j)| \leqq 1 \quad (\forall\, m\in\mathfrak{M},\; j = 1, 2, \dots).$$

Suppose that the series

$$\sum_{m\in\mathfrak{M}} \frac{1-K(m,\, j)}{q^{\lambda(m)}} = \alpha_j \tag{6.3}$$

converges for $j=1,2,\ldots,$

$$\alpha_j \neq 1 \quad (j=1,2,\ldots) \tag{6.4}$$

and that the series

$$\sum \alpha_j \tag{6.5}$$

converges. In (6.3) m *is arranged according to the increase of* $\lambda(m)$. *Then the product*

$$M = \prod (1-\alpha_j) \tag{6.6}$$

exists, $M\neq 0$, *and*

$$\lim_{x\to\infty} x^{-1} \sum_{n\leqq x} g(n) = M. \tag{6.6}$$

If $g(n)=g_t(n)$ *is a family of functions depending on a parameter* t *and conditions* (6.3), (6.5), (6.6) *are satisfied uniformly in* $t\in\mathcal{T}$ *then* (6.6) *holds uniformly in* $t\in\mathcal{T}$.

REMARK. If $\alpha_j=1$, then $K(m;j)=0$ for every $m\in\mathfrak{M}$. Consequently $g(n)=0$ if $\mu(n)\geqq j$. In this case the limit in (6.6) exists and equals zero.

PROOF OF THEOREM 6.1. The existence of M, and the assertion $M\neq 0$ is an immediate consequence of the convergence of (6.5) and (6.4).

First we shall prove some lemmas.

Here and in the sequel $\sum\limits_m$ denotes a summation over $\mathfrak{M}$. The star "$*$" denotes that $m=0$ is omitted.

LEMMA 6.1. *Let* $t(m,j)\geqq 0$ *be defined on* $\mathfrak{M}\setminus\{0\}$ *for* $j=1,2,\ldots,$ $m^{\mu(n)}$ *denote the last digit of* n. *Then*

$$q^{-N} \sum_{1\leqq n<q^N} t\big(m^{\mu(n)}, \mu(n)\big) = \sum_{H\geqq 1}\sum_{t\leqq N} q^{-t} \sum_{m<q^t}^{*} t(m,H)P(Y_{H-1}=N-t).$$

PROOF. Let μ_H denote a general number having H digits $m^1,\ldots; m^H$ and put $\Lambda(\mu_H)=\lambda(m^1)+\ldots+\lambda(m^H)$. Let us consider those $n<q^N$ which can be written as

$$n = \mu_{H-1}+q^{\Lambda(\mu_{H-1})}m$$

with a fixed $m\in\mathfrak{M}$. The number of μ_{H-1} satisfying the condition $\Lambda(\mu_{H-1})=S$ is $q^S P(Y_{H-1}=S)$. Consequently

$$q^{-N} \sum_{1\leqq n<q^N} t\big(m^{\mu(n)}, \mu(n)\big) = \sum_{H\geqq 1}\sum_{S<N} q^{-(N-S)} \sum_{m<q^{N-S}}^{*} t(m,H)P(Y_{H-1}=S).$$

Putting $S=N-t$ we get our lemma.

LEMMA 6.2. *Let* $f(n)$ *be a nonnegative quasi-*$\mathfrak{M}$*-additive function defined by*

$$f(n) = \sum_{j=1}^{\mu(n)-1} t(m^j, j), \tag{6.7}$$

where $t(m,j)\geqq 0$ *for* $m\in\mathfrak{M}$, $j=1,2,\ldots.$ *Let*

$$K = \sum_{j=1}^{\infty}\sum_m \frac{t(m,j)}{q^{\lambda(m)}}$$

be finite. Then

$$q^{-N} \sum_{n<q^N} f(n) \leqq K. \tag{6.8}$$

PROOF. Let us consider those integers $n<q^N$ for which $\mu(n)>j$ and the j-th digit of which is m. The number of these integers is $\leqq q^{N-\lambda(m)}$. Consequently (6.8) follows immediately.

LEMMA 6.3. *We have*

$$q^{-N} \# \left\{ n < q^N | \max_{j \leqq \mu(n)-1} \lambda(m^j) \geqq \frac{1}{4} N^{1/4} \right\} \to 0. \tag{6.9}$$

PROOF. Since $\lambda(m) \geqq 1$ for every $m \in \mathfrak{M}$, therefore $\mu(n) \leqq N$ if $n<q^N$. The frequency of integers $n<q^N$ having m^j as the j-th digit is $\leqq q^{-\lambda(m^j)}$. Consequently (6.9) is smaller than

$$N \sum_{\lambda(m) > \frac{1}{4} N^{1/4}} q^{-\lambda(m)} = NP\left(X_1 > \frac{1}{4} N^{1/4}\right) = \varepsilon_N.$$

From (6.1) we get that the right hand side tends to zero: $\varepsilon_N \to 0$.

Let τ_N be a sequence that tends to zero monotonically and satisfies the relation

$$\sigma \sqrt{H} P(Y_H = r) = s(x_{H,r}; H) + O(\tau_N H^{-1})$$

uniformly for $H \geqq \sqrt{N}$. Its existence is a consequence of (5.30).

LEMMA 6.4. *Let* $0 \leqq N-S \leqq N^{1/4}$, $1 \leqq L \leqq N^{1/4}$,

$$R(H; S, L) = P(Y_H = S-L) - P(Y_H = S). \tag{6.10}$$

Then

$$R(H; S, L) = \frac{-L}{\sigma^2 H} s'(x_{S,H}) + O(L^2 H^{-2}) + O(\tau_N H^{-3/2}), \tag{6.11}$$

where $s(x) = s(x, H)$ *is defined in* (5.31), $s'(X_{S,H})$ *is bounded in* S *and* H.

PROOF. Since the derivatives s', s'' are bounded and

$$s(\xi+\eta) = s(\xi) + \eta s'(\xi) + O(\eta^2 s''(\xi)),$$

by putting

$$\xi = x_{S,H} = \frac{S-Ha}{\sigma \sqrt{H}}, \quad \eta = \frac{-La}{\sigma \sqrt{H}} \quad (\xi + \eta = x_{S-L,H})$$

from (5.30) we get the desired result.

LEMMA 6.5. *For* $x>1$ *let* $N=N_x$ *be defined by* $q^{N-1} \leqq x \leqq q^N$. *Let* $J_N = [H_0 - \varrho_N \sqrt{N}, H_0 + \varrho_N \sqrt{N}]$, *where* $H_0 = \dfrac{N}{a}$ *and* ϱ_N *tends to infinity arbitrarily slowly. Then*

$$x^{-1} \# \{n < x | \mu(n) \notin J_N\} \to 0$$

as $x \to \infty$.

This is an immediate consequence of Theorem 5.2.

LEMMA 6.6. *Let*

$$f(n) = \sum_{j=1}^{\mu(n)-1} \delta(m^j, j), \tag{6.12}$$

where $\delta(m, j)$ are complex valued functions defined on $\mathfrak{M}$ for $j=1, 2, \ldots$ satisfying the condition $|\delta(m^j, j)| \leqq 1$. Let

$$z_j = \sum_m \delta(m, j) q^{-\lambda(m)}, \quad \Delta_j = \sum_m |\delta(m, j)|^2 q^{-\lambda(m)}, \quad w_j = \sum_m \delta(m, j)\lambda(m) q^{-\lambda(m)},$$

$$F_H = \sum_{j \leqq H} w_j; \quad \Gamma_H = \sum_{j \leqq H} z_j; \quad E_H = \sum_{j \leqq H} \Delta_j.$$

Let L_N denote the set of those $n < q^N$ for which

$$\max_{j \leqq \mu(n)-1} \lambda(m^j) \leqq \frac{1}{4} N^{1/4}$$

holds. Let S be an integer in the interval $N - N^{1/4} \leqq S \leqq N$, and

$$\mathscr{E}(S) = q^{-s} \sum_{H \in J_N} \sum_{\substack{\mu(n)=H+1 \\ n \in \mathscr{L}_N}}^{*} |f(n)|^2$$

where the star denotes that we sum up for those n only for which $\mu(m^1)+\ldots+\mu(m^H)=S$ holds. Then

$$\mathscr{E}(S) \ll \max_{H \leqq N} |\Gamma_H|^2 + \varepsilon_N^2 + E_N) + (\varepsilon_N + \max_{H \leqq N} |\Gamma_H|) \varrho_N \sqrt{E_N} + \frac{E_N \varrho_N}{\sqrt{N}} + \varrho_N \tau_N E_N. \tag{6.13}$$

PROOF. For $\lambda(m) \geqq \frac{1}{4} N^{1/4}$ let the value of $\delta(m, j)$ be zero. Let $\tilde{\delta}, \tilde{z}_i, \tilde{\omega}_i, \tilde{\Delta}_i, \tilde{F}_H, \tilde{\Gamma}_H, \tilde{E}_H$ be the corresponding variables. Let

$$\mathscr{E}^{(1)}(H, S) = q^{-s} \sum_{\Lambda(u_H)=S} \left| \sum_{j=1}^{H} \tilde{\delta}(m^j, j) \right|^2,$$

where we sum up over all μ_H which satisfy the condition $\Lambda(u_H) = S$. It is obvious that

$$\mathscr{E}(S) \leqq \sum_{H \in J_N} \mathscr{E}^{(1)}(H, S).$$

Furthermore

$$\mathscr{E}^{(1)}(H, S) = \sum_{\substack{i, j \leqq H \\ i \neq j}} \sum_{m^1, m^2 \in \mathfrak{M}} q^{-\lambda(m^1)-\lambda(m^2)} \tilde{\delta}(m^1 i) \overline{\tilde{\delta}(m^2, j)} P\big(Y_{H-2} = S - \lambda(m^1) - \lambda(m^2)\big)$$

$$+ \sum_{i \leqq H} \Big(\sum_m q^{-\lambda(m)} |\tilde{\delta}(m, i)|^2 P\big(Y_{H-1} = S - \lambda(m)\big)\Big) = \mathscr{F}^{(1)}(H, S) + \mathscr{F}^{(2)}(H, S).$$

Since

$$\sum_H P(Y_{H-2} = S - \lambda(m)) \leqq 1,$$

therefore

$$\sum \mathscr{F}^2(H, S) \ll \sum \Delta_i \leqq E_N.$$

Let us consider the sum: $\mathscr{F}^{(1)}(H, S)$. Observing that

$$P\big(Y_{H-2} = S-\lambda(m^1)-\lambda(m^2)\big) = P(Y_{H-2} = S)+R\big(H-2, S, \lambda(m^1)+\lambda(m^2)\big),$$

by using (6.11) we get

$$\mathscr{E}^{(1)}(H, S) = P(Y_{H-2} = S) \sum_{\substack{i,j\leqq H\\ i\neq j}} \sum_{m^1, m^2} q^{-\lambda(m^1)-\lambda(m^2)}\tilde{\delta}(m^1, i)\overline{\tilde{\delta}(m^2, j)}-$$

$$-\frac{s'(x_{S-2};\, H-2)}{\sigma^2(H-2)} \sum_{\substack{i,j\leqq H\\ i\neq j}} \sum_{m^1, m^2} q^{-\lambda(m^1)-\lambda(m^2)}\big(\lambda(m^1)+\lambda(m^2)\big)\tilde{\delta}(m^1, i)\overline{\tilde{\delta}(m^2, j)}+$$

$$+O(H^{-2}) \sum_{\substack{i,j\leqq H\\ i\neq j}} q^{-\lambda(m^1)-\lambda(m^2)}\big(\lambda(m^1)+\lambda(m^2)\big)^2|\tilde{\delta}(m^1, i)|\,|\tilde{\delta}(m^2 j)|+$$

$$+o(H^{-3/2}) \sum_{\substack{i,j\leqq H\\ i\neq j}} q^{-\lambda(m^1)-\lambda(m^2)}|\delta(m^1, i)|\,|\delta(m^2, j)| =$$

$$= \mathscr{K}_1(H, S)+\mathscr{K}_2(H, S)+\mathscr{K}_3(H, S)+\mathscr{K}_4(H, S).$$

Since

$$\Big(\sum_{i\leqq H}\sum_{m} \frac{|\delta(m, i)|^2}{q^{\lambda(m)}}\Big)^2 \leqq HE_H,$$

therefore

$$\mathscr{K}_4(H, S) = O(\tau_N E_H H^{-1/2})$$

uniformly in H as $N\to\infty$.

Observing that $\big(\lambda(m^1)+\lambda(m^2)\big)^2\leqq 2\big(\lambda(m^1)^2+\lambda(m^2)^2\big)$ and $MX_i^4<\infty$, we get

$$\mathscr{K}_3(H, S) \ll H^{-2}\Big(\sum_{i\leqq H}\sum_{m} q^{-\lambda(m)}|\delta(m, i)|^2\lambda^2(m)\Big)^2 \ll \frac{E_H}{H}.$$

The sum standing after $P(H_{H-2}=S)$ can be written as $|\tilde{\Gamma}_H|^2-\sum_{i\leqq H}|\tilde{z}_i|^2$; consequently

$$\mathscr{K}_1(H, S) = \Big(|\tilde{\Gamma}_H|^2-\sum_{i\leqq H}|\tilde{z}_i|^2\Big)P(Y_{H-2} = S).$$

Similarly, the sum in $\mathscr{K}_2(H, S)$ can be written as

$$\sum_{\substack{i,j\leqq H\\ i\neq j}} (\tilde{z}_i\overline{\tilde{\omega}}_j+\tilde{\omega}_i\overline{\tilde{z}}_j) = 2\,\mathrm{Re}\,\tilde{\Gamma}_H\overline{\tilde{F}}_H-2\,\mathrm{Re}\big(\sum \tilde{z}_i\overline{\tilde{\omega}}_i\big).$$

From Lemma 6.4 the derivative s' is bounded and so

$$\mathscr{K}_2(H, S) \ll H^{-1}\Big(|\tilde{\Gamma}_H\tilde{F}_H|+\sum_{i\leqq H}|\tilde{z}_i|\,|\tilde{\omega}_i|\Big).$$

Now we observe that

$$|\tilde{z}_i|^2 \leqq \big(\sum q^{-\lambda(m)}\big)\big(\sum q^{-\lambda(m)}|\tilde{\delta}(m, i)|^2\big) \leqq \Delta_i,$$

$$|\tilde{\omega}_i|^2 \leqq \big(\sum q^{-\lambda(m)}\lambda^2(m)\big)\Delta_i \leqq \sigma^2(MX_1^2)\Delta_i.$$

Consequently

$$|\sum \tilde{z}_i \bar{\tilde{\omega}}_i| \leqq (\sum |\tilde{z}_i|^2)^{1/2} (\sum |\tilde{\omega}_i|^2)^{1/2} \ll E_H, \quad |\tilde{F}_H| \leqq \sqrt{H} (\sum |\tilde{\omega}_i|^2)^{1/2} \ll \sqrt{E_H H}.$$

Furthermore

$$\Gamma_H - \tilde{\Gamma}_H = \sum_{i \leqq H} \sum_{\lambda(m) \geqq \frac{1}{4} N^{1/4}} q^{-\lambda(m)} \delta(m, j),$$

and so

$$|\Gamma_H - \tilde{\Gamma}_H| \leqq \varepsilon_N; \; \varepsilon_N = NP\left(X_1 \geqq \frac{1}{4} N^{1/4}\right)$$

holds uniformly in H. $\varepsilon_N \to 0$ as $N \to \infty$.

Collecting our results we have

$$\mathcal{K}_1(H, S) \ll (|\Gamma_H|^2 + \varepsilon_N^2 + E_H) P(Y_{H-2} = S),$$

$$\mathcal{K}_2(H, S) \ll H^{-1}\{(\varepsilon_N + |\Gamma_H|) \sqrt{HE_H} + E_H\}.$$

Let now

$$\mathcal{T}_i(S) = \sum_{H \in J_N} \mathcal{K}_i(H, S).$$

Since $\sum_H P(Y_{H-2} = S) \leqq 1$, we have

$$\mathcal{T}_1(S) \ll (\varepsilon_N^2 + E_N + \max_{H \leqq N} |\Gamma_H|^2).$$

Observing that for $H \in J_N$, $H \leqq N$, and that the length of J_N is $2\varrho_N \sqrt{N}$, we have

$$\mathcal{T}_2(S) \ll (\varepsilon_N + \max_{H \leqq N} |\Gamma_H|) \varrho_N \sqrt{E_N} + \frac{E_N + \varrho_N}{\sqrt{N}}.$$

Similarly we have

$$\mathcal{T}_3(S) \ll \frac{E_N \varrho_N}{\sqrt{N}}, \quad \mathcal{T}_4(S) \ll \varrho_N \tau_N E_N.$$

Adding these estimates for $\mathcal{T}_i$ we get the desired inequality.

Lemma 6.7. *Under the conditions stated in Theorem* 6.1 *we have*

$$\sum_j \sum_m q^{-\lambda(m)} |1 - K(m; j)|^2 < \infty.$$

The convergence is uniform if K depends on a parameter $t \in \mathcal{T}$, and so the limit is bounded if t runs over a compact subset of $\mathcal{T}$.

Proof. This is an immediate consequence of the inequality

$$0 \leqq |1 - z|^2 \leqq 2(1 - \operatorname{Re} z) \quad \text{for} \quad |z| \leqq 1.$$

Indeed, $|K(m; j)| \leqq 1$, the real parts of the terms in (6.3) are nonnegative, from the convergence of (6.5) we get the desired result.

Now we consider the sum $\sum_{n \leqq x} g(n)$. Let $q^{M-1} \leqq x < q^M$. Let $R = R_M$ be a monotonically increasing sequence of integers satisfying $R_M = O(\log M)$. Let

$J_R(M)$ be the interval $\left[\frac{M}{a}-\varkappa_R\sqrt{M}, \frac{M}{a}+\varkappa_R\sqrt{M}\right]$, where $\varkappa_R\to\infty$ as $R\to\infty$. For an integer n having the digits $m^1, \dots, m^{\mu(n)}$ let

$$S = S(n) = \sum_{i=1}^{\mu(n)-1} \lambda(m^i)$$

and $n(R)$ be the number composed from $m^1, \dots, m^R$. That is $n(1)=m^1$, $n(j+1)= =n(j)+q^{\Lambda(n(j))}m^{j+1}$.

LEMMA 6.8. *We have*

$$q^{-M}\#\{n<q^M|\mu(n)\notin J_R(M)\}\to 0 \quad (M\to\infty), \tag{6.14}$$

$$q^{-M}\#\{n<q^M|n(R)\geqq 2Ra\}\to 0 \quad (M\to\infty), \tag{6.15}$$

$$q^{-M}\#\left\{n<q^M|\max_{j\leqq\mu(n)-1}\lambda(m^j)\geqq\frac{1}{4}M^{1/4}\right\}\to 0 \quad (M\to\infty), \tag{6.16}$$

$$q^{-M}\#\{n<q^M|M-S(n)\geqq T_M\}\to 0 \quad (T_M\to\infty). \tag{6.17}$$

PROOF. (6.14) is an immediate consequence of Theorem 5.1. To prove (6.15) we observe that the number of n-s satisfying $n(R)=U_R$ is $q^{M-\Lambda(U_R)}$, consequently the left hand side is

$$\sum_{\lambda(U_R)\geqq 2Ra} q^{-\lambda(U_R)} = P(X_1+\dots+X_R>2Ra).$$

From the Chebyshev inequality we get that this tends to zero. (6.16) is obviously $\leqq\varepsilon_M(\to 0)$. (6.17) can be estimated by

$$\sum_{t>T_M}\beta_t(1)\sum_H P(Y_{H-1}=M-t)\ll\sum_{t>T_M}\beta_t(1)\,(\to 0).$$

By this the proof of our lemma is finished.

Let now u_R be fixed. Let us consider the sum

$$\sum_{n<q^M}|g(n)-g(u_R)|, \tag{6.18}$$

where the sum is extended over those integers n for which the following conditions hold:

1. $n(R)=u_R$,
2. $\max_{j\leqq\mu(n)-1}\lambda(m^j)<\frac{1}{4}M^{1/4}$,
3. $M-S(n)<T_M$,
4. $\mu(n)\in J_R(M)$.

LEMMA 6.9. *The sum* (6.18) *extended over the integers satisfying conditions* 1—4 *is* $o(q^M)$ *uniformly for* $\Lambda(u_R)\leqq 2Ra$, *if* τ_R *is suitably chosen.*

PROOF. Let us fix u_R. We put $n=u_R+q^{A(u_R)}v$, $v\leqq q^{M-A(u_R)}$. Let us define

$$G_R(v) = \prod_{j=R+1}^{\mu(n)} K(m^j, j).$$

We have to estimate the sum

$$\sum_{v\leqq q^{M-A(U_R)}} |G_R(v)-1|,$$

satisfying the conditions involved by 1—4. Let $\delta(m, j)=1-K(m, j)$. For $|\delta(m, j)|\leqq 1/2$ we have

$$K(m, j) = e^{\delta(m,j)+O(|\delta^2(m,j)|)},$$

whence

$$|G_R(v)-1| \leqq \left|\sum_{j=R+1}^{\mu(n)} \delta(m^j, j)\right| + O\left(\sum_{j=R+1}^{\mu(n)} |\delta(m^j, j)|^2\right).$$

The last inequality is obviously valid if $|\delta(m^j, j)|\geqq 1/2$ holds for at least one j. Let $M-A(u_R)=N$.

For the sake of convenience we denote the digits of v by $m^1, \dots, m^{\mu(v)}$. In this notation we have the inequality

$$|G_R(v)-1| \leqq \left|\sum_{j=1}^{\mu(v)} \delta(m^j, j+R)\right| + O\left(\sum_{j=1}^{\mu(v)} |\delta(m^j, j+R)|^2\right), \tag{6.19}$$

We now define the functions f_1, f_2, f_3 as follows:

$$f_1(v) = \sum_{j+1}^{\mu(v)-1} \delta(m^j, j+R);\quad f_2(v) = |\delta(m^{\mu(v)}, \mu(v)+R)|;$$

$$f_3(v) = \sum_{j=1}^{\mu(v)-1} |\delta(m^j, j+R)|^2.$$

Observing that $\left|\delta\big(m^{\mu(v)}, \mu(v)\big)\right|^2 \leqq \left|\delta\big(m^{\mu(v)}, \mu(v)\big)\right|$ from (6.19) we have

$$|G_R(v)-1| \ll |f_1(v)|+f_2(v)+f_3(v).$$

To prove the lemma, it is enough to show that the sums

$$q^{-N}\sum_{v<q^N} |f_1(v)|^2,\quad q^{-N}\sum_{v<q^N} f_i(v)\quad (i=2, 3)$$

tend to zero.

From Lemma 6.7 we have $\delta(m, j)\to 0$ as $j\to\infty$ for every fixed m. So

$$\max_{m<q^{T_M}} \sup_{j\geqq 1} |\delta(m^j, j+R_M)| \to 0$$

if T_M is suitably chosen. Hence

$$q^{-N}\sum_{v<q^N} f_2(v) \leqq o_{T_M}(1) \sum q^{-t}\beta_t(1) \sum_H P(Y_{H-1}=N-t) = o_{T_M}(1).$$

By Lemma 6.2 we get

$$q^{-N} \sum_{v<q^N} f_3(v) \ll \sum_{j>R} \sum_m q^{-\lambda(m)} |\delta(m, j)|^2.$$

The right hand side tends to zero as $R \to \infty$.

Now we consider the sum

$$q^{-N} \sum_{v<q^N} |f_1(v)|^2. \tag{6.20}$$

Since $S(n) = S(v) + \Lambda(u_R)$, therefore we may assume that $N - S(v) \leqq T_M$. We shall use Lemma 6.6. Observe that

$$E_N \leqq \sum_{j \geqq R+1} \sum_m q^{-\lambda(m)} |\delta(m, j)|^2 \quad (\overset{\text{def}}{=} K_R),$$

and that

$$\max |\Gamma_H| \leqq \max_{L > R+1} \left| \sum_{j=R+1}^{L} \alpha_j \right| \leqq P_R.$$

The quantities K_R, P_R tend to zero as $R \to \infty$. Let now $\varrho_N = \varkappa_R$ and $\varkappa_R \sqrt{K_R} \to 0$. Then for every fixed S, $\mathscr{E}(S) = o(1)$ uniformly in the interval $N - S \leqq T_M$. Consequently (6.20) is smaller than

$$\sum \beta_t(1) \mathscr{E}(S) \ll o(1).$$

This proves the lemma.

Now we finish the proof of our theorem.

Collecting our results we have

$$\sum_{n \leqq x} g(n) = \sum_{\Lambda(U_R) \leqq 2Ra} g(u_R) \left[\frac{x - u_R}{q^{\Lambda(U_R)}} \right] + o(x) = \sum_{\Lambda(U_R) \leqq 2Ra} g(u_R) q^{-\Lambda(U_R)} + o(x).$$

Finally we observe that

$$\sum_{\Lambda(U_R) > 2Ra} q^{-\Lambda(U_R)} \to 0 \quad (R \to \infty)$$

and

$$\sum g(u_R) q^{-\Lambda(U_R)} = \prod_{j=1}^{R} (1 - \alpha_j) \to M.$$

The proof is complete.

THEOREM 6.2. *Assume that the conditions stated for X_1 in Theorem 6.1 are valid. Let $f(n)$ be a real valued quasi-$\mathfrak{M}$-additive function defined by*

$$f(n) = \sum_{j=1}^{\mu(n)} H(m^j, j).$$

Assume that the series

$$\sum_j \sum_{\substack{m \\ |H(m, j)| \geqq 1}} q^{-\lambda(m)}, \tag{α}$$

$$\sum_j \sum_{\substack{m \\ |H(m, j)| < 1}} H(m, j) q^{-\lambda(m)} \tag{β}$$

$$\sum_j \sum_{\substack{m \\ |H(m, j)| < 1}} H^2(m, j) q^{-\lambda(m)} \tag{γ}$$

converge. Then $f(n)$ has a limit distribution $F(x)$. The characteristic function $\varphi(\tau)$ is

$$\varphi(\tau) = \prod_{j=1}^{\infty}\Big\{\sum_m e^{i\tau H(m,j)} q^{-\lambda(m)}\Big\}. \tag{6.21}$$

Theorem 6.2 is an immediate consequence of Theorem 6.1.

From (6.21) we see that $F(x)$ is the distribution function of $\eta=\sum_{i=1}^{\infty}\xi_i$, where $\xi_1, \xi_2, \dots$ are independent random variables with distribution

$$P(\xi_j = H(m,j)) = q^{-\lambda(m)}.$$

Let

$$d_j = P(\xi_j=0) = \sum_{H(m,j)=0} q^{-\lambda(m)}.$$

A known theorem of P. Levy [10] asserts that $F(x)$ is continous if and only if $\Sigma d_j=\infty$.

Similar theorem holds for vector-valued quasi-$\mathfrak{M}$-additive functions.

THEOREM 6.3. *Assume that the conditions stated in Theorem 6.1 for X_1 are valid. Let $f(n)$ be a quasi-$\mathfrak{M}$-additive function defined by*

$$f(n) = \sum_{j=1}^{\mu(n)} H(m^j, j).$$

Let

$$\delta_j = \sum_{\substack{m\\ |H(m,j)|<1}} H(m,j)q^{-\lambda(m)}, \quad \Gamma_H = \sum_{j\le H}\delta_j.$$

Assume that the series (α), (γ) *in Theorem 6.2 converge. Then $f(n)-\Gamma_{\mu(n)}$ has a limit distribution $F(x)$. The characteristic function of it is*

$$\varphi(\tau) = \prod_{j=1}^{\infty}\Big\{\sum_m e^{i\tau(H(m,j)-\delta_j)} q^{-\lambda(m)}\Big\}. \tag{6.22}$$

Let us assume that $|\Gamma_{H_1}-\Gamma_{H_2}|\to 0$ uniformly for $|H_1-H_2|\le\tau_{H_1}\sqrt{H_1}$, $H_1\to\infty$ where $\tau_H\to\infty$ arbitrary slowly. Let furthermore

$$d_j = \sum_{\substack{H(m,j)=\delta_j\\ m}} q^{-\lambda(m)}, \quad \sum d_j=\infty.$$

Then

$$\lim_x x^{-1}\#\{n<x \mid f(n)-\Gamma_{b(x)}<y\} = F(y)$$

for every real y, where $b(x)=\left[\dfrac{\log x}{a\log q}\right]$.

PROOF. The first part of the theorem follows easily from Theorem 6.2. $f(n)-\Gamma_{\mu(n)}$ is a quasi-$\mathfrak{M}$-additive function:

$$f(n) = \sum_{j=1}^{\mu(n)}(H(m^j,j)-\delta_j).$$

We need to observe only that the conditions $(\alpha)(\beta)$, (γ) in Theorem 6.2 hold for $H_1(m, j)=H(m, j)-\delta_j$. Indeed,

$$\sum_m \frac{H_1(m, j)}{q^{\lambda(m)}} = 0$$

for every j. Since

$$|\delta_j|^2 \leqq \sum_{|H(m, j)|<1} |H(m, j)|^2 q^{-\lambda(m)} = \Delta_j$$

and $\Sigma\Delta_j<\infty$, we have

$$\sum_j \sum_m \min\left(1, |H_1(m, j)|^2\right) q^{-\lambda(m)} \ll \sum \Delta_j + \sum_j \sum_{|H(m, j)|\geqq 1} q^{-\lambda(m)} < \infty.$$

By using P. Levy's theorem, from $\Sigma\delta_j=\infty$ and (6.22) it follows that $F(x)$ is continuous. The number of those n up to x for which $|\mu(n)-b(x)|\geqq\tau_{b(x)}\sqrt{b(x)}$ is $\sigma(x)$. But in the remaining case

$$|\Gamma_{\mu(n)}-\Gamma_{b(x)}| = o(1),$$

and this involves the second part of the theorem.

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(Received September 17, 1981)

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Acta Math. Hung.
41 (3—4) (1983), 279—281.

ON A CLASS OF SCHUR AW*-ALGEBRAS

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Let R be an associative ring with an identity 1. R is called a Schur ring if for all $a, b \in R$, $Ra+Rb=R$ and $ab=ba$ imply $aR+bR=R$. We call an algebra R a Schur algebra if it is a Schur ring. In [5] Herstein and Small showed that all Artinian rings and all rings satisfying a polynomial identity are Schur rings. In [4] Handelman and Raphael proved that complete regular rings with a rank function are Schur rings. When the author was on a short visit to the University of Ottawa in 1979, D. Handelman put the following question: Are the Baer*-rings Schur rings? In this direction we prove the following:

THEOREM. *Let R be a finite AW^*-algebra. Then R is a Schur algebra**.

We follow the terminology of [1] generally.

To begin with we need

LEMMA. *Let R be a finite AW^*-algebra and for $a \in R$, let $r(a)$ denote the right annihilator of a in R. Then $r(a)=0$ implies a is invertible in R.*

PROOF. It is known that $r(a)=eR$ for some projection e in R. Since the maximal algebra Q of quotients of R is regular (in the sense of von Neumann) [7] and a Baer$^*{}_1$ ring, the right annihilator $r(a)_Q$ of a in Q is of the form e_1Q, for some projection e- in Q. From [1], $e_1 \in R$, and so $(r(a)_Q \cap R)=r(a)$ and $e_1=e$. Assume $r(a)=0$, then $r(a)_Q=0$ and a is invertible with inverse a^{-1} in Q. We claim a^{-1} lies in R. If not, then a is not invertible in R, and so it is a topological zero divisor, i.e., we can find a sequence $\{a_n\}$ consisting of elements of R such that $\{aa_n\}$ coverges to zero and $\|a_n\|=1$. Since the singular ideal of R is zero and the set $I=\{x \in R | xa^{-1} \in R\}$ is an essential left ideal in R, for any integer n there corresponds an x in I with the following properties, which are easily checked, $xa_n \neq 0$, $\|x\|=1$ and $0 \neq xa^{-1} \in R$. We take such a x and define t_n as 0 if $xa_n=0$, and $\|xa_n\|^{-1}$ if $xa_n \neq 0$. Then the sequences $\{aa_n t_n\}$ and $\{(xa^{-1})(aa_n t_n)\}=\{xa_n t_n\}$ converge to zero. This is a contradiction and we conclude that $a^{-1} \in R$.

PROOF OF THE THEOREM. Let $a, b \in R$ be such that $ab=ba$ and $Ra+Rb=R$. Then $xa+yb=1$, for some $x, y \in R$. This implies $r\{a, b\}=0$. For any $t \in r(b)$, $b(at)=a(bt)=0$ implies $at \in r(b)$. Since R is an AW^*-algebra we have $r(b)=eR$ $r(a)=fR$ for some projections e, f in R. It follows that $ae \in r(b)=eR$ and $eae=ae$. In the same way $fbf=bf$.

* A Banach algebra with an involution is called an AW^*-algebra, if $|X \cdot X^*|=\|X\|^2$ and every left and right annihilator is a one-sided ideal, generated by an Hermitian idempotent element (see [6]).

If $r(b)=0$, the Lemma implies b is invertible, and there is nothing to do in this case. Assume $r(b)=eR\neq 0$, $r(a)=fR\neq 0$. We claim that e and f can be taken as mutually orthogonal, i.e., so that $ef=0$ holds. Since Q, the maximal algebra of quotients of R, is regular, $r(b)_Q=e_1Q$, $r(a)_Q=f_1Q$ for some idempotents e_1, f_1 in Q. By combining $r(a)_Q\cap r(b)\subset r\{a, b\}_Q=0$ and Corollary 2.12 in [3] we may assume $e_1f_1=0$. Hence $f_1\in(1-e_1)Q$ and $e_1f=0$. This implies $f\in(1-e_1)Q$. From the relation $e_1Q=eQ$, we obtain $(1-e_1)Q=(1-e)Q$. This gives rise to $ef=0$, proving the claim. We now right multiply $xa+yb=1$ by e and use $eae=ae$ to obtain $e=(exe)(eae)$. It follows that $eae=ae$ is invertible in eRe. Similarly $fbf=bf$ is invertible in fRf. As in the proof of Theorem 4 of [4], we let Ω denote the collection of sets $\{(e_i, f_i)\}_{i\in I}$ of pairs of projections satisfying:

(a) I is well ordered,
(b) $\{e_i\}\cup\{f_i\}$ is an orthogonal set of projections,
(c) e_iae_i are invertible in e_iRe_i,
(d) f_ibf_i are invertible in f_iRf_i
(e) $(1-\sup\limits_{r\leqq k} e_r)ae_j=0$ if $j<k$,

(f) $\bigl(1-\sup\limits_{i\in I}(e_i+f_i)\bigr)af_j=0$ for all $j\in I$,

(g) $(1-\sup\limits_{i\in I} e_i)be_i=0$ for all $i\in I$,

(h) $\bigl(1-\sup\limits_{i\in I} e_i+\sup\limits_{j\leqq k} f_j\bigr)bf_n=0$ if $n<k$,

(i) 0 occurs at most once in $\{e_i\}\cup\{f\}$.

The set $\{(e, f)\}$ consisting of the pair chosen earlier shows Ω is nonempty. We order Ω by $\{(e_i, f_i)\}_I\leqq\{(p_j, q_j)\}_J$ if I is an initial segment of J and $e_i=p_i$, $f_i=q_i$ for all $i\in I$. We apply Zorn's lemma to obtain a maximal element $\{(e_i, f_i)\}$ in Ω. We set $e=\sup e_i$, $f=\sup f_i$ and $g=1-e-f$. It can be easily shown that eae is invertible in eRe, fbf is invertible in fRf and that the Pierce matrices of a and b are

$$\begin{bmatrix} eae & eaf & eag \\ 0 & faf & fag \\ 0 & 0 & gag \end{bmatrix} \quad\text{and}\quad \begin{bmatrix} ebe & ebf & ebg \\ 0 & fbf & fbg \\ 0 & 0 & gbg \end{bmatrix}, \quad\text{respectively,}$$

$gaggbg=gbggag$ and $r\{gag, gbg\}=0$ in gRg are now obtained from Lemma 2 of [4]. It is known that gRg is a finite AW^*-algebra. If $r(gag)=0$ in gRg, then gag is invertible and therefore $a(1-f)+bf$ is an upper triangular matrix which is invertible. So $aR+bR=R$. Assume $r(gag)\neq 0$ and $r(gbg)\neq 0$ in gRg. As before, we may find non-zero orthogonal projections p and q such that $r(gag)=pgRg$ and $r(gbg)=qgRg$ in gRg. We add $\{(p, q)\}$ to Ω, extending the partial order so as to yield a larger element $\{(e_i, f_i)\}\cup\{(p, q)\}$ of Ω. This is a contradiction, and so $g=0$. It follows that $ae+bf$ is invertible and this completes the proof.

Remark. In the case of finite AW^*-algebras, Schur property is right-left symmetric.

COROLLARY. *Let R be a finite AW^*-algebra, let A, B, C, D be in R_n, the matrix algebra over R, such that $CD=DC$. Then $X=\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is invertible in R_{2n} if and only if $AD-BC$ is invertible in R_n.*

PROOF. We first prove the sufficiency. $AD-BC$ being invertible in R_n implies that $Y(AD)-Y(BC)=I$, for some Y in R_n. Thus $R_nD+R_nC=R_n$. The matrix algebra R_n is also a finite AW^*-algebra [§ 62, 1], and since $CD=DC$, the Theorem of this note implies that $DR_n+CR_n=R_n$ and so $DV+CU=I$, for some V, U in R_n. The equality

$$\begin{bmatrix} A & B \\ B & D \end{bmatrix} \cdot \begin{bmatrix} D & U \\ -C & V \end{bmatrix} = \begin{bmatrix} AD-BC & AU-BV \\ 0 & I \end{bmatrix}$$

shows that X is invertible. To prove the necessity, we assume

$$\begin{bmatrix} U & V \\ W & Z \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} U & V \\ W & Z \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad \text{where} \quad U, V, W, Z \in R_n.$$

From this we may obtain

(a) $WA+ZC=0$, $WB+ZD=I$, (b) $UA+VC=I$; $UB+VD=0$, (c) $CV+DZ=I$.

Using $CD=DC$, equations (a) and (c) yield the equations (d) $W(AD-BC)= =-C$, $U(AD-BC)=D$. If $AD-BC$ is not invertible, then it it will be a topological zero divisor [2], that is we can find a sequence $\{T_n\}$ in R_n such that $\|T_n\|=1$ and $\{(AD-BC)T_n\}$ is convergent to zero. Equation (c) implies $CR_n+DR_n=R_n$, and from the Theorem we obtain $R_nC+R_nD=R_n$ and so $MC+ND=I$ for some $M, N \in R_n$. From this equality, the fact that $\{(AD-BC)T_n\}$ converges to zero and equations (d) it follows that $\{T_n\}$ converges to zero. This contradicts $\|T_n\|=1$. Hence $AD-BC$ is invertible which completes the proof of the corollary.

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(Received September 23, 1981)

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Acta Math. Hung.
41 (3—4) (1983), 283—290.

CHARACTERIZATIONS OF SCHAUDER DECOMPOSITIONS IN BANACH SPACES

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1. Introduction

The concept of a decomposition is a natural generalization of basis which was initiated in [5] and further studied in [3, 4, 8, 10, 11, 12]. Later on, in view of Enflo's example [2] which exhibits that every separable Banach space does not have an approximation property and hence no basis, the study of decompositions became more interesting and worth studying. It is worth noting that every separable Banach space do have a decomposition, but there are (non-separable) Banach spaces which do not have a Schauder decomposition; consider for instance the Banach space l_∞. Consequently, an attempt was made to obtain a criterion for the existence of Schauder decomposition of a Banach space and in this direction a theorem has been verified which corresponds to the theorem of Nikol'skii for the existence of bases [7]. Motivated by this work very recently Jain and Ahmad [6] obtained certain characterizations of Shcauder decompositions in terms of best approximations in Banach spaces. The purpose of this paper is to obtain some characterizations of Schauder decompositions in Banach spaces, which establish a new proof of the theorem of existence of Schauder decompositions of a Banach space (see [7], p. 93).

