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ON THE THIRD BETTI NUMBER OF SOME COMPACT HOMOGENEOUS MANIFOLDS

H. AZAD

Abstract

We determine closed skew-symmetric covariant tensors of rank 3 on homogeneous spaces of compact simple Lie groups.

Let K be a compact group operating on a manifold M. The orbits of K on M which carry a non-degenerate skew symmetric covariant tensor of rank 2 are classified in [1, 3]; they can be interpreted as phase spaces of homogeneous mechanical systems and are of importance in constructing quantizations of classical systems [3]. Physicists, in connection with the Wess—Zumino term [5, 6], are also interested in understanding homogeneous spaces of Lie groups which admit a skew-symmetric tensor of rank 3 whose integral over a generic three cycle does not vanish. In other words, one is interested in classifying pairs $G \supset H$ such that the 3rd de Rham cohomology group $H^3(G/H)$ does not vanish; the dimension of this group is the 3rd Betti number of G/H. In this note we prove the following:

PROPOSITION. *If L is a positive dimensional closed subgroup of a compact simple Lie group K then $H^3(K/L)$ vanishes.*

From the background on group theory and differential forms the reader might consult [2, 3]. We will follow here the definitions and notations of [3].

PROOF of the Proposition. Let $\pi: K \to K/L$ denote the natural projection. Consider a closed 3-form η on K/L which, by integrating over K, we may also assume to be K-invariant [2]. Let L_a and R_a denote left and right multiplication by the element $a \in K$. The pull-back $\pi^*(\tilde{\eta})$ of $\tilde{\eta}$ is then invariant under left multiplications by K; moreover, as $\pi \circ R_a = \pi (a \in L)$, we see that it is also invariant under right multiplications by L. Denoting the pull-back of $\tilde{\eta}$ by η we also have $\eta(X, Y, Z) = 0$ if one of X, Y, Z is in the Lie-algebra $\dot{L}$ of L. Now $H^3(K)$ is generated by the form ω defined by

$$\omega(X, Y, Z) = ([X, Y], Z),$$

where X, Y, Z are in the Lie algebra $\dot{K}$ of K and round brackets denote the trace form on K [4].

Hence $\eta = \alpha\omega + d\xi$, for some $\alpha \in R$ and ξ a left K-invariant form. As ω is bi-invariant, and η right L-invariant, we may assume that ξ also has the same property. Now for any vector fields X, Y, Z we have the identity

$$(L_X \xi)(Y, Z) = L_X(\xi(Y, Z)) - \xi([X, Y], Z) - \xi(Y, [X, Z])$$

where L_X denotes the Lie derivative with respect to X; and if X, Y, Z are also invariant then

$$(d\xi)(X, Y, Z) = -\xi([X, Y], Z)+\xi([X, Z], Y)-\xi([Y, Z], X)$$

see, e.g. [3]. In particular if $X, Y, Z \in \dot{K}$ then these identities imply that

$$(*) \qquad (d\xi)(X, Y, Z) = \xi(X, [Y, Z]).$$

Hence, for $X \in L$ and $Y, Z \in \dot{K}$ we have

$$0 = \eta(X, Y, Z) = \alpha([XY], Z)+\xi(X, [Y, Z])$$
$$= \alpha(X, [Y, Z])+\xi(X, [Y, Z]).$$

As the commutator of K coincides with K we see that

$$0 = \alpha(X, X)+\xi(X, X) = \alpha(X, X).$$

Hence, if $\dot{L}\neq 0$ we must have $\alpha=0$. In this case we can assume that $\eta=d\xi$, where ξ is left K-invariant and right L-invariant, and from $(*)$ we also have $0=\xi(X, Y)$ for all $X\in\dot{L}$ and $Y\in\dot{K}$. But this means that ξ descends to a 2-form $\tilde{\xi}$ on K/L. Hence $\tilde{\eta}=d\tilde{\xi}$ and therefore $H^3(K/L)=0$ if the connected component of L is positive dimensional.

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MARKOV AND BERNSTEIN TYPE INEQUALITIES FOR CERTAIN CLASSES OF CONSTRAINED TRIGONOMETRIC POLYNOMIALS ON AN INTERVAL SHORTER THAN THE PERIOD

T. ERDÉLYI

Let $n\geqq 0$ be an integer and denote by T_n the set of all real trigonometric polynomials of order at most n. For an arbitrary $p\in T_n$ we have the following Markov type estimate:

$$\|p'\| \leqq \frac{c_1 n^2}{\omega}\|p\| \tag{1}$$

where c_1 (and in what follows $c_2, c_3, \ldots$) is a positive absolute constant and $\|\cdot\|$ denotes the supremum norm over the interval $[-\omega, \omega]$, (see Videnskiĭ [1]). In [2] we introduced and examined thoroughly the following classes of trigonometric polynomials:

$$\mathcal{T}_n(\omega) = \left\{r(t) = \sum_{l=0}^{2n} a_l \sin^l \frac{\omega - t}{2} \sin^{2n-l} \frac{t+\omega}{2} \text{ with all } a_l \geqq 0 \quad \text{or all} \quad a_l \leqq 0\right\}$$

where $0<\omega\leqq\pi$. In case of $n, k\geqq 0$, $0<\omega\leqq\pi$ let

$$\mathcal{H}_{n+k}^k(\omega) = \left\{p(t) = r(t)q(t) \,\middle|\, r\in\mathcal{T}_n(\omega),\ q\in T_k\right\}.$$

Theorem 1 of [3] asserts that

$$\|p^{(m)}\| \leqq c_1(m)\left(\frac{(n+k)(k+1)}{\omega}\right)^m \|p\| \tag{2}$$

$$(p\in\mathcal{H}_{n+k}^k(\omega),\ n, k \geqq 0,\ m \geqq 1,\ 0 < \omega \leqq \pi),$$

where $c_1(m)$ (and in what follows $c_2(m), c_3(m), \ldots$) is a positive constant depending only on m. For an arbitrary $c_2<1$, in case of $0<\omega\leqq c_2\pi/2$, up to the constant $c_1(m)$ inequality (2) proved to be sharp (see [3], Theorem 2). In this paper we give a sharp (both in n and ω) Markov type estimate for the derivatives of polynomials from $\mathcal{T}_n(\omega)$ for all $0<\omega\leqq\pi$. To see how wide $\mathcal{T}_n(\omega)$ is we remark that it contains all the trigonometric polynomials p from T_n for which $p(x+iy)\neq 0$ if $(\operatorname{ch} y)(\cos\omega)<\cos x$, $-\pi\leqq x<\pi$, (see [2], Theorem 1).

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Key words and phrases. Trigonometric polynomials, Markov and Bernstein type inequalities, restrictions for the roots.

THEOREM 1. *Let* $p\in\mathscr{H}^{k}_{n+k}(\omega)$, $n, k\geqq 0$, $0<\omega\leqq\pi$. *Then with a suitable* $c_3\geqq 1$

$$\|p'\|\leqq\frac{c_3(k+1)^2}{\omega}\left(\sqrt{n+1}+(n+1)\left|\frac{\pi}{2}-\omega\right|_+\right)\|p\|,$$

where $|x|_+:=\max\{x, 0\}$.

Of course for an arbitrary $c_2<1$, in case of $0<\omega\leqq c_2\pi/2$ inequality (2) gives a better result, but it does not show any essential improvement if ω tends to $\pi/2$ depending on n.

PROOF. For the sake of brevity in the sequel let

$$r_{ln}(t)=\sin^l\frac{\omega-t}{2}\sin^{2n-l}\frac{t+\omega}{2}\qquad(0\leqq l\leqq 2n). \tag{3}$$

We need a series of lemmas.

LEMMA 1. *Let* $n\geqq 1$,

$$0<\delta\leqq\frac{1}{n\left(\frac{\pi}{2}-\omega\right)}, \tag{4}$$

where

$$\frac{\pi}{4}\leqq\omega\leqq\frac{\pi}{2}-\frac{1}{\sqrt{n}}. \tag{5}$$

Then with a suitable $c_4\geqq 1$

$$|r'_{ln}(t)|\leqq\frac{c_4}{\delta}\max\{r_{ln}(t), r_{ln}(t\pm\delta)\}\qquad(0\leqq l\leqq 2n) \tag{6}$$

if

$$|t|\leqq 2\omega-\frac{\pi}{2}. \tag{7}$$

Compare this with Lemma 2 of [4].

PROOF. Observe that (4), (5) and (7) imply

$$t\pm\delta\in[-\omega, \omega]. \tag{8}$$

It may be supposed that

$$0\leqq t\leqq 2\omega-\frac{\pi}{2}, \tag{9}$$

the other case is similar. Now we distinguish two cases.

Case 1. t is such that $r''_{ln}(\tau)>0$ when $|t-\tau|\leqq\delta$. Hence r'_{ln} is monotone increasing in $[t-\delta, t+\delta]\subset[-\omega, \omega]$ (see (8)), thus if e.g. $r'_{ln}(t)\geqq 0$, then by the mean value theorem, for a suitable $t<\xi<t+\delta$

$$0 \leqq r'_{ln}(t) \leqq r'_{ln}(\xi) = \frac{r_{ln}(t+\delta)-r_{ln}(t)}{\delta} \leqq \frac{r_{ln}(t+\delta)}{\delta}.$$

Similarly if $r'_{ln}(t)<0$, then

$$r'_{ln}(t) \geqq -\frac{r_{ln}(t-\delta)}{\delta}.$$

Hence in this case

$$|r'_{ln}(t)| \leqq \frac{1}{\delta}\max\{r_{ln}(t\pm\delta)\},$$

which gives (6).

Case 2. t is such that $r''_{ln}(\tau^*)<0$ for some $|t-\tau^*|\leqq\delta$. Since

$$4\sin^{2-l}\frac{\omega-\tau^*}{2}\sin^{2+l-2n}\frac{\tau^*+\omega}{2}r''_{ln}(\tau^*) =$$

(10) $$= [(n-l)\sin\omega-(n-1)\sin\tau^*][(n-l)\sin\omega-n\sin\tau^*]-$$

$$-2n\sin\frac{\omega-\tau^*}{2}\sin\frac{\tau^*+\omega}{2}\cos\tau^* \leqq 0.$$

With the notation $x=(n-l)\sin\omega-n\sin\tau^*$, using also (4), (5), (8) and (9), we get

$$x^2+x\sin\tau^* \leqq \frac{n}{2}(\omega-\tau^*)(\tau^*+\omega)\cos\tau^* \leqq$$

$$\leqq \frac{n}{2}(\omega-t+|t-\tau^*|)(|\tau^*-t|+t+\omega)\left(\frac{\pi}{2}-t+|t-\tau^*|\right) \leqq$$

$$\leqq \frac{n}{2}(\omega-t+\delta)(\delta+t+\omega)\left(\frac{\pi}{2}-t+\delta\right) \leqq \frac{n}{2}\,2(\omega-t)\,2\omega 2\left(\frac{\pi}{2}-t\right) \leqq 7n(\omega-t)\left(\frac{\pi}{2}-t\right).$$

Hence

$$|x| \leqq \frac{|\sin\tau^*|+\sqrt{\sin^2\tau^*+28n(\omega-t)\left(\frac{\pi}{2}-t\right)}}{2} \leqq$$

(11) $$\leqq |\sin\tau^*|+\sqrt{7}\sqrt{n(\omega-t)\left(\frac{\pi}{2}-t\right)} \leqq 1+3\sqrt{n(\omega-t)\left(\frac{\pi}{2}-t\right)}.$$

So with a suitable ξ lying between t and τ^*, by using the mean value theorem, (5) and (9) we obtain

$$\frac{|r'_{ln}(t)|}{r_{ln}(t)} = \frac{|(n-l)\sin\omega - n\sin t|}{2\sin\frac{\omega-t}{2}\sin\frac{\omega+t}{2}} \leqq$$

$$(12) \qquad \leqq \frac{|x|+n|\tau^*-t|\cos\xi}{\frac{2}{\pi^2}(\omega-t)(\omega+t)} \leqq 2\pi\frac{|x|+n\delta\left(\frac{\pi}{2}-t+|t-\xi|\right)}{\omega-t} \leqq$$

$$\leqq 2\pi\frac{1+3\sqrt{n(\omega-t)\left(\frac{\pi}{2}-t\right)}+n\delta\left(\frac{\pi}{2}-t\right)+n\delta^2}{\omega-t}.$$

Here by (4), (5), (8) and (9) we have

$$(13) \qquad \frac{n\delta^2}{\omega-t} \leqq \frac{1}{\omega-t} \leqq \frac{1}{\frac{\pi}{2}-\omega} \leqq \sqrt{n} \leqq \frac{1}{\delta},$$

$$(14) \qquad \frac{3\sqrt{n(\omega-t)\left(\frac{\pi}{2}-t\right)}}{\omega-t} = 3\sqrt{n}\frac{\sqrt{\frac{\pi}{2}-t}}{\sqrt{\omega-t}} \leqq 3\sqrt{2}\sqrt{n} \leqq \frac{3\sqrt{2}}{\delta},$$

and

$$(15) \qquad \frac{n\delta\left(\frac{\pi}{2}-t\right)}{\omega-t} \leqq 2n\delta \leqq \frac{2}{\delta}.$$

Now (12)—(15) give that

$$(16) \qquad |r'_{ln}(t)| \leqq \frac{c_4}{\delta} r_{ln}(t),$$

so Lemma 1 is proved. □

LEMMA 2. *Let* $k \geqq 1$, $q \in T_k$, $\varphi > 0$ *and* $\bar{\varphi} := \varphi\left(1+\frac{1}{8k^2}\right) \leqq \frac{\pi}{2}$. Then

$$\text{(i)} \qquad \max_{|\tau| \leqq \bar{\varphi}} |q(\tau)| \leqq 2 \max_{|\tau| \leqq \varphi} |q(\tau)|,$$

$$\text{(ii)} \qquad \max_{|\tau| \leqq \bar{\varphi}} |q'(\tau)| \leqq \frac{8k^2}{\varphi} \max_{|\tau| \leqq \varphi} |q(\tau)|.$$

The proof of this lemma can be found in [4] (see Lemma 1).

LEMMA 3. *Let* $n \geqq 1$,

$$\frac{\pi}{4} \leqq \omega \leqq \frac{\pi}{2} - \frac{1}{\sqrt{n}}, \tag{17}$$

$$0 < \gamma \leqq \frac{1}{6c_4 n\left(\frac{\pi}{2}-\omega\right)} \quad (c_4 \geqq 1 \quad \textit{is the same as in } (6)), \tag{18}$$

$$|t| \leqq 3\omega - \pi, \tag{19}$$

$k \geqq 1$, $q \in T_k$, *and consider the intervals*

$$I_1(t, \gamma, k) = \left[t - \gamma\left(1 + \frac{1}{16k^2}\right), t - \frac{\gamma}{16k^2}\right]$$

and

$$I_2(t, \gamma, k) = \left[t + \frac{\gamma}{16k^2}, t + \gamma\left(1 + \frac{1}{16k^2}\right)\right].$$

Choose $\xi_j \in I_j(t, \gamma, k)$ *for which*

$$|q(\xi_j)| = \max_{\tau \in I_j(t, \gamma, k)} |q(\tau)| \qquad (j = 1, 2).$$

Then for $p_l = r_{ln} q \quad (l = 0, 1, \ldots, 2n)$ *we have*

$$|p_l'(t)| \leqq \frac{c_5 k^2}{\gamma} \max \{|p_l(\xi_1)|, |p_l(\xi_2)|\}.$$

PROOF. Observe that (17), (18) and (19) imply that $I_j(t, \gamma, k) \subset [-\omega, \omega]$ $(j = 1, 2)$. If r_{ln} is monotone in the interval

$$I_3(t, \gamma, k) := \left[t - \gamma\left(1 + \frac{1}{16k^2}\right), t + \gamma\left(1 + \frac{1}{16k^2}\right)\right]$$

(e.g. monotone decreasing, the other case is similar), then

$$0 < r_{ln}(t) \leqq r_{ln}(\xi_1). \tag{20}$$

On the other hand, if there exists an $\eta_l \in \operatorname{int} I_3(t, \gamma, k)$ such that $r_{ln}'(\eta_l) = 0$, then $r_{ln}(\eta_l) = \|r_{ln}\|$. Using the mean value theorem, Lemma 1 with $\delta = \dfrac{1}{n\left(\frac{\pi}{2}-\omega\right)}$, and (18), for all $\xi \in I_3(t, \gamma, k)$ we get

$$r_{ln}(\eta_l) - r_{ln}(\xi) \leqq |\eta_l - \xi| \sup_{\tau \in I_3(t, \delta, k)} |r_{ln}'(\tau)| \leqq 3\gamma c_4 n\left(\frac{\pi}{2} - \omega\right) \|r_{ln}\| \leqq \frac{1}{2} r_{ln}(\eta_l)$$

((17), (18) and (19) imply that $|\tau| \leqq 2\omega - \frac{\pi}{2}$ for all $\tau \in I_3(t, \gamma, k)$, therefore Lemma 1 can be applied, indeed). Hence

$$r_{ln}(t) \leqq \|r_{ln}\| = r_{ln}(\eta_l) < 2r_{ln}(\xi) \qquad (\xi\in I_3(t, \gamma, k)), \tag{21}$$

in particular (21) holds for $\xi=\xi_1$.

If r_{ln} is monotone decreasing in $I_3(t, \gamma, k)$, then using Lemma 1 with $\delta=t-\xi_1\leqq 2\gamma\leqq\frac{1}{3n\left(\frac{\pi}{2}-\omega\right)}$, we obtain

$$|r'_{ln}(t)| \leqq \frac{c_4}{t-\xi_1} r_{ln}(\xi_1) \leqq \frac{16c_4k^2}{\gamma} r_{ln}(\xi_1). \tag{22}$$

On the other hand, if there exists an $\eta_l\in\operatorname{int} I_3(t, \gamma, k)$ such that $r'_{ln}(\eta_l)=0$, then Lemma 1 with $\delta=\frac{1}{n\left(\frac{\pi}{2}-\omega\right)}$, and (21) give

$$|r'_{ln}(t)| \leqq c_4n\left(\frac{\pi}{2}-\omega\right)\|r_{ln}\| \leqq 2c_4n\left(\frac{\pi}{2}-\omega\right) r_{ln}(\xi_1) \leqq \frac{1}{3\gamma} r_{ln}(\xi_1). \tag{23}$$

From Lemma 2, by definition of $I_1(t, \gamma, k)$ and ξ_1 we easily obtain

$$|q(t)| \leqq 2\,|q(\xi_1)| \tag{24}$$

and

$$|q'(t)| \leqq \frac{16k^2}{\gamma}|q(\xi_1)|. \tag{25}$$

Thus (20)—(25) yield

$$|p'_l(t)| \leqq |r'_{ln}(t)q(t)|+|r_{ln}(t)q'(t)| \leqq \frac{c_5k^2}{\gamma}|r_{ln}(\xi_1)q(\xi_1)| = \frac{c_5k^2}{\gamma}|p_l(\xi_1)|,$$

which gives the lemma. □

LEMMA 4. *Let* $0<\varphi\leqq\frac{\pi}{2}$, $k\geqq 1$, $s\in T_k$, $j\geqq 2$, $u(\tau)=\sin^{2j}\frac{\tau+\varphi}{2}s(\tau)$,

$$0<\beta\leqq\frac{\varphi}{\sqrt{j}+j\left(\frac{\pi}{2}-\varphi\right)} \tag{26}$$

and consider the interval $I_4(\varphi, \beta)=[\varphi-\beta, \varphi]$. *Choose a* $\xi_4\in I_4(\varphi, \beta)$ *for which*

$$|s(\xi_4)| = \max_{\tau\in I_4(\varphi,\beta)}|s(\tau)|.$$

Then

$$|u'(\varphi)| \leqq \frac{c_6k^2}{\beta}|u(\xi_4)|.$$

PROOF. We have

$$|u'(\varphi)| \leqq j \sin^{2j-1}\varphi \cos\varphi\, |s(\varphi)| + \sin^{2j}\varphi\, |s'(\varphi)|. \tag{27}$$

A simple calculation shows that $\xi_4 \in I_4(\varphi, \beta)$ and (26) imply

$$\sin\varphi \leqq \left(1+\frac{c_7}{j}\right)\sin\xi_4, \tag{28}$$

so by the definition of ξ_4

$$\begin{aligned} j\sin^{2j-1}\varphi\cos\varphi\,|s(\varphi)| &\leqq j\left(1+\frac{c_7}{j}\right)^{2j}(\sin^{2j}\xi_4)\frac{\frac{\pi}{2}-\varphi}{\sin\varphi}|s(\xi_4)| \leqq \\ &\leqq \frac{c_8}{\varphi}\, j\left(\frac{\pi}{2}-\varphi\right)|u(\xi_4)| \leqq \frac{c_8}{\beta}|u(\xi_4)|. \end{aligned} \tag{29}$$

Further from (28), (1) and the definition of ξ_4 we deduce

$$\sin^{2j}\varphi\,|s'(\varphi)| \leqq \left(1+\frac{c_7}{j}\right)^{2j}\sin^{2j}\xi_4\,\frac{2c_1k^2}{\beta}|s(\xi_4)| \leqq \frac{c_9k^2}{\beta}|u(\xi_4)|. \tag{30}$$

Now (27), (29) and (30) yield the desired result. □

LEMMA 5. *Let* $n \geqq 100$, $k \geqq 1$, $\frac{15\pi}{32} < \psi \leqq \frac{\pi}{2}$, $r \in \mathcal{T}_n(\psi)$, $q \in T_k$, $p = rq$, $4\psi - \frac{3\pi}{2} - \frac{5}{\sqrt{n}} \leqq |t| \leqq \psi$, $0 < \beta \leqq \dfrac{1}{12\left(\sqrt{n}+n\left(\frac{\pi}{2}-\psi\right)\right)}$ *and*

$$I_5(t,\beta) = \begin{cases} I_4(t,\beta) & \textit{for} \quad t \geqq 0 \\ -I_4(t,\beta) & \textit{for} \quad t < 0. \end{cases}$$

Then

$$|p'(t)| \leqq \frac{c_{10}k^2}{\beta}\max_{\tau \in I_5(t,\beta)}|p(\tau)|.$$

PROOF. It may be supposed that $t \geqq 0$, the other case is similar. Observe that under the conditions of the lemma

$$\frac{\pi}{2} \geqq t \geqq \frac{3\pi}{8} - \frac{5}{10} \geqq \frac{1}{2} \tag{31}$$

and

$$0 < \beta \leqq \frac{1}{12\left(\sqrt{n}+n\left(\frac{\pi}{2}-\psi\right)\right)} \leqq \frac{t}{\sqrt{n}+n\left(\frac{\pi}{2}-t\right)}. \tag{32}$$

By Theorem 1 of [2], $r\in\mathcal{T}_n(\psi)$ and $0<t\leqq\psi\leqq\frac{\pi}{2}$ imply $r\in\mathcal{T}_n(t)$, so we have the representation

$$r(\tau)=\sum_{l=0}^{2n} a_l \sin^l \frac{t-\tau}{2}\sin^{2n-l}\frac{\tau+t}{2} \quad \text{with e.g. all} \quad a_l\geqq 0. \tag{33}$$

Using (32), (33), Lemma 4 twice with $\varphi=t$, $j=n$, $s=q\in T_k$ and $\varphi=t$, $j=n-1$, $s(\tau)=q(\tau)\sin\frac{t-\tau}{2}\sin\frac{\tau+t}{2}\in T_{k+1}$ resp., further recalling the conditions of the lemma, we get

$$\begin{aligned} |p'(t)| &\leqq a_0\left|\left(\sin^{2n}\frac{\tau+t}{2}\,q(\tau)\right)'(t)\right|+\\ &+a_1\left|\left(\sin^{2n-2}\frac{\tau+t}{2}\left(q(\tau)\sin\frac{t-\tau}{2}\sin\frac{\tau+t}{2}\right)\right)'(t)\right|\leqq\\ &\leqq\frac{c_6(k^2+(k+1)^2)}{\beta}\max_{\tau\in I_4(t,\beta)}|p(\tau)|,\end{aligned} \tag{34}$$

which yields the lemma. □

The proof of Theorem 1 is now straightforward. Let $p\in\mathcal{H}^k_{n+k}(\omega)$. This means $p=rq$, where $r\in\mathcal{T}_n(\omega)$ is of the form $r(t)=\sum_{l=0}^{2n}a_l r_{l_n}(t)$ with e.g. all $a_l\geqq 0$. First let $\frac{15\pi}{32}\leqq\omega\leqq\frac{\pi}{2}$, $n\geqq 100$, $k\geqq 1$. To estimate $|p'(t)|$ for all $t\in[-\omega,\omega]$ we distinguish three cases.

Case 1. $\omega\leqq\frac{\pi}{2}-\frac{1}{\sqrt{n}}$, $|t|\leqq 3\omega-\pi$. Then using Lemma 3 with $\gamma:=\dfrac{1}{6c_4 n\left(\frac{\pi}{2}-\omega\right)}$ we obtain the theorem.

Case 2. $3\omega-\pi-\frac{3}{\sqrt{n}}\leqq|t|\leqq\omega$. Then Lemma 5 with $\psi=\omega$ and

$$\beta=\frac{1}{12\left(\sqrt{n}+n\left(\frac{\pi}{2}-\omega\right)\right)}$$

gives the theorem.

Case 3. $\frac{\pi}{2}-\frac{1}{\sqrt{n}}\leqq\omega$, $|t|\leqq 3\omega-\pi-\frac{3}{\sqrt{n}}$. Observe that $\frac{\pi}{2}-\frac{1}{\sqrt{n}}\leqq\omega$ and $r\in\mathcal{T}_n(\omega)$ imply $r\in\mathcal{T}_n(\psi)$, where $\psi=\frac{\pi}{2}-\frac{1}{\sqrt{n}}$ (see [2], Theorem 1). Further from

$|t| \leqq 3\omega - \pi - \frac{3}{\sqrt{n}}$ we deduce $|t| \leqq 3\psi - \pi$, so using the just proved Case 1 with $\omega = \psi$, we get

$$|p'(t)| \leqq c_{11} k^2 \left(\sqrt{n} + n \left(\frac{\pi}{2} - \psi \right) \right) \max_{|\tau| \leqq \psi} |p(\tau)| \leqq 2 c_{11} k^2 \sqrt{n} \, \|p\|.$$

Now by Cases 1, 2 and 3 the theorem is true when $\frac{15\pi}{32} \leqq \omega \leqq \frac{\pi}{2}$, $n \geqq 100$, $k \geqq 1$, and from this we obtain the case $\frac{15\pi}{32} \leqq \omega \leqq \frac{\pi}{2}$, $n, k \geqq 0$ trivially. The case $0 < \omega < \frac{15\pi}{32}$, $n, k \geqq 0$ follows from (2). Using the fact that $r \in \mathscr{T}_n(\omega)$ and $\frac{\pi}{2} < \omega \leqq \pi$ imply $s_{\pm}(t) = r\left(t \pm \omega \mp \frac{\pi}{2} \right) \in \mathscr{T}_n\left(\frac{\pi}{2} \right)$ (see [2], Theorem 1), from the case $\omega = \frac{\pi}{2}$ we get the theorem for $\frac{\pi}{2} < \omega \leqq \pi$. By this the proof of Theorem 1 is complete. □

The properties possessed by the trigonometric polynomials from $\mathscr{H}_{n+k}^{k}(\omega)$ are, in general, not inherited by the derivatives of these polynomials. Thus a straightforward generalization of Theorem 1 for higher derivatives seems difficult. We shall prove Markov type inequalities for higher derivatives only for $\mathscr{H}_n^0(\omega) = \mathscr{T}_n(\omega)$ and $T_n^k(\omega)$ $(0 \leqq k,\ 0 < \omega \leqq \pi)$ where $T_n^k(\omega)$ denotes the set of those polynomials from T_n which have all but at most k roots in $[-\pi, \pi] \setminus (-\omega, \omega)$, mod 2π.

THEOREM 2. *Let* $r \in \mathscr{T}_n(\omega)$, $n \geqq 1$, $0 < \omega \leqq \pi$, $m \geqq 1$. *Then*

$$\|r^{(m)}\| \leqq \frac{c_2(m)}{\omega^m} \left(\sqrt{n} + n \left| \frac{\pi}{2} - \omega \right|_+ \right)^m \|r\|.$$

PROOF. The method of the proof will be rather similar to that of Theorem 3 in [4], therefore we shall not give all the details. We need some lemmas.

LEMMA 6. *Let* $n, k, m \geqq 1$, $\frac{\pi}{4} \leqq \omega \leqq \frac{\pi}{2} - \frac{1}{\sqrt{n}}$, $|t| \leqq 4\omega - \frac{3\pi}{2}$, $s \in T_n^k(\omega)$ *and consider the intervals*

$$K_j(t) = \left[t - \frac{j}{16(c_3 + c_4) mn \left(\frac{\pi}{2} - \omega \right)},\ t + \frac{j}{16(c_3 + c_4) mn \left(\frac{\pi}{2} - \omega \right)} \right],$$

where $c_3, c_4 \geqq 1$ *are defined by Theorem 1 and Lemma 1, resp. Then we have*

$$|s^{(j)}(t)| \leqq c(j, m) \left(k^2 n \left(\frac{\pi}{2} - \omega \right) \right)^m \max_{\xi \in K_j(t)} |s(\xi)| \quad (1 \leqq j \leqq m), \tag{35}$$

where $c(j, m)$ *depends only on* j *and* m.

LEMMA 7. *Let* $k, m \geqq 1$, $n \geqq 100$, $\frac{15\pi}{32} \leqq \psi \leqq \frac{\pi}{2}$, $4\psi - \frac{3\pi}{2} - \frac{4}{\sqrt{n}} \leqq |t| \leqq \psi$, $s \in T_n^k(\psi)$ *and consider the intervals*

$$L_j(t) = \begin{cases} \left[t - \dfrac{jt}{4c_3 m\left(\sqrt{n}+n\left(\frac{\pi}{2}-\psi\right)\right)},\ t\right] & (t>0),\\[2ex] \left[t,\ t+\dfrac{jt}{4c_3 m\left(\sqrt{n}+n\left(\frac{\pi}{2}-\psi\right)\right)}\right] & (t<0),\end{cases}$$

where $c_3 \geqq 1$ *is defined by Theorem 1. Then we have*

$$|s^{(j)}(t)| \leqq c'(j,m)\left(k^2\left(\sqrt{n}+n\left(\frac{\pi}{2}-\psi\right)\right)\right)^j \max_{\xi \in L_j(t)} |s(\xi)| \quad (1 \leqq j \leqq m), \tag{36}$$

where $c'(j, m)$ *depends only on* j *and* m.

PROOFS. Observe that $s \in T_n^k(\omega)$ implies $s' \in T_n^{k+1}(\omega)$ and $s \in T_n^k(\omega)$ $\left(0 \leqq k \leqq 2n,\ 0 < \omega \leqq \frac{\pi}{2}\right)$ implies $s = rq$, where $r \in T_{n-k}^0(\omega) \subset \mathcal{T}_{n-k}(\omega) \subset \mathcal{T}_n(\omega)$ (see [2], Theorem 1). Thus using Lemma 3 with $\gamma = \dfrac{1}{32m(c_3+c_4)\,n\left(\frac{\pi}{2}-\omega\right)}$ and Lemma 5 with $\beta = \dfrac{1}{48c_3 m\left(\sqrt{n}+n\left(\frac{\pi}{2}-\psi\right)\right)}$ we obtain Lemmas 6 and 7 by induction on j $(1 \leqq j \leqq m)$. □

The following lemma is an immediate consequence of Theorem 1 and the mean value theorem.

LEMMA 8. *Let* $n \geqq 1$, $0 < \omega \leqq \pi$ *and* $[\tau_1, \tau_2] \subset [-\omega, \omega]$ *be such that* $\tau_2 - \tau_1 < \dfrac{\omega}{4c_3\left(\sqrt{n}+n\left|\frac{\pi}{2}-\omega\right|_+\right)}$, *where* $c_3 \geqq 1$ *is defined by Theorem 1. Then*

$$\max_{\tau_1 \leqq \tau \leqq \tau_2} r_{ln}(\tau) \leqq 2\max\{r_{ln}(\tau_1), r_{ln}(\tau_2)\} \qquad (0 \leqq l \leqq 2n). \tag{37}$$

The proof of Theorem 2 is now straightforward. Let $r \in \mathcal{T}_n(\omega)$. This means $r(t) = \sum_{l=0}^{2n} a_l r_{ln}(t)$ with e.g. all $a_l \geqq 0$. First let $\frac{15\pi}{32} \leqq \omega \leqq \frac{\pi}{2}$, $n \geqq 100$. To estimate $|r^{(m)}(t)|$ we distinguish three cases.

Case 1. $\omega \leqq \frac{\pi}{2} - \frac{1}{\sqrt{n}}$, $|t| \leqq 4\omega - \frac{3\pi}{2}$. Then Lemma 6 yields

$$|r_{ln}^{(m)}(t)| \leqq c_3(m)\left(n\left(\frac{\pi}{2}-\omega\right)\right)^m \max_{\tau\in K_m(t)} r_{ln}(\tau). \tag{38}$$

Thus, if $K_m(t)=[\tau_1(t), \tau_2(t)]$, then (38) and Lemma 8 give

$$|r_{ln}^{(m)}(t)| \leqq c_4(m)\left(n\left(\frac{\pi}{2}-\omega\right)\right)^m \max_{i=1,2} \{r_{ln}(\tau_i(t))\},$$

which together with the representation of r and $\tau_i(t)\in[-\omega, \omega]$ $(i=1, 2)$ give the desired result.

Case 2. $4\omega - \frac{3\pi}{2} - \frac{4}{\sqrt{n}} \leqq |t| \leqq \omega$. Then Lemma 7 yields

$$|r_{ln}^{(m)}(t)| \leqq c_3(m)\left(\sqrt{n}+n\left(\frac{\pi}{2}-\omega\right)\right)^m \max_{\tau\in L_m(t)} r_{ln}(\tau), \tag{39}$$

from which we get the theorem by Lemma 8 and the representation of r, similarly to Case 1.

Case 3. $\frac{\pi}{2}-\frac{1}{\sqrt{n}} \leqq \omega$, $|t| \leqq 4\omega - \frac{3\pi}{2} - \frac{4}{\sqrt{n}}$. Observe that $\frac{\pi}{2}-\frac{1}{\sqrt{n}} \leqq \omega$ and $r\in\mathcal{T}_n(\omega)$ imply $r\in\mathcal{T}_n(\psi)$ with $\psi=\frac{\pi}{2}-\frac{1}{\sqrt{n}}$ (see [2], Theorem 1). Further, from $|t|\leqq 4\omega-\frac{3\pi}{2}-\frac{4}{\sqrt{n}}$ we deduce $|t|\leqq 4\psi-\frac{3\pi}{2}$, so using the just proved Case 1 with $\omega=\psi$ we obtain the theorem.

Now by Cases 1, 2 and 3 the theorem is true when $\frac{15\pi}{32}\leqq\omega\leqq\frac{\pi}{2}$, $n\geqq 100$ and from this we get the case $\frac{15\pi}{32}\leqq\omega\leqq\frac{\pi}{2}$, $n\geqq 0$ trivially. The case $0<\omega<\frac{15\pi}{32}$, $n\geqq 0$ follows from (2). Since $r\in\mathcal{T}_n(\omega)$, $\frac{\pi}{2}<\omega\leqq\pi$ imply $s_\pm(t)=r\left(t\pm\omega\mp\frac{\pi}{2}\right)\in\mathcal{T}_n\left(\frac{\pi}{2}\right)$, (see [2], Theorem 1) the case $\omega=\frac{\pi}{2}$ give the theorem for $\frac{\pi}{2}<\omega\leqq\pi$. By this Theorem 2 is proved. □

Denote by $Q_n^k(\omega)$ the set of those polynomials from T_n which have at most k roots $x+iy\in\mathbf{C}$ such that $(\text{ch}\, y)(\cos\omega)<\cos x$. From Theorem 1 of [2] and Theorems 1 and 2 the following corollaries are straightforward.

COROLLARY 1. *If* $p\in Q_n^{2k}(\omega)$, $0\leqq k\leqq n$, $0<\omega\leqq\pi$, *then*

$$\|p'\|\leqq\frac{c_3(k+1)^2}{\omega}\left(\sqrt{n-k+1}+(n-k+1)\left|\frac{\pi}{2}-\omega\right|_+\right)\|p\|.$$

COROLLARY 2. *If* $p\in Q_n^0(\omega)$, $0<\omega\leqq\pi$, $m\geqq 1$, *then*

$$\|p^{(m)}\|\leqq\frac{c_2(m)}{\omega^m}\left(\sqrt{n}+n\left|\frac{\pi}{2}-\omega\right|_+\right)^m\|p\|.$$

Applying Corollary 2 with $\omega=\frac{\pi}{2}$ on two overlapping subintervals of length π, we obtain

COROLLARY 3. *Let* $0\leqq k\leqq 2n$, $\frac{\pi}{2}\leqq\omega\leqq\pi$. *If* $p\in T_n$ *has at most* k *roots* $x+iy\in\mathbf{C}$ *such that* $|x|<\omega$, *then*

$$\|p'\|\leqq c_3(k+1)^2\sqrt{n+1}\,\|p\|.$$

COROLLARY 4. *Let* $\frac{\pi}{2}\leqq\omega\leqq\pi$, $m\geqq 1$. *If* $p\in T_n$ *has no roots* $x+iy\in\mathbf{C}$ *such that* $|x|<\omega$, *then*

$$\|p^{(m)}\|\leqq c_2(m)\left(\frac{2}{\pi}\right)^m n^{m/2}\|p\|.$$

Using the observation that $p\in T_n^k(\omega)$ implies $p'\in T_n^{k+1}(\omega)$, from Corollary 1, by induction on m we get

COROLLARY 5. *Let* $p\in T_n^{2k}(\omega)$, $0\leqq k\leqq n$, $0<\omega\leqq\pi$, $m\geqq 1$. *Then*

$$\|p^{(m)}\|\leqq\frac{c_6(m)}{\omega^m}\left((k+1)^2\left(\sqrt{n}+n\left|\frac{\pi}{2}-\omega\right|_+\right)\right)^m\|p\|$$

(cf. [5], *Theorem 1, where in case of* $k=0$ *and* $m=1$ *the inequality* $\|p'\|\leqq\frac{c_{12}n}{\omega}\|p\|$ *was proved).*

EXAMPLE 1. Let $p(t)=\sin^{2n}\frac{t+\omega}{2}$, $0<\omega\leqq\pi$ and $n, m\geqq 1$. Then

$$\|p^{(m)}\|\geqq c_7(m)\omega^{-m}\left(\sqrt{n}+n\left|\frac{\pi}{2}-\omega\right|_+\right)^m\|p\|\qquad(n\geqq c_8(m)).\tag{40}$$

To see this we may suppose that $0<\omega\leqq\frac{\pi}{2}-\frac{1}{\sqrt{n}}$, from this the general case $0<\omega\leqq\pi$ is straightforward. For the sake of brevity let $\delta=\frac{\omega}{n\left(\frac{\pi}{2}-\omega\right)}\leqq\frac{\omega}{\sqrt{n}}$. Let

$0<a\leqq\sqrt{n}$. From Theorem 2 it is easy to see that for $\xi=\omega-a\delta\geqq 0$ we have

$$|p^{(m)}(\xi)| \leqq \frac{c_2(m)}{\omega^m}\left(\sqrt{n}+n\left(\frac{\pi}{2}-\frac{\xi+\omega}{2}\right)\right)^m \sin^{2n}\frac{\xi+\omega}{2} \leqq \tag{41}$$

$$\leqq \frac{c_2(m)}{\omega^m}\left[(a+2)n\left(\frac{\pi}{2}-\omega\right)\right]^m \frac{\sin^{2n}\frac{\xi+\omega}{2}}{\sin^{2n}\omega}\|p\|.$$

Further $\xi=\omega-a\delta$ implies

$$\sin\omega-\sin\frac{\xi+\omega}{2} = 2\sin\frac{\omega-\xi}{4}\cos\frac{3\omega+\xi}{4} \geqq$$

$$\geqq \frac{2}{\pi^2}a\delta\left(\frac{\pi}{2}-\omega\right) = \frac{2}{\pi^2}\frac{a\omega}{n} \geqq \frac{2a}{\pi^2 n}\sin\omega.$$

This yields

$$\frac{\sin^{2n}\frac{\xi+\omega}{2}}{\sin^{2n}\omega} \leqq \left(1-\frac{2a}{\pi^2 n}\right)^{2n} \leqq \exp\left(-\frac{4a}{\pi^2}\right). \tag{42}$$

So (41) and (42) give

$$|p^{(m)}(\xi)| \leqq \frac{c_2(m)}{\omega^m}(a+2)^m\exp\left(-\frac{4a}{\pi^2}\right)\left(n\left(\frac{\pi}{2}-\omega\right)\right)^m\|p\|. \tag{43}$$

Now we prove that for each $m\geqq 0$ and $0<\omega\leqq\frac{\pi}{2}-\frac{1}{\sqrt{n}}$ there exist $a_m, b_m>0$ and $t_{n,\omega}$ such that

$$|p^{(m)}(t_{m,\omega})| \geqq b_m\omega^{-m}\left(n\left(\frac{\pi}{2}-\omega\right)\right)^m\|p\| \tag{44}$$

$$(t_{m,\omega}\in[\omega-a_m\delta,\omega],\ n\geqq a_m^2).$$

This will prove (40) with $c_7(m)=\frac{b_m}{2^m}$ and $c_8(m)=a_m^2$. To show (44) we proceed by induction. For $m=0$ it is true with $a_0=0$, $t_{0,\omega}=\omega$, $b_0=1$. Assume that it holds for m and let us choose $a=a_{m+1}$ so large that

$$c_2(m)(a_{m+1}+2)^m\exp\left(-\frac{4a_{m+1}}{\pi^2}\right) \leqq \frac{b_m}{2}.$$

Then by (43)

$$|p^{(m)}(\xi)| \leqq \frac{b_m}{2}\omega^{-m}\left(n\left(\frac{\pi}{2}-\omega\right)\right)^m\|p\| \qquad (\xi=\omega-a_{m+1}\delta,\ n\geqq a_{m+1}^2).$$

Hence and from (44), by the mean value theorem there exists a

$$t_{m+1,\omega}\in[\omega-a_{m+1}\delta,\omega]\subset[0,\omega]$$

such that

$$|p^{(m+1)}(t_{m+1,\omega})| = \frac{|p^{(m)}(t_{m,\omega})-p^{(m)}(\omega-a_{m+1}\delta)|}{|t_{m,\omega}-(\omega-a_{m+1}\delta)|} \geqq$$

$$\geqq \frac{\frac{b_m}{2}\omega^{-m}\left(n\left(\frac{\pi}{2}-\omega\right)\right)^m \|p\|}{(a_m+a_{m+1})\delta} = \frac{b_m}{2(a_m+a_{m+1})}\,\omega^{-(m+1)}\left(n\left(\frac{\pi}{2}-\omega\right)\right)^{m+1}\|p\|,$$

i.e. (44) holds with $m+1$ instead of m if we choose $b_{m+1}=\dfrac{b_m}{2(a_m+a_{m+1})}$ and $n\geqq a_{m+1}^2$. □

The following theorem shows that how far Theorem 1 and Corollary 5 are from being sharp in case of $\omega=\dfrac{\pi}{2}$.

THEOREM 3. *There exist* $p_{nkm}\in T_n^{2k}\left(\frac{\pi}{2}\right)\subset\mathscr{H}_n^k\left(\frac{\pi}{2}\right)$ $(0\leqq k\leqq 2n,\ m\geqq 1)$ *such that*

$$\|p_{nkm}^{(m)}\| \geqq c_9(m)n^{m/2}(k+1)^{3m/2}\|p_{nkm}\| \qquad (n\geqq c_{10}(m)),$$

where $\|\cdot\|$ *denotes the supremum norm over the interval* $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

PROOF. We distinguish three cases.

Case 1. $n\geqq 6k$ and $k\geqq m+1$. Let

$$s(t) = C_{2k-1}\left(\sqrt{\frac{n}{3k}}\sin\frac{t}{2}\right)$$

where $C_{2k-1}(x)=\cos((2k-1)\arccos x)$ is the Chebyshev polynomial of degree $2k-1$ and let

$$q(t) = \cos^{2n-2k+1}\frac{t}{2}\,s(t).$$

We have

$$\max_{-\pi\leqq t\leqq\pi}|q(t)|\leqq 1 \tag{45}$$

(see Example 1 of [4] with $\omega=\pi$). Now define $t_j\in\left[0,\frac{\pi}{2}\right]$ $(1\leqq j\leqq k)$ by

$$\sin\frac{t_j}{2} = \sqrt{\frac{3k}{n}}\sin\left(\frac{2j-1}{2k-1}\frac{\pi}{2}\right) \qquad (j=1,2,\dots,k).$$

Then

$$s(t_j) = (-1)^{j+k} \qquad (j=1,2,\dots,k). \tag{46}$$

Furthermore

$$t_k - t_{k-m} \leqq 2\sqrt{2}\left(\sin\frac{t_k}{2} - \sin\frac{t_{k-m}}{2}\right) =$$

(47) $$= 2\sqrt{2}\sqrt{\frac{3k}{n}}\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{2(k-m)-1}{2k-1}\,\frac{\pi}{2}\right)\right] \leqq$$

$$\leqq 2\sqrt{2}\sqrt{\frac{3k}{n}}\,\frac{\pi}{2}\,\frac{2m}{2k-1}\sin\left(\frac{2m}{2k-1}\,\frac{\pi}{2}\right) \leqq c_{11}(m)n^{-1/2}k^{-3/2}.$$

Now let

(48) $$r(t) = \cos^{2n-2k+1}\frac{t_k - t}{2}\, s(t).$$

Then from (45) obviously

$$|r(t)| \leqq |q(t)| \leqq 1 \qquad (-\pi + t_k \leqq t \leqq 0).$$

Hence and from $\max\limits_{|t|\leqq t_k} |s(t)| = 1$

(49) $$\max_{-\pi + t_k \leqq t \leqq t_k} |r(t)| \leqq 1.$$

From (47) it is clear that

$$\cos^{2n-2k+1}\frac{t_k - t}{2} \geqq \cos^{2n-2k+1}\frac{t_k - t_{k-m}}{2} \geqq$$

(50) $$\geqq \cos^{2n}\left(\frac{c_{11}(m)}{2}n^{-1/2}k^{-3/2}\right) \geqq \left(1 - \frac{c_{12}(m)}{nk^3}\right)^{2n} \geqq c_{13}(m)$$

$$(n \geqq c_{14}(m),\ t_{k-m} \leqq t \leqq t_k).$$

From (46), (48) and (50) we deduce

(51) $$|r(t_j)| \geqq c_{13}(m) \qquad (k-m \leqq j \leqq k,\ n \geqq c_{14}(m))$$

and

(52) $$\operatorname{sgn} r(t_j) = -\operatorname{sgn} r(t_{j+1}) \qquad (k-m \leqq j \leqq k-1).$$

Thus if $\Omega(t) = \prod\limits_{j=k-m}^{k} (t - t_j)$, then by (47)

(53) $$|\Omega'(t_j)| \leqq \frac{c_{11}(m)}{n^{1/2}k^{3/2}} \qquad (k-m \leqq j \leqq k),$$

and evidently

(54) $$\operatorname{sgn}\Omega'(t_j) = -\operatorname{sgn}\Omega'(t_{j+1}) \qquad (k-m \leqq j \leqq k-1).$$

Using (51)—(54) and a well-known relation for the m^{th} order divided differences, we can find a $\xi\in[t_{k-m}, t_k]$ such that

$$(55)\qquad |r^{(m)}(\xi)| = m!\left|\sum_{j=0}^{m}\frac{r(t_j)}{\Omega'(t_j)}\right| = m!\sum_{j=0}^{m}\left|\frac{r(t_j)}{\Omega'(t_j)}\right| \geqq$$

$$\geqq (m+1)!\,c_{11}(m)^{-m}(n^{1/2}k^{3/2})^m = c_{15}(m)\,n^{m/2}k^{3m/2} \qquad (n\geqq c_{14}(m)).$$

Now (49) and (55) give

$$(56)\qquad \max_{t_k-\pi\leqq t\leqq t_k}|r^{(m)}(t)| \geqq c_{15}(m)n^{m/2}k^{3m/2}\max_{t_k-\pi\leqq t\leqq t_k}|r(t)| \qquad (n\geqq c_{14}(m)).$$

Finally let $p_{nkm}(t)=r\left(t-t_k+\frac{\pi}{2}\right)$. Then from (48) it is clear that $p_{nkm}\in T_n^{2k}\left(\frac{\pi}{2}\right)$ and from (56) we get

$$\|p_{nkm}\| \geqq c_{15}(m)\,n^{m/2}k^{3m/2}\|p_{nkm}\| \quad (n\geqq c_{14}(m)).$$

Case 2. $\tilde{n}:=\left[\frac{n}{6}\right]<k\leqq 2n$ and $k\geqq m+1$. Then the polynomials $p_{nkm}:=p_{n\tilde{n}m}$ give the theorem.

Case 3. $0\leqq k\leqq m$. Then Theorem 4 will show the desired result. □

THEOREM 4. *For an arbitrary* $t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, *there exist* $p_{nkmt}\in T_n^{2k}\left(\frac{\pi}{2}\right)\subset \subset\mathscr{H}_n^k\left(\frac{\pi}{2}\right)$ $(0\leqq k\leqq 2n,\ m\geqq 1)$ *such that*

$$|p^{(m)}(t)| \geqq c_{16}(m)\big(n(k+1)\big)^{m/2}\|p\| \qquad (n\geqq c_{17}(m))$$

where $\|\cdot\|$ *denotes the supremum norm over the interval* $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

This theorem shows not only that Theorem 2 is sharp, but even that there is no better Bernstein type estimate for derivatives of polynomials from $\mathscr{T}_n\left(\frac{\pi}{2}\right)$ than the Markov type estimate of Theorem 2.

PROOF. We distinguish two cases.

Case 1. $0\leqq k\leqq m+1$. Let $q_n(\tau)=\cos^{2n}\frac{\tau}{2}$. Then $q_n\in T_n^0(\pi)$ and by Example 2 of [4] there are $\xi_{nm}\in\left[0,\frac{\pi}{2}\right]$ such that

$$(57)\qquad |q_n^{(m)}(\pm\xi_{nm})| \geqq c_{18}(m)\,n^{m/2}\max_{-\pi\leqq\tau\leqq\pi}|q_n(\tau)| \qquad (n\geqq c_{19}(m)).$$

So the polynomials

$$p_{nkmt}(t) := \begin{cases} q_n(t-\xi_{nm}-\tau) & \text{if } \ 0 \leqq t \leqq \frac{\pi}{2}, \quad 0 \leqq k \leqq m+1, \\ q_n(t+\xi_{nm}-\tau) & \text{if } \ -\frac{\pi}{2} \leqq t < 0, \quad 0 \leqq k \leqq m+1 \end{cases}$$

give the desired result.

Case 2. $k > m+1$. Now let

$$q_{nk}(\tau) = \cos^{2n-2k+1}\frac{\tau}{2}\, C_{2k-1}\left(\sin\frac{\tau}{2}\right).$$

Obviously, $q_{nk} \in T_n$ has all but $2k-1$ roots at π, and by Example 1 of [4] there are $\xi_{nkm} \in \left[0, \frac{\pi}{2}\right]$ such that

(58) $$|q_{nk}^{(m)}(\pm\xi_{nkm})| \geqq c_{20}(m)(nk)^{m/2} \max_{-\pi \leqq \tau \leqq \pi} |q_{nk}(\tau)| \qquad (n \geqq c_{21}(m)).$$

Therefore the polynomials

$$p_{nkmt}(\tau) = \begin{cases} q_{nk}(t-\xi_{nkm}-\tau) & \text{if } \ 0 \leqq t \leqq \frac{\pi}{2}, \quad m+2 \leqq k \leqq 2n \\ q_{nk}(t+\xi_{nkm}-\tau) & \text{if } \ -\frac{\pi}{2} \leqq t < 0, \quad m+2 \leqq k \leqq 2n \end{cases}$$

show the theorem. □

Let $\delta(n, \omega, x) = \sqrt{\frac{\omega^2 - x^2}{n}}$ and $\Delta(n, \omega, x) = \delta(n, \omega, x) + \frac{\omega}{n}$ $(|x| \leqq \omega)$. In [6] G. G. Lorentz examined the following class of polynomials:

$$\mathscr{P}_n := \left\{p \mid p(x) = \sum_{i=1}^{n} a_i(1-x)^i(x+1)^{n-i} \text{ with all } a_i \geqq 0 \text{ or all } a_i \leqq 0\right\}.$$

He proved that (see Theorem B of [6])

(59) $$|p^{(m)}(y)| \leqq c_{22}(m)\Delta(n, 1, y)^{-m} \max_{|x| \leqq 1} |p(x)| \qquad (p \in \mathscr{P}_n,\ |y| \leqq 1).$$

By Theorem 1 of [4] and (2) we have

(60) $$|p^{(m)}(t)| \leqq c_{23}(m)\Delta(n, \omega, t)^{-m} \max_{|x| \leqq \omega} |p(x)| \quad (p \in \mathscr{T}_n(\omega),\ |t| \leqq \omega).$$

The following two theorems show that (59) and (60) are sharp.

THEOREM 5. *Let* $y \in [-1, 1]$ *be fixed. Then*

$$\sup_{p \in \mathscr{P}_n} \frac{|p^{(m)}(y)|}{\max_{|x| \leqq 1} |p(x)|} \geqq c_{24}(m)\Delta(n, 1, y)^{-m} \qquad (n \geqq m \geqq 1).$$

THEOREM 6. *Let* $t\in[-\omega,\omega]$ *be fixed,* $m\geqq 1$, $0<\omega\leqq c_{13}\frac{\pi}{2}$, $c_{13}<1$. *Then*

$$\sup_{p\in\mathcal{T}_n(\omega)}\frac{|p^{(m)}(t)|}{\|p\|}\geqq c_{25}(m)\Delta(n,\omega,t)^{-m}\qquad (n\geqq c_{26}(m)).$$

The condition $0<\omega\leqq c_{13}\frac{\pi}{2}$, $c_{13}<1$ in Theorem 6 is essential, see Theorem 2.

PROOF of Theorem 5. Let Π_n be the set of all real algebraic polynomials of degree at most n, P_n^0 be the set of those polynomials from Π_n which have only real roots outside $(-1, 1)$. By a simple observation of G. G. Lorentz

$$P_n^0\subset\mathcal{P}_n. \tag{61}$$

It may be supposed that $2m<n$, otherwise the polynomials $p_{ynm}(x)=(x+1)^m$ give the desired result. For the sake of brevity in the sequel let

$$p_{ln}(x)=(1-x)^{n-l}(x+1)^l\qquad (0\leqq l\leqq n).$$

Observe that

$$\frac{p_{ln}^{(m)}(x)}{p_{ln}(x)}=\frac{q_{lnm}(x)}{(1-x^2)^m}\qquad (|x|<1,\ m\leqq l\leqq n-m) \tag{62}$$

where $\deg q_{lnm}=m$ and the leading coefficient of q_{lnm} is $(-1)^{n-l}\frac{n!}{(n-m)!}$. Further, the repeated application of Rolle's Theorem shows that q_{lnm} has only real roots, so

$$q_{lnm}(x)=(-1)^{n-l}\frac{n!}{(n-m)!}\prod_{j=1}^{m}(x-\alpha_j)$$
$$(\alpha_j\in\mathbf{R},\ j=1,2,\dots,m). \tag{63}$$

Let $x_l:=-1+\frac{2l}{n}$ $(l=1,2,\dots,n)$ and $c:=\min\left\{\frac{1}{4c_{22}(1)},\frac{1}{4}\right\}$ and consider the intervals

$$I_l:=[x_l,\ x_l+c\delta(n,1,x_l)]. \tag{64}$$

First we show that there exist $\xi_l\in I_l$ such that

$$|p_{ln}^{(m)}(\xi_l)|\geqq c_{27}(m)\delta(n,1,\xi_l)^{-m}\max_{|x|\leqq 1}|p_{ln}(x)|$$
$$\left(\left[\frac{n}{2}\right]+1\leqq l\leqq n-m\right). \tag{65}$$

By the definition of x_l, c and I_l it is easy to see that

$$\delta(n,1,\eta)\geqq\frac{1}{2}\delta(n,1,x_l)\qquad \left(\eta\in I_l,\ \left[\frac{n}{2}\right]+1\leqq l\leqq n-m\right), \tag{66}$$

so using the mean value theorem, from (59) and (66), for all $\xi \in I_l$ we get

$$p_{ln}(x_l)-p_{ln}(\xi)=(x_l-\xi)\max_{\eta\in I_l}|p_{ln}'(\eta)|\leqq$$
$$\leqq c\delta(n,1,x_l)c_{22}(1)\max_{\eta\in I_l}\delta(n,1,\eta)^{-1}\max_{|x|\leqq 1}|p_{ln}(x)|\leqq$$
$$\leqq\frac{1}{4c_{22}(1)}\delta(n,1,x_l)c_{22}(1)2\delta(n,1,x_l)^{-1}p_{ln}(x_l)\leqq\frac{1}{2}p_{ln}(x_l),$$

from which

$$p_{ln}(\xi)\geqq\frac{1}{2}p_{ln}(x_l)=\frac{1}{2}\max_{|x|\leqq 1}|p_{ln}(x)| \tag{67}$$
$$\left(\xi\in I_l,\ \left[\frac{n}{2}\right]+1\leqq l\leqq n-m\right).$$

Now let us choose $\xi_l\in I_l$ such that

$$|\xi_l-\alpha_j|\geqq\frac{c}{2m}\delta(n,1,x_l)\qquad\left(1\leqq j\leqq m,\ \left[\frac{n}{2}\right]+1\leqq l\leqq n-m\right), \tag{68}$$

so from (63) and $2m<n$ we obtain

$$|q_{lnm}(\xi_l)|\geqq\frac{n!}{(n-m)!}\left(\frac{c}{2m}\delta(n,1,x_l)\right)^m\geqq\left(\frac{n}{2}\right)^m\left(\frac{c}{2m}\delta(n,1,\xi_l)\right)^m \tag{69}$$
$$\left(\left[\frac{n}{2}\right]+1\leqq l\leqq n-m\right).$$

By (67), (62) and (69) we deduce

$$\frac{|p_{ln}^{(m)}(\xi_l)|}{\max\limits_{|x|\leqq 1}|p_{ln}(x)|}\geqq\frac{1}{2}\frac{|p_{ln}^{(m)}(\xi_l)|}{p_{ln}(\xi_l)}=\frac{1}{2}\frac{|q_{lnm}(\xi_l)|}{(1-\xi_l^2)^m}\geqq$$
$$\geqq\frac{1}{2}\left(\frac{c}{4m}\right)^m\frac{n^m}{(1-\xi_l^2)^m}\delta(n,1,\xi_l)^m=\frac{1}{2}\left(\frac{c}{4m}\right)^m\delta(n,1,\xi_l)^{-m}$$
$$\left(\left[\frac{n}{2}\right]+1\leqq l\leqq n-m\right),$$

thus (65) is proved with $c_{27}(m)=\frac{1}{2}\left(\frac{c}{4m}\right)^m$.

Now we need

LEMMA 9. *Let* $z\in[0,1)$ *and suppose that there exists a* $p_z(x)\in P_n^0$ *such that*

$$|p_z^{(m)}(z)|\geqq c_{27}(m)\delta(n,1,z)^{-m}\max_{|x|\leqq 1}|p_z(x)|. \tag{70}$$

Then for all $y\in\left[\frac{3z-1}{2}, z\right]$ *there exist polynomials* $p_y(x)\in P_n^0$ *such that*

$$|p_y^{(m)}(y)| \geqq c_{28}(m)\delta(n, 1, y)^{-m} \max_{|x|\leqq 1}|p(x)|.$$

PROOF. Consider the polynomials

$$p_{z,\alpha}(x) = p_z\left(\frac{2}{1-\alpha}x - \frac{1+\alpha}{1-\alpha}\right) \qquad (-2 \leqq \alpha \leqq -1).$$

Obviously, $p_{z,\alpha}$ has all its zeros in $\mathbf{R}\setminus(\alpha, 1)$, thus $p_{z,\alpha}\in P_n^0$ $(-2\leqq\alpha\leqq-1)$. Further (70) implies that

$$\left|p_{z,\alpha}^{(m)}\left(\frac{1-\alpha}{2}z+\frac{\alpha+1}{2}\right)\right| \geqq c_{27}(m)\left[\left(\frac{2}{1-\alpha}\right)^m \delta(n, 1, z)^{-m} \max_{\alpha\leqq x\leqq 1}|p_{z,\alpha}(x)|\right] \geqq$$

(71)

$$\geqq c_{27}(m)\left(\frac{2}{3}\right)^m \delta(n, 1, z)^{-m} \max_{|x|\leqq 1}|p_{z,\alpha}(x)| \qquad (-2 \leqq \alpha \leqq -1).$$

Observe that

(72) $$\delta(n, 1, z) \leqq 2\delta(n, 1, y) \qquad \left(y\in\left[\frac{3z-1}{2}, z\right]\right).$$

Now let $f(\alpha):=\frac{1-\alpha}{2}z+\frac{\alpha+1}{2}$. As $f(-1)=z$ and $f(-2)=\frac{3z-1}{2}$, there exists an $\alpha_0\in[-2, 1]$ such that $f(\alpha_0)=y$, if $y\in\left[\frac{3z-1}{2}, z\right]$. Therefore from (71) and (72) we get

$$|p_{z,\alpha_0}^{(m)}(y)| \geqq c_{27}(m)3^{-m}\delta(n, 1, y)^{-m} \max_{|x|\leqq 1}|p_{z,\alpha_0}(x)|. \quad \square$$

A simple calculation shows that

$$[x_{l-1}, x_l] \subset \left[\frac{3\xi_l-1}{2}, \xi_l\right]$$

(73)

$$\left(\left[\frac{n}{2}\right]+1 \leqq l \leqq s := \min\{n-m, n-8\}\right).$$

This together with Lemma 9 and (65) implies that there exist polynomials $p_{ynm}\in P_n^0$ such that

$$\frac{|p_{ynm}^{(m)}(y)|}{\max_{|x|\leqq 1}|p_{ynm}(x)|} \geqq c_{28}(m)\delta(n, 1, y)^{-m} \geqq c_{28}(m)\Delta(n, 1, y)^{-m}$$

$$\left(0 \leqq y \leqq -1+\frac{2s}{n}, 2m < n\right).$$

If $2m<n$, $y\geqq 0$ and $-1+\frac{2s}{n}<y\leqq 1$, then the polynomials $p_{ynm}(x)=$ $=(1+x)^n$ show that

$$\frac{|p_{ynm}^{(m)}(y)|}{\max\limits_{|x|\leqq 1}|p_{ynm}(x)|}\geqq c_{29}(m)n^m\geqq c_{29}(m)\Delta(n,1,y)^{-m}.$$

In case of $-1<y\leqq 0$ the polynomials $p_{ynm}:=p_{-ynm}(-x)\in P_n^0$ give the desired result. □

NOTE. The polynomials p_{ynm} $(|y|\leqq 1,\ 1\leqq m\leqq n)$ which were constructed in the proof of Theorem 1 have only real roots in $[-2,-1]\cup[1,2]$.

PROOF of Theorem 6. Denote by T_n the set of all real trigonometric polynomials. We need

LEMMA 10. *Let p be an algebraic polynomial, $0<\omega\leqq\frac{\pi}{2}$ and $q(t)=p\left(\frac{\sin t}{\sin\omega}\right)$. Then*

$$q^{(m)}(t)=p^{(m)}\left(\frac{\sin t}{\sin\omega}\right)\frac{\cos^m t}{\sin^m\omega}+\sum_{i=0}^{m-1}p^{(i)}\left(\frac{\sin t}{\sin\omega}\right)r_{im}(t)$$

(74)

$$\left(r_{im}\in T_i\quad\text{and}\quad\max_{x\in\mathbf{R}}|r_{im}(x)|\leqq c_{im}\omega^{-i}\right),$$

where c_{im} are constants depending only on m and i.

To prove this is a routine work, so we omit the proof.

By Lemma 10 and Theorem 5 the proof of Theorem 6 is straightforward. Let $t\in[-\omega,\omega]$ be fixed and $y:=\frac{\sin t}{\sin\omega}\in[-1,1]$, so by Theorem 5 there exist polynomials $p_{ynm}\in\mathscr{P}_n$ such that

(75) $$|p_{ynm}^{(m)}(y)|\geqq c_{24}(m)\Delta(n,1,y)^{-m}\max_{|x|\leqq 1}|p_{ynm}(x)|\qquad(1\leqq m\leqq n).$$

By using the relation $T_n^0(\omega)\subset\mathscr{T}_n(\omega)$ it is easy to see that $p_{ynm}\in\mathscr{P}_n$ and $0<\omega<\frac{\pi}{2}$ imply

(76) $$q_{tnm}(\tau):=p_{ynm}\left(\frac{\sin\tau}{\sin\omega}\right)\in\mathscr{T}_n(\omega).$$

Using Lemma 10, we have

(77) $$q_{tnm}^{(m)}(t)=p_{ynm}^{(m)}(y)\frac{\cos^m t}{\sin^m\omega}+\sum_{i=0}^{m-1}p_{tnm}^{(i)}(y)r_{im}(t),$$

where

(78) $$\max_{x\in\mathbf{R}}|r_{im}(x)|\leqq c_{im}\omega^{-i}.$$

Observe that $0<\omega\leqq c_{13}\frac{\pi}{2}$, $c_{13}<1$ imply

$$\frac{c_{14}}{\omega}\Delta(n,\omega,t)\leqq\Delta(n,1,y)\leqq\frac{c_{15}}{\omega}\Delta(n,\omega,t) \tag{79}$$

and

$$\frac{\cos^m t}{\sin^m\omega}\geqq c_{30}(m)\omega^{-m}.$$

Hence and from (75) and (76) we get

$$\left|p_{ynm}^{(m)}(y)\frac{\cos^m t}{\sin^m\omega}\right|\geqq c_{31}(m)\Delta(n,\omega,t)^{-m}\|q_{tnm}\|. \tag{80}$$

Further from (59), (78) and (79) we obtain

$$|p_{ynm}^{(i)}(y)r_{im}(t)|\leqq c_{32}(m)\Delta(n,\omega,t)^{-i}\|q_{tnm}\| \quad (1\leqq i\leqq m-1). \tag{81}$$

Thus

$$\left|\sum_{i=1}^{m-1}p_{ynm}^{(i)}(y)r_{im}(t)\right|\leqq c_{33}(m)\Delta(n,\omega,t)^{1-m}\|q_{tnm}\|. \tag{82}$$

Now (77), (80) and (82) yield

$$|q_{tnm}^{(m)}(t)|\geqq c_{34}(m)\Delta(n,\omega,t)^{-m}\|q_{tnm}\| \qquad (n\geqq c_{26}(m)). \quad \square$$

Several problems remained open. It seems very difficult to establish a Markov type estimate being sharp in n, k and $0<\omega\leqq\pi$, for the derivatives of polynomials from $\mathscr{H}_{n+k}^{k}(\omega)$. For example in case of $\omega=\frac{\pi}{2}$ is it true that

$$\|p^{(m)}\|\leqq c_{35}(m)(n+k)^{m/2}(k+1)^{3m/2}\|p\| \tag{83}$$

$$\left(p\in\mathscr{H}_{n+k}^{k}\left(\frac{\pi}{2}\right),\ n,k\geqq 0,\ m\geqq 1\right)?$$

The answer is not known even in case of $m=1$ and $p\in T_{n+k}^{2k}\left(\frac{\pi}{2}\right)$. Theorem 3 shows that (83) would be sharp both in n and k.

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ON A CLOSEDNESS PROPERTY OF UNBIASED ESTIMATORS WITH MINIMAL RISK

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Abstract

As an extension of a theorem of Schmetterer a closedness property of unbiased Banach-valued estimators with minimal risk is proved. As the proof is given by a method of directional derivatives it does not require the Fréchet-differentiability of the loss function.

In this paper we are concerned with a closedness property of unbiased estimators with minimal risk. Such a result was first presented by Schmetterer in [5]. Schmetterer considers there real-valued estimators and formulates his results basing on loss functions defined by the k-th power for $k \geqq 2$ of the Euclidean norm of the space $\mathbf{R}$ of estimates. The given proof requires the Fréchet differentiability of these special loss functions.

In the sequel we consider Banach-valued estimators, where the differentiability of the k-th power of the norm not necessarily can be assumed. We prove our theorem in a slightly generalized version, i.e. we prove it for a certain type of convex loss function, which need not necessarily be differentiable. Thereby we make use of the method of directional derivatives. The main result remains of course true, too, for a theory of unbiased compact-convex-valued estimators with minimal risk, developed in [4], where the space of estimates — basing on a well-known theorem of Rådström (compare [4]) — will be embedded as a closed convex cone into a Banach space. Note that the case when the loss function is given by $\|\cdot\|^1$ — excluded by Schmetterer — is also covered by the approach presented here.

An estimation problem is determined by a seven-tuple

$$(\mathbf{H}, \mathscr{H}, \Gamma, \mathscr{W}, E, g, v),$$

what we call a (convex) *estimation experiment;* thereby $(\mathbf{H}, \mathscr{H})$ (sample space) is a measurable space. Γ (parameter space) is a non-empty set and $\mathscr{W} := \{P_\gamma(\cdot) | \gamma \in \Gamma\}$ (set of possible sample distributions) is a set of probability measures P_γ on $\mathscr{H}$, such that $\gamma \to P_\gamma(\cdot)$ is one-to-one. E denotes a real separable Banach space with norm $\|\cdot\|$ and $g\colon \Gamma \to E$ (parameter function) is a mapping. Let $\mathscr{B}$ and $\mathscr{B}(E)$ be the Borel σ-algebra on $\mathbf{R}$ and E resp., then we assume for $v\colon \Gamma \times E \to \mathbf{R}_+$ (loss function) that $v(\gamma, \cdot)\colon E \to \mathbf{R}_+$ is $\mathscr{B}(E) - \mathscr{B}$-measurable and convex $(\gamma \in \Gamma)$.

Definition 1. Let an estimation experiment be given. Then any $\mathscr{H} - \mathscr{B}(E)$-measurable mapping $T\colon \mathbf{H} \to E$ is called an *estimator* (to the given estimation ex-

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periment). $\mathcal{T}(g):=\mathcal{T}$ denotes the linear subspace of the set of all estimators for the parameter function g.

The notion of integral to be used in the sequel will be the Bochner-integral [2], as presented for instance in [3], Chapter IV, 10.7, Definition 7. This integral, of course, coincides for real-valued functions with the usual integral well-known from measure theory.

DEFINITION 2. Let an estimation experiment be given.

1. An estimator T is called *unbiased*, iff

$$\int T\,dP_\gamma = g(\gamma) \qquad (\gamma\in\Gamma);$$

the set of all unbiased estimators will be denoted by $\mathcal{T}^u(g):=\mathcal{T}^u$. For $g\equiv 0$ we are speaking of $T\in\mathcal{T}^u(0)$ as a *zero estimator*.

2. For $T\in\mathcal{T}$ the mapping

$$\varrho(\cdot\,,T):\ \Gamma\to[0,\infty]$$

defined by

$$\varrho(\gamma,T):=\int v(\gamma,T)\,dP_\gamma \qquad (\gamma\in\Gamma)$$

is called the *risk* of T.

3. $T\in\mathcal{T}$ is called of *finite risk*, iff

$$\varrho(\gamma,T)<\infty \qquad (\gamma\in\Gamma);$$

the set of all unbiased estimators $T\in\mathcal{T}$ with finite risk is denoted by $\mathcal{T}^{u\varrho}(g):=\mathcal{T}^{u\varrho}$.

4. $\bar{T}\in\mathcal{T}^{u\varrho}$ is called an *unbiased estimator with minimal risk*, iff

$$\varrho(\gamma,\bar{T})\leqq\varrho(\gamma,T) \qquad (\gamma\in\Gamma,\ T\in\mathcal{T}^{u\varrho});$$

the set of all unbiased estimators with minimal risk will be denoted by $\mathcal{T}^{u\varrho}_{\mathrm{Min}}(g):=\mathcal{T}^{u\varrho}_{\mathrm{Min}}$.

We first give a characterization of unbiased estimators with minimal risk by means of directional derivatives. Note, that the convexity of $v(\gamma,\cdot)$ entails the one of $\varrho(\gamma,\cdot)$ on the linear space $\mathcal{T}^u(g)$ of estimators. The directional derivative of $\varrho(\gamma,\cdot)$ at $T\in\mathcal{T}^u(g)$ w.r.t. some $S\in\mathcal{T}^u(0)$ is defined by

$$D_S\varrho(\gamma,T):=\lim_{t\downarrow 0}\frac{1}{t}\big(\varrho(\gamma,T+tS)-\varrho(\gamma,T)\big).$$

The following Lemma will be stated without proof.

LEMMA 3. *The following statements are equivalent:*

(1) $$T\in\mathcal{T}^{u\varrho}_{\mathrm{Min}}(g)$$

(2) $$D_S\varrho(\gamma,T)\geqq 0 \qquad (S\in\mathcal{T}^u(0),\ \gamma\in\Gamma).$$

In the sequel, we consider estimation experiments with the loss function $v:=v_g$ being of the type

$$v_g(\gamma,z):=w(\|g(\gamma)-z\|) \qquad (\gamma\in\Gamma,\ z\in E),$$

where w: $\mathbf{R}_+ \to \mathbf{R}_+$ is convex and increasing (and therefore measurable). In connection with this type of loss function we use the symbol ϱ_g for the risk of an estimator $T \in \mathcal{T}(g)$.

In the following Remarks we quote some properties of the convex function w and the risk ϱ_g.

REMARK 4. 1. We denote the left-hand sided derivative of w at the point $y \in \mathbf{R}_+$ by

$$D_- w(y) = \lim_{h \downarrow 0} \frac{w(y-h) - w(y)}{-h}$$

(extended continuously to $y=0$). Then, $D_- w(y)$ is a lower semicontinuous function on $\mathbf{R}_+$, i.e.

$$y_n \to y \Rightarrow \liminf_{n \to \infty} D_- w(y_n) \geqq D_- w(y).$$

2. For $y_1, y_2 \in \mathbf{R}_+$ the valuation

$$|w(y_1) - w(y_2)| \leqq |y_1 - y_2| \max \left(D_- w(y_1), D_- w(y_2)\right)$$
$$\leqq |y_1 - y_2| \left(D_- w(y_1) + D_- w(y_2)\right)$$

holds.

REMARK 5. From the definition of the directional derivative for ϱ_g one derives the following upper semicontinuity property:

$$\lim_{n \to \infty} \varrho_{g_n}(\gamma, T_n + S) = \varrho_g(\gamma, T_0 + S) \quad S \in \mathcal{T}^u(0)$$
$$\Rightarrow \limsup_{n \to \infty} D_S \varrho_{g_n}(\gamma, T_n) \leqq D_S \varrho_g(\gamma, T_0) \quad S \in \mathcal{T}^u(0)$$

for an arbitrary sequence $(T_n)_{n \in \mathbf{N} \cup \{0\}}$ with finite risk.

THEOREM 6. *Let* $(\mathbf{H}, \mathcal{H}, \mathcal{W}, E, g_n, v_{g_n})_{n \in \mathbf{N}}$ *be a sequence of estimation experiments; and let* $(T_n)_{n \in \mathbf{N}}$ *be a sequence of estimators* $T_n \in \mathcal{T}^{uq}_{\mathrm{Min}}(g_n)$ *satisfying*

(1) $$\limsup_{n \to \infty} N^q_\gamma\left(D_- w(\|T_n + S - g_n(\gamma)\|)\right) < \infty \quad (S \in \mathcal{T}^u(0),\ \gamma \in \Gamma)^1$$

for some $q > 1$. *Moreover, let* T *be an estimator s.t.*

(2) $$\lim_{n \to \infty} N^p_\gamma(\|T - T_n\|) = 0$$

holds for $p > 1$ *with* $p^{-1} + q^{-1} = 1$. *Then, an estimation experiment* $(\mathbf{H}, \mathcal{H}, \mathcal{W}, E, g, v_g)$ *exists s.t.* $T \in \mathcal{T}^{uq}_{\mathrm{Min}}(g)$.

PROOF. According to (2) the parameter function g may be defined by

$$g(\gamma) := \int T\, dP_\gamma = \lim_{n \to \infty} \int T_n\, dP_\gamma = \lim_{n \to \infty} g_n(\gamma) \quad (\gamma \in \Gamma).$$

[1] For any measurable function f: $\mathcal{H} \to \mathbf{R}_+$, we define

$$N^p_\gamma(f) := \left(\int f^p\, dP_\gamma\right)^{1/p} \quad (\gamma \in \Gamma,\ p > 0).$$

We first prove that

(i) $$\lim_{n\to\infty} \varrho_{g_n}(\gamma, T_n+S) = \varrho_g(\gamma, T+S) \qquad (S\in\mathscr{T}^u(0),\ \gamma\in\Gamma)$$

holds true. Observing Remark 4.2 we obtain for $\tilde{T}:=T+S$, $\tilde{T}_n=T_n+S$ with $S\in\mathscr{T}^u(0)$ from Hölder's inequality the valuation

$$|\varrho_g(\gamma,\tilde{T})-\varrho_{g_n}(\gamma,\tilde{T}_n)| = \left|\int w(\|\tilde{T}-g(\gamma)\|)\,dP_\gamma-\int w(\|\tilde{T}_n-g_n(\gamma)\|)\,dP_\gamma\right| \leqq$$
$$\leqq \int \left|w(\|\tilde{T}-g(\gamma)\|)-w(\|\tilde{T}_n-g_n(\gamma)\|)\right| dP_\gamma \leqq$$
$$\leqq \int (\|g(\gamma)-g_n(\gamma)\|+\|T-T_n\|)(D_-w(\|\tilde{T}-g(\gamma)\|)+D_-w(\|\tilde{T}_n-g_n(\gamma)\|))\,dP_\gamma \leqq$$
$$\leqq (N^p_\gamma(\|g(\gamma)-g_n(\gamma)\|)+N^p_\gamma(\|T-T_n\|))\times$$
$$\times(N^q_\gamma(D_-w(\|\tilde{T}-g(\gamma)\|))+N^q_\gamma(D_-w(\|\tilde{T}_n-g_n(\gamma)\|)))$$

for arbitrary $\gamma\in\Gamma$. Since the first factor of this product converges to 0, (i) is proved when we ascertain that the second factor is bounded. As a consequence of [1], Satz 15.5, S. 78, and (2) a subsequence $(\tilde{T}_{n_j})_{j\in\mathbf{N}}$ of $(\tilde{T}_n)_{n\in\mathbf{N}}$ with

$$\lim_{j\to\infty} \tilde{T}_{n_j} = \tilde{T} \quad P_\gamma\text{-a.e.}$$

exists. Hence, using Fatou's Lemma and (1), we verify that the inequalities

$$N^q_\gamma(D_-w(\|\tilde{T}-g(\gamma)\|)) \leqq N^q_\gamma(\liminf_{j\to\infty} D_-w(\|\tilde{T}_{n_j}-g_{n_j}(\gamma)\|)) \leqq$$
$$\leqq \liminf_{j\to\infty} N^q_\gamma(D_-w(\|\tilde{T}_{n_j}-g_{n_j}(\gamma)\|)) \leqq \limsup_{n\to\infty} N^q_\gamma(D_-w(\|\tilde{T}_n-g_n(\gamma)\|)) < \infty$$

are satisfied. Thus, (i) is proved.

From (i) and Remark 5, we derive the inequality

$$0 \leqq \limsup_{n\to\infty} D_S\varrho_{g_n}(\gamma, T_n) \leqq D_S\varrho_g(\gamma, T) \qquad (S\in\mathscr{T}^u(0)),$$

by which — together with Lemma 3 — the Theorem is proved. □

REMARK 7. The case when the loss function is given by

$$v(\gamma, z) = \|z-g(\gamma)\| \qquad (\gamma\in\Gamma,\ z\in\mathbf{R})$$

is not treated in [5]. In this case $w\equiv\mathrm{id}_{\mathbf{R}}$ and therefore $D_-w\equiv 1$, s.t. Condition 6 (1) is automatically fulfilled.

Note that Theorem 6 remains valid for the estimation concept introduced in [4] basing on the Hausdorff distance as loss function which becomes the norm of a separable Banach space in which the estimation problem may be embedded.

2. In the case when the loss function is given by

$$v(\gamma, z) = \|z-g(\gamma)\|^k \qquad (\gamma\in\Gamma,\ z\in\mathbf{R})$$

with $k\in\mathbf{N}$, $k\geqq 2$ and $p=k$ (as treated in [5]) Condition 6 (1) takes the form

$$\limsup_{n\to\infty}\int\|T_n+S-g_n(\gamma)\|^k\,dP_\gamma<\infty\qquad (S\in\mathcal{T}^u(0),\ \gamma\in\Gamma),$$

where $q=k/k-1$. 6 (1) is therefore a consequence of 6 (2).

3. A rechecking of the proof of Theorem 6 shows that it remains valid even if $p=1$ and $p=\infty$.

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ЗАМЕЧАНИЯ О РЕШЕТКАХ КОНГРУЭНЦИЙ УНИВЕРСАЛЬНЫХ АЛГЕБР

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В настоящей заметке доказывается следующее: для всякой универсальной алгебры **A** со счетным множеством операций какой-нибудь арности $n>0$, имеющих общий левый ноль, решетка Con **A** конгруэнций на **A** представима как решетка конгруэнций алгебры сигнатуры $\langle \min\{2, n\}, 1\rangle$; для всякой конечной алгебры **A**, операции которой имеют общий левый ноль, Con **A** представима как решетка конгруэнций конечной алгебры сигнатуры $\langle \min\{2, n\}, 1\rangle$.

Отсюда выводится, в частности, что всякая алгебраическая решетка с не более чем счетным множеством компактных элементов, представимая как решетка конгруэнций группоида с двусторонним нулем, представима как решетка конгруэнций 2-унарной алгебры.

Отсюда же выводится, что для всякой алгебраической решетки L с не более чем счетным множеством компактных элементов решетка $L+1$ представима как решетка конгруэнций 2-унарной алгебры и что для всякой решетки L, представимой как решетка конгруэнций конечной алгебры, $L+1$ представима как решетка конгруэнций конечной 2-унарной алгебры.

В заключение приводятся обобщения и усиления некоторых из этих результатов.

Мы будем использовать следующие обозначения и термины. N будет обозначать множество всех натуральных чисел.

Пусть f — унарная операция на множестве A. Тогда $f^0(a)=a$, $f^{n+1}(a)=f(f^n(a))$ $(a\in A,\ n\in N)$. Для всякой упорядоченной пары $\langle a, b\rangle$ элементов A через $f\langle a, b\rangle$ будет обозначаться пара $\langle f(a), f(b)\rangle$.

Пусть g — n-арная операция на A, где $n>0$. Элемент ω множества A называется нулем (левым нулем) для g, если $g(\bar{a})=\omega$ для всякой упорядоченной n-ки $\bar{a}$ элементов A, у которой какой-нибудь (первый) член есть ω.

Говоря об алгебрах, будем иметь в виду универсальные алгебры без нульарных операций.

Элемент алгебры называется её (левым) нулем, если он есть (левый) ноль для всякой её операции. Нули унарной алгебры будем называть также её неподвижными точками.

Для всякого множества M и всякого $R\subseteq M^2$ через $\mathbf{A}(R)$ будем обозначать наименьшую конгруэнцию на алгебре $\mathbf{A}=\langle A; \ldots\rangle$, включающую $R\cap A^2$.

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§ 1

Пусть $\mathbf{A}=\langle A; \ldots\rangle$ и $\mathbf{B}=\langle B; \ldots, h, \ldots\rangle$ — алгебры такие, что $A\subset B$, h — унарная операция, A — система образующих для $\langle B; h\rangle$, ω — некоторый элемент A, для которого $h(\omega)=\omega$, и пусть $E\subseteq B^2$. Рассмотрим следующие условия, относящиеся к четверке $\lambda=\langle \mathbf{A}, \mathbf{B}, E, \omega\rangle$:

(A) $$E\cap A^2\in \mathrm{Con}\, \mathbf{A},$$

(B) $$h^m\langle a, b\rangle\in E\Rightarrow h^n\langle a, b\rangle\in E,$$

(C) $$m\neq n\wedge\langle h^m(a), h^n(b)\rangle\in E\Rightarrow\langle a, \omega\rangle\in E\wedge\langle b, \omega\rangle\in E,$$

(D) $$\langle a, \omega\rangle\in E\wedge\langle b, \omega\rangle\in E\Rightarrow\langle h^m(a), h^n(b)\rangle\in E,$$

(E) $$\langle h^m(a), h^n(b)\rangle\in E\Rightarrow\langle a, b\rangle\in E,$$

(F) $$\begin{cases} a\neq\omega\wedge m\neq n\Rightarrow h^m(a)\neq h^n(a), \\ \qquad a\neq b\Rightarrow h^m(a)\neq h^n(b)\end{cases}$$

($a, b\in A$; $m, n\in N$). Систему условий (A), (B), (C), (D), (F) будем обозначать через ($\mathfrak{A}$). Следующее предложение очевидно:

(1.1) Из ($\mathfrak{A}$) следует (E).

(1.2) Пусть E — конгруэнция на $\langle B; h\rangle$ и λ удовлетворяет (E). Тогда λ удовлетворяет (B) и (D).

Действительно, пусть $h^m\langle a, b\rangle\in E$ для каких-нибудь $a, b\in A$ и $m\in N$. Тогда, в силу (E), $\langle a, b\rangle\in E$. Отсюда $h^n\langle a, b\rangle\in E$ для всякого $n\in N$ вследствие $E\in\mathrm{Con}\,\langle B; h\rangle$. Таким образом, λ удовлетворяет (B).

Пусть $\langle a, \omega\rangle\in E$ и $\langle b, \omega\rangle\in E$ для каких-нибудь $a, b\in A$. Тогда, в силу $E\in\mathrm{Con}\,\langle B; h\rangle$, для всяких $m, n\in N$ имеет место $h^m\langle a, \omega\rangle\in E$ и $h^n\langle b, \omega\rangle\in E$, то есть $\langle h^m(a), \omega\rangle\in E$ и $\langle h^n(b), \omega\rangle\in E$, а значит, $\langle h^m(a), h^n(b)\rangle\in E$. Таким образом, λ удовлетворяет (D).

(1.3) Пусть λ удовлетворяет ($\mathfrak{A}$). Тогда E — конгруэнция на $\langle B; h\rangle$.

Из (A) следует, что $\langle a, a\rangle\in E$ для всякого $a\in A$. Но тогда, в силу (B), $\langle h^m(a), h^m(a)\rangle\in E$ для всякого $m\in N$. Таким образом, E рефлексивно.

Докажем симметричность E. Пусть $\langle h^m(a), h^m(b)\rangle\in E$ для каких-нибудь $a, b\in A$ и $m\in N$. Тогда, в силу (B), $\langle a, b\rangle\in E$. Отсюда, в силу (A), $\langle b, a\rangle\in E$. Но тогда, в силу (B), $\langle h^m(b), h^m(a)\rangle\in E$.

Пусть $\langle h^m(a), h^n(b)\rangle\in E$ и $m\neq n$. Тогда, в силу (C), $\langle b, \omega\rangle\in E$ и $\langle a, \omega\rangle\in E$. Отсюда, в силу (D), $\langle h^n(b), h^m(a)\rangle\in E$.

Докажем транзитивность E. Пусть $\langle h^m(a), h^n(b)\rangle\in E$ и $\langle h^n(b), h^p(c)\rangle\in E$ (где $a, b, c\in A$). Тогда, в силу (1.1), $\langle a, b\rangle\in E$ и $\langle b, c\rangle\in E$, а значит, $\langle a, c\rangle\in E$ (в силу (A)).

Если $m=n=p$, то $\langle a, c\rangle\in E$ влечет, в силу (B), $\langle h^m(a), h^p(c)\rangle\in E$. Если $m\neq n$, то, в силу (C), $\langle a, \omega\rangle\in E$ и $\langle b, \omega\rangle\in E$. Отсюда и из $\langle b, c\rangle\in E$ следует, в силу (A), $\langle c, \omega\rangle\in E$. Из $\langle a, \omega\rangle\in E$ и $\langle c, \omega\rangle\in E$ следует, в силу (D), $\langle h^m(a), h^p(c)\rangle\in E$.

В случае $n \neq p$ доказательство аналогично. Итак, E — эквивалентность на B. Стабильность E относительно h очевидна.

(1.4) Пусть $E \in \mathrm{Con}\,\mathbf{B}$ и λ удовлетворяет ($\mathfrak{A}$). Тогда $\mathbf{B}(\mathbf{A}(E)) = E$.

Из (A) следует $\mathbf{B}(\mathbf{A}(E)) \subseteq E$. Докажем обратное включение. Пусть пара $\pi = \langle h^m(a), h^n(b)\rangle$, где $a, b \in A$, принадлежит E. Тогда, в силу (1.1), $\langle a, b\rangle \in E$, а значит, $\langle a, b\rangle \in \mathbf{B}(\mathbf{A}(E))$. Отсюда $\langle h^m(a), h^m(b)\rangle \in \mathbf{B}(\mathbf{A}(E))$. Таким образом, в случае $m = n$ имеем $\pi \in \mathbf{B}(\mathbf{A}(E))$.

Пусть $m \neq n$. Тогда, в силу (C), $\langle a, \omega\rangle \in E$ и $\langle b, \omega\rangle \in E$. Отсюда следует, в силу (A), что $\langle a, \omega\rangle$ и $\langle b, \omega\rangle$ принадлежат $\mathbf{B}(\mathbf{A}(E))$. Но тогда $\langle h^m(a), \omega\rangle$ и $\langle h^n(b), \omega\rangle$ принадлежат $\mathbf{B}(\mathbf{A}(E))$. Отсюда $\pi \in \mathbf{B}(\mathbf{A}(E))$.

Для всякой конгруэнции E' на $\mathbf{A}$ будем обозначать через $\mathbf{B}^*(E')$ множество, состоящее из всевозможных пар $\langle h^m(a), h^m(b)\rangle$ таких, что $\langle a, b\rangle \in E'$, и всевозможных пар $\langle h^m(a), h^n(b)\rangle$ таких, что $\langle a, \omega\rangle$ и $\langle b, \omega\rangle$ принадлежат E'. Очевидно, что в предположении (F) условие $E_1 \neq E_2$ равносильно $\mathbf{B}^*(E_1) \neq \mathbf{B}^*(E_2)$ для всяких $E_1, E_2 \in \mathrm{Con}\,\mathbf{A}$.

(1.5) Пусть 1) $\langle \mathbf{A}, \mathbf{B}, E, \omega\rangle$ удовлетворяет ($\mathfrak{A}$) для всякой конгруэнции E на $\mathbf{B}$ и 2) $\mathbf{B}^*(E') = \mathbf{B}(E')$ для всякой конгруэнции E' на $\mathbf{A}$. Тогда $\mathrm{Con}\,\mathbf{A} \cong \mathrm{Con}\,\mathbf{B}$.

Из второго условия следует, что отображение $\mathbf{B}^*$, переводящее всякую конгруэнцию E' на $\mathbf{A}$ в $\mathbf{B}^*(E')$, есть отображение в $\mathrm{Con}\,\mathbf{B}$. Так как по первому условию $\langle B; h\rangle$ удовлетворяет (F), то отображение $\mathbf{B}^*\colon \mathrm{Con}\,\mathbf{A} \to \mathrm{Con}\,\mathbf{B}$ взаимно однозначно. Из первого и второго условий следует, что $\mathbf{B}(E') = \mathbf{B}^*(E')$ для всякой конгруэнции E' на $\mathbf{A}$. Отсюда и из (1.4) следует, что $\mathbf{B}^*$ есть отображение на всю решетку $\mathrm{Con}\,\mathbf{B}$. Итак, $\mathbf{B}^*$ есть взаимно однозначное отображение $\mathrm{Con}\,\mathbf{A}$ на $\mathrm{Con}\,\mathbf{B}$. Так как $\mathbf{B}^*(E_1) \subseteq \mathbf{B}^*(E_2)$ равносильно $E_1 \subseteq E_2$ для всяких E_1, E_2 из $\mathrm{Con}\,\mathbf{A}$, то отсюда следует $\mathrm{Con}\,\mathbf{A} \cong \mathrm{Con}\,\mathbf{B}$.

§ 2

Пусть $\mathbf{A} = \langle A; f_0, \ldots, f_n, \ldots\rangle$ $(n \in N)$ — алгебра с p-арными $(p > 0)$ операциями, имеющая левые нули. Зафиксируем какой-нибудь её левый ноль ω. Рассмотрим семейство $(A_n)_{n \in N}$ равномощных множеств такое, что $A_0 = A$ и $A_m \cap A_n = \{\omega\}$ для всяких m и $n \neq m$. Пусть B — объединение этих множеств, h — взаимно однозначное преобразование B такое, что $h(A_n) = A_{n+1}$ для всякого $n \in N$ и $h(\omega) = \omega$, $\bar{h}$ — преобразование B такое, что $\bar{h}(h(b)) = b$ для всякого $b \in B$ и $\bar{h}(a) = a$ для всякого $a \in A$. Пусть далее g — p-арная операция на B, определяемая так:

$$g(a, h^{q_1}(b_1), \ldots, h^{q_{p-1}}(b_{p-1})) = \omega,$$

$$g(h(a), h^{q_1}(b_1), \ldots, h^{q_{p-1}}(b_{p-1})) = h(a),$$

$$g(h^{n+2}(a), h^{q_1}(b_1), \ldots, h^{q_{p-1}}(b_{p-1})) = f_n(a, b_1, \ldots, b_{p-1})$$

для всяких $a, b_1, \ldots, b_{p-1} \in A$ и всяких $n, q_1, \ldots, q_{p-1} \in N$. Обозначим через $\mathbf{B}$ алгебру $\langle B; g, h, \bar{h}\rangle$. ω — её единственный левый нуль.

2.1. Пусть $E \in \mathrm{Con}\,\mathbf{B}$. Тогда четверка $\lambda = \langle \mathbf{A}, \mathbf{B}, E, \omega \rangle$ удовлетворяет ($\mathfrak{A}$).

Очевидно, что λ удовлетворяет (F). Покажем, что λ удовлетворяет (A). $E \cap A^2$— эквивалентность на A. Пусть пары $\langle a, a' \rangle, \langle b, b_1' \rangle, \ldots, \langle b_{p-1}, b_{p-1}' \rangle$ принадлежат $E \cap A^2$. Тогда $h^{n+2} \langle a, a' \rangle \in E$, и значит,

$$\left\langle g\left(h^{n+2}(a), b_1, \ldots, b_{p-1}\right), \; g\left(h^{n+2}(a'), b_1', \ldots, b_{p-1}'\right)\right\rangle \in E,$$

то есть

$$\left\langle f_n(a, b_1, \ldots, b_{p-1}), f_n(a', b_1', \ldots, b_{p-1}')\right\rangle \in E \cap A^2.$$

Таким образом, $E \cap A^2$ стабильно относительно операций в $\mathbf{A}$.

Пусть $\langle h^m(a), h^n(b) \rangle \in E$ для каких-нибудь $a, b \in A$. Тогда $\tilde{h}^r \langle h^m(a), h^n(b) \rangle \in E$ для всякого $r \in N$. Если $r \geqq \max\{m, n\}$, то последняя пара есть $\langle a, b \rangle$. Таким образом, λ удовлетворяет (E).

Так как λ удовлетворяет (E), то, в силу (1.2), λ удовлетворяет (B) и (D).

Покажем, что λ удовлетворяет (C). Пусть пара $\pi = \langle h^m(a), h^n(b) \rangle$, где $a, b \in A$, принадлежит E и $m < n$. Тогда $\tilde{h}^{n-1} \in E$, то есть $\langle a, h(b) \rangle \in E$. Отсюда $\left\langle g(a, c, \ldots, c), g(h(b), c, \ldots, c)\right\rangle \in E$ для всякого c, то есть $\langle \omega, h(b) \rangle \in E$, а значит, $\tilde{h} \langle \omega, h(b) \rangle \in E$, то есть $\langle \omega, b \rangle \in E$. Из $\langle a, h(b) \rangle \in E$ и $\langle h(b), \omega \rangle \in E$ следует $\langle a, \omega \rangle \in E$. Таким образом, λ удовлетворяет (C).

(2.2) Пусть λ удовлетворяет ($\mathfrak{A}$). Тогда $E \in \mathrm{Con}\,\mathbf{B}$.

Из (1.3) следует, что E — конгруэнция на $\langle B; h \rangle$. Стабильность E относительно $\tilde{h}$ очевидна. Докажем стабильность E относительно g.

Пусть пары $\langle r, r' \rangle, \langle s_1, s_1' \rangle, \ldots, \langle s_{p-1}, s_{p-1}' \rangle$ принадлежат E. Покажем, что тогда пара $\pi = \langle g(r, s_1, \ldots, s_{p-1}), \; g(r', s_1', \ldots, s_{p-1}') \rangle$ принадлежит E. Пусть $r = h^m(a)$, $r' = h^n(a')$, $s_i = h^{t_i}(b_i)$, $s_i' = h^{u_i}(b_i')$ $(1 \leqq i \leqq p-1;\; a, a', b_i, b_i' \in A)$. Если $n = m$, то

$$\pi = \begin{cases} \langle \omega, \omega \rangle, & \text{если} \quad m = 0, \\ \langle h(a), h(a') \rangle, & \text{если} \quad m = 1, \\ \langle f_{m-2}(a, b_1, \ldots, b_{p-1}), f_{m-2}(a', b_1', \ldots, b_{p-1}') \rangle, & \text{если} \quad m > 1. \end{cases}$$

В первом случае $\pi \in E$ в силу рефлексивности E, во втором случае π совпадает с $\langle r, r' \rangle \in E$. Покажем, что и в третьем случае $\pi \in E$. В силу (E) из $\langle r, r' \rangle \in E$ следует $\langle a, a' \rangle \in E$, а из $\langle s_i, s_i' \rangle \in E$ следует $\langle b_i, b_i' \rangle \in E$ $(1 \leqq i \leqq p-1)$. Так как пары $\langle a, a' \rangle, \langle b_1, b_1' \rangle, \ldots, \langle b_{p-1}, b_{p-1}' \rangle$ принадлежат $E \cap A^2$, то, в силу (A), $\langle f_{m-2}(a, b_1, \ldots, b_{p-1}), f_{m-2}(a', b_1', \ldots, b_{p-1}') \rangle \in E$, то есть $\pi \in E$.

Пусть $m < n$. Тогда, в силу (C), $\langle a, \omega \rangle \in E$ и $\langle a', \omega \rangle \in E$. Отсюда, в силу (D), $\langle r, \omega \rangle \in E$ и $\langle r', \omega \rangle \in E$.

Если $m = 0$ и $n = 1$, то $\pi = \langle \omega, h(a') \rangle = \langle \omega, r' \rangle \in E$ (так как $\langle r', \omega \rangle \in E$ и E симметрично).

Если $m = 0$ и $n > 1$, то $\pi = \langle \omega, f_{n-2}(a', b_1', \ldots, b_{p-1}') \rangle$. Так как $\langle \omega, a' \rangle \in E \cap A^2$, то, в силу (A), $\langle f_{n-2}(\omega, b_1', \ldots, b_{p-1}'), f_{n-2}(a', b_1', \ldots, b_{p-1}') \rangle \in E \cap A^2$, то есть $\langle \omega, f_{n-2}(a', b_1', \ldots, b_{p-1}') \rangle \in E \cap A^2$, или $\pi \in E$.

Если $m = 1$, то $\pi = \langle h(a), f_{n-2}(a', b_1', \ldots, b_{p-1}') \rangle$. Так как $\langle a, \omega \rangle \in E$, то, в силу стабильности E относительно h, $\langle h(a), h(\omega) \rangle \in E$, то есть $\langle h(a), \omega \rangle \in E$. Отсюда и из $\langle \omega, f_{n-2}(a', b_1', \ldots, b_{p-1}')\rangle \in E$ (последнее доказано выше) следует

$$\langle h(a), f_{n-2}(a', b_1', \ldots, b_{p-1}')\rangle \in E,$$

то есть $\pi \in E$.

Пусть $m > 1$. Тогда $\pi = \langle f_{m-2}(a, b_1, \ldots, b_{p-1}), f_{n-2}(a', b_1', \ldots, b_{p-1}')\rangle$. Так как $\langle a, \omega\rangle \in E$, то, в силу (A), $\langle f_{m-2}(a, b_1, \ldots, b_{p-1}),\ f_{m-2}(\omega, b_1, \ldots, b_{p-1})\rangle \in E$, то есть $\langle f_{m-2}(a, b_1, \ldots, b_{p-1}),\ \omega\rangle \in E$. Отсюда и из $\langle \omega, f_{n-2}(a', b_1', \ldots, b_{p-1}')\rangle \in E$ следует $\pi \in E$.

Для всякой конгруэнции E' на **A** четверка $\langle \mathbf{A}, \mathbf{B}, \mathbf{B}^*(E'), \omega\rangle$ удовлетворяет ($\mathfrak{A}$), и поэтому, в силу (2.2), $\mathbf{B}^*(E') \in \mathrm{Con}\,\mathbf{B}$. Отсюда, из (2.1) и (1.5) следует

$$\text{(2.3)} \qquad \mathrm{Con}\,\mathbf{A} \cong \mathrm{Con}\,\mathbf{B}.$$

Как показано Р. Маккензи (см. [4]), для всякой алгебры **A** конечной сигнатуры $\mathrm{Con}\,\mathbf{A}$ представима как решетка конгруэнций алгебры сигнатуры $\langle 2, 1\rangle$. Отсюда и из (2.3) следует

Теорема 1. *Пусть* **A** *— алгебра со счетным множеством операций ограниченных арностей имеющая левый ноль. Тогда* $\mathrm{Con}\,\mathbf{A}$ *представима как решетка конгруэнций алгебры сигнатуры* $\langle 2, 1\rangle$.

§ 3

Пусть $\mathbf{A} = \langle A; f_0, \ldots, f_l\rangle$ — унарная алгебра конечной сигнатуры, ω — её неподвижная точка. Рассмотрим равномощные множества $A_0, \ldots, A_{l+2}$ такие, что $A_0 = A$ и $A_m \cap A_n = \{\omega\}$ для всяких m и $n \neq m$ $(m, n \leqq l+2)$. Пусь B — объединение этих множеств, h — взаимно однозначное преобразование B такое, что

$$h(A_m) = A_{m+1} \quad \text{для всякого} \quad m \leqq l+1,$$

$$h(A_{l+2}) = A_0,$$

$$h^{l+3} \text{ — тождественное преобразование},$$

$$h(\omega) = \omega.$$

Пусть далее g — преобразование B такое, что

$$g(a) = \omega,$$

$$g(h(a)) = h(a),$$

$$g(h^{n+2}(a)) = f_n(a) \quad (n \leqq l)$$

для всякого $a \in A$. Обозначим через **B** алгебру $\langle B; g, h\rangle$.

Условие, образуемое из (C) (из (F)) заменой «$m \neq n$» на «$m \not\equiv n \pmod{l+3}$», будем обозначать через (C′) (через (F′)). Систему условий (A), (B), (C′), (D), (F′) будем обозначать через ($\mathfrak{B}$).

Следующее предложение очевидно:

(3.1) Из ($\mathfrak{B}$) следует (E).

(3.2) Пусть E — конгруэнция на $\mathbf{B}$. Тогда четверка $\lambda=\langle \mathbf{A}, \mathbf{B}, E, \omega\rangle$ удовлетворяет ($\mathfrak{B}$).

В самом деле, $E\cap A^2$ — эквивалентность на A. Пусть $\langle a, b\rangle \in E\cap A^2$. Тогда $gh^{n+2}\langle a, b\rangle\in E$ для всякого n. В частности, для n такого, что $0\leqq n\leqq l$, имеем $\langle f_n(a), f_n(b)\rangle\in E$. Следовательно, λ удовлетворяет (A).

Пусть пара $\pi=\langle h^m(a), h^n(b)\rangle$ принадлежит E для каких-нибудь $a, b\in A$ и m, n таких, что $m\leqq n\leqq l+2$. Если $m=n$, то $h^{l+3-m}\pi\in E$, то есть $\langle a, b\rangle\in E$.

Пусть $m<n$. Тогда из $g^2h^{l+4-m}\pi\in E$ следует $\langle h(a), \omega\rangle\in E$, а из $g^2h^{l+4-n}\pi\in E$ следует $\langle\omega, h(b)\rangle\in E$. Отсюда $\langle h(a), h(b)\rangle\in E$, а значит, $h^{l+2}\langle h(a), h(b)\rangle\in E$, то есть $\langle a, b\rangle\in E$. Таким образом, λ удовлетворяет (E), а значит, в силу (1.2), λ удовлетворяет (B) и (D). Из $\langle h(a), \omega\rangle\in E$ следует $h^{l+2}\langle h(a), \omega\rangle\in E$, то есть $\langle a, \omega\rangle\in E$. Из $\langle h(b), \omega\rangle\in E$ следует $\langle b, \omega\rangle\in E$. Таким образом, λ удовлетворяет (C′).

Удовлетворимость (F′) очевидна.

(3.3) Пусть $E\subseteq B^2$ и четверка $\lambda=\langle \mathbf{A}, \mathbf{B}, E, \omega\rangle$ удовлетворяет ($\mathfrak{B}$). Тогда $E\in \mathrm{Con}\,\mathbf{B}$.

Заменяя в доказательстве (1.3) «$m\neq n$» на «$m\not\equiv n \pmod{l+3}$» и ссылку на (C) ссылкой на (C′), получаем доказательство того, что E — эквивалентность на B. Стабильность E относительно h очевидна. Докажем стабильность E относительно g.

Пусть пара $\pi=\langle h^m(a), h^m(b)\rangle$, где $a, b\in A$, принадлежит E. Тогда в силу (3.1), $\langle a, b\rangle\in E$. Если $m\equiv 0 \pmod{l+3}$, то $g\pi=\langle\omega, \omega\rangle\in E$. Если $m\equiv 1 \pmod{l+3}$, то $g\pi=\pi\in E$. Если $m\equiv k \pmod{l+3}$ и $1<k\leqq l+2$, то $g\pi=\langle f_{k-2}(a), f_{k-2}(b)\rangle$. Эта пара принадлежит E в силу (A).

Пусть пара $\varrho=\langle h^m(a), h^n(b)\rangle$, где $a, b\in A$ и $m<n\leqq l+2$, принадлежит E. Тогда, в силу (C′), $\langle a, \omega\rangle\in E$ и $\langle b, \omega\rangle\in E$. Отсюда, в силу (A) и (D), декартов квадрат D множества

$$\{f_p(a)|p\leqq l\}\cup\{f_p(b)|p\leqq l\}\cup\{h^r(a)|r\leqq l+2\}\cup\{h^r(b)|r\leqq l+2\}\cup\{\omega\}$$

включается в E, и так как $g\varrho\in D$, то $g\varrho\in E$.

Очевидным образом модифицируя доказательство (1.4) и применяя (3.2), получаем доказательство следующего предложения:

$$E\in \mathrm{Con}\,\mathbf{B}\Rightarrow \mathbf{B}(\mathbf{A}(E))=E. \tag{3.4}$$

Очевидно, что $\langle \mathbf{A}, \mathbf{B}, \mathbf{B}^*(E'), \omega\rangle$ удовлетворяет ($\mathfrak{B}$) для всякой конгруэнции E' на $\mathbf{A}$[1]. Отсюда и из (3.3) следует, что $\mathbf{B}^*(E')$ принадлежит $\mathrm{Con}\,\mathbf{B}$ для всякой конгруэнции E' на $\mathbf{A}$. Следовательно, отображение $\mathbf{B}^*$, переводящее всякую конгруэнцию E' на $\mathbf{A}$ в $\mathbf{B}^*(E')$, есть взаимно однозначное отображение в $\mathrm{Con}\,\mathbf{B}$. Из (3.2) и (3.3) следует $E'\in \mathrm{Con}\,\mathbf{A}\Rightarrow \mathbf{B}(E')=\mathbf{B}^*(E')$. Отсюда и из (3.4) следует, что $\mathbf{B}^*$ есть отображение на всю решетку $\mathrm{Con}\,\mathbf{B}$. Итак, $\mathbf{B}^*$ — взаимно однозначное отображение $\mathrm{Con}\,\mathbf{A}$ на $\mathrm{Con}\,\mathbf{B}$. Отсюда и из $E_1\subseteq E_2\Leftrightarrow \mathbf{B}^*(E_1)\subseteq \mathbf{B}^*(E_2)$ $(E_1, E_2\in \mathrm{Con}\,\mathbf{A})$ следует

$$\mathrm{Con}\,\mathbf{A}\cong \mathrm{Con}\,\mathbf{B}. \tag{3.5}$$

[1] Определение $\mathbf{B}^*(E')$ см. в конце § 1.

Иначе говоря, справедливо следующее: для всякой унарной алгебры **A** конечной сигнатуры, имеющей неподвижную точку, существует 2-унарная алгебра **B**, имеющая неподвижную точку и такая, что 1) $\mathrm{Con}\,\mathbf{B} \cong \mathrm{Con}\,\mathbf{A}$ и 2) если **A** конечна, то и **B** конечна. Отсюда и из (2.3) следует

Теорема 2. *Для всякой унарной алгебры* **A** *не более чем счетной сигнатуры имеющей неподвижную точку существует 2-унарная алгебра* **B** *такая что* 1) $\mathrm{Con}\,\mathbf{B} \cong \mathrm{Con}\,\mathbf{A}$ *и* 2) *если* **A** *конечна то и* **B** *конечна.*

Пусть $\mathbf{A} = \langle A; f_0, \ldots, f_\alpha, \ldots \rangle$ — унарная алгебра и $\omega \notin A$. Рассмотрим унарную алгебру $\mathbf{C} = \langle A \cup \{\omega\}; f_0^*, \ldots, f_\alpha^*, \ldots; \{g_a | a \in A\} \rangle$ такую, что

$$f_\alpha^*(a) = \begin{cases} f_\alpha(a), & \text{если} \quad a \in A \\ \omega, & \text{если} \quad a = \omega \end{cases}$$

$$g_b(a) = \begin{cases} b, & \text{если} \quad a \in A, \\ \omega, & \text{если} \quad a = \omega. \end{cases}$$

$E \cup \{\langle \omega, \omega \rangle\} \in \mathrm{Con}\,\mathbf{C}$ для всякой конгруэнции E на **A**.

Пусть E — конгруэнция на **C**, содержащая $\langle a, \omega \rangle$ для какого-нибудь $a \in A$. Тогда $g_b \langle a, \omega \rangle = \langle b, \omega \rangle \in E$ для всякого $b \in A$, и следовательно, E — единичная конгруэнция. Таким образом, для всякой счетной (конечной) алгебры **A** существует счетная (конечная) унарная алгебра **C**, имеющая неподвижную точку и такая, что $\mathrm{Con}\,\mathbf{C} \cong \mathrm{Con}\,\mathbf{A} + 1$. Отсюда и из теоремы 2 следует

Теорема 3. *Для всякой счетной (конечной) алгебры* **A** *существует счетная (конечная) 2-унарная алгебра* **B** *такая что*

$$\mathrm{Con}\,\mathbf{B} \cong \mathrm{Con}\,\mathbf{A} + 1.$$

Иначе говоря для всякой алгебраической решетки L *с не более чем счетным множеством компактных элементов решетка* $L+1$ *представима как решетка конгруэнций 2-унарной алгебры. Если при этом* L *представима как решетка конгруэнций конечной алгебры то* $L+1$ *представима как решетка конгруэнций конечной 2-унарной алгебры.*

Пусть A — непустое множество, $\gamma = \langle a, b, c, d \rangle \in A^4$. Унарную операцию f на A назовем совместной с γ, если $f(a) = c$ и $f(b) = d$. Пусть $\mathbf{A} = \langle A; F \rangle$ — унарная алгебра. Кортеж $\gamma \in A^4$ назовем допустимым в **A**, если некоторая термальная операция в **A** совместна с γ. Пусть Γ — множество всех допустимых в **A** кортежей, φ — функция, соотносящая всякому $\gamma \in \Gamma$ совместную с γ термальную операцию в **A**. Рассмотрим алгебру $\mathbf{C} = \langle A; \{\varphi[\gamma] | \gamma \in \Gamma\} \rangle$. Ясно, что $\mathrm{Con}\,\mathbf{A} \subseteq \mathrm{Con}\,\mathbf{C}$. Обратное включение также имеет место. Действительно, пусть E — конгруэнция на **C**, $\langle a, b \rangle \in E$. Рассмотрим кортеж $\gamma = \langle a, b, f(a), f(b) \rangle$, где f — произвольная операция в **A**. Так как $\gamma \in \Gamma$, то $\langle \varphi[\gamma](a), \varphi[\gamma](b) \rangle \in E$. Но $\varphi[\gamma](a) = f(a)$, $\varphi[\gamma](b) = f(b)$. Значит, $\langle f(a), f(b) \rangle \in E$. Таким образом, E — конгруэнция на **A**. Следовательно, $\mathrm{Con}\,\mathbf{A} = \mathrm{Con}\,\mathbf{C}$. Если A бесконечно, то в предположении AC мощность Γ равна мощности A. Отсюда следует: для всякой бесконечной алгебры $\mathbf{A} = \langle A; F \rangle$ существует унарная алгебра $\mathbf{C} = \langle A; G \rangle$ такая, что $\mathrm{Con}\,\mathbf{A} = \mathrm{Con}\,\mathbf{C}$ и $\bar{G} = \bar{A}$. В частности, в предположении счетной аксиомы выбора имеет место следующее: для всякой счетной алгебры $\mathbf{A}=\langle A; F\rangle$ существует счетная унарная алгебра $\mathbf{C}=\langle A; G\rangle$ счетной сигнатуры такая, что $\mathrm{Con}\,\mathbf{A}=\mathrm{Con}\,\mathbf{C}$. Отсюда и из теоремы 2 легко выводится (в предположении счетной аксиомы выбора).

(3.6) Для всякой счетной (конечной) алгебры с нулем $\mathbf{A}$ существует счетная (конечная) 2-унарная алгебра $\mathbf{B}$ такая, что $\mathrm{Con}\,\mathbf{B}\cong\mathrm{Con}\,\mathbf{A}$.

Действительно, пусть $\mathbf{A}=\langle A; F\rangle$ — счетная (конечная) алгебра с нулем. Тогда существует счетная (конечная) унарная алгебра $\mathbf{C}=\langle A; G\rangle$ счетной сигнатуры, имеющая неподвижную точку и такая, что $\mathrm{Con}\,\mathbf{C}=\mathrm{Con}\,\mathbf{A}$. Она удовлетворяет условиям теоремы 2, и потому $\mathrm{Con}\,\mathbf{C}\cong\mathrm{Con}\,\mathbf{D}$ для некоторой счетной (конечной) 2-унарной алгебры $\mathbf{D}$.

(3.7) Пусть $\mathbf{A}=\langle A; F\rangle$ — бесконечная алгебра бесконечной сигнатуры, имеющая не более чем счетное множество компактных конгруэнций. Тогда для всякого счетного множества $C\subseteq A$ существуют счетные $B\subseteq A$ и $G\subseteq F$ такие, что $C\subseteq B$, B замкнуто относительно операций из G и $\mathrm{Con}\,\langle B; G\rangle\cong$ $\cong\mathrm{Con}\,\mathbf{A}$.

Пусть, например, множество компактных конгруэнций на $\mathbf{A}$ бесконечно, $(X_n)_{n\in N}$ — последовательность конечных множеств упорядоченных пар элементов A такая, что $\mathbf{A}(X_n)$ исчерпывают все компактные конгруэнции на $\mathbf{A}$ и $m\neq n\Rightarrow\mathbf{A}(X_m)\neq\mathbf{A}(X_n)$. Каждой паре $\langle m, n\rangle$ такой, что $\mathbf{A}(X_m)\subsetneqq\mathbf{A}(X_n)$, соотнесем какую-нибудь пару $\pi(m, n)$ из $\mathbf{A}(X_m)-\mathbf{A}(X_n)$ и какое-нибудь конечное множество $F_{m,n}\subset F$ такое, что в алгебре $\mathbf{A}_{m,n}=\langle A; F_{m,n}\rangle$ имеет место $\pi(m, n)\in\mathbf{A}_{m,n}(X_m)$. Пусть $G_0=\bigcup F_{m,n}$. Обозначим через B_0 наименьшее из подмножеств A, включающих C, содержащих члены всех пар из всех X_n и всех пар $\pi(m, n)$ и замкнутых относительно операций из G_0. Через $\mathbf{B}_0$ обозначим алгебру $\langle B_0; G_0\rangle$. Ясно, что она счетна.

Пусть X — произвольный конечный набор пар элементов алгебры $\mathbf{B}_n=$ $=\langle B_n; G_n\rangle$. Очевидно, что $\mathbf{B}_n(X)\subseteq\mathbf{B}_n(\mathbf{A}(X))$. Каждой паре $\langle a, b\rangle$ из $\mathbf{B}_n(\mathbf{A}(X))-$ $-\mathbf{B}_n(X)$ соотнесем какой-нибудь конечный набор $\varrho(a, b)$ элементов A и конечный набор $F_{a,b}\subset F$ такие, что всякое подмножество A, включающее $\varrho(a, b)$ и члены всех пар из X и замкнутое относительно операций из $F_{a,b}$, содержит $\langle a, b\rangle$.

Пусть G_{n+1} — множество, образованное присоединением к G_n всевозможных $F_{a,b}$, B_{n+1} — наименьшее из множеств, включающих B_n, всевозможные $\varrho(a, b)$ и замкнутых относительно операций из G_{n+1}. Через $\mathbf{B}_{n+1}$ обозначим алгебру $\langle B_{n+1}; G_{n+1}\rangle$. Из счетности $\mathbf{B}_n$ следует счетность $\mathbf{B}_{n+1}$.

Рассмотрим последовательность $(\mathbf{B}_n)_{n\in N}$. Пусть $B=\bigcup\{B_n|n\in N\}$, $G=\bigcup$ $\bigcup\{G_n|n\in N\}$. Обозначим через $\mathbf{B}$ алгебру $\langle B; G\rangle$. Эта алгебра счетная.

Для всякого конечного набора X пар элементов B имеет место $\mathbf{B}(X)=$ $=\mathbf{B}(\mathbf{A}(X))$. Действительно, пусть $\langle a, b\rangle\in\mathbf{B}(\mathbf{A}(X))$ и пусть a, b и члены всех пар из X входят в B_m. Тогда, по построению $\mathbf{B}_{m+1}$, $\langle a, b\rangle\in\mathbf{B}_{m+1}(X)$, а значит, $\langle a, b\rangle\in\mathbf{B}(X)$.

Так как члены пар $\pi(m, n)$ принадлежат $\mathbf{B}$, то $\mathbf{B}(X_n)$ $(n\in N)$ попарно различны. Отсюда и из $\mathbf{A}(X_m)\subseteq\mathbf{A}(X_n)\Leftrightarrow\mathbf{B}(X_m)\subseteq\mathbf{B}(X_n)$ следует, что верхняя полурешетка конгруэнций $\mathbf{B}(X_n)$ изоморфна версней полурешетке компактных конгруэнций на **A**.

Пусть X — произвольное конечное множество пар элементов **B**. Для некоторого $m \in N$ имеет место $\mathbf{A}(X) = \mathbf{A}(X_m)$. Но тогда $\mathbf{B}(\mathbf{A}(X)) = \mathbf{B}(\mathbf{A}(X_m))$, то есть $\mathbf{B}(X) = \mathbf{B}(X_m)$. Таким образом, $\mathbf{B}(X_n)$ $(n \in N)$ исчерпывают все компактные конгруэнции на **B**. Значит, полурешетка компактных конгруэнций на **B** изоморфна полурешетке компактных конгруэнций на **A**. Отсюда Con $\mathbf{B} \cong$ Con **A**.

Отметим также, что из тех же рассуждений вытекает следующее: Пусть **A** — бесконечная алгебра счетной сигнатуры, у которой множество компактных конгруэнций не более чем счетно. Тогда всякая её счетная подалгебра вложима в такую счетную подалгебру **B**, что Con $\mathbf{B} \cong$ Con **A**.

Теорема 4. *Если алгебраическая решетка с не более чем счетным множеством компактных элементов представима как решетка конгруенций алгебры с нулем то она представима как решетка конгруэнций 2-унарной алгебры.*

В самом деле, пусть **A** — алгебра с нулем такая, что Con $\mathbf{A} \cong L$. Если **A** конечна или счетна, то представимость L решеткой конгруэнций 2-унарной алгебры следует из (3.6). Если **A** — несчетная алгебра, то согласно (3.7) существует счетная алгебра **B** с нулем такая, что Con $\mathbf{B} \cong$ Con **A**. В силу (3.6) Con **B** изоморфна решетке конгруэнций некоторой 2-унарной алгебры.

§ 4

В заключение — несколько замечаний.

1. Из методических соображений мы ограничились в § 1 теоремой 1. Незначительная модификация содержащегося в [4] доказательства теоремы Маккензи дает доказательство следующей более общей теоремы:

Теорема 1*. *Для всякой алгебры* **A** *счетной сигнатуры имеющей одноэлементную подалгебру* Con **A** *представима как решетка конгруенций алгебры сигнатуры* $\langle 2, 1 \rangle$.

2. Если бы удалось доказать, что для всякой конечной алгебры решетка её конгруэнций представима как решетка конгруэнций конечной унарной алгебры с неподвижной точкой, то тем самым была бы доказана, в силу теоремы 2, справедливость следующей гипотезы Маккензи: для всякой конечной алгебры **A** решетка Con **A** представима как решетка конгруэнций конечной 2-унарной алгебры. В Тейлор [7] построил счетную алгебраическую решетку L_0, не представимую как решетка конгруэнций полугруппы. Р. Фриз показал, что L_0 не представима как решетка конгруэнций группоида с единицей[2]. У. Лэмп [5] доказал, что эта решетка не представима как решетка конгруэнций унарной алгебры конечной сигнатуры. Из этого результата и теоремы 4 следует, что L_0 не представима как решетка конгруэнций алгебры с нулем.

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[2] Этот результат Фриза приводится в [5].

Из рассмотрений, содержащихся в работе П. Палфи [6], выводимо существование конечных решеток, не представимых как решетки конгруэнций унарных алгебр а неподвижными точками (таковы решетки M_n для $n \geqq 4$)[3].

Из рассуждений, доказывающих теорему 2, почти непосредственно выводима

Теорема 2*. *Всякая алгебраическая решетка с компактной единицей имеющая не более чем счетное множество компактных элементов представима как решетка конгруэнций частичной 2-унарной алгебры. Всякая конечная решетка представима как решетка конгруенций конечной частичной 2-унарной алгебры.*

Эта теорема как будто усиливает надежду на справедливость гипотезы Маккензи. Однако следующая теорема показывает, что обольщаться подобными аргументами рискованно:

Теорема 1**. *Всякая алгебраическая (конечная) решетка представима как решетка конгруэнций (конечного) частичного группоида.*

С другой стороны, теоремы 1** и 2* показывают, что представления алгебраических решеток решетками конгруенций частичных алгебр заслуживают специального изучения.