2. Notations and terminology

Let E be a Banach space. A sequence (M_i) of subspaces of E is a decomposition of E if and only if for each $x \in E$ there exists a unique sequence (x_i) such that $x_i \in M_i$ for all i and $x = \sum_{i=1}^{\infty} x_i$, the convergence being in the norm topology of E. The uniqueness implies the existence of (not necessarily continuous) associated projections P_i of E onto M_i defined by $P_i(x) = x_i$, where $x = \sum_{i=1}^{\infty} x_i$ with $x_i \in M_i$. These projections are obviously orthogonal i.e. $P_i P_j = \delta_{ij} P_j$, where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$. If, in addition, each P_i is continuous, the decomposition is called Schauder decomposition, and we write it as (M_i, P_i). Let $S_n(x) = \sum_{i=1}^{n} P_i(x)$, $x \in E$. Then S_n is a partial sum operator on E and is continuous.

A sequence (M_i) of subspaces of E is said to be

(a) finitely linearly independent, if every finite subsequence of (M_i) is linearly ndependent:

(b) Ω-linearly independent, if $\sum_{i=1}^{\infty} x_i = 0, x_i \in M_i \Rightarrow x_i = 0$ for each i;

(c) minimal, if $M_j \cap \left[\bigcup_{\substack{i=1 \\ i \neq j}}^{\infty} M_i\right] = \emptyset$.

Obviously, every minimal sequence of subspaces is Ω-linearly independent and every Ω-linearly independent sequence is finitely linearly independent. For finite sequences of subspaces the converse statements are also valid, but the converse need not be true in the case of infinite sequences of subspaces.

The distance between two subspaces F, G of E is given by

$$\operatorname{dist}(F, G) = \inf \{\|x-y\| : x \in F,\ y \in G\}.$$

Let $\sigma_F = \{x \in F : \|x\| = 1\}$. Define the number $(\widehat{F; G}) = \operatorname{dist}(\sigma_F, G)$. Let $L_n = \left[\bigcup_{i=1}^{n} M_i\right]$, $L^n = \left[\bigcup_{i=n+1}^{\infty} M_i\right]$ and $L = \left[\bigcup_{i=1}^{\infty} M_i\right]$, where the bracketed expressions denote the closed linear spans of the sequence (M_i) of subspaces of E.

A sequence (M_i) of non-trivial subspaces of E is said to be complete if $\left[\bigcup_{i=1}^{\infty} M_i\right] = E$.

3. Main results

THEOREM 3.1. *Let (M_i) be a sequence of nontrivial closed subspaces of a Banach space E. Then the following statements are equivalent:*

(a) *(M_i) is minimal.*

(b) *There exists a sequence (P_i) of projections on L such that $P_i P_j = \delta_{ij} P_j$, where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$.*

(c) *There exists a sequence of constants $\mu_i > 0$ $(i = 1, 2, \dots)$ such that*

$$\sum_{i=1}^{n} \|x_i\| \mu_i \leqq \left\|\sum_{i=1}^{n} x_i\right\|,$$

where $x_i \in M_i$ for $i = 1, 2, \dots, n$.

(d) *We have $L = L_n \oplus L^n$.*

(e) *There exists a sequence (U_n) of endomorphisms on E such that*

$$U_n(x) = \begin{cases} x, & x \in L_n, \quad n = 1, 2, \dots \\ 0, & x \in L^n, \quad n = 1, 2, \dots \end{cases}$$

(f) *For each positive integer n, there exists a constant C_n, $1 \leqq C_n < \infty$ such that*

$$\left\|\sum_{i=1}^{n} x_i\right\| \leqq C_n \left\|\sum_{i=1}^{n+m} x_i\right\|$$

for all positive integers n, m and $x_i \in M_i$ for each i.

(g) *We have* $\operatorname{dist}(\sigma_{L_n}, L^n)>0$ $(n=1,2,\ldots)$.

(h) *For each positive integer n there exists a constant* C'_n, $1\leqq C'_n<\infty$, *with the property that for each* $h_0\in L_n$ *there exists an* $f\in E^*$ *such that*

$$f(h_0)=\|h_0\|, \tag{3.1}$$

$$f(y)=0 \quad (y\in L^n), \tag{3.2}$$

$$1\leqq \|f\|\leqq C'_n. \tag{3.3}$$

(i) *The norm of the linear operator* S_n *on* L *is given by*

$$\|S_n\| = \sup_{\substack{h\in L\\ \|h\|\leqq 1}} \|S_n(h)\|<\infty \quad (n=1,2,\ldots).$$

PROOF. (a)$\Rightarrow$(b). Let (a) hold. The nontrivial closed subspaces M_i and $\left[\bigcup_{\substack{j=1\\ j\neq i}}^{\infty} M_j\right]$ of E are such that

$$M_i\cap\left[\bigcup_{\substack{j=1\\ j\neq i}}^{\infty} M_j\right]=\emptyset,$$

so that L can be expressed as

$$L=M_i\oplus\left[\bigcup_{\substack{j=1\\ j\neq i}}^{\infty} M_j\right].$$

Hence, there exists a sequence (P_i) of projections on L such that $P_iP_j=\delta_{ij}P_j$, where $\delta_{ij}=0$ for $i\neq j$ and $\delta_{ij}=1$ for $i=j$.

$(b)\Rightarrow(c)$. Let (b) hold. Put

$$\mu_i=\frac{1}{2^{i+1}\|P_i\|} \quad (i=1,2,\ldots).$$

Then, for all $x_i\in M_i$ $(i=1,2,\ldots,n)$, we have

$$\|x_j\|\mu_j=\frac{\left\|P_j\left(\sum_{i=1}^{n} x_i\right)\right\|}{2^{j+1}\|P_j\|}\leqq\frac{1}{2^{j+1}}\left\|\sum_{i=1}^{n} x_i\right\| \quad (j=1,2,\ldots,n)$$

so that

$$\sum_{i=1}^{n}\|x_i\|\mu_i\leqq\sum_{i=1}^{n}\frac{1}{2^{i+1}}\left\|\sum_{i=1}^{n} x_i\right\|<\frac{1}{2}\left\|\sum_{i=1}^{n} x_i\right\|.$$

(c)$\Rightarrow$(a). It is obvious from the given inequality that the finite subsequence of (M_i) is finitely linearly independent. Consequently, from the definition, the finite subsequence of (M_i) is Ω-linearly independent and hence it is a minimal sequence of subspaces.

(b)$\Rightarrow$(d). Let (b) hold. Then $S_n(L)=L_n$, which shows that S_n is a continuous projection of L onto L_n along L^n. This verifies (d).

(d)$\Rightarrow$(e). Given $L=L_n\oplus L^n$, S_n is a continuous projection of L onto L_n along L^n, so that $S_n(x)=\sum_{i=1}^{n} x_i$. Thus

$$S_n(x)=\begin{cases} x, & x\in L_n, \quad n=1,2,\ldots \\ 0, & x\in L^n, \quad n=1,2,\ldots . \end{cases}$$

Let U_n be an endomorphism on E so that $U_n(x)=\sum_{i=1}^{n} x_i$. Consequently, $S_n(x)=$ $=U_n(x)$ for every $x\in L$. This shows that S_n can be extended to U_n. Hence

$$U_n(x)=\begin{cases} x, & x\in L_n, \quad n=1,2,\ldots \\ 0, & x\in L^n, \quad n=1,2,\ldots . \end{cases}$$

(e)$\Rightarrow$(f). From (e) we have

$$\left\|\sum_{i=1}^{n} x_i\right\| = \left\|U_n\left(\sum_{i=1}^{n+m} x_i\right)\right\| \leqq \|U_n\| \left\|\sum_{i=1}^{n+m} x_i\right\|.$$

This verifies (f), if we put $C_n=\|U_n\|$ $(n=1, 2, \ldots)$.

(f)$\Rightarrow$(g). Let (f) hold. For every $\sum_{i=1}^{n} x_i\in\sigma_{L_n}$ and $x_i\in M_i$, $i=n+1$, $n+2, \ldots,$ $\ldots, n+m$, we have

$$\left\|\sum_{i=1}^{n} x_i-\sum_{i=n+1}^{n+m} x_i\right\| \geqq \left\|\sum_{i=1}^{n+m} x_i\right\| \geqq \frac{1}{C_n}\left\|\sum_{i=1}^{n} x_i\right\| = \frac{1}{C_n}.$$

Hence, $\operatorname{dist}(\sigma_{L_n}, L^n)>0$.

(g)$\Rightarrow$(h). Let $h_0\in L_n$. If $h_0=0$, there exists $f\in E^*$ satisfying (3.1), (3.2) and $\|f\|=$arbitrary constant.

If $h_0\neq 0$, we have $\dfrac{h_0}{\|h_0\|}\in\sigma_{L_n}$, hence by (g),

$$\operatorname{dist}\left(\frac{h_0}{\|h_0\|}, L^n\right) \geqq \operatorname{dist}(\sigma_{L_n}, L^n) = d_n > 0.$$

Then, by the Corollary of Hahn—Banach theorem, there exists a functional $f\in E^*$, satifying (3.1), (3.2) and $\|f\|=\dfrac{1}{d_n}$.

(h)$\Rightarrow$(i). By (3.1), (3.2), and (3.3), for every $h=\sum_{i=1}^{\infty} x_i\in L$, $x_i\in M_i$ for each i, we have

$$S_n(h)=S_n\left(\sum_{i=1}^{\infty} x_i\right)=\sum_{i=1}^{n} x_i = h_0(say),$$

and

$$f(h)=f\left(\sum_{i=1}^{\infty} x_i\right)=f\left(\sum_{i=1}^{n} x_i+\sum_{i=n+1}^{\infty} x_i\right)=f(h_0).$$

Thus,

$$\|S_n(h)\| = \|h_0\| = |f(h_0)| = |f(h)| \leqq C'_n\|h\|.$$

This implies that

$$\|S_n\| = \sup_{\substack{h\in L\\ \|h\|\leqq 1}} \|S_n(h)\| < \infty.$$

(i)$\Rightarrow$(c). Let S_n be a continuous linear operator on L such that

$$\|S_n\| = \sup_{\substack{h\in L\\ \|h\|\leqq 1}} \|S_n(h)\| < \infty \quad (n = 1, 2, \ldots),$$

Put $\mu_n = \dfrac{1}{2^{n+1}(\|S_n\|+\|S_{n-1}\|)}$ $(n = 1, 2, \ldots)$. Then $\mu_n>0$ and for every finite sequence of vectors $x_i \in M_i$, $i=1, 2, \ldots, n$, we have

$$\|x_j\|\mu_j = \frac{\|x_j\|}{2^{j+1}(\|S_j\|+\|S_{j-1}\|)} \leqq \frac{\left\|\sum_{i=1}^{j} x_i\right\| + \left\|\sum_{i=1}^{j-1} x_i\right\|}{2^{j+1}(\|S_j\|+\|S_{j-1}\|)} =$$

$$= \frac{\left\|S_j\left(\sum_{i=1}^{n} x_i\right)\right\| + \left\|S_{j-1}\left(\sum_{i=1}^{n} x_i\right)\right\|}{2^{j+1}(\|S_j\|+\|S_{j-1}\|)} \leqq \frac{1}{2^{j+1}}\left\|\sum_{i=1}^{n} x_i\right\| \quad (j = 1, 2, \ldots, n).$$

Hence, the result follows. This completes the proof of the theorem.

THEOREM 3.2. *Let E be a Banach space and (M_i) a sequence of nontrivial closed subspaces of E such that $\left[\bigcup_{i=1}^{\infty} M_i\right] = E$. Then the following statements are equivalent:*

(a) *(M_i) is a Schauder decomposition of E.*

(b) *There exists a sequence (U_n) of endomorphisms on E satisfying*

$$U_n(x) = \begin{cases} x, & x\in L_n, \quad n = 1, 2, \ldots \\ 0, & x\in L^n, \quad n = 1, 2, \ldots \end{cases}$$

and $1 \leqq C_1 = \sup\limits_{1\leqq n<\infty} \|U_n\| < \infty.$

In this case, the sequence (U_n) is uniquely determined and coincides with the sequence (S_n) of partial sum operators associated to the Schauder decomposition (M_i).

(c) *There exists a constant C_2 with $1\leqq C_2<\infty$ such that*

$$\left\|\sum_{i=1}^{n} x_i\right\| \leqq C_2\left\|\sum_{i=1}^{n+m} x_i\right\|$$

for all positive integers n, m and $x_i\in M_i$ for each i.

(d) *We have* $C_3 = \inf\limits_{1\leqq n<\infty} \operatorname{dist}(\sigma_{L_n}, L^n) > 0.$

(e) *There exists a constant* C_4, $1 \leqq C_4 < \infty$, *with the property that for every* n *and every* $h_0 \in L_n$ *there exists an* $f \in E^*$ *satisfying* (3.1), (3.2) *and*

$$1 \leqq \|f\| \leqq C_4. \tag{3.4}$$

(f) *We have*

$$\sup_{1 \leqq n < \infty} \sup_{\substack{h \in L \\ \|h\| \leqq 1}} \|S_n(h)\| < \infty.$$

For the proof of the theorem we need the following lemma:

LEMMA. *Let* E *be a Banach space and* (M_i) *a sequence of closed subspaces of* E *and* (P_i) *a sequence of projections on* E. *Then the following statements are equivalent:*

(i) (M_i, P_i) *is a Schauder decomposition of* E.

(ii) *For every* $x \in E$, *the expansion* $\sum_{i=1}^{\infty} P_i(x)$ *is convergent and its sum is* x *i.e. we have* $\lim_{n \to \infty} S_n(x) = x$ *for all* $x \in E$.

(iii) (M_i) *is complete and* $\sup_{1 \leqq n < \infty} \|S_n(x)\| < \infty$ *for all* $x \in E$.

(iv) (M_i) *is complete and there exists a constant* $M \geqq 1$ *such that* $\|S_n\| \leqq \leqq M$ $(n = 1, 2, \ldots)$.

PROOF. (i)$\Rightarrow$(ii). If (i) is given, then (ii) is obvious from the definition of decomposition of a space.

(ii)$\Rightarrow$(iii). If (ii) exists, then (M_i) is complete. Also, since the sequence $(S_n(x))$ converges, we have $\sup_{1 \leqq n < \infty} \|S_n(x)\| < \infty$.

(iii)$\Rightarrow$(iv). This implication is a consequence of the principle of uniform boundedness i.e.

$$\sup_{1 \leqq n < \infty} \|S_n(x)\| < \infty \Rightarrow \sup_{1 \leqq n < \infty} \|S_n\| < \infty.$$

Hence there exists a constant $M \geqq 1$ such that $\|S_n\| \leqq M$.

(iv)$\Rightarrow$(i). Since (M_i) is complete in E, there exists a dense subset of finite linear combinations $\sum_{i=1}^{m} x_i$, $x_i \in M_i$ $(i = 1, 2, \ldots, m;\ m = 1, 2, \ldots)$ in E such that

$$\lim_{n \to \infty} S_n\left(\sum_{i=1}^{m} x_i\right) = \sum_{i=1}^{m} x_i.$$

Then $\lim_{n \to \infty} S_n(x) = x$ for all $x \in E$. Hence the result follows from the definition of Schauder decomposition.

PROOF OF THEOREM 3.2. (a)$\Rightarrow$(b). If (M_i, P_i) is a Schauder decomposition of E, then, by the lemma and Theorem 3.1, the sequence (S_n) of partial sum operators associated to the Schauder decomposition uniquely satisfies

$$S_n(x) = \begin{cases} x, & x \in L_n, \quad n = 1, 2, \ldots \\ 0, & x \in L^n, \quad n = 1, 2, \ldots \end{cases}$$

and $1 \leqq C_1 = \sup_{1 \leqq n < \infty} \|S_n\| < \infty$.

(b)$\Rightarrow$(c). From (b) we have

$$\left\|\sum_{i=1}^{n} x_i\right\| = \left\|U_n\left(\sum_{i=1}^{n+m} x_i\right)\right\| \leqq C_1\left\|\sum_{i=1}^{n+m} x_i\right\|.$$

The implications (c)$\Rightarrow$(d)$\Rightarrow$(e)$\Rightarrow$(f) are immediate from Theorem 3.1.

(f)$\Rightarrow$(a). Since (M_i) is a complete sequence of nontrivial closed subspaces of E and $\sup_{1 \leqq n < \infty} \|S_n\| < \infty$, by the implication (iv)$\Rightarrow$(i) of lemma, it follows that (M_i) is a Schauder decomposition of E. This completes the proof.

THEOREM 3.3. *Let (M_i) be a complete sequence of closed subspaces of E such that*

$$\prod_{n=1}^{\infty} \widehat{\left(\left[\bigcup_{i=1}^{n} M_i\right]; M_{n+1}\right)} > 0. \tag{3.5}$$

Then (M_i) is a Schauder decomposition of E.

PROOF. Put

$$\beta_n = \widehat{\left(\left[\bigcup_{i=1}^{n} M_i\right]; M_{n+1}\right)} \quad (n = 1, 2, \ldots)$$

and $\beta = \prod_{n=1}^{\infty} \beta_n$. Then $0 < \beta \leqq 1$. For all positive integers n, m and $x_i \in M_i$ for each i, and by (3.5) we have

$$\left\|\sum_{i=i}^{n+m} x_i\right\| = \left\|\sum_{i=1}^{n+m-1} x_i + x_{n+m}\right\| \geqq \operatorname{dist}\left(\sigma_{\left[\bigcup_{i=1}^{n+m-1} M_i\right]}, M_{n+m}\right)\left\|\sum_{i=1}^{n+m-1} x_i\right\| =$$

$$= \widehat{\left(\left[\bigcup_{i=1}^{n+m-1} M_i\right]; M_{n+m}\right)}\left\|\sum_{i=1}^{n+m-1} x_i\right\| = \beta_{n+m-1}\left\|\sum_{i=1}^{n+m-1} x_i\right\| \geqq$$

$$\geqq \beta_{n+m-1}\beta_{n+m-2}\left\|\sum_{i=1}^{n+m-2} x_i\right\| \geqq \ldots \geqq \beta_{n+m-1}\beta_{n+m-2}\cdots\beta_n\left\|\sum_{i=1}^{n} x_i\right\| \geqq \beta\left\|\sum_{i=1}^{n} x_i\right\|.$$

Hence the result follows by using the theorem 3.2.

Finally, our thanks are due to Dr. P. K. Jain for his consistent help in the preparation of this paper.

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(Received September 24, 1981)

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Acta Math. Hung.
41 (3—4) (1983), 291—307.

UNIFORM APPROXIMATION BY SZÁSZ—MIRAKJAN TYPE OPERATORS

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§ 1

The purpose of this paper is to investigate the uniform approximation problem for the Szász—Mirakjan operators

$$S_n(f;x)=\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)p_{nk}(x) \quad \left(p_{n,k}(x)=e^{-nx}\frac{(nx)^k}{k!},\ x\geqq 0\right)$$

on the whole positive real line. So far, beyond local results, the approximation property of S_n was settled in polynomial weight spaces by M. BECKER [1] (here the order of approximation has the form $(x/n)^{\alpha}$), in exponential weight spaces by M. Becker, D. Kucharski and R. J. Nessel [5] and B. D. Boyanov and V. M. Veselinov [8] have results for the case of uniform approximation when the function f has a finite limit at the infinity.

We shall be working in C_B, the set of bounded and continuous functions defined on $[0,\infty)$. The continuity is a necessary assumption at uniform approximation and the boundedness will turn out to be a convenient one, although many of our considerations remain valid just as well in the unbounded case.

Unlike Bernstein polynomials (which, otherwise, have very similar properties to S_n) for S_n a new question arises: to identify those f for which $S_n(f)$ converge uniformly on $[0,\infty)$ to f. The answer for C_B is given in Theorem 1 below.

Let

$$\Delta_h^2(f;x)=f(x)-2f(x+h)+f(x+2h)$$

and let us introduce the following modified modulus of smoothness which is the appropriate one in our problems:

$$\omega(\delta)=\omega(f;\delta)=\sup_{0\leqq x<\infty,\,0<h\leqq\delta}|\Delta_{h\sqrt{x}}^2(f;x)| \quad (\delta>0)$$

(in the following we use the shorter form $\omega(\delta)$). This ω has the known properties of moduli of smoothness: it is an increasing function, and a standard argument gives that

$$\omega(\lambda\delta)\leqq K\lambda^2\omega(\delta) \quad (\lambda\geqq 1)$$

with an absolute constant K.

Now we have

THEOREM 1. *For an* $f\in C_B$ *the following are equivalent*

(i) $S_n(f)-f=o(1)$ $(n\to\infty)$,

(ii) $\omega(\delta)=o(1)$ $(\delta\to 0)$,

(iii) $f(x+h\sqrt{x})-f(x)=o(1)$ *as* $h\to 0$ *uniformly in* x,
(iv) *the function* $f(x^2)$ *is uniformly continuous.*

We mention that (ii)⇒(i) for any continuous f, but it is an open problem whether (i) implies (ii) without the boundedness assumption. Our proof will show that the equivalence (i)⇔(ii) holds even if $f\in C_B$ is replaced by the weaker assumption: $f\in C[0,\infty)$, $\omega(1)<\infty$.

For the Lipschitz-case we have

THEOREM 2. *Let* $0<\alpha\leqq 1$. *For an* $f\in C_B$
(i) $S_n(f)-f=O(n^{-\alpha})$
and
(ii)

$$\omega(\delta)=O(\delta^{2\alpha}) \tag{1.1}$$

are equivalent.

Here we mention that (1.1) is equivalent to $x^\alpha\Delta_h^2(f;x)\leqq Kh^2$ $(h>0)$, so our result is the mate of M. Becker's one [1, Theorem 1], namely the weight x^α is moved from the approximation to the smoothness condition.

For a similar result in the saturation case $\alpha=1$ see [4, Satz 4.11].

It is interesting to consider the analogous problem for the Kantorovich-type modification of S_n:

$$K_n(f;x)=\sum_{k=0}^{\infty}\left(n\int_{k/n}^{(k+1)/n}f(u)\,du\right)p_{n,k}(x)\quad(x\geqq 0).$$

K_n can be used to approximate f in various integral metrics which we shall do in a forthcoming paper. Here we are interested in uniform approximation and for this we have

THEOREM 3. *For an* $f\in C_B$
(i) $K_n(f)-f=o(1)$ $(n\to\infty)$
and
(ii) $\omega(\delta)=o(1)$ $(\delta\to 0)$
are equivalent.

THEOREM 4. *Let* $0<\alpha\leqq 1$. *For an* $f\in C_B$ (i) *and* (ii) *below are equivalent:*
(i) $K_n(f)-f=O(n^{-\alpha})$,
(ii) (1) $\omega(\delta)=O(\delta^{2\alpha})$, (2) $f(0)-f(h)=O(h^\alpha)$.

Let us mention that conditions (ii) (1) and (2) are independent of each other.

We want to settle also the multidimensional case. We carry over our results only to two dimensions; the higher dimensional problem can be treated similarly.

One possible variant of S_n in two dimensions is the operator

$$S_{n,m}(f;x,y)=e^{-nx-my}\sum_{i,j=0}^{\infty}f\left(\frac{i}{n},\frac{j}{m}\right)\frac{(nx)^i}{i!}\frac{(my)^j}{j!}\quad(x,\,y\geqq 0)$$

with two parameters $n,m=1,2,\ldots$ Let $C_{2,B}$ denote the set of continuous and bounded functions defined on the first quadrant $x\geqq 0$, $y\geqq 0$.

THEOREM 5. *For an* $f\in C_{2,B}$ *the following are equivalent*

(i) $S_{n,m}(f)-f=o(1)$ *as* $n, m\to\infty$,

(ii) $\omega(\delta_1,\delta_2)=o(1)$ *as* $\delta_1,\delta_2\to 0$,

(iii) $f(x+h_1\sqrt{x}, y+h_2\sqrt{y})-f(x,y)=o(1)$ *as* $h_1, h_2\to 0$ *uniformly in* $x\geqq 0$, $y\geqq 0$,

(iv) *the function* $f(x^2, y^2)$ *is uniformly continuous.*

Here

$$\omega(\delta_1,\delta_2)=\sup_{x\geqq 0,\, y\geqq 0,\, 0<h_1\leqq\delta_1,\, 0<h_2<\delta_2}|\Delta^2_{h_1\sqrt{x},h_2\sqrt{y}}(f;x,y)|$$

and

$$\Delta^2_{h_1,h_2}(f;x,y)=f(x,y)+f(x,y+2h_2)+f(x+2h_1,y)+f(x+2h_1,y+2h_2)-$$
$$-4f(x+h_1,y+h_2).$$

THEOREM 6. *Let* $0<\alpha\leqq 1$. *For an* $f\in C_{2,B}$ *the following are equivalent*

(i) $S_{n,m}(f)-f=O(n^{-\alpha}+m^{-\alpha})$,

(ii) $\omega(\delta_1,\delta_2)=O(\delta_1^{2\alpha}+\delta_2^{2\alpha})$.

We mention that in an other paper we shall prove the analogues of Theorems 1, 2 for the Baskakov and Meyer—König and Zeller operators.

The paper is organized in the following way: § 2 contains some necessary formulae. In § 3 we prove Theorem 2, the ideas of which are used in the proof of the other theorems in § 4.

§ 2. Some formulae and estimates

First we give an other expression for $S_n(f)$. Let

$$\Delta_h^k(f;x)=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}f(x+ih)$$

be the k-th difference-function. A simple computation gives that

$$S_n(f;x)=\sum_{k=0}^{\infty}\frac{(-nx)^k}{k!}\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)\frac{(nx)^k}{k!}=\sum_{k=0}^{\infty}\Delta^k_{\frac{1}{n}}(f;0)\frac{(nx)^k}{k!}. \tag{2.1}$$

We have also (see [1, p. 136])

$$S_n''(f;x)=n^2\sum_{k=0}^{\infty}\Delta^2_{\frac{1}{n}}\left(f;\frac{k}{n}\right)p_{n,k}(x), \tag{2.2}$$

$$S_n''(f;x)=\frac{n^2}{x^2}\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)r_{n,k}(x)p_{n,k}(x) \tag{2.3}$$

where

$$r_{n,k}(x) = \left(\frac{k}{n}-x\right)^2 - \frac{k}{n^2},$$

and

$$S_n'(f; x) = n \sum_{k=0}^{\infty} \Delta^1_{\frac{1}{n}}\left(f; \frac{k}{n}\right) p_{n,k}(x).$$

For $p_{n,k}$ we shall often use the following estimates. Let $p_k(x)=e^{-x}\frac{x^k}{k!}$. For a given x the largest of the $p_k(x)$ is $p_M(x)$ for $M=[x]$ (the integral part of x) and the sequence $\{p_k(x)\}_{k=0}^{\infty}$ is increasing from $k=0$ to M and decreasing for $k \geqq M$. With the notation $k=M+h$ we have for $|h| \leqq 2x^{3/5}$ the asymptotic expression

$$p_k(x) = \frac{1}{\sqrt{2\pi x}} e^{-h^2/(2x)}\left(1+O\left(\frac{|h|+1}{x}\right)+O\left(\frac{|h|^3}{x^2}\right)\right). \tag{2.4}$$

We also have

$$\sum_{|h|>x^{3/5}} p_k(x) = O(e^{-x^{\eta}}) \tag{2.5}$$

with $\eta < \frac{1}{5}$, and for $0<\zeta<1$

$$\sum_{|h|>\zeta x} p_k(x) = O(e^{-\tau x}) \tag{2.6}$$

with $\tau = \frac{1}{5}\zeta^2$.

For all of these results see [7, p. 200].

We shall also frequently use without any further reference the identities

$$\frac{k}{n} p_{n,k}(x) = x p_{n,k-1}(x) \quad (n,\, k = 1, 2, \ldots).$$

§ 3. Proof of Theorem 2

I. *Proof of* (ii)⇒(i). Let $X>0$, $\delta = \sqrt{\frac{x}{n}}$, and

$$f_\delta(t) = \left(\frac{2}{\delta}\right)^2 \iint_0^{\delta/2} \big(2f(t+u+v) - f\big(t+2(u+v)\big)\big)\,du\,dv. \tag{3.1}$$

For this (see [1, p. 135])

$$f(t)-f_\delta(t) = \left(\frac{2}{\delta}\right)^2 \iint_0^{\delta/2} \Delta^2_{u+v}(f; t)\,du\,dv; \; f_\delta''(t) = \delta^{-2}\big(8\Delta^2_{\frac{\delta}{2}}(f; t) - \Delta^2_\delta(f; t)\big),$$

and so

$$|f(t)-f_\delta(t)| \leqq \omega\left(\sqrt{\frac{x}{nt}}\right), \quad |f_\delta''(t)| \leqq K\frac{n}{x}\omega\left(\sqrt{\frac{x}{nt}}\right)$$

by which, using the inequality $\omega(\lambda t) \leqq K\lambda^2\omega(t)$ we obtain

$$\tag{3.2} |f(t)-f_\delta(t)| \leqq K\begin{cases} \dfrac{x}{t}\omega\left(\dfrac{1}{\sqrt{n}}\right) & \text{for } t \leqq x \\ \omega\left(\dfrac{1}{\sqrt{n}}\right) & \text{for } t > x \end{cases}$$

and

$$\tag{3.3} |f_\delta''(t)| \leqq K\begin{cases} \dfrac{n}{t}\omega\left(\dfrac{1}{\sqrt{n}}\right) & \text{for } t \leqq x \\ \dfrac{n}{x}\omega\left(\dfrac{1}{\sqrt{n}}\right) & \text{for } t > x. \end{cases}$$

Let

$$S_n^*(f; x) = \sum_{k=1}^{\infty} f\left(\frac{k}{n}\right) p_{n,k}(x).$$

S_n^* is defined also for functions which do not have any value at the origin.

Now

$$|S_n(f; x)-f(x)| \leqq |f(x)-f_\delta(x)|+|S_n(f_\delta; x)-f_\delta(x)|+|S_n^*(f_\delta-f; x)|+$$
$$+e^{-nx}|f(0)-f_\delta(0)| = A+B+C+D.$$

By (3.2) $A \leqq K\omega\left(\frac{1}{\sqrt{n}}\right)$, and taking into account the positivity of S_n, S_n^* we obtain from (3.2) and Lemma 1 below that $C \leqq K\omega\left(\frac{1}{\sqrt{n}}\right)$ and from (3.3) and Lemma 2 that $B \leqq K\omega\left(\frac{1}{\sqrt{n}}\right)$. Finally, for D we have by Lemma 3 below that

$$e^{-nx}|f_\delta(0)-f(0)| \leqq Ke^{-nx}\left(\sqrt{\frac{x}{n}}\right)^\alpha \leqq Kn^{-\alpha}e^{-nx}(nx)^{\alpha/2} \leqq Kn^{-\alpha}.$$

Thus,

$$|S_n(f; x)-f(x)| \leqq K\left(n^{-\alpha}+\omega\left(\frac{1}{\sqrt{n}}\right)\right) \leqq Kn^{-\alpha}$$

with a K independent of x and our proof will be complete after the justification of Lemmas 1—3 below.

LEMMA 1. *If* $x>0$ *and*

$$g(t) = \begin{cases} \dfrac{x}{t} & \textit{for} \quad t \leqq x \\ 1 & \textit{for} \quad t \geqq x \end{cases}$$

then $|S_n^*(g; x)| \leqq 3$.

PROOF.

$$S_n^*(g;x) = \sum_{k=1}^{[nx]} \frac{nx}{k} p_{n,k}(x) + \sum_{k=[nx]+1}^{\infty} p_{n,k}(x) \leqq 2\sum_{k=1}^{[nx]} \frac{nx}{k+1} p_{nk}(x)+1 =$$

$$= 2\sum_{k=1}^{[nx]} p_{n,k+1}(x)+1 \leqq 3.$$

LEMMA 2. *If* $x>0$, $g\in C_B$ *is twice continuously differentiable on* $(0;\ \infty)$ *and*

$$|g''(t)| \leqq \begin{cases} \dfrac{1}{t} & \textit{for} \quad t \leqq x \\ \dfrac{1}{x} & \textit{for} \quad t > x \end{cases}$$

then

$$|S_n(g;x)-g(x)| \leqq \frac{K}{n} \quad (n=1, 2, \ldots)$$

with a K *independent of* x.

PROOF. Let

$$h(t) = \begin{cases} t\log t & \text{for} \quad t \leqq x \\ x\log x + (t-x)(\log x+1) + \dfrac{1}{2x}(t-x)^2 & \text{for} \quad t > x. \end{cases}$$

h is continuous on $[0, \infty)$ and twice continuously differentiable on $(0; \infty)$ with second derivative

$$h''(t) = \begin{cases} \dfrac{1}{t} & \text{for} \quad t \leqq x \\ \dfrac{1}{x} & \text{for} \quad t > x. \end{cases}$$

An easy consideration gives that $S_n(r;\ x) \geqq r(x)$ for convex r, so, by our assumption, $S_n(h\pm g;\ x) \geqq (h\pm g)(x)$ and together with this $|S_n(g)-g| \leqq S_n(h)-h$. Thus it has remained to show that $S_n(h)-h \leqq \dfrac{K}{n}$ with a K independent of x. Using that S_n $S_n\big((t-x)^2;\ x\big)=x/n$ (see [1]) and that the linear functions are reproduced by S_n, this is equivalent to $|S_n(r;\ x)| \leqq \dfrac{K}{n}$, where

$$r(t) = \begin{cases} t\log t - t\log x - (t-x) & \text{for} \quad t \leqq x \\ 0 & \text{for} \quad t > x. \end{cases}$$

Now (with the convention $0\cdot\log 0=0$)

$$S_n(r;x) = \sum_{k=0}^{[nx]} \left[\frac{k}{n}\log\frac{k}{nx} - \left(\frac{k}{n}-x\right)\right] p_{n,k}(x) =$$

$$= \frac{1}{n}\sum_{k=0}^{[nx]} \left(k\log\frac{k}{nx} - (k-nx)\right) p_{nk}(x) \stackrel{\text{def}}{=} \frac{1}{n} R(nx).$$

Let $u=nx$. For $n<2$ clearly $|R(u)|\leqq K$, and for $u\geqq 2$ we have

$$R(u)=\left(\sum_{k=0}^{u-[u^{3/5}]}+\sum_{k=u-[u^{3/5}]+1}^{u}\right)\left(k\log\frac{k}{u}-(k-u)\right)e^{-u}\frac{u^k}{k!}=R_1(u)+R_2(u).$$

By (2.5)

$$|R_1(u)|\leqq Ku^2e^{-u^\eta}\leqq K.$$

For $u\geqq k>u-u^{3/5}$ let $\tau_k=u-k$. With this we have (see (2.4))

$$e^{-u}\frac{u^k}{k!}\leqq\frac{K}{\sqrt{u}}e^{-\tau_k^2/(2u)}$$

and

$$\left|k\log\frac{k}{u}-(k-u)\right|=\left|(u-\tau_k)\log\left(1-\frac{\tau_k}{u}\right)+\tau_k\right|=$$

$$=\left|(u-\tau_k)\left(\frac{-\tau_k}{u}+O\left(\left(\frac{\tau_k}{u}\right)^2\right)\right)+\tau_k\right|=O\left(\frac{\tau_k^2}{u}\right)$$

hence

$$|R_2(u)|\leqq K\sum_{l=0}^{[u^{3/5}]}\frac{(l+1)^2}{u^{3/2}}e^{-l^2/2u}\leqq K\sum_{i=0}^{\infty}\sum_{l=(2^i-1)[\sqrt{u}]}^{(2^{i+1}-1)[\sqrt{u}]}(\dots)\leqq$$

$$\leqq K\sum_{i=0}^{\infty}e^{-2^{2i-2}}2^{i+1}\sqrt{u}(2^{i+1}\sqrt{u})^2u^{-3/2}\leqq K$$

and collecting the above inequalities our proof is complete.

LEMMA 3. *If* $0<\alpha\leqq 1$, $f\in C_B$ *and* $\omega(\delta)\leqq K\delta^{2\alpha}$ *then* $|\Delta_h^2(f;0)|\leqq Kh^\alpha$.

PROOF. Let $v(\delta)=\sup\limits_{0<h\leqq\delta}|\Delta_h^2(f;0)|$. For an arbitrary $h>0$ let $\delta=h/2$. Using the functions f_δ introduced in (3.1) we have

$$|\Delta_h^2(f;0)|\leqq|\Delta_h^2(f_\delta;0)|+|f_\delta(0)-f(0)|+2|f_\delta(h)-f(h)|+|f_\delta(2h)-f(2h)|.$$

Here the last two terms are less than $K\omega(\sqrt{h})\leqq Kh^\alpha$, furthermore,

$$|f_\delta(0)-f(0)|=\left|\left(\frac{4}{h}\right)^2\iint_0^{h/4}\Delta^2_{t+s}(f;0)dsdt\right|\leqq v\left(\frac{h}{2}\right),$$

and since $|f_\delta''(t)|\leqq\dfrac{K}{h^2}\left(\dfrac{h}{\sqrt{t}}\right)^{2\alpha}$,

$$|\Delta_h^2(f_\delta;0)|=\left|\iint_0^h f_\delta''(t+s)dtds\right|\leqq K\frac{h^{2\alpha}}{h^2}\iint_0^h(t+s)^{-\alpha}dsdt\leqq Kh^{2\alpha-2}|\Delta_h^2(t^{2-\alpha};0)|\leqq$$

$$\leqq Kh^{2\alpha-2}h^{2-\alpha}\leqq Kh^\alpha$$

if $0<\alpha<1$ and

$$|\Delta_h^2(f_\delta;0)|\leqq K|\Delta_h^2(t\log t;0|\leqq Kh$$

if $\alpha=1$.

We have proved that $v(h) \leqq Kh^\alpha + v(h/2)$, which implies easily $v(h) \leqq Kh^\alpha$ and this is what we wanted to prove.

II. *Proof of* (i)$\Rightarrow$(ii) *in the nonoptimal case* $0<\alpha<1$. We use the ideas of the so called elementary method of inverse results developed by G. G. Lorentz, H. Berens, M. Becker and R. J. Nessel (see [1, 2, 3, 6]).

The essence of our result is contained in Lemma 4 below, by which the proof of (i)$\Rightarrow$(ii) is easy: Let $0<\delta, h \leqq 1$, and let us choose n so that $(n+1)^{-1/2}<\delta \leqq n^{-1/2}$ be satisfied. We have

$$|\Delta^2_{h\sqrt{x}}(f;x)| \leqq |\Delta^2_{h\sqrt{x}}(S_n(f);x)| + \Delta^2_{h\sqrt{x}}(S_n(f)-f;x) \leqq$$

$$\leqq \iint_0^{h\sqrt{x}} |S_n''(f;x+s+t)|\,ds\,dt + Kn^{-\alpha} \leqq K\left((n^{-\alpha}+h^2x\left(n^{-\alpha}+\omega\left(\frac{1}{\sqrt{n}}\right)\right)\frac{n}{x}\right)$$

i.e.

$$\omega(h) \leqq K\left(\delta^{2\alpha}+\frac{h^2}{\delta^2}\left((\delta^{2\alpha}+\omega(\delta)\right)\right).$$

Adding to this the trivial estimate $h^{2\alpha} \leqq \delta^{2\alpha}+\frac{h^2}{\delta^2}(\delta^{2\alpha}+\omega(\delta))$, we get for $\Omega(h)=h^{2\alpha}+\omega(h)$ the inequality

$$\Omega(h) \leqq K\left(\delta^{2\alpha}+\frac{h^2}{\delta^2}\,\Omega(\delta)\right)$$

which already implies $\Omega(h)=O(h^{2\alpha})$ (see [3, Lemma]) and this is exactly (ii).