3. Имеет место

Теорема 3*. *Для всякой счетной (конечной) алгебры* **A** *решетка* Con **A**$+1$ *представима как решетка конгруэнций (конечной) 2-унарной алгебры не имеющей собственных подалгебр.*

Из этого усиления теоремы 3* выводится, что всякая алгебраическая решетка с не более чем счетным множеством компактных элементов представима как интервал решетки конгруэнций полугруппы с тремя образующими. Естественен вопрос: всякая ли алгебраическая решетка с не более чем счетным множеством компактных элементов представима как интервал решетки конгруэнций полугруппы с двумя образующими? Положительный ответ на этот вопрос давал бы некоторое усиление теоремы Ежека, утверждающей, что всякая такая решетка представима как интервал решетки эквациональных теорий 2-унарных алгебр.

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[2] Jónsson, B., *Topics in universal algebra*, Lecture Notes in Mathematics, Vol. 250, Springer-Verlag, Berlin—New York, 1972. *MR* **49** # 10625.

[3] Jónsson, B., Congruence varieties, *Algebra Universalis* **10** (1980), 355—394. *MR* **81e**: 08004.

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[3] Приносим благодарность рецензенту, обратившему наше внимание на статью Палфи [6] и на отмеченный факт, связанный с нею. Мы благодарны ему и за указание пропуска в § 1, восполненного в настоящем варианте заметки.

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EXTENSIONS OF QUASI-UNIFORMITIES FOR PRESCRIBED BITOPOLOGIES I

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Abstract

(For Parts I and II)

Given a quasi-uniformity $\mathscr{U}$ and an extension of the induced bitopology, we are looking for conditions guaranteeing that there is an extension of $\mathscr{U}$ compatible with the bitopological extension. The results will be formulated in terms of trace filter pairs. Some related problems will also be considered.

Császár [10] investigated the following problem: let $\mathscr{U}$ be a quasi-uniformity, $\mathscr{T}$ the topology induced by $\mathscr{U}$, and $\mathscr{S}$ an extension of $\mathscr{T}$; describe now those quasi-uniformities extending $\mathscr{U}$ that are compatible with $\mathscr{S}$, or give at least necessary and/or sufficient conditions for the existence of such quasi-uniform extensions. The answer to the analogous question for uniformities is well-known. We are going to consider a third problem of this type: let $\mathscr{U}$ be a quasi-uniformity, $(\mathscr{T}^{-1}, \mathscr{T}^1)$ the bitopology induced by $\mathscr{U}$, $(\mathscr{S}^{-1}, \mathscr{S}^1)$ an extension of $(\mathscr{T}^{-1}, \mathscr{T}^1)$; find out as much as possible about those quasi-uniformities extending $\mathscr{U}$ that are compatible with $(\mathscr{S}^{-1}, \mathscr{S}^1)$. (All the necessary definitions will be given in § 0.) Similarly to the case of extensions for topologies, a lot of problems will remain open.

It is easy to give a complete solution to the above problem if we require in the definition of a bitopological extension that the original bitopological space should be dense for the supremum of the two topologies; therefore we shall only assume density for the topologies separately.

The results will be formulated in terms of trace filter pairs (i.e. trace filters taken for both topologies). We shall have to define several notions for filter pairs, most of them straightforward generalizations of similar notions for filters (e.g. Cauchy or round).

The following weaker form of our problem will also be considered: given a quasi-uniform space $(X, \mathscr{U})$ and some filter pairs in X, find an extension $\mathscr{V}$ of $\mathscr{U}$ such that it is compatible with the given filter pairs (i.e. the bitopology of $\mathscr{V}$ induces them as trace filter pairs).

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§ 0. Preliminaries

A. Notations, terminology, basic definitions

0.1. X will always denote a non-empty set. A *filter* means a proper filter. The filter in X generated by the filter (sub)base $\mathfrak{b}$ is denoted by $\mathrm{fil}_X\,\mathfrak{b}$. $\mathfrak{p}(X)$ is the power set of X. If $\mathfrak{a}\subset\mathfrak{p}(X)$ then $\sec\mathfrak{a}$ denotes the collection of those subsets of X that intersect each element of $\mathfrak{a}$. For families of sets $\mathfrak{a}$ and $\mathfrak{c}$,

$$\mathfrak{a}(\cap)\mathfrak{c}=\{A\cap C\colon\ A\in\mathfrak{a},\ C\in\mathfrak{c}\}.$$

R is the set of the reals, **N** the set of the positive integers, $\mathscr{E}$ the Euclidean topology on **R**. In a topological space $(X,\mathscr{T})$, $\mathscr{T}|A$ denotes the subspace topology on $A\subset X$.

If $s,t\in\mathbf{R}$ and $s>t$ then $]s,t[$ means the interval $]t,s[$.

An *entourage* U is a reflexive relation on X; we shall write $(x, y)\in U$ also as $x\, U\, y$. U^{-1} is the inverse of U. The symbol $\circ$ is used for denoting the composition of relations. If $A\subset X$ and $x\in X$ then let

$$U[A] = \{y\in X\colon \exists z\in A,\ z\, U\, y\};\quad Ux = U[\{x\}];\quad U|A = U\cap(A\times A).$$

Δ_X is the diagonal of $X\times X$.

0.2. A *quasi-semiuniformity* (on X) [22] is a filter in $X\times X$ whose elements are entourages. A *(sub)base* for a quasi-semiuniformity is to be understood in the sense of a filter (sub)base. A quasi-semiuniformity $\mathcal{U}$ is a *quasi-uniformity* [29, 4] if some base $\mathcal{B}$ for $\mathcal{U}$ (equivalently: $\mathcal{B}=\mathcal{U}$) satisfies the following condition:

(1) for each $U\in\mathcal{B}$ there is a $V\in\mathcal{B}$ with $V^2\subset U$,

where $V^2=V\circ V$. If a subbase $\mathcal{B}$ for a quasi-semiuniformity $\mathcal{U}$ satisfies (1) then $\mathcal{U}$ is a quasi-uniformity [28].[1]

If $\mathcal{U}$ is a quasi-semiuniformity then so is $\mathcal{U}^{-1}=\{U^{-1}\colon U\in\mathcal{U}\}$. A quasi-uniformity $\mathcal{U}$ is a *uniformity* if $\mathcal{U}^{-1}=\mathcal{U}$ (equivalently: if it has a base consisting of symmetric entourages). If $\mathcal{U}$ is a quasi-uniformity then

$$\mathcal{U}^s = \mathcal{U}(\cap)\mathcal{U}^{-1} = \mathrm{fil}_{X\times X}\{U\cap U^{-1}\colon U\in\mathcal{U}\}$$

is a uniformity. The *weight* of the quasi-semiuniformity $\mathcal{U}$, denoted by $w(\mathcal{U})$, is the smallest infinite cardinal $\varkappa$ for which there exists a base (equivalently: a subbase) for $\mathcal{U}$ of cardinality $\leq\varkappa$ ("rank" in [6]).

The entourage U will sometimes be written as U^1, and the quasi-semiuniformity $\mathcal{U}$ as $\mathcal{U}^1$.

Let $\mathcal{U}$ be a quasi-uniformity. In the topology $\mathcal{U}^{tp}$ induced by $\mathcal{U}$, $\{Ux\colon U\in\mathcal{U}\}$ is the neighbourhood filter of $x\in X$; a neighbourhood (sub)base is obtained if U is taken from a (sub)base only. In other words: $\mathcal{U}$ is compatible with the topology $\mathcal{U}^{tp}$. These expressions will also be used for quasi-*semi*uniformities, provided that the neighbourhood filters $\{Ux\colon U\in\mathcal{U}\}$ define a topology. $\mathcal{U}^{-tp}=(\mathcal{U}^{-1})^{tp}$.

For more information on quasi-uniformities, see [21], and also [5, 28].

0.3. A filter $\mathfrak{f}$ in a quasi-uniform space $(X, \mathcal{U})$ is

a) *round* ($\mathcal{U}$-round) [24, 5, 7, 10] if for any $S\in\mathfrak{f}$ there are an $S_0\in\mathfrak{f}$ and a $U\in\mathcal{U}$ with $U[S_0]\subset S$;

b) *Cauchy* ($\mathcal{U}$-Cauchy) [2, 33] if for any $U\in\mathcal{U}$, we have $Ux\in\mathfrak{f}$ with a suitable $x\in X$.

$\mathfrak{f}$ is $\mathcal{U}^s$-Cauchy iff for any $U\in\mathcal{U}$ there is an $S\in\mathfrak{f}$ with $S\times S\subset U$ ([21] 3.2). (Caution! "Cauchy" is often used in the sense "$\mathcal{U}^s$-Cauchy", e.g. in [6, 10].)

A *bitopology* on X [23] is an ordered pair of topologies on X. $(\mathcal{U}^{-tp}, \mathcal{U}^{tp})$ is the bitopology induced by the quasi-uniformity (or more generally, by the suitable quasi-semiuniformity) $\mathcal{U}$ [25, 20]. In other words: $\mathcal{U}$ is compatible with the bitopology $(\mathcal{U}^{-tp}, \mathcal{U}^{tp})$. (The two topologies are usually taken in reverse order, but our definition, which seems to be more convenient in the present context, is not unprecedented either, see [3, 26].)

[1] We are not interested in quasi-semiuniformities as such, but they will be of use in formulating some statements concerning quasi-uniformities.

0.4. The topological space $(Y, \mathcal{S})$ is an *extension* of the topological space $(X, \mathcal{T})$ (in other words, $\mathcal{S}$ is an extension of $\mathcal{T}$)[2] if $(X, \mathcal{T})$ is a dense subspace of $(Y, \mathcal{S})$. The *trace filter* $\mathfrak{f}(a)$ of the point $a \in Y$ is the trace on X of the $\mathcal{S}$-neighbourhood filter $\mathfrak{n}(a)$ of a, i.e.

$$\mathfrak{f}(a) = \mathfrak{n}(a)|X = \{S \cap X:\ S \in \mathfrak{n}(a)\}.$$

If $x \in X$ then $\mathfrak{f}(x)$ is the $\mathcal{T}$-neighbourhood filter of x; for any $a \in Y$, $\mathfrak{f}(a)$ is a $\mathcal{T}$-open filter (i.e. it has a base consisting of open sets). Conversely, if we prescribe $\mathcal{T}$-open filters $\mathfrak{f}(a)$ for each $a \in Y$ such that $\mathfrak{f}(x)$ is the $\mathcal{T}$-neighbourhood filter of x whenever $x \in X$ then there are extensions of $\mathcal{T}$ with just these trace filters; there exist a finest and a coarsest one among such extensions [1], called the *loose extension*, respectively the *strict extension* associated with the trace filter system $\{\mathfrak{f}(a):\ a \in Y\}$. (Strictly speaking, a trace filter system is not just a system of filters, but a function defined on Y, since the same filter is allowed to belong to different points.) If $\mathcal{S}$ is a loose extension then

(1) $$\mathfrak{n}(a) = \mathrm{fil}_Y \{S \cup \{a\}:\ S \in \mathfrak{f}(a)\};$$

if $\mathcal{S}$ is a strict extension then

(2) $$\mathfrak{n}(a) = \mathrm{fil}_Y \{\{b:\ S \in \mathfrak{f}(b)\}:\ S \in \mathfrak{f}(a)\}.$$

In both cases, it is enough to take S from a subbase for $\mathfrak{f}(a)$.

0.5. The bitopological space $(Y; \mathcal{S}^{-1}, \mathcal{S}^1)$ is an *extension* of the bitopological space $(X; \mathcal{T}^{-1}, \mathcal{T}^1)$ if $\mathcal{S}^i$ is an extension of $\mathcal{T}^i$ $(i = \pm 1)$. In other words, X is required to be dense for both topologies; the problems to be investigated are much easier when X is dense for the topology $\sup\{\mathcal{S}^{-1}, \mathcal{S}^1\}$. A *filter pair* is an ordered pair of filters in the same set. The *trace filter pair* of $a \in Y$ belonging to (or induced by) the extension $(\mathcal{S}^{-1}, \mathcal{S}^1)$ is just the pair $(\mathfrak{f}^{-1}(a), \mathfrak{f}^1(a))$ where $\mathfrak{f}^i(a)$ is the trace filter of a belonging to the extension $\mathcal{S}^i$ of $\mathcal{T}^i$ $(i = \pm 1)$. Each trace filter pair is open (i.e. $\mathfrak{f}^i(a)$ is $\mathcal{T}^i$-open for $i = \pm 1$).

Conversely, if we are given open filter pairs $(\mathfrak{f}^{-1}(a), \mathfrak{f}^1(a))$ for each $a \in Y$ such that $\mathfrak{f}^i(x)$ is the $\mathcal{T}^i$-neighbourhood filter of x whenever $x \in X$ and $i = \pm 1$ then there are extensions for these filter pairs. The finest, respectively the coarsest one can be obtained by taking loose, respectively strict extensions in both topologies [$(\mathcal{S}_1^{-1}, \mathcal{S}_1^1)$ is called finer than $(\mathcal{S}_2^{-1}, \mathcal{S}_2^1)$ if $\mathcal{S}_1^i$ is finer than $\mathcal{S}_2^i$ $(i = \pm 1)$]; these extensions will be referred to as the *doubly loose*, respectively the *doubly strict* extension associated with the system $\{(\mathfrak{f}^{-1}(a), \mathfrak{f}^1(a)):\ a \in Y\}$ of trace filter pairs.

0.6. The bitopological space $(X; \mathcal{T}^{-1}, \mathcal{T}^1)$ is

a) *regular* [23] if any $\mathcal{T}^i$-neighbourhood of x contains a $\mathcal{T}^{-i}$-closed $\mathcal{T}^i$-neighbourhood of x $(x \in X,\ i = \pm 1)$;

b) *completely regular* [19, 25] if it can be induced by a quasi-uniformity;

c) *zero-dimensional* [32] if any $\mathcal{T}^i$-neighbourhood of x contains a $\mathcal{T}^{-i}$-closed $\mathcal{T}^i$-open $\mathcal{T}^i$-neighbourhood of x $(x \in X,\ i = \pm 1)$.

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[2] In similar situations, only one of such pairs of parallel definitions will be given, the other being understood.

The original definition of bitopological complete regularity was given in terms of semicontinuous real functions, and the property chosen here as a definition is an equivalent characterization, cf. [19, 25, 20, 8]. Any zero-dimensional bitopology is completely regular; any completely regular bitopology is regular.

A bitopology of the form $(\mathcal{T}, \mathcal{T})$ is regular, completely regular, respectively zero-dimensional iff the topology $\mathcal{T}$ has the same property.

See [13] for more information and further references on bitopological separation properties.

0.7. Any real function d defined on a subset of $X\times X$ induces a quasi-semi-uniformity $\mathcal{U}(d)$ on X as follows:

$$\mathcal{U}(d) = \mathrm{fil}_{X\times X}\,\{U_{(\varepsilon)}\colon\ \varepsilon > 0\}$$

where

$$U_{(\varepsilon)} = U_{(\varepsilon)}(d) = \Delta\cup\{(x, y)\colon\ d(x, y) < \varepsilon\}. \tag{1}$$

The function d will be called a *distance* if

$$d(x, y)+d(y, z) \geqq d(x, z),$$

in the sense that if $d(x, y)$ and $d(y, z)$ are both defined then so is $d(y, z)$, and the inequality holds. If d is a distance then $\mathcal{U}(d)$ is a quasi-uniformity.

A *quasi-pseudometric* [31, 35] is a non-negative distance defined on the whole $X\times X$ such that $d(x, x)=0$ $(x\in X)$. (In this case Δ can be dropped from (1).) If d is a distance then a quasi-pseudometric e inducing the same quasi-uniformity is defined by

$$e(x, y) = \begin{cases} 0 & \text{if}\quad x = y \quad\text{or}\quad d(x, y) \leqq 0,\\ d(x, y) & \text{if}\quad 0 < d(x, y) < 1,\\ 1 & \text{otherwise.}\end{cases}$$

The following quasi-uniformities will be needed in several simple counter-examples.

EXAMPLES. On $X=(\mathbf{R}\setminus\{0\})\times\mathbf{R}$, let

$$d_1((x', x''), (y', y'')) = y'-x' \quad\text{if}\quad x' < y',$$

$$d_2((x', x''), (y', y'')) = y'-x'+|y''-x''| \quad\text{if}\quad x' < y'.$$

Define d_{01} and d_{02} similarly, with $x'<y'$ replaced by $x'<0<y'$. Let $\mathcal{U}_k=\mathcal{U}(d_k)$ $(k=1, 2, 01, 02)$. Each d_k is a distance, thus each $\mathcal{U}_k$ is a quasi-uniformity.

The following filters in X will later be needed: if $\mathcal{T}$ is a topology on $\mathbf{R}$, $t\in\mathbf{R}$ and $i=\pm 1$ then let

$$\mathfrak{g}^i(t, \mathcal{T}) = \mathrm{fil}_X\,\{]0, i\varepsilon[\times H\colon\ \varepsilon > 0,\ H \text{ is a } \mathcal{T}\text{-neighbourhood of } t\}.$$

0.8. Now we are going to describe the main purpose of this paper more precisely than in the introduction. Let $(X, \mathcal{U})$ be a quasi-uniform space, $\mathcal{S}$ an extension of $\mathcal{U}^{tp}$, $(\mathcal{S}^{-1}, \mathcal{S}^1)$ an extension of $(\mathcal{U}^{-tp}, \mathcal{U}^{tp})$. We are looking for quasi-uniformities $\mathcal{V}$ such that $\mathcal{V}|X=\mathcal{U}$, and $\mathcal{V}$ is compatible with $(\mathcal{S}^{-1}, \mathcal{S}^1)$ (the analogous case when $\mathcal{V}$ is required to be compatible with $\mathcal{S}$ was investigated in [10, 11, 27, 14]).

Such a $\mathscr{V}$ will be called an *extension of* $\mathscr{U}$ *compatible with* $(\mathscr{S}^{-1}, \mathscr{S}^1)$ (an "extension of a quasi-uniformity" is always supposed to be a quasi-uniformity, and not just a quasi-semiuniformity).

The following weaker problem will also be considered: assume that $(X, \mathscr{U})$ is a quasi-uniform space, $Y \supset X$, and let us be given filter pairs $(\mathfrak{f}^{-1}(a), \mathfrak{f}^1(a))$ in X for $a \in Y$, with the usual assumption that $(\mathfrak{f}^{-1}(x), \mathfrak{f}^1(x))$ is the neighbourhood filter pair of x for each $x \in X$ (in the bitopology induced by $\mathscr{U}$)[3]; look for quasi-uniformities $\mathscr{V}$ on Y such that $\mathscr{V}|X = \mathscr{U}$, and $(\mathscr{V}^{-tp}, \mathscr{V}^{tp})$ is an extension of $(\mathscr{U}^{-tp}, \mathscr{U}^{tp})$ with the given trace filter pairs. Such a $\mathscr{V}$ will be called an *extension of* $\mathscr{U}$ *compatible with the trace filter pairs* $\{(\mathfrak{f}^{-1}(a), \mathfrak{f}^1(a)) : a \in Y\}$ (or: with the trace filter pairs $\{(\mathfrak{f}^{-1}(p), \mathfrak{f}^1(p)) : p \in Y \setminus X\}$). We shall, however, not stick to this wording, and other self-explanatory expressions will be used as well, e.g. that $\mathscr{V}$ is an extension with (or belonging to) the trace filter pairs, etc.

Let us further agree to call the quasi-uniform space $(Y, \mathscr{V})$ an *extension* of the quasi-uniform space $(X, \mathscr{U})$ if $\mathscr{V}|X = \mathscr{U}$, and X is $\mathscr{V}^{-tp}$-dense and $\mathscr{V}^{tp}$-dense[4] in Y. The extension $(Y; \mathscr{S}^{-1}, \mathscr{S}^1)$ of the bitopological space $(X; \mathscr{T}^{-1}, \mathscr{T}^1)$ is *firm* if X is $\sup\{\mathscr{S}^{-1}, \mathscr{S}^1\}$-dense; the extension $\mathscr{V}$ of the quasi-uniformity $\mathscr{U}$ is *firm* if $(\mathscr{V}^{-tp}, \mathscr{V}^{tp})$ is a firm extension of $(\mathscr{U}^{-tp}, \mathscr{U}^{tp})$.

B. Results on extensions for prescribed topologies

We list here some results on extensions of (quasi-)uniformities for prescribed topologies, in order that the reader might compare them with our results on extensions for bitopologies. The case of uniformities is well-known, see e.g. in [9]; it will be familiar to the reader at least in the case when all the non-convergent round Cauchy filters (=non-convergent minimal Cauchy filters) are assigned to the new points ("completion of a uniformity").

0.9. Let $(X, \mathscr{U})$ be a uniform space, $(Y, \mathscr{S})$ and extension of $(X, \mathscr{U}^{tp})$, with trace filters $\mathfrak{f}(a)$ $(a \in Y)$. Then we have:

a) If there is a *uniformity* that is an extension of $\mathscr{U}$ compatible with $\mathscr{S}$ then $\mathscr{S}$ is completely regular, it is a strict extension of $\mathscr{U}^{tp}$ (because regular extensions are strict), and each $\mathfrak{f}(a)$ is round and Cauchy.

b) Conversely, if $\mathscr{S}$ is regular (or $\mathscr{S}$ is a strict extension) and each $\mathfrak{f}(p)$ is a round Cauchy filter $(p \in Y \setminus X)$ then there is exactly one uniformity $\mathscr{V}$ that is an extension of $\mathscr{U}$ for $\mathscr{S}$; moreover, $w(\mathscr{V}) = w(\mathscr{U})$.

It is essential in both cases that the extension is a uniformity, and not simply a quasi-uniformity.

0.10. Let X, $\mathscr{U}$, $\mathscr{S}$, $\mathfrak{f}(a)$ be as above, but with $\mathscr{U}$ only a quasi-uniformity. Now the following hold:

a) [12, 10]. If there is an extension of $\mathscr{U}$ compatible with $\mathscr{S}$ then each $\mathfrak{f}(a)$ is round $(a \in Y)$.

b) [10]. If $\mathscr{S}$ is a loose extension and each $\mathfrak{f}(p)$ is round $(p \in Y \setminus X)$ then there is at least one extension $\mathscr{V}$ of $\mathscr{U}$ compatible with $\mathscr{S}$. The finest (=largest) $\mathscr{V}$ sat-

[3] In the sequel, this condition will always be assumed implicitly.

[4] Or only $\mathscr{V}^{tp}$-dense when we are dealing with the one-sided case.

isfying this condition can be obtained as follows ([10]; in a special case also [34]):

$$\{V(f, U)\colon f\in\Phi,\ U\in\mathscr{U}\}$$

is a base for $\mathscr{V}$, where Φ denotes the family of all those functions $f\colon Y\setminus X\to\mathfrak{p}(X)$ for which $f(p)\in\mathfrak{f}(p)$ $(p\in Y\setminus X)$, and

$$V(f, U)p = \{p\}\cup U[f(p)] \qquad (p\in Y\setminus X),$$

$$V(f, U)x = Ux \qquad (x\in X).$$

c) [12, 10]. If there is at least one extension of $\mathscr{U}$ compatible with $\mathscr{S}$ then there is a finest one among such extensions.

d) [14]. In general, there is no coarsest extension of $\mathscr{U}$ compatible with $\mathscr{S}$, not even in the case when $\mathscr{S}$ is a loose and strict extension at the same time.

We do not cite here those results of [10] that concern strict extensions.

§ 1. Necessary conditions

Let $(X, \mathscr{U})$ be a quasi-uniform space, $(\mathscr{S}^{-1}, \mathscr{S}^{1})$ and extension of $(\mathscr{U}^{-tp}, \mathscr{U}^{tp})$; denote by $(\mathfrak{n}^{-1}(a), \mathfrak{n}^{1}(a))$ the neighbourhood filter pair, and by $(\mathfrak{f}^{-1}(a), \mathfrak{f}^{1}(a))$ the trace filter pair of the point $a\in Y$.

1.1. Definition. The filter pair $(\mathfrak{f}^{-1}, \mathfrak{f}^{1})$ is

a) *round* if $\mathfrak{f}^{i}$ is $\mathscr{U}^{i}$-round $(i=\pm 1)$ (cf. 0.3 a));

b) (in [26] for a special case) *Cauchy* if for any $U\in\mathscr{U}$ there are $S_i\in\mathfrak{f}^{i}$ with $S_{-1}\times S_{1}\subset U$.

Remark. In the terminology of [16], $(\mathfrak{f}^{-1}, \mathfrak{f}^{1})$ is $\mathscr{U}$-Cauchy iff $\mathfrak{f}^{-1}\times\mathfrak{f}^{1}$ is $\mathscr{U}^{c}$-micromeric. We could work in the present paper with products of filters ("bifilter" in [3]) since $(\mathfrak{f}^{-1}, \mathfrak{f}^{1})$ can be recovered from $f=\mathfrak{f}^{-1}\times\mathfrak{f}^{1}$ as $(\operatorname{dom} f, \operatorname{ran} f)$.

Theorem. *If there is an extension of $\mathscr{U}$ compatible with $(\mathscr{S}^{-1}, \mathscr{S}^{1})$ then*

a) *$(\mathscr{S}^{-1}, \mathscr{S}^{1})$ is completely regular;*

b) *the trace filter pairs are round;*

c) *the trace filter pairs are Cauchy.*

Proof. a) This is evident from the definition given in 0.6 b).

b) If $\mathscr{V}$ is an extension of $\mathscr{U}$ for $(\mathscr{S}^{-1}, \mathscr{S}^{1})$ then $\mathscr{V}^{i}$ is an extension of $\mathscr{U}^{i}$ for $\mathscr{S}^{i}$, thus 0.10 a) can be applied.

c) Take a $U\in\mathscr{U}$; we are looking for $S_i\in\mathfrak{f}^{i}(a)$ with $S_{-1}\times S_{1}\subset U$. As $\mathscr{U}=\mathscr{V}|X$, there is a $V\in\mathscr{V}$ with $V|X=U$. Choose now $V_0\in\mathscr{V}$ such that $V_0^2\subset V$; then $S_i= =V_0^{i}a\cap X$ will do. Indeed: if $x\in S_{-1}$ and $y\in S_1$ then $x\,V_0\,a\,V_0\,y$, thus $x\,V\,y$, i.e. $x\,U\,y$; $S_i\in\mathfrak{f}^{i}(a)$ follows from $V_0^{i}a\in\mathfrak{n}^{i}(a)$. □

Corollary. *Neighbourhood filter pairs in a quasi-uniform space are always round and Cauchy.*

Proof. Apply the theorem to the case $Y=X$. □

Consequently, when saying that the trace filter pairs are round and/or Cauchy, we do not have to specify whether all the trace filter pairs are meant or only those belonging to the new points.

1.2. The conditions in Theorem 1.1 are not sufficient:

Example. Take a completely regular non-normal topology $\mathcal{T}$ on $\mathbf{R}$. Let $X=(\mathbf{R}\setminus\{0\})\times\mathbf{R}$, $Y=\mathbf{R}^2$, $(\mathcal{T}^{-1},\mathcal{T}^1)$ the discrete bitopology on X, $\mathfrak{f}^i((0,p))=\mathfrak{g}^i(p,\mathcal{T})$ (see in Examples 0.7). These trace filter pairs are evidently open, thus we can take the doubly strict extension $(\mathcal{S}^{-1},\mathcal{S}^1)$ associated with them. The points of X are isolated in $\mathcal{S}^i$, while

$$\mathfrak{n}^i((0,p))=\mathrm{fil}_Y\,\{[0,i\varepsilon[\times G\colon\ G \text{ is } \mathcal{T}\text{-open},\ p\in G,\ \varepsilon>0\}.$$

In particular, denoting by $\mathcal{T}'$ the image of the topology $\mathcal{T}$ under the bijection $p\mapsto(0,p)$, we have

$$(1)\qquad (\mathcal{S}^{-1},\mathcal{S}^1)|Y\setminus X=(\mathcal{T}',\mathcal{T}').$$

Let us first prove that $(\mathcal{S}^{-1},\mathcal{S}^1)$ is completely regular. Take on $\mathbf{R}$ a uniformity $\mathcal{W}$ compatible with $\mathcal{T}$. On Y, define entourages $Z(\varepsilon,W)$ for $\varepsilon>0$ and $W\in\mathcal{W}$ as follows (with $x'\neq0\neq y'$, $i=\pm1$):

$$(0,p)\,Z(\varepsilon,W)\,(0,q)\quad\text{iff}\quad p\,W\,q,$$

$$(x',x'')\,Z(\varepsilon,W)\,(y',y'')\quad\text{iff}\quad x'<0<y'<x'+\varepsilon\quad\text{or}\quad (x',x'')=(y',y''),$$

$$(0,p)\,Z^i(\varepsilon,W)\,(x',x'')\quad\text{iff}\quad x'\in]0,i\varepsilon[,\quad p\,W^i\,x''.$$

It is easy to check that these entourages form a base for a quasi-semiuniformity $\mathcal{Z}$ on Y. $\mathcal{Z}$ is a quasi-uniformity, since if $W_0^2\subset W$ then $Z(\varepsilon/2,W_0)^2\subset Z(\varepsilon,W)$. $\mathcal{Z}$ is clearly compatible with $(\mathcal{S}^{-1},\mathcal{S}^1)$, thus this bitopology is completely regular indeed.

For each $\mathcal{T}$-open covering $\mathfrak{c}$ of $\mathbf{R}$, and $\varepsilon>0$, take an entourage $U(\varepsilon,\mathfrak{c})$ on X as follows:

$$(x',x'')\,U(\varepsilon,\mathfrak{c})\,(y',y'')\quad\text{iff either}\quad (x',x'')=(y',y'')$$
$$\text{or}\quad x'<0<y'<x'+\varepsilon,\quad \exists G\in\mathfrak{c},\quad x'',y''\in G.$$

As $U(\varepsilon,\mathfrak{c})\cap U(\varepsilon',\mathfrak{c}')\supset U(\min\{\varepsilon,\varepsilon'\},\mathfrak{c}(\cap)\mathfrak{c}')$, these entourages form a base for a quasi-semiuniformity $\mathcal{U}$. $\mathcal{U}$ is a quasi-uniformity, since $U(\varepsilon,\mathfrak{c})^2\subset U(\varepsilon,\mathfrak{c})$. $\mathcal{U}$ is evidently compatible with $(\mathcal{T}^{-1},\mathcal{T}^1)$, and the trace filter pairs are round. They are also Cauchy: if $U=U(\varepsilon,\mathfrak{c})$ and $p\in\mathbf{R}$ then $S_{-1}\times S_1\subset U$ and $S_i\in\mathfrak{f}^i((0,p))$ hold with $S_i=]0,i\varepsilon/2[\times G$ if G is chosen to satisfy $p\in G\in\mathfrak{c}$.

And yet there is no extension of $\mathcal{U}$ compatible with $(\mathcal{S}^{-1},\mathcal{S}^1)$.

Assume the existence of an extension $\mathcal{V}$. Given $U\in\mathcal{U}$, take a $V\in\mathcal{V}$ with $V|X=U$, and then a $V_0\in\mathcal{V}$ such that $V_0^4\subset V$. As $\mathcal{T}$ is not normal, we can choose disjoint $\mathcal{T}$-closed sets $A_i\subset\mathbf{R}$ $(i=\pm1)$ that are not contained by disjoint $\mathcal{T}$-open sets. Let $A_i'=\{0\}\times A_i$. It follows from (1) that $V_0[A_{-1}']\cap V_0^{-1}[A_1']\neq\emptyset$, so there are $p_i\in A_i$ such that $(0,p_{-1})\,V_0^2\,(0,p_1)$. Denoting $V_0^i(0,p_i)\cap X$ by S_i, we have

$$(2)\qquad S_i\in\mathfrak{f}^i((0,p_i))$$

and $S_{-1}\times S_1\subset V_0^4\subset V$, thus

(3) $$S_{-1}\times S_1 \subset U.$$

There are, however, no such points p_i and sets S_i for $U=U(1, \mathfrak{c}_0)$ where $\mathfrak{c}_0=\{\mathbf{R}\setminus A_{-1}, \mathbf{R}\setminus A_1\}$, since (2) implies that there are $x<0$ and $y>0$ with $(x, p_{-1})\in S_{-1}$, $(y, p_1)\in S_1$, thus $(x, p_{-1})\, U\, (y, p_1)$ would follow from (3), a contradiction, because p_{-1} and p_1 cannot be contained by the same element of $\mathfrak{c}_0$.

1.3. The reasoning in the last but one paragraph of Example 1.2 gives another necessary condition. Let us agree to extend Definition 1.1 b) to arbitrary pairs $(\mathfrak{a}^{-1}, \mathfrak{a}^1)$ $(\mathfrak{a}^i\subset\mathfrak{p}(X))$: $(\mathfrak{a}^{-1}, \mathfrak{a}^1)$ is *Cauchy* if for any $U\in\mathcal{U}$, there are $A_i\in\mathfrak{a}^i$ with $A_{-1}\times A_1\subset U$. (It is enough to take U from a base in general, and from a subbase for filter pairs.)

THEOREM. *If there is an extension of $\mathcal{U}$ compatible with $(\mathcal{S}^{-1}, \mathcal{S}^1)$ then for any $A, B\subset Y$,*

(1) $$\Big(\bigcup_{a\in A}\mathfrak{f}^{-1}(a),\ \bigcup_{b\in B}\mathfrak{f}^1(b)\Big)\quad \text{is not Cauchy}$$

implies

(2) there are disjoint $\mathcal{S}^i$-open sets G_i with $A\subset G_1$, $B\subset G_{-1}$. □

REMARKS. a) If the above condition is fulfilled then the trace filter pairs are Cauchy (take $A=B=\{a\}$).

b) If (1)⇒(2) is known to hold for $\mathcal{S}^{-1}$-closed sets A and $\mathcal{S}^1$-closed sets B then it holds for arbitrary A and B (the proof is left to the reader).

1.4. In 0.9, the complete regularity of $\mathcal{S}$ implied that it was a strict extension. Now the condition that $(\mathcal{S}^{-1}, \mathcal{S}^1)$ should be completely regular is not as strong as that; we shall later see that there may belong different completely regular bitopological extensions to the same system of trace filter pairs. We shall also see that the doubly strict extension may fail to be completely regular even when there do exist completely regular bitopological extensions with the given trace filter pairs.

Theorem 1.1 a) suggests that it would be of use to get some information on completely regular bitopological extensions. Instead, we shall deal with the simpler problem of regular extensions in the next section; the results obtained there will be strong enough to meet the case.

§ 2. Regular extensions of bitopologies

Let $(X; \mathcal{T}^{-1}, \mathcal{T}^1)$ be a bitopological space, $Y\supset X$, and let us be given trace filter pairs $(\mathfrak{f}^{-1}(a), \mathfrak{f}^1(a))$ for $a\in Y$. We shall formulate a necessary and sufficient condition for the existence of a regular extension with these trace filter pairs; if the condition is satisfied, we construct the finest regular extension. The corresponding result on regular extensions of topologies was given in [18].

2.1. NOTATIONS. The closure [interior] in the topology $\mathcal{T}^i$, respectively $\mathcal{S}^i$, will be denoted by cl^i [int^i], respectively Cl^i [Int^i].

THEOREM. *There is a regular extension compatible with the trace filter pairs* $(\mathfrak{f}^{-1}(a), \mathfrak{f}^1(a))$ *iff*

(1) *for any* $a\in Y$, $i=\pm 1$, *and* $S\in\mathfrak{f}^i(a)$, *there is an* $S_0\in\mathfrak{f}^i(a)$ *such that* $S\in\mathfrak{f}^i(b)$ *whenever* $b\in Y$ *and* $S_0\in\sec\mathfrak{f}^{-i}(b)$.

REMARKS. a) It is not necessary to assume that the trace filters are open, since (1) applied to $b\in X$ implies

(2) $$\mathrm{cl}^{-i} S_0 \subset \mathrm{int}^i S,$$

thus the trace filter pairs are automatically open.

b) (1) implies (through (2) applied when $a\in X$) the regularity of $(\mathcal{T}^{-1}, \mathcal{T}^1)$ as well.

PROOF. 1° Let $(\mathcal{S}^{-1}, \mathcal{S}^1)$ be a regular extension with neighbourhood filters $\mathfrak{n}^i(a)$. Given an $S\in\mathfrak{f}^i(a)$, there is an $H\in\mathfrak{n}^i(a)$ with $S=H\cap X$. By the regularity, we can choose an $H_0\in\mathfrak{n}^i(a)$ such that

(3) $$\mathrm{Cl}^{-i} H_0 \subset \mathrm{Int}^i H.$$

Now $S_0=H_0\cap X$ will do in (1).

Indeed, if $S_0\in\sec\mathfrak{f}^{-i}(b)$ for some $b\in Y$ then $H_0\in\sec\mathfrak{n}^{-i}(b)$, which means $b\in\mathrm{Cl}^{-i}H_0$, thus (3) gives $b\in\mathrm{Int}^i H$, in other words $H\in\mathfrak{n}^i(b)$, so $S\in\mathfrak{f}^i(b)$.

2° Conversely, we are going to construct a regular extension $(\mathcal{S}^{-1}, \mathcal{S}^1)$ for trace filter pairs satisfying (1).

For $a\in Y$, $i=\pm 1$ and $S\in\mathfrak{f}^i(a)$, define

(4) $$N_S^i(a) = \{a\}\cup\{b\colon S\in\sec\mathfrak{f}^{-i}(b)\}.$$

Observe that if $S\subset T$ then $N_S^i(a)\subset N_T^i(a)$, thus ($\mathfrak{f}^i(a)$ being a filter)

(5) $$\mathfrak{b}^i(a) = \{N_S^i(a)\colon S\in\mathfrak{f}^i(a)\}$$

is a filter base. (An equivalent filter base is obtained if we take S from a base for $\mathfrak{f}^i(a)$ only.) Let

(6) $$\mathfrak{n}^i(a) = \mathrm{fil}_Y\,\mathfrak{b}^i(a) \qquad (a\in Y,\ i=\pm 1).$$

Define $\mathcal{S}^i$ with these neighbourhood filters.

a) *We have defined topologies.* Let a, i and $S\in\mathfrak{f}^i(a)$ be fixed; pick an S_0 according to (1). It is enough to show that if $c\in N_{S_0}^i(a)$ then $N_S^i(a)\in\mathfrak{n}^i(c)$. In fact

(7) $$N_S^i(a) \supset N_S^i(c)\in\mathfrak{n}^i(c).$$

This is evident for $c=a$. Otherwise, (4) applied to S_0 instead of S implies $S_0\in\sec\mathfrak{f}^{-i}(c)$, thus $S\in\mathfrak{f}^i(c)$ by (1), i.e. $N_S^i(c)\in\mathfrak{n}^i(c)$. Now (7) holds, since if $b\in N_S^i(c)$ then either $b=c$, and then $b\in N_{S_0}^i(a)\subset N_S^i(a)$ [the inclusion follows from (2) and the observation made after (4)], or $b\neq c$, and then $b\in N_S^i(a)$ again, since we have $N_S^i(a)\setminus\{a\}=N_S^i(c)\setminus\{c\}$ from (4).

b) $(\mathscr{S}^{-1}, \mathscr{S}^1)$ *is regular*. We are going to show that if a, i, S and S_0 are as in (1) then

$$\text{Cl}^{-i} N^i_{S_0}(a) \subset N^i_S(a). \tag{8}$$

Assume

$$c \notin N^i_S(a). \tag{9}$$

Then $S \notin \sec \mathfrak{f}^{-i}(c)$, i.e. there is a

$$T \in \mathfrak{f}^{-i}(c) \tag{10}$$

such that

$$T \cap S = \emptyset. \tag{11}$$

It is now enough to prove that

$$N^{-i}_T(c) \cap N^i_{S_0}(a) = \emptyset, \tag{12}$$

since (10) and (12) imply $c \notin \text{Cl}^{-i} N^i_{S_0}(a)$, showing the validity of (8).

Let us prove (12) indirectly: assume that b belongs to both sets. As mentioned earlier, $N^i_{S_0}(a) \subset N^i_S(a)$, thus

$$b \in N^i_{S_0}(a) \tag{13}$$

and (9) give $b \neq c$. Hence, according to (4), $b \in N^{-i}_T(c)$ implies

$$T \in \sec \mathfrak{f}^i(b). \tag{14}$$

Furthermore, (4) and (13) give $b = a$ or $S_0 \in \sec \mathfrak{f}^{-i}(b)$, and then (1) implies

$$S \in \mathfrak{f}^i(b). \tag{15}$$

Now (14) and (15) contradict (11), thus proving (12).

c) $(\mathscr{S}^{-1}, \mathscr{S}^1)$ *is compatible with the given trace filter pairs.* To prove $\mathfrak{n}^i(a)|X = \mathfrak{f}^i(a)$, it is enough to show, according to (2), that

$$N^i_S(a) \cap X = \text{cl}^{-i} S \qquad (S \in \mathfrak{f}^i(a)).$$

First assume $y \in \text{cl}^{-i} S$. This means $S \in \sec \mathfrak{f}^{-i}(y)$, thus $y \in N^i_S(a)$ by (4).

Conversely, if $y \in N^i_S(a) \cap X$ then either $y = a$ and then $y \in S$ is evident from $S \in \mathfrak{f}^i(a)$, or $S \in \sec \mathfrak{f}^{-i}(y)$, i.e. $y \in \text{cl}^{-i} S$. □

2.2. Theorem. *If* 2.1 (1) *is fulfilled then the extension constructed in* 2.1 (4)—(6) *is the finest regular extension compatible with the given trace filter pairs.*

Proof. Let $(\mathscr{S}^{-1}, \mathscr{S}^1)$ be the extension constructed in 2.1 (4)—(6), and $(\mathscr{Q}^{-1}, \mathscr{Q}^1)$ another regular extension with the same trace filter pairs. We have to show that if C is a $\mathscr{Q}^i$-neighbourhood of a point $a \in Y$ then it is also an $\mathscr{S}^i$-neighbourhood of a.

By the regularity of $(\mathscr{Q}^{-1}, \mathscr{Q}^1)$, there is a $\mathscr{Q}^i$-neighbourhood D of a such that $\text{CL}^{-i} D \subset C$ where CL^{-i} denotes the $\mathscr{Q}^{-i}$-closure. Put $S = D \cap X$; then $S \in \mathfrak{f}^i(a)$.

To complete the proof, it is enough to show that

$$N_S^i(a) \subset C. \tag{1}$$

If $b \in N_S^i(a)$ then either $b=a$ and then $b \in C$ is evident, or $S \in \sec \mathfrak{f}^{-i}(b)$, therefore $b \in CL^{-i} S \subset CL^{-i} D \subset C$, proving (1). □

Definition. The extension constructed in 2.1 (4)—(6) will be called the *fine regular extension* associated with the trace filter pairs $(\mathfrak{f}^{-1}(a), \mathfrak{f}^1(a))$ $(a \in Y)$.

2.3. The next example shows that in general there is no coarsest regular extension for prescribed trace filter pairs. (In particular, there may exist a regular extension without the doubly strict extension being regular.)

Example. Let $X=(\mathbf{R} \setminus \{0\}) \times \mathbf{R}$, $Y=\mathbf{R}^2$, $\mathscr{T}^{-1}=\mathscr{T}^1$ the discrete topology on X, $\mathfrak{f}^1((0, t))=\mathfrak{g}^1(t, \mathscr{E})$, $\mathfrak{f}^{-1}((0, t))=\mathfrak{g}^{-1}(t, \mathscr{C})$ where $\mathscr{C}$ is the co-finite topology on $\mathbf{R}$. Denote by $\mathscr{S}^i$ the loose, and by $\mathscr{Q}^i$ the strict extension of $\mathscr{T}^i$ associated with the trace filters $\mathfrak{f}^i(a)$. Now $(\mathscr{S}^{-1}, \mathscr{Q}^1)$ and $(\mathscr{Q}^{-1}, \mathscr{S}^1)$ are zero-dimensional (the points of X are isolated in each topology in question, so the condition in 0.6 c) is to be checked only for the points of $Y \setminus X$; the argument is essentially the same as the one needed for proving that $(\mathscr{D}, \mathscr{E})$ and $(\mathscr{C}, \mathscr{D})$ are zero-dimensional where $\mathscr{D}$ is the discrete topology on $\mathbf{R}$), but $(\mathscr{Q}^{-1}, \mathscr{Q}^1)$ is not even regular, since its trace on $Y \setminus X$ is homeomorphic with $(\mathscr{C}, \mathscr{E})$, which is not regular (the $\mathscr{C}$-closure of a non-empty $\mathscr{E}$-open set is the whole $\mathbf{R}$).

Lemma. *If the filter pairs* $(\mathfrak{f}^{-1}(a), \mathfrak{f}^1(a))$ *are round and Cauchy in some quasi-uniform space then they satisfy* 2.1 (1).

Proof. For $S \in \mathfrak{f}^i(a)$ there are an $S_0 \in \mathfrak{f}^i(a)$ and a $U \in \mathscr{U}$ with $U^i[S_0] \subset S$; 2.1 (1) will be fulfilled with this S_0.

Indeed, assume that for some $b \in Y$,

$$S_0 \in \sec \mathfrak{f}^{-i}(b), \tag{1}$$

and choose sets $F_j \in \mathfrak{f}^j(b)$ with $F_{-1} \times F_1 \subset U$. According to (1), there exists a point $x \in S_0 \cap F_{-i}$, thus $F_i \subset U^i x \subset U^i[S_0] \subset S$, therefore $S \in \mathfrak{f}^i(b)$. □

§ 3. Extensions compatible with a fine regular extension

3.1. Theorem. *Let* $\mathscr{U}$ *be a quasi-uniformity,* $(\mathscr{S}^{-1}, \mathscr{S}^1)$ *a fine regular extension of* $(\mathscr{U}^{-tp}, \mathscr{U}^{tp})$ *associated with round and Cauchy trace filter pairs. Then there is an extension of* $\mathscr{U}$ *compatible with* $(\mathscr{S}^{-1}, \mathscr{S}^1)$.

Remark. This means, together with Theorem 1.1, that there is an extension of $\mathscr{U}$ compatible with the fine regular extension associated with some trace filter pairs iff these filter pairs are round and Cauchy. (And if the trace filter pairs satisfy these conditions then the fine regular extension associated with them does exist by Lemma 2.3.)

PROOF. Let Φ^i denote the family of those functions $f\colon Y\to \mathfrak{p}(X)$ for which $f(a)\in\mathfrak{f}^i(a)$ whenever $a\in Y$ (where X, Y and $\mathfrak{f}^i(a)$ are as usual). Consider the system

(1) $$\mathcal{B} = \{V(f^{-1}, f^1, U)\colon f^i\in\Phi^i \quad (i=\pm 1),\ U\in\mathcal{U}\}$$

where

(2) $$a\,V(f^{-1}, f^1, U)\,b \quad \text{iff either} \quad a=b$$
$$\text{or} \quad \exists x\in f^1(a),\ \exists y\in f^{-1}(b),\quad x\,U\,y.$$

The following holds for $i=\pm 1$ (for $i=1$ it is just (2), for $i=-1$ it follows from (2) by interchanging the points):

(2′) $$a\,V(f^{-1}, f^1, U)^i\,b \quad \text{iff either} \quad a=b$$
$$\text{or} \quad \exists x\in f^i(a),\ \exists y\in f^{-i}(b),\quad x\,U^i y.$$

We claim that $\mathcal{B}$ is a base for an extension of $\mathcal{U}$ compatible with $(\mathcal{S}^{-1}, \mathcal{S}^1)$. We shall write

(3) $$^0\mathcal{U} = {}^0\mathcal{U}(\mathfrak{f}^{-1}, \mathfrak{f}^1) = \mathrm{fil}_{Y\times Y}\,\mathcal{B}.$$

1° *$\mathcal{B}$ is a filter base.* If $V_k=V(f_k^{-1}, f_k^1, U_k)$ $(k=1,2)$ then $V_1\cap V_2\supset\supset V(f_3^{-1}, f_3^1, U_1\cap U_2)$ where $f_3^i(a)=f_1^i(a)\cap f_2^i(a)$.

2° *$^0\mathcal{U}$ is a quasi-semiuniformity.* Evident.

3° *$^0\mathcal{U}$ is a quasi-uniformity.* Let $V=V(f^{-1}, f^1, U)$. Choose $U_0\in\mathcal{U}$ with $U_0^3\subset U$, then $f_0^i(a)\in\mathfrak{f}^i(a)$ $(a\in Y,\ i=\pm 1)$ such that

(4) $$f_0^{-1}(a)\times f_0^1(a)\subset U_0,$$

and also

(5) $$f_0^i(a)\subset f^i(a).$$

We are going to show that $V_0^2\subset V$ where $V_0=V(f_0^{-1}, f_0^1, U_0)$.

Assume $a\,V_0\,b\,V_0\,c$. If $a=b$ or $b=c$ then $a\,V\,c$ follows from $V_0\subset V$ implied by (5), so we may assume $a\neq b\neq c$. By the definition of V_0, there are $x\in f_0^1(a)$ and $y\in f_0^{-1}(b)$ with $x\,U_0\,y$, and also $z\in f_0^1(b)$ and $w\in f_0^{-1}(c)$ with $z\,U_0\,w$. Furthermore, $y\,U_0\,z$ follows from (4), thus $x\,U\,w$; now $x\in f^1(a)$ and $w\in f^{-1}(c)$ by (5), hence $a\,V\,c$.

4° *$\mathcal{U}$ is the trace of $^0\mathcal{U}$.* It is enough to show that $V_0|X\subset U\subset V|X$ where U, V and V_0 are as in 3°.

a) If $x\,U\,y$ then $x\,V\,y$ follows from $x\in f^1(x)$ and $y\in f^{-1}(y)$.

b) If $x\,V_0\,y$ and $x, y\in X$ then take $z\in f_0^1(x)$ and $w\in f_0^{-1}(y)$ such that $z\,U_0\,w$. Since $x\in f_0^{-1}(x)$, we have $x\,U_0\,z$ from (4); similarly, $y\in f_0^1(y)$ implies $w\,U_0\,y$, thus $x\,U\,y$.

5° *$^0\mathcal{U}^{itp}$ is coarser than $\mathcal{S}^i$.* In fact,

$$N_S^i(a)\subset V(f^{-1}, f^1, U)^i a$$

holds with $S=f^i(a)$: if $b\in N_S^i(a)$, $b\neq a$ then $S\in\sec\mathfrak{f}^{-i}(b)$, thus $S\cap f^{-i}(b)\neq 0$, i.e. the condition in the second line of (2′) is fulfilled for some $x=y$.

6° ${}^0\mathcal{U}^{itp}$ *is finer than* $\mathcal{S}^i$. Fix i, a and some $S\in\mathfrak{f}^i(a)$. A $V\in{}^0\mathcal{U}$ is needed with

$$V^i a\subset N_S^i(a). \tag{6}$$

As $\mathfrak{f}^i(a)$ is $\mathcal{U}^i$-round, there are $S_0\in\mathfrak{f}^i(a)$ and $U\in\mathcal{U}$ such that

$$U^i[S_0]\subset S. \tag{7}$$

For $b\notin N_S^i(a)$ we have $S\notin\sec\mathfrak{f}^{-i}(b)$, i.e. there is a $T_b\in\mathfrak{f}^{-i}(b)$ with $T_b\cap S=\emptyset$. Thus we can define functions $f^j\in\Phi^j$ $(j=\pm 1)$ such that

$$\text{if}\quad b\notin N_S^i(a)\quad\text{then}\quad f^{-i}(b)\cap S=\emptyset, \tag{8}$$

$$f^i(a)=S_0. \tag{9}$$

Now (6) will hold with $V=V(f^{-1},f^1,U)$.

Indeed, let $b\in V^i a$ and, the case $b=a$ being trivial, assume $b\neq a$. According to (2′), there are $x\in f^i(a)$ and $y\in f^{-i}(b)$ such that $x\,U^i\,y$, in other words, $U^i[f^i(a)]\cap\cap f^{-i}(b)\neq\emptyset$, thus (9) and (7) imply $S\cap f^{-i}(b)\neq\emptyset$, hence $b\in N_S^i(a)$ follows from (8); this proves (6). □

3.2. Corollary. *Let trace filter pairs be prescribed in a bitopological space, and assume that there exists a completely regular extension inducing these trace filter pairs. Then the fine regular extension associated with them is completely regular, too.*

Proof. Let us denote by $(Y;\mathcal{Q}^{-1},\mathcal{Q}^1)$ the completely regular extension of $(X;\mathcal{T}^{-1},\mathcal{T}^1)$. Take a quasi-uniformity $\mathcal{W}$ compatible with $(\mathcal{Q}^{-1},\mathcal{Q}^1)$. $\mathcal{W}$ is an extension of $\mathcal{U}=\mathcal{W}|X$, thus each trace filter pair is round and Cauchy (Theorem 1.1 b) and c)). The fine regular extension $(\mathcal{S}^{-1},\mathcal{S}^1)$ does exist by Lemma 2.3. Now ${}^0\mathcal{U}$ is compatible with $(\mathcal{S}^{-1},\mathcal{S}^1)$, thus this bitopology is completely regular indeed. □

One can, however, not conclude from the existence of a completely regular extension that *all* the other regular extensions with the same trace filter pairs are completely regular:

Example. Let $X=(\mathbf{R}\setminus\{0\})\times\mathbf{R}$, $Y=\mathbf{R}^2$, $(\mathcal{T}^{-1},\mathcal{T}^1)$ the discrete bitopology on X, and $\mathcal{Q}$ a regular, not completely regular topology on $\mathbf{R}$. Consider the trace filters $\mathfrak{f}^i((0,t))=\mathfrak{g}^i(t,\mathcal{Q})$ (see in Examples 0.7). Now the doubly loose extension associated with these trace filter pairs is zero-dimensional, while the doubly strict extension is regular, but not completely regular. (Proof: similar to the argument used in Example 2.3.)

3.3. Theorem. *Under the conditions of Theorem 3.1, the quasi-uniformity* ${}^0\mathcal{U}$ [defined by 3.1 (1)—(3)] *is the finest extension of* $\mathcal{U}$ *compatible with* $(\mathcal{S}^{-1},\mathcal{S}^1)$.

Remark. Observe for later application that the following proof uses only the trace filter pairs instead of the bitopology $(\mathcal{S}^{-1},\mathcal{S}^1)$.

Proof. Assume that $\mathcal{W}$ is an extension of $\mathcal{U}$ compatible with $(\mathcal{S}^{-1},\mathcal{S}^1)$. Given a $W\in\mathcal{W}$, we have to find a $V\in{}^0\mathcal{U}$ such that $V\subset W$.

Take $W_0 \in \mathscr{W}$ with $W_0^3 \subset W$, and let $U = W_0|X$, $f^i(a) = W_0^i a \cap X$; then $V \subset W$ will hold with $V = V(f^{-1}, f^1, U)$.

Indeed, if $a\,V\,b$ and $a \neq b$ then there are $x \in f^1(a) \subset W_0 a$ and $y \in f^{-1}(b) \subset \subset W_0^{-1} b$ such that $x\,U\,y$, so we have $a\,W_0\,x\,W_0\,y\,W_0\,b$, i.e. $a\,W\,b$. □

$w({}^0\mathscr{U})$ can be larger than $w(\mathscr{U})$, even when there exists an extension $\mathscr{W}$ such that $w(\mathscr{W}) = w(\mathscr{U})$:

Example. Take X and $\mathscr{U} = \mathscr{U}_{02}$ from Examples 0.7, and let $Y = \mathbf{R}^2$. Consider the trace filters $\mathfrak{f}^i((0, t)) = \mathfrak{g}^i(t, \mathscr{E})$, which make up round and Cauchy filter pairs. The fine regular extension is now the doubly loose one. It is easy to check that $w({}^0\mathscr{U})$ is uncountable, while $w(\mathscr{U}) = \omega$.

On the other hand, there is an extension $\mathscr{W}$ of $\mathscr{U}$ compatible with the same bitopology such that $w(\mathscr{W}) = \omega$: let $\mathscr{W} = \mathscr{U}(d^*)$ where

$$d^*((a', a''), (b', b'')) = b' - a' + |b'' - a''| \quad \text{if} \quad a' < 0 \leqq b' \quad \text{or} \quad a' \leqq 0 < b'.$$

3.4. It can also occur that there is no extension with the same weight:

Examples. a) Take X and $\mathscr{U} = \mathscr{U}_{01}$ from Examples 0.7, and let $Y = X \cup \{p\}$ with some $p \notin X$,

$$\mathfrak{f}^i(p) = \mathrm{fil}_X \{]0, i\varepsilon] \times H:\ \varepsilon > 0,\ H \subset \mathbf{R},\ |\mathbf{R} \setminus H| < \omega\}.$$

$(\mathfrak{f}^{-1}(p), \mathfrak{f}^1(p))$ is round and Cauchy, so there is an extension of $\mathscr{U}$ compatible with the fine regular extension (which is now doubly loose as well as doubly strict). $w(\mathscr{U}) = \omega$, but the weight of the extension cannot be countable, since the point p has no countable $\mathscr{S}^i$-neighbourhood base.

b) (A less trivial example, with a first countable fine regular extension.) Let $X = (\mathbf{R} \setminus \{0, 1\}) \times \mathbf{R}$, $Y = \mathbf{R}^2$. Take a bijection $t \mapsto \varphi_t$ from $\mathbf{R}$ onto the family of the functions $\mathbf{N} \to \{2^{-n}: n \in \mathbf{N}\}$. Consider the following distance on X:

$$d((x', x''), (y', y'')) =$$
$$= \begin{cases} y' - x' & \text{if} \quad x'' = y'',\ x' < 0 < y' < 1, \\ y' - x' - 1 & \text{if} \quad x' < 0,\ 1 < y', \\ y' - x' + \varphi_t(k) - 1 & \text{if} \quad x'' = k,\ y'' = t,\ \dfrac{\varphi_t(k)}{2} < x' < \varphi_t(k),\ 1 < y'. \end{cases}$$

d is indeed a distance, since if $d((x', x''), (y', y''))$ and $d((y', y''), (z', z''))$ are both defined then $x' < 0 < y' < 1 < z'$, so $d((x', x''), (z', z''))$ is defined as well, and the Triangle Inequality can be easily checked. It may be of assistance in visualizing the space if we write the values of d as $(y' - 0) + (0 - x')$, $(y' - 1) + (0 - x')$, respectively $(y' - 1) + (\varphi_t(k) - x')$. The quasi-uniformity $\mathscr{U} = \mathscr{U}(d)$ induces the discrete bitopology on X. Define the trace filters as follows:

$$\mathfrak{f}^{-1}((0, t)) = \mathrm{fil}_X \{]-\varepsilon, 0[\times \{t\}:\ \varepsilon > 0\},$$
$$\mathfrak{f}^1((0, t)) = \mathrm{fil}_X \{(]0, \varepsilon[\times \{t\}) \cup (]1, 1+\varepsilon[\times \mathbf{R}):\ \varepsilon > 0\},$$
$$\mathfrak{f}^1((1, t)) = \mathrm{fil}_X \{]1, 1+\varepsilon[\times \{t\}:\ \varepsilon > 0\},$$
$$\mathfrak{f}^{-1}((1, t)) = \mathrm{fil}_X \{(]-\varepsilon, 0[\times \mathbf{R}) \cup H(\varepsilon, t):\ \varepsilon > 0\}$$

where

$$H(\varepsilon, t) = \left\{(s, k):\ k\in\mathbf{N},\ s\in\mathbf{R},\ \frac{\varphi_t(k)}{2} < s < \varphi_t(k),\ \varphi_t(k)-\varepsilon < s\right\}.$$

These filter pairs are round and Cauchy, so we can take the fine regular extension $(\mathcal{S}^{-1}, \mathcal{S}^1)$ associated with them. $w(\mathcal{U})=\omega$, and each $\mathfrak{f}^i(a)$ has a countable base, but if $\mathcal{V}$ is an extension of $\mathcal{U}$ compatible with $(\mathcal{S}^{-1}, \mathcal{S}^1)$ then $w(\mathcal{V})>\omega$.

Indeed, assume indirectly that there is a base $\{V_k:\ k\in\mathbf{N}\}$ for $\mathcal{V}$. Choose $W_k\in\mathcal{V}$ with $W_k^2\subset V_k$ $(k\in\mathbf{N})$. For each k, there is a $\varphi(k)\in\{2^{-n}:\ n\in\mathbf{N}\}$ such that

$$]0, \varphi(k)[\times\{k\}\subset W_k(0, k).$$

Take now the number $t\in\mathbf{R}$ for which $\varphi=\varphi_t$. Then

$$H(\varepsilon, t)\cap W_k(0, k)\neq\emptyset \qquad (\varepsilon>0),$$

i.e. $W_k^{-1}(1, t)\cap W_k(0, k)\neq\emptyset$, implying $(0, k)\,V_k\,(1, t)$. This means that any $\mathcal{S}^{-1}$-neighbourhood of $(1, t)$ meets $\{0\}\times\mathbf{N}$, a contradiction, since

$$(0, k)\notin N_S^{-1}((1, t)) \qquad (k\in\mathbf{N})$$

holds with $S=(]-1, 0[\times\mathbf{R})\cup H(1, t)$ [because

$$\left(\left(\left]0, \frac{\varphi_t(k)}{2}\right[\times\{k\}\right)\cup(]1, 2[\times\mathbf{R})\right)\cap S=\emptyset,$$

and so $S\notin\sec\mathfrak{f}^1((0, k))$].

3.5. Consider on $\mathbf{R}$ the following two important quasi-uniformities: the *Sorgenfrey quasi-uniformity* $\mathcal{U}_{so}=\mathcal{U}(d_{so})$ and the *quasi-uniformity of semicontinuities* $\mathcal{U}_{se}=\mathcal{U}(d_{se})$ where

$$d_{so}(s, t)=t-s \quad \text{if} \quad s<t,$$

$$d_{se}(s, t)=t-s.$$

Example. Let X be an $\mathcal{E}$-dense subset of $Y=\mathbf{R}$, and $\mathcal{U}=\mathcal{U}_s|X$ where $\mathcal{U}_s=\mathcal{U}_{so}$ or $\mathcal{U}_s=\mathcal{U}_{se}$. Now if we take the trace filter pairs in $(X, \mathcal{U})$ induced by the bitopological extension $(\mathcal{U}_s^{-tp}, \mathcal{U}_s^{tp})$ then we can recover $\mathcal{U}_s$ from $\mathcal{U}$: $\mathcal{U}_s={}^0\mathcal{U}$.