Thus, it is left to prove

LEMMA 4. *If* $f \in C_B$ *and* $S_n(f)-f=O(n^{-\alpha})$ $(0<\alpha \leqq 1)$ *then*

$$|S_n''(f;x)| \leqq K\frac{n}{x}\left(n^{-\alpha}+\omega\left(\frac{1}{\sqrt{n}}\right)\right) \quad (x>0,\ n=1,2,\ldots).$$

PROOF. Let $x>0$, $\delta=\sqrt{\dfrac{x}{n}}$. With the function (3.1) we have

$$|S_n''(f;x)| \leqq |S_n''(f-f_\delta;x)|+|S_n''(f_\delta;x)|. \tag{3.4}$$

We shall estimate the terms on the right hand side separately.

a) Using (2.3) we have (if $x<1/n$ then $\Sigma_2=0$ below)

$$|S_n''(f-f_\delta;x)| \leqq \frac{n^2}{x^2}\sum_{k=0}^{\infty}|r_{n,k}(x)|\left|f\left(\frac{k}{n}\right)-f_\delta\left(\frac{k}{n}\right)\right|p_{n,k}(x) =$$

$$=\sum_{k=0}+\sum_{k=1}^{[nx]}+\sum_{k=[nx]+1}^{\infty}=\Sigma_1+\Sigma_2+\Sigma_3.$$

α) To estimate Σ_1 we prove first

LEMMA 5. *If* $f \in C_B$ *and* $S_n(f)-f=O(n^{-\alpha})$ $(0<\alpha \leqq 1)$ *then*

$$|\Delta_h^2(f;0)| \leqq K(h^\alpha+\omega(\sqrt{h})) \quad (h>0).$$

PROOF. Using the expression of S_n given in (2.1) we get

$$S_n\left(f;\frac{1}{n}\right)-f\left(\frac{1}{n}\right)=\left\{f(0)+\left(f\left(\frac{1}{n}\right)-f(0)\right)+\sum_{k=2}^{\infty}\Delta_{\frac{1}{n}}^{k}(f;0)\frac{1}{k!}\right\}-f\left(\frac{1}{n}\right)=$$

$$=\sum_{k=2}^{\infty}\frac{1}{k!}\sum_{i=0}^{k-2}(-1)^{k-i}\binom{k-2}{i}\Delta_{\frac{1}{n}}^{2}\left(f;\frac{i}{n}\right)=\Delta_{\frac{1}{n}}^{2}(f;0)\left(\frac{1}{2!}-\frac{1}{3!}+\dots\right)+$$

$$+\sum_{k=2}^{\infty}\frac{1}{k!}\sum_{i=1}^{k-2}(-1)^{k-i}\binom{k-2}{i}\Delta_{\frac{1}{n}}^{2}\left(f;\frac{i}{n}\right)=e^{-1}\Delta_{\frac{1}{n}}^{2}(f:0)+$$

$$+O\left(\omega\left(\frac{1}{\sqrt{n}}\right)\sum_{k=2}^{\infty}\frac{1}{k!}\sum_{i=1}^{k-2}\binom{k-2}{i}\right)=e^{-1}\Delta_{\frac{1}{n}}^{2}(f:0)+O\left(\omega\left(\frac{1}{\sqrt{n}}\right)\right)$$

and since the left hand side is $O(n^{-\alpha})$ we get

$$|\Delta_{\frac{1}{n}}^{2}(f;0)|\leq K\left(n^{-\alpha}+\omega\left(\frac{1}{\sqrt{n}}\right)\right)\quad (n=1,2,\dots). \tag{3.5}$$

Let now $0<h\leq 1$ be arbitrary, and let n be defined by $\frac{1}{n+1}<h\leq\frac{1}{n}$. Then

$$|\Delta_{h}^{2}(f;0)-\Delta_{\frac{1}{n}}^{2}(f;0)|\leq 2\left|f\left(\frac{1}{n}\right)-f(h)\right|+\left|f\left(\frac{2}{n}\right)-f(2h)\right| \tag{3.6}$$

and here (see (3.2))

$$\left|f\left(\frac{1}{n}\right)-f(h)\right|\leq\left|f\left(\frac{1}{n}\right)-f_{\frac{1}{n}}\left(\frac{1}{n}\right)\right|+|f(h)-f_{\frac{1}{n}}(h)|+\left|f_{\frac{1}{n}}\left(\frac{1}{n}\right)-f_{\frac{1}{n}}(h)\right|\leq$$

$$\leq\omega\left(\frac{1}{\sqrt{n}}\right)+K\omega\left(\frac{1}{\sqrt{n}}\right)+\left|\frac{1}{n}-h\right|\left|f_{\frac{1}{n}}'(\xi)\right|$$

for some $\xi\in\left(\frac{1}{n+1};\,\frac{1}{n}\right)$. Since $f_{\frac{1}{n}}$ is bounded, $|f_{\frac{1}{n}}'(x)|\leq Kn$ for all $x\geq 0$, by which $\left|f_{\frac{1}{n}}\left(\frac{1}{n}\right)-f_{\frac{1}{n}}(h)\right|\leq Kn^{-1}\leq Kh$, i.e. $\left|f\left(\frac{1}{n}\right)-f(h)\right|\leq K(h+\omega(\sqrt{h}))$. A similar estimate can be given on the second term in (3.6), and we get the desired result from (3.5)—(3.6).

Let us turn to Σ_1. Using that $r_{n,0}(x)=x^2$, we obtain from Lemma 5

$$\Sigma_1\leq Kn^2(\delta+\omega(\sqrt{\delta}))e^{-nx}=\frac{n}{x}n^{-\alpha}(nx)^{1+\alpha/2}e^{-nx}+$$

$$+\omega\left(\frac{1}{\sqrt{n}}\sqrt[4]{nx}\right)n^2e^{-nx}\leq K\frac{n}{x}\left(n^{-\alpha}+\omega\left(\frac{1}{\sqrt{n}}\right)\right)(1+nx)^2e^{-nx}\leq K\frac{n}{x}\left(n^{-\alpha}+\omega\left(\frac{1}{\sqrt{n}}\right)\right).$$

β) In Σ_2 we have

$$\left|f\left(\frac{k}{n}\right)-f_\delta\left(\frac{k}{n}\right)\right| \leqq K\omega\left(\sqrt{\frac{x}{k}}\right) \leqq K\omega\left(\frac{1}{\sqrt{n}}\right)\frac{nx}{k} \quad (1 \leqq k \leqq nx),$$

hence

$$\Sigma_2 \leqq K\frac{n^2}{x^2}\,\omega\left(\frac{1}{\sqrt{n}}\right)\sum_{k=1}^{[nx]}\frac{nx}{k}\,|r_{n,k}(x)|p_{n,k}(x) = \sum_{k=1}^{[nx/2]} + \sum_{j=[nx/2]+1}^{[nx]} \leqq$$

$$\leqq K\frac{n^2}{x^2}\,\omega\left(\frac{1}{\sqrt{n}}\right)\left[x^2e^{-\tau nx}\,nx+2S_n((t-x)^2;x)+\frac{x}{n}\sum_{k=1}^{[nx]}p_{n,k}(x)\right] \leqq$$

$$\leqq K\frac{n^2}{x^2}\,\omega\left(\frac{1}{\sqrt{n}}\right)\left[\frac{x}{n}\left(1+(nx)^2e^{-\tau nx}\right)\right] \leqq K\frac{n}{x}\,\omega\left(\frac{1}{\sqrt{n}}\right)$$

where we used (2.6), and the relations

$$S_n((t-x)^2;x) = \frac{x}{n}, \quad |\tau_{n,k}(x)| \leqq \left(\frac{k}{n}-x\right)^2+\frac{k}{n^2}.$$

γ) In Σ_3 we have (see β)) $\left|f\left(\frac{k}{n}\right)-f_\delta\left(\frac{k}{n}\right)\right| \leqq \omega\left(\frac{1}{\sqrt{n}}\right)$ by which

$$\Sigma_3 \leqq \frac{n^2}{x^2}\,\omega\left(\frac{1}{\sqrt{n}}\right)\sum_{k=[nx]+1}^{\infty}|r_{n,k}(x)|p_{n,k}(x) \leqq \frac{n^2}{x^2}\,\omega\left(\frac{1}{\sqrt{n}}\right)\left[S_n((t-x)^2;x)+\right.$$

$$\left.+\sum_{k=[nx]+1}^{\infty}\frac{k}{n^2}\,p_{n,k}(x)\right] \leqq \frac{n^2}{x^2}\,\omega\left(\frac{1}{\sqrt{n}}\right)\left[\frac{x}{n}+\frac{x}{n}\sum_{k=[nx]+1}^{\infty}p_{n,k-1}(x)\right] \leqq 2\frac{n}{x}\,\omega\left(\frac{1}{\sqrt{n}}\right).$$

b) Using (2.2) we have $\left(\text{again, } \Sigma_2=0 \text{ if } x<\frac{1}{n}\right)$

$$S_n''(f_\delta;x) = n^2\sum_{k=0}^{\infty}\Delta_{\frac{1}{n}}^2\left(f_\delta;\frac{k}{n}\right)p_{n,k}(x) = \sum_{k=0}+\sum_{k=1}^{[nx]}+\sum_{k=[nx]+1}^{\infty} = \Sigma_1+\Sigma_2+\Sigma_3.$$

α) By Lemma 5

$$|\Sigma_1| \leqq e^{-nx}\left(n^2|\Delta_{\frac{1}{n}}^2(f;0)|+n^2|\Delta_{\frac{1}{n}}^2(f-f_\delta;0)|\right) \leqq K\left\{n^2\left(n^{-\alpha}+\omega\left(\frac{1}{\sqrt{n}}\right)\right)\cdot\right.$$

$$\left.\cdot\, e^{-nx}+n^2|f(0)-f_\delta(0)|e^{-nx}+3n^2\omega(\sqrt{x})e^{-nx}\right\} \leqq$$

$$\leqq K\frac{n}{x}\left(n^{-\alpha}+\omega\left(\frac{1}{\sqrt{n}}\right)\right)\left\{(nx)e^{-nx}+(nx)^{1+\alpha/2}e^{-nx}+\left(1+(nx)\right)^2e^{-nx}\right\} \leqq$$

$$\leqq K\frac{n}{x}\left(n^{-\alpha}+\omega\left(\frac{1}{\sqrt{n}}\right)\right).$$

β) In Σ_2 we have

$$\left|\Delta^2_{\frac{1}{n}}\left(f_\delta;\frac{k}{n}\right)\right| \leqq \iint_0^{1/n} \left|f''_\delta\left(\frac{k}{n}+u+v\right)\right| dudv \leqq \iint_0^{1/n} \delta^{-2}\left(8\left|\Delta^2_{\frac{\delta}{2}}\left(f;\frac{k}{n}+u+v\right)\right|+\right.$$

$$\left.+\left|\Delta^2_\delta\left(f;\frac{k}{n}+u+v\right)\right|\right) dudv \leqq \frac{K}{n^2}\delta^{-2}\omega\left(\sqrt{\frac{x}{k}}\right) \leqq \frac{K}{k}\,\omega\left(\frac{1}{\sqrt{n}}\right) \quad (1 \leqq k \leqq nx);$$

hence

$$|\Sigma_2| \leqq K\frac{n}{x}\,\omega\left(\frac{1}{\sqrt{n}}\right)\sum_{k=1}^{[nx]}\frac{nx}{k+1}\,p_{n,k}(x) \leqq K\frac{n}{x}\,\omega\left(\frac{1}{\sqrt{n}}\right).$$

γ) For Σ_3 we get similarly

$$|\Sigma_3| \leqq \frac{n}{x}\,\omega\left(\frac{1}{\sqrt{n}}\right)\sum_{k=[nx]+1}^{\infty} p_{nk}(x) \leqq \frac{n}{x}\,\omega\left(\frac{1}{\sqrt{n}}\right).$$

(3.4) and a) b) prove Lemma 4.

III. *Proof of* (i)$\Rightarrow$(ii) *in the saturation case* $\alpha=1$. First we prove

LEMMA 6. *For the function* $h(x)=x\log x$ *we have*

$$S_n(h;\,x)-h(x) \geqq C\frac{1}{n} \quad \textit{for} \quad x \leqq \frac{1}{n}, \quad \textit{where} \quad C>0 \quad \textit{is a constant.}$$

PROOF. A simple computation gives

$$S_n(h;x)-h(x) = x\sum_{k=0}^{\infty}\left(\log\frac{k+1}{n}-\log x\right)p_{n,k}(x) =$$

$$= \frac{1}{n}\,u\sum_{k=0}^{\infty}\left(\frac{k}{u}\log\frac{k}{u}\right)e^{-u}\frac{u^k}{k!} = \frac{1}{n}\,uT(u)$$

with $u=nx$. Thus, it is enough to show that $uT(u)\geqq C$ for $u\geqq 1$.

$T(u)$ is the y-coordinate of the weight point of the weight distribution: the weight $p_k=p_k(u)=e^{-u}\dfrac{u^k}{k!}$ is placed in the point $\left(k;\,\dfrac{k}{u}\log\dfrac{k}{u}\right)$. Since for large u and $|k-u|\leqq\sqrt{u}$ we have (see (2.4))

$$p_k = \frac{1}{\sqrt{2\pi u}}\,e^{-|[u]-k|^2/(2u)}\left(1+O\left(\frac{|u-k|+1}{u}\right)+O\left(\frac{|u-k|^3}{u^2}\right)\right),$$

an easy consideration gives the following: there are constants $0<\beta,\ \gamma<1$ independent of u such that for large enough u we can replace the weight-system $\{p_k\}$ by two such systems $\{p_k^{(1)}\}$, $\{p_k^{(2)}\}$, i.e. $p_k=p_k^{(1)}+p_k^{(2)}$, $p_k^{(1)}$, $p_k^{(2)}\geqq 0$ in such a way that $\sum_{k=0}^{\infty}p_k^{(1)}=\gamma$, $\gamma^{-1}\sum_{k=0}^{\infty}kp_k^{(1)}=u$ and $p_k^{(2)}=0$ for $|k-u|\leqq\beta\sqrt{u}$ be satisfied. Then we have automatically $(1-\gamma)^{-1}\sum_{k=0}^{\infty}kp_k^{(2)}=u$, i.e. the weight points S, $S^{(1)}$, $S^{(2)}$

of the weight-systems $\{p_k\}$, $\{p_k^{(1)}\}$, $\{p_k^{(2)}\}$ have u as their first coordinate. Let $S=(u;\ T(u))$, $S^{(1)}=(u;\ T^{(1)}(u))$, $S^{(2)}=(u;\ T^{(2)}(u))$. We have $T(u)=\gamma T^{(1)}(u)+(1-\gamma)T^{(2)}(u)$, and by the convexity of the function $h_u(x)=\frac{x}{u}\log\frac{x}{u}$ we have $T^{(1)}(u)\geqq 0$ and $T^2(u)\geqq\frac{1}{2}\left(h_u(u+\beta\sqrt{u})+h_u(u-\beta\sqrt{u})\right)$ since the positive weights in $\{p_k^{(2)}\}$ are all above the line joining the points $(u-\beta\sqrt{u};\ h_u(u-\beta\sqrt{u}))$ and $(u+\beta\sqrt{u};\ h_u(u+\beta\sqrt{u}))$. Since

$$h_u(u+\beta\sqrt{u})+h_u(u-\beta\sqrt{u})=\log\left(1-\frac{\beta^2}{u}\right)+\frac{\beta}{\sqrt{u}}\log\left(1+\frac{2\beta}{\sqrt{u}-\beta}\right)\geqq\frac{1}{2}\frac{\beta^2}{u}$$

for large u, we have for $u\geqq u_0$ $uT(u)\geqq\frac{1}{4}\beta^2$. However, $h_u(x)$ is strictly convex in x and continuous in u which give easily that $uT(u)\geqq c$ also for $1\leqq u\leqq u_0$ with a constant $c>0$.

We have proved our lemma.

After this we turn to the proof of the implication (i)$\Rightarrow$(ii). We shall use an idea of A. Grundmann (see [1]), namely the possibility of the application of a convexity argument.

Since $S_n(f)-f=O(n^{-1})$, Lemma 6 gives that with a suitably chosen constant C we have for the functions

$$f^{\pm}(x)=Cx\log x\pm f(x)$$

the inequalities (see also the statement and the proof of Lemma 2)

$$S_n(f^{\pm};x)\geqq f^{\pm}(x)\quad\text{for}\quad x\geqq\frac{1}{n};\ S_n(f^{\pm})-f^{\pm}=O(n^{-1}).$$

Now at this point we could use the ideas of C. A. Michelli to conclude that the functions $f^{\pm}$ are convex but for the sake of completeness and also to remain on an elementary level we prove this in

LEMMA 7. *If* $f(x)=Kx\log x+f_1$ *where* $f_1\in C_B$, $S_n(f;\ x)\geqq f(x)$ *for* $x\geqq\frac{1}{n}$ *and* $S_n(f)-f=O\left(\frac{1}{n}\right)$ *then* f *is convex.*

Assuming the validity of this lemma the proof can be completed as follows. The convexity of f^+ gives $\Delta^2_{h\sqrt{x}}(f^+;\ x)\geqq 0$, i.e.

$$\Delta^2_{h\sqrt{x}}(f;x)\geqq -c\Delta^3_{h\sqrt{x}}(t\log t;x)\geqq -ch^2.$$

Similarly, from f^- we get $\Delta^2_{h\sqrt{x}}(f;\ x)\leqq ch^2$ and these together give (ii) for $\alpha=1$.

It is left to prove Lemma 7. Let $[a, b]$ be an arbitrary finite interval with $a>0$. It is enough to show that f is convex in (a, b). By $S_n(f)-f=O(n^{-1})$, $n(S_n(f)-f)$ is a uniformly bounded sequence on $[a, b]$ so, by weak compactness, there is a $g\in L^\infty[a, b]$ and a subsequence $\{n_k\}$ for which $n_k(S_{n_k}(f)-f)$ converges weakly to

g on $[a, b]$. Since $S_n(f)-f\geqq 0$ on $[a, b]$ for all large n, g is non-negative. Let F be a primitive function of f. By the above proved part of our theorem $S_n(f)-f=O(n^{-1})$ implies $\omega(\delta)=O(\delta^{3/2})$ i.e. $\Delta_h^2(f; x)=O((h/\sqrt{a})^{3/2})$ on $\left[\frac{a}{2}, 2b\right]$ and so (as it is well-known) f is continuously differentiable and $f'\in\operatorname{Lip}\frac{1}{2}$, $|f'|\leqq Ka^{-3/4}$ on $\left[\frac{a}{2}, 2b\right]$. By a simple Voronowskaja-type result $n(S_n(F; x)-F(x))\to\frac{x}{2}f'(x)$ uniformly on $[a, b]$. Finally, let

$$K_n(f; x)=\sum_{k=0}^{\infty}\left(n\int_{k/n}^{(k+1)/n} f(u)du\right)p_{n,k}(x)$$

be the associated Kantorowich operator. Since $f'\in\operatorname{Lip}\frac{1}{2}$ on $\left[\frac{a}{2}, 2b\right]$ we have

$$f(\tau)=f\left(\frac{k}{n}\right)+f'\left(\frac{k}{n}\right)\left(\tau-\frac{k}{n}\right)+O\left(n^{-\frac{1}{2}}\left(\tau-\frac{k}{n}\right)\right)\quad\left(\tau\in\left(\frac{k}{n}; \frac{k+1}{n}\right)\right)$$

and so

$$n\int_{k/n}^{(k+1)/n} f(u)du=f\left(\frac{k}{n}\right)+\frac{1}{2n}f'\left(\frac{k}{n}\right)+O(n^{-3/2})\quad\left(\frac{k}{n}\in\left(\frac{a}{2}, 2b\right)\right)$$

by which, using (2.6), we get that $n(K_n(f; x)-S_n(f; x))-\frac{1}{2}S_n(f'; x)$ tends uniformly to 0 on $\left[\frac{3}{4}a; \frac{3}{2}b\right]$, and since f' is continuous in $[a, b]$ and $|f'(x)|\leqq\leqq Kx^{-3/4}$ for all x we get easily that $n(K_n(f; x)-S_n(f; x))$ tends uniformly to $\frac{1}{2}f'(x)$ on $[a, b]$.

Let now h be any continuously differentiable function with compact support in (a, b). Integration by parts yields

$$\int_a^b\left(\frac{h'(x)}{x}-\frac{h(x)}{x^2}\right)n(S_n(F; x)-F(x)dx=-\int_a^b\frac{h(x)}{x}n(S_n(f; x)-f(x))dx-$$

$$-\int_a^b\frac{h(x)}{x}n(K_n(f; x)-S_n(f; x))dx.$$

Putting here $n=n_k$ and letting k tend to the infinity we obtain by the above considerations

$$\int_a^b\left(\frac{h'(x)}{x}-\frac{h(x)}{x^2}\right)\frac{1}{2}xf'(x)dx=-\int_a^b\frac{h(x)}{x}g(x)dx-\int_a^b\frac{h(x)}{x}\frac{1}{2}f'(x)dx,$$

i. e.

$$\int_a^b h'(x)f'(x)dx=-\int_a^b h(x)\frac{2g(x)}{x}dx=\int_a^b h'(x)G(x)dx$$

where $G(x)$ denotes a primitive function of $\frac{2g(x)}{x}$. Since $g(x) \geqq 0$, G is increasing, and since the above equation holds for all h of the above kind we can deduce easily that $f'(x) = G(x) + c$ on $[a, b]$ with a constant c, i.e. f' is increasing, thus f is convex.

The proof of Theorem 2 is complete.

§ 4. The proof of Theorems 1 and 3—6

PROOF OF THEOREM 1. (ii)$\Rightarrow$(i). Since $S_n(f;\, x) - f(x) = O(1)$ uniformly on finite intervals, we may assume $x \geqq 1$. But for $x \geqq 1$ the proof of Theorem 2 gives $S_n(f;\, x) - f(x) = O\left(\omega\left(\frac{1}{\sqrt{n}}\right)\right) = o(1)$ uniformly. Thus (ii) implies (i).

(i)$\Rightarrow$(ii). We follow the proof of (i)$\Rightarrow$(ii) in Theorem 1 in the case $0 < \alpha < 1$. Again, by the continuity of f we may assume in the estimation of $\Delta^2_{h\sqrt{x}}(f;\, x)$ that $x \geqq 1$. Let $0 < h, \delta < 1$ and $(n+1)^{-1/2} < \delta \leqq n^{-1/2}$. Let $o_\delta(1)$ denote a quantity which tends to zero together with δ. By Part II of § 3 we have

$$|\Delta^2_{h\sqrt{x}}(f;\, x)| \leqq o_{\frac{1}{n}}(1) + \int_0^{h\sqrt{x}}\!\!\int |S_n''(f;\, x+s+t)|\, ds\, dt.$$

The proof of Lemma 4 gives for $x \geqq 1$

$$|S_n''(f;\, x)| \leqq K\left(e^{-nx} n^2 + \frac{n}{x}\,\omega\left(\frac{1}{\sqrt{n}}\right)\right) \leqq K\frac{n}{x}\left(o_{\frac{1}{n}}(1) + \omega\left(\frac{1}{\sqrt{n}}\right)\right)$$

and so

$$\omega(h) \leqq o_\delta(1) + K\left(\frac{h}{\delta}\right)^2 \left(o_\delta(1) + \omega(\delta)\right). \tag{4.1}$$

Since the boundedness of f gives $\omega(1) < \infty$, (4.1) implies easily $\omega(h) = o_h(1)$.

(ii)$\Rightarrow$(iii) is standard: it is enough to consider the case $x \geqq 1$, and let m and M be defined by $2^m \leqq x < 2^{m+1}$ and $M = h\sqrt{2^m}/\sqrt{\omega(h)}$. For $g(t) = f_{h\sqrt{2^m}}(t)$ (see (3.1)) we have

$$|g(x) - f(x)| + |g(x+h\sqrt{x}) - f(x+h\sqrt{x})| \leqq K\omega(h).$$

By the boundedness of f we obtain for some $\xi \in (x, x+M)$

$$|g'(\xi)| = \left|\frac{g(x+M) - g(x)}{M}\right| \leqq \frac{2\sup|f|}{M} \leqq \frac{K}{M}$$

and since

$$g''(t) \leqq \frac{K}{(h\sqrt{2^m})^2}\left(|\Delta^2_{\frac{h\sqrt{2^m}}{2}}(f;\, t)| + |\Delta^2_{h\sqrt{2^m}}(f;\, t)|\right) \leqq \frac{K}{h^2 2^m}\,\omega\left(\frac{h\sqrt{2^m}}{\sqrt{t}}\right) \leqq \frac{K}{h^2 2^m}\,\omega(h) (t \geqq x)$$

we have

$$|g'(t)| \leqq |g'(\xi)| + \left|\int_\xi^t g''(\tau)\, d\tau\right| \leqq \frac{K}{M} + K\frac{\omega(h)}{2^m h^2} M \quad (t \in (x, x+M))$$

and so

$$|g(x)-g(x+h\sqrt{x})| = \left|\int\limits_{x}^{x+h\sqrt{x}} g'(t)dt\right| \leqq h\sqrt{x}\left(\frac{K}{M}+K\frac{\omega(h)}{2^m h^2}M\right) \leqq K\sqrt{\omega(h)}$$

which proves (iii).

The other implications in Theorem 1 can be proved very easily.

PROOF OF THEOREM 3. (ii) implies easily the uniform continuity of $f\in C_B$ (see Theorem 1) by which $S_n(f)-K_n(f)=o(1)$ uniformly, thus Theorem 1 can be applied.

Conversely, if $K_n(f)-f=o(1)$, then, since $|K_n'(f;\,x)|\leqq 2n\sup|f|$ we have that f is uniformly continuous and we can apply again Theorem 1 to derive (ii).

PROOF OF THEOREM 4. **I.** *Proof of* (ii)$\Rightarrow$(i). We show that (ii) implies $|f(x+h)-$ $-f(x)|\leqq Kh^\alpha$ uniformly in x, by which $K_n(f)-S_n(f)=O(n^{-\alpha})$ and (i) follows from Theorem 2.

Let first $0<\alpha<1$. For $0<x<h$ (ii) (2) gives the desired estimate, and on the interval (h,∞) we get from (ii) (1) $\Delta_h^2(f;\,x)\leqq K\omega(\sqrt{h})\leqq Kh^\alpha$ and it is well-known that this implies $|f(x+h)-f(x)|\leqq Kh^\alpha$ $(0<\alpha<1)$.

For $\alpha=1$ we argue as follows. It is well-known that (ii) (1) implies that f has an absolutely continuous derivatitive on $(0;\infty)$ with $|f''(x)|\leqq K/x$. (see also § 3, III). (ii) (2) gives $|f(t)-f(0)|\leqq Kh$ for $h\leqq t\leqq 2h$ by which $|f'(\xi)|\leqq K$ for some $\xi\in$ $\xi\in(h,2h)$. Since $|f''(t)|\leqq K/h$ for $t\in(h,2h)$, we can infer $|f'(t)|\leqq K$ $(h\leqq t\leqq 2h)$ and hence $|f'(t)|\leqq K$ $(t>0)$ $(h>0$ was arbitrary) and the proof is complete.

II. *Proof of* (i)$\Rightarrow$(ii) *in the non-optimal case* $0<\alpha<1$. To prove (ii) (1), we follow the argument of § 3, II. If we write there K_n instead of S_n and apply Lemma 8 below instead of Lemma 5, everything remains valid. (ii) (2) follows from (ii) (1) and Lemma 8.

LEMMA 8. *If* $0<\alpha\leqq 1$ *and for* $f\in C_B$ *we have* $K_n(f)-f=O(n^{-\alpha})$ *then* $|f(0)-$ $-f(h)|\leqq K(h^\alpha+\omega(\sqrt{h}))$ $(h>0)$.

PROOF. Let $F_{n,k}=n\int\limits_{k/n}^{(k+1)/n} f(u)du$ and $\Delta_{n,k}=F_{n,k}-2F_{n,k+1}+F_{n,k+2}$. Exactly as in the proof of Lemma 5 we get that

$$K_n\left(f;\frac{1}{n}\right) = F_{n,1}+\Delta_{n0}\left(\frac{1}{2!}-\frac{1}{3!}+\dots\right)+\sum_{k=2}^{\infty}\frac{1}{k!}\sum_{i=1}^{k-2}(-1)^{k-i}\binom{k-2}{i}\Delta_{n,i}$$

and so, using our assumption, we can infer the relations

$$A_n\colon f(0) = F_{n,0}+O(n^{-\alpha}),\quad B_n\colon f\left(\frac{1}{n}\right) = F_{n,1}+\Delta_{n,0}e^{-1}+O\left(n^{-\alpha}+\omega\left(\frac{1}{\sqrt{n}}\right)\right).$$

The expression of $\frac{1}{2}A_{2n}-A_{3n}+\frac{1}{2}A_{6n}$ gives $0=n\left(\int\limits_0^{3/6n}-3\int\limits_0^{2/6n}+3\int\limits_0^{1/6n}\right)f(u)du+$ $+O(n^{-\alpha})=\Delta_{6n,0}+O(n^{-\alpha})$ and $A_{3n}-A_{6n}$ gives that $F_{6n,0}-F_{6n,1}=O(n^{-\alpha})$. Using these we get from $B_{6n}-A_{6n}$ the estimate $f(0)-f\left(\frac{1}{n}\right)\leqq K\left(n^{-\alpha}+\omega\left(\frac{1}{\sqrt{n}}\right)\right)$ by which we can argue for arbitrary $h>0$ as in Lemma 5.

III. *Proof of* (i)⇒(ii) (1) *in the saturation case* $\alpha=1$. Above we showed that $\omega(\delta)=O(\delta^{3/2})$, and so (see § 3, III)

$$S_n(f;x)-f(x)=\big(K_n(f;x)-f(x)\big)+\big(S_n(f;x)-K_n(f;x)\big)=O\left(\frac{1}{n}\right)$$

uniformly on each interval $[a,b]$ with $0<a<b<\infty$. Hence $\Delta_h^2(f;x)\leqq K_{a,b}h^2(x\in(a,b))$ with a constant $K_{a,b}$ depending on a and b (see the proof in § 3, III which, using (2.6), works also locally). It is well-known that the last relation implies that f has an absolutely continuous derivative f' on (a,b), and since a,b were arbitrary, f' is absolutely continuous on $(0,\infty)$.

By our assumption $n(K_n(f)-f)$ is bounded, and

$$n(K_n(f;\,x)-f(x))=n(S_n(f;\,x)-f(x))+n(K_n(f;\,x)-S_n(f;\,x))$$

tends to $\dfrac{1}{2}xf''(x)+\dfrac{1}{2}f'(x)$ almost everywhere (see § 3, III and the fact that $n(S_n(f;\,x)-f(x))$ tends to $\dfrac{1}{2}xf''(x)$ a.e.), thus $xf''(x)+f'(x)=(x\cdot f'(x))'=$ $=g(x)$ is bounded. Since this gives that $xf'(x)=c+\int\limits_0^x g(\tau)\,d\tau$ i.e. $f'(x)=\dfrac{c}{x}+$ $+\dfrac{1}{x}\int\limits_0^x g(\tau)\,d\tau$, and since here the last term is bounded and f is continuous at 0 we get $c=0$ and together with this the boundedness of f'. Thus $xf''(x)=g(x)-$ $-f'(x)$ is also bounded and an application of Lemma 2 and Theorem 2 (i)⇒(ii) gives the desired result.

For the proof of Theorem 5 see that of Theorems 1 and 6 below.

Proof of Theorem 6. First of all let us notice that (ii) is equivalent to the fact that (1.1) holds for the functions $f(\cdot,y)$ and $f(x,\cdot)$ uniformly in y and x, respectively. In fact, putting $h_2=0$ into (ii) we obtain (1.1) for $f(\cdot,y)$, and conversely, by adding the inequalities

$$\Delta^2_{h_1\sqrt{x}}(f(\cdot,y+h_2\sqrt{y}\pm h_2\sqrt{y});x)=O(h_1^{2\alpha}),\quad 2\Delta^2_{h_1\sqrt{x}}(f(\cdot,y+h_2\sqrt{y});x)=O(h_1^{2\alpha}),$$

$$\Delta^2_{h_1\sqrt{y}}(f(x+h_1\sqrt{x}\pm h_1\sqrt{x};\cdot);y)=O(h_2^{2\alpha}),\quad 2\Delta^2_{h_2\sqrt{y}}(f(x+h_1\sqrt{x};\cdot);y)=O(h_2^{2\alpha})$$

we get (ii).

Thus, (ii)⇒(i) follows easily from Theorem 2, since

$$S_{n,m}(f;x,y)-f(x,y)=S_n(f(\cdot,y);x)-f(x,y)+S_n(S_m(f(t;\cdot);y)-$$
$$-f(t;y);x)=O(n^{-\alpha}+m^{-\alpha}).$$

To prove (i)⇒(ii), for a fixed n let m tend to the infinity in (i). Since f is continuous and S_m cowerges for bounded continuous functions, we get for each i

$$\lim_{m\to\infty}\sum_{j=5}^{\infty}f\left(\frac{i}{n},\frac{j}{m}\right)e^{-my}\frac{(my)^j}{j!}=f\left(\frac{i}{n};y\right).$$

This gives by

$$S_n(f(\cdot\,;y);x)-f(x,y)=\sum_{i=0}^{[2nx+n\sqrt{n}]}\left(f\left(\frac{i}{n},y\right)-f(x,y)\right)p_{n,i}(x)+O(n^{-1})$$

(in fact, (2.6) gives $\sum_{k=[2nx]}^{\infty} p_{n,k}(x)=O(e^{-\tau nx})=O(n^{-1})$ for $x\geqq 1/\sqrt{n}$ and $\big((nx)^{i+1}/$ $/(i+1)!\big)/\big((nx)^i/i!\big)\leqq\sqrt{n}/i$ for $0\leqq x\leqq 1\sqrt{n}$ by which $\sum_{k=[n\sqrt{n}]}^{\infty} p_{n,k}(x)\leqq K/n$)

$$\left|S_n(f(\cdot\,,y);x)-f(x,y)\right|=\left|\lim_{m\to\infty}\sum_{i=0}^{[2nx+n\sqrt{n}]}\sum_{j=0}^{\infty}f\left(\frac{i}{n},\frac{j}{m}\right)e^{-nx-my}\cdot\right.$$

$$\left.\cdot\frac{(nx)^i}{i!}\frac{(my)^j}{j!}-f(x,y)+O(n^{-1})\right|=\left|\lim_{m\to\infty}\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\right)-f(x,y)+O(n^{-1})\right|\leqq$$

$$\leqq K\lim_{m\to\infty}(n^{-\alpha}+m^{-\alpha}+n^{-1})\leqq Kn^{-\alpha}$$

and so we have, by Theorem 2, that $f(\cdot\,,y)$ satisfies (1.1) uniformly in y.

We have completed our proof.

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(Received September 29, 1981)

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Acta Math. Hung.
41 (3—4) (1983), 309—324.

ON ALMOST DIVISIBILITY PROPERTIES OF SEQUENCES OF INTEGERS. I

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1. Throughout this paper we put $e^{2\pi i\alpha}=e(\alpha)$. We write $\{\alpha\}=\alpha-[\alpha]$ and $\|\alpha\|=\min(\{\alpha\}, 1-\{\alpha\})$ (i.e., $\|\alpha\|$ denotes the distance from α to the nearest integer). $c, c_1, c_2, \dots$ denote positive absolute constants.

We may say that the positive real number b is *almost divisible* by the positive real number a if $\left\|\frac{b}{a}\right\|$ is "small". More exactly, we may say that if $\varepsilon>0$, $\left\|\frac{b}{a}\right\|<\varepsilon$ then b is *ε-divisible* by a and a is an *ε-divisor* of b; in this case, we write $a|_\varepsilon b$.

The aim of this series is to study the ε-divisibility properties of sequences of integers. In particular, in this paper we study *the ε-divisibility by consecutive integers.*

2. In Section 3, we show that if t is not much greater than n, then there exists an integer j such that

$$1 \leqq j \leqq n \tag{1}$$

and $(n+j)|_\varepsilon t$. In fact, Theorem 2 in Section 3 contains this assertion in a sharper form, namely the interval (1) is replaced there by a smaller interval of the form

$$1 \leqq j \leqq P(n, t) \tag{2}$$

(where $P(n, t)$ is much less than n).

Theorem 2 will be derived from Theorem 1 below; this section is devoted to the proof of Theorem 1.

THEOREM 1. *There exists a positive absolute constant c_1 such that the following assertion holds:*

Let $\varepsilon>0$, n a positive integer satisfying $n>n_0(\varepsilon)$, t a real number such that

$$n^2 \leqq t < \exp\left(\frac{(\log n)^{5/4}}{\log\log n}\right). \tag{3}$$

Let us write

$$k=\begin{cases}\left[2\dfrac{\log t}{\log n}\right]-3 & \text{if } 2\leqq \dfrac{\log t}{\log n}<c_1\\[2ex] \left[\dfrac{\log t}{\log n}+\dfrac{1}{2}\right] & \text{if } \dfrac{\log t}{\log n}\geqq c_1,\end{cases} \tag{4}$$

$$P=\begin{cases}[n^{1-1/2^{k+2}}] & \text{if } 2\leqq \dfrac{\log t}{\log n}<c_1\\[2ex] \left[\left(\dfrac{n^{k+5/2}}{t}\right)^{1/(k+2)}\right] & \text{if } \dfrac{\log t}{\log n}\geqq c_1\end{cases} \tag{5}$$

(note that for $\dfrac{\log t}{\log n} \geqq c_1$*, i.e.,* $t \geqq n^{c_1}$*, we have* $\dfrac{1}{2} n^{2/(k+2)} < P \leqq n^{3/(k+2)}$ *by (32) and (33)) and*

(6) $$N(\alpha, \beta) = \sum_{\substack{1 \leqq j \leqq P \\ \alpha \leqq \left\{\frac{t}{n+j}\right\} < \beta}} 1 \quad (\textit{for } 0 \leqq \alpha < \beta \leqq 1).$$

Then we have

(7) $$|N(\alpha, \beta) - (\beta - \alpha)P| < \varepsilon P \quad \textit{for} \quad 0 \leqq \alpha < \beta \leqq 1.$$

Proof of Theorem 1. The proof will be based mostly on Vinogradov's ideas; see [3] and [4]. We need three lemmas.

Lemma 1. *Let α, β, Δ be real numbers satisfying*

(8) $$0 < \Delta < 1/2, \quad \Delta \leqq \beta - \alpha \leqq 1 - \Delta.$$

Then there exists a periodic function $\psi(x)$, with period 1, satisfying

(i) $\psi(x) = 1$ *in the interval* $\alpha + \frac{1}{2}\Delta \leqq x \leqq \beta - \frac{1}{2}\Delta$,

(ii) $\psi(x) = 0$ *in the interval* $\beta + \frac{1}{2}\Delta \leqq x \leqq 1 + \alpha - \frac{1}{2}\Delta$,

(iii) $0 \leqq \psi(x) \leqq 1$ *in the remainder of the interval* $\alpha - \frac{1}{2}\Delta \leqq x \leqq 1 + \alpha - \frac{1}{2}\Delta$,

(iv) $\psi(x)$ *has an expansion in Fourier series of the form*

$$\psi(x) = (\beta - \alpha) + \sum_{m=1}^{+\infty} (a_m \cos 2\pi m x + b_m \sin 2\pi m x)$$

where

$$|a_m| \leqq \frac{2}{\pi m}, \quad |b_m| \leqq \frac{2}{\pi m},$$

$$|a_m| \leqq 2(\beta - \alpha), \quad |b_m| \leqq 2(\beta - \alpha),$$

$$|a_m| < \frac{2}{\pi^2 m^2 \Delta}, \quad |b_m| < \frac{2}{\pi^2 m^2 \Delta}.$$

This lemma is identical with the special case $r = 1$ of Lemma 12 in [4], p. 32.

Lemma 2. *Let r, M, M' be positive integers, u, w real numbers such that*

(9) $$u \geqq 2^{r+3},$$

(10) $$0 \leqq w \leqq 1,$$

(11) $$M^{\frac{r+3}{2}} \leqq u \leqq M^{r+2}$$

and

(12) $$M \leqq M' \leqq 2M.$$

Then we have

$$\left|\sum_{m=M}^{M'} e\left(\frac{u}{m+w}\right)\right| < c_2 M^{1-1/2^{r-1}-1/2^{r-1}(r+1)} u^{1/2^{r-1}(r+1)} \log u \tag{13}$$

where c_2 is an absolute constant (independent of r, M, M', u, w).

This lemma can be proved by using Weyl's method and it is identical with Theorem 1 in [5], p. 22.

LEMMA 3. *There exists an absolute constant c_3 such that if k, P are positive integers, Q is an integer, $\alpha, \alpha_k, \dots, \alpha_0$ are real numbers,*

$$k \geqq c_3 \tag{14}$$

and

$$0 < 2(k+1)P|\alpha| \leqq 1, \tag{15}$$

then writing

$$f(x) = \alpha x^{k+1} + \alpha_k x^k + \dots + \alpha_1 x + \alpha_0,$$

we have

$$\left|\sum_{n=Q+1}^{Q+P} e\big(f(n)\big)\right| \leqq 2e^{15k\log^2 k} P^{1-1/6k^2\log k} \log P + 2|\alpha|^{-1/k}. \tag{16}$$

This lemma can be derived from an estimate of Hua (see [1]), and it is identical with Theorem 4.2 in [2], p. 286.

Now we are going to show that the assertion of Theorem 1 holds with

$$c_1 = \max\left(c_3 + \frac{1}{2},\ \ 20\right)$$

(where c_3 is defined in Lemma 3).