Moreover, $(\mathcal{U}_s^{-tp}, \mathcal{U}_s^{tp})$ is a doubly strict extension of $(\mathcal{U}^{-tp}, \mathcal{U}^{tp})$, thus there is no other regular extension with the same trace filter pairs. One can also check that $\mathcal{U}_s$ is the only extension of $\mathcal{U}$ compatible with $(\mathcal{U}_s^{-tp}, \mathcal{U}_s^{tp})$.

3.6. Besides the fine regular extensions, there is another (very special) class of bitopological extensions for which the conditions in Theorem 1.1 are sufficient, namely the finite extensions (i.e. when $Y\setminus X$ is finite). (See 10.8 for a third class having this property.) We shall need

Lemma. *If* $(Y; \mathcal{S}^{-1}, \mathcal{S}^1)$ *is a regular extension then for* $a\in Y$, $X\cap\mathrm{Cl}^i\{a\}=\bigcap\mathfrak{f}^{-i}(a)$.

Proof. By the regularity, $x\in\mathrm{Cl}^i\{a\}$ iff $a\in\mathrm{Cl}^{-i}\{x\}$; for $x\in X$, the latter is equivalent to $x\in\bigcap\mathfrak{f}^{-i}(a)$. □

3.7. Lemma. *A finite regular extension* $(Y; \mathcal{S}^{-1}, \mathcal{S}^1)$ *is completely determined by the trace filter pairs and* $(\mathcal{S}^{-1}, \mathcal{S}^1)|Y\setminus X$.

Proof. For each $\mathcal{S}^i$-neighbourhood filter $\mathfrak{n}^i(a)$, $\mathfrak{n}^i(a)|X=\mathfrak{f}^i(a)$ is known; so it is enough to show that $\mathfrak{r}^i(a)=\mathfrak{n}^i(a)|Y\setminus X$ (which is not necessarily a filter) is uniquely determined.

If $p\in Y\setminus X$ then $\mathfrak{r}^i(p)$ is the neighbourhood filter of p in the topology $\mathcal{S}^i|Y\setminus X$, which was assumed to be known.

If $x\in X$ then

$$\mathfrak{r}^i(x) = \text{fil}_{Y\setminus X}\{\{p\in Y\setminus X\colon\ x\in \text{Cl}^i\{p\}\}\}$$

(it is here that we use the finiteness of $Y\setminus X$), so Lemma 3.6 implies that

$$\mathfrak{r}^i(x) = \text{fil}_{Y\setminus X}\{\{p\in Y\setminus X\colon\ x\in \bigcap \mathfrak{f}^{-i}(p)\}\},$$

i.e. $\mathfrak{r}^i(x)$ is determined by the trace filter pairs. □

Corollary. *Any regular bitopological extension with only one new point is fine regular.* □

A special case of the promised result follows now immediately from Theorem 3.1: a quasi-uniformity can be extended to a prescribed regular one-point extension of the induced bitopology iff the trace filter pair of the new point is round and Cauchy. This reasoning does not work for all the finite extensions, because a finite regular extension is not necessarily fine regular.

3.8. Theorem. *Let* $(X, \mathcal{U})$ *be a quasi-uniform space, and* $(Y; \mathcal{S}^{-1}, \mathcal{S}^1)$ *a finite regular extension of* $(X; \mathcal{U}^{-tp}, \mathcal{U}^{tp})$. *Then there is an extension of* $\mathcal{U}$ *compatible with* $(\mathcal{S}^{-1}, \mathcal{S}^1)$ *iff the trace filter pairs are round and Cauchy; if so then there is only one extension with this property.*

Proof. a) *Necessity.* Theorem 1.1.

b) *Sufficiency.* Let Ψ^i denote the family of those functions $f\colon Y\setminus X\to \mathfrak{p}(X)$ for which $f(p)\in\mathfrak{f}^i(p)$ whenever $p\in Y\setminus X$. Consider the system

$$\mathcal{B} = \{V(f^{-1}, f^1, U)\colon f^i\in\Psi^i\ (i=\pm 1),\ U\in\mathcal{U}\}$$

where, with V standing for $V(f^{-1}, f^1, U)$,

(1) $x\,V\,y$ iff $x\,U\,y$ (for $x, y\in X$),

(2) $x\,V\,p$ iff $x\in f^{-1}(p)$ (for $x\in X$, $p\in Y\setminus X$),

(3) $p\,V\,x$ iff $x\in f^1(p)$ (for $x\in X$, $p\in Y\setminus X$),

(4) $p\,V\,q$ iff $p\in \text{Cl}^1\{q\}$ (for $p, q\in Y\setminus X$).

$\mathcal{B}$ will be a base for the required extension.

1° $\mathcal{B}$ is a filter base (just like in 1° in the proof of Theorem 3.1). The elements of $\mathcal{B}$ are clearly, entourages, thus

$$\mathcal{V} = \text{fil}_{Y\times Y}\mathcal{B}$$

is a quasi-semiuniformity.

2° $\mathscr{V}$ *is a quasi-uniformity.* For $V=V(f^{-1},f^1,U)$ we need a

$$V_0 = V(f_0^{-1}, f_0^1, U_0)\in\mathscr{B}$$

with $V_0^2\subset V$. As $(\mathfrak{f}^{-1}(p), \mathfrak{f}^1(p))$ is Cauchy, there are $S_i(p)\in\mathfrak{f}^i(p)$ such that

$$S_{-1}(p)\times S_1(p) \subset U \qquad (p\in Y\setminus X). \tag{5}$$

By the regularity of $(\mathscr{S}^{-1}, \mathscr{S}^1)$, we may also assume, making the sets $S_i(p)$ small enough, that

$$\text{if}\quad p\notin \mathrm{Cl}^1\{q\}\quad\text{then}\quad S_1(p)\cap S_{-1}(q)=\emptyset \qquad (p,q\in Y\setminus X); \tag{6}$$

the finiteness of $Y\setminus X$ will guarantee that (6) can be achieved for all pairs simultaneously. The trace filter pairs are round, so there are $U_0\in\mathscr{U}$ and $T_i(p)\in\mathfrak{f}^i(p)$ with

$$U_0^i[T_i(p)]\subset f^i(p) \qquad (p\in Y\setminus X,\ i=\pm 1) \tag{7}$$

(the same U_0 will do for each p and i, again by the finiteness of $Y\setminus X$); we may also assume that $U_0^2\subset U$. Now we are going to show that $V_0^2\subset V$ holds with this U_0 and

$$f_0^i(p) = S_i(p)\cap\bigcap\{T_i(q)\colon q\in Y\setminus X,\ T_i(q)\in\mathfrak{f}^i(p)\}. \tag{8}$$

It is evident that $f_0^i(p)\in\mathfrak{f}^i(p)$, so $V_0\in\mathscr{B}$. To prove $V_0^2\subset V$, eight cases are to be considered; x, y and z will denote points of X, p, q and r points of $Y\setminus X$.

If $x\,V_0\,y\,V_0\,z$ then $x\,V\,z$ follows from (1) and $U_0^2\subset U$.

If $p\,V_0\,x\,V_0\,y$ then $x\in T_1(p)$ by (3) and (8), $x\,U_0\,y$ by (1), so $p\,V\,y$ follows from (7) and (3).

The case $x\,V_0\,y\,V_0\,p$ is analogous.

If $x\,V_0\,p\,V_0\,y$ then $x\,V\,y$ follows from (2), (3), (8), (5) and (1).

If $x\,V_0\,p\,V_0\,q$ then from (4) we have $p\in\mathrm{Cl}^1\{q\}$, which is equivalent to $q\in\mathrm{Cl}^{-1}\{p\}$, i.e. each $\mathscr{S}^{-1}$-neighbourhood of q contains p, and therefore $\mathfrak{f}^{-1}(q)\subset\subset\mathfrak{f}^{-1}(p)$. This means that $T_{-1}(q)\in\mathfrak{f}^{-1}(p)$, and so (8) implies $T_{-1}(q)\supset f_0^{-1}(p)$; hence $x\in T_{-1}(q)$ by (2). $x\,V\,q$ follows now from (7) and (2).

The case $p\,V_0\,q\,V_0\,x$ is analogous.

If $p\,V_0\,x\,V_0\,q$ then, by (2), (3) and (8), $x\in S_1(p)\cap S_{-1}(q)$, thus (6) implies $p\in\mathrm{Cl}^1\{q\}$, i.e. $p\,V\,q$ by (4).

Finally, if $p\,V_0\,q\,V_0\,r$ then $p\,V\,r$ is evident from (4).

3° According to (1), $\mathscr{V}|X=\mathscr{U}$.

4° $\mathscr{V}$ *is compatible with* $(\mathscr{S}^{-1}, \mathscr{S}^1)$. $(\mathscr{V}^{-tp}, \mathscr{V}^{tp})$ is an extension of $\mathscr{U}$ for the same trace filter pairs by (2) and (3); according to (4), its trace on $Y\setminus X$ coincides with that of $(\mathscr{S}^{-1}, \mathscr{S}^1)$; so Lemma 3.7 guarantees that $(\mathscr{V}^{-tp}, \mathscr{V}^{tp})=(\mathscr{S}^{-1}, \mathscr{S}^1)$.

c) *Unicity.* Let $\mathscr{W}$ and $\mathscr{Z}$ be extensions of $\mathscr{U}$ for $(\mathscr{S}^{-1}, \mathscr{S}^1)$, and $Z\in\mathscr{Z}$. Take a $W\in\mathscr{W}$ with $W|X=Z|X$, and for each $p\in Y\setminus X$ and $i=\pm 1$ choose a

$W_{i,p}\in\mathscr{W}$ such that $W_{i,p}^{i}p\subset Z^{i}p$. Then with

$$W_0=W\cap\bigcap\{W_{i,p}\colon\ p\in Y\setminus X,\ i=\pm 1\}$$

we have $Z\supset W_0\in\mathscr{W}$, thus $\mathscr{Z}\subset\mathscr{W}$. □

Remarks. a) In the analogous case of extending a quasi-uniformity for a finite strict extension of the induced topology ([10] § 5), the unicity does not hold (but see also [11] § 4).

b) The above proof is very similar to that of [10] 5.1 (where the word "strict" can be dropped, see in [15]).

c) Regularity can be replaced in the theorem by a weaker bitopological separation property (called R_1 in [36], S_2 in [13]); Lemma 3.7 holds with an even weaker axiom (R_0 in [28], S_1 in [13]).

§ 4. Finest and coarsest extensions

4.1. THEOREM. *If there is at least one extension of a quasi-uniformity compatible with an extension of the induced bitopology then there is a finest one among all such extensions.*

PROOF. Just like in the proof of [10] 3.1, take the supremum of all the compatible extensions. □

4.2. On the other hand, there is in general no coarsest extension, not even for a fine regular extension:

EXAMPLE. Let $X=(\mathbf{R}\setminus\{0\})\times\mathbf{N}$, $Y=\mathbf{R}\times\mathbf{N}$, $\mathscr{U}=\mathscr{U}(d)$ where

$$d((s,n),(t,k))=\sqrt{-st}\quad\text{if}\quad s<0<t,\quad n=k;$$

$$\mathfrak{f}^{i}((0,n))=\operatorname{fil}_X\{]0,i\varepsilon[\times\{n\}\colon\ \varepsilon>0\}.$$

The filter pairs $(\mathfrak{f}^{-1}(a),\mathfrak{f}^{1}(a))$ are round and Cauchy, and the doubly loose extension $(\mathscr{S}^{-1},\mathscr{S}^{1})$ associated with them is zero-dimensional, therefore it coincides with the fine regular extension, and $\mathscr{U}$ can be extended to $(\mathscr{S}^{-1},\mathscr{S}^{1})$. (As a matter of fact, there is no other regular extension of $(\mathscr{U}^{-tp},\mathscr{U}^{tp})$ with the same filter pairs.)

Define now $\mathscr{U}^*=\mathscr{U}(d^*)$ on Y where $d^*=d\cup e_1$,

$$e_1((s,n),(t,k))=\begin{cases}-ns & \text{if}\quad s<0=t,\ n=k,\\ \dfrac{t}{n} & \text{if}\quad s=0<t,\ n=k.\end{cases}$$

d^* is a distance since if $s<0<t$ then

$$d^*((s,n),(0,n))+d^*((0,n),(t,n))=-ns+\frac{t}{n}\geqq 2\sqrt{-st}>d^*((s,n),(t,n)),$$

and this is the only possibility of $d^*(a, b)+d^*(b, c)$ being defined. Now $\mathcal{U}^*|X=\mathcal{U}$, and it is easy to check that $\mathcal{U}^*$ is compatible with $(\mathcal{S}^{-1}, \mathcal{S}^1)$. One can similarly define the "mirror image" $\mathcal{U}^{**}=\mathcal{U}(d^{**})$ of $\mathcal{U}^*$ with $d^{**}=d\cup e_2$,

$$e_2((s, n), (t, k)) = \begin{cases} -\dfrac{s}{n} & \text{if} \quad s < 0 = t, \; n = k, \\ nt & \text{if} \quad s = 0 < t, \; n = k. \end{cases}$$

$\mathcal{U}^{**}$ is also an extension of $\mathcal{U}$ compatible with $(\mathcal{S}^{-1}, \mathcal{S}^1)$. We are going to show that $\mathcal{U}^*\cap\mathcal{U}^{**}$ does not contain an extension of $\mathcal{U}$.

Assume that $\mathcal{V}$ is a quasi-uniformity on Y, $\mathcal{V}|X=\mathcal{U}$, $\mathcal{V}\subset\mathcal{U}^*\cap\mathcal{U}^{**}$. Take a $V\in\mathcal{V}$ with $V|X=U_{(1)}$, then a $V_0\in\mathcal{V}$ such that $V_0^2\subset V$. As $V_0\in\mathcal{U}^*\cap\mathcal{U}^{**}$, there is an $\varepsilon>0$ with

(1) $$U^*_{(2\varepsilon)} \subset V_0, \quad U^{**}_{(2\varepsilon)} \subset V_0$$

(the same ε is good in both cases if it is small enough). For each $n\in\mathbf{N}$, we have

$$d^{**}((-n\varepsilon, n), (0, n)) < 2\varepsilon, \quad d^*((0, n), (n\varepsilon, n)) < 2\varepsilon,$$

thus (1) implies

$$(-n\varepsilon, n) \; V_0 \; (0, n) \; V_0 \; (n\varepsilon, n)$$

and thus from $V_0^2\subset V$ and $V|X=U_{(1)}$ it follows that $d((-n\varepsilon, n), (n\varepsilon, n))<1$, i.e. $n\varepsilon<1$, which is not true if n is large enough.

§ 5. Extensions of quasi-proximities

The reader unfamiliar with quasi-proximities and totally bounded quasi-uniformities may consult [21], or skip this section and 6.2.

$\mathcal{U}^t$ is the quasi-proximity induced by the quasi-uniformity $\mathcal{U}$; δ^p is the topology, $((\delta^{-1})^p, \delta^p)=(\delta^{-p}, \delta^p)$ the bitopology induced by the quasi-proximity δ.

5.1. Definition. Let $\mathfrak{a}, \mathfrak{b}\subset\mathfrak{p}(X)$. $(\mathfrak{a}, \mathfrak{b})$ is *compressed* in the quasi-proximity space (X, δ) if $A\in\sec\mathfrak{a}$, $B\in\sec\mathfrak{b}$ imply $A \,\delta\, B$.

Compare this notion with that of a compressed filter [6, 9].

Lemma. *Let* $(X, \mathcal{U})$ *be a quasi-uniform space,* $\mathfrak{a}, \mathfrak{b}\subset\mathfrak{p}(X)$.

a) *If* $(\mathfrak{a}, \mathfrak{b})$ *is* $\mathcal{U}$*-Cauchy then it is* $\mathcal{U}^t$*-compressed.*

b) *If* $(\mathfrak{a}, \mathfrak{b})$ *is a* $\mathcal{U}^t$*-compressed filter pair and* $\mathcal{U}$ *is totally bounded then* $(\mathfrak{a}, \mathfrak{b})$ *is* $\mathcal{U}$*-Cauchy.*

Proof. a) Assume that $(\mathfrak{a}, \mathfrak{b})$ is $\mathcal{U}$-Cauchy, and let $A \,\overline{\mathcal{U}^t}\, B$. Then

$$U_{A,B} = X\times X\setminus A\times B\in\mathcal{U},$$

thus there are $A_0\in\mathfrak{a}$ and $B_0\in\mathfrak{b}$ with $A_0\times B_0\subset U_{A,B}$, i.e. either $A_0\cap A=\emptyset$ or $B_0\cap B=\emptyset$, therefore $A\notin\sec\mathfrak{a}$ or $B\notin\sec\mathfrak{b}$.

b) Assume now that $\mathscr{U}$ is totally bounded, and $(\mathfrak{a}, \mathfrak{b})$ is a $\mathscr{U}^t$-compressed filter pair. $\{U_{A,B}\colon A\, \overline{\mathscr{U}^t}\, B\}$ is a subbase for $\mathscr{U}$ ([21] 1.33), so [since $(\mathfrak{a}, \mathfrak{b})$ is a filter pair] it is enough to find for each $U_{A,B}$ sets $A_0 \in \mathfrak{a}$ and $B_0 \in \mathfrak{b}$ such that $A_0 \times B_0 \subset$ $\subset U_{A,B}$.

As $(\mathfrak{a}, \mathfrak{b})$ is compressed, $A\, \overline{\mathscr{U}^t}\, B$ implies that there is either an $A_0 \in \mathfrak{a}$ with $A_0 \cap A = \emptyset$, thus $A_0 \times X \subset U_{A,B}$, or a $B_0 \in \mathfrak{b}$ with $B_0 \cap B = \emptyset$, and then $X \times B_0 \subset$ $\subset U_{A,B}$. □

REMARK. A filter pair of the form $(\mathfrak{f}, \mathfrak{f})$ is compressed iff $\mathfrak{f}$ is compressed, and it is $\mathscr{U}$-Cauchy iff $\mathfrak{f}$ is $\mathscr{U}^s$-Cauchy, thus it follows from [6] (19.25) that the condition in b) characterizes the totally bounded quasi-uniformities.

5.2. DEFINITION. a) [5]. The filter $\mathfrak{f}$ in the quasi-proximity space (X, δ) is *round* if for each $S \in \mathfrak{f}$ there is an $S_0 \in \mathfrak{f}$ such that $S_0 \bar{\delta} (X \setminus S)$.

b) The filter pair $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ is *round* if $\mathfrak{f}^i$ is δ^i-round $(i = \pm 1)$.

LEMMA. *A filter pair is $\mathscr{U}$-round for a quasi-uniformity $\mathscr{U}$ iff it is $\mathscr{U}^t$-round.* □

THEOREM. *If δ is a quasi-proximity, $(\mathscr{S}^{-1}, \mathscr{S}^1)$ is an extension of (δ^{-p}, δ^p) and there is an extension of δ compatible with $(\mathscr{S}^{-1}, \mathscr{S}^1)$ then each trace filter pair is round and compressed.*

PROOF. Take an arbitrary quasi-uniformity $\mathscr{V}$ compatible with the extension of δ. Then $\mathscr{U} = \mathscr{V}|X$ is compatible with δ ([6] (8.24) or [21] 1.30). The trace filter pairs are $\mathscr{U}$-round and Cauchy by Theorem 1.1 b)—c), thus they are δ-round (Lemma 5.2) and compressed (Lemma 5.1 a)). □

REMARK. It would be, of course, equally simple to give a direct proof not using quasi-uniformities.

5.3. THEOREM. *If δ is a quasi-proximity, $(\mathscr{S}^{-1}, \mathscr{S}^1)$ is a fine regular extension of (δ^{-p}, δ^p), and each trace filter pair is round and compressed then there is an extension of δ compatible with $(\mathscr{S}^{-1}, \mathscr{S}^1)$.*

PROOF. Let $\mathscr{U}$ be the totally bounded quasi-uniformity compatible with δ. By Lemmas 5.1 b) and 5.2, the trace filter pairs are $\mathscr{U}$-round and Cauchy, thus there is an extension $\mathscr{V}$ of $\mathscr{U}$ inducing $(\mathscr{S}^{-1}, \mathscr{S}^1)$ (Theorem 3.1). Now $\mathscr{V}^t$ is an extension of δ compatible with $(\mathscr{S}^{-1}, \mathscr{S}^1)$. □

Taking $\mathscr{V} = {}^0\mathscr{U}$ in the above proof, we obtain the following extension ${}^0\delta$ of δ:

$$A\, \overline{{}^0\delta}\, B \text{ iff } A \cap B = \emptyset \text{ and there are } A', B' \subset X \text{ such that } A'\, \bar{\delta}\, B',\ A' \in \mathfrak{f}^1(a)\ (a \in A) \text{ and } B' \in \mathfrak{f}^{-1}(b)\ (b \in B).$$

This construction can be generalized to arbitrary syntopogenous structures; we shall return to the problem of syntopogenous extensions in [17]. ${}^0\delta$ is the finest extension of δ compatible with the prescribed fine regular extension (a stronger result will be proved in § 6).

5.4. The conditions in Theorem 5.3 are not sufficient if $(\mathscr{S}^{-1}, \mathscr{S}^1)$ is an arbitrary completely regular extension since with $\mathscr{U}$ from Example 1.2, $\delta = \mathscr{U}^t$ satisfies these conditions, but it fails to satisfy the following necessary condition with $A = A'_{-1}$

and $B=A_1'$. (Indeed, with $C_i=]0, i[\times A_i$ we have $C_i\in\sec \bigcup_{a\in A_i'} \mathfrak{f}^i(a)$ $(i=\pm 1)$ and $C_{-1}\,\bar{\delta}\,C_1$, so (1) is fulfilled, but 1.3 (2) does not hold.)

THEOREM. *If there is an extension of δ compatible with $(\mathscr{S}^{-1}, \mathscr{S}^1)$ then*

(1) $$\Big(\bigcup_{a\in A} \mathfrak{f}^{-1}(a),\ \bigcup_{b\in B} \mathfrak{f}(b)\Big) \text{ is not compressed}$$

implies 1.3 (2).

PROOF. Let $\mathscr{V}$ be a quasi-uniformity compatible with the extension of δ; then $\mathscr{U}=\mathscr{V}|X$ is compatible with δ, thus (1) implies 1.3 (1) by Lemma 5.1 a). Now Theorem 1.3 can be applied. □

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EXTENSIONS OF QUASI-UNIFORMITIES FOR PRESCRIBED BITOPOLOGIES II

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§§ 0 to 5 can be found in the first part [11], to which the reader is referred for notations, terminology, and a description of the purpose of this paper.

§ 6. Extensions for prescribed trace filter pairs

6.1. THEOREM. *There exists an extension of a quasi-uniformity $\mathcal{U}$ compatible with a given system of trace filter pairs iff each trace filter pair is round and Cauchy. If so then ${}^0\mathcal{U}$ is the finest one among all such extensions.*

PROOF. Necessity: Theorem 1.1 b) and c).
Sufficiency: Remark 3.1.
Finest: Remark 3.3. □

6.2. THEOREM. *There exists an extension of a quasi-proximity δ compatible with a given system of trace filter pairs iff each trace filter pair is round and compressed. If so then ${}^0\delta$ is the finest one among all such extensions.*

PROOF. Necessity: Theorem 5.2.
Sufficiency: Lemmas 5.1 b), 5.2 and 2.3, Theorems 2.1 and 5.3.
Finest: Assume that δ' is an extension of δ for the given trace filter pairs, and denote by $\mathcal{U}$ and $\mathcal{U}'$ the totally bounded quasi-uniformity compatible with δ, respectively with δ'. Now $\mathcal{U}'$ is an extension of $\mathcal{U}$ compatible with the given trace filter pairs [$(\mathcal{U}'|X)^t=\delta$ is clear, and $\mathcal{U}'|X$ is totally bounded, so it has to coincide with $\mathcal{U}$; $\mathcal{U}'$ is compatible with the given trace filter pairs because δ' was so]. By Theorem 6.1, ${}^0\mathcal{U}$ is finer than $\mathcal{U}'$, thus ${}^0\delta=({}^0\mathcal{U})^t$ is finer than $\delta'=\mathcal{U}'^t$. □

6.3. The problem of the existence of a quasi-uniform extension preserving the weight can be solved completely if the extension is required to be compatible with some trace filter pairs only, and not with a given bitopological extension.

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THEOREM. *Let* $(X, \mathcal{U})$ *be a quasi-uniform space,* $Y \supset X$, $\varkappa$ *an infinite cardinal. There is an extension* $\mathcal{V}$ *of* $\mathcal{U}$ *with* $w(\mathcal{V}) \leqq \varkappa$ *for some trace filter pairs* $(\mathfrak{f}^{-1}(a), \mathfrak{f}^{1}(a))$ $(a \in Y)$ *iff these filter pairs are round and Cauchy,* $\mathfrak{f}^{i}(p)$ *has a base of cardinality* $\leqq \varkappa$ $(i = \pm 1,\ p \in Y \setminus X)$, *and* $w(\mathcal{U}) \leqq \varkappa$.

REMARKS. a) We could have written "$\mathfrak{f}^{i}(a)$ has a base of cardinality $\leqq \varkappa$ $(i = \pm 1,\ a \in Y)$", since $\{\mathcal{U}^{i}x\colon U \in \mathcal{B}\}$ is a base for $\mathfrak{f}^{i}(x)$ if $x \in X$ and $\mathcal{B}$ is a base for $\mathcal{U}$.

b) The condition for the existence of an extension preserving the weight can be obtained by putting $\varkappa = w(\mathcal{U})$.

PROOF. The necessity of the conditions is clear, so we shall only prove the sufficiency. For each ordinal $\alpha < \varkappa$, take an entourage $U_{\alpha,1}$ and sets $S_{\alpha}^{i}(a)$ such that $\{U_{\alpha,1}\colon \alpha < \varkappa\}$ is a base for $\mathcal{U}$ and $\{S_{\alpha}^{i}(a)\colon \alpha < \varkappa\}$ is a base for $\mathfrak{f}^{i}(a)$ $(a \in Y,\ i = \pm 1)$. For $\beta < \varkappa$ and $n \in \mathbf{N}$, take $U_{\beta,n+1} \in \mathcal{U}$ such that $U_{\beta,n+1}^{2} \subset U_{\beta,n}$ $(n \in \mathbf{N})$. The filter pairs are Cauchy, so we can choose sets $T_{\beta,n}^{i}(a) \in \mathfrak{f}^{i}(a)$ satisfying

$$(1) \qquad T_{\beta,n}^{-1}(a) \times T_{\beta,n}^{1}(a) \subset U_{\beta,n} \qquad (\beta < \varkappa,\ n \in \mathbf{N},\ a \in Y).$$

Put now

$$(2) \qquad f_{\alpha,\beta,n}^{i}(a) = S_{\alpha}^{i}(a) \cap \bigcap_{k \leqq n} T_{\beta,k}^{i}(a) \qquad (\alpha, \beta < \varkappa,\ n \in \mathbf{N},\ a \in Y,\ i = \pm 1).$$

Then, with the notations in the proof of Theorem 3.1, we have $f_{\alpha,\beta,n}^{i} \in \Phi^{i}$, thus

$$W_{\alpha,\beta,n} = V(f_{\alpha,\beta,n}^{-1}, f_{\alpha,\beta,n}^{1}, U_{\beta,n}) \in {}^{0}\mathcal{U} \qquad (\alpha, \beta < \varkappa,\ n \in \mathbf{N}),$$

therefore

$$\mathcal{W} = \mathrm{fil}_{Y \times Y} \{W_{\alpha,\beta,n}\colon \alpha, \beta < \varkappa,\ n \in \mathbf{N}\}$$

is a quasi-semiuniformity coarser than ${}^{0}\mathcal{U}$. Clearly, $w(\mathcal{W}) \leqq \varkappa$.

If $V = W_{\alpha,\beta,n}$ and $V_0 = W_{\alpha,\beta,n+1}$ then the conditions in 3° of the proof of Theorem 3.1 are fulfilled with $U = U_{\beta,n}$, $U_0 = U_{\beta,n+1}$, $f^{i} = f_{\alpha,\beta,n}^{i}$, $f_0^{i} = f_{\alpha,\beta,n+1}^{i}$, so (as proved there) $V_0^{2} \subset V$, i.e. $\mathcal{W}$ is a quasi-uniformity. Furthermore, from 4° of the same proof we know that $W_{\alpha,\beta,n+1}|X \subset U_{\beta,n} \subset W_{\alpha,\beta,n}|X$, hence $\mathcal{W}|X = \mathcal{U}$ (since the entourages $U_{\beta,n}$ form a base).

$\mathcal{W} \subset {}^{0}\mathcal{U}$ implies that the trace filters induced by $\mathcal{W}$ are coarser than the original ones; it will complete the proof if we show that they are finer as well.

Let $a \in Y$ and $i \in \{-1, 1\}$ be fixed, and take an $S \in \mathfrak{f}^{i}(a)$. Since $\mathfrak{f}^{i}(a)$ is round, there are $\alpha, \beta < \varkappa$ such that

$$(3) \qquad U_{\beta,1}^{i2}[S_{\alpha}^{i}(a)] \subset S.$$

We are going to show that

$$(4) \qquad W_{\alpha,\beta,1}^{i} a \cap X \subset S.$$

Indeed, if $x \in W_{\alpha,\beta,1}^{i} a \cap X$ and $x \neq a$ then 3.1 (2′) implies that there are $y, z \in X$ such that $y \in f_{\alpha,\beta,1}^{i}(a)$, $z \in f_{\alpha,\beta,1}^{-i}(x)$ and $y\ U_{\beta,1}^{i}\ z$. From (2) we have $y \in S_{\alpha}^{i}(a)$, and from (1) and (2) $z\ U_{\beta,1}^{i}\ x$, thus $x \in U_{\beta,1}^{i2}[S_{\alpha}^{i}(a)]$, i.e. (4) follows from (3). □

6.4. The above proof does not yield a well-defined extension, since $\mathscr{W}$ depends on the choice of $U_{\beta,n}$, $S^i_\alpha(a)$ and $T^i_{\alpha,n}(a)$. In fact, the supremum of all such extensions $\mathscr{W}$ is equal to ${}^0\mathscr{U}$, since given some $V=V(f^{-1},f^1,U)$, we can take $S^i_0(a)=f^i(a)$, $U_{0,1}=U$, and in this case $W_{0,0,1}\subset V$.

There is no finest one among the extensions preserving the weight, since we have just seen that such a finest extension would coincide with ${}^0\mathscr{U}$, but Example 3.4 b) shows that we cannot expect to have a weight-preserving extension compatible with the fine regular extension even when the conditions of Theorem 6.3 are fulfilled (with $\varkappa=w(\mathscr{U})=\omega$ in the example).

We shall continue the investigation of extensions preserving the weight in § 8; for this purpose, some special properties of filter pairs in a quasi-uniform space will be introduced in § 7.

6.5. A similar, but simpler proof gives the following analogue of Theorem 6.3. (χ denotes the character of a topology.)

THEOREM. *There is an extension $\mathscr{V}$ of a quasi-uniformity $\mathscr{U}$ compatible with a loose extension $\mathscr{S}$ of $\mathscr{U}^{tp}$ such that $w(\mathscr{V})\leqq\varkappa$ iff the trace filters are round, $\chi(\mathscr{S})\leqq\varkappa$ and $w(\mathscr{U})\leqq\varkappa$.*

PROOF. Take $U_{\beta,n}$ as in the proof of Theorem 6.3, and for each trace filter $\mathfrak{f}(p)$ $(p\in Y\setminus X)$, choose a base $\{f_\alpha(p)\colon \alpha<\varkappa\}$. Now the entourages (cf. 0.10 b))

$$V(f_\alpha, U_{\beta,n}) \qquad (\alpha,\beta<\varkappa,\ n\in\mathbf{N})$$

form a subbase for the required extension. □

§ 7. Special properties of filter pairs

(Linked, concentrated, weakly concentrated, stable)

Throughout this section, let $(\mathfrak{f}^{-1},\mathfrak{f}^1)$ be a filter pair in a quasi-uniform space $(X,\mathscr{U})$.

7.1. DEFINITION. $(\mathfrak{f}^{-1},\mathfrak{f}^1)$ is *linked* if $S_i\in\mathfrak{f}^i$ $(i=\pm1)$ implies that $S_{-1}\cap S_1\neq\emptyset$.

In [25], "pair filter" means a linked filter pair.

LEMMA. *$(\mathfrak{f}^{-1},\mathfrak{f}^1)$ is linked iff $\mathfrak{f}^{-1}(\cap)\mathfrak{f}^1$ is a filter.* □

7.2. LEMMA. *A bitopological extension is firm iff the trace filter pairs are linked.* □

7.3. NOTATIONS. Let

$$\mathfrak{f}^\times=\mathfrak{f}^\times(\mathfrak{f}^{-1},\mathfrak{f}^1)=\{S_{-1}\times S_1\colon S_i\in\mathfrak{f}^i\ (i=\pm1)\}.$$

We shall write $\mathfrak{f}^\times_\alpha$ for $\mathfrak{f}^\times(\mathfrak{f}^{-1}_\alpha,\mathfrak{f}^1_\alpha)$, $\mathfrak{g}^\times$ for $\mathfrak{f}^\times(\mathfrak{g}^{-1},\mathfrak{g}^1)$, etc. For $K\subset X^2$, let $K_{-1}=\mathrm{dom}\,K$, $K_1=\mathrm{ran}\,K$. (Thus $\mathfrak{f}^i=\{K_i\colon K\in\mathfrak{f}^\times\}$.)

DEFINITION. $(\mathfrak{f}^{-1},\mathfrak{f}^1)$ is *concentrated* if for any $K\in\mathfrak{f}^\times$ there is a $U\in\mathscr{U}$ such that $L\subset K$ whenever $L\in\mathfrak{f}^\times$ and $L\subset U$.

LEMMA. $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ *is concentrated iff the following condition is fulfilled both for* $i=-1$ *and* $i=1$:

(1) *given* $S\in\mathfrak{f}^i$, *there is a* $U\in\mathcal{U}$ *such that* $B\subset S$ *whenever* $B\in\mathfrak{f}^i$ *and there is an* $A\in\mathfrak{f}^{-i}$ *with* $A\times B\subset U^i$. □

7.4. LEMMA. *Any non-Cauchy filter pair is concentrated.* □

7.5. LEMMA. *A Cauchy filter pair* $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ *is concentrated iff the following condition is fulfilled both for* $i=-1$ *and* $i=1$:

(1) *given* $S\in\mathfrak{f}^i$, *there is a* $U\in\mathcal{U}$ *such that* $x\in S$ *whenever* $x\in X$ *and* $U^{-i}x\in\mathfrak{f}^{-i}$.

PROOF. The sufficiency is evident (even without assuming that the filter pair is Cauchy).

To prove the necessity, choose U according to 7.3 (1); then the same U will do in (1), too. Indeed, let $U^{-i}x\in\mathfrak{f}^{-i}$; as $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ is Cauchy, we can choose an $M\in\mathfrak{f}^\times$ with $M\subset U$; now

$$M_{-i}\cap U^{-i}x\in\mathfrak{f}^{-i},\quad M_i\cup\{x\}\in\mathfrak{f}^i,\quad (M_{-i}\cap U^{-i}x)\times(M_i\cup\{x\})\subset U^i,$$

so it follows from 7.3 (1) that $M_i\cup\{x\}\subset S$, i.e. $x\in S$. □

7.6. DEFINITION. $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ is *weakly concentrated* if for any $U\in\mathcal{U}$ there is a $U_0\in\mathcal{U}$ such that $K, L\in\mathfrak{f}^\times$, $K, L\subset U_0$ imply $K_{-1}\times L_1\subset U$.

LEMMA. *Any non-Cauchy filter pair is weakly concentrated.* □

7.7. LEMMA. *A Cauchy filter pair* $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ *is weakly concentrated iff the following condition is satisfied:*

(1) *given a* $U\in\mathcal{U}$, *there is a* $U_0\in\mathcal{U}$ *such that* $x_{-1}\,U\,x_1$ *whenever* $U_0^i x_{-i}\in\mathfrak{f}^i$ $(i=\pm 1)$.

PROOF. Assume that U_0 is chosen according to Definition 7.6, and $U_0^i x_{-i}\in\mathfrak{f}^i$ $(i=\pm 1)$. Take $M\in\mathfrak{f}^\times$ such that $M\subset U_0$ and $M_i\subset U_0^i x_{-i}$ $(i=\pm 1)$. Let $K_{-1}=M_{-1}\cup\{x_{-1}\}$, $K_1=M_1$, $L_{-1}=M_{-1}$, $L_1=M_1\cup\{x_1\}$. □

7.8. The terminology is justified by

LEMMA. *Any concentrated filter pair is weakly concentrated.*

PROOF. By Lemma 7.6, we may suppose that $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ is Cauchy. Given a $U\in\mathcal{U}$, choose $M\in\mathfrak{f}^\times$ with $M\subset U$; take a $U_0\in\mathcal{U}$ according to Definition 7.3. Now if $K, L\in\mathfrak{f}^\times$, $K, L\subset U_0$ then $K, L\subset M$, i.e. $K_{-1}\subset M_{-1}$, $L_1\subset M_1$, thus $K_{-1}\times L_1\subset M\subset U$, showing that $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ is weakly concentrated. □

7.9. LEMMA. a) *Any linked filter pair is weakly concentrated; in fact, the condition in Definition 7.6 is fulfilled whenever* $U_0^2\subset U$.

b) *Any round linked filter pair is concentrated. In particular, neighbourhood filter pairs are concentrated.*

PROOF. a) Given $U\in\mathcal{U}$, take $U_0\in\mathcal{U}$ such that $U_0^2\subset U$. Now if K and L are as in Definition 7.6, take an $x\in K_{-1}\cap K_1\cap L_{-1}\cap L_1$; from $K_{-1}\subset U_0^{-1}x$ and $L_1\subset U_0x$ we have $K_{-1}\times L_1\subset U$.

b) Assume that $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ is Cauchy (Lemma 7.4). Let i and $S \in \mathfrak{f}^i$ be fixed. As $\mathfrak{f}^i$ is $\mathcal{U}^i$-round, there is an $S_0 \in \mathfrak{f}^i$ and a $U \in \mathcal{U}$ with $U^i[S_0] \subset S$. This U will do in 7.5 (1). Indeed: if $U^{-i}x \in \mathfrak{f}^{-i}$ then take a $y \in U^{-i}x \cap S_0$; now $x \in U^i y \subset U^i[S_0] \subset S$. □

REMARK. Since all the filter pairs occurring in the theory of quasi-uniform extensions are Cauchy, 7.5 (1) and 7.7 (1) could have been chosen as the definition of a concentrated, respectively a weakly concentrated filter pair. Lemma 7.9 would then remain valid, but not Lemma 7.8.

EXAMPLES. Round Cauchy filter pairs that are

a) *Not weakly concentrated.* $(\mathfrak{f}^{-1}((0, 0)), \mathfrak{f}^1((0, 0)))$ in Example 4.2.

b) *Weakly concentrated but not concentrated.* Take

(1) $$\mathfrak{f}^i = \text{fil}_X \{]0, i\varepsilon[: \varepsilon > 0\} \quad (i = \pm 1)$$

in $(X, \mathcal{U}) = (\mathbf{R}, \mathcal{U}_{s0})$ (7.5 (1) does not hold for $S =]0, 1[\in \mathfrak{f}^1$, since for any $U \in \mathcal{U}$, $U^{-1}0 \in \mathfrak{f}^{-1}$, and yet $0 \notin S$).

c) *Concentrated but not linked.* The filter pair defined by (1) in $(X, \mathcal{U}) = (\mathbf{R} \setminus \{0\}, \mathcal{U}_{s0} | \mathbf{R} \setminus \{0\})$.

7.10. Let us introduce a partial order $\leqq$ between filter pairs as follows:

$$(\mathfrak{g}^{-1}, \mathfrak{g}^1) \leqq (\mathfrak{f}^{-1}, \mathfrak{f}^1) \quad \text{iff} \quad \mathfrak{g}^{-1} \subset \mathfrak{f}^{-1} \quad \text{and} \quad \mathfrak{g}^1 \subset \mathfrak{f}^1,$$

i.e. iff $\mathfrak{g}^\times \subset \mathfrak{f}^\times$. If this condition is fulfilled then we shall say that $(\mathfrak{g}^{-1}, \mathfrak{g}^1)$ is *smaller* than (or is *contained* by) $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ (although it would be perhaps more natural to use the word "coarser").

LEMMA. a) *If a filter pair is (weakly) concentrated then any smaller filter pair has the same property.*

b) *If a filter pair is Cauchy then any larger filter pair is Cauchy, too.* □

7.11. NOTATION.

(1) $$M_i(U) = M_i(U, \mathfrak{f}^{-1}, \mathfrak{f}^1) = \bigcup \{K_i : K \in \mathfrak{f}^\times, K \subset U\}.$$

LEMMA. *If $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ is weakly concentrated and Cauchy then $(\mathfrak{g}^{-1}, \mathfrak{g}^1)$ is the smallest Cauchy filter pair contained by $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$, where*

(2) $$\mathfrak{g}^i = \text{fil} \{M_i(U) : U \in \mathcal{U}\} \qquad (i = \pm 1).$$

PROOF. 1° For $U \in \mathcal{U}$ fixed, there is a $K \in \mathfrak{f}^\times$ such that $K \subset U$ (because $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ is Cauchy), so $M_i(U) \supset K_i \in \mathfrak{f}^i$, hence $\{M_i(U) : U \in \mathcal{U}\}$ is a filter subbase (in fact, a filter base), and $(\mathfrak{g}^{-1}, \mathfrak{g}^1) \leqq (\mathfrak{f}^{-1}, \mathfrak{f}^1)$.

2° If U_0 is chosen according to Definition 7.6 then $M_{-1}(U_0) \times M_1(U_0) \subset U$, thus $(\mathfrak{g}^{-1}, \mathfrak{g}^1)$ is Cauchy.

3° Let $(\mathfrak{h}^{-1}, \mathfrak{h}^1)$ be another Cauchy filter pair contained by $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$, and $S \in \mathfrak{g}^i$ (with i fixed). Pick $U \in \mathcal{U}$ such that $M_i(U) \subset S$. Since $(\mathfrak{h}^{-1}, \mathfrak{h}^1)$ is Cauchy, there is a $K \in \mathfrak{h}^\times$ such that $K \subset U$. Now $K \in \mathfrak{f}^\times$, so $K_i \subset M_i(U) \subset S$, therefore $S \in \mathfrak{h}^i$, showing that $\mathfrak{g}^i \subset \mathfrak{h}^i$. □

7.12. DEFINITION. If $\mathfrak{f}$ is a filter and $i = \pm 1$ then define

$$\mathfrak{f}^{\circ(i)} = \{U^i[S] : S \in \mathfrak{f}, U \in \mathcal{U}\}$$

(cf. [10] 1.9); for a filter pair $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$, let

$$(\mathfrak{f}^{-1}, \mathfrak{f}^1)^\circ = ((\mathfrak{f}^{-1})^{\circ(-1)}, (\mathfrak{f}^1)^{\circ(1)}).$$

We shall write $\mathfrak{f}^{\circ\times}$ for $\mathfrak{f}^\times((\mathfrak{f}^{-1}, \mathfrak{f}^1)^\circ)$.

LEMMA. a) $(\mathfrak{f}^{-1}, \mathfrak{f}^1)^\circ$ *is the largest round filter pair smaller than* $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$. *In particular,* $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ *is round iff it is equal to* $(\mathfrak{f}^{-1}, \mathfrak{f}^1)^\circ$.

b) *If* $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ *is Cauchy then so is* $(\mathfrak{f}^{-1}, \mathfrak{f}^1)^\circ$.

c) *Any minimal Cauchy filter pair is round.* □

EXAMPLES. a) The round Cauchy filter pair in Example 7.9 b) is not minimal Cauchy.

b) *A Cauchy filter pair that does not contain a minimal Cauchy filter pair*. We take essentially the same filter pair as in Example 7.9 a), but eliminate the superfluous part of the fundamental set in order to simplify the notations; that is to say, let $X=\mathbf{R}\setminus\{0\}$, $\mathcal{U}=\mathcal{U}(d)$ where

$$d(x, y) = \sqrt{-xy} \quad \textit{if} \quad x < 0 < y.$$

Define $\mathfrak{f}^i$ by 7.9 (1). $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ is clearly Cauchy.

Assume that $(\mathfrak{g}^{-1}, \mathfrak{g}^1)$ is a Cauchy filter pair smaller than $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$; we are going to show that $(\mathfrak{g}^{-1}, \mathfrak{g}^1)$ cannot be minimal Cauchy. $\mathfrak{g}^{-1}\subsetneqq\mathfrak{f}^{-1}$ and $\mathfrak{g}^1\subsetneqq\mathfrak{f}^1$ do not hold simultaneously, since they would imply the existence of an $\varepsilon>1$ with $]0, i\varepsilon[\notin\mathfrak{g}^i$ $(i=\pm 1)$, contradicting the Cauchy property. So we may assume that, say, $\mathfrak{g}^{-1}=\mathfrak{f}^{-1}$. As $(\mathfrak{f}^{-1}, \mathfrak{g}^1)$ is Cauchy, there has to be a bounded element of $\mathfrak{g}^1$, i.e. there is a $t>0$ such that $]0, t[\in\mathfrak{g}^1$. Now with

$$\mathfrak{h}^1 = \{S\cup]0, t+1[\colon S\in\mathfrak{g}^1\},$$

$(\mathfrak{f}^{-1}, \mathfrak{h}^1)$ is a Cauchy filter pair strictly smaller than $(\mathfrak{f}^{-1}, \mathfrak{g}^1)=(\mathfrak{g}^{-1}, \mathfrak{g}^1)$.

c) *A Cauchy filter pair that contains more than one minimal Cauchy filter pair.* Let everything be as above, but with $X=]-1, 0[\cup]0, 1[$. Now $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ contains two minimal Cauchy filter pairs, namely $(\mathfrak{m}^{-1}, \mathfrak{f}^1)$ and $(\mathfrak{f}^{-1}, \mathfrak{m}^1)$ where $\mathfrak{m}^i=\text{fil}\,\{]0, i[\}$.

7.13. LEMMA. *A filter pair is concentrated Cauchy iff it is weakly concentrated and minimal Cauchy.*

PROOF. By Lemmas 7.8 and 7.11, it is enough to see that a Cauchy filter pair $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ is concentrated iff $\mathfrak{f}^i\subset\mathfrak{g}^i$ $(i=\pm 1)$ where $\mathfrak{g}^i$ is defined by 7.11 (2). This is, however, nothing but a rewording of Definition 7.3.[1] □

REMARK. By Lemmas 7.11 and 7.13, any weakly concentrated Cauchy filter pair contains exactly one concentrated Cauchy filter pair, which can be given by 7.11 (1)—(2).

COROLLARY. *Every concentrated Cauchy filter pair is round.* □

[1] The Cauchy property has to be assumed here, since otherwise 7.11 (2) would not define a filter.

EXAMPLE. *A minimal Cauchy filter pair that is not (weakly) concentrated.* Let

$$X =]0, 1[\times\{-1, 0, 1\}, \quad \mathcal{U} = \mathcal{U}(d),$$

$$d((s, n), (t, k)) = \begin{cases} s+t & \text{if} \quad n=-1, \ k=1, \\ t & \text{if} \quad s<t, \ n=-1, \ k=0, \end{cases}$$

$$\mathfrak{f}^i = \text{fil}\,\{]0, \varepsilon[\times\{i\}: \ 0<\varepsilon<1\} \quad (i=\pm 1).$$

Now $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ is a minimal Cauchy filter pair, but it is not concentrated, since 7.5 (1) does not hold for $S=]0, 1[\times\{1\}\in\mathfrak{f}^1$. Indeed,

$$U_{(\varepsilon)}^{-1}(\varepsilon/2, 0) \supset]0, \varepsilon/2[\times\{-1\}\in\mathfrak{f}^{-1},$$

but $(\varepsilon/2, 0)\notin S$.

7.14. LEMMA. *The following conditions are equivalent for a filter pair:*
(i) *it is linked round Cauchy;*
(ii) *it is linked minimal Cauchy;*
(iii) *it is minimal linked Cauchy.*

PROOF. (i)⇒(ii). Lemmas 7.9 b) and 7.13.
(ii)⇒(iii). Evident.
(iii)⇒(ii). If a filter pair is linked then so is any smaller filter pair.
(ii)⇒(i). Lemma 7.12 c). □

7.15. DEFINITION. A family $\mathfrak{F}$ of filter pairs is
a) *uniformly weakly concentrated* if for any $U\in\mathcal{U}$ there is a $U_0\in\mathcal{U}$ such that $(\mathfrak{f}^{-1}, \mathfrak{f}^1)\in\mathfrak{F}$, $K, L\in\mathfrak{f}^\times$, $K, L\subset U_0$ imply $K_{-1}\times L_1\subset U$;
b) *uniformly concentrated* if it is uniformly weakly concentrated, and each element of it is concentrated.

LEMMA. a) *If $\mathfrak{F}$ is uniformly weakly concentrated then each element of $\mathfrak{F}$ is weakly concentrated.*
b) *A family $\mathfrak{F}$ of Cauchy filter pairs is uniformly weakly concentrated iff*

(1) *given a $U\in\mathcal{U}$, there is a $U_0\in\mathcal{U}$ such that $x_{-1}\,U\,x_1$ whenever $U_0^i x_{-i}\in\mathfrak{f}^i$ $(i=\pm 1)$ holds for some $(\mathfrak{f}^{-1}, \mathfrak{f}^1)\in\mathfrak{F}$.*

PROOF. b) Just like the proof of Lemma 7.7. □

EXAMPLE. *A family of concentrated Cauchy filter pairs that is not uniformly concentrated.* Let $X=(\mathbf{R}\setminus\{0\})\times\mathbf{N}$, $\mathcal{U}=\mathcal{U}(d)$,

$$d((s, n), (t, k)) = (t-s)^n \quad \text{if} \quad n=k, \quad s<0<t,$$

(2) $$\mathfrak{f}_n^i = \text{fil}\,\{]0, \varepsilon i[\times\{n\}: \ \varepsilon>0\} \quad (i=\pm 1),$$

and take $\mathfrak{F} = \{(\mathfrak{f}_n^{-1}, \mathfrak{f}_n^1): n\in\mathbf{N}\}$.

7.16. LEMMA. *If $\mathfrak{F}_1$ is uniformly (weakly) concentrated, and each element of $\mathfrak{F}_2$ is linked and round [in particular, if $\mathfrak{F}_2$ is the system of the neighbourhood filter pairs] then $\mathfrak{F}_1\cup\mathfrak{F}_2$ is also uniformly (weakly) concentrated.*

PROOF. For each $U \in \mathscr{U}$, take a U_0 satisfying the condition in Definition 7.15 a) with $\mathfrak{F}_1$; if $U_0' \in \mathscr{U}$, $U_0' \subset U_0$ and $U_0'^2 \subset U$ then U_0' will satisfy the same condition with $\mathfrak{F}_1 \cup \mathfrak{F}_2$ (Lemma 7.9 a)). The elements of $\mathfrak{F}_2$ are concentrated (Lemma 7.9 b)). □

7.17. DEFINITION. a) [20, 21, 9]. A filter $\mathfrak{f}$ is *$\mathscr{U}$-stable* if for any $U \in \mathscr{U}$ there is an $S \in \mathfrak{f}$ such that $S \subset U[A]$ whenever $A \in \mathfrak{f}$.
b) $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ is *stable* if $\mathfrak{f}^i$ is $\mathscr{U}^i$-stable $(i = \pm 1)$.

LEMMA. *Linked Cauchy filter pairs are stable.* □

EXAMPLES. a) *A concentrated Cauchy filter pair that is not stable.* Take the filter pair $(\mathfrak{f}^{-1}((0, 0)), \mathfrak{f}^1((0, 0)))$ from Example 3.3. (To obtain a simpler example, consider the trace of the quasi-uniformity and of the filter pair on $(\mathbf{R} \setminus \{0\}) \times \{0\}$.)
b) *A stable Cauchy filter pair that is not weakly concentrated.* Let $X = (\mathbf{R} \setminus \{0\}) \times \times \{0, 1\}$, $\mathscr{U} = \mathscr{U}(d)$,

$$d((s, n), (t, k)) = t - s \quad \text{if either} \quad n + k < 2, \quad s < 0 < t$$
$$\text{or} \quad n = k = 0, \quad s < t,$$

and take $(\mathfrak{f}_0^{-1}, \mathfrak{f}_0^1)$ defined by 7.15 (2). With $x_i = (i\varepsilon/2, 1)$ we have $U_{(\varepsilon)}^i x_{-i} \in \mathfrak{f}_0^i$, but $x_{-1} U_{(1)} x_1$ does not hold, so it follows from Lemma 7.7 that $(\mathfrak{f}_0^{-1}, \mathfrak{f}_0^1)$ is not weakly concentrated.
c) The Cauchy filter pair in Example 7.9 c) is concentrated as well as stable, and yet it is not linked.

7.18. LEMMA (partly contained by [9] 4.6). $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ *is stable iff* $(\mathfrak{f}^{-1}, \mathfrak{f}^1)^\circ$ *is stable.* □

§ 8. Extensions preserving the weight

The most natural way of constructing an extension of $(X, \mathscr{U})$ of the same weight is the following: assign to each $U \in \mathscr{U}$ an entourage $\tilde{U}$ on $Y \supset X$ such that $\{\tilde{U}: U \in \mathscr{U}\}$ is a base for an extension $\tilde{\mathscr{U}}$ of $\mathscr{U}$ (but $\tilde{U}|X = U$ is not required to hold), and

$$V \subset U \Rightarrow \tilde{V} \subset \tilde{U}. \tag{1}$$

Indeed, (1) guarantees $w(\tilde{\mathscr{U}}) = w(\mathscr{U})$, since $\{\tilde{U}: U \in \mathscr{B}\}$ will be a base for $\tilde{\mathscr{U}}$ if $\mathscr{B}$ is a base for $\mathscr{U}$. (The extension constructed in the proof of Theorem 6.3 is not of this type.)

8.1. In case of uniformities, the extension mentioned in 0.9 b) has the above property; $\tilde{U}$ can be defined in several slightly different ways (the different constructions do not assign the same entourage to U, but they yield the same uniformity $\tilde{\mathscr{U}}$):
(i) $a \tilde{U} b$ iff $A \in \mathfrak{f}(a)$ and $B \in \mathfrak{f}(b)$ imply $U[A] \cap B \neq \emptyset$;
(ii) $a \tilde{U} b$ iff there are $A \in \mathfrak{f}(a)$ and $B \in \mathfrak{f}(b)$ such that $A \times A \subset U$, $B \times B \subset U$ and $A \cap B \neq \emptyset$;
(iii) $a \tilde{U} b$ iff there are $A \in \mathfrak{f}(a)$ and $B \in \mathfrak{f}(b)$ such that $A \times A \subset U$, $B \times B \subset U$ and $U[A] \cap B \neq \emptyset$;

(iv) $a\,\tilde{U}\,b$ iff there are $A\in\mathfrak{f}(a)$ and $B\in\mathfrak{f}(b)$ such that $A\times B\subset U$;

(v) $a\,\tilde{U}\,b$ iff there is an $S\in\mathfrak{f}(a)\cap\mathfrak{f}(b)$ such that $S\times S\subset U$.

See e.g. [1, 2, 3, 8, 27, 29]. We shall try to generalize these constructions to the asymmetrical case; for this purpose, observe that the following more complicated version of (ii) [and an analogous modification of (iii)] could also be used:

(ii)′ $a\,\tilde{U}\,b$ iff there are $A, A'\in\mathfrak{f}(a)$ and $B, B'\in\mathfrak{f}(b)$ such that $A\times A'\subset U$, $B\times B'\subset U$ and $A\cap B\neq\emptyset$.

8.2. Let us be given now a quasi-uniform space $(X, \mathcal{U})$, a set $Y\supset X$, and round Cauchy trace filter pairs $(\mathfrak{f}^{-1}(a), \mathfrak{f}(a))$ for $a\in Y$. (From now on, trace filter pairs will always be assumed to be round and Cauchy.)

Definition. Assign to $U\in\mathcal{U}$ relations ${}^kU$ $(1\leqq k\leqq 4)$ on Y as follows:

$a\,{}^1U\,b$ iff either $a=b$ or for any $A\in\mathfrak{f}^1(a)$ and $B\in\mathfrak{f}^{-1}(b)$ we have $U[A]\cap\cap B\neq\emptyset$;

$a\,{}^2U\,b$ iff either $a=b$ or there are $K\in\mathfrak{f}^\times(a)$ and $L\in\mathfrak{f}^\times(b)$ such that $K, L\subset U$ and $K_1\cap L_{-1}\neq\emptyset$;

$a\,{}^3U\,b$ iff either $a=b$ or there are $K\in\mathfrak{f}^\times(a)$ and $L\in\mathfrak{f}^\times(b)$ such that $K, L\subset U$ and $U[K_1]\cap L_{-1}\neq\emptyset$;

$a\,{}^4U\,b$ iff there are $A\in\mathfrak{f}^{-1}(a)$ and $B\in\mathfrak{f}^1(b)$ such that $A\times B\subset U$.

Let ${}^k\mathcal{U}={}^k\mathcal{U}(\mathfrak{f}^{-1}, \mathfrak{f}^1)=\mathrm{fil}_{Y\times Y}\,\{{}^kU\colon U\in\mathcal{U}\}$ $(1\leqq k\leqq 4)$.

Lemma. *For* $k=1, 2, 3, 4$ *we have:*

If $U\in\mathcal{U}$ *then* ${}^kU$ *is an entourage on* Y. *If* $V\subset U$ *then* ${}^kV\subset{}^kU$. ${}^k\mathcal{U}$ *is a quasi-semiuniformity on* Y. $\{{}^kU\colon U\in\mathcal{B}\}$ *is a base for* ${}^k\mathcal{U}$ *if* $\mathcal{B}$ *is a base for* $\mathcal{U}$. $w({}^k\mathcal{U})=$ $=w(\mathcal{U})$.

Proof. ${}^4U$ is reflexive because the trace filter pairs are Cauchy. □

Remarks. a) There seems to be no sensible generalization of the very simple construction in 8.1 (v).

b) If we had not disposed of the case $a=b$ separately in the definition of ${}^kU$ $(k=1, 2, 3)$ then ${}^kU$ would not be an entourage; this can be easily seen in Example 3.5 with $\mathcal{U}_s=\mathcal{U}_{s0}$.

c) It will be of help in the proofs if we observe that, with ${}^kU^i$ denoting $({}^kU)^i$, the following statements hold for $i=\pm 1$:

$a\,{}^1U^i\,b$ iff either $a=b$ or for any $A\in\mathfrak{f}^i(a)$ and $B\in\mathfrak{f}^{-i}(b)$ we have $U^i[A]\cap\cap B\neq\emptyset$;

$a\,{}^2U^i\,b$ iff either $a=b$ or there are $K\in\mathfrak{f}^\times(a)$ and $L\in\mathfrak{f}^\times(b)$ such that $K, L\subset U$ and $K_i\cap L_{-i}\neq\emptyset$;

$a\,{}^3U^i\,b$ iff either $a=b$ or there are $K\in\mathfrak{f}^\times(a)$ and $L\in\mathfrak{f}^\times(b)$ such that $K, L\subset U$ and $U^i[K_i]\cap L_{-i}\neq\emptyset$;

$a\,{}^4U^i\,b$ iff there are $A\in\mathfrak{f}^{-i}(a)$ and $B\in\mathfrak{f}^i(b)$ such that $A\times B\subset U^i$.

8.3. Lemma. *If* $U, V\in\mathcal{U}$ *and* $V^2\subset U$ *then* ${}^1U\subset{}^3U$, ${}^2U\subset{}^3U$, ${}^3V\subset{}^2U$, ${}^2V\subset{}^4U$. *Consequently,* ${}^1\mathcal{U}\supset{}^2\mathcal{U}={}^3\mathcal{U}\supset{}^4\mathcal{U}$.

Proof. ${}^1U\subset{}^3U$ because the trace filter pairs are Cauchy. ${}^2U\subset{}^3U$ is evident. If $a\,{}^3V\,b$, $a\neq b$, and K and L are chosen according to the definition of ${}^3V$ then $K_{-1}\times V[K_1]\subset U$ and $L\subset U$, i.e. the conditions in the definition of ${}^2U$ are fulfilled with $V[K_1]$ substituted for K_1. Hence ${}^3V\subset{}^2U$.

If $a\,{}^2V\,b$, $a\neq b$, and K and L are chosen according to the definition of ${}^2V$ then with $A=K_{-1}$ and $B=L_1$ we have $A\in\mathfrak{f}^{-1}(a)$, $B\in\mathfrak{f}^1(b)$ and $A\times B\subset V^2\subset U$. Hence ${}^2V\subset{}^4U$. □

Because of the equality ${}^2\mathcal{U}={}^3\mathcal{U}$, we shall only consider ${}^1\mathcal{U}$, ${}^2\mathcal{U}$ and ${}^4\mathcal{U}$.

8.4. Lemma. ${}^0\mathcal{U}\subset{}^1\mathcal{U}$.

Proof. With the notation of 3.1 (2), ${}^1U\subset V(f^{-1},f^1,U)$. □

8.5. Lemma. a) *If* $U\in\mathcal{U}$ *then* ${}^4U|X\subset U\subset{}^1U|X$.
b) *Consequently*, ${}^k\mathcal{U}|X=\mathcal{U}$ $(k=1,2,4)$.

Proof. a) Recall that the trace filter pairs of the points of X coincide with the neighbourhood filter pairs.
b) Apply a) and Lemma 8.3. □

8.6. Lemma. ${}^1\mathcal{U}$ *is a quasi-uniformity.*

Proof. Let $U,V\in\mathcal{U}$, $V^3\subset U$; we claim that $({}^1V)^2\subset{}^1U$.
Assume $a\,{}^1V\,b\,{}^1V\,c$. The case $a=b$ or $b=c$ being trivial, let $a\neq b\neq c$. Choose $K\in\mathfrak{f}^\times(b)$ with $K\subset V$. Now if $A\in\mathfrak{f}^1(a)$ and $C\in\mathfrak{f}^{-1}(c)$ then $V[A]\cap K_{-1}\neq\emptyset\neq$ $\neq V[K_1]\cap C$, thus $U[A]\cap C\neq\emptyset$. Hence $a\,{}^1U\,c$. □

8.7. Lemma. ${}^1\mathcal{U}$ *is compatible with the prescribed trace filter pairs iff they are stable.*

Proof. Denote by $(\mathfrak{g}^{-1}(a),\mathfrak{g}^1(a))$ $(a\in Y)$ the trace filter pairs induced by the extension ${}^1\mathcal{U}$. It follows from Lemma 8.4 and Theorem 6.1 that $\mathfrak{f}^i(a)\subset\mathfrak{g}^i(a)$. Consequently, it is enough to prove the following:

$$\mathfrak{g}^i(a)\subset\mathfrak{f}^i(a) \quad \text{iff} \quad \mathfrak{f}^i(a) \quad \text{is } \mathcal{U}^i\text{-stable}.$$

1° Assume first that $\mathfrak{f}^i(a)$ is $\mathcal{U}^i$-stable, and $S\in\mathfrak{g}^i(a)$. Take a $U\in\mathcal{U}$ such that ${}^1U^i a\cap X\subset S$. As $\mathfrak{f}^i(a)$ is $\mathcal{U}^i$-stable, there is a $T\in\mathfrak{f}^i(a)$ such that $T\subset U^i[A]$ whenever $A\in\mathfrak{f}^i(a)$. In order to prove $S\in\mathfrak{f}^i(a)$, it is enough to show that $T\subset S$.
Let $x\in T$, $A\in\mathfrak{f}^i(a)$ and $B\in\mathfrak{f}^{-i}(x)$. Now $T\subset U^i[A]$ implies $x\in U^i[A]$, i.e. $U^i[A]\cap B\neq\emptyset$. This means that $a\,{}^1U^i\,x$, therefore $x\in S$.
2° To prove the converse, assume now that $\mathfrak{f}^i(p)$ is not $\mathcal{U}^i$-stable for some i and $p\in Y\setminus X$ (the trace filter pairs belonging to the points of X are always stable by Lemma 7.17). We have to produce an element of $\mathfrak{g}^i(p)\setminus\mathfrak{f}^i(p)$.
Choose a $U\in\mathcal{U}$ such that for any $S\in\mathfrak{f}^i(p)$ there is an $S_0\in\mathfrak{f}^i(p)$ with

$$(1)\qquad S\not\subset U^i[S_0].$$

Take a $V\in\mathcal{U}$ with $V^2\subset U$. Clearly, ${}^1V^i p\cap X\in\mathfrak{g}^i(p)$. On the other hand, the assumption $S={}^1V^i p\cap X\in\mathfrak{f}^i(p)$ will lead to a contradiction.
Indeed, choose S_0 to this S as above. If $x\in S$ then $p\,{}^1V^i\,x$ and $p\neq x$, thus for any $A\in\mathfrak{f}^i(p)$ and $B\in\mathfrak{f}^{-i}(x)$ we have

$$(2)\qquad V^i[A]\cap B\neq\emptyset.$$

With $A=S_0$ and $B=V^{-i}x$, (2) implies $x\in U^i[S_0]$, contradicting (1). □

THEOREM. *${}^1\mathcal{U}$ is an extension of $\mathcal{U}$ for the prescribed trace filter pairs iff they are stable. If so then ${}^1\mathcal{U}={}^0\mathcal{U}$ (thus ${}^1\mathcal{U}$ is compatible with the fine regular extension) and $w({}^1\mathcal{U})=w(\mathcal{U})$.*

PROOF. The first statement follows from Lemmas 8.5 b), 8.6 and 8.7. ${}^0\mathcal{U}\subset{}^1\mathcal{U}$ by Lemma 8.4; ${}^1\mathcal{U}\subset{}^0\mathcal{U}$ by Theorem 6.1. The last statement is contained by Lemma 8.2. □

COROLLARY. *If the trace filter pairs are stable then $w({}^0\mathcal{U})=w(\mathcal{U})$.* □

REMARK. $w({}^0\mathcal{U})=w(\mathcal{U})$ may hold without the trace filter pairs being stable. (Modify Example 3.3, taking one new point only.)

8.8. LEMMA. *If $K\in\mathfrak{k}^\times(a)$, $K\subset U\in\mathcal{U}$ and $x\in K_i$ then $a\,{}^2U^i x$.*

PROOF. Choose a $V\in\mathcal{U}$ with $V^2\subset U$. Let $L_j=V^jx$ $(j=\pm1)$. Now $L\in\mathfrak{k}^\times(x)$, $L\subset U$ and $x\in K_i\cap L_{-i}$, so $a\,{}^2U^i x$. □

8.9. NOTATIONS. Denote by $\mathfrak{h}^i_k(a)$ the filter $\{W^ia\colon W\in{}^k\mathcal{U}\}$ $(i=\pm1,\ k=2,4,\ a\in Y)$; in other words,

$$\mathfrak{h}^i_k(a)=\mathrm{fil}_Y\,\{{}^kU^ia\colon U\in\mathcal{U}\}. \tag{1}$$

$(\mathfrak{s}^{-1}(a),\mathfrak{s}^1(a))$ is the neighbourhood filter pair of a in the doubly strict extension associated with the given trace filter pairs.

LEMMA. a) $\mathfrak{h}^i_4(a)|X\subset\mathfrak{h}^i_2(a)|X\subset\mathfrak{k}^i(a)$.
b) $\mathfrak{h}^i_4(a)\subset\mathfrak{s}^i(a)$.

PROOF. a) The first inclusion follows from ${}^4\mathcal{U}\subset{}^2\mathcal{U}$ (Lemma 8.3). To prove the second one, take an $S\in\mathfrak{h}^i_2(a)|X$; choose a $U\in\mathcal{U}$ such that ${}^2U^ia\cap X\subset S$. Taking a $K\in\mathfrak{k}^\times(a)$ with $K\subset U$, we have $K_i\subset{}^2U^ia$ (Lemma 8.8), so $K_i\subset S$, i.e. $S\in\mathfrak{k}^i(a)$.
b) According to (1), it is enough to prove that

$${}^4U^ia\in\mathfrak{s}^i(a)\qquad(U\in\mathcal{U}). \tag{2}$$

For a and U fixed, take a $K\in\mathfrak{k}^\times(a)$ with $K\subset U$. Now (2) follows from

$$K_i\in\mathfrak{k}^i(b)\Rightarrow a\,{}^4U^ib, \tag{3}$$

(this is shown by the sets $A=K_{-i}$ and $B=K_i$), since (3) means

$$\{b\colon K_i\in\mathfrak{k}^i(b)\}\subset{}^4U^ia, \tag{4}$$

and the left-hand side of (4) belongs to $\mathfrak{s}^i(a)$, because $K_i\in\mathfrak{k}^i(a)$ (cf. 0.4 (2)). □

8.10. LEMMA. *The following are equivalent:*
(i) $\mathfrak{h}^i_2(a)|X=\mathfrak{k}^i(a)$ $(i=\pm1,\ a\in Y)$;
(ii) $h^i_4(a)|X=\mathfrak{k}^i(a)$ $(i=\pm1,\ a\in Y)$;
(iii) *${}^4\mathcal{U}$ is compatible with the doubly strict extension associated with the trace filter pairs* $((\mathfrak{k}^{-1}(a),(\mathfrak{k}^1(a)))$ $(a\in Y)$;
(iv) *the filter pairs* $(\mathfrak{k}^{-1}(a),\mathfrak{k}^1(a))$ *are concentrated.*

PROOF. (iv)⇒(iii). According to Lemma 8.9 b), it is enough to prove that

(1) $$\mathfrak{s}^i(a) \subset \mathfrak{h}_4^i(a).$$

Given an $S\in\mathfrak{f}^i(a)$, take a $U\in\mathcal{U}$ satisfying 7.3 (1). Now (1) will follow from

(2) $${}^4U^i a \subset \{b\colon S\in\mathfrak{f}^i(b)\},$$

because the sets on the right-hand side of (2) form a base for $\mathfrak{s}^i(a)$, and ${}^4U^i a\in\mathfrak{h}_4^i(a)$.

To prove (2), assume $a\,{}^4U^i\,b$. Then there are $A\in\mathfrak{f}^{-i}(a)$ and $B\in\mathfrak{f}^i(b)$ with $A\times B\subset U^i$. By 7.5 (1) we have $B\subset S$, thus $S\in\mathfrak{f}^i(b)$.

(iii)⇒(ii). Evident.

(ii)⇒(i). Lemma 8.9 a).

(i)⇒(iv). Take an $S\in\mathfrak{f}^i(a)$ and choose $U\in\mathcal{U}$ such that ${}^2U^i a\cap X\subset S$. Now if $K\in\mathfrak{f}^\times(a)$ and $K\subset U$ then, by Lemma 8.8, $K_i\subset{}^2U^i a$, thus $K_i\subset S$. This means that 7.3 (1) is satisfied, so $(\mathfrak{f}^{-1}(a), \mathfrak{f}^1(a))$ is concentrated by Lemma 7.3. □

8.11. LEMMA. *${}^2\mathcal{U}$ is a quasi-uniformity iff the trace filter pairs are uniformly weakly concentrated.*

PROOF. *Sufficiency.* For $U\in\mathcal{U}$, take a $U_0\subset U$ such that the condition in Definition 7.15 a) is fulfilled. We are going to show that

(1) $$({}^2U_0)^2 \subset {}^3U.$$

Assume $a\,{}^2U_0\,b\,{}^2U_0\,c$. If $a=b$ or $b=c$ then $a\,{}^2U_0\,c$, thus $a\,{}^3U\,c$ follows from $U_0\subset U$ (so ${}^2U_0\subset{}^2U$) and ${}^2U\subset{}^3U$. If $a\neq b\neq c$ then pick $J\in\mathfrak{f}^\times(a)$, $K, L\in\mathfrak{f}^\times(b)$ and $M\in\mathfrak{f}^\times(c)$ such that $J, K, L, M\subset U_0$ and $J_1\cap K_{-1}\neq\emptyset\neq L_1\cap M_{-1}$. Now $K_{-1}\times L_1\subset U$, thus $U[J_1]\cap M_{-1}\neq\emptyset$. We have also $J, M\subset U_0\subset U$, therefore $a\,{}^3U\,c$.

It follows from (1) and Lemma 8.3 that ${}^2\mathcal{U}$ is a quasi-uniformity.

Necessity. Given a $U\in\mathcal{U}$, choose first a $V\in\mathcal{U}$ with $V^2\subset U$, then a $U_0\in\mathcal{U}$ such that $({}^2U_0)^2\subset{}^2V$. We claim that the condition in Definition 7.15 a) is satisfied with this U_0.

Let $K, L\in\mathfrak{f}^\times(a)$, $K, L\subset U_0$. Now if $x\in K_{-1}$ and $y\in L_1$ then $x\,{}^2U_0\,a\,{}^2U_0\,y$ by Lemma 8.8, thus $x\,{}^2V\,y$. Therefore $x\,{}^4U\,y$ (Lemma 8.3), and $x\,U\,y$ (Lemma 8.5 a)). This means that $K_{-1}\times L_1\subset U$. □

THEOREM. *${}^2\mathcal{U}$ is an extension of $\mathcal{U}$ for the prescribed trace filter pairs iff they are uniformly concentrated.* $w({}^2\mathcal{U})=w(\mathcal{U})$.

PROOF. Lemmas 8.11, 8.10, 8.5 b) and 8.2. □

EXAMPLE. *${}^2\mathcal{U}$ is not always compatible with the fine regular extension.* Let

$$p_n = \frac{1}{2^n},\quad C_n = \left]p_n - \frac{1}{2^{n+2}},\, p_n\right[,\quad D_n = \left]p_n,\, p_n + \frac{1}{2^{n+2}}\right[,$$

$$X = \{0\}\cup\bigcup_{n\in\mathbf{N}}(C_n\cup D_n),\quad Y = X\cup\{p_n\colon n\in\mathbf{N}\},$$

$\mathscr{U}=\mathscr{U}(d)$ where $d(x,y)=y-x$ if there is an $n\in\mathbf{N}$ such that either $x,y\in C_n\cup D_n$, $x<y$, or $x=0$, $y\in D_n$. Let

$$\mathfrak{f}^i(p_n)=\mathrm{fil}_X\left\{]p_n, p_n+i\varepsilon[:\ 0<\varepsilon<\frac{1}{2^{n+2}}\right\}.$$

The trace filter pairs are uniformly concentrated as well as stable. The points p_n converge to 0 in ${}^2\mathscr{U}^{tp}$, but not in ${}^0\mathscr{U}^{tp}={}^1\mathscr{U}^{tp}$.

8.12. LEMMA. *If $\mathscr{V}$ is an extension of $\mathscr{U}$ for the prescribed trace filter pairs then ${}^4\mathscr{U}\subset\mathscr{V}$.*

PROOF. Given a $U\in\mathscr{U}$, we need a $V\in\mathscr{V}$ with $V\subset{}^4U$.

Choose first a $W\in\mathscr{V}$ such that $W|X=U$, then a $V\in\mathscr{V}$ with $V^3\subset W$. Assume $a\,V\,b$, and let $A=V^{-1}a\cap X$, $B=Vb\cap X$. As $\mathscr{V}$ is compatible with the given trace filter pairs, we have $A\in\mathfrak{f}^{-1}(a)$, $B\in\mathfrak{f}^1(b)$. Moreover, $A\times B\subset V^3\subset W$, thus (the sets A and B being in X) $A\times B\subset U$. Therefore $a\,{}^4U\,b$. □

We can now completely describe the family of the quasi-uniform extension for given *concentrated* trace filter pairs:

THEOREM. *The quasi-uniformity $\mathscr{V}$ is an extension of $\mathscr{U}$ compatible with some prescribed concentrated trace filter pairs iff ${}^4\mathscr{U}\subset\mathscr{V}\subset{}^0\mathscr{U}$.*

REMARK. This theorem does not assert that ${}^4\mathscr{U}$ itself is also an extension (i.e. that ${}^4\mathscr{U}$ is a quasi-uniformity), cf. Lemma 8.13.

PROOF. *Necessity.* Lemma 8.12 and Theorem 6.1.
Sufficiency. Lemma 8.5 b), Theorem 6.1 and Lemma 8.10. □

8.13. LEMMA. *${}^4\mathscr{U}$ is a quasi-uniformity iff the trace filter pairs are uniformly weakly concentrated.*

PROOF. *Sufficiency.* For $U\in\mathscr{U}$ take a U_0 satisfying 7.15 (1). It is enough to show that $({}^4U_0)^2\subset{}^4U$.

Assume $a\,{}^4U_0\,b\,{}^4U_0\,c$. Then there are $A\in\mathfrak{f}^{-1}(a)$, $K\in\mathfrak{f}^\times(b)$ and $C\in\mathfrak{f}^1(c)$ such that $A\times K_1\subset U_0$ and $K_{-1}\times C\subset U_0$. Now $A\times C\subset U$ follows from 7.15 (1). Therefore $a\,{}^4U\,c$.

Necessity. Let $U\in\mathscr{U}$, and choose a $V\in\mathscr{U}$ such that $({}^4V)^2\subset{}^4U$, then a $U_0\in\mathscr{U}$ with $U_0^2\subset V$. We claim that the condition in Definition 7.15 a) is satisfied with this U_0.

Let $K,L\in\mathfrak{f}^\times(a)$, $K,L\subset U_0$. If $x\in K_{-1}$ then $U_0^{-1}x\times K_1\subset U_0^2\subset V$, thus $x\,{}^4V a$; similarly, if $y\in L_1$ then $a\,{}^4V\,y$. Hence $x\,{}^4U\,y$, and $x\,U\,y$ by Lemma 8.5 a), i.e. $K_{-1}\times L_1\subset U$. □

THEOREM. *${}^4\mathscr{U}$ is an extension of $\mathscr{U}$ for the prescribed trace filter pairs iff they are uniformly concentrated. If so then ${}^4\mathscr{U}$ is the coarsest extension for these trace filter pairs; it is compatible with the doubly strict extension; $w({}^4\mathscr{U})=w(\mathscr{U})$.*

PROOF. Lemmas 8.13, 8.10, 8.5 b), 8.12 and 8.2. □

EXAMPLE. *${}^2\mathcal{U}$ and ${}^4\mathcal{U}$ can induce different bitopologies.* With $\mathcal{U}$ and $\mathfrak{f}^i(a)$ from Example 3.3, the trace filter pairs are uniformly concentrated, ${}^2\mathcal{U}$ induces the doubly loose extension, while ${}^4\mathcal{U}$ induces the doubly strict one (and these two bitopological extensions are now evidently different).

REMARKS. a) Let us observe (for an application in another paper) that if the trace filter pairs are only uniformly weakly concentrated then ${}^4\mathcal{U}$ is still an extension of $\mathcal{U}$, but it induces a bitopology coarser than the doubly strict extension associated with the given trace filter pairs [use Lemma 8.9 b) instead of Lemma 8.10].

b) Assume that the trace filter pairs are uniformly weakly concentrated, and let $(\mathfrak{g}^{-1}(a), \mathfrak{g}^1(a))$ be the concentrated Cauchy filter pair contained by $(\mathfrak{f}^{-1}(a), \mathfrak{f}^1(a))$ (cf. Remark 7.13). It is easy to check that the new filter pairs are uniformly concentrated. We are going to show that ${}^4\mathcal{U}(\mathfrak{f}^{-1}, \mathfrak{f}^1) = {}^4\mathcal{U}(\mathfrak{g}^{-1}, \mathfrak{g}^1)$.

1° For $U \in \mathcal{U}$, let ${}^4U(\mathfrak{f})$, respectively ${}^4U(\mathfrak{g})$ denote the entourage ${}^4U$ defined with $(\mathfrak{f}^{-1}(a), \mathfrak{f}^1(a))$ $(a \in Y)$, respectively with $(\mathfrak{g}^{-1}(a), \mathfrak{g}^1(a))$ $(a \in Y)$. As $\mathfrak{g}^i(a) \subset \mathfrak{f}^i(a)$, ${}^4U(\mathfrak{g}) \subset {}^4U(\mathfrak{f})$ follows directly from the definition of ${}^4U$.

2° Conversely, if U_0 is chosen for U according to 7.15 (1) and $U_{00} \subset U_0$ is chosen for U_0 in the same way then ${}^4U_{00}(\mathfrak{f}) \subset {}^4U(\mathfrak{g})$.

Indeed, assume that $a\,{}^4U_{00}(\mathfrak{f})\,b$. Then there are $A \in \mathfrak{f}^{-1}(a)$ and $B \in \mathfrak{f}^1(b)$ such that $A \times B \subset U_{00}$. To prove $a\,{}^4U(\mathfrak{g})\,b$, it is enough to show that (cf. 7.11):

$$M_{-1}(U_{00}, \mathfrak{f}^{-1}(a), \mathfrak{f}^1(a)) \times M_1(U_{00}, \mathfrak{f}^{-1}(b), \mathfrak{f}^1(b)) \subset U. \tag{1}$$

In order to prove (1), take a pair (x, y) from the left-hand side of it. Now there are $K \in \mathfrak{f}^\times(a)$ and $L \in \mathfrak{f}^\times(b)$ such that $K, L \subset U_{00}$, $x \in K_{-1}$ and $y \in L_1$. For any $z \in B$, we have $U_{00}^{-1}z \supset A \in \mathfrak{f}^{-1}(a)$ and $U_{00}x \supset K_1 \in \mathfrak{f}^1(a)$, thus $x\,U_0\,z$ $(z \in B)$ by 7.15 (1), i.e. $U_0 x \supset B \in \mathfrak{f}^1(b)$. On the other hand, $U_0^{-1}y \supset U_{00}^{-1}y \supset L_{-1} \in \mathfrak{f}^{-1}(b)$, hence $x\,U\,y$, again by 7.15 (1).

§ 9. Extensions of quasi-pseudometrics

This section contains a theorem on quasi-pseudometric extensions, in the proof of which we can use a theorem on quasi-uniform extensions. (The reverse process is more usual: e.g. the existence of a uniform completion is deduced in several textbooks from the existence of a metric completion, see [22, 21, 14].)

9.1. The filter pair $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ in the quasi-pseudometric space (X, d) is *Cauchy, round,* respectively *stable* if it has the same property in the quasi-uniform space $(X, \mathcal{U}(d))$, e.g. $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ is Cauchy iff for any $\varepsilon > 0$ there is a $K \in \mathfrak{f}^\times$ such that $d(x, y) < \varepsilon$ whenever $(x, y) \in K$.

NOTATIONS. $U_{[\varepsilon]}(d) = \{(x, y):\ d(x, y) \leqq \varepsilon\}$,

$$d(A, B) = \inf\{d(x, y):\ x \in A,\ y \in B\} \quad (A, B \subset X).$$

THEOREM. *Let (X, d) be a quasi-pseudometric space, $Y \supset X$, and let us be given stable round Cauchy trace filter pairs $(\mathfrak{f}^{-1}(a), \mathfrak{f}^1(a))$ $(a \in Y)$. Then*

$${}^1d(a, b) = \begin{cases} \sup\{d(A, B):\ A \in \mathfrak{f}^1(a),\ B \in \mathfrak{f}^1(b)\} & \text{if } a \neq b, \\ 0 & \text{if } a = b \end{cases}$$

defines a quasi-pseudometric extension of d compatible with the fine regular extension associated with the given trace filter pairs.

Moreover, if e is another quasi-pseudometric extension of d for the same trace filter pairs then $e \leqq {}^1d$.

PROOF. Using the stableness of the trace filter pairs, one can easily check that ${}^1d(a, b)$ is finite. A straightforward computation shows also that ${}^1d$ is a quasi-pseudometric and ${}^1d|X=d$. Observe that

$$ {}^1U_{[\varepsilon]}(d) = U_{[\varepsilon]}({}^1d), $$

so ${}^1d$ is compatible with the fine regular extension by Theorem 8.7. $e \leqq {}^1d$ follows easily from the requirement that e is to satisfy the Triangle Inequality. □

§ 10. Refined extensions

Matolcsy [26] has found a method for modifying a given extension of a quasi-uniformity (more generally, of a syntopogeneous structure) in order to obtain other (finer) extensions for the same trace filters. An analogous construction will yield new extensions from old ones for the same trace filter *pairs.*

10.1. Let $(X, \mathscr{U})$ be a quasi-uniform space. As a special case of a construction given in [26] for syntopogeneous spaces, define

$$ U*Q = \Delta_X \cup (U \cap (X \times Q)), $$

(1) $$ \mathscr{U}*\mathfrak{q} = \mathrm{fil}_{X\times X}\{U*Q\colon\ U\in\mathscr{U},\ Q\in\mathfrak{q}\} $$

where $Q \subset X$ and $\emptyset \neq \mathfrak{q} \subset \mathfrak{p}(X)$. Now $\mathscr{U}*\mathfrak{q}$ is a quasi-uniformity finer than $\mathscr{U}$; if $(Y, \mathscr{V})$ is an extension of $(X, \mathscr{U})$ and $X \subset Q$ for each $Q \in \mathfrak{q} \subset \mathfrak{p}(Y)$ then $\mathscr{V}*\mathfrak{q}$ is also an extension of $\mathscr{U}$ for the same trace filters ([26] 2.2 and 2.3).

Let us now describe the same construction in a more complicated way:

$$ U*\mathfrak{q} = \bigcap_{Q\in\mathfrak{q}} (U*Q), $$

(2) $$ \mathscr{U}*\mathfrak{Q} = \mathrm{fil}_{X\times X}\{U*\mathfrak{q}\colon\ U\in\mathscr{U},\ \mathfrak{q}\in\mathfrak{Q}\}, $$

where $\mathfrak{Q} \subset \mathfrak{p}(\mathfrak{p}(X))$, $\emptyset \notin \mathfrak{Q} \neq \emptyset$. (2) is not more general than (1), since

$$ U*\mathfrak{q} = U*\bigcap\mathfrak{q}; $$

nevertheless, this modification of (1) is not entirely pointless in the more general case of syntopogeneous structures (see in [13]), and our analogous construction (to be defined in 10.2) will be modelled on (2).

10.2. For $Q \subset X$, $\mathfrak{q} \subset \mathfrak{p}(X)$ and $\mathfrak{Q} \subset \mathfrak{p}(\mathfrak{p}(X))$ (with $\mathfrak{q} \neq \emptyset \notin \mathfrak{Q} \neq \emptyset$), define

$$ U**Q = \Delta_X \cup \{(x, y)\colon \exists w\in Q,\ x\,U\,w\,U\,y\}, $$

$$ U**\mathfrak{q} = \bigcap_{Q\in\mathfrak{q}} (U**Q), $$

$$ \mathscr{U}**\mathfrak{Q} = \mathrm{fil}_{X\times X}\{U**\mathfrak{q}\colon\ U\in\mathscr{U},\ \mathfrak{q}\in\mathfrak{Q}\}. $$

LEMMA. *If U is an entourage then so is* $U**\mathfrak{q}$, *and*

(1) $$U\cap((\cap\mathfrak{q}\times X)\cup(X\times\cap\mathfrak{q}))\subset U**\mathfrak{q}\subset U^2. \quad \square$$

10.3. LEMMA. *If* $V^3\subset U$ *then* $(V**\mathfrak{q})^3\subset U**\mathfrak{q}$. $\square$

10.4. LEMMA. *If* $\mathcal{U}$ *is a quasi-uniformity then* $\mathcal{U}**\mathfrak{Q}$ *is a quasi-uniformity finer than* $\mathcal{U}$.

PROOF. $\mathcal{U}**\mathfrak{Q}$ is a quasi-uniformity by Lemma 10.3; it is finer than $\mathcal{U}$ by the second inclusion in 10.2 (1). $\square$

10.5. LEMMA. *If* $R=\cap\cup\mathfrak{Q}$ *and* $Z=(R\times X)\cup(X\times R)$ *then* $(\mathcal{U}**\mathfrak{Q})(\cap)\{Z\}=$ $=\mathcal{U}(\cap)\{Z\}$. *In particular,* $(\mathcal{U}**\mathfrak{Q})|R=\mathcal{U}|R$.

PROOF. Lemma 10.2. $\square$

THEOREM. *If* $(Y,\mathcal{V})$ *is an extension of the quasi-uniform space* $(X,\mathcal{U})$, $\mathfrak{Q}\subset\mathfrak{p}(\mathfrak{p}(Y))$, $\emptyset\notin\mathfrak{Q}\neq\emptyset$, *and* $X\subset Q$ *for each* $Q\in\mathfrak{q}\in\mathfrak{Q}$ *then* $\mathcal{V}**\mathfrak{Q}$ *is another (finer) extension of* $\mathcal{U}$ *for the same trace filter pairs.*
$\mathcal{V}**\{\{X\}\}$ *is finer than* $\mathcal{V}**\mathfrak{Q}$. $\square$

10.6. LEMMA. *If* $U\in\mathcal{U}$ *then* ${}^4U|(Y\times X)\cup(X\times Y)\subset{}^2U$.

PROOF. Assume that $a\in Y$, $x\in X$ and $a\,{}^4U^i\,x$. Then there are $A\in\mathfrak{f}^{-i}(a)$ and $B\in\mathfrak{f}^i(x)$ with $A\times B\subset U^i$. Choose an $L\in\mathfrak{f}^\times(a)$ with $L\subset U$, and apply Lemma 8.8 to $K_{-i}=A\cap L_{-i}$ and $K_i=B\cup L_i$. $\square$

10.7. LEMMA. *If* $V^2\subset U$ *and* $V\in\mathcal{U}$ *then*

$${}^2V\subset{}^4U**X\subset({}^2U)^2.$$

PROOF. 1° Assume that $a\,{}^2V\,b$, $a\neq b$. Then there are $K\in f^\times(a)$ and $L\in f^\times(b)$ such that $K, L\subset V$ and $K_1\cap L_{-1}\neq\emptyset$. Take a point $x\in K_1\cap L_{-1}$; now $a\,{}^2V\,x\,{}^2V\,b$ by Lemma 8.8, thus $a\,{}^4U\,x\,{}^4U\,b$ by Lemma 8.3.
2° The second inclusion follows from Lemma 10.6. $\square$

THEOREM. *If the trace filter pairs are uniformly concentrated then* ${}^4\mathcal{U}**\{\{X\}\}=$ $={}^2\mathcal{U}$. $\square$

EXAMPLES. a) Take $\mathcal{U}=\mathcal{U}_2$ from Examples 0.7, $Y=\mathbf{R}^2$, $\mathfrak{f}^i((0,t))=\mathfrak{g}^i(t,\mathscr{E})$. Now ${}^4\mathcal{U}**\{\{X\cup\{(0,0)\}\}\}$ is an extension of $\mathcal{U}$ compatible with a bitopological extension for which no earlier theorem guarantees the existence of a quasi-uniform extension.

b) Let $\mathcal{U}$ and the trace filter pairs be as above. Define a distance d^* on Y by

$$d^*((a',a''),(b',b''))=\begin{cases} b'-a'+|b''-a''| & \text{if } a'<b',\\ b''-a'' & \text{if } a'=b'=0,\ a''<b''.\end{cases}$$

Now $\mathcal{V}=\mathcal{U}(d^*)$ is an extension of $\mathcal{U}$, and no extension of $\mathcal{U}$ compatible with $(\mathcal{V}^{-tp},\mathcal{V}^{tp})$ can be obtained through the operation $**$ from the extensions of $\mathcal{U}$ constructed earlier. This is clear for ${}^0\mathcal{U}={}^1\mathcal{U}={}^2\mathcal{U}$, whose bitopology is already strictly finer than that of $\mathcal{V}$. As to ${}^4\mathcal{U}$, observe that if $\mathcal{W}={}^4\mathcal{U}**\mathfrak{Q}$ then $\mathcal{W}|Y\setminus X$ is symmetrical, thus $\mathcal{W}^{-tp}|Y\setminus X=\mathcal{W}^{tp}|Y\setminus X$, while $\mathcal{V}^{-tp}|Y\setminus X\neq\mathcal{V}^{tp}|Y\setminus X$.

c) It follows from Theorem 10.7 and the last statement in Theorem 10.5 that in Example 8.11, there is no $\mathfrak{Q}$ for which ${}^4\mathscr{U}**\mathfrak{Q}$ is compatible with the fine regular extension. (The situation is quite different for the operation $*$, see [26], the Remark on p. 118.)

d) Let $\mathscr{U}=\mathscr{U}_{02}$ from Examples 0.7, $Y=\mathbf{R}^2$, and the trace filter pairs as in a). Denote by $\mathscr{V}$ the finest extension of $\mathscr{U}$ for the doubly strict extension. (The existence of $\mathscr{V}$ is guaranteed by Theorems 8.13 and 4.1.) Now $\mathscr{V}$ cannot be written in the form ${}^4\mathscr{U}**\mathfrak{Q}$, because $\mathscr{V}$ is not coarser than ${}^2\mathscr{U}$ (cf. Theorems 10.7 and 10.5).

Remark. The above examples show that the power of the operation $**$ is very restricted. It would impose a further (real) restriction on this operation if we only allowed systems $\mathfrak{Q}$ of the form $\{\mathfrak{q}\}$ or $\{\{Q\}: Q\in\mathfrak{q}\}$ (the latter case corresponds to 10.1 (1) instead of 10.1 (2)); see [13] for examples.

10.8. It can be proved, with the help of a more effective but less constructive method, that if the trace filter pairs are uniformly concentrated then the conditions in Theorem 1.1 are sufficient. This is a very special case of a result to be published in [12].

§ 11. Firm extensions

Recall that, by definition, the bitopological extension $(Y; \mathscr{S}^{-1}, \mathscr{S}^1)$ of $(X; \mathscr{T}^{-1}, \mathscr{T}^1)$ is *firm* if X is $\sup\{\mathscr{S}^{-1}, \mathscr{S}^1\}$-dense in Y. According to Lemma 7.2, an extension is firm iff all the trace filter pairs are linked.

11.1. Lemma. *Any firm regular extension is doubly strict.*

Proof. If there exists a regular extension $(\mathscr{S}^{-1}, \mathscr{S}^1)$ for the linked trace filter pairs $(\mathfrak{f}^{-1}(a), \mathfrak{f}^1(a))$ then there exists the fine regular extension associated with these trace filter pairs, which is finer than $(\mathscr{S}^{-1}, \mathscr{S}^1)$, while the doubly strict extension associated with the same trace filter pairs is coarser than $(\mathscr{S}^{-1}, \mathscr{S}^1)$, so it is enough to prove that firm fine regular extensions are doubly strict. The latter statement follows from

$$N_S^i(a) \supset \{b:\ S\in\sec\mathfrak{f}^{-i}(b)\} \supset \{b:\ S\in\mathfrak{f}^i(b)\}$$

where the first inclusion holds by the definition of $N_S^i(a)$, and the second one is evident for linked trace filter pairs. □

11.2. Lemma. *If the trace filter pairs are linked then* ${}^4U\subset{}^1U$ *and* ${}^1\mathscr{U}\subset{}^4\mathscr{U}$. □

Theorem. *Let $\mathscr{U}$ be a quasi-uniformity, and $(\mathscr{S}^{-1}, \mathscr{S}^1)$ a firm extension of $(\mathscr{U}^{-tp}, \mathscr{U}^{tp})$.*

a) *If there is an extension of $\mathscr{U}$ compatible with $(\mathscr{S}^{-1}, \mathscr{S}^1)$ then $(\mathscr{S}^{-1}, \mathscr{S}^1)$ is completely regular, and it is a doubly strict extension associated with linked round Cauchy filter pairs.*

b) *If $(\mathscr{S}^{-1}, \mathscr{S}^1)$ is a regular or doubly strict extension associated with round Cauchy trace filter pairs then there is exactly one extension $\mathscr{V}$ of $\mathscr{U}$ compatible with $(\mathscr{S}^{-1}, \mathscr{S}^1)$. Moreover, $\mathscr{V}={}^0\mathscr{U}={}^1\mathscr{U}={}^2\mathscr{U}={}^4\mathscr{U}$ and $w(\mathscr{V})=w(\mathscr{U})$.*

Proof. a) Theorem 1.1 and Lemmas 7.2 and 11.1.

b) $(\mathcal{S}^{-1}, \mathcal{S}^1)$ is doubly strict by Lemma 11.1, so ${}^4\mathcal{U}$ is an extension of $\mathcal{U}$ compatible with $(\mathcal{S}^{-1}, \mathcal{S}^1)$ by Lemma 7.16 [applied with $\mathfrak{F}_1=\emptyset$] and Theorem 8.13. On the other hand, ${}^1\mathcal{U}$ is also an extension for the same trace filter pairs by Lemma 7.17 and Theorem 8.7. From Theorem 8.7 and Lemmas 8.3 and 11.2 we have ${}^0\mathcal{U}={}^1\mathcal{U}={}^2\mathcal{U}={}^4\mathcal{U}$, so Theorems 6.1 and 8.13 imply that this is the only extension compatible with $(\mathcal{S}^{-1}, \mathcal{S}^1)$. □

Remark. Most of the statements in the above theorem can be proved directly, without using the results of §§ 6 to 8. One proves e.g. the unicity of $\mathcal{V}$ as follows: Let $\mathcal{V}$ and $\mathcal{W}$ be two extensions, and $V\in\mathcal{V}$. Choose first a $V_0\in\mathcal{V}$ with $V_0^3\subset V$, then a $W\in\mathcal{W}$ such that $W|X=V_0|X$, and finally a $W_0\in\mathcal{W}$ with $W_0^3\subset W$. It is now straightforward to check that $W_0\subset V$.

Similarly, it is not necessary to invoke the fine regular extension in the proof of Lemma 11.1.

§ 12. The problem of completeness

There are several notions of quasi-uniform completeness; the earliest one is the following [6, 23]: the quasi-uniformity $\mathcal{U}$ is called complete if the uniformity $\mathcal{U}^s$ is complete. A completion of $\mathcal{U}$, having all the good properties of the usual completion of uniformities, can be obtained through any of the Constructions 8.1 (i), (iii) and (iv) [but not (ii) or (v)] applied to the family of all the minimal $\mathcal{U}^s$-Cauchy (=round $\mathcal{U}^s$-Cauchy) filters. (Essentially contained by [7], with Construction 8.1 (i).)

In [25], completeness is defined in terms of filter pairs, and this definition is shown to be equivalent to the $\mathcal{U}^s$-completeness; although the authors promise an "alternate construction" (p. 56), they also use the minimal $\mathcal{U}^s$-Cauchy filters (Construction 8.1 (iv)); see also [17] 3.33. The main purpose of this section is to describe the completion with filter pairs, not using either single filters or the uniformity $\mathcal{U}^s$.

There are several other notions of quasi-uniform completeness, related to the "one-sided" problem outlined in 0.7: one can e.g. require that the ($\mathcal{U}$-round) $\mathcal{U}$-Cauchy filters should converge, or have a cluster point, in $\mathcal{U}^{tp}$, see [4, 5, 10, 15—19, 30—32].

12.1. Definition. Let $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ be a filter pair in a bitopological space $(X; \mathcal{T}^{-1}, \mathcal{T}^1)$, and $x\in X$.

a) $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ *converges* to x if $\mathfrak{f}^i$ $\mathcal{T}^i$-converges to x $(i=\pm 1)$.

b) x is a *cluster point* of $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ if it is a $\mathcal{T}^i$-cluster point of $\mathfrak{f}^i$ $(i=\pm 1)$.

In a quasi-uniform space, these notions are to be understood with respect to the induced bitopology.

12.2. Definition ([25]; equivalent to definitions given in [6, 23]). A quasi-uniform space is *complete* if each linked Cauchy filter pair is convergent.

Lemma. *The following conditions are equivalent for a quasi-uniformity $\mathcal{U}$:*

(i) *$\mathcal{U}$ is complete;*

(ii) *each linked round Cauchy filter pair is convergent;*

(iii) *each linked Cauchy filter pair has a cluster point.*

PROOF. (i)⇒(ii) and (i)⇒(iii). Evident.

(ii)⇒(i). If the linked filter pair $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ is Cauchy then so is the linked filter pair $(\mathfrak{f}^{-1}, \mathfrak{f}^1)^\circ$ (Lemma 7.12); if $(\mathfrak{f}^{-1}, \mathfrak{f}^1)^\circ$ is convergent then $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ is convergent, too.

(iii)⇒(i). Let $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ be a linked Cauchy filter pair. Then

$$(\mathfrak{f}^{-1}(\cap)\mathfrak{f}^1, \mathfrak{f}^{-1}(\cap)\mathfrak{f}^1) \tag{1}$$

is also a linked Cauchy filter pair. Let x be a cluster point of (1); we claim that $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ converges to x.

For $U\in\mathcal{U}$, take $V\in\mathcal{U}$ with $V^2\subset U$, then choose $K\in\mathfrak{f}^\times$ such that $K\subset V$. Since $K_i\in\mathfrak{f}^{-1}(\cap)\mathfrak{f}^1$, and x is a cluster point of (1), there are $y_i\in K_i$ such that $x\,V^i\,y_{-i}$. Thus, by $K\subset V$, $K_i\subset V^i y_{-i}\subset V^i[V^i x]\subset U^i x$. □

REMARK. In (ii), "round" can be replaced by "minimal" (Lemma 7.14).

12.3. [25] Corollary 8 states that if a linked Cauchy filter pair has a cluster point then it is convergent. This is, however, false; in fact we have:

EXAMPLE. *A non-complete quasi-uniform space in which every linked round Cauchy filter pair has a cluster point.* Let $X=\mathbf{R}^2\setminus\{(0,0)\}$,

$$d((x', x''), (y', y'')) = \begin{cases} y'-x' & \text{if } x'<0\leq y' \text{ or } x'\leq 0<y', \\ |y''-x''| & \text{if } x'=y'=0, \end{cases}$$

$\mathcal{U}=\mathcal{U}(d)$. Each linked round Cauchy filter pair has a cluster point (if such a filter pair is not the neighbourhood filter pair of a point then any $(0, x'')$ is a cluster point of it).

On the other hand, $(\mathfrak{f}, \mathfrak{f})$ is a non-convergent linked Cauchy filter pair, where

$$\mathfrak{f} = \mathrm{fil}\,\{\{0\}\times]0, \varepsilon[: \varepsilon>0\},$$

so $\mathcal{U}$ is not complete.

THEOREM. *Let $(X, \mathcal{U})$ be a quasi-uniform space, $\mathfrak{F}$ the system of the non-convergent linked round Cauchy filter pairs, $Y=X\cup\mathfrak{F}$. Assign to each point $p\in Y\setminus X$ the filter pair p. Then the quasi-uniformity given by Theorem 11.2 is complete, and it is a firm extension of $\mathcal{U}$.*

PROOF. It follows from Lemma 7.14 that the convergent linked round Cauchy filter pairs are the same as the neighbourhood filter pairs in X. Thus $\{(\mathfrak{f}^{-1}(a), \mathfrak{f}^1(a)) : a\in Y\}$ is the system of all the linked round Cauchy filter pairs (those belonging to the points of X may occur more than once). Let $\mathcal{V}$ denote the extension furnished by Theorem 11.2.

Take a linked round Cauchy filter pair $(\mathfrak{g}^{-1}, \mathfrak{g}^1)$ in Y. One can easily check that its trace on X is also linked, round and Cauchy; thus

$$(\mathfrak{g}^{-1}, \mathfrak{g}^1)|X = (\mathfrak{f}^{-1}(a), \mathfrak{f}^1(a))$$

with some $a\in Y$. Now $(\mathfrak{g}^{-1}, \mathfrak{g}^1)$ will converge to a, showing that $\mathcal{V}$ is complete.

Indeed, if $V\in\mathcal{V}$ then, by Theorem 11.2, there is a $U\in\mathcal{U}$ such that ${}^4U\subset V$. Select a $K\in\mathfrak{f}^\times(a)$ with $K\subset U$, and an $M\in\mathfrak{g}^\times$ with $M_i\cap X\subset K_i$ $(i=\pm 1)$. Fix

$i=-1$ or 1. As $\mathfrak{g}^i$ is $\mathscr{V}^i$-round, there are a $W\in\mathscr{V}$ and an $S\in\mathfrak{g}^i$ such that $W^i[S]\subset \subset M_i$. If $b\in S$ then $W^i b\cap X\subset M_i\cap X\subset K_i$, i.e. there is a $B\in\mathfrak{f}^i(b)$ with $B\subset K_i$. Now $K_{-i}\times B\subset K_{-i}\times K_i\subset U^i$, thus $a\,{}^4U^i\,b$. This means that $S\subset{}^4U^i a\subset V^i a$, therefore $\mathfrak{g}^i$ $\mathscr{V}^{itp}$-converges to a. □

Definition. This extension, denoted by $\hat{\mathscr{U}}$, will be called the *completion* of $\mathscr{U}$.

12.4. Lemma. *A one-to-one correspondence between linked round Cauchy filter pairs and $\mathscr{U}^s$-round $\mathscr{U}^s$-Cauchy filters can be defined by*

$$(1)\qquad (\mathfrak{f}^{-1},\mathfrak{f}^1)\mapsto\mathfrak{f}^{-1}(\cap)\mathfrak{f}^1,\quad \mathfrak{f}\mapsto(\mathfrak{f},\mathfrak{f})^\circ.$$

Proof. 1° Assume that $(\mathfrak{f}^{-1},\mathfrak{f}^1)$ is a linked round Cauchy filter pair. Then $\mathfrak{f}=\mathfrak{f}^{-1}(\cap)\mathfrak{f}^1$ is clearly a filter. For $U\in\mathscr{U}$, there is $K\in\mathfrak{f}^\times$ with $K\subset U$; now $S=K_{-1}\cap \cap K_1\in\mathfrak{f}$, $S\times S\subset U\cap U^{-1}$; thus $\mathfrak{f}$ is $\mathscr{U}^s$-Cauchy.

If $S\in\mathfrak{f}$ then $S=K_{-1}\cap K_1$ with some $K\in\mathfrak{f}^\times$. There are $T_i\in\mathfrak{f}^i$ and $U_i\in\mathscr{U}$ such that $U_i^i[T_i]\subset K_i$ $(i=\pm1)$. Taking $S_0=T_{-1}\cap T_1$ and $U=U_{-1}\cap U_1$, we have $(U\cap U^{-1})[S_0]\subset S$, thus $\mathfrak{f}$ is $\mathscr{U}^s$-round, too.

2° If $\mathfrak{f}$ is $\mathscr{U}^s$-Cauchy then $(\mathfrak{f},\mathfrak{f})$ is evidently linked and Cauchy. $(\mathfrak{f},\mathfrak{f})^\circ$ is also linked; it is round and Cauchy by Lemma 7.12.

3° It remained to be proved that the operations in (1) are inverse to each other, i.e. that

$$(2)\qquad \mathfrak{f}=\mathfrak{f}^{-1}(\cap)\mathfrak{f}^1\Rightarrow(\mathfrak{f},\mathfrak{f})^\circ=(\mathfrak{f}^{-1},\mathfrak{f}^1)$$

if $(\mathfrak{f}^{-1},\mathfrak{f}^1)$ is linked, round and Cauchy, and

$$(3)\qquad (\mathfrak{f}^{-1},\mathfrak{f}^1)=(\mathfrak{f},\mathfrak{f})^\circ\Rightarrow\mathfrak{f}^{-1}(\cap)\mathfrak{f}^1=\mathfrak{f}$$

if $\mathfrak{f}$ is $\mathscr{U}^s$-round and $\mathscr{U}^s$-Cauchy.

Proof of (2). $(\mathfrak{f},\mathfrak{f})$ is a linked Cauchy filter pair larger than $(\mathfrak{f}^{-1},\mathfrak{f}^1)$. By Lemmas 7.14 and 7.12 a), b), $(\mathfrak{f},\mathfrak{f})^\circ$ is the only round Cauchy filter pair contained by $(\mathfrak{f},\mathfrak{f})$.

Proof of (3). $\mathfrak{f}^i\subset\mathfrak{f}$ $(i=\pm1)$, thus $\mathfrak{f}^{-1}(\cap)\mathfrak{f}^1\subset\mathfrak{f}$ is clear. To show the converse, take an $S\in\mathfrak{f}$. As $\mathfrak{f}$ is $\mathscr{U}^s$-round, we can choose an $S_0\in\mathfrak{f}$ and a $U\in\mathscr{U}$ such that

$$(4)\qquad (U\cap U^{-1})[S_0]\subset S.$$

$\mathfrak{f}$ is $\mathscr{U}^s$-Cauchy, so, taking a $U_0\in\mathscr{U}$ with $U_0^2\subset U$, we may also assume that

$$(5)\qquad S_0\times S_0\subset U_0.$$

Now

$$(6)\qquad U_0[S_0]\cap U_0^{-1}[S_0]\subset(U\cap U^{-1})[S_0],$$

since if x belongs to the left-hand side of (6) then there are $y,z\in S_0$ such that $y\,U_0\,x\,U_0\,z$; by (5), $z\,U_0\,y$, thus $x\in U_0^{-1}z\subset U^{-1}z$ and $x\in U_0^2z\subset Uz$, therefore x is in the right-hand side of (6). It follows from $S_0\in\mathfrak{f}$ and the definition of $(\mathfrak{f},\mathfrak{f})^\circ$ that $U_0^i[S_0]\in\mathfrak{f}^i$ $(i=\pm1)$, thus (6) and (4) imply that $S\in\mathfrak{f}^{-1}(\cap)\mathfrak{f}^1$. □

12.5. LEMMA. *A filter $\mathfrak{f}$ is $\mathcal{U}^{stp}$-convergent iff* $(\mathfrak{f}, \mathfrak{f})^\circ$ *is convergent.* □

THEOREM ([25] 11). *A quasi-uniformity is complete iff the uniformity $\mathcal{U}^s$ is complete in the usual sense.* □

REMARK. The proof of this theorem needs only a weaker form of Lemma 12.4 (see in [25]). The full force of Lemma 12.4 is, however, required if we want to see clearly the connexion between the different constructions of the completion (described with filters on the one hand, and with filter pairs on the other).

It is, of course, possible to prove the fundamental properties of the completion with the help of filter pairs, but such a method would only mean a transcription of the usual proofs through Lemmas 12.4 and 12.5.

12.6. $\mathcal{U}_{so}$ and $\mathcal{U}=\mathcal{U}_{so}|\mathbf{R}\setminus\{0\}$ are both complete in the sense of Definition 12.2; neither is complete in the senses mentioned before 12.1. It would be desirable to find a notion of completeness that makes $\mathcal{U}_{so}$ complete, but $\mathcal{U}$ incomplete. There arise two difficulties:

a) One ought to find appropriate notions of compactness and precompactness to match the new notion of completeness.

b) Disregarding a), there is still the problem of completion. We could try e.g. calling a quasi-uniformity *S-complete* if each stable Cauchy filter pair is convergent (or a similar definition with some other class of Cauchy filter pairs containing that of the linked ones; or assume only the existence of a cluster point). Now we can assign a limit point to each round stable Cauchy filter pair using Theorem 8.7 (by Lemma 7.18, this will make also the non-round ones converge), but, unfortunately, there may appear in Y new stable Cauchy filter pairs that do not even have a cluster point. (A counterexample similar to the space in Example 8.11 can be constructed.)

ADDED IN PROOF (1989). a) Doitchinov [36, 37] calls a quasi-uniformity $\mathcal{U}$ *complete* if (in our terminology) the second element of each Cauchy filter pair is $\mathcal{U}^{tp}$-convergent, respectively *quiet* if (again in our terminology) the family of all the Cauchy filter pairs is uniformly weakly concentrated. ($\mathcal{U}^{tp}$ is assumed to be T_1 in [36], but not in [37]; nets are used instead of filters in [37].) Then he constructs a completion $\mathcal{U}^*$ for each quiet quasi-uniformity $\mathcal{U}$, and shows that this completion has several good properties. $\mathcal{U}^*$ is the same as ${}^4\mathcal{U}$ taken with the family of all the concentrated Cauchy filter pairs.

$\mathcal{U}_{so}$ is complete in this sense, while $\mathcal{U}_{so}|\mathbf{R}\setminus\{0\}$ is incomplete; nevertheless, Problem 12.6 remains unsolved, since the construction of completion is restricted to the class of quiet spaces. (A good solution of the problem would also fulfill the condition that $\mathcal{U}$ is complete iff $\mathcal{U}^{-1}$ is so; this holds for Doitchinov's completeness in the realm of quiet spaces, see [37] Proposition 24.)

See also [34, 35, 38, 39, 40].

b) Throughout both parts of this paper, the assumption $Y=\mathrm{cl}^{-1} X=\mathrm{cl}^1 X$ can be replaced by $Y=\mathrm{cl}^{-1} X\cup\mathrm{cl}^1 X$, see [33] § 2.

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CHARACTERIZATIONS OF SEMIMODULARITY

MANFRED STERN

1. Introduction

This paper deals with some conditions characterizing upper semimodularity in lattices of finite length. Upper semimodular lattices are also called semimodular lattices for short. In lattices of finite length, semimodularity is usually defined as follows:

DEFINITION. A lattice L of finite length is called (upper) semimodular if

$$a\wedge b \prec a \quad \text{implies} \quad b \prec a\vee b \qquad (a, b\in L).$$

Here by $x\prec y$ we mean that x is covered by y.

We shall not deal with examples here. Let us only remark that classically semimodular lattices arose out of certain closure operators (cf. Crawley—Dilworth [2], p. 25). For theory and examples of semimodular lattices (of finite as well as of infinite length) we refer to the books Birkhoff [1], Crawley—Dilworth [2], Dubreil—Jacotin et al. [5], Grätzer [9], Maeda—Maeda [10], Szász [15] and to the paper Croisot [3].

There are many characterizations of semimodularity in lattices of finite length. In Stern [14] there is a survey of some important characterizations published so far. In the present paper we give three necessary and sufficient conditions for semimodularity in lattices of finite length. These conditions are distinguished by the fact that they all involve join-irreducible elements.

2. Statement and proof of the results

We recall first that an element u of a lattice L of finite length is join-irreducible if it has exactly one lower cover in L which will be denoted by u'. By $J(L)$ we mean the set of all join-irreducible elements of L.

As a preparation we shall need the following

LEMMA (s. Stern [11]). *Let L be a lattice of finite length. If* $x\prec y$ $(x, y\in L)$, *then there exists a join-irreducible element* $u\in J(L)$ *such that* $u\leqq y$, $u\nleqq x$ *and* $u\wedge x=u'$.

We remark that for semimodular lattices this lemma was proved in Faigle [6].

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Now we state and prove the main result:

THEOREM. *Let L be a lattice of finite length. Then the following conditions are equivalent:*

(i) *L satisfies the implication*

$$a\wedge b \prec a \Rightarrow b \prec a\vee b \qquad (a, b\in L),$$

that is, L is semimodular;

(ii) *L has the following exchange property for join-irreducible elements: for $u, v\in J(L)$ and $b\in L$ $v\leqq b\vee u$ and $v\nleqq b\vee u'$ imply $u\leqq b\vee v\vee u'$;*

(iii) $u\wedge b=u'\prec u\Rightarrow b\prec u\vee b$ $(u\in J(L),\ b\in L)$;

(iv) $u\wedge b=u'\prec u\Rightarrow (u, b)M^*$ *(for this notation cf. Remark below)*, $(u\in J(L), b\in L)$.

REMARK. By $(u, b)M^*$ we mean that (u, b) is a dual-modular pair in the sense of Maeda—Maeda [10], that is,

$$c \geqq b \quad \text{implies} \quad (c\wedge u)\vee b = c\wedge(u\vee b) \qquad (c\in L).$$

We observe that the equivalence of conditions (i) and (ii) was proved in Stern [12]. Condition (ii) is closely related to the exchange property $\triangle 5$ of Finkbeiner [8] (s. also Faigle [7] and Dilworth [4]). The equivalence of conditions (i) and (iii) was shown in Stern [13] (and independently in Teo [16]). Condition (iv) appears to be new.

PROOF of the Theorem. (i)$\Rightarrow$(ii). Let (i) be satisfied and assume that for $u, v\in J(L)$ and $b\in L$ we have

$$v \leqq b\vee u \quad \text{but} \quad v \nleqq b\vee u'.$$

Then

$$b\vee u' < b\vee u$$

(since from $b\vee u'=b\vee u$ we get $v\leqq b\vee u'$), and therefore

$$b\vee u' \ngeqq u$$

(for if $b\vee u'\geqq u$, then $b\vee u\leqq b\vee u'$). Thus we have

$$u\wedge(b\vee u') = u' \prec u$$

which yields by (i) that

$$b\vee u' \prec (b\vee u')\vee u = b\vee u.$$

From this and from $v\nleqq b\vee u'$ we get $v\vee b\vee u'=b\vee u$ implying

$$u \leqq b\vee v\vee u'.$$

(ii)$\Rightarrow$(iii). Assume that (iii) does not hold. Then there exists an element $c\in L$ such that $u\vee b>c\succ b$ and an element $v\in J(L)$ such that

$$v \leqq c \quad \text{but} \quad v \nleqq b.$$

It follows that $v\leqq b\vee u$ and $v\nleqq b=b\vee u'$. On the other hand, we have $c=b\vee v$ and therefore $c=b\vee v\vee u'$. This means that $u\nleqq b\vee v\vee u'$, that is, (ii) does not hold.

(iii)⇒(iv). Let condition (iii) be satisfied and assume that

$$u \wedge b = u' \prec u.$$

Then we obtain from Maeda—Maeda [10, 7.5.2, p. 31] that

$$(u, b) M^*$$

holds which shows that (iv) is satisfied.

(iv)⇒(i). Assume that condition (iv) holds and let $a, b \in L$ be elements for which $a \wedge b \prec a$. Without loss of generality we may suppose that a, b are incomparable elements. We show that then $b \prec a \vee b$, which means that condition (i) holds.

Now if $a \wedge b \prec a$, then by the preceding lemma there exists a $u \in J(L)$ such that

$$u \wedge b = u' \quad \text{and} \quad a = (a \wedge b) \vee u.$$

The first of these equalities implies by (iv) that

$$(u, b) M^*$$

holds. This together with $u \wedge b = u' \prec u$ yields

$$b \prec u \vee b$$

by Maeda—Maeda [10, 7.5.4, p. 31]. Since $a \vee b = (a \wedge b) \vee u \vee b = u \vee b$, we obtain

$$b \prec a \vee b$$

and the proof is finished.

We remark that this theorem was announced without proof in Stern [14].

If the lattice is atomistic, that is, if each join-irreducible element is an atom, then we get immediately the following

COROLLARY. *Let L be an atomistic lattice of finite length. Then the following conditions are equivalent:*

(i) *L is semimodular;*

(ii) *L satisfies the Steinitz—MacLane exchange property, that is,*

$$p \leqq b \vee q \quad \text{and} \quad p \nleqq b \Rightarrow q \leqq b \vee p$$

for all atoms $p, q \in L$ and for arbitrary $b \in L$;

(iii) *L has the covering property*

$$(C) \quad p (\in L) \ \text{atom}, \quad b \in L, \quad p \wedge b = 0 \Rightarrow b \prec b \vee p;$$

(iv) *$p (\in L)$ atom, $b \in L \Rightarrow (p, b) M^*$.*

We remark that the assertion of the corollary is contained in Maeda—Maeda [10, Theorem 7.6, p. 31 and Theorem 7.10, p. 32].

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QUASI-UNIFORM EXTENSIONS FOR FINER TOPOLOGIES

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Abstract

Main result: if a quasi-uniformity can be extended to an extension of the induced topology [bitopology] then it can be extended to any finer topological [completely regular bitopological] extension belonging to the same system of trace filters [filter pairs]; in particular, conditions known to be sufficient in case of strict topological extensions are always sufficient. The problem of extensions with small quasi-uniform weight will also be considered.

This note contains results on two closely related problems: given a compatible quasi-uniformity $\mathcal{U}$ on a subspace of a topological space, find a compatible extension of $\mathcal{U}$ to the whole space (§ 1), and the analogous problem in a bitopological space (§ 2). On the whole, we shall follow the terminology and notations of our papers [5, 6], which deal with the bitopological case, but those who are interested only in § 1 can read it knowing only Császár [3].

NOTATIONS AND TERMINOLOGY. A *quasi-semiuniformity* on X is a filter in $X \times X$ of entourages (=reflexive relations) on X. Instead of $U(x)$ and $U(A)$ used in [3] (0.4), (0.5), we write Ux and U[A]. The neighbourhood filters $\{Ux: U \in U\}$ (where $\mathcal{U}$ is a quasi-semiuniformity) may or may not define a topology on X; if they do (in particular, if $\mathcal{U}$ is a quasi-uniformity) then this topology (the topology induced by $\mathcal{U}$; the topology $\mathcal{U}$ is compatible with) is denoted by $\mathcal{U}^{tp}$. $(x, y) \in U$ is usually written as $x\,U\,y$. $U|X = U \cap (X \times X)$. $w(\mathcal{U})$, the *weight* of the quasi-semiuniformity $\mathcal{U}$, is the smallest infinite cardinal $\varkappa$ for which $\mathcal{U}$ possesses a base of cardinality $\leqq \varkappa$. The *fine quasi-uniformity* of a (bi)topology is the finest one of all the quasi-uniformities compatible with it, cf. [2, 4]. $\mathrm{fil}\,\mathfrak{b} = \mathrm{fil}_X\,\mathfrak{b}$ denotes the filter in X generated by the filter subbase $\mathfrak{b}$.

In the present paper, the expression "*$\mathcal{V}$ is an extension of $\mathcal{U}$*" (where $\mathcal{V}$ and $\mathcal{U}$ are quasi-uniformities) will simply mean that $\mathcal{U} = \mathcal{V}|X$ with some set X, i.e. no kind of density is required (so in this respect we deviate from [5] 0.8). On the other hand, an extension of a topological [bitopological] space is always assumed to contain the original space as a dense subspace [dense in both topologies].

The method we shall use can also be applied to the problem of extending uniformities: in [8], a generalization of a result due to Úry [11] will be proved.

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§ 1. Quasi-uniformity and topology

Given a topological space $(Y, \mathcal{T})$ and a compatible quasi-uniformity $\mathcal{U}$ on a subspace $X \subset Y$, we are looking for an extension of $\mathcal{U}$ compatible with $\mathcal{T}$. If there exists such an extension, it can be obtained in two steps: (i) extending to the closure, (ii) extending from a closed subspace. Császár [3] investigated (i); we are going to supplement his results as well as give a complete solution of (ii): a quasi-uniformity can always be extended from a closed subspace (in such a way that the weight of the extension is the smallest that can be expected at all).

We begin with a construction and some lemmas.