In order to prove (7), we may assume that $\varepsilon < 1$ and let η, ϱ be arbitrary real numbers satisfying $0 \leqq \eta < \varrho \leqq 1$ and

$$\frac{\varepsilon}{4} \leqq \varrho - \eta \leqq 1 - \frac{\varepsilon}{4}. \tag{17}$$

Then writing

$$\alpha = \eta - \frac{\varepsilon}{16}, \quad \beta = \varrho + \frac{\varepsilon}{16}, \quad \Delta = \frac{\varepsilon}{8}, \tag{18}$$

we have $0 < \Delta = \frac{\varepsilon}{8} < \frac{1}{2}$ and

$$\Delta < \frac{\varepsilon}{4} \leqq \varrho - \eta < \beta - \alpha = \left(\varrho + \frac{\varepsilon}{16}\right) - \left(\eta - \frac{\varepsilon}{16}\right) = (\varrho - \eta) + \frac{\varepsilon}{8} \leqq$$

$$\leqq \left(1 - \frac{\varepsilon}{4}\right) + \frac{\varepsilon}{8} = 1 - \frac{\varepsilon}{8} = 1 - \Delta,$$

so that (8) holds and thus Lemma 1 can be applied with the numbers α, β, Δ defined by (18). We obtain that there exists a periodic function $F(x)$ with period 1, satisfying

$$F(x)=1 \text{ for } \eta \leqq x \leqq \varrho, \tag{19}$$

$$F(x)=0 \text{ for } \varrho+\frac{\varepsilon}{8} \leqq x \leqq 1+\eta-\frac{\varepsilon}{8}; \tag{20}$$

$$0 \leqq F(x) \leqq 1 \quad \text{for all } x, \tag{21}$$

and such that it has an expansion in Fourier series of the form

$$F(x)=\left(\varrho-\eta+\frac{\varepsilon}{8}\right)+\sum_{m=1}^{+\infty}(a_m\cos 2\pi mx+b_m\sin 2\pi mx)= \tag{22}$$

$$=\left(\varrho-\eta+\frac{\varepsilon}{8}\right)+\sum_{m=1}^{+\infty}\operatorname{Re}(a_m-ib_m)e(mx)=\left(\varrho-\eta+\frac{\varepsilon}{8}\right)+\sum_{m=1}^{+\infty}\operatorname{Re} d_m e(mx)$$

where

$$|d_m|=|a_m-ib_m|=(|a_m|^2+|b_m|^2)^{1/2}\leqq\frac{2\sqrt{2}}{\pi m}<\frac{1}{m}, \tag{23}$$

$$|d_m|=|a_m-ib_m|=(|a_m|^2+|b_m|^2)^{1/2}\leqq 2\sqrt{2}(\beta-\alpha) \tag{24}$$

and

$$|d_m|=|a_m-ib_m|=(|a_m|^2+|b_m|^2)^{1/2}<\frac{2\sqrt{2}}{\pi^2m^2\Delta}=\frac{16\sqrt{2}}{\pi^2}\frac{1}{\varepsilon m^2}<\frac{3}{\varepsilon m^2}. \tag{21}$$

Let

$$m_0=\left[\frac{48}{\varepsilon^2}\right]+1.$$

Then by (19), (21), (22), (23) and (25), we have

$$N(\varrho,\eta)=\sum_{\substack{1\leqq j\leqq P\\ \eta\leqq\left\{\frac{t}{n+j}\right\}<\varrho}}1\leqq\sum_{1\leqq j\leqq P}F\left(\left\{\frac{t}{n+j}\right\}\right)= \tag{26}$$

$$=\sum_{j=1}^{P}F\left(\frac{t}{n+j}\right)=\sum_{j=1}^{P}\left(\varrho-\eta+\frac{\varepsilon}{8}+\sum_{m=1}^{+\infty}\operatorname{Re} d_m e\left(m\frac{t}{n+j}\right)\right)=$$

$$=\left(\varrho-\eta+\frac{\varepsilon}{8}\right)P+\sum_{m=1}^{m_0}\operatorname{Re}\left(d_m\sum_{j=1}^{P}e\left(m\frac{t}{n+j}\right)\right)+\sum_{j=1}^{P}\sum_{m=m_0+1}^{+\infty}\operatorname{Re} d_m e\left(m\frac{t}{n+j}\right)\leqq$$

$$\leqq\left(\varrho-\eta+\frac{\varepsilon}{8}\right)P+\sum_{m=1}^{m_0}|d_m|\left|\sum_{j=1}^{P}e\left(\frac{mt}{n+j}\right)\right|+\sum_{j=1}^{P}\sum_{m=m_0+1}^{+\infty}|d_m|\leqq$$

$$\leqq\left(\varrho-\eta+\frac{\varepsilon}{8}\right)P+\sum_{m=1}^{m_0}\frac{1}{m}\left|\sum_{j=1}^{P}e\left(\frac{mt}{n+j}\right)\right|+P\sum_{m=m_0+1}^{+\infty}\frac{3}{\varepsilon m^2}<$$

$$<\left(\varrho-\eta+\frac{\varepsilon}{8}\right)P+\sum_{m=1}^{m_0}\frac{1}{m}\left|\sum_{j=1}^{P}e\left(\frac{mt}{n+j}\right)\right|+\frac{3}{\varepsilon}P\sum_{m=m_0+1}^{+\infty}\frac{1}{(m-1)m}=$$

$$= \left(\varrho-\eta+\frac{\varepsilon}{8}\right)P+\sum_{m=1}^{m_0}\frac{1}{m}\left|\sum_{j=1}^{P}e\left(\frac{mt}{n+j}\right)\right|+\frac{3}{\varepsilon}P\sum_{m=m_0+1}^{+\infty}\left(\frac{1}{m-1}-\frac{1}{m}\right)=$$

$$= \left(\varrho-\eta+\frac{\varepsilon}{8}\right)P+\sum_{m=1}^{m_0}\frac{1}{m}\left|\sum_{j=1}^{P}e\left(\frac{mt}{n+j}\right)\right|+\frac{3}{\varepsilon}P\frac{1}{m_0}<$$

$$< \left(\varrho-\eta+\frac{\varepsilon}{8}\right)P+\sum_{m=1}^{m_0}\frac{1}{m}\left|\sum_{j=1}^{P}e\left(\frac{mt}{n+j}\right)\right|+\frac{3}{\varepsilon}P\frac{1}{48/\varepsilon^2}=$$

$$= \left(\varrho-\eta+\frac{3\varepsilon}{16}\right)P+\sum_{m=1}^{m_0}\frac{1}{m}\left|\sum_{j=1}^{P}e\left(\frac{mt}{n+j}\right)\right|.$$

Now we are going to show that

$$\sum_{m=1}^{m_0}\frac{1}{m}\left|\sum_{j=1}^{P}e\left(\frac{mt}{n+j}\right)\right|<\frac{\varepsilon}{16}P. \tag{27}$$

We have to distinguish two cases.

Case 1. Assume first that $\frac{\log t}{\log n}<c_1$ (i.e., $t<n^{c_1}$). In this case, we are going to apply Lemma 2 with $k, n, n+P-1, mt$ and 1 in place of $r, M\ M', u$ and w respectively. In fact, for large n, (9), (10) and (12) hold trivially. (Note that $k>0$ follows from (3).) Furthermore, we have

$$u = mt \geqq t = n^{\frac{\log t}{\log n}} = n^{\frac{1}{2}\left(\left(2\frac{\log t}{\log n}-3\right)+3\right)} \geqq n^{\frac{1}{2}(k+3)}$$

and for large n,

$$u = mt \leqq m_0 t < \frac{49}{\varepsilon^2}t = \frac{49}{\varepsilon^2}n^{\frac{\log t}{\log n}} = \frac{49}{\varepsilon^2}n^{\frac{1}{2}\left(2\frac{\log t}{\log n}-3\right)+\frac{3}{2}} \leqq \frac{49}{\varepsilon^2}n^{k+\frac{3}{2}} < n^{k+2}$$

so that also (11) holds. Thus we may apply Lemma 2. We obtain for lage n that

$$\sum_{m=1}^{m_0}\frac{1}{m}\left|\sum_{j=1}^{P}e\left(\frac{mt}{n+j}\right)\right| < \sum_{m=1}^{m_0}\frac{1}{m}c_2 n^{1-1/2^{k-1}-1/2^{k-1}(k+1)}(mt)^{1/2^{k-1}(k+1)}\log mt <$$

$$< \sum_{m=1}^{m_0}c_2\frac{1}{m}n^{1-1/2^{k-1}}\left(\frac{t}{n}\right)^{1/2^{k-1}(k+1)} m\log mt <$$

$$< c_2 m_0 n^{1-1/2^{k-1}} n^{\left(\frac{\log t}{\log n}-1\right)/2^{k-1}(k+1)}\log m_0 t <$$

$$< \frac{c_4}{\varepsilon^2}n^{1-1/2^{k-1}+\left(\frac{1}{2}\left(2\frac{\log t}{\log n}-3\right)+\frac{1}{2}\right)/2^{k-1}(k+1)}\log\frac{49}{\varepsilon^2}n^{c_1} <$$

$$< \frac{c_5}{\varepsilon^2}n^{1-1/2^{k-1}+\left(\frac{1}{2}(k+1)+\frac{1}{2}\right)/2^{k-1}(k+1)}\log n =$$

$$= \frac{c_5}{\varepsilon^2}n^{1-1/2^{k-1}+1/2^k+1/2^k(k+1)}\log n \leqq \frac{c_5}{\varepsilon^2}n^{1-1/2^{k-1}+1/2^k+1/2^{k+1}}\log n =$$

$$= \frac{c_5}{\varepsilon^2}n^{1-1/2^{k+1}}\log n = \frac{c_5}{\varepsilon^2}n^{1-1/2^{k+2}}n^{-1/2^{k+2}}\log n < \frac{\varepsilon}{16}\left[n^{1-1/2^{k+2}}\right] = \frac{\varepsilon}{16}P$$

which completes the proof of (27) in this case.

Case 2. Assume that

$$\frac{\log t}{\log n} \geqq c_1 = \max\left(c_3 + \frac{1}{2},\ 20\right). \tag{28}$$

Let us write

$$f_m(x) = \sum_{l=0}^{k+1} (-1)^l \frac{mt}{n^{l+1}} x^l.$$

Then by the well-known inequality

$$|1 - e(\alpha)| \leqq 2\pi|\alpha|,$$

we have

$$\sum_{m=1}^{m_0} \frac{1}{m} \left| \sum_{j=1}^{P} e\left(m \frac{t}{n+j}\right) \right| = \sum_{m=1}^{m_0} \frac{1}{m} \left| \sum_{j=1}^{P} e\left(\frac{mt}{n} \frac{1}{1+\frac{j}{n}}\right) \right| = \tag{29}$$

$$= \sum_{m=1}^{m_0} \frac{1}{m} \left| \sum_{j=1}^{P} e\left(\sum_{l=0}^{+\infty} (-1)^l \frac{mt}{n^{l+1}} j^l\right) \right| =$$

$$= \sum_{m=1}^{m_0} \frac{1}{m} \left| \sum_{j=1}^{P} e\left(f_m(j) + \sum_{l=k+2}^{+\infty} (-1)^l \frac{mt}{n^{l+1}} j^l\right) \right| \leqq$$

$$\leqq \sum_{m=1}^{m_0} \frac{1}{m} \left(\left| \sum_{j=1}^{P} e\big(f_m(j)\big) \right| + \sum_{j=1}^{P} \left| e\left(f_m(j) + \sum_{l=k+2}^{+\infty} (-1)^l \frac{mt}{n^{l+1}} j^l\right) - e\big(f_m(j)\big) \right| \right) \leqq$$

$$\leqq \sum_{m=1}^{m_0} \frac{1}{m} \left(\left| \sum_{j=1}^{P} e\big(f_m(j)\big) \right| + \sum_{j=1}^{P} 2\pi \left| \frac{mt}{n} \sum_{l=k+2}^{+\infty} (-1)^l \left(\frac{j}{n}\right)^l \right| \right) \leqq$$

$$\leqq \sum_{m=1}^{m_0} \frac{1}{m} \left(\left| \sum_{j=1}^{P} e\big(f_m(j)\big) \right| + \sum_{j=1}^{P} 2\pi \frac{mt}{n} \left(\frac{j}{n}\right)^{k+2} \right) \leqq$$

$$\leqq \sum_{m=1}^{m_0} \left(\left| \sum_{j=1}^{P} e\big(f_m(j)\big) \right| + 2\pi t \frac{P^{k+3}}{n^{k+3}} \right) = \sum_{m=1}^{m_0} \left| \sum_{j=1}^{P} e\big(f_m(j)\big) \right| + 2\pi m_0 t \frac{P^{k+3}}{n^{k+3}} <$$

$$< \sum_{m=1}^{m_0} \left| \sum_{j=1}^{P} e\big(f_m(j)\big) \right| + 2\pi \frac{49}{\varepsilon^2} t \frac{P^{k+3}}{n^{k+3}} < \sum_{m=1}^{m_0} \left| \sum_{j=1}^{P} e\big(f_m(j)\big) \right| + \frac{400}{\varepsilon^2} t \frac{P^{k+3}}{n^{k+3}}.$$

Now we are going to estimate the parameters k, P. Firs we note that in this case, (4) can be rewritten in the form

$$k \leqq \frac{\log t}{\log n} + \frac{1}{2} < k+1, \tag{30}$$

i.e.,

$$n^{k-1/2} \leqq t < n^{k+1/2}. \tag{31}$$

Furthermore, with respect to (31) we have

$$\text{(32)} \qquad P = \left[\left(\frac{n^{k+5/2}}{t}\right)^{1/(k+2)}\right] \leqq \left(\frac{n^{k+5/2}}{t}\right)^{1/(k+2)} \leqq \left(\frac{n^{k+5/2}}{n^{k-1/2}}\right)^{1/(k+2)} = n^{3/(k+2)}$$

and

$$\text{(33)} \qquad P = \left[\left(\frac{n^{k+5/2}}{t}\right)^{1/(k+2)}\right] > \left(\frac{n^{k+5/2}}{t}\right)^{1/(k+2)} - 1 >$$

$$> \left(\frac{n^{k+5/2}}{n^{k+1/2}}\right)^{1/(k+2)} - 1 = n^{2/(k+2)} - 1 > \frac{1}{2} n^{2/(k+2)}$$

(note that $n^{2/(k+2)} \to +\infty$ follows easily from (3) and (30)).

For large n, the last term in (29) can be estimated in the following way:

$$\text{(34)} \qquad \frac{400}{\varepsilon^2} t \frac{P^{k+3}}{n^{k+3}} = \frac{400}{\varepsilon^2} t \left[\left(\frac{n^{k+5/2}}{t}\right)^{1/(k+2)}\right]^{k+2} \frac{P}{n^{k+3}} \leqq$$

$$\leqq \frac{400}{\varepsilon^2} t \frac{n^{k+5/2}}{t} \frac{P}{n^{k+3}} = \frac{400}{\varepsilon^2} \frac{1}{n^{1/2}} P < \frac{\varepsilon}{32} P.$$

Finally, we estimate the sum $\left|\sum_{j=1}^{P} e\big(f_m(j)\big)\right|$ by using Lemma 3 with 0 and $f_m(x)$ in place of Q and $f(x)$, respectively. In fact, by (28) and (30) we have

$$k > \frac{\log t}{\log n} - \frac{1}{2} \geqq c_1 - \frac{1}{2} \geqq \left(c_3 + \frac{1}{2}\right) - \frac{1}{2} = c_3$$

so that (14) holds. Furthermore, with respect to (30) we have

$$\text{(35)} \qquad |\alpha| = \left|(-1)^{k+1} \frac{mt}{n^{k+2}}\right| = m \frac{t}{n^{k+2}} = mn^{\frac{\log t}{\log n} - k - 2} \leqq$$

$$\leqq m_0 n^{\left(\frac{\log t}{\log n} + \frac{1}{2}\right) - (k+1) - \frac{3}{2}} < \frac{49}{\varepsilon^2} n^{-3/2}.$$

(3), (30), (32) and (35) yield for large n that

$$0 < 2(k+1) P |\alpha| < 2\left(\frac{\log t}{\log n} + \frac{3}{2}\right) n^{3/(k+2)} \frac{49}{\varepsilon^2} n^{-3/2} <$$

$$< 2\left(\frac{1}{\log n} \frac{(\log n)^{5/4}}{\log\log\log n} + \frac{3}{2}\right) n \frac{49}{\varepsilon^2} n^{-3/2} < (\log n)^{1/4} n^{-1/2} < 1$$

so that also (15) holds. Thus Lemma 2 can be applied, and we obtain that

$$\left|\sum_{j=1}^{P} e\big(f_m(j)\big)\right| \leqq 2e^{15k\log^2 k}P^{1-1/6k^2\log k}\log P+2|\alpha|^{-1/k}. \tag{36}$$

First we estimate the first term on the right hand side. By (3), (30), (32) and (33), for large n we have

$$2e^{15k\log^2 k}P^{1-1/6k^2\log k}\log P = \tag{37}$$

$$= 2P\exp\left(15k\log^2 k-\frac{\log P}{6k^2\log k}+\log\log P\right)<$$

$$< 2P\exp\left(15\left(\frac{\log t}{\log n}+\frac{1}{2}\right)\log^2\left(\frac{\log t}{\log n}+\frac{1}{2}\right)-\frac{\log\frac{1}{2}n^{2/(k+2)}}{6k^2\log k}+\log\log n^{3/(k+2)}\right)<$$

$$< 2P\exp\left(30\frac{\log t}{\log n}\log^2\left(\frac{\log t}{\log n}\right)-\frac{\log n}{6k^2(k+2)\log k}+\log\log n\right)<$$

$$< 2P\exp\left(30\frac{1}{\log n}\frac{(\log n)^{5/4}}{\log\log n}\log^2\left(\frac{1}{\log n}\frac{(\log n)^{5/4}}{\log\log n}\right)-\frac{\log n}{18k^3\log k}+\log\log n\right)<$$

$$< 2P\exp\left(30\frac{(\log n)^{1/4}}{\log\log n}(\log\log n)^2-\frac{\log n}{18\left(\frac{\log t}{\log n}+\frac{1}{2}\right)^3\log\left(\frac{\log t}{\log n}+\frac{1}{2}\right)}+\log\log n\right)<$$

$$< 2P\exp\left(31(\log n)^{1/4}\log\log n-\frac{\log n}{100\left(\frac{\log t}{\log n}\right)^3\log\left(\frac{\log t}{\log n}\right)}\right)<$$

$$< 2P\exp\left(31(\log n)^{1/4}\log\log n-\frac{\log n}{100\left(\frac{(\log n)^{1/4}}{\log\log n}\right)^3\log\frac{(\log n)^{1/4}}{\log\log n}}\right)<$$

$$< 2P\exp\left(31(\log n)^{1/4}\log\log n-\frac{\log n}{100\frac{(\log n)^{3/4}}{(\log\log n)^3}\log\log n}\right)=$$

$$= 2P\exp\left(31(\log n)^{1/4}\log\log n-\frac{1}{100}(\log n)^{1/4}(\log\log n)^2\right)<$$

$$< 2P\exp\left(-\frac{1}{101}(\log n)^{1/4}(\log\log n)^2\right)< P\exp\left(-(\log n)^{1/5}\right).$$

With respect to $k \geqq c_1 \geqq 20$, (3), (28), (30), (31), (33) and (35), for large n the second term on the right hand side of (36) can be estimated in the following way:

$$(38) \qquad 2|\alpha|^{-1/k} = 2\left(\frac{mt}{n^{k+2}}\right)^{-1/k} = 2\left(\frac{n^{k+2}}{mt}\right)^{1/k} \leqq 2\left(\frac{n^{k+2}}{t}\right)^{1/k} =$$

$$= 2\left(\frac{n^{k+5/2}}{t}\right)^{1/k} n^{-1/2k} = 2\left(\frac{n^{k+5/2}}{t}\right)^{1/(k+2)} \left(\frac{n^{k+5/2}}{t}\right)^{1/k-1/(k+2)} n^{-1/2k} \leqq$$

$$\leqq 4P\left(\frac{n^{k+5/2}}{t}\right)^{2/k(k+2)} n^{-1/2k} \leqq 4P\left(\frac{n^{k+5/2}}{n^{k-1/2}}\right)^{2/k(k+2)} n^{-1/2k} =$$

$$= 4Pn^{6/k(k+2)-1/2k} = 4Pn^{(10-k)/2k(k+2)} \leqq$$

$$\leqq 4Pn^{(k/2-k)/2k(k+2)} = 4Pn^{-1/4(k+2)} < 4Pn^{-1/12k} =$$

$$= 4P\exp\left(-\frac{\log n}{12k}\right) \leqq 4P\exp\left(-\frac{\log n}{12\left(\frac{\log t}{\log n}+\frac{1}{2}\right)}\right) <$$

$$< 4P\exp\left(-\frac{\log n}{12\left(\frac{(\log n)^{1/4}}{\log\log n}+\frac{1}{2}\right)}\right) < P\exp\left(-\frac{\log n}{(\log n)^{1/4}}\right) = P\exp\left(-(\log n)^{3/4}\right).$$

(29), (34), (36), (37) and (38) yield for large n that

$$\sum_{m=1}^{m_0}\frac{1}{m}\left|\sum_{j=1}^{P} e\left(m\frac{t}{n+j}\right)\right| < \sum_{m=1}^{m_0}\left|\sum_{j=1}^{P} e(f_m(j))\right| + \frac{400}{\varepsilon^2}t\frac{P^{k+3}}{n^{k+3}} <$$

$$< \sum_{m=1}^{m_0}\left(P\exp\left(-(\log n)^{1/5}\right)+P\exp\left(-(\log n)^{3/4}\right)\right)+\frac{\varepsilon}{32}P <$$

$$< 2m_0P\exp\left(-(\log n)^{1/5}\right)+\frac{\varepsilon}{32}P =$$

$$= 2\left(\left[\frac{48}{\varepsilon^2}\right]+1\right)P\exp\left(-(\log n)^{1/5}\right)+\frac{\varepsilon}{32}P < \frac{\varepsilon}{32}P+\frac{\varepsilon}{32}P = \frac{\varepsilon}{16}P$$

which proves that (27) holds also in Case 2.

(Note that like Case 1, also Case 2 could be treated in a simpler way by replacing Lemma 3 by Theorem 1 in [5], p. 47; in fact in this way we can show that the exponent 5/4 in the upper bound in (3) can be replaced by the greater on 3/2, but, on the other hand, this methods yields the much worse estimate $P \sim n^{1-c(\log t/\log n)^{-2}} \sim n^{1-c/k^2}$ for P; this is why we have preferred the more complicated way based on Lemma 3.)

We obtain from (26) and (27) that

$$N(\eta, \varrho) < \left(\varrho-\eta+\frac{3\varepsilon}{16}\right)P+\frac{\varepsilon}{16}P = \left(\varrho-\eta+\frac{\varepsilon}{4}\right)P$$

provided that (17) holds:

$$N(\eta, \varrho) < \left(\varrho-\eta+\frac{\varepsilon}{4}\right) P \quad \text{for} \quad \frac{\varepsilon}{4} \leqq \varrho-\eta \leqq 1-\frac{\varepsilon}{4}. \tag{39}$$

Assume now that $0 \leqq \varrho-\eta < \varepsilon/4$. Then (39) yields (with $\eta+\frac{\varepsilon}{4}$ in place of ϱ) that

$$N(\eta, \varrho) \leqq N\left(\eta, \varrho+\left(\frac{\varepsilon}{4}-(\varrho-\eta)\right)\right) = N\left(\eta, \eta+\frac{\varepsilon}{4}\right) < \tag{40}$$

$$< \left(\left(\eta+\frac{\varepsilon}{4}\right)-\eta+\frac{\varepsilon}{4}\right) P = \frac{\varepsilon}{2} P \leqq \left(\varrho-\eta+\frac{\varepsilon}{2}\right) P \quad \text{for} \quad 0 \leqq \varrho-\eta < \varepsilon/4.$$

Finally, let $1-\frac{\varepsilon}{4} < \varrho-\eta = 1$. Then we have

$$N(\eta, \varrho) \leqq P = \left(\left(1-\frac{\varepsilon}{4}\right)+\frac{\varepsilon}{4}\right) P < \left(\varrho-\eta+\frac{\varepsilon}{4}\right) P \quad \text{for} \quad 1-\frac{\varepsilon}{4} < \varrho-\eta \leqq 1. \tag{41}$$

(39), (40) and (41) yield that

$$N(\eta, \varrho) < \left(\varrho-\eta+\frac{\varepsilon}{2}\right) P \quad \text{for all} \quad 0 \leqq \varrho-\eta \leqq 1. \tag{42}$$

On the other hand, by using (42) repeatedly, we obtain that

$$N(\alpha, \beta) = N(0, 1)-N(0, \alpha)-N(\beta, 1) = P-N(0, \alpha)-N(\beta, 1) > \tag{43}$$

$$> P-\left(\alpha+\frac{\varepsilon}{2}\right) P-\left(1-\beta+\frac{\varepsilon}{2}\right) P = (\beta-\alpha-\varepsilon) P \quad \text{for all} \quad 0 \leqq \alpha < \beta \leqq 1.$$

(42) and (43) yield (7) and this completes the proof of Theorem 1.

3. In this section, we prove the following consequence of Theorem 1:

THEOREM 2. *Let* $\varepsilon > 0$, *n a positive integer satisfying* $n > n_1(\varepsilon)$, *t a real number such that*

$$n < t < \exp\left(\frac{(\log n)^{5/4}}{\log\log n}\right).$$

Let us define k by (4) *(where c_1 denotes the constant defined in Theorem* 1*), and write*

$$P = \begin{cases} n \ \textit{if} \ n < t < n^2 \\ [n^{1-1/2^{k+2}}] \ \textit{if} \ n^2 \leqq t < n^{c_1} \\ \left[\left(\frac{n^{k+5/2}}{t}\right)^{1/(k+2)}\right] \ \textit{if} \ n^{c_1} \leqq t. \end{cases}$$

Then there exists a positive integer j such that

$$1 \leqq j \leqq P \tag{44}$$

and

$$(n+j)|_\varepsilon t. \tag{45}$$

PROOF. We have to distinguish three cases.
Case 1. Let

$$n < t < \varepsilon n^2. \tag{46}$$

If $n<t<2n+2$, then (45) holds with

$$j = \begin{cases} 1 \text{ for } n<t<n+1 \\ [t-n] \text{ for } n+1 \leqq t<2n+1 \\ n \text{ for } 2n+1 \leqq t<2n+2 \end{cases}$$

(for large n). Thus we may assume that $2n+2 \leqq t$; hence

$$\left[\frac{t}{n+1}\right] \geqq 2. \tag{47}$$

Let us write t in the form

$$t = \left[\frac{t}{n+1}\right](n+1)+r \quad \text{where} \quad 0 \leqq r < n+1 \tag{48}$$

and

$$t = \left[\frac{t}{2n}\right](2n)+s \quad \text{where} \quad 0 \leqq s < 2n, \tag{49}$$

respectively. (48) and (49) yield that

$$2\left(\left[\frac{t}{n+1}\right]-\left[\frac{t}{2n}\right]\right)n = \left[\frac{t}{n+1}\right](n-1)-r+s. \tag{50}$$

By (47), (48) and (49), we have

$$\left[\frac{t}{n+1}\right](n-1)-r+s \geqq 2(n-1)-r > 2(n-1)-(n+1) = n-3 > 0 \tag{51}$$

for $n>3$. (50) and (51) yield (for $n>3$) that

$$\left[\frac{t}{n+1}\right] > \left[\frac{t}{2n}\right].$$

Thus there exists an integer j such that $1 \leqq j \leqq n-1$ $(=P-1)$ and

$$\left[\frac{t}{n+j}\right] > \left[\frac{t}{n+j+1}\right]. \tag{52}$$

We are going to show that this integer j satifies also (45).

Let us write $q=\left[\frac{t}{n+j}\right]$. Then by (52), we have $\frac{t}{n+j}\geqq q>\frac{t}{n+j+1}$, thus with respect to (46)

$$0\leqq\frac{t}{n+j}-q<\frac{t}{n+j}-\frac{t}{n+j+1}=\frac{t}{(n+j)(n+j+1)}<\frac{t}{n^2}<\varepsilon$$

which implies (45) and this completes the proof of the theorem in this case.

Case 2. Let

(53) $$\varepsilon n^2\leqq t<n^2.$$

For $j=1,2,\dots,n-1$, let

$$d_j=\frac{t}{n+j}-\frac{t}{n+j+1}=\frac{t}{(n+j)(n+j+1)}.$$

Then obviously,

(54) $$0<d_{n-1}<d_{n-2}<\dots<d_2<d_1.$$

By (53), for sufficiently large n we have

(55) $$d_1-d_{n-[n/3]}=\frac{t}{(n+1)(n+2)}-\frac{t}{(2n-[n/3])(2n-[n/3]+1)}>$$
$$>\frac{t}{\left(\frac{4}{3}n\right)^2}-\frac{t}{\left(\frac{5}{3}n\right)^2}=\frac{81}{400}\frac{t}{n^2}>\frac{1}{5}\frac{t}{n^2}\geqq\frac{\varepsilon}{5}.$$

Let p denote number such that

(56) $$\frac{10}{\varepsilon}<p<\frac{20}{\varepsilon}.$$

(It is well-known that for $x\geqq 2$, there exists a prime number q such that $x<q<2x$.) (56) yields that

(57) $$\frac{1}{p}<\frac{\varepsilon}{10}.$$

(55) and (57) imply that there exists an integer a such that

(58) $$d_{n-[n/3]}<\frac{a}{p}<\frac{a+1}{p}<d_1.$$

Then either

(59) $$(a,p)=1$$

or $(a+1,p)=1$ holds; we may assume that (59).

(54) and (58) imply that there exists an integer u such that

(60) $$1\leqq u\leqq n-[n/3]-1<\frac{2n}{3}$$

and

$$d_{u+1} \leqq \frac{a}{p} < d_u. \tag{61}$$

By (53), for $j=1, 2, \ldots, n-2$ we have

$$0 < d_j - d_{j+1} = \frac{t}{(n+j)(n+j+1)} - \frac{t}{(n+j+1)(n+j+2)} = \tag{62}$$

$$= \frac{2t}{(n+j)(n+j+1)(n+j+2)} < \frac{2t}{n^3} < \frac{2}{n} \quad (\text{for } j = 1, 2, \ldots, n-2).$$

(61) and (62) yield that

$$0 < d_u - \frac{a}{p} \leqq d_u - d_{u+1} < \frac{2}{n}$$

hence

$$\left| d_u - \frac{a}{p} \right| < \frac{2}{n}. \tag{63}$$

Obviously, there exists an integer b such that

$$\left| \frac{t}{n+u} - \frac{b}{p} \right| < \frac{1}{2p}. \tag{64}$$

Define the integer l by

$$al \equiv b \pmod{p} \tag{65}$$

(such an l exists by (59)) and

$$1 \leqq l \leqq p. \tag{66}$$

Put $q = \dfrac{b-al}{p}$ and $j=u+l$. (56,) (60) and (66) yield for large n that

$$(1 \leqq) j = u+l < \frac{2n}{3} + p < \frac{2n}{3} + \frac{20}{\varepsilon} < \frac{2n}{3} + \frac{n}{3} = n. \tag{67}$$

For $i=1, 2, \ldots, n-1-u$, we have

$$d_{u+i} = d_u + (d_{u+1} - d_u) + (d_{u+2} - d_{u+1}) + \ldots + (d_{u+i} - d_{u+i-1})$$

thus by (62) and (63),

$$\left| d_{u+i} - \frac{a}{p} \right| \leqq |d_{u+i} - d_u| + \left| d_u - \frac{a}{p} \right| \leqq \tag{68}$$

$$\leqq |d_{u+1} - d_u| + |d_{u+2} - d_{u+1}| + \ldots + |d_{u+i} - d_{u+i-1}| + \frac{2}{n} < i\frac{2}{n} + \frac{2}{n} = \frac{2(i+1)}{n}$$

$$(\text{for } 0 \leqq i \leqq n-1-u).$$

Furthermore, we have

$$\frac{t}{n+j}=\frac{t}{n+u+l}=\frac{t}{n+u}-\left(\frac{t}{n+u}-\frac{t}{n+u+1}\right)-\left(\frac{t}{n+u+1}-\frac{t}{n+u+2}-\dots-\right.$$

$$\left.-\left(\frac{t}{n+u+l-1}-\frac{t}{n+u+l}\right)=\frac{t}{n+u}-d_u-d_{u+1}-\dots-d_{u+l-1}=\right.$$

$$=\left(\frac{t}{n+u}-\frac{b}{p}\right)+\frac{b-la}{p}-\sum_{i=0}^{l-1}\left(d_{u+i}-\frac{a}{p}\right)=\left(\frac{t}{n+u}-\frac{b}{p}\right)+q-\sum_{i=0}^{l-1}\left(d_{u+i}-\frac{a}{p}\right)$$

thus with respect to (56), (57), (64), (66) and (68)

$$\left|\frac{t}{n+j}-q\right|\leqq\left|\frac{t}{n+u}-\frac{b}{p}\right|+\sum_{i=0}^{l-1}\left|d_{u+i}-\frac{a}{p}\right|<\frac{1}{2p}+\sum_{i=0}^{l-1}\frac{2(i+1)}{n}\leqq$$

$$\leqq\frac{1}{2p}+\frac{2l^2}{n}\leqq\frac{1}{2p}+\frac{2p^2}{n}<\frac{\varepsilon}{20}+\frac{800}{\varepsilon^2 n}<\frac{\varepsilon}{20}+\frac{\varepsilon}{2}<\varepsilon$$

which implies that

(69) $$\left\|\frac{t}{n+j}\right\|<\varepsilon.$$

(67) and (69) show that (44) and (45) hold also in Case 2.

Case 3. Let

$$n^2\leqq t<\exp\left(\frac{(\log n)^{5/4}}{\log\log n}\right).$$

Then by using Theorem 1 with $\frac{\varepsilon}{2}$ in place of ε, we obtain for large n that

$$\sum_{\substack{1\leqq j\leqq P\\ (n+j)|_\varepsilon t}}1=\sum_{\substack{1\leqq j\leqq P\\ \left\|\frac{t}{n+j}\right\|<\varepsilon}}1\geqq\sum_{\substack{1\leqq j\leqq P\\ 0\leqq\left\{\frac{t}{n+j}\right\}<\varepsilon}}1=N(0,\varepsilon)=$$

$$=\varepsilon P+\big(N(0,\varepsilon)-\varepsilon P\big)\geqq\varepsilon P-|N(0,\varepsilon)-\varepsilon P|>\varepsilon P-\frac{\varepsilon}{2}P=\frac{\varepsilon}{2}P>1$$

which shows that there exists an integer j satisfying (44) and (45), and this completes the proof of Theorem 2.

4. In this section, we show that if t is large (in terms of n) then it may occur that there does not exist an integer j satisfying $1\leqq j\leqq n$ and $(n+j)|_\varepsilon t$.

THEOREM 3. *Let* $\frac{1}{4}>\varepsilon>0$, $\delta>0$. *Then for* $n>n_2(\varepsilon)$, *there exists a real number* t *such that*

(70) $$n<t<\exp\left((2+\delta)n\right)$$

and there does not exist an integer j satisfying $1\leqq j\leqq n$ *and*

$$(n+j)\,|_{\varepsilon}\, t. \tag{71}$$

PROOF. Let $t=[1,2,\ldots,2n]+\dfrac{n}{2}$ (where $[1,2,\ldots,2n]$ denotes the least common multiple of the numbers $1,2,\ldots,2n$); then $n<t$ holds trivially. For $p\leqq 2n$, define the positive integer α_p by

$$p^{\alpha_p}\leqq 2n<p^{\alpha_p+1}.$$

Then by the prime number theorem, we have

$$\log\,[1,2,\ldots,2n]=\log\Big(\prod_{p\leqq 2n}p^{\alpha_p}\Big)=\sum_{p\leqq 2n}\log p^{\alpha_p}\sim 2n$$

so that for large n,

$$t=[1,2,\ldots,2n]+\frac{n}{2}<\exp\left(\left(2+\frac{\delta}{2}\right)n\right)+\frac{n}{2}<\exp\left((2+\delta)n\right)$$

which proves (70).

Furthermore, if $1\leqq j\leqq n$ then

$$\left\{\frac{t}{n+j}\right\}=\left\{\frac{[1,2,\ldots,2n]+n/2}{n+j}\right\}=\left\{\frac{[1,2,\ldots,2n]}{n+j}+\frac{n}{2(n+j)}\right\}=\left\{\frac{n}{2(n+j)}\right\}.$$

Here we have

$$\frac{1}{4}=\frac{n}{4n}\leqq\frac{n}{2(n+j)}<\frac{n}{2n}=\frac{1}{2}$$

hence

$$\frac{1}{4}\leqq\left\{\frac{t}{n+j}\right\}=\left\{\frac{n}{2(n+j)}\right\}=\frac{n}{2(n+j)}<\frac{1}{2}$$

which implies that

$$\frac{1}{4}\leqq\left\|\frac{t}{n+j}\right\|=\left\{\frac{t}{n+j}\right\}.$$

Thus (71) does not hold which completes the proof of Theorem 3.

5. Note that there is a considerable gap between Theorems 2 and 3. In fact, let $f(n,\varepsilon)$ denote the infimum of the real numbers t such that $n<t$ and there does not exist an integer j such that $1\leqq j\leqq n$ and $(n+j)\,|_{\varepsilon}\,t$. Then for $n>n_0(\varepsilon)$, Theorem 2 shows that

$$\exp\left(\frac{(\log n)^{5/4}}{\log\log n}\right)\leqq f(n,\varepsilon) \tag{72}$$

and on the other hand, Theorem 3 yields that

$$f(n,\varepsilon)\leqq\exp\left((2+\delta)n\right); \tag{73}$$

we guess that both the lower estimate (72) and the upper estimate (73) are far from the best possible.

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(Received October 2, 1981)

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Acta Math. Hung.
41 (3—4) (1983), 325—336.

BEMERKUNG ZU DER ARBEIT „ÜBER DIE LEBESGUESCHEN FUNKTIONEN. II"

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1. Es sei $\lambda=\{\lambda_k\}_1^\infty$ eine monoton nichtabnehmende Folge von positiven Zahlen. Für ein orthonormiertes System $\varphi=\{\varphi_k(x)\}_1^\infty$ im Intervall $(0,1)$ seien

$$L_n(\varphi;x)=\int_0^1\left|\sum_{k=1}^n\varphi_k(x)\varphi_k(t)\right|dt,\quad L_n^*(\varphi;x)=\int_0^1\max_{1\leqq\nu\leqq n}\left|\sum_{k=1}^\nu\varphi_k(x)\varphi_k(t)\right|dt,$$

$$L_n\left(\frac{\varphi}{\sqrt{\lambda}};x\right)=\int_0^1\left|\sum_{k=1}^n\frac{\varphi_k(x)\varphi_k(t)}{\lambda_k}\right|dt,\quad L_n^*\left(\frac{\varphi}{\sqrt{\lambda}};x\right)=\int_0^1\max_{1\leqq\nu\leqq n}\left|\sum_{k=1}^\nu\frac{\varphi_k(x)\varphi_k(t)}{\lambda_k}\right|dt$$

$(n=1,2,\ldots)$ die verschiedenen Lebesgueschen Funktionen.

Offensichtlich gelten

$$L_n(\varphi;x)\leqq L_n^*(\varphi;x),\quad L_n\left(\frac{\varphi}{\sqrt{\lambda}};x\right)\leqq L_n^*\left(\frac{\varphi}{\sqrt{\lambda}};x\right)\quad (x\in(0,1);\,n=1,2,\ldots).$$

Weiterhin kann man leicht zeigen, daß die Implikationen

$$L_n\left(\frac{\varphi}{\sqrt{\lambda}};x\right)=O(1)\quad (x\in(0,1);\,n=1,2,\ldots)\Rightarrow L_n(\varphi;x)=O(\lambda_n)$$
$$(x\in(0,1);\,n=1,2,\ldots),$$

$$L_n^*\left(\frac{\varphi}{\sqrt{\lambda}};x\right)=O(1)\quad (x\in(0,1);\,n=1,2,\ldots)\Rightarrow L_n^*(\varphi;x)=O(\lambda_n)$$
$$(x\in(0,1);\,n=1,2,\ldots)$$

bestehen. Aus der Bedingung

$$L_n^*\left(\frac{\varphi}{\sqrt{\lambda}};x\right)=O(1)\quad (x\in(0,1);\,n=1,2,\ldots) \tag{1}$$

folgen also die Bedingungen

$$L_n(\varphi_i;x)=O(\lambda_n)\quad (x\in(0,1);\,n=1,2,\ldots), \tag{2}$$

$$L_n^*(\varphi_i;x)=O(\lambda_n)\quad (x\in(0,1);\,n=1,2,\ldots), \tag{3}$$

$$L_n\left(\frac{\varphi}{\sqrt{\lambda}};x\right)=O(1)\quad (x\in(0,1);\,n=1,2,\ldots). \tag{4}$$

In der Arbeit [3] haben wir den folgenden Satz bewiesen.

SATZ A. *Es sei λ eine von unten konkave, monoton wachsend gegen Unendlich strebende Folge von positiven Zahlen mit $\lambda_n = O(\log^2 n)$. Gilt für eine im absoluten Betrag monoton nichtwachsende Folge $\{a_k\}$*

$$\sum_{k=1}^{\infty} a_k^2 \lambda_k = \infty,$$

so gibt es ein orthonormiertes System $\varphi = \{\varphi_k(x)\}_1^\infty$ im Inervalle $(0, 1)$ mit (2) derart, daß die Reihe

$$\sum_{k=1}^{\infty} a_k \varphi_k(x) \tag{5}$$

in $(0, 1)$ fast überall divergiert.

In der Arbeit [4] haben wir dieses Resultat verschärft.

SATZ B. *Unter den Bedingungen der Satzes A gibt es ein orthonormiertes System $\varphi = \{\varphi_k(x)\}_1^\infty$ im Intervall $(0, 1)$ mit (4) derart, daß die Reihe (5) in $(0, 1)$ fast überall divergiert.*

In dieser Arbeit beweisen wir den folgenden Satz, der alle erwähnten Sätze verschärft.

SATZ. *Unter den Bedingungen der Satzes A gibt es ein orthonormiertes System $\varphi = \{\varphi_k(x)\}_1^\infty$ in $(0, 1)$ mit (1) derart, daß die Reihe (5) in $(0, 1)$ fast überall divergiert.*

BEMERKUNG. Aus diesem Satz folgt die folgende Behauptung. Unter den Bedingungen des Satzes A gibt es ein orthonormiertes System $\{\varphi_k(x)\}_1^\infty$ in $(0, 1)$ mit (3) derart, daß die Reihe (5) in $(0, 1)$ überall divergiert.

Diese Behauptung verschärft einen Satz von L. Csernyák [1].

2. Zum Beweis des Satzes benötigen wir den folgenden Hilfssatz.

HILFSSATZ. *Es seien $p(\geqq 2)$, q positive ganze Zahlen. Dann gibt es ein in $(0, 1)$ orthonormiertes System von Treppenfunktionen $g_l(p, q; x)$ $(l = 1, \ldots, 2pq)$ mit den folgenden Eigenschaften.*

Es gilt

$$\int_0^1 \max_{1 \leqq \nu \leqq 2pq} \left| \sum_{l=1}^{\nu} g_l(p, q; x) g_l(p, q; t) \right| dt \leqq C_1 \log^2 p \quad (x \in (0, 1);\ C_1 \geqq 1),$$

und es gibt eine einfache Menge $E(\subseteqq (0, 1))$ (d. h. ist E die Vereinigung endlichvieler Intervalle) mit $\operatorname{mes}(E) = \dfrac{1}{5}$ derart, daß für jedes $x \in E$ ein Index $m(x)(<2pq)$ existiert, für welchen $g_l(p, q; x) \geqq 0$ $(l = 1, \ldots, m(x))$ und

$$\sum_{l=1}^{m(x)} g_l(p, q; x) \geqq C_2 \sqrt{2pq} \log p$$

erfüllt sind, wobei C_2 eine positive Konstante bezeichnet.

Beweis des Hilfssatzes. Der Beweis kann mit der in [2] angewandten Methode geführt werden. Es sei

$$\bar{f}_l(p;x)=\frac{1}{k-p-l-1/2},\quad \text{für}\quad x\in\left(\frac{k-1}{p},\frac{k}{p}\right)\quad (k=1,\ldots,4p;\ l=1,\ldots,2p).$$

Dann ist

$$\int_0^4 \bar{f}_l^2(p;x)\,dx=\frac{1}{p}\sum_{k=1}^{4p}\frac{1}{(k-p-l-1/2)^2}\quad (l=1,\ldots,2p),$$

woraus

$$\frac{C_3}{p}\le\int_0^4 \bar{f}_l^2(p;x)\,dx\le\frac{C_4}{p}\quad (l=1,\ldots,2p) \tag{6}$$

folgt. ($C_3, C_4, \ldots$ bezeichnen positiven Konstante.) Wir setzen

$$\alpha_{ij}=\int_0^4 \bar{f}_i(p;x)\bar{f}_j(p;x)\,dx\quad (1\le i,j\le 2p;\ i\ne j).$$

Durch einfache Rechnung folgt für $i>j$

$$\alpha_{ij}=\frac{1}{p}\sum_{k=1}^{4p}\frac{1}{k-p-i-1/2}\cdot\frac{1}{k-p-j-1/2}=$$

$$=\frac{1}{p(i-j)}\sum_{k=1}^{4p}\left(\frac{1}{k-p-i-1/2}-\frac{1}{k-p-j-1/2}\right)=$$

$$=\frac{1}{p(i-j)}\left(\sum_{l=1-p-i}^{3p-i}\frac{1}{l-1/2}-\sum_{l=1-p-j}^{3p-j}\frac{1}{l-1/2}\right)=$$

$$=\frac{1}{p(i-j)}\left(\sum_{l=1-p-i}^{1-p-j-1}\frac{1}{l-1/2}-\sum_{l=3p-i+1}^{3p-j}\frac{1}{l-1/2}\right),$$

und so gilt

$$|\alpha_{i,j}|\le\frac{C_5}{p^2}\quad (i,j=1,\ldots,2p;\ i\ne j). \tag{7}$$

Im Intervall [4, 5] definieren wir die Funktionen $\bar{f}_l(p;x)$ folgenderweise. Wir teilen das Intervall [4, 5] in $N=2p(2p-1)$ paarweise disjunkte Teilintervalle $I_{i,j}$ $(i,j=1,\ldots$ $\ldots,N;\ i\ne j)$ gleicher Länge. Dann sei für $l=1,\ldots,2p$

$$\bar{f}_l(p;x)=\begin{cases}\sqrt{\frac{1}{2}N|\alpha_{l,j}|}, & \text{für}\quad x\in I_{l,j};\ j=1,\ldots,2p,\ j\ne l,\\ -\sqrt{\frac{1}{2}N|\alpha_{l,j}|}\ \operatorname{sign}\alpha_{l,j}, & \text{für}\quad x\in I_{j,l};\ j=1,\ldots,2p,\ j\ne l,\\ 0 & \text{sonst.}\end{cases}$$

Die so definierten Treppenfunktionen $\bar{f}_l(p; x)$ $(l=1, \ldots, 2p)$ bilden im Intervall $[0, 5]$ offenbar ein orthogonales System. Wir setzen

$$\alpha_l^2 = \int_0^5 \bar{f}_l^2(p; x)\,dx = \int_0^4 \bar{f}_l^2(p; x)\,dx + \sum_{\substack{j=1 \\ j \neq l}}^{N} \alpha_{l,j}.$$

Aus (6) und (7) folgt

$$\frac{C_6}{p} \leqq \alpha_l^2 \leqq \frac{C_7}{p} \quad (l = 1, \ldots, 2p). \tag{8}$$

Wir betrachten die orthonormierten Funktionen

$$f_l(p; x) = \frac{1}{\alpha_l} \bar{f}_l(p; x) \quad (l = 1, \ldots, 2p).$$

Diese sind offenbar Treppenfunktionen in $[0, 5]$, weiterhin gibt es für jedes $x \in [2, 3)$ eine von x abhängige positive Zahl $m(x) < (2p)$ derart, daß die Ungleichungen

$$f_l(p; x) > 0 \; (l = 1, \ldots, m(x)), \quad \sum_{l=1}^{m(x)} f_l(p; x) \geqq C_8 \sqrt{2p} \log p \tag{9}$$

erfüllt sind.

Offensichtlich gilt

$$\int_0^5 \max_{1 \leqq \nu \leqq 2p} \left| \sum_{l=1}^{\nu} f_l(p; x) f_l(p; t) \right| dt \leqq \int_0^5 \sum_{l=1}^{2p} |f_l(p; x) f_l(p; t)|\,dt = \tag{10}$$

$$= \int_0^4 \max_{1 \leqq \nu \leqq 2p} \left| \sum_{l=1}^{\nu} f_l(p; x) f_l(p; t) \right| dt + \int_4^5 \max_{1 \leqq \nu \leqq 2p} \left| \sum_{l=1}^{\nu} f_l(p; x) f_l(p; t) \right| dt =$$

$$= R_1(x) + R_2(x).$$

Es sei $0 \leqq x < 4p$. Dann gibt es einen Index $k(x)$ $\left(1 \leqq k(x) \leqq 4p\right)$ mit $x \in \left[\frac{k(x)-1}{p}; \frac{k(x)}{p}\right)$. Nach der Definition von $f_l(p; x)$, auf Grund von (8) gilt

$$R_1(x) = \sum_{k=1}^{4p} \int_{\frac{k-1}{p}}^{\frac{k}{p}} \left(\sum_{l=1}^{2p} \frac{1}{\alpha_l^2} \frac{1}{|(k(x)-p-l-1/2)(k-p-l-1/2)|} \right) dt \leqq$$

$$\leqq C_7 \sum_{k=1}^{4p} \sum_{l=1}^{2p} \frac{1}{|(k(x)-p-l-1/2)(k-p-l-1/2)|} \leqq$$

$$\leqq C_9 + C_7 \sum_{\substack{k=1 \\ k \neq k(x)}}^{4p} \frac{1}{|k-k(x)|} \sum_{k=1}^{2p} \left(\frac{1}{|k(x)-p-l-1/2|} + \frac{1}{|k-p-l-1/2|} \right).$$

Daraus folgt:

$$R_1(x) \leqq C_{10} (\log p)^2 \quad (x \in [0, 4)). \tag{11}$$

Weiterhin, bekommen wir auf Grund von (8)

$$R_2(x) \leqq C_{11} p \sum_{l=1}^{2p} \frac{1}{|k(x)-p-l-1/2|} \int_4^5 |\bar{f}_l(p;x)|dx \leqq$$

$$\leqq C_{11} p \sum_{l=1}^{2p} \frac{1}{|k(x)-p-l-1/2|} \left(\sum_{\substack{j=1 \\ j \neq l}}^{2p} \operatorname{mes}(I_{l,j}) \sqrt{\frac{1}{2} N|\alpha_{l,j}|} + \sum_{\substack{j=1 \\ j \neq l}}^{2p} \operatorname{mes}(I_{l,j}) \sqrt{\frac{1}{2} N|\alpha_{l,j}|} \right) \leqq$$

$$\leqq C_{12} \sum_{l=1}^{2p} \frac{1}{|k(x)-p-l-1/2|},$$

woraus

$$R_2(x) \leqq C_{13} \log p \ (x \in [0, 4)) \tag{12}$$

folgt. Aus (10), (11) und (12) erhalten wir

$$\int_0^5 \max_{1 \leqq v \leqq 2p} \left| \sum_{l=1}^{v} f_l(p;x) f_l(p;t) \right| dt \leqq C_{14} (\log p)^2 \ (x \in [0, 4)). \tag{13}$$

Es sei nun $x \in [4, 5]$. Dann gibt es Indizes $i(x), j(x)$ $(1 \leqq i(x)), \ j(x) \leqq 2p, i(x) \neq j(x)$ mit $x \in I_{i(x)j(x)}$. Es gelten $f_l(p;x) = 0(l = 1, ..., 2p, l \neq i(x), l \neq j(x))$ und $|\bar{f}_{i(x)}(p;x)|, |\bar{f}_{j(x)}(p;x)| \leqq C_{15}$. Somit ist

$$\int_0^5 \max_{1 \leqq v \leqq 2q} \left| \sum_{l=1}^{v} f_l(p;x) f_l(p;t) \right| dt \leqq C_{16} p \left\{ \int_0^5 |\bar{f}_{i(x)}(p;t)|dt + \int_0^5 |\bar{f}_{j(x)}(p;t)|dt \right\}. \tag{14}$$

Da

$$\int_0^5 |\bar{f}_l(p;t)|dt = \frac{1}{p} \sum_{k=1}^{4p} \frac{1}{|k-p-l-1/2|} + \sum_{\substack{j=1 \\ j \neq l}}^{2p} \operatorname{mes}(I_{l,j}) \sqrt{\frac{1}{2} N|\alpha_{l,j}|} +$$

$$+ \sum_{\substack{j=1 \\ j \neq l}}^{2p} \operatorname{mes}(I_{j,l}) \sqrt{\frac{1}{2} N|\alpha_{l,j}|} \leqq C_{17} \left(\frac{\log p}{p} + \frac{1}{p} \right)$$

besteht, erhalten wir aus (14):

$$\int_0^5 \max_{1 \leqq v \leqq 2p} \left| \sum_{l=1}^{v} f_l(p;x) f_l(p;t) \right| dt \leqq C_{18} \log p \quad (x \in [4, 5]).$$

Daraus und aus (13) ergibt sich

$$\int_0^5 \max_{1 \leqq v \leqq 2p} \left| \sum_{l=1}^{v} f_l(p;x) f_l(p;t) \right| dt \leqq C_{19} (\log p)^2 \quad (x \in [0, 5]). \tag{15}$$

Es sei

$$\bar{g}_l(p;x) = \sqrt{5} f_l(p;5x) \quad (x \in (0, 1); l = 1, ..., 2p).$$

Offentsichtlich bilden die Treppenfunktionen $\bar{g}_l(p;\ x)$ $(l=1, \dots, 2p)$ ein orthonormiertes System in (0,1). Aus (15) folgt

$$\int_0^1 \max_{1\leqq\nu\leqq 2p}\left|\sum_{l=1}^{\nu}\bar{g}_l(p;x)\bar{g}_l(p;t)\right|dt \leqq C_{19}(\log p)^2 \quad (x\in(0,1)). \tag{16}$$

Weiterhin, folgt aus (9), daß für jedes $x\in\bar{E}=\left[\frac{2}{5}, \frac{3}{5}\right)$ ein von x abhängiger Index $m(x)(<2p)$ derart existiert, daß die Ungleichungen

$$\bar{g}_l(p;x)>0\,(l=1, \dots, m(x)), \quad \sum_{l=1}^{m(x)}\bar{g}_l(p;x)\geqq C_{20}\sqrt{2p}\log p \tag{17}$$

erfüllt sind.

Für eine in $(0,1)$ definierte Funktion $f(x)$ und für ein Intervall $I=(a,b)$ $(\subseteqq(0,1))$ setzen wir

$$f(I;x)=\begin{cases} f\left(\dfrac{x-a}{b-a}\right) & (x\in I),\\ 0 & \text{sonst.}\end{cases}$$

Für eine Menge $H(\subseteqq(0,1))$ sei weiterhin $H(I)$ die Menge, die aus H unter der Transformation $y=(b-a)x+a$ entsteht.

Es sei $I_i=\left(\frac{i-1}{q}, \frac{i}{q}\right)$ $(i=1, \dots, q)$, und wir setzen

$$g_{l+2(i-1)p}(p,q;x)=\bar{g}_l(I_i;p;x)\sqrt{q} \quad (l=1, \dots, 2p;\, i=1, \dots, q), \quad E=\bigcup_{i=1}^{q}\bar{E}(I_i).$$

Auf Grund von (16) und (17) ist es klar, daß für diese Funktionen $g_l(p,q;x)$ $(l=1, \dots, 2pq)$ und für diese Menge E alle Forderungen des Hilfssatzes erfüllt sind.

3. Beweis des Satzes. Der Beweis des Satzes erfolgt mit der in [4] angewandten Methode. Auf Grund der Annahmen des Satzes gibt es eine nichtabnehmende, gegen Unendlich strebende, von unten konkave Folge $\mu=\{\mu(k)\}_1^\infty$ mit $\mu(1)\geqq 1$ und

$$\frac{\mu(k)}{\lambda_k}\searrow 0 \quad (k\nearrow\infty), \tag{18}$$

für die

$$\sum_{k=1}^{\infty}a_k^2\mu(k)=\infty \tag{19}$$

besteht.

Nach der Voraussetzung über λ und nach (18) gibt es eine positive Konstante $C_{21}(\leqq 1)$ mit

$$\mu(k)\leqq C_{21}\log^2 k \quad (k=2,3,\dots). \tag{20}$$

Wir definieren eine Indexfolge $\{m_k\}_1^\infty$ folgenderweise: Es seien $m_1=1$ und m_{k+1} die kleinste natürliche Zahl mit $\mu(m_{k+1})>2\mu(m_k)$ $(k=1,2,\dots)$. Wegen der Konkavität gilt

$$\frac{\mu(2m_k)-\mu(m_k+1)}{m_k-1}\leqq\frac{\mu(m_k+1)-\mu(m_1)}{m_k-m_1+1},$$

woraus

$$\mu(2m_k)-\mu(m_k+1) \leqq \frac{m_k-1}{m_k}\mu(m_k+1) \leqq \mu(m_k+1)$$

folgt. Nach der Definition von m_{k+1} gilt also $m_{k+1}>2m_k$. Daraus erhalten wir nach (20)

$$C_{21}\log^2(m_{k+1}-m_k) > C_{21}\log^2 m_k \geqq \mu(m_k) \quad (k = 2, 3, \ldots).$$

Ist k genügend groß $(k>k_0)$, so gelten die Ungleichungen

$$\mu(m_k) \leqq \mu(m_k-1)+1, \quad \mu(m_k)/C_1C_{21} \geqq 8,$$

und es gibt eine positive ganze Zahl $\bar{q}_k$ mit $m_{k+1}-m_k>2\bar{q}_k$ und

$$4C_1C_{21} \leqq \frac{\mu(m_k)}{2} \leqq C_1C_{21}\log^2\left[\frac{m_{k+1}-m_k}{2\bar{q}_k}\right] \leqq \mu(m_k).$$

(Hier bezeichnet $[\alpha]$ den ganzen Teil von α, und C_1 ist die Konstante aus dem Hilfssatz.) Es seien $n_0=1$, $n_k=m_{k+k_0}$, $q_k=\bar{q}_{k+k_0}$ $(k=1, 2, \ldots)$. Dann gelten die Bezichnungen

$$n_{k+1} > 2n_k \quad (k = 1, 2, \ldots), \tag{21}$$

$$\mu(n_{k+1}) \leqq 4\mu(n_k-1) \quad (k = 1, 2, \ldots), \tag{22}$$

$$4C_1C_{21} \leqq \frac{\mu(n_k)}{2} \leqq C_1C_{21}\log^2\left[\frac{n_{k+1}-n_k}{2q_k}\right] \leqq \mu(n_k) \quad (k = 1, 2, \ldots). \tag{23}$$

Ohne Beschränkung der Allgemeinheit können wir $a_k \geqq 0$ $(k=1, 2, \ldots)$ voraussetzen.

Wir definieren erstens für jedes k $(k \geqq 2)$ ein in $(0, 1)$ orthonormiertes System von Treppenfunktionen $\varphi_l((k; x)$ $(l=n_k, \ldots, n_{k+1}-1)$ derart, daß für jedes k

$$\int_0^1 \max_{n_k \leqq \nu < n_{k+1}} \left| \sum_{l=n_k}^{\nu} \frac{\varphi_l(k; x)\varphi_l(k; t)}{\lambda_l} \right| dt \leqq C_{22}\frac{\mu(n_k)}{\lambda_{n_k}} \quad (x \in (0, 1)), \tag{24}$$

$$\max_{n_k \leqq n \leqq m < n_{k+1}} |a_n\varphi_n(k; x)+\ldots+a_m\varphi_m(k; x)| \geqq C_{23}A_k \quad (x \in E_k) \tag{25}$$

gelten, wobei

$$A_k = \left\{\sum_{i=1}^{c(k)}(n_k-n_{k-1})a^2_{n_k+i(n_k-n_{k-1})}\mu(n_k)\right\}^{1/2}, \quad c(k) = \left[\frac{n_{k+1}-n_k}{n_k-n_{k-1}}\right]$$

bedeutet, und für die einfache Menge E_k $(\subseteqq (0, 1))$

$$\operatorname{mes}(E_k) = \frac{1}{10} \tag{26}$$

besteht.

Wir wenden zu diesem Zweck den Hilfssatz im Falle

$$p = \left[\frac{n_k-n_{k-1}}{2q_{k-1}}\right], \quad q = q_{k-1}$$

an. Die entsprechenden Funktionen bezeichnen wir mit $g_s(x)$ $(s=1, \ldots, 2pq)$. Dann gelten auf Grund des Hilfssatzes und der Ungleichung (23)

$$\int_0^1 \max_{1 \leqq v \leqq 2pq} \left| \sum_{s=1}^{v} g_s(x) g_s(t) \right| dt \leqq \mu(n_k) \quad (x \in (0, 1)), \tag{27}$$

$$\max_{1 \leqq n \leqq m < 2pq} |a_{n_k+(i-1)(n_k-n_{k-1})+n} g_n(x) + \ldots + a_{n_k+(i-1)(n_k-n_{k-1})+m} g_m(x)| \geqq \tag{28}$$

$$\geqq C_{24} \sqrt{n_k - n_{k-1}}\, a_{n_k+i(n_k-n_{k-1})} \sqrt{\mu(n_k)} \quad (x \in E;\ i = 1, \ldots, c(k)).$$

Aus (27) folgt, auf Grund von (18),

$$\int_0^1 \max_{1 \leqq v \leqq 2pq} \left| \sum_{s=1}^{v} \frac{g_s(x) g_s(t)}{\lambda_{n_k+s-1}} \right| dt \leqq \tag{29}$$

$$\leqq \sum_{s=1}^{2pq-1} \left(\frac{1}{\lambda_{n_k+s-1}} - \frac{1}{\lambda_{n_k+s}} \right) \int_0^1 \max_{1 \leqq v \leqq 2pq} \left| \sum_{s=1}^{v} g_s(x) g_s(t) \right| + dt$$

$$+ \frac{1}{\lambda_{n_k}} \int_0^1 \max_{1 \leqq v \leqq 2pq} \left| \sum_{s=1}^{v} g_s(x) g_s(t) \right| dt \leqq 2 \frac{\mu(n_k)}{\lambda_{n_k}} \quad (x \in (0, 1)).$$

Es seien $s_0 = 0$,

$$s_i = \sum_{j=1}^{i} a^2_{n_k+j(n_k-n_{k-1})} \Big/ 2 \sum_{j_0=1}^{c(k)} a^2_{n_k+j_0(n_k-n_{k-1})} \quad (i = 1, \ldots, c(k)),$$

$s_{c(k)+1} = 1$ und $I_i = (s_{i-1}, s_i)$ $(i = 1, \ldots, c(k)+1)$. Wir setzen

$$\varphi_n(k; x) = \frac{1}{\sqrt{\operatorname{mes}(I_i)}} g_{n-(n_k+(i-1)(n_k-n_{k-1}))+1}(I_i; x)$$

für $n_k + (i-1)(n_k - n_{k-1}) \leqq n < n_k + i(n_k - n_{k-1}) + 2pq$, und $i = 1, \ldots, c(k)$. Weiterhin sei

$$E_k = \bigcup_{i=1}^{c(k)} E(I_i).$$

Aus der Definition der Menge E_k und aus dem Hilfssatz folgt (26). Endlich seien $J_1, J_2, \ldots$ paarweise disjunkte Intervalle in $(s_{c(k)}, s_{c(k)+1}) \left(= \left(\frac{1}{2}, 1 \right) \right)$ und die Funktionen $\varphi_m(k; x)$ für $n_k + (i-1)(n_k - n_{k-1}) + 2pq \leqq n < n_k + i(n_k - n_{k-1})$; $i = 1, \ldots$ $\ldots, c(k)$ und für $n_k + c(k)(n_k - n_{k-1}) + 2pq < n < n_{k+1}$ seien der Reihe nach gleich mit den Funktionen

$$f_n(x) = \begin{cases} \dfrac{1}{\sqrt{\operatorname{mes}(J_n)}} & (x \in J_n), \\ 0 & \text{sonts.} \end{cases}$$

Aus dem Hilfssatz und aus (29) erhalten wir durch einfache Rechnung

$$\int_0^1 \max_{n_k \leqq \nu \leqq n_{k+1}} \left| \sum_{l=n_k}^{\nu} \frac{\varphi_l(k;x)\varphi_l(k;t)}{\lambda_l} \right| dt \leqq \begin{cases} 2\dfrac{\mu(n_k)}{\lambda_{n_k}}, & x \in \bigcup_{i=1}^{c(k)} I_i, \\ \dfrac{1}{\lambda_{n_k}}, & x \in I_{c(k)+1}. \end{cases}$$

Daraus ergibt sich (24) wegen $\mu(n) \geqq 1 \, (n \geqq 1)$. Endlich folgt auf Grund der Definition der Funktionen $\varphi_l(k; x)$ und aus (28) auch die Ungleichung (25).

Aus (19), (21), (22), aus der Monotonität der Folge $\{a_k\}_1^\infty$ und aus der Definition der $c(k)$ folgt

$$\text{(30)} \qquad \sum_{k=2}^{\infty} A_k^2 = \sum_{k=2}^{\infty} \sum_{i=1}^{c(k)} (n_k - n_{k-1}) a^2_{n_k + i(n_k - n_{k-1})} \mu(n_k) \geqq C_{25} \sum_{n=n_2}^{\infty} a_n^2 \mu(n) = \infty.$$

Wir definieren eine Indexfolge $\{k_r\}_0^\infty$ folgenderweise: Es sei $k_0 = 2$; wenn $k_r (r \geqq 0)$ schon definiert ist, dann sei k_{r+1} die kleinste natürliche Zahl (größer als k_r), für die

$$\text{(31)} \qquad \sqrt{\sum_{k=k_r+1}^{k_{r+1}} A_k^2} \geqq r+1,$$

und

$$\text{(32)} \qquad \frac{\mu(n)}{\lambda_n} \leqq \frac{1}{2^r} \quad (k_{r+1} \leqq n)$$

erfüllt sind. Wegen (18) und (30) existiert eine solche Indexfolge.

Dann definieren wir für jeden Index $r (\geqq 1)$ ein orthonormiertes System von Treppenfunktionen $\psi_n(r; x)$ $(n = n_{k_r+1}, \ldots, n_{k_{r+1}+1} - 1)$ mit folgenden Eigenschaften: Es gilt

$$\text{(33)} \qquad \int_0^1 \max_{n_{k_r+1} \leqq \nu < n_{k_{r+1}+1}} \left| \sum_{l=n_{k_r+1}}^{\nu} \frac{\psi_l(r;x)\psi_l(r;t)}{\lambda_l} \right| dt \leqq C_{22} \frac{1}{2^r} \quad (x \in (0,1)),$$

und

$$\text{(34)} \qquad \max_{n_{k_r+1} \leqq n \leqq m < n_{k_{r+1}+1}} |a_n \psi_n(r;x) + \ldots + a_m \psi_m(r;x)| \geqq C_{23}(r+1) (x \in H_r),$$

wobei für die einfache Menge $H_r (\subseteqq (0,1))$

$$\text{(35)} \qquad \operatorname{mes}(H_r) \geqq \frac{1}{10}$$

besteht.

Es seien nähmlich $\tilde{s}_0 = 0$,

$$\tilde{s}_i = \sum_{j=1}^{i} A^2_{k_r+j} \Big/ \sum_{j_0=k_r+1}^{k_{r+1}} A^2_{j_0} \quad (i = 1, \ldots, k_{r+1} - k_r),$$

und $\bar{I}_i = (\tilde{s}_{i-1}, \tilde{s}_i)$ $(i = 1, \ldots, k_{r+1} - k_r)$. Wir setzen

$$\psi_n(r;x) = \frac{1}{\sqrt{\operatorname{mes}(\bar{I}_i)}} \varphi_n(\bar{I}_i; k; x) (n_{k_r+i} \leqq n < n_{k_r+i+1}; \ i = 1, \ldots, k_{r+1} - k_r),$$

und

$$H_r = \bigcup_{i=1}^{k_{r+1}-k_r} E_{k_r+i}(\tilde{I}_i).$$

Aus (26) folgt (35). Aus der Definition der Funktionen $\psi_n(r; x)$ und aus (25), (31) erhalten wir (34). Endlich bekommen wir aus der Definition der Funktionen $\psi_n(r; x)$ und aus (18), (24), (32) auch (33).

Endlich definieren wir durch Induktion ein orthonormiertes System von Treppenfunkctionen $\varphi_n(x)$ $(n=1, 2, \ldots)$ im Intervall $(0, 1)$ und eine Folge von einfachen Mengen $G_r(\subseteqq(0, 1))$ $(r=1, 2, \ldots)$, für welche die folgenden Bedingungen erfüllt sind: Es gilt

$$\int_0^1 \max_{1 \leqq \nu < n_{k_1+1}} \left| \sum_{l=1}^{\nu} \frac{\varphi_l(x)\varphi_l(t)}{\lambda_l} \right| dt \leqq C_{26} \quad (x \in (0, 1)), \tag{36}$$

die Mengen G_r $(r=1, 2, \ldots)$ sind stochestisch unabhängig, und für jede natürliche Zahl r gilt

$$\operatorname{mes}(G_r) = \frac{1}{10}. \tag{37}$$

Ferner bestehen für jede positive ganze Zahl r die Abschätzungen

$$\int_0^1 \max_{n_{k_r+1} \leqq \nu < n_{k_{r+1}+1}} \left| \sum_{l=n_{k_r+1}}^{\nu} \frac{\varphi_l(x)\varphi_l(t)}{\lambda_l} \right| dt \leqq C_{22} \frac{1}{2^r} \quad (x \in (0, 1)), \tag{38}$$

$$\max_{n_{k_r+1} \leqq n \leqq m < n_{k_{r+1}+1}} |a_n\varphi_n(x) + \ldots + a_m\varphi_m(x)| \geqq C_{23}(r+1) \quad (x \in G_r). \tag{39}$$

Es seien J_n^* $(n=1, \ldots, n_{k_1+1}-1)$ paarweise disjunkte Intervalle in $(0, 1)$, und wir setzen für $n=1, \ldots, n_{k_1+1}-1)$

$$\varphi_n(x) = \begin{cases} \dfrac{1}{\sqrt{\operatorname{mes}(J_n^*)}} & (x \in J_n^*), \\ 0 & \text{sonts.} \end{cases}$$

Dann besteht (36) offensichtlich.

Es sei r_0 eine positive ganze Zahl. Wir nehmen an, daß die Treppenfunktionen $\varphi_n(x)$ $(n=1, \ldots, n_{k_{r_0}+1}-1)$ und die einfachen Mengen $G_1, \ldots, G_{r_0-1}$ $(\in(0, 1))$ schon derart definiert sind, daß diese Funktionen in $(0, 1)$ ein orthonormiertes System bilden, diese Mengen stochastisch unabhängig sind, ferner für $r=1, \ldots, r_0-1$ (37), (38) und (39) bestehen.

Dann gibt es eine Einteilung des Intervalls $(0, 1)$ in endlichviele disjunkte Intervalle I_s^* $(s=1, \ldots, \sigma)$ derart, dass jede Funktion $\varphi_n(x)$ $(n=1, \ldots, n_{k_{r_0}+1}-1)$ in jedem Intervall I_s^* konstant, und jede Menge $G_r(r=1, \ldots, r_0-1)$ die Vereinigung gewisser I_s^* ist. Die zwei Hälften von I_s^* bezeichnen wir mit $I_s^{*\prime}$, bzw. mit $I_s^{*\prime\prime}$.

Wir setzen

$$\varphi_n(x) = \sum_{s=1}^{\sigma} \psi_n(I_s^{*\prime}; x) - \sum_{s=1}^{\sigma} \psi_n(I_s^{*\prime\prime}; x) \quad (n = n_{k_{r_0}+1}, \ldots, n_{k_{r_0+1}+1}-1),$$

und

$$G_{r_0} = \bigcup_{s=1}^{\sigma} \left(H_{r_0}(I_s^{*\prime}) \cup H_{r_0}(I_s^{*\prime\prime})\right).$$

Offensichtlich bilden die Treppenfunktionen $\varphi_n(x)$ $(n=1, \ldots, n_{k_{r_0}+1}-1)$ ein orthonormiertes System in $(0, 1)$. Die einfachen Mengen $G_1, \ldots, G_{r_0}$ sind stochastisch unabhängig, ferner besteht (37) auch für $r=r_0$, auf Grund von (35). Aus (34) erhalten wir sofort (39) für $r=r_0$. Endlich bekommen wir aus (33), (38) für $r=r_0$ durch einfache Rechnung.

Das ganze Funktionensystem $\{\varphi_n(x)\}_1^\infty$ und die Mengenfolge $\{G_r\}_1^\infty$ mit den geforderten Eigenschaften erhalten wir also durch Induktion.

Aus (37) folgt

$$\sum_{r=1}^{\infty} \operatorname{mes}(G_r) = \infty.$$

Da die Mengen G_r stochastisch unabhängig sind, erhalten wir durch Anwendung des zweiten Borel—Cantellischen Lemmas, daß

$$\operatorname{mes}(\overline{\lim_{r\to\infty}}\, G_r) = 1.$$

Daraus und aus (39) ergibt sich, daß die Reihe (5) im Intervall (0,1) fast überall divergiert derart, daß

$$\overline{\lim_{N\to\infty}} \left| \sum_{n=1}^{N} a_n \varphi_n(x) \right| = \infty$$

in $(0, 1)$ fast überall gilt.

Endlich sei n eine positive ganze Zahl. Ist $1 \leqq n < n_{k_1+1}$, so gilt

$$L_n^*\left(\frac{\varphi}{\sqrt{\lambda}}\,;x\right) \leqq C_{26} \quad (x\in(0,1)) \tag{40}$$

auf Grund von (36). Ist aber $n_{k_{r_0}+1} \leqq n < n_{k_{r_0+1}+1}$ mit einem Index $r_0(\geqq 1)$, so gilt

$$L_n^*\left(\frac{\varphi}{\sqrt{\lambda}}\,;x\right) \leqq \int_0^1 \max_{1\leqq \nu < n_{k_1+1}} \left| \sum_{l=1}^{\nu} \frac{\varphi_l(x)\varphi_l(t)}{\lambda_l} \right| dt + \tag{41}$$

$$+\sum_{r=1}^{r_0-1} \int_0^1 \max_{n_{k_r+1} \leqq \nu < n_{k_{r+1}+1}} \left| \sum_{l=n_{k_r+1}}^{\nu} \frac{\varphi_l(x)\varphi_l(t)}{\lambda_l} \right| dt +$$

$$+\int_0^1 \max_{n_{k_{r_0}+1}\leqq \nu < n} \left| \sum_{l=n_{k_{r_0}+1}}^{\nu} \frac{\varphi_l(x)\varphi_l(t)}{\lambda_l} \right| dt \leqq C_{27}\left(1+\frac{1}{2}+\ldots+\frac{1}{2^{r_0}}\right) \leqq C_{28} \quad (x\in(0,1)).$$

Aus (40) und (41) ergibt sich:

$$L_n^*\left(\frac{\varphi}{\sqrt{\lambda}}\,;x\right) = O(1) \quad (x\in(0,1);\; n=1,2,\ldots).$$

Damit haben wir unseren Satz bewiesen.

Auf Grund der Beweisführung kann man sehen, daß für dieses Funktionensystem $\{\varphi_k(x)\}_1^\infty$ auch

$$\int_0^1 \left(\sum_{k=1}^{\infty} \frac{|\varphi_k(x)\varphi_k(t)|}{\lambda_k} \right) dt \leqq C_{29} \quad (x\in(0,1))$$

gilt.

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(Eingegangen am 5. Oktober 1981)

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Acta Math. Hung.
41 (3—4) (1983), 337—346.

A GENERAL MOMENT INEQUALITY FOR THE MAXIMUM OF THE RECTANGULAR PARTIAL SUMS OF MULTIPLE SERIES

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1. A preliminary result

Let (X, A, μ) be a positive measure space and let $\{\xi_{k_1}=\xi_{k_1}(x)\colon k_1\in Z_1\}\subset \subset L^\gamma(X, A, \mu)$ where $Z_1=\{1, 2, \dots\}$ and γ is a fixed real, $\gamma\geqq 1$. Studying the a.e. convergence of the single series

$$\sum_{k_1=1}^{\infty} \xi_{k_1}, \tag{1.1}$$

denote by $S(\mathcal{I})$ and $M(\mathcal{I})$ the partial sum of (1.1) extended over the integers contained in the interval $\mathcal{I}=(b_1, b_1+m_1]$ and the maximum of the consecutive partial sums extended also over $\mathcal{I}$, respectively. That is,

$$S(\mathcal{I}) = S(b_1, m_1) = \sum_{k_1\in\mathcal{I}} \xi_{k_1} = \sum_{k_1=b_1+1}^{b_1+m_1} \xi_{k_1}$$

and

$$M(\mathcal{I}) = M(b_1, m_1) = \max_{1\leqq p_1\leqq m_1} |S(b_1, p_1)|.$$

Here and in the sequel $b_1\in Z_+=\{0, 1, \dots\}$ and $p_1, m_1\in Z_1$; further, $m_1=|\mathcal{I}|$ denotes the number of the integers contained in the interval $\mathcal{I}$. We note that clearly

$$M(\mathcal{I}) \leqq \max_{J\subseteq\mathcal{I}} |S(J)| \leqq 2M(\mathcal{I}).$$

A nonnegative function $f(\mathcal{I})$ of the interval $\mathcal{I}$ with integral endpoints is said to be superadditive if for every $\mathcal{I}$ and for every disjoint representation $\mathcal{I}_1\cup\mathcal{I}_2=\mathcal{I}$ we have the inequality $f(\mathcal{I}_1)+f(\mathcal{I}_2)\leqq f(\mathcal{I})$. Further, let $\varphi(t_1, m_1)$ be also a nonnegative function defined on $R_+\times Z_1$ where R_+ is the set of the nonnegative reals.

A recent result by the present author (1980) reads as follows.

THEOREM 1 ([7]). *Let $\gamma\geqq 1$ be given. Suppose that there exist a nonnegative and superadditive function $f(\mathcal{I})$ of the interval $\mathcal{I}$, and a nonnegative function $\varphi(t_1, m_1)$, nondecreasing in both variables, such that for every $\mathcal{I}$ we have*

$$\int |S(\mathcal{I})|^\gamma d\mu \leqq f(\mathcal{I})\varphi^\gamma(f(\mathcal{I}), m_1), \quad m_1 = |\mathcal{I}|.$$

* This research was conducted while the author was on leave from Szeged University and a visiting professor at the Johns Hopkins University, Baltimore. The author gratefully acknowledges the support received from the United States Office of Naval Research under contract N00014—79—C—0801.

Then for every $\mathcal{I}$ *we have both*

$$\int M^{\gamma}(\mathcal{I})\,d\mu \leqq 3^{\gamma-1} f(\mathcal{I}) \left\{ \sum_{k_1=0}^{[\log m_1]-1} \varphi\left(\frac{f(\mathcal{I})}{2^{k_1}}, \left[\frac{m_1}{2^{k_1+1}}\right]\right)\right\}^{\gamma} \tag{1.2}$$

and

$$\int M^{\gamma}(\mathcal{I})\,d\mu \leqq \frac{5}{2} f(\mathcal{I}) \left\{ \sum_{k_1=0}^{[\log m_1]} \varphi\left(\frac{f(\mathcal{I})}{2^{k_1}}, \left[\frac{m_1}{2^{k_1}}\right]\right)\right\}^{\gamma}.$$

In this paper the integrals are taken over the whole space X, $[t_1]$ is the integral part of t_1, and the logarithms are with base 2. Furthermore, in the case $m_1=1$ we agree to take $[\log m_1]-1$ to be equal to 0 and $[m_1/2^{k_1+1}]$ to be equal to 1 on the right-hand side of (1.2).

2. The main result

Let Z_+^d be the set of all d-tuples $k=(k_1, \ldots, k_d)$ with nonnegative integers for coordinates, where the dimension d is a fixed positive integer. As usual, $k \leqq m$ iff $k_j \leqq m_j$ for each j, and we write $1=(1, \ldots, 1)$. If all the coordinates k_j are positive integers, we write $k \in Z_1^d$.