1.1. Let $\emptyset \neq X \subset Y$, $\mathcal{V}$ a quasi-uniformity on Y, and $\mathcal{U}$ a quasi-uniformity on X. For $V \in \mathcal{V}$ and $U \in \mathcal{U}$ define

(1) $$V + U = V \cup V \circ U \circ V.$$

Evidently,

(2a) $$V \subset V + U,$$ (2b) $$U \subset V + U.$$

If $V_0 \subset V$ and $U_0 \subset U$ then $V_0 + U_0 \subset V + U$, so, $\mathcal{V}$ and $\mathcal{U}$ being filters, the entourages in (1) form a filter base. Now

(3) $$\mathcal{V} + \mathcal{U} = \mathrm{fil}_{Y \times Y} \{V + U \colon V \in \mathcal{V},\ U \in \mathcal{U}\}$$

is a quasi-semiuniformity on Y. It is enough in (3) to take V from a base for $\mathcal{V}$ and U from a base for $\mathcal{U}$, therefore

(4) $$w(\mathcal{V} + \mathcal{U}) \leqq \max \{w(\mathcal{U}), w(\mathcal{V})\}.$$

From (2) we have

(5a) $$\mathcal{V} + \mathcal{U} \subset \mathcal{V},$$ (5b) $$\mathcal{V} + \mathcal{U} | X \subset \mathcal{U}.$$

REMARKS. a) Compare the definition of $\mathcal{V} + \mathcal{U}$ with the following one due to Bing [1]: given a metric d on X and a metric e on Y, define a function f on $Y \times Y$ by

$$f(a, b) = \min \{e(a, b),\ \inf \{e(a, x) + d(x, y) + e(y, b) \colon x, y \in X\}\}.$$

Under suitable conditions, f is a metric on Y, and $f|X = d$; see also [11] 2.1.

b) A similar operation + can be defined for syntopogenous structures, see in [7].

1.2. LEMMA. *$(\mathcal{V} + \mathcal{U})^{tp} = \mathcal{V}^{tp}$ iff the $\mathcal{V}^{tp}$-trace filters in X are $\mathcal{U}$-round.*

REMARKS. a) The points outside $\mathcal{V}^{tp}$-cl X are to be understood to have no trace filter.

b) If the conditions of the lemma are satisfied then for each $x \in X$, the $\mathcal{V}^{tp}$-trace filter of x is $\mathcal{U}$-round, hence $\mathcal{U}^{tp}$-open; this implies that $\mathcal{V}^{tp}|X$ is coarser than $\mathcal{U}^{tp}$.

PROOF. 1° Assume that

(1) $$(\mathcal{V} + \mathcal{U})^{tp} = \mathcal{V}^{tp}$$

and let $a \in \mathscr{V}^{tp}$-cl X. If $S \in \mathfrak{f}(a)$, where $\mathfrak{f}(a)$ denotes the $\mathscr{V}^{tp}$-trace filter of a, then by (1) there are $V \in \mathscr{V}$ and $U \in \mathscr{U}$ with $(V+U)a \cap X \subset S$, thus

$$U[Va \cap X] = (U \circ V)a \subset (V+U)a \cap X \subset S,$$

and $Va \cap X \in \mathfrak{f}(a)$, hence $\mathfrak{f}(a)$ is $\mathscr{U}$-round.

2° Assume now conversely that the trace filters are round. By 1.1 (5a), (1) will follow if we show that for any $V \in \mathscr{V}$ and $a \in Y$ there are $V_0 \in \mathscr{V}$ and $U_0 \in \mathscr{U}$ such that

(2) $$(V_0 + U_0)a \subset Va.$$

a) If $a \notin \mathscr{V}^{tp}$-cl X then choose a $V_0 \in \mathscr{V}$ such that $V_0 \subset V$ and $V_0 a \cap X = \emptyset$. Now (2) holds with this V_0 and an arbitrary $U_0 \in \mathscr{U}$.

b) Let $a \in \mathscr{V}^{tp}$-cl X. Pick a $V_1 \in \mathscr{V}$ with $V_1^2 \subset V$. As $\mathfrak{f}(a)$ is round, there are $V_0 \in \mathscr{V}$ and $U_0 \in \mathscr{U}$ such that

(3) $$U_0[V_0 a \cap X] \subset V_1 a,$$

and we may also assume that $V_0 \subset V_1$. To prove (2), take a $b \in (V_0 + U_0)a$. Then either $a\, V_0\, b$ or

(4) $$a\, V_0 \circ U_0 \circ V_0\, b.$$

In the first case $b \in Va$ follows from $V_0 \subset V_1 \subset V$. On the other hand, if (4) holds then there are x and y such that $a\, V_0\, x\, U_0\, y\, V_0\, b$, so (3) implies $a\, V_1\, y\, V_0\, b$, and then $a\, V\, b$ follows from $V_0 \subset V_1$ and $V_1^2 \subset V$. □

1.3. Lemma. *If X is $\mathscr{V}^{tp}$-closed and $\mathscr{V}^{tp}|X = \mathscr{U}^{tp}$ then $(\mathscr{V} + \mathscr{U})^{tp} = \mathscr{V}^{tp}$.*

Proof. The $\mathscr{V}^{tp}$-trace filters coincide now with the $\mathscr{V}^{tp}|X$-neighbourhood filters, i.e. with the $\mathscr{U}^{tp}$-neighbourhood filters, which are $\mathscr{U}$-round, thus Lemma 1.2 can be applied. □

1.4. Lemma. *$\mathscr{V} + \mathscr{U}|X = \mathscr{U}$ iff $\mathscr{U} \subset \mathscr{V}|X$.*

Proof. 1° If $\mathscr{V} + \mathscr{U}|X = \mathscr{U}$ then $\mathscr{U} \subset \mathscr{V}|X$ follows from 1.1 (5a).

2° Assume conversely that

(1) $$\mathscr{U} \subset \mathscr{V}|X.$$

According to 1.1 (5b), it is enough to show that $\mathscr{U} \subset \mathscr{V} + \mathscr{U}|X$, i.e. that for any $U \in \mathscr{U}$ there are $V_0 \in \mathscr{V}$ and $U_0 \in \mathscr{U}$ such that

(2) $$V_0 + U_0|X \subset U.$$

Pick a $U_0 \in \mathscr{U}$ with

(3) $$U_0^3 \subset U.$$

By (1), there is a $V_0 \in \mathscr{V}$ such that

(4) $$V_0|X = U_0.$$

To prove (2), take $x, y \in X$ such that $x(V_0+U_0)y$. Now either xV_0y, and then (4) and (3) imply xUy, or there are x' and y' such that $xV_0x'U_0y'V_0y$. It is clear from $x'U_0y'$ that $x', y' \in X$, too. Therefore xUy follows again from (4) and (3). □

Remark. If $\mathcal{U} \supset \mathcal{V}|X$ then $\mathcal{V}+\mathcal{U}=\mathcal{V}$. (Proof: for $V \in \mathcal{V}$, choose $V_0 \in \mathcal{V}$ such that $V_0^3 \subset V$; now we have $V_0+(V_0|X) \subset V$.)

1.5. Lemma. *If either* $\mathcal{V}|X \subset \mathcal{U}$ *or* $\mathcal{U} \subset \mathcal{V}|X$ *then* $\mathcal{V}+\mathcal{U}$ *is a quasi-uniformity.*

Proof. 1° If $\mathcal{V}|X \subset \mathcal{U}$ then $\mathcal{V}+\mathcal{U}$ is a quasi-uniformity by Remark 1.4.
2° Assume now that

(1) $$\mathcal{U} \subset \mathcal{V}|X.$$

For $V \in \mathcal{V}$ and $U \in \mathcal{U}$, we need $V_0 \in \mathcal{V}$ and $U_0 \in \mathcal{U}$ such that

(2) $$(V_0+U_0)^2 \subset V+U.$$

Pick a $U_0 \in \mathcal{U}$ with

(3) $$U_0^3 \subset U.$$

By (1), there is a $V_1 \in \mathcal{V}$ such that

(4) $$V_1|X = U_0.$$

Choose now a $V_0 \in \mathcal{V}$ satisfying

(5) $$V_0^2 \subset V_1 \cap V.$$

To prove (2) holds, we have to show the following inclusions:

(6) $$V_0^2 \subset V+U,$$

(7) $$V_0^2 \circ U_0 \circ V_0 \cup V_0 \circ U_0 \circ V_0^2 \subset V+U,$$

(8) $$V_0 \circ U_0 \circ V_0^2 \circ U_0 \circ V_0 \subset V+U.$$

(6) is clear from (5) and $V \subset V+U$; (7) follows easily from (5), (3) and $V \circ U \circ V \subset \subset V+U$, so we have only to prove (8).

If $x\, U_0 \circ V_0^2 \circ U_0\, y$ then (5) implies that there are x' and y' with $xU_0x'V_1y'U_0y$. From xU_0x' and $y'U_0y$ we have $x', y' \in X$, thus xUy by (4) and (3). This means that $U_0 \circ V_0^2 \circ U_0 \subset U$, so (5) implies that the left-hand side of (8) is in $V \circ U \circ V$. □

1.6. Lemma. *If* $\mathcal{W}$ *is another quasi-uniformity on* Y, $\mathcal{W} \subset \mathcal{V}$ *and* $\mathcal{W}|X \subset \mathcal{U}$ *then* $\mathcal{W} \subset \mathcal{V}+\mathcal{U}$.

Proof. Take a $W \in \mathcal{W}$, and choose a $V \in \mathcal{W}$ such that $V^3 \subset W$. From $\mathcal{W} \subset \mathcal{V}$ we have $V \in \mathcal{V}$. On the other hand, $V \in \mathcal{W}$ and $\mathcal{W}|X \subset \mathcal{U}$ imply that $U \in \mathcal{U}$ where $U=V|X$. Now $V+U \subset W$ is evident. □

Remark. A special case of this lemma was already stated in Remark 1.4 and used in the proof of Lemma 1.5.

1.7. THEOREM. *If* $(X, \mathscr{U})$ *is a quasi-uniform space,* $(X, \mathscr{U}^{tp})$ *is a subspace of* $(Y, \mathscr{T})$, *and* $\mathscr{V}$ *is the fine quasi-uniformity of* $\mathscr{T}$ *then the following statements are equivalent:*

(i) $\mathscr{U}$ *can be extended to* $\mathscr{T}$;

(ii) *the* $\mathscr{T}$*-trace filters in* X *are* $\mathscr{U}$*-round, and there exists a quasi-uniformity finer than* $\mathscr{U}$ *that can be extended to some topology coarser than* $\mathscr{T}$;

(iii) *the* $\mathscr{T}$*-trace filters in* X *are* $\mathscr{U}$*-round, and* $\mathscr{U} \subset \mathscr{V}|X$;

(iv) $\mathscr{V} + \mathscr{U}$ *is an extension of* $\mathscr{U}$ *compatible with* $\mathscr{T}$;

(v) $\mathscr{V} + \mathscr{U}$ *is the finest extension of* $\mathscr{U}$ *compatible with* $\mathscr{T}$.

PROOF. (v)⇒(iv)⇒(i)⇒(ii). Evident.

(ii)⇒(iii). Let $\mathscr{V}'$ be a quasi-uniformity on Y such that $\mathscr{U}' = \mathscr{V}'|X$ is finer than $\mathscr{U}$ and $\mathscr{V}'^{tp}$ is coarser than $\mathscr{T}$. $\mathscr{V}' \subset \mathscr{V}$, because $\mathscr{V}$ is the fine quasi-uniformity of $\mathscr{T}$, so $\mathscr{U} \subset \mathscr{U}' = \mathscr{V}'|X \subset \mathscr{V}|X$.

(iii)⇒(v). Lemmas 1.2, 1.4, 1.5 and 1.6. □

REMARK. It would be enough to assume in (iv) [but not in (v)] that $\mathscr{V} + \mathscr{U}$ is only a quasi-semiuniform extension of $\mathscr{U}$ compatible with $\mathscr{T}$, since Lemmas 1.4 and 1.5 imply that $\mathscr{V} + \mathscr{U}$ is in this case a quasi-uniformity. On the other hand, it is not enough to assume that $\mathscr{V} + \mathscr{U}$ is a quasi-uniformity compatible with $\mathscr{T}$, since it does not follow from this condition that $\mathscr{V} + \mathscr{U}|X = \mathscr{U}$.

COROLLARY. *If* $\mathscr{T}$ *is an extension of* $\mathscr{U}^{tp}$ *and the trace filters are stable*[1] *and round then* $\mathscr{U}$ *can be extended to* $\mathscr{T}$.

PROOF. [3] 6.3 and (ii)⇒(i). □

1.8. COROLLARY. *Let* $(Y, \mathscr{T})$ *be an extension of* $(X, \mathscr{U}^{tp})$ *and assume that the trace filters are round and pseudo-Cauchy*[2]. *If the system*

$$\{\mathfrak{f}(a) \colon a \in Y\} \setminus \{\mathfrak{f}(x) \colon x \in X\}$$

is finite (in particular, if $Y \setminus X$ *is finite) then* $\mathscr{U}$ *can be extended to* $\mathscr{T}$.

PROOF. [3] 5.1 states that $\mathscr{U}$ can be extended to Y if $\mathscr{T}$ is a strict extension, the trace filters are round and pseudo-Cauchy, and $\{\mathfrak{f}(p) \colon p \in Y \setminus X\}$ is finite. Therefore $\mathscr{U}$ can be extended to the strict extension on

$$Y_0 = X \cup \{p \colon \mathfrak{f}(p) \neq \mathfrak{f}(x) \ (x \in X)\}.$$

Denote such a quasi-uniform extension to Y_0 by $\mathscr{V}_0$, and take for each $V \in \mathscr{V}_0$ an entourage V' on Y as follows: $a\, V'\, b$ iff $\varphi(a)\, V\, \varphi(b)$, where $\varphi(a) = a$ $(a \in Y_0)$, and for $p \in Y \setminus Y_0$, let $\varphi(p) \in X$ be chosen such that $\mathfrak{f}(\varphi(p)) = \mathfrak{f}(p)$. $\{V' \colon V \in \mathscr{V}_0\}$ is now a base for an extension to the strict extension on Y,[3] so Theorem 1.7 (ii)⇒(i) can be applied. □

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[1] Defined in [3] p. 126, line 9.

[2] Defined in [3] p. 126, line 4.

[3] In fact, [3] 5.1 can be reduced in the same way to the special case when $Y \setminus X$ is finite. It is essential in the reasoning that the extension should be strict, therefore a similar reduction of the corollary seems to be impossible.

1.9. Generalizing the terminology introduced in [11] for uniformities, a quasi-uniformity in a topological space $(X, \mathcal{T})$ will be called *continuous* if $\mathcal{U}^{tp}$ is coarser than $\mathcal{T}$.

THEOREM. *Let $(Y, \mathcal{T})$ be a topological space, $X \subset Y$. Then the following are equivalent:*

(i) *all those compatible quasi-uniformities on X for which the trace filters are round have a compatible extension;*

(ii) *all those compatible quasi-uniformities on X for which the trace filters are round have a continuous extension;*

(iii) *all the compatible quasi-uniformities on X have a continuous extension;*

(iv) *all the continuous quasi-uniformities on X have a continuous extension;*

(v) $\mathcal{V}_Y|X = \mathcal{V}_X$ *where $\mathcal{V}_A$ denotes the fine quasi-uniformity of $\mathcal{T}|A$.*

PROOF. (v)⇒(iv). Let $\mathcal{U}$ be continuous on X. Then $\mathcal{U} \subset \mathcal{V}_X = \mathcal{V}_Y|X$; $\mathcal{V}_Y + \mathcal{U}$ is a quasi-uniformity by Lemma 1.5; it is an extension of $\mathcal{U}$ by Lemma 1.4; it is continuous by 1.1 (5a).

(iv)⇒(iii). Evident.

(iii)⇒(i). Theorem 1.7.

(i)⇒(ii). Evident.

(ii)⇒(v). Clearly, $\mathcal{V}_Y|X \subset \mathcal{V}_X$. The trace filters are $\mathcal{V}_Y|X$-round, so they are $\mathcal{V}_X$-round, too. According to (ii), there is a continuous extension $\mathcal{W}$ of $\mathcal{V}_X$. Then $\mathcal{W} \subset \mathcal{V}_Y$, hence $\mathcal{V}_X \subset \mathcal{V}_Y|X$. □

1.10. LEMMA. *If a quasi-uniformity has a continuous extension then it has a continuous extension preserving the weight.*

PROOF. Take a base $\mathcal{B}$ for $\mathcal{U}$ with $|\mathcal{B}| \leqq w(\mathcal{U})$, and let $\mathcal{V}$ be a continuous extension of $\mathcal{U}$. For each $U \in \mathcal{B}$ and $n \in \mathbf{N}$, choose $V(U, n) \in \mathcal{V}$ such that $V(U, 1)|X = U$ and $V(U, n+1)^2 \subset V(U, n)$. Now

$$\{V(U, n)\colon\ U \in \mathcal{B},\ n \in \mathbf{N}\}$$

is a subbase for the required extension. □

THEOREM. *Let $(X, \mathcal{U})$ be a quasi-uniform space, $(X, \mathcal{U}^{tp})$ a subspace of $(Y, \mathcal{T})$, and $\varkappa$ an infinite cardinal. Assume that $w(\mathcal{U}) \leqq \varkappa$, $\mathcal{U}$ can be extended to $\mathcal{T}$, and there exists a quasi-uniformity of weight $\leqq \varkappa$ compatible with $\mathcal{T}$. Then there is an extension $\mathcal{W}$ of $\mathcal{U}$ compatible with $\mathcal{T}$ such that $w(\mathcal{W}) \leqq \varkappa$.*

PROOF. According to the lemma, $\mathcal{U}$ has a *continuous* extension $\mathcal{V}_1$ such that $w(\mathcal{V}_1) \leqq \varkappa$. Let $\mathcal{V}_2$ be a quasi-uniformity with $\mathcal{V}_2^{tp} = \mathcal{T}$ and $w(\mathcal{V}_2) \leqq \varkappa$. Take $\mathcal{V} = \sup\{\mathcal{V}_1, \mathcal{V}_2\}$. Clearly, $\mathcal{V}^{tp} = \mathcal{T}$, $w(\mathcal{V}) \leqq \varkappa$, and $\mathcal{U} \subset \mathcal{V}|X$. By Lemmas 1.5 and 1.4, $\mathcal{W} = \mathcal{V} + \mathcal{U}$ is a quasi-uniformity, and $\mathcal{W}|X = \mathcal{U}$. It follows from the existence of a compatible extension that the trace filters are round, thus $\mathcal{W}^{tp} = \mathcal{T}$ by Lemma 1.2. $w(\mathcal{W}) \leqq \varkappa$ by 1.1 (4). □

1.11. THEOREM. *Any compatible [continuous] quasi-uniformity on a closed subspace of a topological space has a compatible [continuous] extension.*

PROOF. By Theorem 1.9, it is enough to show that each compatible quasi-uniformity has a continuous extension. A base for such an extension is furnished by

$$\{U \cup Y \times (Y \setminus X):\ U \in \mathcal{U}\}.\quad \square$$

REMARKS. a) The above proof could have been based on the fact that (v) of Theorem 1.9 holds if X is closed ([9], Corollary 2.19).

b) Combining Theorems 1.10 and 1.11, we obtain extensions with small weight.

1.12. THEOREM. *A compatible quasi-uniformity can be extended from an open subspace of a topological space iff the trace filters are round.*

REMARK. Since the original space is open in a loose extension, this theorem generalizes [3] 2.2.

FIRST PROOF. With the usual notations,

$$\{U \cup (Y \setminus X) \times Y:\ U \in \mathcal{U}\}$$

is a base for a continuous extension, so there is a compatible extension by Theorem 1.7 (ii)⇒(i).

SECOND PROOF (essentially the same as the first one). It follows from [9] Corollary 2.19 that (v) of Theorem 1.9 holds.

THIRD PROOF (which does not use the fine quasi-uniformity). Let $\mathcal{V}$ be an arbitrary compatible quasi-uniformity on $(Y, \mathcal{T})$, e.g. the Pervin quasi-uniformity ([9], 2.2). For $U \in \mathcal{U}$ and $V \in \mathcal{V}$, define an entourage $W = W(U, V)$ on Y as follows:

$$p\,W\,q \quad \text{iff} \quad p\,V\,q \qquad (\text{for } p, q \in Y \setminus X),$$

$$x\,W\,y \quad \text{iff} \quad x\,U\,y \qquad (\text{for } x, y \in X),$$

$$p\,W\,x \quad \text{iff} \quad p\,U \circ V\,x \quad (\text{for } p \in Y \setminus X,\ x \in X),$$

and $x\,W\,p$ does not hold if $x \in X$, $p \in Y \setminus X$. Now

$$\mathcal{W} = \operatorname{fil}\,\{W(U, V):\ U \in \mathcal{U},\ V \in \mathcal{V}\}$$

will be the required extension. ($\mathcal{W}$ is a quasi-uniformity because $W(U, V)^2 \subset \subset W(U^2, V^2)$; $\mathcal{W}|X = \mathcal{U}$ is evident; $\mathcal{W}^{tp} = \mathcal{T}$ can be easily checked, using that the trace filters are round.) $\square$

1.13. Császár [3] 9.9 raised the following problem: let $\mathcal{T}$ be a strict extension of $\mathcal{U}^{tp}$, and suppose that $\mathcal{U}$ can be extended to $\mathcal{T}$; construct the finest extension of $\mathcal{U}$ compatible with $\mathcal{T}$. Our Theorem 1.7 gives a sort of answer: $\mathcal{V} + \mathcal{U}$ is the finest extension where $\mathcal{V}$ denotes the fine quasi-uniformity of $\mathcal{T}$ (and it is not necessary to assume that $\mathcal{T}$ is a strict extension). It is, however, questionable whether this can be regarded as a "construction", since the fine quasi-uniformity is rather elusive: it can be characterized [4, 9], but cannot be constructed in such an elementary way as e.g. the Pervin quasi-uniformity. (This is the reason why we thought it worthwhile to give the third proof of Theorem 1.12.) The aim of the next argument is to show that we cannot expect to have a more constructive solution, not even in the case of strict extensions associated with stable trace filters.

Assume that the finest quasi-uniform extension can be constructed for any strict extension associated with round and stable trace filters, and let $(Z, \mathcal{S})$ be a topological space, $Z \neq \emptyset$; we are going to construct the fine quasi-uniformity of $\mathcal{S}$.

On $X = \mathbf{N} \times Z$, consider the quasi-uniformity

$$\mathcal{U} = \mathrm{fil}\,\{U(n, G):\ n \in \mathbf{N},\ G \text{ is } \mathcal{S}\text{-open}\}$$

where

$$U(n, G) = \Delta_X \cup (S_n \times G) \times (S_n \times G) \cup (S_n \times (Z \setminus G)) \times (S_n \times Z),$$

$S_n = \{k \in \mathbf{N}:\ k > n\}$, and Δ_X is the diagonal of $X \times X$. Clearly, $\mathcal{U}^{tp}$ is the discrete topology. In order to simplify the notations, let us assume that $Z \cap X = \emptyset$. Taking $Y = Z \cup X$, we assign to the point $p \in Z$ the trace filter

$$\mathfrak{f}(p) = \mathrm{fil}_X\,\{S_n \times G:\ n \in \mathbf{N},\ p \in G,\ G \text{ is } \mathcal{S}\text{-open}\}.$$

$\mathfrak{f}(p)$ is round, because $U(n, G)[S_n \times G] \subset S_n \times G$. To prove that $\mathfrak{f}(p)$ is stable, take a basic entourage

$$U = \bigcap\,\{U(n_j, G_j):\ 1 \leq j \leq k\}.$$

Now with $n = \max\,\{n_1, \ldots, n_k\}$ and $x = (n+1, p)$, we have

$$Ux \in \mathfrak{f}(p), \quad Ux \subset U[F] \quad (F \in \mathfrak{f}(p)).$$

Let $\mathcal{T}$ be the strict extension of $\mathcal{U}^{tp}$ associated with these trace filters. It is easy to check that $\mathcal{T}|Z = \mathcal{S}$. $\mathcal{U}$ can be extended to $\mathcal{T}$ ([3] 6.3); let $\mathcal{W}$ be the finest one of such extensions. According to our assumption, $\mathcal{W}$ can be constructed. We claim that $\mathcal{W}|Z$ is the fine quasi-uniformity of $\mathcal{S}$.

Denote the fine quasi-uniformity of $\mathcal{S}$ by $\mathcal{V}$. Z being $\mathcal{T}$-closed, we can take the continuous extension of $\mathcal{V}$ given in the proof of Theorem 1.11, i.e. let

$$\mathcal{V}' = \mathrm{fil}_{Y \times Y}\,\{V \cup Y \times X:\ V \in \mathcal{V}\}.$$

Put $\mathcal{V}'' = \sup\,\{\mathcal{W}, \mathcal{V}'\}$. Clearly, $\mathcal{V}''^{tp} = \mathcal{T}$. $\mathcal{V}''$ is an extension of $\mathcal{U}$, because $\mathcal{W}|X = \mathcal{U}$, and $\mathcal{V}'|X$ is indiscrete. Hence $\mathcal{V}'' = \mathcal{W}$ (as $\mathcal{W}$ was the finest extension), and $\mathcal{W}|Z = \mathcal{V}''|Z$ is finer than $\mathcal{V}'|Z = \mathcal{V}$, so $\mathcal{W}|Z = \mathcal{V}$ (as $(\mathcal{W}|Z)^{tp} = \mathcal{T}|Z = \mathcal{S}$, and $\mathcal{V}$ is fine).

§ 2. Quasi-uniformity and bitopology

Using the lemmas of § 1, the following theorems can be proved similarly to the analogous results on extensions for prescribed topologies.

2.1. THEOREM. *If $(X, \mathcal{U})$ is a quasi-uniform space, $(X; \mathcal{U}^{-tp}, \mathcal{U}^{tp})$ is a subspace of the completely regular bitopological space $(Y; \mathcal{T}^{-1}, \mathcal{T}^{1})$, and $\mathcal{V}$ is the fine quasi-uniformity of $(\mathcal{T}^{-1}, \mathcal{T}^{1})$ then the following conditions are equivalent:*

(i) *$\mathcal{U}$ can be extended to $(\mathcal{T}^{-1}, \mathcal{T}^{1})$;*

(ii) *the* $\mathcal{T}^i$*-trace filters in* X *are* $\mathcal{U}^i$*-round* $(i=\pm 1)$, *and there exists a quasi-uniformity finer than* $\mathcal{U}$ *that can be extended to a bitopology coarser than* $(\mathcal{T}^{-1}, \mathcal{T}^1)$;

(iii) *the* $\mathcal{T}^i$*-trace filters in* X *are* $\mathcal{U}^i$*-round* $(i=\pm 1)$, *and* $\mathcal{U}\subset\mathcal{V}|X$;

(iv) $\mathcal{V}+\mathcal{U}$ *is an extension of* $\mathcal{U}$ *compatible with* $(\mathcal{T}^{-1}, \mathcal{T}^1)$;

(v) $\mathcal{V}+\mathcal{U}$ *is the finest extension of* $\mathcal{U}$ *compatible with* $(\mathcal{T}^{-1}, \mathcal{T}^1)$. □

Corollary. *Let* $(X, \mathcal{U})$ *be a quasi-uniform space,* $(Y; \mathcal{T}^{-1}, \mathcal{T}^1)$ *a completely regular extension of* $(X; \mathcal{U}^{-tp}, \mathcal{U}^{tp})$, *and assume that the trace filter pairs are uniformly weakly concentrated, round, and Cauchy. Then* $\mathcal{U}$ *can be extended to* $(\mathcal{T}^{-1}, \mathcal{T}^1)$.

Proof. By [6] Remark 8.13 a), $\mathcal{U}$ can be extended to a bitopology coarser than the doubly strict extension associated with the given trace filter pairs, which is in turn coarser than $(\mathcal{T}^{-1}, \mathcal{T}^1)$. □

Remark. We cannot expect to have an extension preserving the weight, since the trace filter pairs in [5] Example 3.4 b) are uniformly concentrated.

2.2. Theorem. *Let* $(X, \mathcal{U})$ *be a quasi-uniform space,* $(X; \mathcal{U}^{-tp}, \mathcal{U}^{tp})$ *a subspace of* $(Y; \mathcal{T}^{-1}, \mathcal{T}^1)$, *and* $\varkappa$ *an infinite cardinal. Assume that* $w(\mathcal{U})\leqq\varkappa$, $\mathcal{U}$ *can be extended to* $(\mathcal{T}^{-1}, \mathcal{T}^1)$, *and there exists a quasi-uniformity of weight* $\leqq\varkappa$ *compatible with* $(\mathcal{T}^{-1}, \mathcal{T}^1)$. *Then there is an extension* $\mathcal{W}$ *of* $\mathcal{U}$ *compatible with* $(\mathcal{T}^{-1}, \mathcal{T}^1)$ *such that* $w(\mathcal{W})\leqq\varkappa$. □

Remark. An analogue of Theorem 1.9 could also be formulated.

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NECESSARY AND SUFFICIENT CONDITIONS FOR UNIFORM CONVERGENCE OF QUASI-HERMITE—FEJÉR AND EXTENDED HERMITE—FEJÉR INTERPOLATION

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1. Introduction

1.1. Let $\alpha, \beta > -1$ and denote by

$$-1 < x_{nn}^{(\alpha,\beta)} < x_{n-1,n}^{(\alpha,\beta)} < \ldots < x_{2n}^{(\alpha,\beta)} < x_{1n}^{(\alpha,\beta)} < 1 \tag{1.1}$$

the roots of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ of degree n with the normalization

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}. \tag{1.2}$$

It is well-known that for every continuous function $f(x)$ in $[-1, 1]$ ($f \in C[-1, 1]$) there exists the uniquely determined Hermite—Fejér interpolatory polynomial $H_n^{(\alpha,\beta)}(f, x)$ of degree $\leqq 2n-1$ such that

$$H_n^{(\alpha,\beta)}(f, x_{kn}^{(\alpha,\beta)}) = f(x_{kn}^{(\alpha,\beta)}), \qquad \frac{d}{dx} H_n^{(\alpha,\beta)}(f, x)\big|_{x=x_{kn}} = 0, \tag{1.3}$$

$$k = 1, 2, \ldots, n.$$

Denote

$$\Delta_n^{(\alpha,\beta)}(f, x) = f(x) - H_n^{(\alpha,\beta)}(f, x)$$

and

$$\|f\| = \max_{-1 \leqq x \leqq 1} |f(x)|.$$

Szegő ([1], Theorem 15.6) proved that

$$\lim_{n\to\infty} \|\Delta_n^{(\alpha,\beta)}(f, x)\| = 0 \tag{1.4}$$

for all continuous functions $f(x)$ provided that $-1 < \alpha, \beta < 0$. Moreover, $H_n^{(\alpha,\beta)}(f, 1)$ generally does not tend to $f(1)$ as $n \to \infty$ if $\alpha \geqq 0$. In the particular case $\alpha = \beta = 0$ we have the following theorem:

THEOREM A. *Let f be a continuous function on* $[-1, 1]$ *and let* $H_n^{(0,0)}(f, x)$ *be defined by* (1.3). *In order that* $\lim_{n\to\infty} \|H_n^{(0,0)}(f, x) - f(x)\| = 0$ *uniformly on* $[-1, 1]$ *it is necessary and sufficient that*

$$f(1) = f(-1) = \frac{1}{2}\int_{-1}^{1} f(x)\, dx. \tag{1.5}$$

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The necessity of (1.5) had been proved by L. Fejér [9]. The sufficiency part was proved by Egerváry and Turán [4] and independently also by A. Schönhage [10]. For further result for this particular case we refer to the interesting paper of J. Szabados [2]. Later G. Freud [11] gave necessary conditions for the uniform convergence on $[-1, 1]$ for a wide family of nodes. Further the general cases $\alpha \geqq 0$, $\beta > -1$ were extensively investigated by J. Szabados [2], [3].

In paper [5] P. Vértesi proved the following:

Let p be a positive integer, $\beta > -1$, $\alpha \in [p-1, p)$, $\varepsilon > 0$ is fixed. Then

$$\lim_{n\to\infty} \|\Delta_n^{(\alpha,\beta)}(f, x)\|_{[-1+\varepsilon, 1]} = 0 \tag{1.6}$$

iff

$$\begin{gathered} \lim_{n\to\infty} H_n^{(\alpha,\beta)}(f, 1) = f(1), \quad \text{moreover if} \quad \alpha \geqq 1, \\ [H_n^{(\alpha,\beta)}(f, x)]_{x=1}^{(r)} = o(n^{2r}), \quad r = 1, 2, \ldots, p-1. \end{gathered} \tag{1.7}$$

Remark. The corresponding theorems for Hermite—Fejér interpolation based on the roots of

a) the generalized Jacobi polynomials,

b) the polynomials orthogonal to the weight

$$|x|^{2\alpha}(1-x^2)^\beta, \quad \alpha, \beta > -1,$$

c) the Laguerre polynomials $L_n^{(\alpha)}(x)$, $\alpha > -1$, can be found in P. Névai, P. Vértesi [6], T. Hermann [7] and Barbara Háy [8], respectively.

1.2. The notion of quasi-Hermite—Fejér interpolation has been defined by P. Szász [17], E. Egerváry and P. Turán [4]. The quasi-Hermite—Fejér interpolatory polynomial $Q_n^{(\alpha,\beta)}(f, x)$ of degree $\leqq 2n+1$ can be uniquely defined by

$$\begin{gathered} Q_n^{(\alpha,\beta)}(f, x_{kn}^{(\alpha,\beta)}) = f(x_{kn}^{(\alpha,\beta)}), \; Q_n^{(\alpha,\beta)\prime}(f, x_{kn}^{(\alpha,\beta)}) = 0, \quad k = 1, 2, \ldots, n, \\ Q_n^{(\alpha,\beta)}(f, 1) = f(1), \quad Q_n^{(\alpha,\beta)}(f, -1) = f(-1) \quad (f \in C[-1, 1]). \end{gathered} \tag{1.8}$$

Later J. Sánta [14] proved that if $f \in C[-1, 1]$ and $0 \leqq \alpha < 1$, $0 \leqq \beta < 1$ then $\{Q_n^{(\alpha,\beta)}(f, x)\}$ converges to $f(x)$ uniformly on $[-1, 1]$. For the case $\alpha = \beta = 0$ Prasad and Varma [13] proved that if $f \in C[-1, 1]$

$$|Q_n^{(0,0)}(f, x) - f(x)| \leqq c_1 \sum_{i=1}^{n} \frac{1}{i^2} w\left(f, \frac{i \sin\theta}{n}\right) \tag{1.9}$$

where c_1 is a positive constant independent of n, x, f, $w(f, \delta)$ being the modulus of continuity of f, $x = \cos\vartheta$.

1.3. Another interesting modification of the process $H_n^{(\alpha,\beta)}$ was studied by D. L. Berman [15]. Berman considers the polynomial $R_n^{(\alpha,\beta)}(f, x)$ of degree $\leqq 2n+3$ satisfying the following conditions:

$$\begin{gathered} R_n^{(\alpha,\beta)}(f, x_{kn}^{(\alpha,\beta)}) = f(x_{kn}^{(\alpha,\beta)}), \; R_n^{(\alpha,\beta)\prime}(f, x_{kn}^{(\alpha,\beta)}) = 0, \quad k = 1, 2, \ldots, n, \\ R_n^{(\alpha,\beta)}(f, 1) = f(1), \; R_n^{(\alpha,\beta)}(f, -1) = f(-1), \\ R_n^{(\alpha,\beta)\prime}(f, 1) = 0, \; R_n^{(\alpha,\beta)\prime}(f, -1) = 0. \end{gathered} \tag{1.10}$$

The surprising discovery of Berman [15] (who considered the case $\alpha=\beta=-1/2$) was that the inclusion of the end points ± 1 may completely change the nature of the Hermite—Fejér process. In case $\alpha=\beta=-1/2$ both $H_n(f, x)$ and $Q_n(f, x)$ converges uniformly to $f(x)$ on $[-1, 1]$ (when $n\to\infty$) whenever $f\in C[-1, 1]$ (see [1], [5]). But, as Berman [15] proved, $\{R_n^{(-1/2,-1/2)}(f, x)\}$ does not converge to $f(x)$ at any point of $(-1, 1)$ even for the simple function $f(x)=x^2$. A systematic study of the above process was carried out by R. Bojanić [16]. He proved the following

THEOREM B (R. Bojanić). *Let f be a continuous function on $[-1, 1]$ such that $f_L'(1)$, $f_R'(-1)$ exist and let $R_n^{(-1/2,-1/2)}(f, x)$ be defined by* (1.10). *In order that*

$$\lim_{n\to\infty} R_n^{(-1/2,-1/2)}(f, x) = f(x)$$

uniformly on $[-1, 1]$ it is necessary and sufficient that

$$f_L'(1) = 0, \quad f_R'(-1) = 0.$$

1.4. The object of this paper is to give those necessary and sufficient conditions for arbitrary $\alpha=\beta>-1$ which ensure the uniform convergence of the corresponding quasi-Hermite—Fejér process and extended Hermite—Fejér process for any $f\in C[-1, 1]$.

2. Main results

Let $\alpha>-1$, and $f\in C[-1, 1]$. It turns out that the necessary and sufficient conditions under which $Q_n^{(\alpha,\alpha)}(f, x)$ converges uniformly to $f(x)$ on $[-1, 1]$ are

(1) integral conditions,
(2) no conditions,
(3) differential conditions at ± 1,

depending on whether α is in the range $-1<\alpha<-1/2$, $-1/2\leqq\alpha<1$, $1\leqq\alpha$. We shall prove the following

THEOREM 2.1. *Let $f\in C[-1, 1]$. Then*

$$\lim_{n\to\infty} \|Q_n^{(\alpha,\alpha)}(f, x)-f(x)\| = 0, \tag{2.1}$$

iff

$$\begin{cases} \text{a)} \ \lim\limits_{n\to\infty} \int\limits_{-1}^{1} (Q_n^{(\alpha,\alpha)}(f, x)-f(x))\,dx = 0, \\ \text{b)} \ \lim\limits_{n\to\infty} \int\limits_{-1}^{1} x(Q_n^{(\alpha,\alpha)}(f, x)-f(x))\,dx = 0, \end{cases} \tag{2.2}$$

whenever $-1<\alpha<-1/2$;
if $-1/2\leqq\alpha<1$ (2.1) *holds for arbitrary* $f\in C[-1, 1]$ *(i.e. no conditions are needed); for* $\alpha\in[p-1, p)$, $p\geqq 2$, *(integer)* (2.1) *holds iff*

$$[Q_n^{(\alpha,\alpha)}(f, x)]_{x=\pm 1}^{(r)} = o(n^{2r}), \quad r = 1, 2, \dots, p-1. \tag{2.3}$$

REMARK. If $\alpha\in[-1/2, 1)$, the result is well-known (see J. Sántha [14] and P. Vértesi [5] when $\alpha\in[0, 1)$ and $[-1/2, 0)$, respectively).

Our next theorem concerns analogous problems related to the extended Hermite—Fejér process. We formulate them as follows:

THEOREM 2.2. *Let* $f\in C[-1, 1]$. *In the case when* $\alpha\in[-1/2, 1/2)$ *the necessary and sufficient conditions for*

$$\lim_{n\to\infty}\|R_n^{(\alpha,\alpha)}(f,x)-f(x)\| = 0 \tag{2.4}$$

is given by

$$\text{a)}\ \lim_{n\to\infty}\int_{-1}^{1}(R_n^{(\alpha,\alpha)}(f,x)-f(x))\,dx = 0,$$
$$\text{b)}\ \lim_{n\to\infty}\int_{-1}^{1}x(R_n^{(\alpha,\alpha)}(f,x)-f(x))\,dx = 0. \tag{2.5}$$

If $\alpha\in[1/2, 2)$, (2.4) *holds true for arbitrary* $f\in C[-1, 1]$ *(i.e. no conditions are needed).*

In the case $\alpha\in[p-1, p)$, $p\geqq 3$ *(p integer) the necessary and sufficient conditions for the validity of* (2.4) *is given by*

$$[R_n^{(\alpha,\alpha)}(f,x)]_{x=\pm1}^{(r)} = o(n^{2r}),\qquad r = 1, 2, \dots, p-1. \tag{2.6}$$

REMARKS. a) If $\alpha\in[1/2, 2)$ the result is well-known (see Vértesi [5]).

b) In the case when $-1<\alpha<1/2$ (2.5) and (2.2) are sufficient conditions for the validity of (2.4). It is also easy to see that (2.5) is also necessary, but we are unable to say the same for (2.2).

3. Some properties

Here we list some of the properties concerning $P_n^{(\alpha,\alpha)}(x)$ which we shall use in the proofs. If $x=\cos\theta$, $x_{kn}^{(\alpha,\alpha)}=\cos\theta_{kn}^{(\alpha,\alpha)}$ and $x_{jn}^{(\alpha,\alpha)}$ is the nearest root to $x=\cos\theta$, then

$$|P_n^{(\alpha,\alpha)}(x)|\sim|\theta-\theta_{jn}^{(\alpha,\alpha)}|\,\theta_{jn}^{-\alpha-(1/2)}n^{1/2},\qquad 0\leqq\theta<\pi/2, \tag{3.1}$$

$$0<c_1(\alpha)\leqq n(\theta_{k+1,n}^{(\alpha,\alpha)}-\theta_{kn}^{(\alpha,\alpha)})<c_2(\alpha),\qquad k = 1, 2, \dots, n;\quad n = 1, 2, \dots \tag{3.2}$$

(we assume $\theta_0=0$, $\theta_{n+1}=\pi$).

$$|P_n^{(\alpha,\alpha)}(x)| = |P_n^{(\alpha,\alpha)}(-x)| \tag{3.3}$$

(see e.g. [18], [20] and [1], respectively). Especially by (3.1) and (3.3)

$$|P_n^{(\alpha,\alpha)}(\pm1)|\sim n^{\alpha}. \tag{3.4}$$

4. Proof of Theorem 2.1

First we consider the case when $-1<\alpha<-1/2$. It is obvious that if (2.1) is satisfied then (2.2) follows at once. So we turn to prove (2.1) under the assumption (2.2). For the sake of simplicity we shall denote $Q_n^{(\alpha,\alpha)}(f,x)$, $H_n^{(\alpha,\alpha)}(f,x)$, $P_n^{(\alpha,\alpha)}(x)$ by $Q_n(f,x)$, $H_n(f,x)$, $P_n(x)$, respectively. Now by a simple consideration

$$(4.1)\qquad Q_n(f,x)-H_n(f,x)=\frac{(1+x)P_n^2(x)}{2P_n^2(1)}(f(1)-H_n(f,1))+$$

$$+\frac{(1-x)P_n^2(x)}{2P_n^2(-1)}(f(-1)-H_n(f,-1)).$$

Next we set

$$(4.2)\qquad A_n(f)=\frac{f(1)-H_n(f,1)}{2P_n^2(1)},\quad B_n(f)=\frac{f(-1)-H_n(f,-1)}{2P_n^2(-1)},$$

and note that

$$(4.3)\qquad \int_{-1}^{1}(Q_n(f,x)-H_n(f,x))\,dx=(A_n(f)+B_n(f))\int_{-1}^{1}P_n^2(x)\,dx$$

and

$$(4.4)\qquad \int_{-1}^{1}x(Q_n(f,x)-H_n(f,x))\,dx=(A_n(f)-B_n(f))\int_{-1}^{1}x^2P_n^2(x)\,dx.$$

We denote by

$$\lambda_n=\int_{-1}^{1}P_n^2(x)\,dx,\qquad \mu_n=\int_{-1}^{1}x^2P_n^2(x)\,dx.$$

First we show that

$$(4.5)\qquad \lambda_n\sim\mu_n\sim 1/n.$$

Indeed, by (3.1)—(3.3)

$$\int_{-1}^{1}P_n^2(x)\,dx=2\int_{0}^{1}P_n^2(x)\,dx=2\int_{0}^{\pi/2}P_n^2(\cos\theta)\sin\theta\,d\theta\sim$$

$$\sim\sum_{j=1}^{n}\left[\frac{1}{n^2}\left(\frac{j}{n}\right)^{-2\alpha-1}n\right]\frac{j}{n}\frac{1}{n}\sim\frac{1}{n};$$

$\mu_n\sim 1/n$ can be proved similarly. By these relations

$$\frac{P_n^2(x)}{\lambda_n}\leqq c\left(\frac{j}{n}\right)^{-2\alpha-1}\leqq c\quad\text{(see (3.1))}.$$

Similarly, $\dfrac{|xP_n^2(x)|}{\mu_n}\leqq c.$

Therefore, we can express

$$Q_n(f,x)-f(x) = H_n(f,x)-f(x)+\frac{P_n^2(x)}{\lambda_n}\left(\int_{-1}^{1}(Q_n(f,x)-H_n(f,x))\,dx+\right.$$

$$\left.+\frac{xP_n^2(x)}{\mu_n}\int_{-1}^{1}x(Q_n(f,x)-H_n(f,x))\,dx\right).$$

Now, by previous estimations, (2.2) and using that if $-1<\alpha=\beta<0$ then $\lim_{n\to\infty} H_n(f,x)=f(x)$ uniformly on $[-1,1]$ we have $\lim_{n\to\infty} Q_n(f,x)=f(x)$ (for $f\in C[-1,1]$) uniformly on $[-1,1]$.

This completes the proof of the first part of the Theorem 2.1.

Next, we turn to prove that (2.3) implies (2.1). For this purpose we need the process $H_{n,p}^{(\alpha,\alpha)}(f,x)\equiv H_{n,p}(f,x)$, the Hermite—Fejér-type interpolatory polynomials of degree $\leqq 2n+2p-1$ based on the zeros of $P_n^{(\alpha,\alpha)}(x)$ and ± 1. It is uniquely determined by the conditions

$$\text{(4.6)}\quad \begin{aligned} &H_{n,p}(f,x_{kn})=f(x_{kn}), && H'_{n,p}(f,x_{kn})=0, && k=1,2,\dots,n,\\ &H_{n,p}(f,\pm 1)=f(\pm 1), && H^{(j)}_{n,p}(f,\pm 1)=0, && j=1,2,\dots,p-1.\end{aligned}$$

Especially, $H_{n,0}=H_n$, $H_{n,1}=Q_n$ and $H_{n,2}=R_n$.

First, let $1\leqq\alpha<2$ then

$$\text{(4.7)}\quad \begin{aligned} H_{n,2}(f,x)-Q_n(f,x) = \frac{(1-x^2)P_n^2(x)}{2P_n^2(1)}\{Q_n'(f,1)-Q_n'(f,-1)+\\ +x(Q_n'(f,1)+Q_n'(f,-1))\}.\end{aligned}$$

From (2.3) it follows that for $1\leqq\alpha<2$ we have

$$\text{(4.8)}\qquad Q_n'(f,1)=o(n^2),\qquad Q_n'(f,-1)=o(n^2).$$

Also from (3.1) and (3.2) we have

$$\text{(4.9)}\qquad \frac{(1-x^2)P_n^2(x)}{2P_n^2(1)}\leqq c_3 n^{-2},\quad \alpha\geqq 1.$$

From (4.7)—(4.9) it follows that

$$\text{(4.10)}\qquad |H_{n,2}(f,x)-Q_n(f,x)|=o(1),\qquad \alpha\geqq 1.$$

Further, by [5], Theorem 3.2 we obtain

$$\text{(4.11)}\qquad \lim_{n\to\infty}\|H_{n,1}(f,x)-f(x)\|=0\quad \text{if}\quad 1\leqq\alpha<2.$$

Therefore we may easily conclude from (4.10) and (4.11) that

$$\lim_{n\to\infty}\|Q_n(f,x)-f(x)\|=0$$

provided (4.8) is satisfied. Next, let us assume that $2 \leqq \alpha < 3$. In this case (4.11) is not valid. But for $2 \leqq \alpha < 3$, again by [5], Theorem 3.2,

$$\lim_{n\to\infty} \|H_{n,3}(f, x) - f(x)\| = 0 \tag{4.12}$$

for $f \in C[-1, 1]$. Thus $H_{n,3}(f, x)$ will serve as an approximating polynomial in this case $(2 \leqq \alpha < 3)$. Next, by (4.10), using Markov inequalities for polynomials we get

$$\|H_{n,2}^{(j)}(f, x) - Q_n^{(j)}(f, x)\| = o(n^{2j}), \qquad j = 1, 2, \ldots . \tag{4.13}$$

Also we note that for $2 \leqq \alpha < 3$ we have from (2.3)

$$|Q_n^{(j)}(f, \pm 1)| = o(n^{2j}), \qquad j = 1, 2. \tag{4.14}$$

Therefore from (4.13), (4.14) we also have

$$|H_{n,2}^{(j)}(f, \pm 1)| = o(n^{2j}), \quad j = 1, 2. \tag{4.15}$$

Now from a simple consideration we have

$$H_{n,3}(f, x) - H_{n,2}(f, x) = \frac{(1-x^2)^2 P_n^2(x)}{4P_n^2(1)} [H_{n,2}''(f, 1) - H_{n,2}''(f, -1) + \\ + x(H_{n,2}''(f, 1) + H_{n,2}''(f, -1))]. \tag{4.16}$$

On using (4.15), (4.16) and

$$\frac{(1-x^2)^2 P_n^2(x)}{4P_n^2(1)} \leqq c_4 n^{-4} \tag{4.17}$$

we obtain

$$\|H_{n,3}(f, x) - H_{n,2}(f, x)\| = o(1). \tag{4.18}$$

Now, on combining (4.18), (4.12), (4.10) and

$$f(x) - Q_n(f, x) = f(x) - H_{n,3}(f, x) + H_{n,3}(f, x) - H_{n,2}(f, x) + H_{n,2}(f, x) - Q_n(f, x),$$

we get

$$\|f(x) - Q_n(f, x)\| = o(1).$$

Thus we have shown that (2.3) implies (2.1) also in the case $2 \leqq \alpha < 3$. The proof for the case $\alpha > 3$ follows on the same lines so we omit the details. To finish the proof we must show that (2.1) also imply (2.3). Indeed by Stečkin theorem [19] we know that for any polynomial $P_m(x)$ of degree $\leqq m$, $|p_m'(x)| \leqq c_5 m^2 w(p_m, 1/m)$. Then by Markov inequality

$$|P_m^{(r)}(x)| \leqq c_r m^{2r} \left[w\left(P_m - f, \frac{1}{m}\right) + w\left(f, \frac{1}{m}\right)\right] \leqq$$

$$\leqq c_r m^{2r} \left[\|p_m - f\| + w\left(f, \frac{1}{m}\right)\right]$$

which gives (2.3) if we choose $P_m(x) = Q_n(f, x)$ and $f \in C[-1, 1]$. Thus we have proved Theorem 2.1.

5. Proof of Theorem 2.2

It is similar to Theorem 2.1. First let $\alpha\in(-1/2, 1/2)$. Then we use

$$R_n(f, x)-Q_n(f, x) = \frac{(1-x^2)P_n^2(x)}{4P_n^2(1)}\{(1+x)Q_n'(f, 1)-(1-x)Q_n'(f, -1)\}. \tag{5.1}$$

Integrating both sides and using the fact that

$$\int_{-1}^{1}(1-x^2)P_n^2(x)\,dx\sim\frac{1}{n} \quad \text{if} \quad \alpha\leqq 1 \tag{5.2}$$

(compare (4.5)), from (5.1), (5.2) and $P_n^2(1)\sim n^{2\alpha}$ we have

$$n^{2\alpha+1}\int_{-1}^{1}(R_n(f, x)-Q_n(f, x))\,dx\sim(Q_n'(f, 1)-Q_n'(f, -1)). \tag{5.3}$$

Similarly

$$n^{2\alpha+1}\int_{-1}^{1}x(R_n(f, x)-Q_n(f, x))\,dx\sim(Q_n'(f, 1)+Q_n'(f, -1)). \tag{5.4}$$

By (5.1)—(5.4) we obtain

$$\begin{aligned}|R_n(f, x)-Q_n(f, x)| \leqq \frac{c_5 n^{2\alpha+1}}{n^{2\alpha}}(1-x^2)P_n^2(x)\Big\{\Big|\int_{-1}^{1}(R_n(f, x)-Q_n(f, x))\,dx\Big|+\\ +\Big|\int_{-1}^{1}x(R_n(f, x)-Q_n(f, x))\,dx\Big|\Big\}.\end{aligned} \tag{5.5}$$

If $\alpha\leqq 1/2$, by (3.1) we get

$$(1-x^2)P_n^2(x)\leqq\frac{c_6}{n}. \tag{5.6}$$

Further by Theorem 2.1, if $-1/2\leqq\alpha<1/2$ then

$$\lim_{n\to\infty}\|Q_n(f, x)-f(x)\| = 0 \tag{5.7}$$

for $f\in C[-1, 1]$. Now, using (5.5)—(5.7), and (2.5) we obtain (2.4). It is obvious that (2.4)→(2.5). The proof of Theorem 2.2 in the case $\alpha\in[p-1, p), p\geqq 3$, p integer, is similar to that given by the corresponding parts of Theorem 2.1. So we omit the details.

In the case $-1<\alpha<1/2$ let (2.5) and (2.2) be satisfied. Then by using Theorem 2.1 for $-1<\alpha<1/2$ we have $\lim\limits_{n\to\infty}\|Q_n(f, x)-f(x)\|=0$. Thus we choose $Q_n(f, x)$ as the approximating polynomial for $R_n(f, x)$. The rest of the proof in this case can now be carried out like in Theorem 2.1.

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ON THE PACKING DENSITY OF TRANSLATES OF A DOMAIN

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Let w be a domain in the Euclidean plane. We shall denote the density of the densest packing of translates of w by $d(w)$ and the density of the densest lattice packing of translates of w by $d'(w)$.

It has been shown ([1], [2], [3]) that if w is convex then

(1) $$d(w) = d'(w).$$

In a recent paper L. Fejes Tóth [4] started an interesting field of research trying to extend the validity of (1) to more general domains. A simply connected domain is called semi-convex if there are two points A and B of its boundary lying on opposite parallel support-lines such that the domain can be made convex by replacing one of its two boundary arcs connecting A and B with another arc BA. Choose an arbitrary point C on the convex arc AB and translate the arc AC through the vector CB and the arc BC through the vector CA. The above defined semi-convex domain is called limited semi-convex if the "non-convex" arc BA is lying in the domain defined by the convex arc AB and the two translated arcs.

Fejes Tóth has proved that if w is limited semi-convex [4] or [5] if it is the union of two intersecting equal circles then equation (1) is still valid.

The range of validity of this property has been curbed by constructions of A. Bezdek and G. Kertész [6]. They have constructed

(i) a domain consisting of 5 convex domains,
(ii) a domain that is semi-convex,
(iii) a domain that is direction-convex

such that each can be arranged to have higher density if you do not require the packing to be latticelike.

Fejes Tóth's conjecture [5] is that this cannot be done with a domain that is the union of two convex domains with a point in common. The analogous question has been raised for star-shaped domains as well.

In the present paper we are going to give a construction for a domain u with the following properties:

(i) u is the union of three convex domains,
(ii) u is star-shaped,
(iii) $d(u) > d'(u)$, i.e. the densest packing of u is not latticelike.

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To describe the domain u and to show its properties we shall use the 2-dimensional coordinate system. In what follows $A(x)$ will denote the area of x, and the sum of a domain and a vector denotes the translate of the domain by that vector. The three components that we use to construct our domain u are two rhombs R_1 and R_2 and a hexagon H. We define them by listing their corners as follows (Fig. 1):

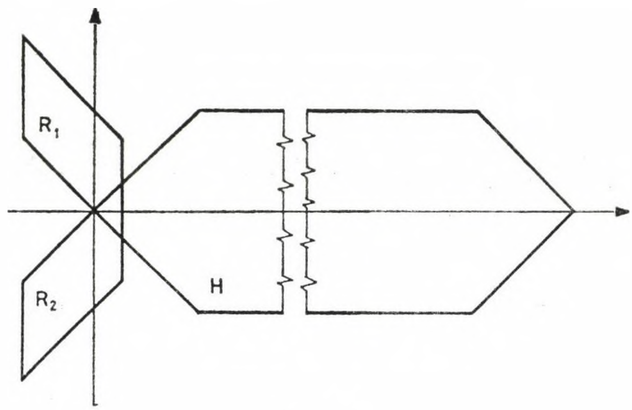

Fig. 1

R_1 $(a, -a)$, $(a, 1-a)$, $(-1+a, 2-a)$, $(-1+a, 1-a)$

R_2 (a, a), $(a, -1+a)$, $(-1+a, -2+a)$, $(-1+a, -1+a)$

H $(0, 0)$, $(1, 1)$, $(1+L, 1)$, $(2+L, 0)$, $(1+L, -1)$, $(1, -1)$.

Here L denotes a sufficiently large and $a>0$ denotes a sufficiently small number.

The union u of R_1, R_2 and H is clearly star-shaped with respect to any point of the triangle $(0, 0)$, (a, a), $(-a, -a)$ thus u shares properties (i) and (ii).

Consider now the translate $u_1=u+(1, 2)$ and the union v of u and u_1. On the one hand u and u_1 have no interior point in common, on the other hand the vectors $(0, 4)$ and $(L+3, 2)$ define a latticelike arrangement of v that is a packing (Fig. 2). Thus we have a packing of translates of u of density $A(u)/(2*L+6)$.

Although we are convinced that the best lattice packing is generated by the vectors $(1, 2)$ and $(L+3-a, -1+a)$ (Fig. 3) we need not prove that to reach our goal. All we are left to show is that any lattice-packing of u has a smaller density than $A(u)/(2*L+6)$.

Let us consider a lattice-packing of u. First we define the "side strip" and the "neck" of u. The side strip of u is a rhomb of area $b*(L+1)$ given by its corners: $(0, 1)$, $(L+1, 1)$, $(L+1-b, 1+b)$, $(-b, 1+b)$, and the neck is a triangle given again by its corners (a, a), $(a+b, a+b)$, $(a, a+b)$; where b is a sufficiently small but positive number.

We distinguish two cases. First we assume that in the lattice-packing the hexagonal parts of the neighbouring domains are not close to each other, more precisely, we assume that the side strip of u does not contain a point of the hexagonal part of a translate. Then — considering that no more than a single rhombic part of

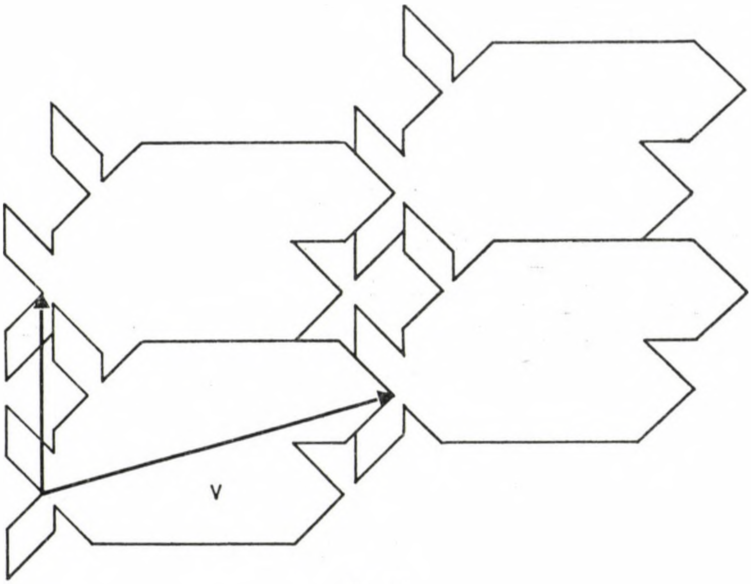

Fig. 2

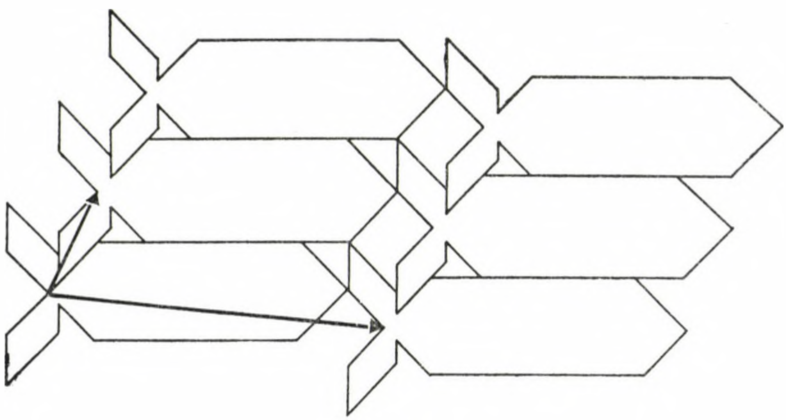

Fig. 3

the whole packing can have a point in common with the side strip of u, and that the area of that common part is certainly smaller than b, to each translate there belongs an uncovered part of area $>b*L$. Since $A(u)=2*L+4-2*a*a$, to any prefixed a and b L can be chosen so that $A(u)+b*L>2*L+6$.

In the other case the sides of certain pairs of hexagons are closer than b. Of the logically symmetric two subcases we assume that the rhombic part of a translate u_2 of u enters the neck triangle of u. Then $u_2=u+(1+t_1, 2+t_2)$, where $0\leq t_1\leq t_2<b$. The domains u and u_2 define a stripe of the lattice packing, and the whole lattice is

defined by two neighbouring stripes. Since the closest position of two such stripes is defined by the translation $(L+3-a-t_2, -1+t_2)$, the area of the fundamental domain of the lattice of the densest such lattice packing is $2*L+7+(L+3)*t_2-2*(a+t_2)-(t_2-t_1)-t_1*t_2$. For suitably chosen a and b this area is $>2*L+6.5$, and this is what we wanted to show.

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ON SUPER LEHMER PSEUDOPRIMES

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Let $U=U(L, M)=\{U_n\}_{n=0}^{\infty}$ be a Lehmer sequence defined by fixed rational integers L, M and by recursion

$$U_n = \begin{cases} LU_{n-1}-MU_{n-2} & \text{for } n \text{ odd} \\ U_{n-1}-MU_{n-2} & \text{for } n \text{ even,} \end{cases}$$

where the initial values are $U_0=0$, $U_1=1$. The terms of U are called Lehmer numbers. We shall denote the roots of the characteristic polynomial $f(x)=x^2-\sqrt{L}\,x+M$ by α and β. We may assume that $|\alpha|\geqq|\beta|$ and the sequence is not degenerate, that is $LM\neq 0$, $K:=L-4M\neq 0$ and α/β is not a root of unity. In this case, as it is well-known, the terms of the sequence U can be expressed as

$$(1)\qquad U_n = \begin{cases} (\alpha^n-\beta^n)/(\alpha-\beta) & \text{for } n \text{ odd} \\ (\alpha^n-\beta^n)/(\alpha^2-\beta^2) & \text{for } n \text{ even.} \end{cases}$$

We can assume, without any essential loss of generality, that $L>0$ and $(L, M)=1$.

A. Rotkiewicz [5] gave a proper generalization of pseudoprimes for Lehmer sequences. A composite number n is called a Lehmer pseudoprime with respect to the sequence U if $(n, LMK)=1$ and $U_{n-(LK/n)}\equiv 0 \pmod n$, where (LK/n) is the Jacobi symbol. A number n is called a super Lehmer pseudoprime with respect to the sequence U if each divisor of it is a prime or a Lehmer pseudoprime with respect to the sequence U. The Lehmer and super Lehmer pseudoprimes are generalizations of pseudoprimes and super pseudoprimes to base c; if $(n, c)=1$ and $c^{n-1}\equiv 1 \pmod n$ we say n is a super pseudoprime to base c if each divisor d of it is a prime or a pseudoprime to base c, i.e. $c^{d-1}\equiv 1 \pmod d$.

It is known that there are infinitely many super pseudoprime numbers to base 2. For example, K. Szymiczek [9] proved that there exist infinitely many super pseudoprimes to base 2 which are products of exactly three distinct primes. This result was extended by A. Rotkiewicz [4], J. Fehér and P. Kiss [3] and B. M. Phong [1]. In [2] B. M. Phong obtained a similar result for super Lehmer pseudoprimes, proving that for any non-degenerate Lehmer sequence $U(L, M)$ there exist a positive in-

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teger w_0 such that for infinitely many primes p of the form $ax+b$, where $(a, b)=1$ and $b\equiv 1 \pmod{(w_0, a)}$, there are primes q and r such that pqr is a super Lehmer pseudoprime with respect to the sequence $U(L, M)$. The constant w_0 is effectively computable in terms of L and M.

The aim of this note is to give a lower bound for the counting function of super Lehmer pseudoprimes.

In the following we shall use some notations. For a non-degenerate Lehmer sequence $U(L, M)$ and an integer n with $(M, n)=1$ we denote by $u(n)$ the least positive integer m such that $n|U_m$ but $n\nmid U_1, U_2, \ldots, U_{m-1}$. If n is an integer, then denote $k(n)$ the square-free kernel of n. Finally, denote by $\psi_3^*(U)$ the set of all super Lehmer pseudoprimes with respect to the sequence U which are products of exactly three distinct primes.

We shall prove the following

THEOREM. *Let* $U=U(L, M)$ *be a non-degenerate Lehmer sequence and* $K=L-4M$. If $\Delta:=k(M\cdot\max(L, K))\equiv\pm 1 \pmod 4$, *then for all large* x *we have*

$$\# \{n \leqq x:\ n\in\psi_3^*(U)\} > (4\Delta \log|\alpha|)^{-1}\log x.$$

For the proof we need some lemmas.

LEMMA 1. *Let* $U=U(L, M)$ *be a non-degenerate Lehmer sequence and let* p, q, r *be distinct primes. The number* pqr *is a super Lehmer pseudoprime with respect to* U *if and only if* $u(pqr)|(p-(LK/p), q-(LK/q), r-(LK/r))$, *where* $(x, y, \ldots)$ *denotes the G.C.D. of* $x, y, \ldots$.

LEMMA 2. *Let* $U=U(L, M)$ *be a non-degenerate Lehmer sequence. Then for any* $n>n_0=e^{452}\cdot 4^{67}$ *there exist at least one prime number* p *such that* $u(p)=n$.

LEMMA 3. *Let* $U=U(L, M)$ *be a non-degenerate Lehmer sequence. Let* $\Delta:=k(M\cdot\max(L, K))$ *and*

$$\Omega = \begin{cases} 1 & \text{if } \Delta\equiv 1 \pmod 4 \\ 2 & \text{if } \Delta\equiv 2 \text{ or } 3 \pmod 4). \end{cases}$$

There exists an absolute constant n_1 *such that if* $n>n_1$ *and* $n/\Delta\cdot\Omega$ *is an odd integer then there are at least two distinct primes* p *and* q *for which* $u(p)=u(q)=n$.

Lemma 1 was proved by B. M. Phong in [2], Lemma 2 is a known result of A. Schinzel [7] and C. L. Stewart [8] and the Lemma 3 follows from a theorem of A. Schinzel [6].

PROOF of the Theorem. Let $U=U(L, M)$ be a non-degenerate Lehmer sequence with condition

$$\Delta = k(M\max(L, K)) \equiv \pm 1 \pmod 4.$$

Using (1), for any integer $n\geqq 0$, we have

$$|U_n| < 2|\alpha|^n. \tag{2}$$

Since U is non-degenerate, we have $|\alpha|>1$. Let m_0 be the least odd integer with the condition

(3) $$m_0 > \max(2n_0, n_1, |LMK|),$$

where n_0, n_1 are constants in Lemma 2 and 3, respectively. Let M_0 be a positive number such that if $x > M_0$ and m_x is an odd integer with

(4) $$2|\alpha|^{2\Delta m_x} \leqq x < 2|\alpha|^{2\Delta(m_x+2)},$$

then $m_x > m_0$.

Let x be a real number with $x > M_0$ and let m_x be an odd integer giving by (4). If m is an odd integer and

(5) $$m_0 < m \leqq m_x,$$

then by Lemma 3 there exist primes p_m and q_m such that $u(p_m)=\Delta\Omega m$ and $u(q_m)= =\Delta\Omega m$, where

$$\Omega = \begin{cases} 1 & \text{if} \quad \Delta \equiv 1 \pmod 4 \\ 2 & \text{if} \quad \Delta \equiv -1 \pmod 4. \end{cases}$$

Since $m_0 > 2n_0$, using Lemma 2, there exists a prime r_m such that

$$u(r_m) = \begin{cases} 2\Delta\Omega m & \text{if} \quad \Omega = 1 \\ \Delta\Omega m/2 & \text{if} \quad \Omega = 2. \end{cases}$$

Since Δ and m are odd integers and $u(p)|(p-(LK/p))$ for any prime p with $(p, M)=1$, we have

$$u(p_m) = u(q_m) = \Delta\Omega m|(p_m-(LK/p_m))/2,$$

$$(q_m-(LK/q_m))/2 \quad \text{if} \quad \Omega = 1$$

and

$$u(p_m) = u(q_m) = \Delta\Omega m|(r_m-(LK/r_m))/2 \quad \text{if} \quad \Omega = 2.$$

In both cases we get $u(p_m q_m r_m)=2\Delta m$ and

$$2\Delta m|(p_m-(LK/p_m),\ q_m-(LK/q_m),\ r_m-(LK/r_m)).$$

By Lemma 1 it follows that $p_m q_m r_m$ is a super Lehmer pseudoprime with respect to the sequence U.

Thus for any odd integer m satisfying (5) there exists a number $p_m q_m r_m$ such that $p_m q_m r_m \in \psi_3^*(U)$, and $u(p_m q_m r_m)=2\Delta m$. Using (2), (4) and (5) we obtain $p_m q_m r_m \leqq |U_{2\Delta m}| < 2|\alpha|^{2\Delta m} \leqq 2|\alpha|^{2\Delta m_x} \leqq x$, from which we get

(6) $$T_1(x) := \#\{n \leqq x:\ n\in\psi_3^*(U) \text{ and } 2\,\|\,u(n)\} \geqq \frac{m_x-m_0}{2}.$$

On the other hand, by Theorem 1 of [2], we have

(7) $$T_2(x) := \#\ \{n \leqq x:\ n\in\psi_3^*(U) \text{ and } 8|u(n)\} \to \infty \quad (x\to\infty).$$

By (4), (6) and (7) we have

$$\#\{n \leqq x:\ n \in \psi_3^*(U)\} \geqq T_1(x)+T_2(x) \geqq \frac{1}{2}\,m_x - \frac{m_0}{2} + T_2(x) >$$

$$> \frac{1}{4\Delta \log|\alpha|}\log x - \left(\frac{m_0}{2} + \frac{\log 2}{4\Delta \log|\alpha|} + 1\right) + T_2(x) > \frac{1}{4\Delta \log|\alpha|}\log x$$

for all large x, which proves our theorem.

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Γ-RINGS AND KÖTHE'S PROBLEM

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Abstract

One of the longest standing open questions in ring theory is the problem of Köthe: Does the nil radical of a ring contain all its nil one-sided ideals? In this note, we find some equivalent formulations of Köthe's problem in terms of Γ-rings.

1. Fundamentals

Throughout this paper, the term "Γ-ring" will mean a Γ-ring in the sense of Barnes [1]. For all definitions pertaining to Γ-rings and their operator rings, we refer to [3] and [4]. The nil radical of a Γ-ring M was defined by Coppage and Luh [3]. If I is an ideal of M, then I is a nil ideal of M, if for each $x \in I$ and $\gamma \in \Gamma$, there exists a positive integer $n = n(x, \gamma)$ such that $(x\gamma)^n x = x\gamma x\gamma \dots \gamma x = 0$. The nil radical of M, $\mathcal{N}(M)$, is the sum of all the nil ideals of M. Let L and R denote the left and right operator rings, respectively, of M. In [3], Theorem 6.2, it is shown that $\mathcal{N}(R)^* \subseteq \mathcal{N}(M)$, where $\mathcal{N}(R)^*$ denotes the nil radical of the ring R. By symmetry, we also have that $\mathcal{N}(L)^+ \subseteq \mathcal{N}(M)$. It is not known whether in general the reverse inclusions hold, or whether $\mathcal{N}(R)^* = \mathcal{N}(L)^+$. However, the following result has been proved by the present author.

PROPOSITION 1. *Suppose that it is true that, for every ring A, the nil radical of A contains all the nil one-sided ideals of A. Let M be a Γ-ring with left and right operator rings L and R, respectively. Then*

$$\mathcal{N}(L)^+ = \mathcal{N}(M) = \mathcal{N}(R)^*.$$

For a proof of this Proposition, we refer to [2], Proposition 3.4, and the discussion in Section 4 of that paper.

Throughout this note, if A is a ring and m is a positive integer A_m will denote the ring of $m \times m$ matrices over A. Let B be any set and let p, q be positive integers. Then $B_{p,q}$ will denote the set of $p \times q$ matrices with entries from B.

Suppose M is a Γ-ring, and p, q are positive integers. Let $(a_{ij}), (b_{ij}) \in M_{p,q}$ and $(\gamma_{ij}) \in \Gamma_{q,p}$. We define

$$(a_{ij})(\gamma_{ij})(b_{ij}) = (c_{ij})$$

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Key words and phrases. Γ-ring, nil ideal, nil radical.

where

$$c_{rs} = \sum_m \sum_n a_{rm} \gamma_{mn} b_{ns}$$

$$(1 \leqq r \leqq m, \ 1 \leqq s \leqq n).$$

PROPOSITION 2. *Let M be a Γ-ring with left and right operator rings L and R, respectively. Let L and R denote, respectively, the left and right operator rings of the $\Gamma_{q,p}$-ring $M_{p,q}$. Then L' is isomorphic to L_p and R' is isomorphic to R_q.*

For a proof of this result, see the discussion on page 376 of [4].

2. A Γ-ring formulation of Köthe's problem

Let A be a ring. Then A is a Γ-ring with $\Gamma = A$, and the ordinary addition and multiplication operations on A. Let L denote the left operator ring of A. Define a mapping $f\colon A^2 \to L$ by

$$f(\sum_i x_i y_i) = \sum_i [x_i, y_i] \quad (x_i, y_i \in A),$$

where $[x, y]$ denotes, as usual, the mapping defined by $[x, y]z = xyz$ for all $z \in A$. It is easily seen that this mapping is a surjective ring homomorphism, and that its kernel is $A^2 \cap \ell(A)$ where $\ell(A) = \{a \in A\colon aA = 0\}$. Hence, we have that L is isomorphic to the ring $A^2/(A^2 \cap \ell(A))$. Similarly, if R denotes the right operator ring of A, then R is isomorphic to $A^2/(A^2 \cap r(A))$ where $r(A) = \{a \in A\colon \ Aa = 0\}$. Since the class of nil rings is a hereditary radical class, it follows that if A is a nil ring, then both L and R are nil rings.

Suppose now that Köthe's problem has a negative answer, i.e. the nil radical of a ring need not contain all its nil one-sided ideals. Then by [5], Theorem 10, there exists a nil ring N such that the ring N_2 is not nil. Now let $M = N_{1,2}$, and let $\Gamma = N_{2,1}$. Then M is a Γ-ring with the usual operations of matrix addition and multiplication. Let L' and R' denote the left and right operator rings, respectively, of M in this case. By Proposition 2 and the previous discussion, L' is isomorphic to $N^2/N^2 \cap \ell(N)$ and R' is isomorphic to R_2, where $R = N^2/(N^2 \cap r(N))$. From the previous discussion, L' is a nil ring and hence $\mathcal{N}(L')^+ = M$. We claim that R' is not a nil ring.

Suppose that R' is a nil-ring. Then

$$N^2/(N^2 \cap r(N)) \approx (N^2)_2/(N^2 \cap r(N))_2$$

is a nil ring. But $N^2 \cap r(N)$ is a ring with zero multiplication, and hence so is $(N^2 \cap r(N))_2$. Consequently, $(N^2 \cap r(N))_2$ is a nil ring. Since the class of nil rings is a radical class, and hence closed under extensions, it follows that $(N^2)_2$ is a nil ring. If $x \in N_2$, $x^2 \in (N^2)_2$ and hence $(x^2)^n = 0$, i.e. $x^{2n} = 0$ for some positive integer n. Hence N_2 is a nil ring, contradicting our original choice of N. Thus, R is not a nil ring and hence $\mathcal{N}(R') \neq R'$. Now the nil radical is special in the variety of rings, and hence $\mathcal{N}(R')$ is the intersection of a family of prime ideals of R'. It follows (cf. [4], Theorem 1) that $\mathcal{N}(R')=(\mathcal{N}(R')^*)^{*'}$. Suppose the $\mathcal{N}(R')^*=M$. Then $\mathcal{N}(R')=(\mathcal{N}(R')^*)^{*'}=M^{*'}=R'$. Thus we have that $\mathcal{N}(R')^*\neq M$.

We are now in a position to state the main result of this note.

THEOREM. *Let M be an arbitrary Γ-ring with left and right operator rings L and R, respectively. Then the following statements are equivalent:*

(a) *If A is an arbitrary ring, then the nil radical of A contains all the nil one-sided ideals of A.*

(b) $\mathcal{N}(L)^+ = \mathcal{N}(R)^*$

(c) $\mathcal{N}(L)^+ = \mathcal{N}(M)$

(d) $\mathcal{N}(R)^* = \mathcal{N}(M)$.

PROOF.

(a)⇒(b), (a)⇒(c) and (a)⇒(d) follows from Proposition 1.
(b)⇒(a) follows from the example constructed above.

In the example above, we have $\mathcal{N}(L')^+=M$, whence $\mathcal{N}(M)=M$, but $\mathcal{N}(R')^*\neq M$. Hence it follows that (d)⇒(a). A similar example may be constructed to show that (c)⇒(a). (Let N be the ring of the example, let $M=N_{2,1}$ and $\Gamma=N_{1,2}$.) This concludes the proof.

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MEAN CONVERGENCE OF QUASI HERMITE—FEJÉR INTERPOLATION

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1. Introduction. Notations. Preliminary results

1.1. The aim of this paper is twofolded. First, we investigate the weighted L^p convergence, $0<p<\infty$, of quasi Hermite—Fejér (qHF, in short) interpolating processes based on the roots of generalized Jacobi polynomials (cf. Theorem 2.1), further we get the order of convergence, too (cf. Theorem 2.4).

1.2. The qHF interpolation has a long history. Since 1958, when E. Egerváry and P. Turán [2] discovered that the qHF process can be better than the HF one, there have been many papers dealing with uniform convergence. On the other hand, mean convergence was considered only by J. Prasad and A. K. Varma in a very recent paper [7].

In this paper we prove several theorems on weighted mean convergence of qHF interpolation. The theorems and the applied methods in many cases are related to the ones developed in papers P. Nevai [4], P. Nevai, P. Vértesi [6], and P. Vértesi, Y. Xu [11].

1.3. First let us consider some basic notations and facts.

N denotes the set of positive integers. The symbol "const" (or "c") denotes some constant which is positive and independent of the variables and indices. Whenever "const" is used it will always be clear what variables and indices it is independent of. In each formula "const" may take a different value. The symbol "$\sim$" is used as follows. If A and B are two expressions depending on some variables and indices then

$$A\sim B \Leftrightarrow |AB^{-1}|\leqq \text{const} \quad\text{and}\quad |A^{-1}B|\leqq \text{const}.$$

Orthogonal polynomials. Let w be a nonnegative integrable function in $[-1, 1]$ such that

$$\int_{-1}^{1} w>0.$$

The corresponding set of orthogonal polynomials is denoted by $\{p_n(w)\}$:

$$p_n(w)=\gamma_n(w)x^n+\text{lower degree terms},\quad \gamma_n(w)>0$$

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and

$$\int_{-1}^{1} p_n(w)\, p_m(w) w = \delta_{nm}.$$

The zeros of $p_n(w)$ are denoted by $x_{kn}(w)$ and they are indexed so that

(1.1) $$-1 < x_{nn}(w) < x_{n-1,n}(w) < \ldots < x_{1n}(w) < 1.$$

The reproducing kernel $K_n(w)$ is defined by

$$K_n(w, x, t) = \sum_{k=0}^{n-1} p_k(w, x) p_k(w, t).$$

According to the Christoffel—Darboux formula [9, p. 43],

$$K_n(x, t) = \frac{\gamma_{n-1}(w)}{\gamma_n(w)} [p_n(w, x) p_{n-1}(w, t) - p_{n-1}(w, x) p_n(w, t)] (x-t)^{-1}.$$

The Christoffel function $\lambda_n(w, x)$ is defined by

$$\lambda_n(w, x)^{-1} = K_n(w, x, x).$$

It is well-known [9, Theorem 3.1.3] that

(1.2) $$\lambda_n(w, x) = \min \int_{-1}^{1} |P(t)|^2 w(t)\, dt$$

where the minimum is taken over all polynomials P of degree less than n such that $P(x)=1$. The number $\lambda_{kn}(w)$ defined by

$$\lambda_{kn}(w) = \lambda_n(w, x_{kn}(w))$$

are the Cotes numbers. By the Gauss—Jacobi quadrature formula [9, p. 47]

$$\sum_{k=1}^{n} P(x_{kn}(w)) \lambda_{kn}(w) = \int_{-1}^{1} Pw$$

holds for every polynomial P of degree less than $2n$. By Szegő's theorem [9, p. 309]

(1.3) $$0 < \lim_{n\to\infty} \gamma_n(w) 2^{-n} < \infty.$$

Lagrange interpolation. The Lagrange interpolating polynomials corresponding to the distribution w and bounded function f are denoted by $L_n(w, f)$. They satisfy

$$L_n(w, f, x_{kn}(w)) = f(x_{kn}(w)),$$

$n \in N$, $1 \leqq k \leqq n$. The polynomial $L_n(w, f)$ can be written as

(1.4) $$L_n(w, f) = \sum_{k=1}^{n} f(x_{kn}(w)) l_{kn}(w)$$

where the fundamental polynomials $l_{kn}(w)$ are defined by

$$l_{kn}(w, x) = \frac{p_n(w, x)}{p_n(w, x_{kn}(w))(x - x_{kn}(w))}, \quad 1 \leq k \leq n.$$

It is well-known [9, p. 48] that

$$l_{kn}(w, x) = \frac{\gamma_{n-1}(w)}{\gamma_n(w)} \lambda_{kn}(w) p_{n-1}(w, x_{kn}(w)) \frac{p_n(w, x)}{x - x_{kn}(w)}. \tag{1.5}$$

Hermite—Fejér interpolation. Consider those uniquely defined polynomials $H_n(w, f, x)$ of degree at most $2n-1$, for which (using again the nodes (1.1))

$$\begin{cases} H_n(w, f, x_{kn}(w)) = f(x_{kn}(w)), & 1 \leq k \leq n, \\ H_n'(w, f, x_{kn}(w)) = 0, & 1 \leq k \leq n, \end{cases}$$

$n \in N$. These *Hermite—Fejér interpolating polynomials* as it was noticed by G. Freud [3, p. 113] can be written as

$$H_n(w, f, x) = \sum_{k=1}^{n} f(x_{kn}(w)) \left[1 + \frac{\lambda_n'(w, x_{kn}(w))}{\lambda_n(w, x_{kn}(w))} (x - x_{kn}(w))\right] l_{kn}(w, x)^2. \tag{1.6}$$

If P is a polynomial of degree at most $2n-1$, then

$$P(x) = H_n(w, P, x) + G_n(w, P', x) \tag{1.7}$$

where

$$G_n(w, f, x) = \sum_{k=1}^{n} f(x_{kn}(w))(x - x_{kn}(w)) l_{kn}^2(w, x). \tag{1.8}$$

Here (1.7) is the Hermite interpolation formula ([9, p. 331]).

Quasi Hermite—Fejér interpolation. The quasi Hermite—Fejér interpolating polynomial is defined to be the unique polynomial of degree at most $2n+1$, satisfying (using nodes (1.1))

$$\begin{cases} Q_n(w, f, x_{kn}(w)) = f(x_{kn}(w)) & 1 \leq k \leq n, \\ Q_n'(w, f, x_{kn}(w)) = 0, & 1 \leq k \leq n, \\ Q_n(\pm 1) = f(\pm 1). \end{cases} \tag{1.9}$$

It is easy to see that by $v(x) = (1-x^2)^{-1}$

$$\begin{aligned} Q_n(w, f, x) &= v^{-1}(x) H_n(w, fv, x) + v^{-1}(x) G_n(w, fv', x) + \\ &+ f(1) \frac{p_n^2(w, x)}{p_n^2(w, 1)} \frac{1+x}{2} + f(-1) \frac{p_n^2(w, x)}{p_n^2(w, -1)} \frac{1-x}{2} \end{aligned} \tag{1.10}$$

and for any polynomial $P(x)$ of degree $\leq 2n+1$

$$P(x) = Q_n(w, P, x) + v^{-1}(x) G_n(w, P' v, x) \tag{1.11}$$

(cf. [10, p. 88]).

L^p and L^p_u spaces. If $0<p<\infty$ then $f\in L^p$ if $\|f\|_p<\infty$ where

$$\|f\|_p=\Big[\int_{-1}^{1}|f(t)|^p\,dt\Big]^{1/p},\quad 0<p<\infty,$$

and

$$\|f\|_\infty=\operatorname*{ess\,sup}_{t\in[-1,1]}|f(t)|.$$

If $u\geqq 0$ and $0<p<\infty$ then $f\in L^p_u$ if $\|f\|_{u,p}<\infty$ where

$$\|f\|_{u,p}=\Big[\int_{-1}^{1}|f(t)|^p u(t)\,dt\Big]^{1/p}. \tag{1.12}$$

Naturally, when $0<p<1$, $\|\cdot\|_{u,p}$ and $\|\cdot\|_p$ are not norms, nevertheless we retain this notation for convenience.

Jacobi weights. The function u is called a *Jacobi weight* function if u can be written in the form

$$u(x)=(1-x)^a(1+x)^b \tag{1.13}$$

for $-1\leqq x\leqq 1$ and $u(x)=0$ for $|x|>1$. In this paper we do not necessarily assume that u is integrable.

Generalized Jacobi polynomials. Let w be a nonnegative integrable function defined in $[-1,1]$. We say that w is a *generalized Jacobi weight* function ($w\in GJ$) if w can be written in the form

$$w(x)=g(x)(1-x)^a(1+x)^b,\quad -1\leqq x\leqq 1, \tag{1.14}$$

where $a,b>-1$ and $g^{\pm 1}\in L^\infty$. If, in addition, $g(x)$ is continuous and its modulus of continuity $\omega(g,t)$ satisfies

$$\int_0^1\frac{\omega(t)}{t}\,dt<\infty,$$

then w is a *generalized smooth Jacobi weight* ($w\in GSJ$). If, moreover, $g'\in\mathrm{Lip}\,1$ in $[-1,1]$, then w is a *very smooth generalized Jacobi weight* ($w\in GCJ$).

a and b in (1.13) and (1.14) are called the *parameters* of the corresponding weight.

Orthogonal polynomials corresponding to generalized Jacobi weight functions are *generalized Jacobi polynomials* (see the works V. Badkov [1] and P. Nevai [5]).

1.4. For the HF interpolation polynomial, in [6, Theorem 5, p. 55] P. Nevai and P. Vértesi proved as follows (the original statement is slightly stronger).

THEOREM 1.1. *Let $w\in GCJ$, $p>0$ and let u be a Jacobi weight function. Then*

$$\lim_{n\to\infty}\|H_n(w,f)-f\|_{u,p}=0\quad\forall f\in C[-1,1] \tag{i}$$

holds iff

$$w^{-p}u\in L^1. \tag{ii}$$

In a very recent paper, we considered the rate of convergence, too (P. Vértesi, Y. Xu [11, Theorem 2.1]).

THEOREM 1.2. *Let* $w\in GCJ$, $W(x):=w(x)\sqrt{1-x^2}$, $p>0$ *and let* u *be an integrable Jacobi weight function. Then*

(a) $\quad \|H_n(w,f)-f\|_{u,p} \leqq \text{const.}\,\omega\left(f,\frac{1}{n}\right) \qquad \forall f\in C[-1,1],$

holds iff

(b) $\quad W^{-p}u\in L^1,$

(b*) $\quad W^{-p^*}u\in L^1 \qquad \forall p^*<p.$

Then (b)⇒(c), *and* (a)⇒(b*).

(The case $u(x)=w(x)=(1-x^2)^{-1/2}$ was previously considered by J. Prasad and A. K. Varma [7, Theorem 1].)

For the qHF interpolation polynomial, J. Prasad and A. K. Varma [7, Theorem 1.3] obtained

THEOREM 1.3. *If* $u(x)=(1-x^2)^{-1/2}$, $w(x)=(1-x^2)^{\pm 1/2}$ *and* $p>0$, *then*

$$\|Q_n(w,f)-f\|_{u,p} \leqq \text{const.}\,\omega\left(f,\frac{1}{n}\right) \qquad \forall f\in C[-1,1].$$

2. Results

2.1. First we state the analogy of Theorem 1.1.

THEOREM 2.1. *Let* $p>0$ *and* $w\in GCJ$ *with parameters*

$$a, b \geqq 0$$

or

$$-\frac{1}{2} \leqq a, b \quad \text{and} \quad |a-b| \leqq 1.$$

Further, let u be an integrable Jacobi weight function. Then

(i) $$\lim_{n\to\infty} \|Q_n(w,f)-f\|_{u,p} = 0 \quad \forall f\in C[-1,1]$$

holds iff with $w_1(x):=w(x)/(1-x^2)$

(ii) $$w_1^{-p}u\in L^1.$$

As a simple consequence of Theorem 2.1, we may consider the following special cases. Let $u(x)=(1-x^2)^{\gamma}$, $w(x)=g(x)(1-x^2)^{\alpha}$, $\alpha\geqq-\frac{1}{2}$, $\gamma>-1$. Now $a=b=\alpha$, so $|a-b|\leqq 1$ holds naturally. Since $w_1^{-p}u\in L^1$ iff $\gamma-\alpha p+p>-1$, then if $-\frac{1}{2}\leqq$

$\leqq\alpha\leqq 1$, $w_1^{-p}u\in L^1$ for any $\gamma>-1$, $p>0$; and if $\alpha>1$, then $w_1^{-p}u\in L^1$ iff $p<\dfrac{\gamma+1}{\alpha-1}$. So we conclude

Corollary 2.2. *If* $u(x)=(1-x^2)^\gamma$ *and* $w(x)=g(x)(1-x^2)^\alpha$, $g>0$, $g'\in\text{Lip}\,1$, $\alpha\geqq-\dfrac{1}{2}$, $\gamma>-1$, *then*

(1) *if* $-\dfrac{1}{2}\leqq\alpha\leqq 1$, *then* (i) *holds for any* $p>0$ *and* $\gamma>-1$;

(2) *if* $\alpha>1$, *then* (i) *holds iff* $p<\dfrac{\gamma+1}{\alpha-1}$, *the last inequality being* $p<1+\dfrac{2}{\alpha-1}$, *when* $\alpha=\gamma$.

In the "irregular case" $|a-b|>1$, we can prove the following

Theorem 2.3. *Let* $w\in GCJ$ *with parameters* $a,b\geqq-\dfrac{1}{2}$ *and* $p>0$. *Further let* u *be an integrable Jacobi weight function. If we introduce the notations*

(i) $$\lim_{n\to\infty}\|Q_n(w,f)-f\|_{u,p}=0\qquad\forall f\in C[-1,1],$$

(ii) $$\left(\frac{w_1(x)}{(1-x^2)^a}\right)^{-p}u(x)\in L^1,$$

(ii*) $$\left(\frac{w_1(x)}{(1-x^2)^a}\right)^{-p^*}u(x)\in L^1\quad\forall p^*<p,$$

(iii) $$\left(\frac{w_1(x)}{(1-x^2)^b}\right)^{-p}u(x)\in L^1,$$

(iii*) $$\left(\frac{w_1(x)}{(1-x^2)^b}\right)^{-p^*}u(x)\in L^1\quad\forall p^*<p,$$

then

(1) *if* $-1/2\leqq a<0$ *and* $b-a>1$, *then* (ii)$\Rightarrow$(i) *and* (i)$\Rightarrow$(ii*),
(2) *if* $-1/2\leqq b<0$ *and* $a-b>1$, *then* (iii)$\Rightarrow$(i) *and* (i)$\Rightarrow$(iii*).

2.2. For the rate of convergence we can prove (cf. Theorem 1.3).

Theorem 2.4. *Let* $w\in GCJ$, *with parameters*

$$a,b\geqq 0$$

or

$$-\frac{1}{2}\leqq a,b\quad\text{and}\quad|a-b|\leqq 1,$$

and let $W_1(x):=w(x)/\sqrt{1-x^2}$ *and* $p>0$. *Further let* u *be an integrable Jacobi weight* *function. If*

(a) $$\|Q_n(w,f)-f\|_{u,p} \leqq \text{const.}\, \omega\left(f, \frac{1}{n}\right) \quad \forall f \in C[-1,1],$$

(b) $$W_1^{-p} u \in L^1,$$

(b*) $$W_1^{-p^*} u \in L^1 \quad \forall p^* < p,$$

then (b)⇒(a) *and* (a)⇒(b*).

Again, let us consider some special cases.

COROLLARY 2.5. *If* $u(x)=(1-x^2)^\gamma$ *and* $w(x)=g(x)(1-x^2)^\alpha$, $g>0$, $g' \in \text{Lip}\, 1$, $\alpha \geqq -\frac{1}{2}$, $\gamma > -1$, *then*

(1) *if* $-\frac{1}{2} \leqq \alpha \leqq \frac{1}{2}$, (a) *holds true for any* $p>0$ *and* $\gamma>-1$;

(2) *if* $\alpha > \frac{1}{2}$, (a) *holds true if* $p < \dfrac{\gamma+1}{\alpha-\frac{1}{2}}$; *and*

whenever (a) *holds true,* $p \leqq \dfrac{\gamma+1}{\alpha-\frac{1}{2}}$, *the last inequality being* $p \leqq 1+\dfrac{2}{2\alpha-1}$ *if* $\alpha=\gamma$.

In the "irregular case" $|a-b|>1$, we can prove the following

THEOREM 2.6. *Let* $w \in GCJ$ *with parameters* $a, b \geqq -\frac{1}{2}$, *and* $p>0$. *Further let u be an integrable Jacobi weight function. If we introduce the notations*

(a) $$\|Q_n(w,f)-f\|_{u,p} \leqq \text{const.}\, \omega\left(f, \frac{1}{n}\right) \quad \forall f \in C[-1,1],$$

(b) $$\left(\frac{W_1(x)}{(1-x^2)^a}\right)^{-p} u(x) \in L^1,$$

(b*) $$\left(\frac{W_1(x)}{(1-x^2)^a}\right)^{-p^*} u(x) \in L^1 \quad \forall p^* < p$$

(c) $$\left(\frac{W_1(x)}{(1-x^2)^b}\right)^{-p} u(x) \in L^1,$$

(c*) $$\left(\frac{W_1(x)}{(1-x^2)^b}\right)^{-p^*} u(x) \in L^1 \quad \forall p^* < p,$$

then

(1) *if* $-\frac{1}{2} \leqq a < 0$ *and* $b-a>1$, *then* (ii)⇒(i) *and* (i)⇒(ii*),

(2) *if* $-\frac{1}{2} \leqq b < 0$ *and* $a-b>1$, *then* (iii)⇒(i) *and* (i)⇒(iii*).

2.3. Remark. *It follows from our proof that in Theorems 2.4 and 2.6,* $\omega\left(f, \frac{1}{n}\right)$ *generally cannot be replaced by* $\omega_2\left(f, \frac{1}{n}\right)$, *so they, in certain sense, are the best possible.* (Indeed, if $f_1(x)=x$, by (3.15),

$$\|Q_n(w, f_1)-f_1\|_{u,p}^p \geqq \int_{(1+2x_{1n})/3}^{(2+x_{1n})/3} \left|\sum_{k=1}^{n} \frac{1-x^2}{1-x_{kn}^2(w)}(x-x_{kn}(w))\, l_{kn}^2(w, x)\right|^p u(x)\, dx >$$

$$> \omega_2\left(f_1, \frac{1}{n}\right) = 0.$$

A similar behaviour was noticed concerning HF interpolation. On the other hand for Lagrange interpolation, the situation is as good as possible: if we have mean convergence, the order is $E_n(f)$ where $E_n(f)$ is the best uniform approximation of f with polynomials of degree $\leqq n$. (See [11, 2.2 and 2.3].)

3. Proofs

3.1. The proofs of Theorems 2.1, 2.3, 2.4, and 2.5 are based on the close connection among $Q_n(f, x)$, $H_n(f, x)$ and $L_n(f, x)$ and some basic inequalities proved below (cf. Lemma 3.1). The cases $\min\{a, b\}<0$, $|a-b|>1$ demanded special considerations, too. (The conditions $a, b \geqq -\frac{1}{2}$ are used when we apply Lemma 3.2 (cf. (3.16)).)

3.2. First let us consider some basic facts.

Let $w \in GSJ$, and let $x_{kn}(w)=\cos\theta_{kn}$ ($x_{on}=1$, $x_{n+1,n}=-1$, $0 \leqq \theta_{kn} \leqq \pi$). Then

(3.1)
$$\theta_{k+1,n}-\theta_{kn} \sim \frac{1}{n}$$

uniformly for $0 \leqq k \leqq n$, $n \in N$,

(3.2)
$$\lambda_{kn}(w) \sim \frac{1}{n} w(x_{kn}(w)) \sqrt{1-x_{kn}(w)^2}$$

uniformly for $1 \leqq k \leqq n$, $n \in N$,

(3.3)
$$|\lambda_n'(w, x_{kn}(w))| \leqq \text{const.} \frac{1}{n} w(x_{kn}(w))(1-x_{kn}(w)^2)^{-1/2}$$

uniformly for $1 \leqq k \leqq n$ and $n \in N$,

(3.4)
$$|p_{n-1}(w, x_{kn}(w))| \sim w(x_{kn}(w))^{-1/2}(1-x_{kn}(w)^2)^{1/4}$$

uniformly for $1 \leqq k \leqq n$, $n \in N$,

(3.5)
$$|p_n(w, x)| \leqq \text{const.} \begin{cases} [w(x)(1-x^2)^{1/2}]^{-1/2}, & |x| \leqq 1-n^{-2}, \\ \sqrt{n}\,[w(1-n^{-2})]^{-1/2}, & 1-cn^{-2} \leqq x \leqq 1, \\ \sqrt{n}\,[w(-1+n^{-2})]^{-1/2}, & -1 \leqq x \leqq -1+cn^{-2}, \end{cases}$$

uniformly for $n \in N$, and,

(3.6)
$$|p_n(w,x)| \sim \begin{cases} n|x-x_{mn}(w)|[w(x)(1-x^2)^{3/2}]^{-1/2}, & -1+x_{nn}(w) \leqq 2x \leqq 1+x_{1,n}(w), \\ n^{1/2}[w(1-n^{-2})]^{-1/2}, & 1+x_{1n}(w) \leqq 2x \leqq 2, \\ n^{1/2}[w(-1+n^{-2})]^{-1/2}, & -2 \leqq 2x \leqq -1+x_{nn}(w), \end{cases}$$

uniformly for $n \in N$ where $x_{mn}(w)$ is (one of) the closest node(s) to x.

3.3. The following inequalities are the fundamental tools in our proof. (Similar ones were proved in [6, Lemma 4].)

LEMMA 3.1. *Let $w \in GJ$ with parameters $a, b \geqq -\frac{1}{2}$. Then, with $0<\sigma<1$,*

(3.7)
$$\sum_{k=1}^{n} \frac{1-x^2}{1-x_{kn}(w)^2} l_{kn}(w,x)^2 \leqq \text{const.}\left[1+\frac{\log n}{n}\frac{(1-x^2)^{1/2}}{w(x)}\right]$$

uniformly for $n \geqq 2$, $|x| \leqq 1-\sigma n^{-2}$. Further

(3.8)
$$\sum_{k=1}^{n} \frac{1-x^2}{1-x_{kn}(w)^2} |x-x_{kn}(w)|\, l_{kn}(w,x)^2 \leqq \text{const.}\left[\frac{\log n}{n}+\frac{1}{n}\frac{(1-x^2)^{1/2}}{w(x)}\right]$$

uniformly for $n \geqq 2$, $|x| \leqq 1-\sigma n^{-2}$. Finally

(3.9)
$$\sum_{k=1}^{n} \frac{1-x^2}{(1-x_{kn}(w)^2)^2} |x-x_{kn}(w)|\, l_{kn}(w,x)^2 \leqq$$

$$\leqq \text{const.}\begin{cases} \left[1+\frac{(1-x^2)^{1/2}}{w(x)}\frac{\log n}{n}\right] & \textit{if} \quad a,b \geqq 0 \quad \textit{or} \quad -\frac{1}{2} \leqq a,b \quad \textit{and} \quad |a-b| \leqq 1, \\ \left[1+\frac{(1-x^2)^{1/2}}{w(x)}\frac{1}{n^{1+2a}}\right] & \textit{if} \quad -\frac{1}{2} \leqq a < 0 \quad \textit{and} \quad b-a>1, \\ \left[1+\frac{(1-x^2)^{1/2}}{w(x)}\frac{1}{n^{1+2b}}\right] & \textit{if} \quad -\frac{1}{2} \leqq b < 0 \quad \textit{and} \quad a-b>1, \end{cases}$$

uniformly for $n \geqq 2$, $|x| \leqq 1-\sigma n^{-2}$.

PROOF. We only prove (3.9). (3.7) and (3.8) can be proved similarly. Let $x_{mn}(w)$ be as above. Write $x=\cos\theta$, $x_{kn}(w)=\cos\theta_{kn}(w)$. Then by (1.5), (3.1)—(3.5),

$$\frac{1-x^2}{(1-x_{mn}(w)^2)^2}[x-x_{mn}(w)]\, l_{kn}(w,x)^2 \leqq \text{const.}\frac{w(x_{mn}(w))(1-x_{mn}(w)^2)^{1/2}}{w(x)(1-x^2)n} \leqq$$

$$\leqq \text{const.}\frac{(1-x_{mn}(w)^2)^{-1/2}}{n} \leqq \text{const.},$$

and if $x \geqq 0$, say,

$$\sum_{k \neq m} \frac{1-x^2}{(1-x_{kn}(w)^2)^2} |x-x_{kn}(w)| \, l_{kn}(w, x)^2 \leqq$$

$$\leqq \text{const.} \sum_{k \neq m} \frac{1-x^2}{1-x_{kn}(w)^2} \cdot \frac{w(x_{kn}(w))}{w(x)} \cdot \frac{1}{n^2 |x-x_{kn}(w)|} \sim$$

$$\sim \frac{1}{m^{2a-1}} \sum_{\substack{k=1 \\ k \neq m}}^{[(n+m)/2]} \frac{k^{2a-1}}{|k-m| \, |k+m|} + \frac{1}{m^{2a-1} n^{2+2b-2a}} \sum_{k=[(n+m)/2]}^{n} (n-k+1)^{2b-1} :=$$

$$:= I_1 + I_2.$$

Let us write I_1 as

$$I_1 = \sum_{1 \leqq k \leqq [m/2]} + \sum_{[m/2] < k \leqq [(3m)/2]} + \sum_{[(3m)/2] < k \leqq [(n+m)/2]} .$$

By estimating the sums we get

$$I_1 \leqq \text{const.} \left[1 + \left(\frac{n}{m} \right)^{2a-1} \frac{\log n}{n} \right] \leqq \text{const.} \left[1 + \frac{\log n}{n} \frac{(1-x^2)^{1/2}}{w(x)} \right]$$

for $a \geqq -\frac{1}{2}$. Further, $I_2 \leqq \dfrac{\text{const.}}{m^{2a-1} n^{2+2b-2a}} \sum_{k=1}^{n} k^{2b-1}$.
Since

$$\sum_{k=1}^{n} k^{2b-1} \sim \begin{cases} n^{2b}, & b > 0, \\ \log n, & b = 0, \\ 1, & -1/2 \leqq b < 0, \end{cases} \tag{3.10}$$

so if $b \geqq 0$, we have for any $a \geqq -\frac{1}{2}$,

$$I_2 \leqq \text{const.} \left(\frac{n}{m} \right)^{2a-1} \frac{\log n}{n} \leqq \text{const.} \frac{\log n}{n} \frac{(1-x^2)^{1/2}}{w(x)}.$$

Further, if $-\frac{1}{2} \leqq b < 0$, then $I_2 \leqq \text{const.} \left(\frac{n}{m} \right)^{2a-1} \frac{1}{n^{1+2b}}$.

Therefore if $2a-1 \leqq 0$ i.e. $a \leqq 1/2$, we get $I_2 \leqq \text{const.}$

On the other hand, if $2a-1>0$ $\left(\text{or } a > \frac{1}{2}\right)$

$$I_2 \leqq \text{const.} \, n^{2(a-b-1)} \leqq \begin{cases} \text{const.} & \text{if } a-b \leqq 1, \\ \text{const.} \dfrac{(1-x^2)^{1/2}}{w(x)} \cdot \dfrac{1}{n^{1+2b}} & \text{if } a-b > 1. \end{cases}$$

Summarizing, we get that

$$I_1+I_2 \leqq \text{const.}\left[1+\frac{\log n}{n}\frac{(1-x^2)^{1/2}}{w(x)}\right]$$

when $b \geqq 0$ and $a \geqq -1/2$ or when $-\frac{1}{2} \leqq b < 0$ and $a-b \leqq 1$; further

$$I_1+I_2 \leqq \text{const.}\left[1+\frac{1}{n^{1+2b}}\frac{(1-x^2)^{1/2}}{w(x)}\right]$$

if $-1/2 \leqq b < 0$ and $a-b > 1$.

Similarly, supposing $x<0$, we get the corresponding relations under the conditions $a \geqq 0, b \geqq -\frac{1}{2}$, or $-\frac{1}{2} \leqq a < 0$ and $b-a \leqq 1$; or $-\frac{1}{2} \leqq a < 0$ and $b-a > 1$. These complete the proof.

3.4. We shall use the following

LEMMA 3.2 [4, Theorem 1, p. 680].[2] *Let $w \in GSJ$ and $p>0$. Let v, V be two not necessarily integrable Jacobi weight functions such that $V \in L^p$, $vV \in L^p$, $V/W^{1/2} \in L^p$ and $vW^{1/2} \in L^1$, where as above, $W(x)=w(x)\sqrt{1-x^2}$. Then for any given bounded functions f_n, $n \in N$,*

$$\|L_n(w, vf_n)V\|_p \leqq \text{const.}\, \|f_n\|_\infty, \quad n \in N,$$

with some constant independent of $\{f_n\}$.

3.5. The following lemma is interesting in itself.

LEMMA 3.3. *Let $w \in GCJ$ with parameters $a, b \geqq -\frac{1}{2}$, further let $p>0$. If u is an integrable Jacobi weight function, then for any bounded functions g_n, $g_n(\pm 1)=0$, $n \in N$,*

$$\|Q_n(w, g_n)\|_{u,p} \leqq \text{const.}\, \|g_n\|_\infty$$

holds true if either of the following holds

(i) $\quad w_1^{-p}u \in L^1$, *if* $a, b \geqq 0$, *or* $|a-b| \leqq 1$,

(ii) $\quad \left(\frac{w_1(x)}{(1-x^2)^a}\right)^{-p} u(x) \in L^1$, *if* $a<0$ *and* $b-a>1$,

(iii) $\quad \left(\frac{w_1(x)}{(1-x^2)^b}\right)^{-p} u(x) \in L^1$, *if* $b<0$ *and* $a-b>1$.

---

[2] The original theorem is stated for a fixed function f. On the other hand, one can see from its proof that it can be stated in the above mentioned form, too.

PROOF. By (1.9) we have

$$Q_n(w, g_n, x)-g_n(x) = v^{-1}H_n(w, g_n v, x)+v^{-1}G_n(w, g_n v^{-1}, x). \tag{3.11}$$

For the second term on the right-hand side

$$v^{-1}G_n(w, g_n v^{-1}, x) =$$

$$=(1-x^2)\sum_{k=1}^{n} g_n(x_{kn}(w))\frac{-2x_{kn}(w)}{(1-x_{kn}(w)^2)^2}(x-x_{kn}(w))\,l_{kn}(w, x)^2.$$

By Theorem 6.3.14 in [5, p. 113], for every $0<p<\infty$ and Jacobi weight u there exists a constant $\sigma=\sigma(p, u)>0$ such that for every polynomial P of degree at most $2n$

$$\int_{-1}^{1}|P(t)|^p u(t)\,dt \leqq 2\int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}}|P(t)|^p u(t)dt. \tag{3.12}$$

Using this for polynomial $v^{-1}G_n(w, gv, x)$, if (i) holds, we get by (3.9)

$$\|v^{-1}G_n(w, g_n v')\|_{u,p}^p \leqq 2\|g_n\|_\infty^p \int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} |v^{-1}G_n(w, g_n v', x)^p u(x)\,dx \leqq$$

$$\leqq \text{const.}\,\|g_n\|_\infty^p \int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}}\left[1+\frac{\log n}{n}\,\frac{(1-x^2)^{1/2}}{w(x)}\right]^p u(x)\,dx \leqq$$

$$\leqq \text{const.}\,\|g_n\|_\infty^p \int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}}[1+w_1(x)^{-1}(1+|\log(1-x^2)|)]^p u(x)\,dx.$$

Since $w_1\in GCJ$ and u is a Jacobi weight function, if $w_1^{-1}\in L_u^p$, then also $w_1^{-1}(x)|\log(1-x^2)|\in L_u^p$. So we get

$$\|v^{-1}G_n(w, g_n v')\|_{u,p} \leqq \text{const.}\,\|g_n\|_\infty.$$

If instead of (i), (ii) or (iii) holds, then we can use the other part of (3.9) and the fact that $\frac{1}{n}\leqq\sqrt{1-x^2}$ in the corresponding interval to get the same conclusion.

By (1.6) the first term on the right-hand side of (3.11) is

$$v^{-1}H_n(w, g_n v, x) = \sum_{k=1}^{n} g_n(x_{kn}(w))\frac{1-x^2}{1-x_{kn}(w)^2}\,l_{kn}(w, x)^2+$$

$$+\sum_{k=1}^{n} g_n(x_{kn}(w))\frac{1-x^2}{1-x_{kn}(w)^2}\cdot\frac{\lambda_n'(x_{kn}(w))}{\lambda_n(x_{kn}(w))}(x-x_{kn}(w))\,l_{kn}(w,x)^2.$$

Here the first term can be estimated by (3.7) and the above method. The second term, by (3.2) and (3.3), is bounded by

$$\text{const.}\,\|g_n\|_\infty \sum_{k=1}^{n}\frac{1-x^2}{(1-x_{kn}(w)^2)^2}\,|x-x_{kn}(w)|\,l_{kn}(w, x)^2,$$

therefore can be estimated as what we did for the second term on the right of (3.11). Put all these estimation together, the proof is completed.

3.6. By a well-known theorem of S. B. Stečkin [8], there exists a polynomial $R_n(x)$ of degree at most n such that for all x, $-1 \leqq x \leqq 1$,

$$|f(x)-R_n(x)| \leqq \text{const.}\, \omega\left(f, \frac{\sqrt{1-x^2}}{n}\right) \tag{3.13}$$

and

$$(1-x^2)^{1/2}|R_n'(x)| \leqq \text{const.}\, n\omega\left(f, \frac{1}{n}\right). \tag{3.14}$$

For this $R_n(x)$ we prove as follows.

Lemma 3.4. *Let $w \in GCJ$ with parameters $\geqq -\frac{1}{2}$, $p>0$. Further let u be an integrable Jacobi weight function. Then*

$$\|Q_n(w, R_n)-R_n\|_{u,p} \leqq \text{const.}\, \omega\left(f; \frac{1}{n}\right) \quad \forall f \in C[-1, 1]$$

holds true if $W_1^{-p} u \in L^1$.

Proof. By (1.11), (1.8) and (1.5)

$$R_n(x)-Q_n(w, R_n, x) = \sum_{k=1}^{n} R'(x_{kn}(w)) \frac{1-x^2}{1-x_{kn}(w)^2}(x-x_{kn}(w)) l_{kn}^2(w, x) = \tag{3.15}$$

$$= \frac{\gamma_{n-1}(w)}{\gamma_n(w)} \sum_{k=1}^{n} R'(x_{kn}(w)) \frac{1-x^2}{1-x_{kn}(w)^2} \lambda_{kn}(w) p_{n-1}(w, x_{kn}(w)) p_n(w, x) l_{kn}(w, x).$$

Using (3.2) and (3.3), we have

$$\lambda_{kn}(w) p_{n-1}(w, x_{kn}(w)) = c_{kn} w(x_{kn}(w))^{1/2}(1-x_{kn}(w)^2)^{3/4}/n$$

where c_{kn} are constants bounded uniformly for k and n. Define the continuous function $c_n(x)$ such that

$$c_n(x_{kn}(w)) = c_{kn}$$

and $c_n(x)$ uniformly bounded for $-1 \leqq x \leqq 1$ and n. If the continuous function $q_n(x)$ is defined by

$$q_n(x) = c_n(x) R'(x) \sqrt{1-x^2}/n,$$

then by (3.14)

$$\|q_n\|_\infty \leqq \text{const.}\, \omega\left(f; \frac{1}{n}\right). \tag{3.16}$$

So we can rewrite (3.15) as

$$R_n(x)-Q_n(w, R_n, x) = \frac{\gamma_{n-1}(w)}{\gamma_n(w)} v(x)^{-1} p_n(w, x) L_n(w, q_n v^{1/2} W_1^{1/2}, x). \tag{3.17}$$

Using (3.12), considering the corresponding interval $|x|\leqq 1-\sigma n^{-2}$ as in the proof of Lemma 3.3, we can get by (3.5)

$$\|R_n-Q_n(w, R_n)\|_{u,p} \leqq \text{const.}\, \|L_n(w, q_n v^{1/2} W_1^{1/2}) v^{-1/2} W_1^{-1/2}\|_{u,p},$$

where $v(x)=(1-x^2)^{-1}$, as before. Now we apply Lemma 3.2 with $v=(vW_1)^{1/2}$, $V=(vW_1)^{-1/2}u^{1/p}$. It is easy to check that the conditions of Lemma 3.2 hold true under the assumption. So we get from (3.16), (3.17) and Lemma 3.2 that

$$\|R_n-Q_n(w, R_n)\|_{u,p} \leqq \text{const.}\, \|q_n\|_\infty \leqq \text{const.}\, \omega\left(f, \frac{1}{n}\right),$$

which was to be proved.

3.7. Lemma 3.5. *Let $w\in GCJ$ with parameters $\geqq -\frac{1}{2}$ and $p>0$, fixed. Further let u be an integrable Jacobi weight function. Then*

(i) $\lim_{n\to\infty} \|Q_n(w, s)-s\|_{u,p}=0$ *for every polynomial s of degree at most $2n+1$,*

iff

(ii) $w_1^{-p}u\in L^1$.

Proof. (ii)⇒(i). By (1.12), (3.12) and Lemma 3.1, (3.8)

$$\|s-Q_n(w, s)\|_{u,p}^p \leqq$$

$$\leqq \text{const.}\, \|s'\|_\infty^p \int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} \left(\frac{\log n}{n}+\frac{\sqrt{1-x^2}}{nw(x)}\right)^p u(x)\,dx \leqq$$

$$\leqq \text{const.}\, \|s'\|_\infty^p \left[\left(\frac{\log n}{n}\right)^p + \int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} \left|\frac{\sqrt{1-x^2}}{nw(x)}\right|^p u(x)\,dx\right].$$

If $w_1^{-p}u\in L^1$, then there exists a $\delta=\delta(p)>0$, such that $\left(\frac{(1-x^2)^{1-\delta}}{w(x)}\right)^p u(x)\in L^1$. So

$$\int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} \left|\frac{\sqrt{1-x^2}}{nw(x)}\right|^p u(x)\,dx = \frac{1}{n^{\delta p}} \int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} \left|\frac{(\sqrt{1-x^2})^{1-\delta}}{n^{1-\delta}w(x)}\right|^p (\sqrt{1-x^2})^{p\delta}u(x)\,dx \leqq$$

$$\leqq n^{-p\delta} \int_{-1}^{1} \left(\frac{(1-x^2)^{1-\delta}}{w(x)}\right)^p u(x)\,dx \leqq \text{const.}\, n^{-p\delta}.$$

Therefore

$$\|s-Q_n(w, s)\|_{u,p} \leqq \text{const.}\, \|s'\|_\infty\left(\frac{\log n}{n}+n^{-p\delta}\right) \to 0.$$

(i)⇒(ii). Let $f_1(x)=x$. Then, since

$$f_1(x)-Q_n(w, f_1, x) = \sum_{k=1}^{n} \frac{1-x^2}{1-x_{kn}(x)^2}(x-x_{kn}(w))\, l_{kn}(w, x)^2,$$

for $\frac{1-x_{1n}(w)}{2} \leqq x \leqq 1$, by (1.3), (3.2) and (3.3), we have

$$|f_1(x)-Q_n(w, f_1, x)| \geqq \text{const.}\,(1-x^2)\sum_{k=1}^{n} w(x_{kn}(w))(1-x_{kn}(w)^2)^{1/2}\frac{p_n^2(w, x)}{n^2(1-x_{kn}(w))}.$$

If $w(x)=g(x)(1-x)^a(1+x)^b$, $u(x)=(1-x)^\gamma(1+x)^\delta$, then

$$|f_1(x)-Q_n(w, f_1, x)| \geqq \text{const.}\,\frac{(1-x^2)p_n(w, x)^2}{n^2}A_n \tag{3.18}$$

where

$$A_n = \sum_{k=1}^{n}\frac{w(x_{kn}(w))(1-x_{kn}(w)^2)^{1/2}}{1-x_{kn}(w)} \sim \sum_{k=1}^{[n/2]}\left(\frac{k}{n}\right)^{2a-1}+\sum_{k=[n/2]}^{n}\left(\frac{n-k+1}{n}\right)^{2b+1}.$$

By (3.10), we have

$$A_n \geqq \text{const.}\begin{cases} n, & a>0,\\ n\log n, & a=0,\\ n^{1-2a}, & -\frac{1}{2}\leqq a<0.\end{cases} \tag{3.19}$$

By (i), we can write

$$\int_{(1+x_{1n}(w))/2}^{1} |Q_n(w, f_1, x)-f_1(x)|^p u(x)\,dx \to 0$$

which, using (3.18), gives

$$n^{-2p}A_n^p \int_{(1+x_{1n}(w))/2} |(1-x^2)p_n(w, x)^2|^p u(x)\,dx \to 0.$$

By (3.6), this is equivalent to

$$n^{-p}w(1-n^{-2})^{-p}A_n^p \int_{(1+x_{1n}(w))/2}^{1}(1-x^2)^p u(x)\,dx \to 0,$$

or

$$n^{2ap-p-2(\gamma+p+1)}A_n^p \to 0. \tag{3.20}$$

Using (3.19), we can see that (3.20) is equivalent to $w_1^{-p}u\in L^1[0,1]$. The interval $[-1, 0]$ can be treated similarly.

3.8. Proof of Theorem 2.1. (ii)⇒(i) follows directly from Lemma 3.3, Lemma 3.5 and the inequality

$$\|Q_n(w,f)-f\|_{u,p} \leqq c(\|Q_n(w,f-R_n)\|_{u,p}+\|Q_n(w, R_n)-R_n\|_{u,p}+\|f-R_n\|_{u,p}). \tag{3.21}$$

(i)⇒(ii) can be obtained if we use (i) for polynomials and apply Lemma 3.5.

3.9. Proof of Theorem 2.3. (ii)⇒(i) and (iii)⇒(i) follows from Lemma 3.3, Lemma 3.5 and (3.21), since (ii) (or (iii)) implies $w_1^{-p}u\in L^1$.

Now we prove that (i)⇒(iii*) whenever $-\frac{1}{2}\leqq b<0$ and $a-b>1$. For this aim let $g_n(x)=0$ if $x\in[x_{nn}(w), 1]$, $g_n(-1)=1$ and let g_n be linear if $x\in[-1, x_{nn}(w)]$. It is easy to see that

$$|Q_n(w, g_n, x)-g_n(x)|\geqq \text{const.}\, n^{2(a-b-1)} \quad \text{if} \quad x\in I_n$$

where $I_n:=[(2x_{1n}(w)+1)/3, (x_{1n}(w)+2)/3]$ (see [12; (4.30)]), from where

$$\|Q_n(w, g_n)-g_n\|_{u,p}^p \geqq \int_{I_n} |Q_n(w, g_n, x)-g_n(x)|^p u(x)\,dx \geqq$$

$$\geqq \text{const.}\, n^{2p(a-b-1)-2-2\gamma}.$$

Now, by the proof of [13, Theorem 2.1] with the cast

$$T_n(f, x) = T_n(f) = \Big(\int_{I_n} |Q_n(w, f, x)-f(x)|^p u(x)\,dx\Big)^{1/p},$$

$$U_n(f, x) = U_n(f) = 0,$$

$$\lambda_n^p(x) = \lambda_n^p = n^{2p(a-b-1)-2-2\gamma},$$

$e_n\equiv 1$ and $\delta_n=n^{-2}$ we get for a suitable $h\in C[-1, 1]$ and $\{n_i\}$

$$T_n(h) > \text{const.}\, \omega\left(h, \frac{1}{n^2}\right)\lambda_n, \quad n = n_1, n_2, \ldots. \tag{3.22}$$

(If $0<p<1$, $T_n(f+g)\leqq A_p[T_n(f)+T_n(g)]$ with $A_p\geqq 1$, so instead of [13, 2.10] we suppose $\omega(\delta_{n_1})\leqq q/A_r$, $\omega(\delta_{n_{i+1}})\leqq q\omega(\delta_{n_i})/A_p$ to get (3.20).)

By (3.22), using $\|Q_n(\omega, h)-h\|_{u,p}^p\to 0$, we get

$$2p(a-b-1)-2-2\gamma \leqq 0 \quad \text{or} \quad p(-a+b+1)+\gamma \geqq -1 \tag{3.23}$$

(if $\omega(h, t)=|\log t|^{-1}$, say). This condition is equivalent to (iii*).

Statement (1) can be completed by a similar argument.

3.10. Proof of Theorem 2.4. (a)⇒(b) follows from Lemma 3.3, Lemma 3.4 and (3.21). (a)⇒(b*) can be proved by (3.18), (3.19),

$$\int_{(1+x_{1n}(w))/2}^{1} |Q_n(w, f_1, x)-f_1(x)|^p u(x)\,dx \leqq \text{const.}\, \omega\left(f_1, \frac{1}{n}\right)^p = \text{const.}\, n^{-p}$$

and by the argument used in the second part of 3.7. We omit the details.

3.11. Proof of Theorem 2.6. (b)⇒(a) and (c)⇒(a) follows from Lemma 3.3, Lemma 3.4 and (3.21) since (b) or (c) implies $w^{-p}u\in L^1$ for the corresponding a, b.

To get (a)⇒(c*) (see (2)) we proceed as in 3.9. Again we get a statement analogous to (3.22) from where by (i) we get instead of (3.21),

$$p(-a+b+1/2)+\gamma \geqq -1 \tag{3.23}$$

which is equivalent to (c*).

(a)⇒(c*) can be obtained similarly.

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ON HERMITE—FEJÉR INTERPOLATION: A BIBLIOGRAPHY (1914—1987)[1]

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1. Introduction

The present inventory intends to give a comprehensive survey of the literature which has been published concerning so-called "Hermite—Fejér interpolation" (HFI). It thus supplements and extends earlier compilations on the same subject due to Mills ("Some techniques in approximation theory"), by Eisenberg ("Recent developments in approximation by Hermite—Fejér operators"), and by Shen ("On polynomial interpolation II — Hermite Interpolation").

Weierstraß' original proofs of his first and second theorem concerning the approximation of continuous functions were quite involved (see his paper "Über die analytische Darstellbarkeit sogenannter willkürlicher Funktionen einer reellen Veränderlichen", presented to the Prussian Academy of Sciences in July 1885). It is for this reason that, at the beginning of this century, several authors strived to give simpler proofs of Weierstraß' fundamental theorems. The most famous fruits of these efforts (for the algebraic case) are probably the construction of the polynomials of S. N. Bernstein (invented in about 1911), the polynomials of D. Jackson, and the interpolatory approach which is nowadays attributed to L. Fejér (seemingly having its roots in work of D. Jackson of 1913). There were other contributions as well. Thus, as a very first definition, the term "Hermite—Fejér Interpolation" describes a way of setting up *interpolatory* approximation operators which can be used to prove classical Weierstraß approximation theorems.