Let $\{\xi_k = \xi_k(x)\colon k \in Z_1^d\} \subset L^{\gamma}(X, A, \mu)$ be given and consider the d-multiple series

$$\sum_{k \in Z_1^d} \xi_k = \sum_{k_1=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} \xi_{k_1, \ldots, k_d}. \tag{2.1}$$

In the following, we denote by

$$R = R(b, m) = R(b_1, \ldots, b_d;\ m_1, \ldots, m_d) =$$

$$= \{k \in Z_1^d\colon b_j < k_j \leqq b_j + m_j \quad \text{for each} \quad j,\ 1 \leqq j \leqq d\} = \mathop{\times}_{j=1}^{d} (b_j, b_j+m_j]$$

an arbitrary rectangle in Z_1^d where $b \in Z_+^d$ and $m \in Z_1^d$. The rectangular partial sum $S(R)$ of (2.1) extended over the lattice points contained in R, and the maximum $M(R)$ extended over R to those rectangular partial sums whose left-hand bottom corners coincide with that of R, are defined as follows:

$$S(R) = S(b, m) = S(b_1, \ldots, b_d;\ m_1, \ldots, m_d) =$$

$$= \sum_{k \in R} \xi_k = \sum_{k_1=b_1+1}^{b_1+m_1} \cdots \sum_{k_d=b_d+1}^{b_d+m_d} \xi_{k_1, \ldots, k_d}$$

and

$$M(R) = M(b, m) = M(b_1, \ldots, b_d;\ m_1, \ldots, m_d) =$$

$$= \max_{1 \leqq p \leqq m} |S(b, p)| = \max_{1 \leqq p_1 \leqq m_1} \cdots \max_{1 \leqq p_d \leqq m_d} |S(b_1, \ldots, b_d;\ p_1, \ldots, p_d)|,$$

respectively. Here and in the sequel $b \in Z_+^d$ and $m \in Z_1^d$; further, m_j denotes the number of the lattice points contained in the rectangle R in a row parallel to the j^{th}

axis, $1 \leqq j \leqq d$. We note that clearly

$$M(R) \leqq \max_{Q \subseteq R} |S(Q)| \leqq 2^d M(R).$$

A nonnegative function $f(R)$ of the rectangle R with corner points from Z_+^d is said to be superadditive if we have the inequality

$$f(R_{j1}) + f(R_{j2}) \leqq f(R) \tag{2.2}$$

for every rectangle R and for every j and p_j where $1 \leqq j \leqq d$, $1 \leqq p_j < m_j$, and

$$R_{j1} = R(b_1, \ldots, b_{j-1}, b_j, b_{j+1}, \ldots, b_d;\ m_1, \ldots, m_{j-1}, p_j, m_{j+1}, \ldots, m_d),$$

$$R_{j2} = R(b_1, \ldots, b_{j-1}, b_j + p_j, b_{j+1}, \ldots, b_d;\ m_1, \ldots, m_{j-1}, m_j - p_j, m_{j+1}, \ldots, m_d).$$

In other words, $R_{j1} \cup R_{j2} = R$ is a disjoint decomposition of R by a hyperplane parallel to each axis except the j^{th} axis. For example,

$$f(R) = \sum_{k \in R} u_k$$

is even an additive function of R, where $\{u_k \colon k \in Z_1^d\}$ is a given d-multiple sequence of nonnegative reals. We mention that the nonnegativity of $f(R)$ and (2.2) imply that $f(R) = f(b_1, \ldots, b_d;\ m_1, \ldots, m_d)$ is a nondecreasing function in each variable m_j, $1 \leqq j \leqq d$.

Furthermore, by $\varphi(t_1, m) = \varphi(t_1;\ m_1, \ldots, m_d)$ we denote a nonnegative function defined on $R_+ \times Z_1^d$, which is nondecreasing in each variable, i.e.

$$\varphi(t_1';\ m_1', \ldots, m_d') \leqq \varphi(t_1'';\ m_1'', \ldots, m_d'')$$

whenever

$$0 \leqq t_1' \leqq t_1'' \quad \text{and} \quad 1 \leqq m_j' \leqq m_j'' \quad \text{for each} \quad j,\ 1 \leqq j \leqq d.$$

After these preliminaries we give an upper estimate for the γ^{th} moment of $M(R)$ in the terms of the given "a priori" upper estimate for the γ^{th} moment of $S(R)$ while R runs over all the rectangles in Z_1^d. This generalization of Theorem 1 reads as follows.

THEOREM 2. *Let $\gamma \geqq 1$ and $d \geqq 1$ be given. Suppose that there exist a nonnegative and superadditive function $f(R)$ of the rectangle R in Z_1^d, and a nonnegative function $\varphi(t_1;\ m_1, \ldots, m_d)$, nondecreasing in each variable, such that for every $R = R(b_1, \ldots, b_d;\ m_1, \ldots, m_d)$ we have*

$$\int |S(R)|^\gamma d\mu \leqq f(R) \varphi^\gamma(f(R);\ m_1, \ldots, m_d).$$

Then for every R we have both the inequality

$$\int M^\gamma(R)\, d\mu \leqq 3^{d(\gamma-1)} f(R) \times \tag{2.3}$$

$$\times \left\{ \sum_{k_1=0}^{[\log m_1]-1} \cdots \sum_{k_d=0}^{[\log m_d]-1} \varphi\left(\frac{f(R)}{2^{k_1+\ldots+k_d}};\ \left[\frac{m_1}{2^{k_1+1}}\right], \ldots, \left[\frac{m_d}{2^{k_d+1}}\right] \right) \right\}^\gamma$$

and the inequality

$$\int M^\gamma(R)\,d\mu \leqq \left(\frac{5}{2}\right)^d f(R)\times \tag{2.4}$$

$$\times\left\{\sum_{k_1=0}^{[\log m_1]} \dots \sum_{k_d=0}^{[\log m_d]} \varphi\left(\frac{f(R)}{2^{k_1+\dots+k_d}};\ \left[\frac{m_1}{2^{k_1}}\right], \dots, \left[\frac{m_d}{2^{k_d}}\right]\right)\right\}^\gamma.$$

Again we use the following convention: in case $m_j=1$ for some j, $1\leqq j\leqq d$, we take $[\log m_j]-1$ to be equal to 0 and $[m_j/2^{k_j+1}]$ to be equal to 1 on the right of (2.3).

3. Special cases

Without aiming at completeness we present here some special cases of Theorem 2 of interest in themselves.

Let us take $\varphi(t_1;\ m_1, \dots, m_d)=t_1^{(\alpha-1)/\gamma}$ with real $\alpha, \alpha>1$. Then

$$\tilde{\Phi}_d(t_1;\ m_1, \dots, m_d) = \sum_{k_1=0}^{[\log m_1]} \dots \sum_{k_d=0}^{[\log m_d]} \varphi\left(\frac{t_1}{2^{k_1+\dots+k_d}};\ \left[\frac{m_1}{2^{k_1}}\right], \dots, \left[\frac{m_d}{2^{k_d}}\right]\right) \leqq$$

$$\leqq (1-2^{(1-\alpha)/\gamma})^{-d}\, t_1^{(\alpha-1)/\gamma},$$

independently of $m_1, \dots, m_d$.

Corollary 1. *Let* $\alpha>1$; $\gamma\geqq 1$, *and* $d\geqq 1$ *be given. Suppose that there exists a nonnegative and superadditive function* $f(R)$ *of the rectangle* R *in* Z_1^d *such that for every* R *we have*

$$\int |S(R)|^\gamma\,d\mu \leqq f^\alpha(R).$$

Then for every R *we have*

$$\int M^\gamma(R)\,d\mu \leqq \left(\frac{5}{2}\right)^d (1-2^{(1-\alpha)/\gamma})^{-d\gamma} f^\alpha(R).$$

This result, apart from the factor $(5/2)^d$ on the right, was proved by the present author in [5, Theorem 7]. For $d=1$ see Longnecker and Serfling [3], and [4].

It is instructive to state this corollary for the still more particular case when $f(R)=\sum_{k\in R} u_k$; where $\{u_k\colon k\in Z_1^d\}$ is a d-multiple sequence of nonnegative reals.

Corollary 1a. (The d-multiple version of the Erdős—Stečkin inequality.) *Let* $\alpha>1$, $\gamma\geqq 1$, *and* $\{u_k\geqq 0\colon k\in Z_1^d\}$ *be given. Suppose that for every rectangle* R *in* Z_1^d *we have*

$$\int |S(R)|^\gamma\,d\mu \leqq \left(\sum_{k\in R} u_k\right)^\alpha.$$

Then for every R *we have*

$$\int M^\gamma(R)\,d\mu \leqq \left(\frac{5}{2}\right)^d (1-2^{(1-\alpha)/\gamma})^{-d\gamma} \left(\sum_{k\in R} u_k\right)^\alpha.$$

As to the case $d=1$, see Erdős [1] and Gapoškin [2, pp. 29—31], the latter author making use of the oral communication of S. B. Stečkin.

Now take $\varphi(t_1; m_1, \ldots, m_d)=t_1^{(\alpha-1)/\gamma}w(t_1)$ where again $\alpha>1$ and $w(t_1)$ is a slowly varying positive function, i.e. $w(t_1)$ is defined on R_+, $w(t_1)>0$ for $t_1>0$, and for every positive C we have

$$\frac{w(Ct_1)}{w(t_1)} \to 1 \quad \text{as} \quad t_1 \to \infty.$$

We emphasize that $w(t_1)$ is not necessarily a nondecreasing function, only $t_1^{(\alpha-1)/\gamma}w(t_1)$ has to be nondecreasing. For example,

$$w(t_1) = \{\log(1+t_1)\}^\beta \{\log\log(2+t_1)\}^\delta$$

is a slowly varying function, where β and δ are arbitrary reals. It is easy to check that again we have

$$\tilde{\Phi}_d(t_1; m_1, \ldots, m_d) \leqq C(\alpha, \gamma, d, w)\, t_1^{(\alpha-1)/\gamma} w(t_1),$$

where $C(\alpha, \gamma, d, w)$ denotes a positive constant depending only on α, γ, d, and $w(t_1)$.

COROLLARY 2. *Let $\alpha>1$, $\gamma\geqq 1$, and $d\geqq 1$ be given. Suppose that there exist a nonnegative and superadditive function $f(R)$ of the rectangle R in Z_1^d, and a slowly varying positive function $w(t_1)$, $t_1^{(\alpha-1)/\gamma}w(t_1)$ is nondecreasing, such that for every R we have*

$$\int |S(R)|^\gamma d\mu \leqq f^\alpha(R) w^\gamma(f(R)).$$

Then for every R we have

$$\int M^\gamma(R)\, d\mu \leqq \left(\frac{5}{2}\right)^d C^\gamma(\alpha, \gamma, d, w) f^\alpha(R) w^\gamma(f(R)).$$

Next take $\varphi(t_1; m_1, \ldots, m_d)=\lambda(m_1, \ldots, m_d)$ where $\lambda(m_1, \ldots, m_d)$ is defined on Z_1^d, positive and nondecreasing in each variable.

COROLLARY 3. *Let $\gamma\geqq 1$ and $d\geqq 1$ be given. Suppose that there exist a nonnegative and superadditive function $f(R)$ of the rectangle R in Z_1^d, and a positive and nondecreasing d-multiple sequence $\{\lambda(m)\colon m\in Z_1^d\}$ such that for every $R=R(b_1, \ldots, b_d; m_1, \ldots, m_d)$ we have*

$$\int |S(R)|^\gamma d\mu \leqq f(R)\lambda^\gamma(m_1, \ldots, m_d).$$

Then for every R we have

$$\int M^\gamma(R)\, d\mu \leqq 3^{d(\gamma-1)} f(R) \left\{ \sum_{k_1=0}^{[\log m_1]-1} \cdots \sum_{k_d=0}^{[\log m_d]-1} \lambda\left(\left[\frac{m_1}{2^{k_1+1}}\right], \ldots, \left[\frac{m_d}{2^{k_d+1}}\right]\right)\right\}^\gamma$$

with the same convention as in Theorem 2 in the case $m_j=1$ for some j.

This moment inequality, apart from the factor $3^{d(\gamma-1)}$ on the right, was proved also by the present author in [6, Theorem 1] in a slightly different form.

To illuminate the strength of Corollary 3, we present two particular cases. First, assume that $\{\xi_k\colon k\in Z_1^d\}$ is a d-multiple orthogonal system. Then we obviously have

$$\int S^2(R)\, d\mu = \sum_{k\in R} \sigma_k^2 \quad \text{where} \quad \sigma_k^2 = \int \xi_k^2\, d\mu.$$

COROLLARY 3a. (The d-multiple version of the Rademacher—Menšov inequality.) *If* $\{\xi_k\colon k\in Z_1^d\}$ *is a d-multiple orthogonal system, then for every rectangle* $R=R(b_1, \dots, b_d; m_1, \dots, m_d)$ *we have*

$$\int M^2(R)\,d\mu \leqq 3^d \left(\sum_{k\in R}\sigma_k^2\right)\prod_{j=1}^{d}(\log(m_j+1))^2.$$

As to the case $d=1$, see e.g. [8, p. 83].

Secondly, assume that $\varphi(t_1; m_1, \dots, m_d)=\lambda(m_1, \dots, m_d)$ essentially grows in each variable in the sense that there exist an $m_0\in Z_1$ and a real q, $q>1$, such that for every j, $1\leqq j\leqq d$, and for every $m\in Z_1^d$ with $m_j\geqq m_0$ we have

$$\frac{\lambda(m_1, \dots, m_{j-1}, 2m_j, m_{j+1}, \dots, m_d)}{\lambda(m_1, \dots, m_{j-1}, m_j, m_{j+1}, \dots, m_d)} \geqq q. \tag{3.1}$$

E.g. $\lambda(m)=\prod_{j=1}^{d} m_j^{\alpha_j} w_j(m_j)$ is such a d-multiple sequence where $\alpha_j>0$ and $w_j(m_j)$ is a slowly varying function for each j, $1\leqq j\leqq d$. Now (3.1) implies, in a routine way, that

$$\tilde{\Phi}_d(t_1; m_1, \dots, m_d) \leqq C(q, m_0)\lambda(m_1, \dots, m_d),$$

where the positive constant $C(q, m_0)$ depends only on q and on those values $\lambda(m)$ for which $m_j\leqq m_0$ for each j, $1\leqq j\leqq d$. In particular, $C(q, m_0)=\{q/(q-1)\}^d$ if $m_0=1$.

COROLLARY 3b. *Let* $\gamma\geqq 1$ *and* $d\geqq 1$ *be given. Suppose that there exist a nonnegative and superadditive funtion* $f(R)$ *of the rectangle* R *in* Z_1^d *and a d-multiple positive and nondecreasing sequence* $\{\lambda(m)\colon m\in Z_1^d\}$ *satisfying relation* (3.1) *with a* $q>1$ *such that for every* $R=R(b_1, \dots, b_d; m_1, \dots, m_d)$ *we have*

$$\int |S(R)|^\gamma\,d\mu \leqq f(R)\lambda^\gamma(m_1, \dots, m_d).$$

Then for every R *we have*

$$\int M^\gamma(R)\,d\mu \leqq \left(\frac{5}{2}\right)^d C^\gamma(q, m_0)f(R)\lambda^\gamma(m_1, \dots, m_d).$$

The special case $d=1$, $\gamma>2$, $f(R)=B^{\gamma-2}\sum_{k\in R}\sigma_k^\gamma$, and $\lambda(m_1)=m_1^{(\gamma-2)/\gamma}$ with the assumption that $|\xi_{k_1}|\leqq B$ a.e. ($k_1=1, 2, \dots$) is known as the Menšov—Paley inequality (cf. [9, p. 189]).

Finally, it is worth mentioning that in any case we can conclude the following

COROLLARY 4. *Under the conditions of Theorem* 2, *for every rectangle* $R=R(b_1, \dots, b_d; m_1, \dots, m_d)$ *we have*

$$\int M^\gamma(R)\,d\mu \leqq 3^{d(\gamma-1)}f(R)\varphi^\gamma\left(f(R); \left[\frac{m_1}{2}\right], \dots, \left[\frac{m_d}{2}\right]\right)\prod_{j=1}^{d}(\log(m_j+1))^\gamma,$$

where again in the case $m_j=1$ *for some* j *we take* $[m_j/2]$ *equal to* 1.

4. Proof of Theorem 2

The proof proceeds by induction on d. The case $d=1$ is stated in Theorem 1. Assume now that Theorem 2 holds for $d-1$. We will show that it holds for d. Consequently, the induction hypothesis can be applied to the following "partial" maximum:

$$M_{d-1}(R) = M_{d-1}(b, m) = M_{d-1}(b_1, \ldots, b_d;\ m_1, \ldots, m_{d-1}, m_d) =$$
$$= \max_{1 \leqq p_1 \leqq m_1} \ldots \max_{1 \leqq p_{d-1} \leqq m_{d-1}} |S(b_1, \ldots, b_d;\ p_1, \ldots, p_{d-1}, m_d)|.$$

Proof of (2.3). By the induction hypothesis,

$$(4.1)\qquad \int M_{d-1}^{\gamma}(R)\, d\mu \leqq 3^{(d-1)(\gamma-1)} f(R) \Phi_{d-1}^{\gamma}(f(R);\ m_1, \ldots, m_d),$$

where

$$\Phi_{d-1}(t_1;\ m_1, \ldots, m_{d-1}, m_d) =$$
$$= \sum_{k_1=0}^{[\log m_1]-1} \ldots \sum_{k_{d-1}=0}^{[\log m_{d-1}]-1} \varphi\left(\frac{t_1}{2^{k_1+\ldots+k_{d-1}}};\ \left[\frac{m_1}{2^{k_1+1}}\right], \ldots, \left[\frac{m_{d-1}}{2^{k_{d-1}+1}}\right], m_d\right)$$

and in the case $m_j=1$ we mean 0 by $[\log m_j]-1$ and 1 by $[m_j/2^{k_j+1}]$.

Setting

$$\Phi_d(t_1;\ m_1, \ldots, m_d) = \sum_{k_1=0}^{[\log m_1]-1} \ldots \sum_{k_d=0}^{[\log m_d]-1} \varphi\left(\frac{t_1}{2^{k_1+\ldots+k_d}};\ \left[\frac{m_1}{2^{k_1+1}}\right], \ldots, \left[\frac{m_d}{2^{k_d+1}}\right]\right)$$

with the above convention, inequality (2.3) to be proved can be rewritten as follows:

$$(4.2)\qquad \int M_d(R)\, d\mu \leqq 3^{d(\gamma-1)} f(R) \Phi_d^{\gamma}(f(R);\ m_1, \ldots, m_d),$$

where

$$M_d(R) = M(R) = \max_{1 \leqq p_1 \leqq m_1} \ldots \max_{1 \leqq p_d \leqq m_d} |S(b_1, \ldots, b_d;\ p_1, \ldots, p_d)| = \max_{1 \leqq p_d \leqq m_d} M_{d-1}(R).$$

It is not hard to verify that $\Phi_d(t_1;\ m_1, \ldots, m_d)$ can be also expressed by the aid of $\Phi_{d-1}(t_1;\ m_1, \ldots, m_d)$ as follows:

$$\Phi_d(t_1;\ m_1, \ldots, m_{d-1}, m_d) = \sum_{k_d=0}^{[\log m_d]-1} \Phi_{d-1}\left(\frac{t_1}{2^{k_d}};\ m_1, \ldots, m_{d-1},\ \left[\frac{m_d}{2^{k_d+1}}\right]\right),$$

with the same convention as above concerning the case $m_d=1$. This relation also turns into the following recurrence:

$$(4.3)\qquad \Phi_d(t_1;\ m_1, \ldots, m_d) = \Phi_{d-1}(t_1;\ m_1, \ldots, m_d) \quad \text{for} \quad m_d = 1, 2, 3$$

and

$$(4.4)\qquad \Phi_d(t_1;\ m_1, \ldots, m_{d-1}, m_d) = \Phi_{d-1}\left(t_1;\ m_1, \ldots, m_{d-1}, \left[\frac{m_d}{2}\right]\right) +$$
$$+ \Phi_d\left(\frac{t_1}{2};\ m_1, \ldots, m_{d-1}, \left[\frac{m_d}{2}\right]\right) \quad \text{for} \quad m_d \geqq 4.$$

After these preliminaries we can prove (4.2) by using again an induction but this time on m_d. Both the case of the initial values $m_d=1, 2, 3$ and the induction step are similar to the argument explained in the proof of Theorem 1 in [7]. Therefore, we only sketch the proof.

If $m_d=1$, then (4.2) immediately follows from (4.1) due to (4.3) and the fact that

$$M_d(b;\ m_1, ..., m_{d-1}, 1) = M_{d-1}(b;\ m_1, ..., m_{d-1}, 1).$$

In case $m_d=2$ or 3 one can use the trivial estimate

$$M_d(b, m) \leqq \sum_{k_d=b_d+1}^{b_d+m_d} M_{d-1}(b_1, ..., b_{d-1}, k_d-1;\ m_1, ..., m_{d-1}, 1)$$

and argue as in [7].

Now we assume, as the second induction hypothesis, that inequality (4.2) holds true for all values of the first $2d-1$ arguments $b_1, ..., b_d;\ m_1, ..., m_{d-1}$ and for all values of the $(2d)^{\text{th}}$ argument less than m_d, $m_d \geqq 4$.

The case $f(R)=f(b, m)=0$ can be handled with ease since then $M(R)=0$ a.e. Hence we assume that $f(R)\neq 0$. Then there exists an integer p_d, $1\leqq p_d \leqq m_d$, such that

$$\text{(4.5)} \qquad f(b;\ m_1, ..., m_{d-1}, p_d-1) \leqq \frac{1}{2} f(R) < f(b;\ m_1, ..., m_{d-1}, p_d),$$

the left hand side being 0 in case $p_d=1$. It is also convenient to set $S(b, m)= =M(b, m)=0$ if $m_j=0$ for some j, $1\leqq j\leqq d$.

Applying (2.2) for $j=d$ and taking (4.5) into account we obtain

$$f(b_1, ..., b_{d-1}, b_d+p_d;\ m_1, ..., m_{d-1}, m_d-p_d) \leqq$$

$$\leqq f(R)-f(b;\ m_1, ..., m_{d-1}, p_d) < \frac{1}{2} f(R).$$

The following three cases will be distinguished: $p_d=1, 2\leqq p_d \leqq m_d-1$, and $p_d=m_d$.

Case (i): $2\leqq p_d\leqq m_d-1$. Set

$$p_d' = \left[\frac{p_d-1}{2}\right] \quad \text{and} \quad q_d' = \begin{cases} p_d' & \text{if } p_d-1 \text{ is even,} \\ p_d'+1 & \text{if } p_d-1 \text{ is odd;} \end{cases}$$

$$p_d'' = \left[\frac{m_d-p_d}{2}\right] \quad \text{and} \quad q_d'' = \begin{cases} p_d'' & \text{if } m_d-p_d \text{ is even,} \\ p_d''+1 & \text{if } m_d-p_d \text{ is odd.} \end{cases}$$

It is obvious that

$$p_d'+q_d' = p_d-1 \quad \text{and} \quad p_d''+q_d'' = m_d-p_d.$$

Now, for $1 \leqq k_d \leqq m_d$, we can establish the following upper estimate:

$$M_{d-1}(b;\ m_1, \ldots, m_{d-1}, k_d) \leqq$$

$$\leqq \begin{cases} M_d(b;\ m_1,\ \ldots, m_{d-1}, p_d') & \text{for} \quad 1 \leqq k_d \leqq p_d', \\ M_{d-1}(b;\ m_1, \ldots, m_{d-1}, q_d') + \\ \quad + M_d(b_1, \ldots, b_{d-1}, b_d + q_d';\ m_1, \ldots, m_{d-1}, p_d') & \text{for} \quad q_d' \leqq k_d \leqq p_d - 1, \\ M_{d-1}(b;\ m_1, \ldots, m_{d-1}, p_d) + \\ \quad + M_d(b_1, \ldots, b_{d-1}, b_d + p_d;\ m_1, \ldots, m_{d-1}, p_d'') & \text{for} \quad p_d \leqq k_d \leqq p_d + p_d'', \\ M_{d-1}(b;\ m_1, \ldots, m_{d-1}, p_d + q_d'') + \\ \quad + M_d(b_1, \ldots, b_{d-1}, b_d + p_d + q_d'';\ m_1, \ldots, m_{d-1}, p_d'') & \text{for} \quad p_d + q_d'' \leqq k_d \leqq m_d; \end{cases}$$

whence

$$\begin{aligned} M_d(b, m) \leqq{}& M_{d-1}(b_1, \ldots, b_{d-1}, b_d;\ m_1, \ldots, m_{d-1}, q_d') + \\ &+ M_{d-1}(b_1, \ldots, b_{d-1}, b_d + q_d'; m_1, \ldots, m_{d-1}, p_d - q_d') + \\ &+ M_{d-1}(b_1, \ldots, b_{d-1}, b_d + p_d;\ m_1, \ldots, m_{d-1}, q_d'') + \\ &+ \{M_d^\gamma(b_1, \ldots, b_{d-1}, b_d;\ m_1, \ldots, m_{d-1}, p_d') + \\ &+ M_d^\gamma(b_1, \ldots, b_{d-1}, b_d + q_d'; m_1, \ldots, m_{d-1}, p_d') + \\ &+ M_d^\gamma(b_1, \ldots, b_{d-1}, b_d + p_d;\ m_1, \ldots, m_{d-1}, p_d'') + \\ &+ M_d^\gamma(b_1, \ldots, b_{d-1}, b_d + p_d + q_d'';\ m_1, \ldots, m_{d-1}, p_d'')\}^{1/\gamma} = A_d + B_d, \end{aligned} \tag{4.6}$$

where A_d denotes the sum of the first three terms and B denotes the fourth term on the right-hand side of (4.6).

Case (ii): $p_d = 1$. Setting

$$p_d'' = \left[\frac{m_d - 1}{2}\right] \quad \text{and} \quad q_d'' = \begin{cases} p_d'' & \text{if} \quad m_d - 1 \quad \text{is even}, \\ p_d'' + 1 & \text{if} \quad m_d - 1 \quad \text{is odd}; \end{cases}$$

we can estimate in a simpler way:

$$\begin{aligned} M_d(b, m) \leqq{}& M_{d-1}(b_1, \ldots, b_{d-1}, b_d;\ m_1, \ldots, m_{d-1}, 1) + \\ &+ M_{d-1}(b_1, \ldots, b_{d-1}, b_d + 1;\ m_1, \ldots, m_{d-1}, q_d'') + \\ &+ \{M_d^\gamma(b_1, \ldots, b_{d-1}, b_d + 1;\ m_1, \ldots, m_{d-1}, p_d'') + \\ &+ M_d^\gamma(b_1, \ldots, b_{d-1}, b_d + q_d'';\ m_1, \ldots, m_{d-1}, p_d'')\}^{1/\gamma} = A_d' + B_d'. \end{aligned} \tag{4.7}$$

Case (iii): $p_d = m_d$. Now

$$M_d(b, m) \leqq M_{d-1}(b_1, \ldots, b_{d-1}, b_d; m_1, \ldots, m_{d-1}, q_d'') + \\ + M_{d-1}(b_1, \ldots, b_{d-1}, b_d + m_d - 1; m_1, \ldots, m_{d-1}, 1) + \\ + \{M_d^\gamma(b_1, \ldots, b_{d-1}, b_d; m_1, \ldots, m_{d-1}, p_d'') + \\ + M_d^\gamma(b_1, \ldots, b_{d-1}, b_d + q_d''; m_1, \ldots, m_{d-1}, p_d'')\}^{1/\gamma}, \tag{4.8}$$

where p_d'' and q_d'' are the same as in Case (ii).

The further reasoning closely follows that of the proof of Theorem 1 in [7]. We omit it.

Proof of (2.4). Without entering into details we note that we only have to modify estimates (4.6)—(4.8) in the following manner: in Case (i) $M_d(b, m) \leqq A_d'' + B_d$, where B_d is defined in (4.6) and

$$A_d'' = \{M_{d-1}^\gamma(b; m_1, \ldots, m_{d-1}, q_d') + \\ + M_{d-1}^\gamma(b; m_1, \ldots, m_{d-1}, p_d) + M_{d-1}^\gamma(b; m_1, \ldots, m_{d-1}, p_d + q_d'')\}^{1/\gamma};$$

in Case (ii) $M_d(b, m) \leqq A_d''' + B_d'$, where B_d' is defined in (4.7) and

$$A_d''' = \{M_{d-1}^\gamma(b; m_1, \ldots, m_{d-1}, 1) + M_{d-1}^\gamma(b; m_1, \ldots, m_{d-1}, q_d'')\}^{1/\gamma};$$

and a similar modification of (4.8) in Case (iii).

Thus, by a double induction, one can prove both (2.3) and (2.4) for each $m_d = 1, 2, \ldots$ and for each $d = 1, 2, \ldots$.

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(Received October 5, 1981)

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Acta Math. Hung.
41 (3—4) (1983), 347—357.

A GENERALIZATION OF THE WEYL GROUP

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Some basic concepts in the structure theory of compact Lie groups can be naturally derived by adhibiting results from the theory of compact transformation groups. In fact, if a compact connected Lie group G is given then by considering its adjoint action $\mathrm{Ad}\colon G\times\mathfrak{g}\to\mathfrak{g}$ on its Lie algebra $\mathfrak{g}$ concepts such as Cartan subalgebra, Weyl chamber and Weyl group of G can be obtained by applying results concerning compact transformation groups ([3], pp. 15—23). It has been shown earlier that in case of an orthogonal action $\alpha\colon G\times\mathbf{R}^n\to\mathbf{R}^n$ of a compact connected Lie group such that the isotropy subgroups of α are of maximal rank, concepts can be introduced which generalize the Cartan subalgebra, the Weyl chamber and the Weyl group in a natural way [12].

Some results will be presented below in order to show that such generalizations are possible in a much more general setting. Actually, let M be a complete analytic Riemannian manifold, G a compact connected Lie group whose elements are isometries of M and such that the isotropy subgroup of its action $\alpha\colon G\times M\to M$ are of maximal rank. Then concepts can be introduced which reduce to Cartan subalgebras, Weyl chambers and to Weyl group of G in that special case when α is the adjoint action of G. In deriving the above results some theorems of J. H. C. Whitehead concerning conjugate loci [14] are generalized too in order to provide some auxiliary statements.

1. A generalization of Cartan subalgebras as the fixed point sets of principal isotropy subgroups

Unless otherwise stated Riemannian manifolds and quantities considered below will be supposed to be of class C^∞.

Let M be a complete Riemannian manifold, G a compact connected Lie group whose elements are isometries of M and $\alpha\colon G\times M\to M$ the action of G. Then α is of class C^∞ or analytic according to as M is of class C^∞ or analytic ([7] pp. 203—315). Let G_x be the isotropy subgroup of α at a point $x\in M$ and

$$F(G_x)=\{z|\alpha(g,z)=z \text{ for } g\in G_x \text{ where } z\in M\}$$

the fixed point set of G_x. According to a basic result concerning fixed point sets of isometry groups the components of $F(G_x)$ are totally geodesic submanifolds of M and any two points in different components are conjugate to each other ([4], [5] pp 59—61). In the special case of the adjoint action $\mathrm{Ad}\colon G\times\mathfrak{g}\to\mathfrak{g}$ of a compact connected Lie group G the above fixed point sets are well-known for principal isotropy subgroups. In fact, the orbit of $X\in\mathfrak{g}$ is principal if and only if X is a regular element of $\mathfrak{g}$ and in such case $F(G_X)=\mathfrak{h}\subset\mathfrak{g}$ is a Cartan subalgebra. Actually, G_X is the union of the maximal tori containing $\exp(X)$ for any $X\in\mathfrak{g}$ by some basic algebraic facts ([3] pp 15—16) and consequently $F(G_X)$ is the Cartan subalgebra containing X if this element is regular. In what follows fixed point sets of principal isotropy subgroups will be considered in case of isometric actions.

The following lemma presents a simple observation concerning the fixed point set of a principal isotropy subgroup, where a basic fact is applied. In fact let G_x be a principal isotropy subgroup of an action α and F_x the component of $F(G_x)$ containing x then obviously any slice S_x of the action α at x is included in F_x.

LEMMA 1. *Let M be a complete Riemannian manifold, G a compact connected Lie group whose elements are isometries of M and $\alpha\colon G\times M\to M$ the action of G. If the orbit $G(x)$ of x is principal then $M=G(F_x)$ is valid for the component F_x of the fixed point set $F(G_x)$ containing the point x.*

PROOF. In fact, let $z'\in M$ be an arbitrary point, z a point of $G(x)$ which is at minimal distance from z' and $g\in G$ an element such that $x=\alpha(g,z)$ is valid. Then x is a point of $G(x)$ having minimal distance from $x'=\alpha(g,z')$; consequently a minimal geodesic segment γ joining x to x' is perpendicular to $G(x)$ at x. Since $G(x)$ is principal any point of γ is left fixed by every element of G_x. Therefore, the image of γ is included in F_x and consequently $z'\in\alpha(g^{-1},F_x)$ is valid. Hence the assertion of the lemma obviously follows.

The next lemma expresses such a property of the fixed point sets of principal isotropy subgroups which subsists in that special case when the isotropy subgroups of an isometric action are of maximal rank.

LEMMA 2. *Let M be a complete Riemannian manifold, G a compact connected Lie group whose elements are isometries of M and such that the isotropy subgroups of its action $\alpha\colon G\times M\to M$ are of maximal rank. If the orbit $G(x)$ of $x\in M$ is principal and $T_xM=T_xG(x)\oplus N_xG(x)$ is the orthogonal decomposition of the tangent space at this point, then $F_x=\exp_x(N_xG(x))$ holds for the component F_x containing x of the fixed point set $F(G_x)$.*

PROOF. Since M is complete and $G(x)$ is principal the inclusion

$$\exp_x(N_xG(x))\subset F_x$$

is obviously valid. On the other hand the assumption that the principal isotropy subgroups of α are of maximal rank implies the existence of an $\varepsilon>0$ such that

$$B(x,\varepsilon)\cap F_x=B(x,\varepsilon)\cap\exp_x(N_xG(x))$$

holds for the solid ball $B(x,\varepsilon)$ of radius ε centered at x. In fact, the non-existence of such an ε obviously entails that there is a $v\in T_xM$ such that $v\notin N_xG(x)$ and $v=T_x\alpha_g v$ for $g\in G_x$. But in this case there exists a $v'\in T_xG(x)-\{0_x\}$ such that $v'=T_x\alpha_g v'$ for $g\in G_x$. Since $G(x)$ is equivariantly isomorphic with G/G_x the existence of such a v' is in contradiction with the assumption that $G_x\subset G$ is of maximal rank ([6] pp 66—70). Since M is complete and F_x is totally geodesic, any point of F_x can be joined to x with a minimal geodesic segment γ in F_x. But then the image of γ

is included in $\exp_x(N_x G(x))$ by the preceding observation. Consequently the inclusion $F_x \subset \exp_x(N_x G(x))$ is valid and the assertion of the lemma is established.

The above lemma and a former observation imply that if α is an isometric action such that its isotropy subgroups are of maximal rank and the orbit of the point x is principal then for any slice S_x of α at x the inclusion $S_x \subset \exp_x(N(G_x))$ is valid. This fact has been already obtained earlier by a different argument [11].

The following lemma presents a fact which will be applied subsequently in generalizing the concept of the Weyl group of a compact connected Lie group.

LEMMA 3. *Let M be a complete Riemannian manifold, G a compact connected Lie group whose elements are isometries of M and $\alpha\colon G\times M\to M$ the action of G. Then for any point $x\in M$ the equality*

$$\{g|\alpha(g, F(G_x)) = F(G_x),\ g\in G\} = N(G_x)$$

is valid, where $N(G_x)$ is the normalizer of the isotropy subgroup G_x in G.

PROOF. Consider first an element $g\in G$ such that $\alpha(g, F(G_x))=F(G_x)$ holds and put $z=\alpha(g, x)$. Then $G_z=gG_xg^{-1}$ and $G_x\subset G_z$ imply that $G_x\subset gG_xg^{-1}$ is valid. Since G is compact $G_x=gG_xg^{-1}$ follows ([2] pp 3—4) and consequently $g\in N(G_x)$ is obtained.

Consider secondly an element $g\in N(G_x)$ and a point $z\in F(G_x)$. If $y=\alpha(g, z)$ then $G_x\subset G_z$ and $G_y=g^{-1}G_zg$ imply that

$$G_x = g^{-1}G_xg\subset g^{-1}G_zg = G_y$$

holds and hence $y\in F(G_x)$ is valid. Thus $\alpha(g, F(G_x))\subset F(G_x)$ is obtained. Since $g^{-1}\in N(G_x)$ holds $\alpha(g^{-1}, F(G_x))\subset F(G_x)$ is valid and consequently $\alpha(g, \alpha(g^{-1}, F(G_x))=F(G_x)\subset\alpha(g, F(G_x))$. Therefore $\alpha(g, F(G_x))=F(G_x)$ follows.

2. The focal locus of an analytic submanifold having trivial normal bundle

In what follows some results will be presented concerning the focal locus of a compact analytic submanifold in a complete analytic Riemannian manifold provided that the submanifold has a trivial normal bundle. These results which will be applied subsequently in case of principal orbits of compact isometry groups of complete analytic Riemannian manifolds, can be considered as generalizations in the special case of Riemannian manifolds of some theorems of J. H. C. Whitehead concerning the conjugate locus of a point in a complete analytic Finsler manifold [14].

Let M be an m-dimensional complete analytic Riemannian manifold, $L\subset M$ a k-dimensional compact analytic submanifold and consider the orthogonal decomposition $T_xM=T_xL\oplus N_xL$ for $x\in L$, where N_xL is said to be the *normal subspace* of L at x. The union of normal subspaces yields the *normal bundle* $N(L)=$ $=\cup\{N_xL|x\in L\}$ of L which is an m-dimensional analytic subbundle of the tangent bundle $T(M)$. The exponential map of $T(M)$ restricted to $N(L)$ yields an analytic map $\varepsilon\colon N(L)\to M$ which is surjective since M is complete and L is compact. Let $x\in L$ and $v\in N_xL$ then v is called a *focal point of L in $N(L)$* provided that the tangent linear map $T_v\varepsilon\colon T_vN(L)\to T_zM$ is not injective where $z=\varepsilon(v)$. In this case z is said to be a *focal point of L in M* and the dimension of the kernel of $T_v\varepsilon$ is called the *order* of the focal points v and z. The set of focal points of L in $N(L)$ and in M are called the *focal locus* of L in $N(L)$ and in M, respectively.

A more restricted concept of focal point has been introduced by F. W. Warner on account of the basic relation of focal points and some Jacobi fields [13]. Actually let $x\in L$ and $u\in N_xL$ a unit vector. Then a geodesic $\gamma\colon \mathbf{R}\to M$ is defined by $\gamma(0)=$ $=x$, $\dot\gamma(0)=u$. Consider now along γ a Jacobi field X; $\mathbf{R}\to T(M)$ then X is said to be an *L-Jacobi field* provided that the following conditions are satisfied:

1. $\langle X(\tau), \gamma'(\tau)\rangle = 0$ for $\tau\in\mathbf{R}$;
2. $X(0)\in T_xL$;
3. $X'(0)-\sigma_u X(0)\in N_xL$

where $\sigma_u\colon T_xL\to T_xL$ is the endomorphism defined at x for the direction of u by the second fundamental tensor of the submanifold ([1] pp 220—224). An L-Jacobi field X is said to be a *strong L-Jacobi field* if instead of condition 3 the following more restrictive condition is satisfied:

3* $X'(0)-\sigma_u X(0)=0_x$.

The basic relation of focal points and Jacobi fields is given by the following assertion: A focal point of L is given by $v=\zeta u$ for some $\zeta>0$ if and only if there is a non-trivial L-Jacobi field X along the geodesic γ such that $X(\zeta)=0$ holds ([1] pp 244—226). Accordingly $v=\zeta u$ is called a *strong focal point of L in* $N(L)$ provided that there is a non-trivial strong L-Jacobi field X along the geodesic γ such that $X(\zeta)=0$ holds [13]. In this case the point $z=\varepsilon(v)$ is said to be a *strong focal point of L in M*. The set of strong focal points of L in $N(L)$ and in M are called the *strong focal locus* of L in $N(L)$ and in M, respectively.

Consider as above a unit vector $u\in N_xL$ at some point $x\in L$ and the geodesic γ defined by u and let $P_\tau L$ the parallel translate of T_xL along γ to the point $\gamma(\tau)$ where $\tau\in\mathbf{R}$. We say that the *Jacobi equation splits along* γ *relative to L* provided the following condition is satisfied: The curvature transformation

$$Y\mapsto R_\tau Y = R\big(Y, \gamma'(\tau)\big)\gamma'(\tau), \quad Y\in T_{\gamma(\tau)}M$$

maps the subspace $P_\tau L\subset T_{\gamma(\tau)}M$ into itself where R is the curvature tensor of the Riemannian manifold [13]. If the Jacobi equation splits along γ relative to L then the space of L-Jacobi fields defined along γ is the direct sum of the subspace of Jacobi field defined along γ and vanishing at x and of the subspace of strong L-Jacobi fields defined along γ as some calculations show [13]. This fact has the following obvious consequence given by

LEMMA 4. *Let M be a complete Riemannian manifold and L a compact submanifold such that the Jacobi equation splits along every geodesic γ defined by unit vectors $u\in N_xL$ where x is any point of L. Then the focal locus of L is the union of its strong focal locus and of the conjugate loci of points of L.*

In the special case when the submanifold L is a principal orbit of an isometric action whose isotropy subgroups have maximal rank the validity of the assumption of the above lemma and its conclusion as well have been established earlier [11].

The normal bundle $N(L)$ of a submanifold L of a Riemannian manifold M is said to be trivial provided that there is a vector bundle isomorphism $N(L)\to L\times N_xL$ where $x\in L$ and $L\times N_xL$ is the trivial vector bundle over L with N_xL as typical fibre. The induced Riemannian metric of the submanifold L and the euclidean metric of the subspace $N_xL\subset T_xM$ define a Riemannian metric on the product $L\times N_xL$ which is analytic if M is analytic and L is an analytic submanifold. Thus there is a unique Riemannian metric on the normal bundle $N(L)$ which renders the isomorphism $N(L)\to L\times N_xL$ isometric. Thus $N(L)$ has the structure of an analytic Riemannian manifold. Now the map $T_v\varepsilon\colon T_vN(L)\to T_zM$, where $v\in N(L)$ and $z=\varepsilon(v)$, is a homomorphism of euclidean vector spaces and consequently it defines a change $J(v)$ of volume; i.e., if v_z and v_v are the canonical volume form at z and v, respectively then $v_v=J(v)(T_v\varepsilon)^*v_z$ holds for the reciprocal image $(T_v\varepsilon)^*v_z$ of v_z. Consequently an analytic function $J\colon N(L)\to\mathbf{R}$ is obtained. Now, the focal locus $F(L)$ of L in $N(L)$ is evidently given by

$$F(L)=\{v|J(v)=0,\quad v\in N(L)\}.$$

In general $F(L)$ is not a submanifold of $N(L)$, however, it is an analytic structure in sense of S. Lefshetz and J. H. C. Whitehead [14].

The following lemma generalizes a result of M. Morse concerning the order of conjugate points in a Finsler manifold [18] for the special case of Riemannian manifolds.

LEMMA 5. *Let M be a complete analytic Riemannian manifold and $L\subset M$ a compact analytic submanifold having trivial normal bundle and such that the Jacobi equation splits along any geodesic orthogonal to L. Let $u\in N_xL$ be a unit vector where $x\in L$ and consider the function $D(u,\tau)=J(\tau,u)$, $\tau\in\mathbf{R}$. Then, $v=\varkappa u$ with $\varkappa>0$ is a focal point of L in $N(L)$ if and only if $D(u,\varkappa)=0$ is valid. Moreover, if $v=\varkappa u$ is a focal point of L then the order of v is equal to the multiplicity of the root $\varkappa$ in the equation $D(u,\tau)=0$.*

PROOF. The first assertion of the lemma is evidently valid. In order to prove the second one consider the unit vector $\overline{X}_1\in T_vN(L)$ which corresponds to u under the canonical isomorphism $N_xL\to T_vN_xL$. Observe now that $T_v\varepsilon$ maps the subspace of $T_vN(L)$ orthogonal to $\overline{X}_1$ into a subspace of T_xM orthogonal to $X_1=\gamma'(\varkappa)\in T_v\varepsilon\overline{X}_1$ and the kernel of $T_v\varepsilon$ is included in the subspace of $T_vN(L)$ orthogonal to $\overline{X}_1$. Consider an orthonormal base $(\overline{X}_1,\dots,\overline{X}_m)$ of $T_vN(L)$ such that $(\overline{X}_{m-1+1},\dots,\overline{X}_m)$ is a base of the kernel of $T_v\varepsilon$. Then there is a unique extension $\overline{X}_i(\tau)$, $\tau\in\mathbf{R}$ of $\overline{X}_i$ along the line τu, $\tau\in\mathbf{R}$ for $i=2,\dots,m$ such that $X_i(\tau)=T_{\tau u}\varepsilon\overline{X}_i(\tau)$, $\tau\in\mathbf{R}$ is an L-Jacobi field; in fact, the L-Jacobi fields defined along γ form an $(m-1)$-dimensional vector space ([1] pp 220—224) and thus the same holds for the vector fields defined along the line τu, $\tau\in\mathbf{R}$ which are mapped into L-Jacobi fields by $T\varepsilon$. Moreover, put $X_1(\tau)=\gamma'(\tau)$, $\tau\in\mathbf{R}$. In a sufficiently small neighborhood of $\varkappa\in\mathbf{R}$ the following equalities are obviously valid:

$$X_1(\tau)=\gamma'(\tau);\quad X_i(\tau)=X_i^0(\tau)+(\tau-\varkappa)\tilde{X}_i(\tau)$$

with a non-zero parallel field $X_i^0(\tau)$ where $i=2,\dots,m-1$;

$$X_i(\tau)=(\tau-\varkappa)X_i^1(\tau)+(\tau-\varkappa)^2\hat{X}_i(\tau)$$

where $i = m-1+1, \ldots, m$. Consequently, in the above neighborhood of $\varkappa \in \mathbf{R}$ the following equality is obtained:

$$D(u, \tau) = (\tau-\varkappa)^1 \tilde{D}(u, \tau).$$

Now, it is evidently sufficient to show that $\tilde{D}(u, \varkappa) \neq 0$ holds. Actually, the value of $\tilde{D}(u, \varkappa)$ is given by a determinant whose columns are the coordinates of the following vectors in an orthonormal base of $T_z M$:

$\gamma'(\varkappa)$;

$X_i(\varkappa) = X_i^0(\varkappa)$ where $i=2, \ldots, m-1$;

$X_i'(\varkappa)$ where $i=m-1+1, \ldots, m$.

Therefore, it is sufficient to show that the above vectors are independent. In fact,

$$\langle X_i'(0), X_j(0)\rangle - \langle X_i(0), X_j'(0)\rangle = \langle \sigma_u X_i(0), X_j(0)\rangle - \langle X_i(0), \sigma_u X_j(0)\rangle = 0$$

holds where $i, j=2, \ldots, m$ since X_i, X_j are L-Jacobi fields and σ_u is symmetric. On the other hand, the functions

$$\langle X_i'(\tau), X_j(\tau)\rangle - \langle X_i(\tau), X_j'(\tau)\rangle, \quad \tau \in \mathbf{R}$$

are constant, where $i, j=1, \ldots, m$, on account of a basic identity concerning Jacobi fields. Consequently, $\langle X_i(\varkappa), X_j'(\varkappa)\rangle = 0$ where $i=2, \ldots, m-1$ and $j=m-l+1, \ldots, m$. Since the Jacobi equation splits along γ, even $\langle X_1(\varkappa), X_j'(\varkappa)\rangle = 0$ holds for $j=m-l+1, \ldots, m$. Therefore, it is sufficient to show that each of the two separate systems $(X_1(\varkappa), \ldots, X_{m-1}(\varkappa))$ and $(X'_{m-l+1}(\varkappa), \ldots, X'_m(\varkappa))$ are independent. In fact, the system $(X_1(\varkappa), \ldots, X_{m-1}(\varkappa))$ is evidently independent by its definition. On the other hand, the Jacobi fields $X_{m-1+1}, \ldots, X_m$ are independent by their definition. Consequently, since

$$X_{m-l+1}(\varkappa) = \ldots = X_m(\varkappa) = 0,$$

the system $(X'_{m-1+1}(\varkappa), \ldots, X'_m(\varkappa))$ has to be independent as well.

The following theorem generalizes for the special case of Riemannian manifolds a theorem of J. H. C. Whitehead stating that the conjugate locus of a point x of a complete analytic m-dimensional Finsler manifold in $T_x M = N(x)$ is $(m-1)$-dimensional [14].

THEOREM 1. *Let M be a complete analytic m-dimensional Riemannian manifold and $L \subset M$ a compact analytic k-dimensional submanifold having trivial normal bundle $N(L)$ and such that the Jacobi equation splits along the geodesic orthogonal to L. Then the intersection of the focal locus $F(L)$ of L in $N(L)$ with the normal space $N_x L$ is $(m-k-1)$-dimensional for $x \in L$.*

PROOF. Let $u \in N_x L$ be a unit vector and assume that $v = \varkappa u$, where $\varkappa > 0$, is a focal point of order l of L in $N(L)$. Fix a value $\zeta > \varkappa$ such that ζu is not a focal point of L. Then the geodesic curve $\gamma(\tau) = \varepsilon(\tau u)$, $\tau \in [0, \zeta]$ has a non-degenerate index form which is given by

$$I(X, Y) = \int_0^\zeta \left(\langle X'(\tau), Y'(\tau)\rangle - \langle R(\gamma'(\tau), X(\tau))\gamma'(\tau), Y(\tau)\rangle\right) d\tau + \langle \sigma_u X(0), Y(0)\rangle$$

and the index of the geodesic γ is equal to its augmented index which in turn is equal to the sum of orders of focal points of L on γ according to some basic results ([1], pp. 232—235). Moreover, if the unit vector $\tilde{u}\in N_x L$ is sufficiently near to u and $\tilde{\zeta}>0$ is sufficiently near to ζ then the index of the geodesic $\tilde{\gamma}(\tau)=\varepsilon(\tau\tilde{u})$, $\tau\in[0, \tilde{\zeta}]$ is equal to that of γ on account of some fundamental results ([1], pp. 236—237).

Therefore in this case the sum of orders of focal points of L on $\tilde{\gamma}$ is not less than the sum of orders of focal points of L on γ. On the other hand, the set $F(L)\cap N_x L$ in the neighborhood $\mathfrak{M}$ of the point $v=\varkappa u$ defined by the above neighborhoods of u and of $\varkappa$ is given by the equation $J(\tau\tilde{u})=D(\tilde{u}, \tau)=0$. Since the function D is analytic some fundamental results of the theory of analytic functions ([9] vol. II pp. 83—92) yield the existence of a neighborhood $\tilde{W}$ of the point $(u, \varkappa)$ where the function $D(\tilde{u}, \tau)$ is given by

$$D(\tilde{u}, \tau) = \tau^l+B_1(\tilde{u})\tau^{l-1}+\ldots+B_l(\tilde{u})$$

where l is the order of the focal point $v=\varkappa u$ and the functions $B_1(\tilde{u}), \ldots, B_l(\tilde{u})$ are analytic. Therefore the preceding lemma yields that the order of a focal point of L on a geodesic $\tilde{\gamma}$ specified above which is in a neighborhood of $v=\varkappa u$ defined by the neighborhood $\tilde{W}$ cannot be greater than l. Since the sum of orders of focal points of L on $\tilde{\gamma}$ is equal to the sum of orders of focal points of L on γ, the order of a focal point of L in the above neighborhood of $v=\varkappa u$ has to be l. But then any ray $\tau\tilde{u}$, $\tau\geqq 0$ corresponding to a geodesic $\tilde{\gamma}$ specified above intersects $F(L)$ in the neighborhood of $v=\varkappa u$. Therefore the set $F(L)\cap N_x L$ must have codimension 1 in the normal space $N_x L$ and thus the assertion of the theorem follows.

Generalizing a definition due to J. H. C. Whitehead [14] under the assumptions made above a point $c\in N_x L$ of the focal locus $F(L)\subset N(L)$ of the submanifold $L\subset M$ is said to be an *ordinary point* provided that the set $F(L)\cap N_x L$ is given by a cell of codimension 1 in $N_x L$ in a sufficiently small neighborhood of the point c, otherwise the point c is said to be a *branch point*. If the set of ordinary points of $F(L)$ in $N_x L$ is A_x and $A_x=C_1\cup\ldots\cup C_q$ is the decomposition of A_x into its components then $C_1, \ldots, C_q$ are evidently submanifolds of codimension 1 in $N_x L$.

3. A generalization of the Weyl chambers and of the Weyl group

Let G be a compact connected Lie group and consider its adjoint action Ad: $G\times\mathfrak{g}\to\mathfrak{g}$ on its Lie algebra $\mathfrak{g}$. If the orbit of $X\in\mathfrak{g}$ is principal then $F(G_X)=\mathfrak{h}\subset\mathfrak{g}$ is a Cartan subalgebra and the union of the walls of the Weyl chambers of G in $F(G_X)$ is the set of those points of $F(G_X)=\mathfrak{h}$ which have singular orbits ([3], pp. 17—23). On the other hand, the set of those points of $F(G_X)=\mathfrak{h}$ which have singular orbits can be obtained as the intersection of $F(G_X)$ with the focal locus of a principal orbit [10]. Consequently, the union of the walls of the Weyl chambers of G in $F(G_X)=\mathfrak{h}$ is the intersection of $F(G_X)=\mathfrak{h}$ with the focal locus of a principal orbit of the action Ad. This observation has been already generalized for orthogonal actions with isotropy subgroups of maximal rank [12]. In what follows the generalization will be carried out in a more general setting.

Let M be a complete Riemannian manifold, G a compact connected Lie group whose elements are isometries of M and α: $G\times M\to M$ the action of G. If the orbit

$G(x)$ of $x \in M$ is principal then the normal bundle of the compact submanifold $G(x) \subset M$ is known to be trivial. Moreover, if the isotropy subgroups of the action α are of maximal rank then the Jacobi equation splits along the geodesics orthogonal to the principal orbit $G(x)$ by an earlier result [11]. Actually, some of Warner's results have been derived in the special case of principal orbits [11], where in unawareness of this results [13] terminology of Warner has not been applied yet. Thus the following lemma follows on account of some former observations.

LEMMA 6. *Let M be a complete Riemannian manifold, G a compact connected Lie group whose elements are isometries of M such that the isotropy subgroups of the action α: $G \times M \to M$ are of maximal rank. If the orbit $G(x)$ of $x \in M$ is principal then the focal locus of $G(x)$ is the union of the strong focal locus of $G(x)$ and of the conjugate loci of points of $G(x)$.*

The following theorem yields a description of the union of singular orbits in terms of the focal locus of principal ones.

THEOREM 2. *Let M be a complete Riemannian manifold, G a compact connected Lie group whose elements are isometries of M and such that the isotropy subgroups of the action α: $G \times M \to M$ are of maximal rank. Let S be the union of the singular orbits of α. If the orbit $G(x)$ of x is principal then S is equal to the strong focal locus of $G(x)$.*

PROOF. Let the orbit $G(z)$ of $z \in M$ be singular. There is no loss of generality by assuming that x is a nearest point of $G(x)$ to z. Consequently, a unit vector $u \in N_x G(x)$ exists such that for the geodesic γ: $\mathbf{R} \to M$ defined by $\gamma(0)=x$, $\gamma'(0)=u$ there is a $\zeta>0$ with $z=\gamma(\zeta)$. Now $G_x \subset G_z$ is obviously valid since $G(x)$ is principal and $G_x \neq G_z$ follows from the assumption that $G(z)$ is singular. Consequently, there is an infinitesimal isometry $X \in \mathfrak{g}$ such that $X(x) \neq 0$ and $X(z)=0$ is valid. The restriction of X to γ is obviously a $G(x)$-Jacobi field ([1] pp. 222—223) which is strong by a former result [11]. Therefore z is a strong focal point of $G(x)$.

Conversely, let a strong focal point $z \in M$ of $G(x)$ be given on a geodesic γ: $\mathbf{R} \to M$ which is defined by the unit vector $u \in N_x G(x)$ with $\gamma(0)=x$ and $\gamma'(0)=u$; and let X: $\mathbf{R} \to TM$ be a strong non-trivial $G(x)$-Jacobi field along γ with $X(\zeta)=0$ where $z=\gamma(\zeta)$ and $\zeta>0$. There is an infinitesimal isometry $\tilde{X} \in \mathfrak{g}$ such that $\tilde{X}(x)=X(0)$ holds. The restriction of $\tilde{X}$ to γ is a strong $G(x)$-Jacobi field by a former result [11]. Therefore, $\Delta_u X(0)-\sigma_u X(0)=0$ and $\Delta_u \tilde{X}(x)-\sigma_u \tilde{X}(x)=0$, i.e. $\Delta_u X(0)=\Delta_u \tilde{X}(x)$. Consequently, X coincides with the restriction of the infinitesimal isometry considered. Thus the orbit $G(z)$ is singular.

Now Theorems 1 and 2 yield the following obvious corollary which will be applied subsequently.

COROLLARY. *Let M be a complete analytic Riemannian manifold, G a compact connected Lie group whose elements are isometries of M and such that the isotropy subgroups of the action α: $G \times M \to M$ are of maximal rank. If the orbit $G(x)$ of $x \in M$ is principal then the set $S \cap F(G_x)$ is of codimension 1 in $F(G_x)$ where S is the union of the singular orbits of α.*

Let M be a complete analytic Riemannian manifold, G a compact connected Lie group whose elements are isometries of M and such that the isotropy subgroups of its action $\alpha\colon G\times M\to M$ are of maximal rank. Consider a point $x\in M$ such that $G(x)$ is principal and let F_x be the component of $F(G_x)$ containing the point x. Then $F_x=\exp_x\big(N_xG(x)\big)$ holds according to Lemma 2; moreover, Theorem 2 yields that the orbit $G(z)$ of a point $z\in F_x$ is singular if and only if $z=\exp_x(v)$ holds with $v\in N_xG(x)$ such that $v\in N_xG(x)$ is a strong focal point of $G(x)$. In other words,

$$S\cap F_x=\exp_x\big(F^*(G(x))\cap N_xG(x)\big)$$

where $F^*\big(G(x)\big)$ is the strong focal locus of $G(x)$ in $N\big(G(x)\big)$. Consider now a point $z\in S\cap F_x$; then any neighborhood of z contains a point $x'\in F_x$ such that $G(x')$ is principal. Consequently, there is a neighborhood of the zero vector in $N_{x'}\big(G(x')\big)$ on which the restricted exponential map $\varepsilon_{x'}$ is diffeomorphic and the image of the neighborhood is a neighborhood of x' in F_x which contains x'. Accordingly the point z is called an *ordinary point* of the set $S\cap F_x$ if z has a neighborhood in $S\cap F_x$ which is a cell of codimension 1 in F_x, otherwise z is called a *branch point*. Let now $\tilde{C}_1\cup\ldots\cup\tilde{C}_q$ be the decomposition of the set of ordinary points of $S\cap F_x$ into its components, then the subsets $\tilde{C}_1, \ldots, \tilde{C}_q$ are submanifolds of codimension 1 in F_x.

Let G be a compact connected Lie group and consider its adjoint action $\mathrm{Ad}\colon G\times\mathfrak{g}\to\mathfrak{g}$ on its Lie algebra $\mathfrak{g}$. Let the orbit of $X\in\mathfrak{g}$ be principal, then $\mathfrak{h}=F(G_X)\subset\mathfrak{g}$ is a Cartan subalgebra where the Weyl chambers are bounded by flats $F\subset\mathfrak{h}$ of codimension 1 called the walls of the Weyl chambers. Moreover, to each wall F there is an element g such that Ad_g restricted to $\mathfrak{h}$ is equal to the reflection of $\mathfrak{h}$ on F and the group generated by the restricted actions of such g as F runs through the set of walls of Weyl chambers in $\mathfrak{h}$ is called the Weyl group of G. In a former paper generalization of the Weyl group has been given for orthogonal action with isotropy subgroups of maximal rank [12]. It is shown below that such generalization works in a much more general setting.

THEOREM 3. *Let M be a complete analytic Riemannian manifold, G a compact connected Lie group whose elements are isometries of M such that the isotropy subgroups of its action $\alpha\colon G\times M\to M$ are of maximal rank. If the orbit $G(x)$ of $x\in M$ is principal and its focal point c is an ordinary point of the focal locus $F\big(G(x)\big)$ then there is an element $g\in G$ such that α_g restricted to the component $F_x\subset F(G_x)$ is an involution and maps $F_x\cap F\big(G(x)\big)$ onto itself keeping a sufficiently small neighborhood of c in this set pointwise fixed.*

PROOF. In fact, since c is an ordinary point of $F\big(G(x)\big)$ there is a unique geodesic $\gamma\colon\mathbf{R}\to M$ in the totally geodesic submanifold $F(G_x)$ with $\gamma(0)=c$ which is orthogonal to $F\big(G(x)\big)$ on account of Theorem 2 and of the preceding Corollary. If z is a point of γ such that $G(z)$ is principal then z is a point of $G(z)$ at minimal distance from c provided that z is sufficiently near to c. In fact assume that z is not at minimal distance from c, then c is a point of $G(c)$ which is not at minimal distance from z and then there is a point $c'\in G(c)$ which is at minimal distance from z, but then there is a cut point of $G(c)$ on the minimal geodesic segment joining t to c. But this yields a contradiction if z is sufficiently near to c. Let now y be a point of γ such that zcy holds and let $\bar{z}$ be a point of $G(z)$ at minimal distance from y. If y is in a sufficiently small strongly convex neighborhood U of c ([1] pp. 246—250) then $F\big(G(x)\big)\cap U$

has only ordinary points and then z is on the same side of $F(G(x))\cap U$ in U as y; in fact a minimal geodesic segment joining y and $\bar{z}$ cannot intersect the focal locus of $G(z)$. Let now z' be a limit point of $\bar{z}$ as y tends to c on γ. There is a $g\in G$ such that $z'=\alpha(g, z)$ is valid. Now, α_g maps S onto itself by definition of S and α_g maps $F(G_x)$ onto itself as well by Lemma 3. Consequently, α_g maps $S\cap F(G_x)$ onto itself and therefore interchanges the submanifolds $\tilde{C}_1, \ldots, \tilde{C}_q$ among themselves. Let now $\tilde{C}$ be that one of the $\tilde{C}_1, \ldots, \tilde{C}_q$ which contains the point c. Since c is ordinary α_g maps $\tilde{C}$ onto itself if z is sufficiently near to c. Consequently, α_g maps γ onto a geodesic passing through z' and orthogonal to $\tilde{C}$ and therefore α_g maps γ onto itself if z is sufficiently near to c. Consequently, α_g leaves c fixed and maps z onto z'. Assume now that α_g does not leave all points of $\tilde{C}$ fixed, then the set of fixed points of α_g is nowhere dense in $\tilde{C}$, since the existence of an open subset of fixed points of $\tilde{C}$ would imply that every point of $\tilde{C}$ is left fixed by α_g. Thus there is an ordinary point c_1 of $\tilde{C}$ which is not left fixed by α_g. The above construction of g applied now to c_1 yields an element $g_1\in G$ with analogous properties. If α_{g_1} does not leave all points of $\tilde{C}$ fixed then there is an ordinary point c_2 of $\tilde{C}$ which is not left fixed by α_g and α_{g_1}. Thus the above construction of g applied to c_2 yields an element $g_2\in G$. The repetition of the above process yields a sequence $\{g_i|\ i\in\mathbf{N}\}$ of elements of G. Since α_{g_i} maps $F(G_x)$ onto itself g_i is an element of $N(G_x)$ by Lemma 3 for $i\in\mathbf{N}$. On the other hand the restriction of α_{g_i} and α_{g_j} to $F(G_x)$ is different if $i\neq j$ by the above construction. Therefore the natural homomorphism $N(G_x)\to N(G_x)/G_x$ maps different elements of the sequence onto different elements. Since G_x is of maximal rank the group $N(G_x)/G_x$ is finite and thus a contradiction is obtained. Therefore α_g has to leave all points of $\tilde{C}$ fixed. Thus the assertion of the theorem follows.

Since the submanifolds $\tilde{C}_1, \ldots, \tilde{C}_q$ consist of fixed points of isometries they are open subsets of totally geodesic submanifolds $F_1, \ldots, F_r$ of codimension 1 in F_x. The components of the set

$$F_x - \cup\{F_i|i = 1, \ldots, r\}$$

are called *generalized Weyl chambers* of the action α and the totally geodesic submanifolds $F_1, \ldots, F_r$ are called walls of the *generalized Weyl chambers*. Moreover, to each F_i there is an element $g\in N(G_x)$ such that the restriction of α_g to F_x is an involution and every point of F_i is left fixed by α_g. Let now $\hat{W}\subset N(G_x)$ be the subgroups generated by elements g which correspond to the totally geodesic submanifolds $F_1, \ldots, F_r$ in the above way, then the group $W=\hat{W}/G_g$ is called the *generalized Weyl group* of the action α.

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(Received October 7, 1981)

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Acta Math. Hung.
41 (3—4) (1983), 359—370.

ON THE DIVERGENCE OF VILENKIN—FOURIER SERIES

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Introduction

Throughout this paper we are concerned with the divergence of the Fourier series with respect to the Vilenkin systems [17]. In the theory of the trigonometric series a remarkable theorem of A. N. Kolmogoroff [8], says that there exists an integrable function with everywhere divergent Fourier series. In Kolmogoroff's proof this function is constructed by means of a rather complicated method. It is worth mentioning that by a theorem of Y. Katznelson [4] it is enough to prove the existence of an integrable function with almost everywhere divergent Fourier series [7]; from this fact it follows already that there exists a function with everywhere divergent Fourier series.

The analogous question for the Walsh—Paley system was investigated first by E. M. Stein [16]. He proved the existence of an integrable function the Walsh—Fourier series of which diverges almost everywhere. Later F. Schipp [12] [13], using Kolmogoroff's method, constructed an integrable function with everywhere divergent Walsh—Fourier series. We remark that the analogue of the above mentioned result of Y. Katznelson was verified for the Walsh—Paley system by D. C. Harris—W. R. Wade [3] and Sh. V. Kheladze [5].

For the Vilenkin systems — with certain boundedness criterion — Sh. V. Kheladze [6] proved the existence of an integrable function with everywhere divergent Vilenkin—Fourier series by means of Katznelson's idea. He proved that the existence of such a function follows already from the existence of an integrable function with divergent Vilenkin—Fourier series on a set of positive measure. This follows from the well-known theorem of S. V. Bockariev [1].

In this connection we remark that P. D. Getsadze [2] sharpened Bockariev's result in the following way:

Suppose $(f_k, k\geqq 1)$ is a bounded functional orthonormal system satisfying the condition

$$\liminf_{n\to\infty} \sum_{k=nl}^{(n+1)l-1} |f_k(t)|^2 < +\infty \quad \text{for a.e. } t,$$

where l is a natural number independent of t. Then there is a function the Fourier series of which for the system $(f_k, k\geqq 1)$ is a.e. divergent.

Since the above conditions are true for all Vilenkin systems (e.g. $l=1$) the existence of the a.e. divergence follows. The above mentioned Katznelson's idea is not known for all Vilenkin systems.

In this paper we construct an integrable function the Vilenkin—Fourier series of which is everywhere divergent. We give the proof on the analogy of a beautiful simple construction of a function, for the Walsh—Paley system by F. Schipp.[1] In addition to the general case we investigate the so-called "unbounded" case and give a construction in this case, which is useful in the proof of certain statements on the divergence. By means of these assertions for the divergence we shall illustrate the sharp contrast between the "bounded" and the "unbounded" case. We are concerned also with the growth of the partial sums of the Vilenkin—Fourier series.

§ 1

In this section we establish the notation and terminology to be used in the sequel. Let

$$m=(m_0, m_1, \ldots, m_k, \ldots) \quad (2 \leqq m_k,\ m_k \in \mathbf{N},\ k \in \mathbf{N}:=\{0, 1, \ldots,\})$$

be a sequence of natural numbers and denote by Z_{m_k} the m_k^{th} discrete cyclic group, i.e.

$$Z_{m_k} := \{0, 1, \ldots, m_k-1\} \quad (k \in \mathbf{N}).$$

If we define the group G_m as the direct product of the groups Z_{m_k}, then G_m is a compact Abelian group. Thus the elements of G_m are of the form $x=(x_0, x_1, \ldots, x_k, \ldots)$ with $0 \leqq x_k < m_k (k \in \mathbf{N})$ and for x, y in G_m their sum $x+y$ is obtained by adding the n^{th} coordinates of x and y modulo m_n. The topology of G_m is determined completely by the following sets in G_m:

$$I_n(x) := \{y \in G_m:\ y=(x_0, \ldots, x_{n-1}, y_n, \ldots)\} \quad (x \in G_m,\ n \in \mathbf{N}).$$

Next, let $\hat{G}_m := \{\psi_n:\ n \in \mathbf{N}\}$ (the so-called Vilenkin system) denote the character group of G_m. We enumerate the elements of $\hat{G}_m$ as follows. For $k \in \mathbf{N}$ let r_k be the function defined by

$$r_k(x) := \exp \frac{2\pi i x_k}{m_k} \quad (x \in G_m,\ i := \sqrt{-1}).$$

If we define the sequence $(M_k,\ k \in \mathbf{N})$ by $M_0:=1$, $M_{k+1}:=m_k M_k$ $(k \in \mathbf{N})$, then each $n \in \mathbf{N}$ has a unique representation of the form

$$n = \sum_{k=0}^{\infty} n_k M_k,$$

where $0 \leqq n_k < m_k$, $n_k \in \mathbf{N}$. For such $n \in \mathbf{N}$ we define the function ψ_n by

$$\psi_n := \prod_{k=0}^{\infty} (r_k)^{n_k}.$$

We remark that $\hat{G}_m$ is a complete orthonormal system with respect to the normalized Haar measure on G_m [17]. Furthermore, if $m_n=2$ $(n \in \mathbf{N})$ then $\hat{G}_m$ is the Walsh—Paley system.

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[1] Personal communication.

For a function $f \in L^1(G_m)$ let

$$\hat{f}(k) := \int_{G_m} f \bar{\psi}_k \quad (k \in \mathbf{N}),$$

$$S_n(f) := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k \quad (n \in \mathbf{N}), \quad \sigma_n(f) := \frac{1}{n} \sum_{k=1}^{n} S_k(f) \ (n \in \mathbf{N} \setminus \{0\}),$$

$$D_n := \sum_{k=0}^{n-1} \psi_k \ (n \in \mathbf{N}), \quad K_n := \frac{1}{n} \sum_{k=1}^{n} D_k (n \in \mathbf{N} \setminus \{0\}), \quad \tau_y(f)(x) := f(x \dot{-} y) \ (x, y \in G_m).$$

($\dot{-}$ denotes the inverse of the group-operation).

The modulus of continuity and the integral modulus of continuity are defined by

$$\omega(f, \delta) := \sup_{\lambda(x) < \delta} \|f - \tau_x(f)\|_\infty \quad (f \in C(G_m)),$$

$$\omega_1(f, \delta) := \sup_{\lambda(x) < \delta} \|f - \tau_x(f)\|_1 \quad (f \in L^1(G_m)),$$

where $\delta > 0$ and $\lambda(x) := \sum_{i=0}^{\infty} \frac{x_i}{M_{i+1}}$ $(x = (x_0, x_1, \ldots) \in G_m)$, and $\|f\|_p$ denotes the L^p-norm of f $(1 \leqq p \leqq \infty)$.

§ 2

In the proof of the statements on the divergence mentioned in the introduction the following "polynomials" of $\hat{G}_m$ play an important role.

Let us assume that $m_{n_k} = 2$ $(k = 0, 1, \ldots)$ in the original occurence and introduce the following notations:

$$\Delta_k := \begin{cases} \left[\dfrac{m_k - 1}{2}\right] & (m_k > 2) \\[2ex] \dfrac{1 + (-1)^s}{2} & (m_k = m_{n_s} = 2) \end{cases} \quad (k \in \mathbf{N}),$$

([ ] denotes the entire part),

$$\tilde{\mathbf{N}} := \{n \in \mathbf{N} \colon \Delta_n \neq 0\}, \ N_n := \sum_{k=0}^{n} \Delta_k M_k \quad (n \in \tilde{\mathbf{N}}),$$

$$h_k := \begin{cases} \displaystyle\sum_{j=1}^{\Delta_k} \frac{r_k^j - \overline{r_k^j}}{j} & (m_k > 2) \\[2ex] (-1)^s r_k & (m_k = m_{n_s} = 2) \end{cases} \quad (k \in \mathbf{N}), \ P_n := \psi_{N_n} \sum_{k=0}^{n-1} \frac{1}{M_k} h_k D_{M_k} \quad (n \in \tilde{\mathbf{N}}).$$

For the partial sums of the Vilenkin polinomials P_k the following statements are true [14]:

(i) $|S_{N_n}(P_n)(x)| \geqq C \log M_n \quad (x \in I_n(0))$,

(ii) $\|P_n\|_\infty \leqq 3\pi \quad (n \in \tilde{\mathbf{N}})$.

($C>0$ will denote an absolute, although not always the same, constant.)

By means of the polynomials P_n we define the following functions:

$$Q_n := \prod_{k=0}^{M_n-1} \left(1 + \frac{r_{n+k+1}\tau_{e_{k,n}}(P_n) + \overline{r_{n+k+1}}\tau_{e_{k,n}}(\bar{P}_n)}{6\pi}\right),$$

where for $k = \sum_{i=0}^{n-1} k_i M_i \in \{0, 1, \ldots, M_n - 1\}$, $e_{k,n}$ is defined by

$$e_{k,n} := (k_0, k_1, \ldots, k_{n-1}, 0, 0, \ldots) \in G_m.$$

It is immediate that the values of Q_n are real and $Q_n \geqq 0$. Furthermore, it is easy to prove that $\|Q_n\|_1 = \int_{G_m} Q_n = 1$. Indeed, if we observe that

$$r_{n+k+1}\tau_{e_{k,n}}(P_n) = \sum_{i=M_{n+k+1}}^{M_{n+k+1}+(\Delta_n+1)M_n-1} \gamma_{ik}\psi_i,$$

$$\overline{r_{n+k+1}}\tau_{e_{k,n}}(\bar{P}_n) = \sum_{i=(m_{n+k+1}-1)M_{n+k+1}}^{(m_{n+k+1}-1)M_{n+k+1}+(m_n-\Delta_n+1)M_n-1} \gamma_{ik}\psi_i$$

(with some coefficients γ_{ik}) and

$$M_{n+k+1} + (\Delta_n+1)M_n - 1 < M_{n+k+1}(m_{n+k+1}-1) < (m_{n+k+1}-1)M_{n+k+1} + (m_n - \Delta_n + 1)M_n - 1 < M_{n+k+2},$$

then we can prove without difficulty the relations

$$\int_{G_m} \prod_{t=1}^{s} \frac{r_{n+j_t+1}\tau_{e_{j_t},n}(P_n) + \overline{r_{n+j_t+1}}\tau_{e_{j_t},n}(\bar{P}_n)}{6\pi} = 0$$

($1 \leqq s \leqq M_n$, $0 \leqq j_1 < j_2 < \ldots < j_s < M_n$).

From the above facts it follows that

$$\delta_k(Q_n) := S_{M_{n+k+1}+N_n}(Q_n) - S_{M_{n+k+1}}(Q_n) = = S_{M_{n+k+1}+N_n}\left(\frac{r_{n+k+1}\tau_{e_{k,n}}(P_n) + \overline{r_{n+k+1}}\tau_{e_{k,n}}(\bar{P}_n)}{6\pi}\right) = \frac{1}{6\pi} r_{n+k+1}\tau_{e_{k,n}}(S_{N_n}(P_n)),$$

i.e. for $x \in I_n(e_{k,n})$ ($k = 0, \ldots, M_n - 1$) we have

$$|\delta_k(Q_n)(x)| = \frac{1}{6\pi} |S_{N_n}(P_n)(0)| \geqq C \log M_n.$$

On the basis of the properties of the Vilenkin polynomials Q_n we can construct already an integrable function with everywhere divergent Vilenkin—Fourier series using standard arguments. Let namely $\varepsilon_k>0$ $(k\in\mathbf{N})$ be a sequence for which $\sum_{k=0}^{\infty}\varepsilon_k<+\infty$ holds. Then for all sequences of the indices $(n_k, k\in\mathbf{N})$ the function

$$f:=\sum_{k=0}^{\infty}\varepsilon_k r_{M_{n_k+1}}Q_{n_k} \tag{1}$$

belongs to $L^1(G_m)$ and with $p_j:=M_{M_{n_j+1}}+M_{n_j+k_j+1}$, $q_j:=p_j+N_{n_j}$ $(j\in\mathbf{N},\ k_j=0, \dots, \dots, M_{n_j-1})$ the equality

$$|S_{p_j}(f)-S_{q_j}(f)| = \varepsilon_j|\delta_{k_j}(Q_{n_j})|$$

holds. For an element $x\in G_m$ let the number k_j be chosen such that $x\in I_{n_j}(e_{k_j,n_j})$ (i.e. $k_j=\sum_{i=0}^{n_j-1}x_iM_i$.) Therefore

$$|S_{p_j}(f)(x)-S_{q_j}(f)(x)| = \varepsilon_j|\Delta_{k_j}(Q_{n_j})(x)| \geqq C\varepsilon_j\log M_{n_j} \tag{2}$$

and if $\limsup\varepsilon_j\log M_{n_j}>0$ then the Vilenkin—Fourier series of the above function f diverges everywhere. Thus we proved

THEOREM 1. *For all sequences m there exists an integrable function with everywhere divergent Vilenkin—Fourier series.*

Later we shall discuss also the "unbounded" case, i.e. when $\limsup m=+\infty$. If the sequence m increases rapidly enough, we can construct an example with the condition of Theorem 1 in a very simple way. We have to observe only that for $m_n>2$

$$|S_{\Delta_nM_n+1}(h_n)(x)| = \sum_{j=1}^{\Delta_n}1/j \geqq C\log m_n \quad (x_n=0).$$

We assume $m_{n+1}\geqq 2^{2m_n}$ $(n\in\mathbf{N})$ and consider the following Vilenkin polynomials:

$$R_n:=\prod_{k=0}^{m_n-1}\left(1+\frac{r_{n+1}^{2^{2k+1}}+\overline{r_{n+1}^{2^{2k+1}}}}{8\pi i}\tau z_{k,n}(h_n)\right) \quad (n\in\mathbf{N}),$$

where $z_{k,n}:=(0, 0, \dots, 0, \overset{n}{\breve{k}}, 0, \dots)\in G_m$ $(k=0, \dots, m_n-1)$. By a similar argument as above it follows that $R_k\geqq 0$, $\|R_n\|_1=\int_{G_m}R_n=1$,

$$|S_{2^{2k+1}M_{n+1}+\Delta_nM_n+1}(R_n)(x)-S_{2^{2k+1}M_{n+1}}(R_n)(x)| =$$

$$=\frac{1}{8\pi}|\tau_{z_{k,n}}(S_{\Delta_nM_n+1}(h_n)(x))| \geqq C\log m_n \quad (x_n=k=0, \dots, m_n-1;\ n\in\mathbf{N}).$$

For a sequence $\alpha_k>0$ $(k\in\mathbf{N})$, $\sum_{k=0}^{\infty}\alpha_k<+\infty$ let

$$g:=\sum_{n=0}^{\infty}\alpha_n(R_n-1). \tag{3}$$

Then obviously, g belongs to $L^1(G_m)$ and for all $n\in\mathbf{N}$, $x\in G_m$, $k:=x_n$ we have

$$|S_{2^{2k+1}M_{n+1}+\Delta_n M_n+1}(g)(x)-S_{2^{2k+1}M_{n+1}}(g)(x)| =$$
$$= \alpha_n|S_{2^{2k+1}M_{n+1}+\Delta_n M_n+1}(R_n)(x)-S_{2^{2k+1}M_{n+1}}(R_n)(x)| \geqq C\alpha_n \log m_n .$$

Since $m_{n+1}\geqq 2^{2m_n}$ $(n\in\mathbf{N})$, the sequence $(\alpha_k;\ k\in\mathbf{N})$ can be selected such that $\limsup \alpha_n\cdot\log m_n>0$ and therefore the function g has an everywhere divergent Vilenkin—Fourier series.

In the next theorem we give a lower bound for the partial sums S_n. We assume that the following assumption for the sequence m is fulfilled:

$$\limsup \frac{\log M_n}{\log\log M_{M_{n+1}}} > 0. \tag{4}$$

Then the following sheorəm is valid.

THEOREM 2. *Let m be a sequence with property* (4). *Then for all sequences $w_n= =o(\log\log n)$ $(n\to\infty)$ there exists $f\in L^1(G_m)$ such that*

$$\lim_n \sup \frac{|S_n(f)(x)|}{w_n}>0 \quad (x\in G_m).$$

We remark that for a bounded sequence m (4) is trivially true. On the other hand, there is a sequence m with property (4) which is not bounded. (So e.g. $m_n:=2^{n+1}$ $(n\in\mathbf{N})$.) But (4) is not valid for all m. (E.g. $m_0:=2$, $m_n:=2^{2^{M_n}}(n\in\mathbf{N})$.) Theorem 2 for the Walsh—Paley system was proved by F. Schipp [12], [13].

In [15] we showed that for a function $f\in L^1(G_m)$ each of the following conditions implies the a.e. convergence of $(S_n(f),\ n\in\mathbf{N})$:

$$\left\{\begin{array}{l}
\text{(i)}\ \ \sum\limits_{k=0}^{\infty} m_k \int\limits_{G_m}\int\limits_{G_m} |f(x\dotplus u)-f(x)|D_{M_k}(u)\,dx\,du < +\infty \\
\text{(ii)}\ \ \sum\limits_{k=0}^{\infty} m_k\omega_1\left(f,\dfrac{1}{M_k}\right) < +\infty \\
\text{(iii)}\ \ \omega_1\left(f,\dfrac{1}{M_k}\right) = o(\log M_k)^{-1-\varepsilon} \quad (k\to\infty)
\end{array}\right. \tag{5}$$

and

$$\sum_{k=1}^{\infty} m_k(\log M_k)^{-1-\varepsilon} < +\infty \quad (\varepsilon>0). \tag{$*$}$$

It is evident that for a bouneed m we can omit the factors m_k and the condition $(*)$. We shall prove that conditions (5) (i) and (5) (ii) without the factors m_k generally do not imply the a.e. convergence. Since (i) follows from (ii), it is enough to prove the following theorem.

THEOREM 3. *There exist a sequence m and a function $g\in L^1(G_m)$ such that $\sum\limits_{k=0}^{\infty}\omega_1\left(g,\dfrac{1}{M_k}\right)<+\infty$ and $(S_n(g),\ n\in\mathbf{N})$ is everywhere divergent.*

The following problem due to A. Zygmund [18] is well-known in the theory of the trigonometric series: does the condition $\omega_1(f, \delta)=O(\log 1/\delta)^{-1}$ $(\delta\to 0)$ for an integrable function f imply the a.e. convergence of the Fourier series of f?. This problem — and the analogous question for $\hat{G}_m$ — is open. In this paper we shall investigate the problem for $\hat{G}_m$.

If the sequence m is bounded, then for a function $f\in L^1(G_m)$ the relation $\omega_1(f,\delta)=O(\omega(\delta))$ $(\delta\to 0$, ω is a modulus of continuity) is equivalent to the restriction $\omega_1\left(f, \frac{1}{M_k}\right)=O\left(\omega\left(\frac{1}{M_k}\right)\right)$ $(k\to\infty)$. Therefore, in this case the question of A. Zygmund has the following form: does the condition

$$\omega_1\left(f, \frac{1}{M_k}\right) = O(\log M_k)^{-1} \quad (k\to\infty) \tag{6}$$

imply the a.e. convergence of $(S_n(f),\ n\in\mathbf{N})$? In the next theorem we prove that for unbounded sequences m condition (6) is generally not sufficient for the a.e. convergence.

THEOREM 4. *There exist a sequence m and a function $f\in L^1(G_m)$ such that* (6) *is true and $(S_n(f),\ n\in\mathbf{N})$ is everywhere divergent.*

We shall see in the proof that the sequence m in Theorem 4 is not bounded. Therefore the complete solution of A. Zygmund's problem for $\hat{G}_m$ remains open both in the bounded and in the unbounded case. If we repalace "$\log 1/\delta$" by (weaker) "$\log\log 1/\delta$", then the following theorem is true.

THEOREM 5. (i) *For every bounded sequence m there exists a function $f\in L^1(G_m)$ for which*

$$\omega_1(f,\delta) = O(\log\log 1/\delta)^{-1} \quad (\delta\to 0) \tag{7}$$

holds and $(S_n(f),\ n\in\mathbf{N})$ diverges everywhere.

(ii) *There exist an unbounded sequence m and a function $g\in L^1(G_m)$ such that* (7) *is true for g and the Vilenkin—Fourier series of g diverges everywhere.*

We remark that part i) of Theorem 5 for the Walsh—Paley system was proved by F. Schipp [13].

Related to the conditions (5) we make the following remarks. If for a modulus ω the estimation $\sum_{k=0}^{\infty} m_k\omega(1/M_k)<+\infty$ holds, then from

$$\omega_1\left(f, \frac{1}{M_k}\right) = O(\omega(1/M_k)) \quad (k\to\infty,\ f\in L^1(G_m))$$

the a.e. convergence of $(S_n(f),\ n\in\mathbf{N})$ follows (so e.g. in the case $\omega(1/M_k)=O(m_k^{-1}\cdot(\log M_k)^{-1-\varepsilon})$ $(k\to\infty,\ \varepsilon>0)$). However, $\omega(1/M_k)=(\log M_k)^{-1}$ $(k\to\infty)$ cannot be chosen, since for every m the relation $\sum_{k=1}^{\infty} m_k(\log M_k)^{-1}=+\infty$ holds.

In [15] we showed that the condition $\omega_1\left(f, \frac{1}{M_k}\right)=o(m_k\log M_k)^{-1}$ $(k\to\infty)$

implies the L_1-convergence of $(S_n(f),\ n\in\mathbf{N})$ $(f\in L^1(G_m))$. If we modify the proof of this statement, i.e. we follow the method of the proof of the sufficiency for a.e. convergence of (5) (i) (see [15]), then — taking into consideration the relation

$$M_s \int\limits_{I_s(0)} \left| \sum_{j=1}^{n} r_s^j \right| = O(\log n) = O(\log m_s)$$

$(n=1, \ldots, m_s-1,\ s\to\infty)$ — we obtain the following assertion:

Any of the following conditions implies the L^1-convergence of $(S_n(f), n\in\mathbf{N})$ $(f\in L^1(G_m))$:

$$\text{(i)}\quad \sum_{k=0}^{\infty} (\log m_k)\omega_1\left(f, \frac{1}{M_k}\right) < +\infty,$$

$$\text{(8)}\qquad \text{(ii)}\quad \omega_1\left(f, \frac{1}{M_k}\right) = O(\log M_k)^{-1-\varepsilon} \quad (k\to\infty, \varepsilon>0) \text{ and}$$

$$\sum_{k=1}^{\infty} (\log m_k)(\log M_k)^{-1-\varepsilon} < +\infty.$$

Consequently, if (5) (ii) or (5) (iii) is true, then $(S_n(f),\ n\in\mathbf{N})$ converges a.e. and in L^1-norm. It is an open question whether the weaker conditions (8) are sufficient for the a.e. convergence. Unfortunately, this will not bring us closer to Zygmund's problem, since $\sum\limits_{k=1}^{\infty} \dfrac{\log m_k}{\log M_k} = +\infty$ for every m.

We remark that for certain sequences m condition (8) (i) is weaker than the above mentioned $\omega_1(f, 1/M_k) = o(m_k \log M_k)^{-1}$ $(k\to\infty)$, e.g. for $m_n := 2^{n+1}$ $(n\in\mathbf{N})$. By means of a counterexample, it was proved in [15] that $\omega_1(f, 1/M_k) = o(\log M_k)^{-1}$ $(k\to\infty)$ is generally not sufficient for the L^1-convergence. If we put $m_k := 2^{M_k}$ $(k\in\mathbf{N})$ in this counterexample, then we attain the growth condition $\omega_1(f, 1/M_k) = O(1/M_k)$ $(k\to\infty)$.

Finally, we remark that the analogous statements to the above mentioned assertions for the L^1-convergence are true for the uniform convergence if we use $f\in C(G_m)$ and $\omega(f,\delta)$.

We shall investigate again the $(C, 1)$-summation of continuous functions. It is known [9], [10] that for bounded sequences m the $(C, 1)$-means $\sigma_n(f)$ of a continuous function are uniform convergent. However, this is no more true in the unbounded case, since J. Price [11] showed the existence of a continuous function with divergent $(C, 1)$-means. In this work we prove Price's theorem by means of a construction.

THEOREM 6. (J. Price [11]). *For any unbounded sequence m there exists a continuous function $f\in C(G_m)$ such that* $\limsup |\sigma_k(f)(0)| = +\infty$.

We remark that Price showed the relation $\limsup \|K_n\|_1 = +\infty$ in the unbounded case, from which Theorem 6 follows. On the other hand, from Theorem 6 $\limsup \|K_n\|_1 = +\infty$ follows.

If the sequence m increases rapidly enough then we can prove that the function f in Theorem 6 has a certain smoothness condition. So e.g. if a sequence $(n_k,\ k\in\mathbf{N})$

of indices can be selected with the properties

$$\text{(9)} \qquad \begin{aligned} &\text{(i) } m_{n_0} < m_{n_1} < \dots \\ &\text{(ii) } \limsup (\log m_{n_k}) \cdot M_{n_k}^{-1} = +\infty, \end{aligned}$$

then the following corollary is valid.

COROLLARY (J. Price [11]). *If (9) is true then we can choose the function f in Theorem 6 with the following property:*

$$\omega(f, 1/M_k) = O(1/M_k) \quad (k \to \infty).$$

It is evident that (9) cannot be true for all unbounded sequences m, e.g. for $m_n := n+2$ $(n \in \mathbf{N})$ we have

$$\lim \frac{\log m_{n_k}}{M_{n_k}} = \lim \frac{\log (n_k+2)}{(n_k+1)!} = 0.$$

§ 3. Proofs

PROOF OF THEOREM 2. We can assume that $w_n = \beta_n \log\log n$ $(n \to \infty)$, where $\beta_0 \geqq \beta_1 \geqq \dots$ and $\lim \beta_n = 0$, $\lim w_n = +\infty$. Let $(n_k, k \in \mathbf{N})$ be the sequence of indices defined by $\sum_{k=0}^{\infty} \beta_{n_k} < \infty$ and consider the function f in (1) with $\varepsilon_k := \beta_{n_k}$ $(k \in \mathbf{N})$. For $x \in G_m$ and for $j \in \mathbf{N}$ let

$$k_j := \sum_{i=0}^{n_j-1} x_i M_i, \; q_j := M_{M_{n_j+1}} + M_{n_j+k_j+1}, \; p_j := q_j + N_{n_j}.$$

Then we have by (4)

$$\lim_j \sup \frac{|S_{p_j}(f)(x) - S_{q_j}(f)(x)|}{\beta_{p_j} \log\log p_j} \geqq C \lim_j \sup \frac{\varepsilon_j \log M_{n_j}}{\beta_{n_j} \log\log M_{M_{n_j+1}}} > 0.$$

From this $\left(\text{taking into consideration the relation } \limsup \frac{w_{q_j}}{w_{p_j}} > 0\right)$ Theorem 2 follows.

PROOF OF THEOREM 3. We consider the function g in (3). Then for every $k \in \mathbf{N}$ we have $\omega_1(g, 1/M_k) \leqq C \sum_{n=k-1}^{\infty} \alpha_n$, i.e. $\sum_{k=0}^{\infty} \omega_1(g, 1/M_k) \leqq C \sum_{n=0}^{\infty} n\alpha_n$. Since the growth condition $m_{n+1} \geqq 2^{2^{m_n}}$ $(n \in \mathbf{N})$ holds, evidently the sequence $\alpha_n > 0$ $(n \in \mathbf{N})$ can be selected with the properties $\sum_{n=0}^{\infty} n\alpha_n < +\infty$ and $\limsup \alpha_n \log m_n > 0$.

PROOF OF THEOREM 4. Let us define the sequence m in the following way: $m_k := 4^{M_k}$ $(k \in \mathbf{N})$. Then $m_{k+1} \geqq 2^{2^{m_k}}$ $(k \in \mathbf{N})$ is evidently true and the Vilenkin—Fourier series of the function $g:=\sum_{n=0}^{\infty}\frac{R_n-1}{\log m_n}\in L^1(G_m)$ in (3) is everywhere divergent. On the other hand

$$\omega_1(g,1/M_k)\leqq\sum_{n=k-1}^{\infty}\frac{1}{\log m_n}\leqq\frac{C}{\log m_{k-1}}\quad(k=1,2,...).$$

Since $\log M_k=\log M_{k-1}+\log m_{k-1}=\log m_{k-1}+\log\log m_{k-1}<2\log m_{k-1}$, the estimation $\frac{1}{\log m_{k-1}}<\frac{2}{\log M_k}$ is true, i.e. $\omega_1(g,1/M_k)\leqq C\frac{1}{\log M_k}$ follows.

PROOF OF THEOREM 5. (i) For a bounded sequence m we introduce the following notations: $M:=\sup m$, $n_j:=2^j$, $\varepsilon_j:=2^{-j}$ $(j\in\mathbf{N})$ and we consider the function $f:=\sum_{k=0}^{\infty}\varepsilon_k r_{M_{n_k}+1}Q_{n_k}\in L^1(G_m)$ in (1). Therefore the Vilenkin—Fourier series of f diverges everywhere and

$$\omega_1(f,1M_k)\leqq\sum_{k\leqq M_{n_j+1}}\varepsilon_j\leqq\sum_{j=j_0-1}^{\infty}\varepsilon_j,$$

where the natural number j_0 is defined by

$$M^{2^{j_0-1}+1}<k\leqq M^{2^{j_0+1}}.$$

From this it follows that

$$\omega_1(f,1/M_k)\leqq\sum_{j=j_0-1}^{\infty}2^{-j}=2^{2-j_0}\leqq\frac{C}{\log k}\leqq\frac{C}{\log\log M_k}\quad(k\to\infty),$$

i.e. on the basis of our earlier remark on the integral modulus we have

$$\omega_1(f,\delta)=O(\log\log 1/\delta)^{-1}\quad(\delta\to 0).$$

(ii) We consider the function g in the proof of Theorem 4. It is enough to prove that for this function (7) is fulfilled. Indeed, let $k\in\mathbf{N}$ be the index for which $\frac{1}{M_k}<\delta\leqq\frac{1}{M_{k-1}}$ is true. Then

$$\omega_1(g,\delta)\leqq\omega_1\left(g,\frac{1}{M_{k-1}}\right)\leqq\frac{C}{\log m_{k-2}}\leqq\frac{C}{\log M_{k-1}}\leqq\frac{C}{\log\log M_k}\leqq\frac{C}{\log\log 1/\delta}.$$

PROOF OF THEOREM 6. We assume $\limsup m=+\infty$ and select the sequence $(n_k,\ k\in\mathbf{N})$ of indices with the following properties:

(i) $m_{n_0}<m_{n_1}<...$

(ii) there exist numbers $\beta_k>0$ $(k\in\mathbf{N})$ such that $\limsup\beta_k\log m_{n_k}=+\infty$ and $\sum_{k=0}^{\infty}\beta_k<+\infty$.

If we consider the function $f := \sum_{k=0}^{\infty} \beta_k h_{n_k} \in C(G_m)$, then — on the basis of the definition of h_n's — we have

$$\sigma_{\Delta_{n_k} M_{n_k}+1}(f)(0) = \sum_{j=0}^{\infty} \beta_j \sigma_{\Delta_{n_k} M_{n_k}+1}(h_{n_j})(0) = \beta_k \sigma_{\Delta_{n_k} M_{n_k}+1}(h_{n_k})(0) +$$

$$+ \sum_{j=0}^{k-1} \beta_j \sigma_{\Delta_{n_k} M_{n_k}+1}(h_{n_j})(0) =: A_k + B_k \quad (k \in \mathbf{N}).$$

It is enough to give suitable estimations for A_k and for B_k.

1^0 $$|A_k| = \beta_k \sum_{j=1}^{\Delta_{n_k}} \left(1 - \frac{jM_{n_k}}{\Delta_{n_k} M_{n_k}+1}\right)\frac{1}{j} > \beta_k \left(\sum_{j=1}^{\Delta_{n_k}} \frac{1}{j} - 1\right) \geqq C\beta_k \log m_{n_k}$$

i.e. $\limsup A_k = +\infty$.

2^0 $$|B_k| = \left| \sum_{j=0}^{k-1} \frac{\beta_j}{\Delta_{n_k} M_{n_k}+1} \sum_{t=1}^{\Delta_{n_k} M_{n_k}+1} S_t(h_{n_j})(0) \right| =$$

$$= \left| \sum_{j=0}^{k-1} \frac{\beta_j}{\Delta_{n_k} M_{n_k}+1} \sum_{t=M_{n_j}+1}^{M_{n_j+1}-M_{n_j}} S_t(h_{n_j})(0) \right| \leqq \sum_{j=0}^{k-1} \frac{\beta_j M_{n_j+1} \log m_{n_j}}{\Delta_{n_k} M_{n_k}+1} \leqq \sum_{j=0}^{\infty} \beta_j < +\infty.$$

Therefore, $\limsup |\sigma_{\Delta_{n_k} M_{n_k}}(f)(0)| \geqq \limsup |A_k| - \limsup |B_k| = +\infty$.

Proof of Corollary. Let (9) be true and define $\beta_k := 1/M_{n_k}$ $(k \in \mathbf{N})$. Then

$$\sum_{k=0}^{\infty} \beta_k \leqq \sum_{k=0}^{\infty} 2^{-n_k} < +\infty$$

and

$$\limsup \beta_k \log m_{n_k} = \limsup \frac{\log m_{n_k}}{M_{n_k}} = +\infty.$$

Therefore, for the function $f := \sum_{k=0}^{\infty} \beta_k h_{n_k} \in C(G_m)$ the relation $\limsup |\sigma_n(f)(0)| = +\infty$ holds by Theorem 6. Furthermore, we have for $j \in \mathbf{N}$

$$\omega(f, 1/M_j) = \sum_{n_k \geqq j} \beta_k \leqq C/M_{n_{k_j}},$$

where $k_j \in \mathbf{N}$ is defined by $n_{k_j - 1} < j \leqq n_{k_j}$. Hence $\omega(f, 1/M_j) \leqq C/M_j$ follows.

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(Received October 7, 1981)

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Acta Math. Hung.
41 (3—4) (1983), 371—377.

A NOTE ON THE SHARPNESS OF J. L. WALSH'S THEOREM AND ITS EXTENSIONS FOR INTERPOLATION IN THE ROOTS OF UNITY

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§ 1. Introduction and statements of new results

Let A_ϱ denote the collection of functions analytic in $|z|<\varrho$ and having a singularity on the circle $|z|=\varrho$, where it is assumed that $1<\varrho<\infty$. Next, for each positive integer n, let $p_{n-1}(z;f)$ denote the Lagrange polynomial interpolant, of degree at most $n-1$, of $f(z)\in A_\varrho$ in the n-th roots of unity, i.e.,

$$p_{n-1}(\omega;f)=f(\omega) \tag{1.1}$$

where ω is any n-th root of unity, and let

$$P_{n-1}(z;f):=\sum_{k=0}^{n-1}a_k z^k \tag{1.2}$$

be the $(n-1)$-st partial sum of $f(z)=\sum_{k=0}^{\infty}a_k z^k$. Letting

$$D_\tau:=\{z\in\mathbf{C}:\ |z|<\tau\}, \tag{1.3}$$

then a beautiful result of J. L. Walsh [2, p. 153] can be stated as

THEOREM A. *For each $f(z)\in A_\varrho$, the interpolating polynomials of* (1.1) *and* (1.2) *satisfy*

$$\lim_{n\to\infty}\{p_{n-1}(z;f)-P_{n-1}(z;f)\}=0,\quad \textit{for all}\quad z\in D_{\varrho^2}. \tag{1.4}$$

Moreover, the result of (1.4) *is best possible in the sense that there is some $\hat f(z)\in A_\varrho$ and some $\hat z$ with $|\hat z|=\varrho^2$ for which the sequence $\{p_{n-1}(\hat z;\hat f)-P_{n-1}(\hat z;\hat f)\}_{n=1}^{\infty}$ does not tend to zero as $n\to\infty$.*

Note that in Theorem A, *no* sharpness assertions are made for *arbitrary* functions $f(z)\in A_\varrho$; in particular, no statement is made on the behavior of the sequence

$$\{p_{n-1}(z;f)-P_{n-1}(z;f)\}_{n=1}^{\infty} \tag{1.5}$$

in $|z|>\varrho^2$. One of the aims of this note is to in fact address this behavior in $|z|>\varrho^2$. As a special case of Theorem 1 below, we prove that, for *any* $f(z)\in A_\varrho$, the sequence in (1.5) can be bounded in at most *one* point in $|z|>\varrho^2$. This fact is of special interest in the case when $f(z)$ in A_ϱ is also continuous in the disk $|z|\leqq\varrho$; for such functions, it has been shown in [1, Thm. 2] that (1.4) is valid for *all* $|z|\leqq\varrho^2$.

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[1] Research supported in part by the National Science Foundation.
[2] Research supported in part by the Air Force Office of Scientific Research, and by the Department of Energy.

For our own purposes below, we need a recent extension of Theorem A. For additional notation, set

$$(1.6)\qquad P_{n-1,j}(z;f):=\sum_{k=0}^{n-1} a_{k+jn}z^k,\quad j=0,1,\ldots.$$

Then, the following result of Cavaretta, Sharma, and Varga [1, Thm. 1], which gives Theorem A as the special case $l=1$, can be stated as

THEOREM B. *For each $f(z)\in A_\varrho$, and for each positive integer l, there holds*

$$(1.7)\qquad \lim_{n\to\infty}\left\{p_{n-1}(z;f)-\sum_{j=0}^{l-1} P_{n-1,j}(z;f)\right\}=0,\quad \textit{for all}\quad z\in D_{\varrho^{l+1}},$$

the convergence being uniform and geometric on any closed subset of $D_{\varrho^{l+1}}$. Moreover, the result of (1.7) *is best possible in the sense that there is some $\tilde f(z)\in A_\varrho$ and some $\tilde z$ with $|\tilde z|=\varrho^{l+1}$ for which the sequence*

$$(1.8)\qquad \left\{p_{n-1}(z;f)-\sum_{j=0}^{l-1} P_{n-1,j}(z;f)\right\}_{n=1}^{\infty}$$

with $z=\tilde z$ and $f=\tilde f$, does not tend to zero as $n\to\infty$.

Our first new result is

THEOREM 1. *For each $f(z)\in A_\varrho$, and for each positive integer l, the sequence* (1.8) *can be bounded in at most l distinct points in $|z|>\varrho^{l+1}$. This result is sharp, in the sense that, given any l distinct points $\{\eta_k\}_{k=1}^{l}$ in the annulus $\varrho^{l+1}<|z|<\varrho^{l+2}$, there is an $\hat f(z)\in A_\varrho$ for which*

$$(1.9)\qquad \lim_{n\to\infty}\left\{p_{n-1}(\eta_k;\hat f)-\sum_{j=0}^{l-1} P_{n-1,j}(\eta_k;\hat f)\right\}=0,\quad k=1,2,\ldots,l.$$

There is an extension of Theorem 1 which we can also state. Note, of course, that Theorem A involves only the Lagrange interpolation of f in the n-th roots of unity. For r a fixed positive integer, Theorem B can be extended using Hermite interpolation. For notation, let $h_{rn-1}(z;f)$ denote the Hermite polynomial interpolant, of degree at most $rn-1$, to $f, f', \ldots, f^{(r-1)}$ in the n-th roots of unity, i.e.,

$$(1.10)\qquad h_{rn-1}^{(j)}(\omega;f)=f^{(j)}(\omega),\quad j=0,1,\ldots,r-1,$$

where again ω is any n-th root of unity. If $f(z)=\sum_{j=0}^{\infty} a_j z^j$, we set

$$(1.11)\qquad H_{rn-1,0}(z;f):=\sum_{k=0}^{rn-1} a_k z^k,$$

and we set

$$(1.12)\qquad H_{rn-1,j}(z;f):=\hat\beta_j(z^n)\sum_{k=0}^{n-1} a_{k+n(r+j-1)}z^k,\quad j=1,2,\ldots,$$

where

$$(1.13)\qquad \hat\beta_j(z):=\sum_{k=0}^{r-1}\binom{r+j-1}{k}(z-1)^k,\quad j=1,2,\ldots.$$

Then, the following result of Cavaretta, Sharma, and Varga [1, Thm. 3], which gives Theorem B as the special case $r=1$, can be stated as

THEOREM C. *For each* $f(z)\in A_\varrho$, *and for each pair of positive integers* r *and* l, *there holds*

$$\lim_{n\to\infty}\left\{h_{rn-1}(z;\,f)-\sum_{j=0}^{l-1}H_{rn-1,j}(z;\,f)\right\}=0,\quad \textit{for all}\quad z\in D_{\varrho^{1+(l/r)}}, \tag{1.14}$$

the convergence being uniform and geometric for any closed subset of $D_{\varrho^{1+(l/r)}}$. *Moreover, the result of* (1.14) *is best possible in the sense that there is some* $\hat{f}(z)\in A_\varrho$ *and some* $\hat{z}$ *with* $|\hat{z}|=\varrho^{1+(l/r)}$ *for which the sequence*

$$\left\{h_{rn-1}(z;\,f)-\sum_{j=0}^{l-1}H_{rn-1,j}(z;\,f)\right\}_{n=1}^{\infty}, \tag{1.15}$$

with $z=\hat{z}$ *and* $f=\hat{f}$, *does not tend to zero as* $n\to\infty$.

Our second new result, which sharpens Theorem C and gives Theorem 1 as the special case $r=1$, can be stated as

THEOREM 2. *For each* $f(z)\in A_\varrho$, *and for each pair of positive integers* r *and* l, *the sequence* (1.15) *can be bounded in at most* $r+l-1$ *distinct points in* $|z|>\varrho^{1+(l/r)}$. *This result is sharp, in the sense that, given any* $r+l-1$ *distinct points* $\{\eta_k\}_{k=1}^{r+l-1}$ *in the annulus* $\varrho^{1+(l/r)}<|z|<\min\left\{\varrho^{l+2};\ \varrho^{1+\frac{l}{r-1}}\right\}$, *there is an* $\tilde{f}(z)\in A_\varrho$ *for which*

$$\lim_{n\to\infty}\left\{h_{rn-1}(\eta_k;\,\tilde{f})-\sum_{j=0}^{l-1}H_{rn-1,j}(\eta_k;\,\tilde{f})\right\}=0,\quad k=1,2,\,\dots,\,r+l-1. \tag{1.16}$$

Since the proof of Theorem 2 is completely analogous to the proof of Theorem 1, we shall give only the proof of Theorem 1.

§ 2. Proof of Theorem 1

To establish the first part of Theorem 1, consider any (fixed $f\in A_\varrho$, consider any fixed positive integer l, and suppose that there are $(l+1)$ distinct points $\{y_k\}_{k=1}^{l+1}$ in $|z|>\varrho^{l+1}$ for which

$$\left|p_{n-1}(y_k;\,f)-\sum_{j=0}^{l-1}P_{n-1,j}(y_k;\,f)\right|\leqq M,\ \forall n\geqq 1,\ \forall 1\leqq k\leqq l+1. \tag{2.1}$$

If $f(z):=\sum_{j=0}^{\infty}a_jz^j$, then the hypothesis that f is analytic in $|z|<\varrho$ with a singularity on $|z|=\varrho$ gives us that

$$\overline{\lim_{n\to\infty}}\,|a_n|^{1/n}=\frac{1}{\varrho}. \tag{2.2}$$

Thus, for any $\varepsilon>0$ with $1<\varrho-\varepsilon$ and with

$$(\varrho-\varepsilon)^{l+2}>\varrho^{l+1}, \tag{2.3}$$

there is an $n_0(\varepsilon)$ for which

$$|a_n|\leqq\frac{1}{(\varrho-\varepsilon)^n},\quad \forall n\geqq n_0(\varepsilon). \tag{2.4}$$

Next, since all the points $\{y_k\}_{k=1}^{l+1}$ lie in $|z|>\varrho^{l+1}$, then

$$\varrho^{l+1}<\sigma_1:=\min_{1\leqq k\leqq l+1}|y_k|\leqq\max_{1\leqq k\leqq l+1}|y_k|=:\sigma_2, \tag{2.5}$$

and we choose the least positive integer m for which

$$\sigma_2<\varrho^{m+1},\quad (\text{where } l<m). \tag{2.6}$$

Applying Theorem B (with l chosen as m), we have that the sequence $\left\{p_{n-1}(z;f)-\sum_{j=0}^{m-1}P_{n-1,j}(z;f)\right\}_{n=1}^{\infty}$ converges to zero for all $z\in D_{\varrho^{m+1}}$. In particular, as the points $\{y_k\}_1^{l+1}$ all lie in $D_{\varrho^{m+1}}$ from (2.5) and (2.6), then there exists a constant M_1 such that

$$\left|p_{n-1}(y_k;f)-\sum_{j=0}^{m-1}P_{n-1,j}(y_k;f)\right|\leqq M_1,\ \forall n\geqq 1,\ \forall 1\leqq k\leqq l+1. \tag{2.7}$$

Using the hypothesis of (2.1), this in turn implies that

$$\left|\sum_{j=l}^{m-1}P_{n-1,j}(y_k;f)\right|\leqq M_2,\ \forall n\geqq 1,\ \forall 1\leqq k\leqq l+1. \tag{2.8}$$

Recalling from (1.6) the definition of $P_{n-1,j}(z;f)$, then it follows from (2.4) that

$$|P_{n-1,j}(z;f)|\leqq\sum_{k=0}^{n-1}\frac{|z|^k}{(\varrho-\varepsilon)^{k+jn}}=\frac{1}{(\varrho-\varepsilon)^{jn}}\sum_{k=0}^{n-1}\left(\frac{|z|}{\varrho-\varepsilon}\right)^k,\quad \forall n\geqq n_0(\varepsilon).$$

Thus,

$$|P_{n-1,j}(z;f)|\leqq\frac{n|z|^n}{(\varrho-\varepsilon)^{(j+1)n}},\quad \forall n\geqq n_0(\varepsilon),\ \forall |z|>\varrho,\ \forall j\geqq 1. \tag{2.9}$$

This can be used as follows. From (2.9), we see that, if $l+1\leqq m-1$, then

$$\left|\sum_{j=l+1}^{m-1}P_{n-1,j}(z;f)\right|\leqq\frac{(m-l-1)n|z|^n}{(\varrho-\varepsilon)^{(l+2)n}},\quad \forall n\geqq n_0(\varepsilon),\ \forall |z|>\varrho. \tag{2.10}$$

Hence, from (2.8) and (2.10),

(2.11)

$$|P_{n-1,l}(y_k;f)|\leqq M_2+\frac{(m-l-1)n|y_k|^n}{(\varrho-\varepsilon)^{(l+2)n}},\quad \forall n\geqq n_0(\varepsilon),\ \forall 1\leqq k\leqq l+1.$$

Now, because of (2.11), it further follows that

$$|y_k^l P_{n,l}(y_k; f) - P_{n-1,l}(y_k; f)| \leqq M_3 + \frac{M_4 n |y_k|^n}{(\varrho-\varepsilon)^{(l+2)n}}, \tag{2.12}$$

for all $n \geqq n_0(\varepsilon)$, all $1 \leqq k \leqq l+1$. Next, because of the definition of $P_{n-1,j}(z; f)$, it can be verified that

$$z^l P_{n,l}(z; f) - P_{n-1,l}(z; f) = \sum_{j=n}^{l+n} a_{ln+j} z^j - \sum_{j=0}^{l-1} a_{ln+j} z^j. \tag{2.13}$$

Obviously, the last term in (2.13) is bounded, independent of n, in the points $\{y_k\}_{k=1}^{l+1}$, whence from (2.12) and (2.13),

$$\left|\sum_{j=n}^{l+n} a_{ln+j} y_k^j\right| \leqq M_5 + \frac{M_4 n |y_k|^n}{(\varrho-\varepsilon)^{(l+2)n}}. \tag{2.14}$$

On dividing through by $|y_k|^n$ in (2.14), we obtain

$$\left|\sum_{j=0}^{l} a_{n(l+1)+j} y_k^j\right| \leqq \frac{M_5}{|y_k|^n} + \frac{M_4 n}{(\varrho-\varepsilon)^{(l+2)n}}, \tag{2.15}$$

and so, from the definition of σ_1 in (2.5), there follows

$$\left|\sum_{j=0}^{l} a_{n(l+1)+j} y_k^j\right| \leqq \frac{M_5}{\sigma_1^n} + \frac{M_4 n}{(\varrho-\varepsilon)^{(l+2)n}}, \tag{2.16}$$

for all $n \geqq n_0(\varepsilon)$, all $1 \leqq k \leqq l+1$. If, for convenience, we set

$$\tau := \max\left\{\frac{1}{\sigma_1};\ \frac{1}{(\varrho-\varepsilon)^{l+2}}\right\}, \tag{2.17}$$

then it follows from (2.3) and (2.5) that

$$\tau < \frac{1}{\varrho^{l+1}}. \tag{2.18}$$

Next, we write a system of $(l+1)$ linear equations in the "unknowns" $a_{(l+1)n+j}$, i.e.,

$$\sum_{j=0}^{l} y_k^j a_{(l+1)n+j} =: f_{k,n}, \quad k = 1, 2, \ldots, l+1 \tag{2.19}$$

where, from (2.16) and (2.17),

$$|f_{k,n}| \leqq M_6 n \tau^n, \quad \forall n \geqq n_0(\varepsilon), \quad \forall 1 \leqq k \leqq l+1. \tag{2.20}$$

In matrix notation, we can write the system of equations (2.19) as

$$\begin{bmatrix} 1 & y_1 & \cdots & y_1^l \\ 1 & y_2 & \cdots & y_2^l \\ \vdots & & & \vdots \\ 1 & y_{l+1} & \cdots & y_{l+1}^l \end{bmatrix} \cdot \begin{bmatrix} a_{(l+1)n} \\ a_{(l+1)n+1} \\ \vdots \\ a_{(l+1)n+l} \end{bmatrix} = \begin{bmatrix} f_{1,n} \\ f_{2,n} \\ \vdots \\ f_{l+1,n} \end{bmatrix}. \tag{2.21}$$

The coefficient matrix, Δ, in (2.21) is a Vandermonde matrix, and, as the points $\{y_k\}_{k=1}^{l+1}$ are *distinct* by hypothesis, then Δ is nonsingular. Using Cramer's rule, it is easy to see from (2.20) and the fact that the $\{y_k\}_1^{l+1}$ are fixed distinct points, that

$$(2.22) \qquad |a_{(l+1)n+j}| \leqq M_7 n\tau^n, \quad \forall n \geqq n_0(\varepsilon), \quad \forall 0 \leqq j \leqq l.$$

However, (2.22) implies that

$$(2.23) \qquad \overline{\lim_{n\to\infty}} |a_n|^{1/n} \leqq \tau^{1/(l+1)} < \frac{1}{\varrho},$$

the last inequality coming from (2.18). As this contradicts (2.2), then there can be at most l distinct points $\{\eta_k\}_{k=1}^{l}$ in $|z|>\varrho^{l+1}$ for which the sequence (1.8) is bounded, completing thefirst part of the proof.

To establish the second part of Theorem 1, let $w_l(z)$ be any monic polynomial of degree l with precisely l distinct zeros in the annulus $\varrho^{l+1}<|z|<\varrho^{l+2}$, i.e.,

$$(2.24) \qquad w_l(z) = \prod_{k=1}^{l} (z-\eta_k) =: \sum_{j=0}^{l} \beta_j z^j,$$

where

$$(2.25) \qquad \varrho^{l+1} < |\eta_k| < \varrho^{l+2} \quad \text{for} \quad k = 1, 2, \ldots, l.$$

Consider then the particular function

$$(2.26) \qquad \hat{f}(z) := \frac{w_l(z)}{\varrho^{l+1}-z^{l+1}}.$$

Clearly, $\hat{f}\in A_\varrho$, and $\hat{f}$ has $l+1$ poles on $|z|=\varrho$. We now show that with these definitions, (1.9) of Theorem 1 is satisfied. From Theorem B, we know that

$$(2.27) \qquad \lim_{n\to\infty} \left\{p_{n-1}(z;\, \hat{f}) - \sum_{j=0}^{l} P_{n-1,j}(z;\, \hat{f})\right\} = 0, \quad \forall z\in D_{\varrho^{l+2}}.$$

We claim that

$$(2.28) \qquad \lim_{n\to\infty} P_{n-1,l}(\eta_k;\, \hat{f}) = 0, \quad \forall 1 \leqq k \leqq l.$$

To establish (2.28), write $\hat{f}(z) := \sum_{k=0}^{\infty} \hat{a}_k z^k$. It follows from (2.24) and (2.26) that

$$(2.29) \qquad \hat{a}_{m(l+1)+j} = \frac{\beta_j}{\varrho^{(m+1)(l+1)}}, \quad \forall 0 \leqq j \leqq l, \quad \forall m \geqq 0.$$

Next, by definition,

$$(2.30) \qquad P_{n-1,l}(z;\, \hat{f}) = \sum_{k=0}^{n-1} \hat{a}_{ln+k} z^k,$$

and we consider the case when n is a multiple of $(l+1)$, i.e., $n=(l+1)s$. On regrouping terms in (2.30) for such n, $P_{n-1,l}(z\colon \hat{f})$ can be expressed as

$$(2.31) \qquad P_{s(l+1)-1,l}(z;\, \hat{f}) = \sum_{k=0}^{s-1} z^{k(l+1)} \sum_{j=0}^{l} \hat{a}_{(l+1)[sl+k]+j} z^j.$$

But, the inner sum of (2.31) can be seen from (2.29) and (2.24) to be

$$\sum_{j=0}^{l} \hat{a}_{(l+1)[sl+k]+j} z^j = \frac{w_l(z)}{\varrho^{(l+1)[sl+k+1]}}. \tag{2.32}$$

Since $w_l(\eta_k)=0$ by definition, it follows from (2.31) that

$$P_{s(l+1)-1,l}(\eta_k;\hat{f}) = 0, \quad \forall 1 \leqq k \leqq l, \quad \forall s \geqq 1. \tag{2.33}$$

Having just considered the case when n is a multiple of $(l+1)$, we now suppose that $n=s(l+1)+t$, where $1 \leqq t \leqq l$. On similarly regrouping the terms in (2.30) and using the fact that $w_l(\eta_k)=0$, it can be shown that

$$P_{s(l+1)+t-1,l}(\eta_k;\hat{f}) = \sum_{j=0}^{t-1} \hat{a}_{sl(l+1)+lt+j}\eta_k^j. \tag{2.34}$$

Since the $\{\eta_k\}_{k=1}^{l}$ are fixed, and t does not exceed l, then, as $|\hat{a}_n| \to 0$ as $n \to \infty$ from (2.29), we have from (2.33) and (2.34) that

$$\lim_{n\to\infty} P_{n-1,l}(\eta_k;\hat{f}) = 0, \quad \forall 1 \leqq k \leqq l, \tag{2.35}$$

as claimed in (2.28). Thus, with (2.27) and the first part of Theorem 1, the sequence

$$\left\{ p_{n-1}(z;\hat{f}) - \sum_{j=0}^{l-1} P_{n-1,j}(z;\hat{f}) \right\}_{n=1}^{\infty} \tag{2.36}$$

is convergent (to zero), only in the points $\{\eta_k\}_{k=1}^{l}$ and unbounded for all other points in $\{z \in \mathbf{C}: |z| > \varrho^{l+1}\}$.

Added in proof. (April 14, 1983) The second part of Theorem 1 remains valid if any l distinct points $\{\eta_k\}_{k=1}^{l}$ are arbitrarily chosen in $|z| > \varrho^{l+1}$, with a similar improvement holding for Theorem 2. This has been shown by the author and, more generally by T. Hermann, "Some remarks on an extension of a Theorem of Walsh", *J. Approx. Th.* (to appear).

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(Received December 27, 1981)

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[2] A. Zygmund, Smooth functions, *Duke Math. J.,* **12** (1945), 47—76.

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