Although invented about 75 years ago, HFI is still quite a topical object in Approximation Theory. As an indication for this we mention the fact that there are more than 50 Chinese papers dealing with HFI and which appeared in 1980 or later. Hermite—Fejér interpolation plays its role in Computer Aided Design as well. We have, however, made no attempt to systematically list all papers dealing with the CAD approach.

Here are a few historical remarks: In 1910 G. Faber's paper "Über stetige Funktionen (Zweite Abhandlung)" appeared (Math. Annalen 69 (1910), 372—443). He showed the existence of a continuous and 2π-periodic function f such that the sequence of trigonometric Lagrange interpolation polynomials constructed on the basis of an odd number of equidistant knots does not uniformly converge to f. In the sequel, D. Jackson (in his paper "A formula of trigonometric interpola-

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tion", presented on August 24, 1913) considered the corresponding analogue of the Fejér means. He defined the trigonometric polynomials M_n given by

$$M_n(t) = \sum_{\nu=0}^{n-1} f_\nu \left(\frac{\sin \frac{n(t-t_\nu)}{2}}{n \sin \frac{t-t_\nu}{2}} \right)^2$$

where $t_\nu = 2\pi\nu/n$, $0 \leqq \nu \leqq n-1$. Jackson proved that for any continuous and 2π-periodic f the uniform convergence

$$\lim_{n\to\infty} M_n f = f$$

holds, if $M_n f$ is the above polynomial with $f_\nu = f(t_\nu)$, $0 \leqq \nu \leqq n-1$. Sometimes $M_n f$ is called the Jackson polynomial (of degree $\leqq n$) of f. Independently from Jackson, L. Fejér investigated the same polynomials M_n. The minutes of the "Göttinger Mathematische Gesellschaft" meeting of October 28, 1913 contain the following note concerning Fejér's work (see Jahresber. Deutsch. Math.-Verein. 22 (1913), Mitteilungen und Nachrichten, p. 206): "...; further, Pólya reported on a new trigonometric interpolation of Fejér. The interpolating functions always lie between the extrema of the given ordinates. The method converges for any given continuous function." In his paper „Über Interpolation" L. Fejér returned to the subject and discussed certain interpolatory properties of the polynomials M_n, namely

$$M_n(t_\nu) = f_\nu, \quad M_n'(t_\nu) = 0, \quad 0 \leqq \nu \leqq n-1.$$

In 1914 G. Faber proved in the paper "Über die interpolatorische Darstellung stetiger Funktionen" (see Jahresber. Deutsch. Math.-Verein. 23 (1914), 192—210) that the divergence statement for the trigonometric Lagrange interpolation polynomials described by him in 1910 remains true if one uses an odd number of arbitrary knots in $[0, 2\pi)$. From this Faber concluded that, given an arbitrary triangular matrix of knots in the compact interval $[a, b]$, there is at least one function $f \in C[a, b]$ such that the sequence of algebraic Lagrange interpolation polynomials does not uniformly converge to f on $[a, b]$.

In his two articles "Interpolatióról" and "Über Interpolation" L. Fejér investigated those interpolation processes which now carry Hermite's and his name. This was done with respect to various knot systems

$$-1 \leqq x_1 < x_2 < \dots < x_n \leqq 1.$$

The interpolation polynomial can be written in the form

$$H_n(x) = \sum_{\nu=1}^{n} f_\nu \left(1 - \frac{\omega''(x_\nu)}{\omega'(x_\nu)}(x-x_\nu)\right)(l_\nu(x))^2,$$

where the fundamental polynomial l_ν of Lagrange and the polynomial ω are defined by

$$\omega(x) = \prod_{k=1}^{n} (x - x_k)$$

and

$$l_\nu(x) = \frac{\omega(x)}{\omega'(x_\nu)(x-x_\nu)}.$$

In the second paper mentioned he proved the algebraic version of the Weierstraß approximation theorem using Hermite—Fejér interpolation at the zeros of the Chebyshev polynomials of the first kind.

The papers listed above were the starting point for the construction of a large number of interpolation processes of Hermite—Fejér type for the approximation of continuous or of continuous and 2π-periodic functions by algebraic and trigonometric polynomials, respectively, and for the corresponding investigation of their various approximation properties. The Hermite—Fejér process has also been serving didactical purposes for quite a number of years: In several textbooks on Approximation Theory the algebraic Weierstraß approximation theorem is proved with the aid of Hermite—Fejér interpolation based on the roots of suitable Jacobi polynomials.

Since "Interpolation" and "Hermite Interpolation" are broad subfields of Numerical Analysis and/or Approximation Theory, and in order to avoid over-comprehensiveness of our inventory, we now try to give a *definition of Hermite—Fejér Interpolation:*

Let a set A (such as a compact interval $[a, b]$, the half axis $[0, \infty)$, the whole real axis $\mathbf{R}$, or the interval $[0, 2\pi)$) and m distinct points $x_i \in A$ be given. Furthermore, let V be a subspace of $C(A, \mathbf{K})$, the vector space of all continuous and $\mathbf{K}$-valued functions defined on $A (\mathbf{K}=\mathbf{R}$, or $\mathbf{K}=\mathbf{C})$. Let

$$E = (e_{i,k})_{\substack{1 \le i \le m \\ 0 \le k \le n}}$$

be an interpolation matrix consisting entirely of Hermite rows (cf. G. G. Lorentz—K. Jetter—S. D. Riemenschneider: Birkhoff Interpolation; Reading, MA: Addison-Wesley 1983). This means that $e_{i,k}=1$ for $k<r_i$ and $e_{i,k}=0$ for $k \ge r_i$, $r_i \in \mathbf{N}$, $i \in \{1, \dots, m\}$. Let $r_i \ge 2$ for at least one i. For $f \in V$, let

$$c_{i,0} := f(x_i), \quad 1 \le i \le m.$$

Furthermore, for $1 \le k < r_i$ and $1 \le i \le m$, let $c_{i,k}$ be given real or complex numbers. We also assume that, for $k \ge 1$, the data $c_{i,k}$ are given in such a fashion that no differentiability of f is required to define them. Usually we shall have $c_{i,k}=0$ in this case. The Hermite—Fejér interpolation problem is to find an algebraic (or trigonometric) polynomial P of minimal degree satisfying the conditions

$$P^{(k)}(x_i) = c_{i,k} \quad \text{iff} \quad e_{i,k} = 1.$$

The assumption $r_i \ge 2$ for at least one $i \in \{1, \dots, m\}$ excludes Lagrange interpolation from our considerations. Since r_i may be greater than 2, so-called Hermite—Fejér interpolation of higher order is included in the above definition. The requirement that, for $k \ge 1$, the data $c_{i,k}$ should not depend upon differentiability properties of f excludes so-called general Hermite interpolation processes for differentiable functions. These were, for instance, considered in H. Esser's and K. Scherer's paper "Eine Bemerkung zur konvergenz Hermitescher Interpolationsprozesse"

(Numer. Math. 21 (1973), 220—222), and by R. Kreß in the article "On general Hermite trigonometric interpolation" (Numer. Math. 20 (1972), 125—138). We have included papers in which a Hermite (and thus a Hermite—Fejér) interpolation problem is solved in a Banach space setting.

In addition we shall not cite papers containing results applicable to numerous HF interpolators, such as papers on general linear positive operators. Furthermore, no papers dealing with the related problems of spline interpolation/approximation are included in this compilation, although this topic is closely related to the more general field of Hermite interpolation.

2. Technical remarks

This inventory contains mainly papers which appeared in mathematical periodicals and conference proceedings, textbooks, and doctoral dissertations. In some instances, we have also included reports and preprints the existence of which became known to us from a reliable source.

Our compilation is ordered alphabetically according to the names of authors. In some (unambiguous) instances, middle names appearing on the papers in question were deleted for technical reasons.

In order to describe the historical development, a chronology is added as an appendix.

All titles are given in their original form when the language used is written with (modifications of) latin characters. In all other instances the titles were translated into English.

The journal abbreviations are those given in Mathematical Reviews; if not listed, they were assimilated. We have striven to add to each publication one or more references to reviews in "Jahrbuch über die Fortschritte der Mathematik" (abbreviated below as "**FdM**"), "Mathematical Reviews ("**MR**"), "Zentralblatt für Mathematik" ("**Zbl**"), and of "Referativnyĭ Žurnal Matematika" "**RŽM**").

In order to ease the reader's access to the items listed, we have striven either to add an Author Institution Code (AIC) as used by "Mathematical Reviews", or, if not available, to give the author's address in its explicit form. This was, however, done systematically only for papers/books which appeared in the 1980's. For articles or books which were published earlier the risk appears too high that a change of address occurred in the meantime. The symbol "#" "indicates that the author in question has passed away. In several instances we are using "—" to indicate that we are not aware of the author's present address.

This compilation of publications concerning Hermite—Fejér interpolation is certainly not yet complete. For this reason we would like to thank in advance all readers who direct our attention to errors, supplements, and missing papers, and who will provide us with new items. We nevertheless believe that our present inventory already gives a reasonably comprehensive survey of what has been published so far on the subject in question. We hope very much that everybody working on Hermite—Fejér interpolation will derive profit from our work.

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Note added in proof

The authors stress the fact that the above inventory solely contains items which were known to them at the compilation date (Sept. 15, 1987).

We have completed bibliographical data if these were available to us at the time of proofreading (January 1990). Moreover, further references to reviews in *MR*, *Zbl* and *RŽM* were added. We have, likewise, updated author institution codes if their most actual version was available to us when checking the proofs (as is the case for the first author of this listing).

The numerous recent papers dealing with Hermite—Fejér interpolation which became known to us after the compilation date will be part of a future supplement of this bibliography.

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THE FULL EMBEDDINGS OF THE CATEGORIES OF UNIFORM SPACES, PROXIMITY SPACES AND RELATED CATEGORIES INTO THEMSELVES AND EACH OTHER I

E. MAKAI, JR.

Abstract

It is proved that each full embedding of the category of proximity spaces into itself is naturally isomorphic to the identity functor. We include a result of J. Pelant and J. Reiterman stating the same for the category of uniform spaces. Further we investigate full embeddings between any two of the categories $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$, $\mathscr{C}_4$ of $T_{3\,1/2}$-spaces, cozero-spaces, proximity spaces and uniform spaces, resp., including a result of M. Hušek and J. Pelant ($\mathscr{C}_3\to\mathscr{C}_4$). Summing up: for $i>j$ there is no full embedding $\mathscr{C}_i\to\mathscr{C}_j$, for $i\leqq j\leqq i+1$ and $i=1$, $j=3$ there is essentially one, while for $i=1$, $j=4$ and $i=2$, $j=4$ there is more than a proper class of essentially different ones. (The case $i=j=1$ was known formerly.) We extend our investigations to the category of set systems as well and prove among others that there are exactly two essentially different full embeddings {cozero-spaces}→{set systems}, and there is essentially one full embedding {proximity spaces X with $\delta dX=0$} (resp. {proximity spaces X with $\delta dX=0$ and the T_0-reflection of X having a metric completion})→{set systems}.

§ 1. Introduction

Let Unif (Prox, $T_{3\,1/2}$) denote the category of all uniform (proximity, $T_{3\,1/2}$) spaces. We do not assume separatedness. For uniform spaces X, Y let $U(X, Y)$ denote the set of all uniformly continuous functions $X\to Y$. The (concrete) embedding functor J: Prox $\hookrightarrow$ Unif is defined by JX=the compatible precompact uniformity on the same underlying set. The (concrete) embedding functor K: $T_{3\,1/2}\hookrightarrow$ Prox is defined by KX= =the proximity on the same underlying set having as a base (for far pairs of sets) all pairs of disjoint zero-sets of X. (A concrete functor between two concrete categories is a functor forming a commutative diagram with the respective underlying set functors.) Whenever necessary we identify $T_{3\,1/2}$-spaces X with these proximity spaces KX and proximity spaces with the compatible precompact uniform spaces. Precompact reflection is denoted by p: Unif→Unif (Prox) ([17], II. 31), while the generated topology functor is denoted by τ: Unif→$T_{3\,1/2}$. Subcategories will always be assumed to be full. The category $\mathscr{S}_0^-$ has for objects all pairs $(X, \mathscr{X})$, X a set, $\{\emptyset, X\}\subset\subset\mathscr{X}\subset 2^X$, and morphisms $f\colon (X_1, \mathscr{X}_1)\to(X_2, \mathscr{X}_2)$ characterized by $f\colon X_1\to X_2$, $f^{-1}(\mathscr{X}_2)\subset\mathscr{X}_1$. The definition and some elementary properties of the category Coz of cozero-spaces will be briefly recalled in §4. For a concrete category $\mathscr{C}$ the underlying

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Key words and phrases. Categories of uniform (proximity, cozero, $T_{3\,1/2}$) spaces, category of set systems, full embeddings, special spaces, inductive (projective) generation, strongly rigid class of objects, spaces $Y_{\mathscr{F}}$, ultraproximities, free ultraspaces.

set functor will be denoted by $U_{\mathscr{C}}$, sometimes by U only (which will be sometimes omitted, if no misunderstanding can arise).

A functor $F: \mathscr{C} \to \mathscr{D}$ is a full embedding if it is a bijection on each hom-set hom (C_1, C_2), $C_1, C_2 \in \mathrm{Ob}\, \mathscr{C}$. F is an equivalence if beside this $\forall D \in \mathrm{Ob}\, \mathscr{D}\ \exists C \in \mathrm{Ob}\, \mathscr{C}$, $FC \cong D$.

In [22] (p. 121, Theorem, p. 125, Corollary) essentially the following has been proved:

THEOREM 1. *Let* F: Unif$\to$Unif (Prox$\to$Prox) *be an equivalence. Then F is naturally isomorphic to the identity functor on* Unif (Prox). *(I.e. for each object X there is an isomorphism* $i_X: GX \to FX$, *such that for each morphism* $f: X_1 \to X_2$ *the diagram*

$$\begin{array}{ccc} GX_1 & \xrightarrow{Gf} & GX_2 \\ \downarrow{\scriptstyle i_{X_1}} & & \downarrow{\scriptstyle i_{X_2}} \\ FX_1 & \xrightarrow{Ff} & FX_2 \end{array}$$

commutes — in notation $F \sim G$ *— where G is the identity functor.)* □

Further in [22] (p. 126, Propositions) it has been observed that results of [15], [30], [33], [27] imply

THEOREM 2. *Let* $F: \mathrm{T}_{3\,1/2} \to \mathrm{T}_{3\,1/2}$ *be a full embedding. Then F is naturally isomorphic to the identity functor.* □

In our paper we will investigate full embeddings between any two of the categories $\mathrm{T}_{3\,1/2}$, Coz, Prox, Unif, $\mathscr{S}_0^-$ and in each case we determine how many essentially different (i.e. not naturally isomorphic) full embeddings exist, resp. if there are more than a proper class of them. Theorems of similar type have been proved e.g. in [30], [3], [27], [2] (cf. also [10]). For our proofs we will use the technics developed by Hušek [15] and rely on methods developed by Rosický—Sekanina [27].

§ 2. Special objects

DEFINITION. A uniform (proximity, $\mathrm{T}_{3\,1/2}$) space X is called special if for any uniform (proximity, $\mathrm{T}_{3\,1/2}$) space Y on the same underlying set hom $(Y, Y) =$ $=$ hom (X, X) implies $Y = X$. (Cf. [33] for the topological case.)

For non-empty X, Y this is equivalent to the following: the existence of a semigroup isomorphism i: hom $(X, X) \to$ hom (Y, Y), for any space $Y (\neq \emptyset)$ implies the existence of an isomorphism $j: X \to Y$ such that $i(f)(y) = j(f(j^{-1}(y)))$ ([21], p. 197).

[23] has shown for a large class of uniform spaces (each containing [0, 1]) that they are special. Among others, each non-degenerate Peano continuum X (i.e. connected, locally connected, compact metric space with $|X| > 1$) or the long line with its unique uniformity has this property. Of course these spaces yield special proximity ($\mathrm{T}_{3\,1/2}$) spaces as well. (The fact that among others non-degenerate Peano continua are special topological spaces has been proved by [33], Theorem 4.6. The fact that [0, 1] is a special T_1-space has been proved by [32], proof of Lemma 3. Related theorems cf. also in [25].)

§ 3. Embeddings into Prox

By the technique developed in [15] we prove

THEOREM 3. *Let* $\mathscr{C}\subset$Unif *and let* $\mathscr{C}$ *contain a space* C_0 *which is a special uniform (proximity,* $T_{3\,1/2}$*) space containing* [0, 1]. *Let* $F: \mathscr{C}\to$Unif(Prox, $T_{3\,1/2}$) *be a full embedding. Then pF is naturally isomorphic to the restriction of p to* $\mathscr{C}$ *(F is naturally isomorphic to the restriction of p, or* τ*, to* $\mathscr{C}$*, resp.). In particular each full embedding* Prox$\to$Prox, *resp.* $T_{3\,1/2}\to$Prox *is naturally isomorphic to the identity functor on* Prox, *resp. to K.*

PROOF. We consider the case $F: \mathscr{C}\to$Unif (the other cases are shown analogously). Let $U_{\mathscr{C}}$, U_{Unif} denote the underlying set functors $\mathscr{C}\to$Set, Unif$\to$Set (Set is the category of all sets). By [22], Corollary to Lemma 2 the functors $U_{\text{Unif}}F$, $U_{\mathscr{C}}: \mathscr{C}\to$Set are naturally isomorphic. (Roughly: we can consider X and FX on the same underlying set.) Let $i: U_{\mathscr{C}}\to U_{\text{Unif}}F$ denote a natural isomorphism.

By hypothesis for $\mathscr{C}_0\in$Ob $\mathscr{C}$ F induces a semigroup isomorphism on $U(C_0, C_0)$. Hence by speciality

$(*)$ $C_0\cong FC_0$, where i_{C_0} is the underlying function of this isomorphism.

By $[0,1]\subset C_0$ we have for each uniform space $X\in$Ob $\mathscr{C}$ $U(X,[0,1])\subset U(X,C_0)$. For any $X\in$Ob $\mathscr{C}$ pX is projectively generated by $U_{\text{Unif}}U(X,[0,1])(\subset U_{\mathscr{C}}U(X,C_0))$. On the other hand for the uniform space FX by $(*)$

$$U_{\text{Unif}}U(FX, i_{C_0}[0,1]) \quad (\subset U_{\text{Unif}}U(FX, FC_0)=U_{\text{Unif}}FU(X,C_0))$$

projectively generates pFX. Thus by the natural isomorphism of $U_{\mathscr{C}}$ and $U_{\text{Unif}}F$ and $(*)$, i_X is the underlying function of an isomorphism $j_X: pX\to pFX$. This gives us the natural isomorphism $j: p\to pF$. (Roughly: pX and pFX have the same set of uniformly continuous maps to $[0,1](\subset C_0)$, hence are equal.)

For the last result consider Prox and $T_{3\,1/2}$ as the subcategories JProx, $JKT_{3\,1/2}$ of Unif. □

COROLLARY 1. *Let* $\mathscr{C}\subset$Unif *and let* $\mathscr{C}$ *contain a space* C_0 *which is a special proximity* ($T_{3\,1/2}$) *space containing* [0, 1]. *Let further* $\mathscr{C}$ *contain spaces* C_1, C_2 *satisfying* $C_1\not\cong C_2$, $pC_1\cong pC_2$ *(*$\tau C_1\cong\tau C_2$*, or let* $\mathscr{C}$ *contain a space C with* $pC\notin$Ob $JKT_{3\,1/2}$*). Then there is no full embedding* $F: \mathscr{C}\to$Prox ($T_{3\,1/2}$). *In particular there are no full embeddings* Unif$\to$Prox, Prox$\to T_{3\,1/2}$ (Unif$\to T_{3\,1/2}$).

PROOF. By Theorem 3 we have $FC_1\cong pC_1\cong pC_2$ (resp. $\cong\tau C_1\cong\tau C_2)\cong FC_2$, while $C_1\not\cong C_2$ (or, identifying $T_{3\,1/2}$ with $JKT_{3\,1/2}$, Ob $JKT_{3\,1/2}\ni FC\cong pC\notin$Ob $JKT_{3\,1/2}$) a contradiction. □

REMARKS 1. There is an evident analogue of Corollary 1 for functors $F: \mathscr{C}\to\mathscr{D}$ ($\subset$Prox) if Ob $p\mathscr{C}\not\subset$ [the isomorphism closed hull of Ob $\mathscr{D}$] or if for some $C_1, C_2\in$Ob $\mathscr{C}$ $U(C_1,C_2)\neq U(pC_1,pC_2)$.

2. The usage of spaces containing [0, 1] was not eventual. Namely, similarly to [12], 19.1.1, [7], 5.13, [24], p. 466 for Prox (or any subcategory $\mathscr{C}$ of Prox containing a space containing [0, 1]) a subcategory $\mathscr{C}_0\subset$Prox ($\mathscr{C}_0\subset\mathscr{C}$) projectively generates

Prox ($\mathscr{C}$) (i.e. the least productive hereditary subcategory of Prox containing $\mathscr{C}_0$ contains the T_0-reflections of all spaces in $\mathscr{C}$) iff $\exists C_0 \in \mathrm{Ob}\,\mathscr{C}_0$, $C_0 \supset [0, 1]$.

3. Analogously to [15], Example 2, by the conclusion of Theorem 3 F is finer than p and coarser than the proximally fine coreflection functor. Hence among the functors $F\colon \mathscr{C} \to \mathrm{Unif}$ (where $\mathscr{C} \subset T_{3\,1/2}$, or $\mathscr{C} \subset p$ {proximally fine spaces}) in Theorem 3 these two are the coarsest and finest ones. (This implies that for X pseudocompact $T_{3\,1/2}$ space FX is uniquely determined. Thus if $\mathscr{C}(\subset T_{3\,1/2})$ contains only a set of non-pseudocompact spaces then there are only a set of not naturally isomorphic full embeddings $F\colon \mathscr{C} \to \mathrm{Unif}$. Compare § 4, Corollary 2.)

4. The word "special" can be deleted from Theorem 3 and Corollary 1, cf. § 7, Remark 1.

§ 4. Cases with many full embeddings

Hušek [15] has shown that there is a proper class of not naturally isomorphic full embeddings $F\colon T_{3\,1/2} \to \mathrm{Unif}$. Another such proper class is (analogously to [27], pp. 545—548) e.g. $\{F_\alpha\}$, F_α=reflection of the fine uniformity in the subcategory of uniform spaces having bases consisting of coverings of cardinalities smaller than α ([17], p. 52 and II. 33). (Note that for X connected $T_{3\,1/2}$ space the embeddings of [15] equal JKX, hence are not equal to F_α, $\alpha > \aleph_0$.) The question arises if there are more not naturally isomorphic full embeddings $F\colon T_{3\,1/2} \to \mathrm{Unif}$. Evidently there cannot be more of them than there are subclasses of a proper class.

We will investigate this question in a more general setting. We will use the term *topological category* in the sense of [13]. If $\mathscr{T}$ is a topological category with underlying set functor U, for $\mathscr{T}$-objects T_λ with $UT_\lambda = X$ $\bigvee_\lambda T_\lambda$ denotes the supremum of the structures T_λ on X. For concrete functors F_λ from some concrete category $\mathscr{C}$ to $\mathscr{T}$ $\bigvee_\lambda F_\lambda$ denotes the concrete functor such that $(\bigvee_\lambda F_\lambda)C = \bigvee_\lambda (F_\lambda C)$ for each $C \in \mathrm{Ob}\,\mathscr{C}$. A class $\{D_\alpha\}$ of objects of a concrete category $\mathscr{C}$ is called *strongly rigid* (cf. e.g. [20]) if any $\mathscr{C}$-morphism $D_{\alpha_1} \to D_{\alpha_2}$ is either constant (as a set map) or an identity. Using the technics of [15] we will prove

Proposition 1. *Let $\mathscr{T}$ be a topological category and let the concrete category $\mathscr{C}$ admit two concrete functors $F', F''\colon \mathscr{C} \to \mathscr{T}$. Let $\forall C_1, C_2 \in \mathrm{Ob}\,\mathscr{C}$* $\hom((F' \vee F'')C_1, F'C_2) \subset \hom(C_1, C_2)$ *(e.g. there exists a concrete functor $H\colon (\mathscr{T} \supset)\{F'C, (F'C) \vee (F''C) | C \in \mathrm{Ob}\,\mathscr{C}\} \to \mathscr{C}$ with $HF' = H(F' \vee F'') = 1_{\mathscr{C}}$). Let further $\mathscr{C}$ contain a strongly rigid class $\{D_\alpha | \alpha \in A\}$ of objects such that $\forall \alpha \in A\ (F'D_\alpha) \vee \vee (F''D_\alpha) \not\cong F'D_\alpha$. Then there are at least as many not naturally isomorphic concrete full embeddings $G\colon \mathscr{C} \to \mathscr{T}$ (actually $\mathscr{C} \to \{T \in \mathrm{Ob}\,\mathscr{T} | T$ is projectively generated by maps into objects $F'C$ and $F''C$, $C \in \mathrm{Ob}\,\mathscr{C}\}$) as there are subclasses of A. For each G given above GC is finer than $F'C$ and coarser than $(F'C) \vee (F''C)$.*

Proof. Let $U_{\mathscr{C}}$, $U_{\mathscr{T}}$ denote the respective underlying set functors. Let $B \subset A$. For any $C \in \mathrm{Ob}\,\mathscr{C}$ define $G_B C$=structure on $U_{\mathscr{C}}C$ projectively generated by all maps $U_{\mathscr{C}} f \in U_{\mathscr{C}} \hom(C, D_\alpha)$, $\alpha \in B$, $U_{\mathscr{C}} D_\alpha$ endowed with the structure $F''D_\alpha$, and by

$U_{\mathscr{C}}1_C \in U_{\mathscr{C}}\hom(C, C)$, $U_{\mathscr{C}}C$ endowed with the structure $F'C$. By [15] $\hom(C_1, C_2) \subset \subset \hom(G_B C_1, G_B C_2)$, i.e. G_B becomes a concrete functor if for $f \in \mathrm{Mor}\,\mathscr{C}$ we let $U_{\mathscr{T}} G_B f = U_{\mathscr{C}} f$. However, by construction, $G_B C$ is finer than $F'C$ and coarser than $(F'C)\vee(F''C)$. Therefore $\hom(G_B C_1, G_B C_2) \subset \hom((F'C_1)\vee(F''C_1), F'C_2) \subset \subset (\hom(H((F'C_1)\vee(F''C_1)), HF'C_2) =) \hom(C_1, C_2)$. Thus G_B is a full embedding into the subcategory of all objects projectively generated by maps into objects $F'C$ and $F''C$, $C \in \mathrm{Ob}\,\mathscr{C}$.

Now let us consider $G_B D_\alpha$. For $\alpha \in B$ $G_B D_\alpha$ equals the structure on $U_{\mathscr{C}} D_\alpha$ projectively generated by $1_{U_{\mathscr{C}} D_\alpha} \in \hom(U_{\mathscr{C}} D_\alpha, U_{\mathscr{C}} D_\alpha)$, $U_{\mathscr{C}} D_\alpha$ endowed with the structures $F'D_\alpha$, resp. $F''D_\alpha$, i.e. $F'D_\alpha \vee F''D_\alpha$. For $\alpha \notin B$ $G_B D_\alpha$ equals the structure on $U_{\mathscr{C}} D_\alpha$ projectively generated by $1_{U_{\mathscr{C}} D_\alpha} \in \hom(U_{\mathscr{C}} D_\alpha, U_{\mathscr{C}} D_\alpha)$, $U_{\mathscr{C}} D_\alpha$ endowed with the structure $F'D_\alpha$, i.e. $F'D_\alpha$. Since $F'D_\alpha \vee F''D_\alpha \not\cong F'D_\alpha$, $B = \{\alpha \in A \mid G_B D_\alpha \cong \cong F'D_\alpha \vee F''D_\alpha\}$ and thus different B's give not naturally isomorphic full embeddings. □

REMARK. Instead of strong rigidity of $\{D_\alpha\}$ it suffices to suppose $D_{\alpha_1} \neq D_{\alpha_2} \Rightarrow \Rightarrow \forall f \in \hom(D_{\alpha_1}, D_{\alpha_2})$ $(F''D_{\alpha_2}) | f(D_{\alpha_1}) \in \mathrm{Ob}\,\mathscr{T}_0$ ($|$ denotes the initial structure w.r.t. the embedding of a subset) and also $\forall C \in \mathrm{Ob}\,\mathscr{C}$ $F'C \in \mathrm{Ob}\,\mathscr{T}_0$, where $\mathscr{T}_0$ is a subcategory of $\mathscr{T}$ closed under suprema, while $\forall \alpha$ $(F'D_\alpha)\vee(F''D_\alpha) \notin \mathrm{Ob}\,\mathscr{T}_0$. If we replace $(F''D_{\alpha_2}) | f(D_{\alpha_1}) \in \mathrm{Ob}\,\mathscr{T}_0$ by "the structure on $U_{\mathscr{C}} D_{\alpha_1}$ projectively generated by $U_{\mathscr{C}} f \in U_{\mathscr{C}} \hom(D_{\alpha_1}, D_{\alpha_2})$, $U_{\mathscr{C}} D_{\alpha_2}$ endowed with the structure $F''D_{\alpha_2}$, belongs to $\mathrm{Ob}\,\mathscr{T}_0$" then the resulting statement will be valid for topological categories over any base category.

COROLLARY 2. *Let $\mathscr{C} \subset T_{3\,1/2}$ and let $\mathscr{C}$ contain a strongly rigid class $\{D_\alpha \mid \alpha \in A\}$ of not pseudocompact spaces. Then there are at least as many not naturally isomorphic full embeddings*

$G\colon \mathscr{C} \to \mathrm{Unif}$ (actually $\mathscr{C} \to \{X \mid X$ is a uniform space projectively generated by $U(X, R)\}$)

as there are subclasses of A. For each G given above and each $C \in \mathrm{Ob}\,\mathscr{C}$ GC is finer than JKC and coarser than the uniformity on C projectively generated by $C(C, R)$. In particular the above statement holds for $\mathscr{C} = T_{3\,1/2}$, with A a proper class.

PROOF. We apply Proposition 1, with $\mathscr{T} = \mathrm{Unif}$, $F' = JK|\mathscr{C}$, $F''C =$ uniformity on $U_{\mathscr{C}}C$ projectively generated by $C(C, R)$ (R taken with the usual uniformity) and $H =$ restriction of the functor $\tau\colon \mathrm{Unif} \to T_{3\,1/2}$. Since D_α is not pseudocompact we have that $F'D_\alpha$ is precompact while $F'D_\alpha \vee F''D_\alpha = F''D_\alpha$ is not, hence they are not isomorphic.

Now we turn to the case $\mathscr{C} = T_{3\,1/2}$. $T_{3\,1/2}$ contains by [20] a strongly rigid proper class of paracompact spaces. Therefore it suffices to show that among the spaces $\tilde{P} = \tilde{P}(X, \mathscr{U})$ given in [20] there are only a set of pseudocompact ones.

We will adhere to the terminology of [20]. If $\mathscr{U}$ is infinite then $\tilde{P}$ is not pseudocompact. Namely on each $D \times \{u\}$, $u \in \mathscr{U}$ one can define a continuous real function f_u, vanishing outside $C_0 \times \{u\}$ and such that $\sup f_u \geqq c_u$, where $\sup\{c_u \mid u \in \mathscr{U}\} = \infty$. Define a real valued function f on $\tilde{P}$, extending these f_u's and vanishing elsewhere. Thus $f \in C(\tilde{P}, R)$ and $\sup f = \infty$.

Now we assert that the class of the spaces $\tilde{P} = \tilde{P}(X, \mathscr{U})$ occurring in the construction of [20], with $\mathscr{U}$ finite, has a cardinality at most 2^c (c = continuum). In fact only such $(X, \mathscr{U})$'s $((X_i, \mathscr{U}_i)$'s, $i=1, 2)$ occur in the construction, for which $\mathscr{U}\subset A^X$, $|A|=c$ and which satisfy 1) $f\colon X\to X$, $A^f\mathscr{U}\subset\mathscr{U}\Rightarrow f=1_X$, 2) $(X_1, \mathscr{U}_1)\neq(X_2, \mathscr{U}_2)\Rightarrow$ $\Rightarrow\nexists f\colon X_1\to X_2$, $A^f\mathscr{U}_2\subset\mathscr{U}_1$. Here $A^f\colon A^X\to A^X$ $(A^{X_2}\to A^{X_1})$ is defined by $A^f u=uf$. Let now $\mathscr{U}$ be finite. If $|X|>|A|^{|\mathscr{U}|}=c$ then $\exists x_1, x_2\in X$, $x_1\neq x_2$ such that $\forall u\in\mathscr{U}$ $u(x_1)=u(x_2)$. Define $f\colon X\to X$ by $f(x_1)=x_2$, $f(x_2)=x_1$ and $f(x)=x$ otherwise. Then $\forall u$ $A^f u=u$, thus $A^f\mathscr{U}=\mathscr{U}$, however $f\neq 1_X$, violating condition 1). Thus the $(X,\mathscr{U})$'s with $\mathscr{U}$ finite, occurring in the construction satisfy $|X|\leqq c$, hence $|A^X|\leqq 2^c$. Therefore the cardinality of the set of all possible finite $\mathscr{U}$'s, with X fixed, hence also the cardinality of the set of all $(X,\mathscr{U})$'s in question, is up to isomorphy $\leqq 2^c$. (Isomorphy of $(X_1, \mathscr{U}_1)$ and $(X_2, \mathscr{U}_2)$ means $\exists f\colon X_1\to X_2$, f is a bijection, $A^f\mathscr{U}_2=\mathscr{U}_1$.) If, however, $(X_1, \mathscr{U}_1)\neq(X_2, \mathscr{U}_2)$ occurring in the construction are isomorphic then condition 2) is violated. Thus in fact the class (set) of all $(X, \mathscr{U})$'s with $\mathscr{U}$ finite, occurring in the construction, has a cardinality $\leqq 2^c$. Hence in fact among the spaces $\bar{P}$ occurring in the construction there are at most 2^c pseudocompact ones. □

Another remarkable category which can be considered as a subcategory of Unif is the category Coz of cozero-spaces (cf. [9], § 41, [1], [5], [6], [8]). Based on these papers we recall their definition and some of their properties. Shortly: a *cozero-space* is a pair $(X, \mathscr{X})$, X a set, $\mathscr{X}\subset 2^X$, such that $\mathscr{X}$ is the collection of cozero-sets of all uniformly continuous real functions (equivalently: functions to [0, 1]) w.r.t. some uniformity on X. Elements of $\mathscr{X}$ are called *cozero-sets*, their complements *zero-sets*. For cozero-spaces $(X_i, \mathscr{X}_i)$ f is a *cozero-map* $(X_1, \mathscr{X}_1)\to(X_2, \mathscr{X}_2)$ iff $f\colon X_1\to X_2$ and $f^{-1}(\mathscr{X}_2)\subset\mathscr{X}_1$. The cozero-spaces and cozero-maps constitute a topological category ([13]) denoted Coz. Initial structures can be obtained as follows: for $f_\alpha\colon X\to$ $\to X_\alpha$, X_α endowed with the cozero-structure $(X_\alpha, \mathscr{X}_\alpha)$ the initial (=weak) structure is $\left(X, \left\{\text{countable unions of finite intersections of elements of } \bigcup_\alpha f_\alpha^{-1}(\mathscr{X}_\alpha)\right\}\right)$. In particular, subspaces of a cozero-space $(X, \mathscr{X})$ are given by $(Y, \mathscr{X}|Y)$ $(Y\subset X)$, and $\Pi(X_\alpha, \mathscr{X}_\alpha)=$ $=\left(\Pi X_\alpha, \left\{\text{countable unions of sets of form } \prod_\alpha C_\alpha, \text{ where } C_\alpha\in\mathscr{X}_\alpha, \text{ and with the exception of finitely many } \alpha\text{'s } C_\alpha=X_\alpha\right\}\right)$.

$(X, \mathscr{X})(\mathscr{X}\subset 2^X)$ is a cozero-space iff 1) $\mathscr{X}$ is closed under countable unions and finite intersections, 2) any two disjoint zero-sets (i.e. complements of cozero-sets) can be included into two disjoint cozero-sets, (i.e. elements of $\mathscr{X}$), 3) any cozero-set is a countable union of zero-sets. For any cozero-space $(X, \mathscr{X})$ we have $\mathscr{X}=\{f^{-1}(0, 1]\mid f\colon (X, \mathscr{X})\to[0, 1]$ is a cozero-map$\}$, [0, 1] taken with its usual cozero-sets. Arbitrary unions of cozero-sets of a cozero space constitute the open sets of a completely regular topology, this gives a concrete functor Coz$\to T_{3\,1/2}$ called the generated topology functor. Compact T_2 spaces are generated by a unique cozero-structure, namely by that given by all topological cozero-sets. Also $T_{3\,1/2}$ can be considered as a full coreflective subcategory of Coz—whose isomorphism closed hull is $\neq$Coz—, by the concrete full embedding given by $C\to(UC$, {topological cozero-sets of $C\})$ $(C\in\text{Ob}\,T_{3\,1/2})$.

The category Coz admits several nice concrete full embeddings in Unif, namely F', F''' given by all finite, resp. all countable cozero covers (F' gives a full embedding in Prox, which is onto a coreflective subcategory of Prox — whose isomorphism closed hull is $\neq$Prox —, with a coreflection given by the concrete functor c, where for $C\in$Ob Prox $cC=F'(UC, \{f^{-1}(0, 1]\mid f\in U(C, [0, 1])\})$, the underlying functions of the coreflection maps being identities) or F'' given by the uniformity on the same underlying set projectively generated by all cozero-maps to R (R taken with its usual cozero-sets, and endowed with its standard uniformity).

LEMMA 1. *Let* H_0: Unif$\to$Coz *denote the concrete functor defined for each* $C\in$Ob Unif *by* $H_0C=(U_{\text{Unif}}C, \{f^{-1}(0, 1]\,|\, f\in U(C, [0, 1])\})$ *and let* F: Coz$\to$Unif *be a concrete full embedding. Then* $H_0F=1_{\text{Coz}}$.

PROOF. We have by $H_0p=H_0$, Theorem 3 and $\{[0, 1]\}\subset F'$ Coz$\subset$ Prox $pF=F'$, thus $H_0F=H_0pF=H_0F'=1_{\text{Coz}}$ (where the last equality follows from [5], 4.3, 3.5). □

Analogously to Corollary 2, by letting D_α=the cozero space consisting of the underlying set and the collection of (topological) cozero-sets of the space D_α in Corollary 2 used for $\mathscr{C}=\mathrm{T}_{3\,1/2}$, and using the functors F' and F'' given above and setting H=the restriction of the functor H_0: Unif$\to$Coz, we obtain by Proposition 1 and Lemma 1

COROLLARY 3. *There are at least as many not naturally isomorphic full embeddings* G: Coz$\to$Unif *(actually* Coz$\to\{X|X$ *is a uniform space projectively generated by* $U(X, R)\}$*) as there are subclasses of a proper class (actually already their restrictions to the above mentioned embedded image of* $\mathrm{T}_{3\,1/2}$ *in* Coz *— via the topological cozero sets — being not naturally isomorphic).* □

In view of $\mathrm{T}_{3\,1/2}\subsetneqq F'$Coz $(\cong$Coz$)\subsetneqq$ Prox the other questions about all full embeddings between Coz and any of the categories $\mathrm{T}_{3\,1/2}$, Prox, Unif (or conversely) are covered by the results of § 3. Namely the above inclusions and their compositions give the only full embeddings between any mentioned pair of these categories, up to natural isomorphism, and there are no full embeddings Prox (Unif)$\to$Coz, or Coz$\to\mathrm{T}_{3\,1/2}$.

[27], pp. 545—548 has shown that among others the categories of completely regular spaces, of 0-dimensional spaces, or of free ultraspaces (i.e. spaces X for which $D\subset X\subset\beta D$, $|X\setminus D|=1$ for some discrete space D) have a proper class of not naturally isomorphic concrete full embeddings F into the category $\mathscr{S}^-$ with objects all pairs $(X, \mathscr{X})$, where X is a set and $\mathscr{X}\subset 2^X$, and with morphisms f: $(X_1, \mathscr{X}_1)\to(X_2, \mathscr{X}_2)$ characterized by f: $X_1\to X_2$, $f^{-1}(\mathscr{X}_2)\subset\mathscr{X}_1$.

On the other hand by [27], Proposition 12 if {perfectly normal free ultraspaces} (={free ultraspaces X with $vD\not\supset X$})$\subset\mathscr{C}$, $\mathscr{C}\setminus${perfectly normal spaces} is a set then there are only a set of concrete full embeddings $\mathscr{C}\to\mathscr{S}^-$. $\mathscr{S}^-$ is not a topological category, but its full subcategory $\mathscr{S}_0^-=\{(X, \mathscr{X})|\{\emptyset, X\}\subset\mathscr{X}\}$ is a topological category, for which the above results on embeddings, stated for $\mathscr{S}^-$, hold and which we will consider henceforward. Using Proposition 1 we can construct for the first case even more concrete full embeddings (the other cases remain open).

COROLLARY 4. *Let* $\mathscr{C}\subset\mathrm{T}_{3\,1/2}$ *and let* $\mathscr{C}$ *contain a strongly rigid class* $\{D_\alpha|\alpha\in A\}$ *of not perfectly normal spaces. Then there are at least as many not naturally isomorphic concrete full embeddings* $G:\mathscr{C}\to\mathscr{S}_0^-$ *as there are subclasses of* A*. For each* $C\in$Ob $\mathscr{C}$ *and each* G *given above* GC *is given by an open base of* C *containing the cozero-sets of* C*. In particular the above statements hold for* $\mathscr{C}=\mathrm{T}_{3\,1/2}$, *with* A *a proper class.*

PROOF. Apply Proposition 1 with $\mathscr{T}=\mathscr{S}_0^-$, $F'C=(U_{\mathscr{C}}C$, {cozero sets of C}), $F''C=(U_{\mathscr{C}}C$, {open sets of C}) and $H(X, \mathscr{X})$=topology on X with open subbase $\mathscr{X}$.

Since D_α is not perfectly normal we have that $F'D_\alpha \vee F''D_\alpha = F''D_\alpha$ is given by a set system closed under arbitrary unions while the same does not hold for $F'D_\alpha$, hence they are not isomorphic.

For the case $\mathscr{C} = T_{3\,1/2}$ we use the same class given by [20] as above. Analogously to Corollary 2 it suffices to show that among the spaces $\bar{P}$ given there there are only a set of perfectly normal ones. However, using the terminology of [20], A^X is a closed subspace of $\bar{P}$. For X uncountable no point of A^X is a G_δ-set, therefore in this case $\bar{P}$ is not perfectly normal. Similarly to Corollary 2 the cardinality of the class (set) of the $(X, \mathscr{U})$'s with $|X| \leqq \aleph_0$, occurring in the construction, is $\leqq 2^c$. Hence in fact among the spaces $\bar{P}$ occurring in the construction there are at most 2^c perfectly normal ones. □

For the zero-dimensional case it would suffice an analogue of the class of [20], consisting of non-discrete zero-dimensional T_0 spaces D_α such that for each $f\colon D_{\alpha_1} \to D_{\alpha_2}$, $\alpha_1 \neq \alpha_2$ and each $G \subset D_{\alpha_2}$ open $f^{-1}(G)$ is clopen (use $F'C = (U_{\mathscr{C}}C$, {clopen sets of C}), $F''C = (U_{\mathscr{C}}C$, {open sets of C}), H as above).

Remark. Our above results suggest the question about the full embeddings $\mathrm{Coz} \to \mathscr{S}_0^-$; cf. § 6. The category Prox does not admit a full embedding in $\mathscr{S}_0^-$; in § 5 and § 6 we will find some maximal (proper) subcategories of Prox admitting full embeddings in $\mathscr{S}_0^-$. There is no full embedding $\mathscr{S}_0^- \to \mathrm{Unif}$. Actually even {$T_3$-spaces} $(\subset \mathscr{S}_0^-)$ does not admit a full embedding F in Unif. In fact by the results of § 3 $pFX =$ = proximity generated by $C(X, [0, 1])$, and there are regular spaces X with $|X| > 1$ and $C(X, [0, 1]) = \{$constant maps $X \to [0, 1]\}$ (e.g. [11]). Hence FX is indiscrete, thus $C(X, X) = U(FX, FX) = (UX)^{UX}$, implying X discrete or indiscrete ([33], proof of Theorem 4.1), a contradiction. Similarly $T_{3\,1/2}$ is a maximal subcategory of Top, admitting a full embedding F into Prox. Namely, supposing F concrete, for a non-$T_{3\,1/2}$ space X and its $T_{3\,1/2}$-reflection $rX \neq X$ (supposing U(reflection map) $= 1_{UX}$) we have $FX = [$proximity on UX generated by $C(X, [0, 1])] = FrX$, a contradiction. Also one sees like in [15], Example 4, [29] or [27], Corollary 1 that any full embedding $F\colon \mathscr{S}_0^- \to \mathscr{S}_0^-$ is naturally isomorphic to the identity or to the concrete functor i given by $i(X, \mathscr{X}) = (X, \{X \setminus A \mid A \in \mathscr{X}\})$. (We have, supposing F concrete, for $C_0 = (\{0, 1\}, \{\emptyset, \{0\}, \{0, 1\}\})$ $FC_0 = C_0$ or $FC_0 = iC_0$; in any of these cases $\forall (X, \mathscr{X})$ $F(X, \mathscr{X}) = (X, \mathscr{X}')$ with $\mathscr{X}' = \{A \subset X \mid \exists f \in \hom((X, \mathscr{X}'), C_0) = \hom(F(X, \mathscr{X}), F^2C_0) = \hom((X, \mathscr{X}), FC_0),\ A = f^{-1}(0)\}$.)

Remark added in proof (November 2, 1990). G. C. L. Brümmer kindly drew the author's attention to his papers Initial quasi-uniformities, *Indag. Math.* **31** (1969), 403—409 (*MR* **41** # 2617), and Topological functors and structure functors, *Categorical topology* (Proc. Conf. Mannheim), Lecture notes in mathematics, Vol. 540, Springer-Verlag, Berlin, 1976, 109—135 (*MR* **56** # 440), and his paper with A. W. Hager, Functorial uniformization of topological spaces, *Topology Appl.* **27** (1987), 113—127 (*MR* **89c**: 54 025), where some results similar to our Theorem 3 and Corollary 2 have been proved.

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VERTEX-DEGREES IN A RANDOM SUBGRAPH OF A REGULAR GRAPH

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Abstract

The asymptotic probability distribution of the number of vertices of a given degree in a random subgraph of a regular graph is given. In particular the degree distribution of the n-cube is obtained when $n\to\infty$.

1. Introduction

In this paper a *random graph* is a pair $(\mathcal{G}, P)$, where $\mathcal{G}$ is the family of all spanning subgraphs of a given graph called *initial graph* and denoted further by ING, and P is the probability distribution on $\mathcal{G}$ so that for every graph $G\in\mathcal{G}$

$$\mathsf{P}(G) = p^{e(G)}(1-p)^{e(ING)-e(G)},$$

where $e(\cdot)$ denotes the number of edges in a given graph and $0\leqq p\leqq 1$.

There are many papers containing asymptotic results on vertex-degrees in a random graph with ING being a complete graph (see e.g, [1], [2], [5]). The degree distribution in the case when ING is a complete bipartited graph is investigated in [6]. In this paper the initial graph ING is a *regular graph* with common vertex-degree relatively small to n. In that sense the paper may be be considered as a sequel to [8].

Let $G=G(n)$ be a d-regular simple graph on n labelled vertices. Consider G_p, a random subgraph of G obtained by removing edges each with the same probability $q=1-p$, $0\leqq p\leqq 1$, independently of all other edges (i.e. each edge remains in G_p with probability p). We can identify G_p with a random graph $(\mathcal{G}, \mathsf{P})$ described above, where $\text{ING}=G$. A wide variety of results devoted to probability distributions of the maximum and minimum degree of G_p with respect to different values of the edge probability p and degree of regularity d are presented in [7].

We will be concerned here with the asymptotic distribution of $X_i=X_i(G_p)$, the number of vertices in G_p having degree i, $0\leqq i\leqq d$, where i and d may depend on n. We will emphasize this fact by writing $i=i(n)$ and $d=d(n)$.

In the next section some conditions for which the random variable X_i has asymptotically Poisson and normal distribution will be formulated. The last section contains an application of our results to the special case of a random graph with the n-cube as the initial graph. In particular we generalize a result about isolated vertices in a random n-cube proved by Erdős and Spencer [4].

Throughout this paper the notation $X\rightsquigarrow\mathcal{N}(0,1)$ and $X\rightsquigarrow\mathcal{P}(\lambda)$ means that the distribution of the random variable $X=X(n)$ is tending for $n\to\infty$ to the standard

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normal distribution and a Poisson distribution with expected value λ, respectively. The symbols $\sim$, o and O are used with respect to $n\to\infty$. Also we write $(d)_i=d(d-1)\dots(d-i+1)$. A sequence $\omega(n)$ is called *slow* if for any positive ε, $\omega(n)=o(n^\varepsilon)$.

2. Main results

First of all we formulate a result about the distribution of the random variable X_i when i is fixed (i.e. i does not depend on n), which was proved in [8].

THEOREM 1. *Let* $d=d(n)$ *be a slow sequence.*
If $pdn\to c>0$ *then* $X_1/2 \rightsquigarrow \mathscr{P}(c)$;
If $p[(d)_i n]^{1/i}\to c>0$ *then* $X_i \rightsquigarrow \mathscr{P}(c^i/i!)$, $i=2,3,\dots$;
If $p[(d)_i n]^{1/i}=\omega(n)\to\infty$ *and* $\omega(n)$ *is slow then*

$$\frac{X_i-a_n}{\sqrt{a_n}} \rightsquigarrow \mathscr{N}(0,1), \qquad i=1,2,\dots$$

with $a_n=\omega(n)^i/i!$. □

Note that under assumptions of Theorem 1, $\mathsf{E}(X_i)$ tends either to a positive constant or slowly to infinity. The next two results are generalizations of Theorem 1 to the case $i=i(n)$. Also $d=d(n)$ need not to be slow.

THEOREM 2. *Let* $d=d(n)$, $p=p(n)$ *and* $i=i(n)\geqq 2$ *be such that* $d=o(n)$, $0<p\leqq\frac{1}{2}$ *and* $\left(\frac{d}{n}p^{-i}\right)^r p=o(1)$ *for any* $r=1,2,\dots$. *If* $\mathsf{E}(X_i)\to c>0$ *then* $X_i\rightsquigarrow\mathscr{P}(c)$.

REMARK. It would be enough to assume that there is a positive ε so that $p\leqq 1-\varepsilon$ for every n. On the other hand $X_i(G_p)=X_{d-i}(G_{1-p})$, so the assumption $p\leqq\frac{1}{2}$ is no restriction at all.

PROOF. Consider the k-th factorial moment of X_i, i.e. $\mathsf{E}_k(X_i)=\mathsf{E}((X_i)_k)$, $k=1,2,\dots$. It is easily seen that

$$\mathsf{E}_k(X_i) = \mathsf{E}'_k+\mathsf{E}''_k,$$

where E'_k is the expected number of ordered k-tuples of vertices of degree i in G_p, which induce an independent set in the initial graph G. Notice that E''_k counts the k-tuples of vertices of degree i in G_p which induce at least one edge in G. Obviously,

$$[n-(k-1)(d+1)]^k\binom{d}{i}^k p^{ik}q^{(d-i)k} \leqq \mathsf{E}'_k \leqq n^k\binom{d}{i}^k p^{ik}q^{(d-i)k}$$

so

$$\mathsf{E}'_k = c^k+o(1). \tag{1}$$

Let $(v_1,\dots,v_k)$ be a k-tuple counted in E''_k which induces in G a graph with t nontrivial (i.e. with at least one edge) components and assume that those nontrivial components contain h vertices in total, $1\leqq t\leqq\left[\frac{h}{2}\right]$, $2t\leqq h\leqq k$. Then the number of edges in a subgraph of G_p induced by the vertices $v_1, v_2, \ldots, v_k$ and their neighbours is at least

$$t(2i-1)+(k-h)i \geqq ki-(h-t)i+1$$

since $i \geqq 2$ and $t \geqq 1$. Thus

$$\mathsf{E}_k'' = O\left(\sum_{t,h} n^{k-h+t} d^{h-t} \binom{d}{i}^k p^{ki-(h-t)i+1} q^{k(d-i)-k(k-1)/2}\right) =$$

$$= O\left(\sum_{t,h} \mathsf{E}(X_i)^k \left(\frac{d}{n} p^{-i}\right)^{h-t} p\right) = o(1)$$

by the assumptions. This together with (1) gives $\mathsf{E}_k(X_i) = c^k + o(1)$, $k=1, 2, \ldots$. From this it follows that X_i has asymptotically Poisson distribution with mean c (see [3, p. 99]). □

With regard to the approximation of X_i by a normal distribution the following is true.

THEOREM 3. *Let* $d=d(n)$, $p=p(n)$, $i=i(n) \geqq 2$ *and* ε *be such that* $0<\varepsilon<1$, $d=o(n^\varepsilon)$, $0<p\leqq\frac{1}{2}$ *and* $\left(\frac{d}{n} p^{-i}\right)^r p = o(n^{-\varepsilon})$ *for any* $r=1, 2, \ldots$. *If* $\mathsf{E}(X_i) = \omega(n) \to \infty$ *and* $\omega(n)$ *is slow then*

$$\frac{X_i - \omega(n)}{\sqrt{\omega(n)}} \rightsquigarrow \mathcal{N}(0, 1).$$

PROOF (Sketch). Since the normal distribution $\mathcal{N}(0, 1)$ is uniquely determined by its moments it suffices to prove

$$\lim_{n\to\infty} \mathsf{E}\left(\frac{X_i-\omega(n)}{\sqrt{\omega(n)}}\right)^r = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u^r e^{-u^2/2}\, du \tag{2}$$

where $r=1, 2, \ldots$. Similarly as in the proof of Theorem 2 we are able to show that

$$\mathsf{E}_k(X_i) = \omega(n)^k + o(n^{-\varepsilon'}) \tag{3}$$

where $0<\varepsilon'<\varepsilon$. Since the r-th moment of X_i satisfies

$$\mathsf{E}(X_i^r) = \sum_{k=1}^{r} S(r, k)\mathsf{E}_k(X_i), \qquad r \geqq 1$$

where $S(r, k)$ are the Stirling numbers of the second kind, so by (3),

$$\mathsf{E}(X_i^r) = o(n^{-\varepsilon'}) + \sum_{k=1}^{r} S(r, k)\omega(n)^k, \qquad r \geqq 1$$

which gives (for details see e.g. [9, p. 53])

$$\mathsf{E}\left(\frac{X_i-\omega(n)}{\sqrt{\omega(n)}}\right)^r = o(1) + \omega^{-r/2} \sum_{k=0}^{\infty} \frac{\omega^k}{k!} e^{-\omega}(k-\omega)^r. \tag{4}$$

Because $\omega=\omega(n)\to\infty$, the right-hand side of (4), which is the r-th moment of a standardized Poisson distribution with mean ω, converges to the r-th moment of $\mathcal{N}(0, 1)$ and (2) is proved. □

REMARK. In many applications the initial graph of a random graph is almost regular in the sense that there are few vertices with the degree smaller than d. As an example take the k-dimensional lattice $L_n(k)$ with vertex set $\{\mathbf{x}=(x_1, \ldots, x_k)\colon 0\leqq x_j\leqq n-1,\ x_j \text{ integer},\ j=1, \ldots, k\}$ and with an edge between $\mathbf{x}$ and $\mathbf{y}$ if and only if

$$\sum_{j=1}^{k} |x_j-y_j| = 1.$$

Obviously, there are n^k vertices in $L_n(k)$ and $n^k-(n-2)^k$ of them have degree $2k$ (these are the "interior" vertices) whereas the "boundary" vertices have the degree smaller than $2k$. But there are only $o(n^k)$ such vertices.

We are still able to cover the "almost regular" case by adding some assumptions to Theorems 2 and 3 which ensure us that the number of "different" vertices is small in a sense. Let $m=m(n)$ be the number of vertices in G with the degree smaller than $d=\Delta(G)$. The extra assumptions are:

$$m = o(n) \quad \text{and} \quad \left(\frac{m}{n}\right)^r q^{\delta(G)-d} = o(1) \quad \text{for any} \quad r = 1, 2, \ldots$$

in Theorem 2, and

$$m = o(n^\varepsilon) \quad \text{and} \quad \left(\frac{m}{n}\right)^r q^{\delta(G)-d} = o(n^{-\varepsilon}) \quad \text{for any} \quad r = 1, 2, \ldots$$

in Theorem 3.

3. Applications to the n-cube

We define *n-cube* Q_n as the n-dimensional binary lattice, i.e. $Q_n=L_2(n)$. Notice that Q_n is an n-regular graph with 2^n vertices and $n2^{n-1}$ edges. Let $Q_{n,p}$ stand for a random subgraph of the n-cube Q_n. We will present some available results about the distribution of the random variable $X_i=X_i(Q_{n,p})$ with respect to different values of the edge probability p. We think of p increasing from $p=0$ to $p=1$.

To begin with let

$$p = c(n2^{n/i})^{-1} \tag{5}$$

where $i=1, 2, \ldots$ is fixed and $c>0$ is a real number. Then by Theorem 1, $X_1/2 \rightsquigarrow \mathcal{P}(\lambda)$ and $X_i\rightsquigarrow\mathcal{P}(\lambda)$, $i\geqq 2$, where

$$\lambda = c^i/i!. \tag{6}$$

Now let us assume that $i=i(n)\to\infty$ but $i=o(n)$ and put

$$p = i(ne2^{(n-\log\sqrt{i}-c)/i})^{-1} \tag{7}$$

where $-\infty < c < \infty$. Logarithms will always be to base 2. Then by Theorem 2, $X_i \rightsquigarrow \mathcal{P}(\lambda)$ where

$$\lambda = 2^c/\sqrt{2\pi}. \tag{8}$$

As a matter of fact, taking into account that for $i = o(n)$

$$\binom{n}{i} \sim \frac{1}{\sqrt{2\pi i}} \left(\frac{en}{i}\right)^i$$

we get by (7)

$$\mathsf{E}(X_i) \sim 2^n \frac{1}{\sqrt{2\pi i}} \left(\frac{enp}{i}\right)^i q^{n-i} \sim 2^c/\sqrt{2\pi}.$$

In both cases considered above the edge probability p is very small, i.e. tends to zero very rapidly as $n \to \infty$. As the next step of our considerations assume that for a given c and ε, where $0 < c < \infty$, $0 < \varepsilon < 1$, the probability $p = p(n, c, \varepsilon)$ is a solution of the equation

$$p^\varepsilon (1-p)^{1-\varepsilon} = h(n, c, \varepsilon) \tag{9}$$

where

$$h(n, c, \varepsilon) = \frac{1}{2} \varepsilon^\varepsilon (1-\varepsilon)^{1-\varepsilon} (c\sqrt{n})^{1/n}. \tag{10}$$

Then putting $i = i(n, \varepsilon) = \lfloor \varepsilon n \rfloor$, by Theorem 2, we have $X_i \rightsquigarrow \mathcal{P}(\lambda)$ where

$$\lambda = c/\sqrt{2\pi\varepsilon(1-\varepsilon)}. \tag{11}$$

Since it does not follow immediately we sketch the proof. Taking into account that

$$\binom{n}{\lfloor \varepsilon n \rfloor} \sim \frac{\varepsilon^{-\varepsilon n}(1-\varepsilon)^{-(1-\varepsilon)n}}{\sqrt{2\pi\varepsilon(1-\varepsilon)n}}$$

we obtain

$$\mathsf{E}(X_{\lfloor \varepsilon n \rfloor}) \sim \frac{1}{\sqrt{2\pi\varepsilon(1-\varepsilon)n}} \left(\frac{2p^\varepsilon(1-p)^{1-\varepsilon}}{\varepsilon^\varepsilon(1-\varepsilon)^{1-\varepsilon}}\right)^n.$$

Now we want $\mathsf{E}(X_{\lfloor \varepsilon n \rfloor})$ to tend to λ given by (11). Thus it suffices to find a solution of the equation (9).

Some information about possible solutions $p_1(n, c, \varepsilon)$ and $p_2(n, c, \varepsilon)$ of (9) can be given at once. If we take into account a function

$$f(p) = p^\varepsilon(1-p)^{1-\varepsilon} - h$$

where $h = h(n, c, \varepsilon)$ is given by (10) then it is obvious that the maximum of $f(p)$ is reached for $p = \varepsilon$. Therefore one can check that for sufficiently large n, $p_1(n, c, \varepsilon) \to c_1$ and $p_2(n, c, \varepsilon) \to c_2$ where $c_1 < \frac{\varepsilon}{2}$ and $c_2 > \frac{1+\varepsilon}{2}$. For instance, if $i = \left\lfloor \frac{n}{2} \right\rfloor$ then $c_1 = \frac{1}{2} - \frac{\sqrt{3}}{4}$, $c_2 = \frac{1}{2} + \frac{\sqrt{3}}{4}$.

Finally let us put

$$p = 2^{-1-(j \log n - c)/n} \tag{12}$$

where $j=0, 1, 2, \ldots$ is fixed and $-\infty < c < +\infty$ is a real number. Then for $i=n-j$, by Theorem 2, $X_i \rightsquigarrow \mathscr{P}(\lambda)$, where this time

$$\lambda = 2^c/j!. \tag{13}$$

In particular, if $j=c=0$ then for $p=\frac{1}{2}$ we have $X_n \rightsquigarrow \mathscr{P}(1)$. Of course, by symmetry, the same assertion is true if we replace X_n by X_0. This special case was proved by Erdős and Spencer [4].

Now if we replace in (5), (7), (10) and (12) c by $\omega(n)$ in such a way that $\lambda=\lambda(\omega)$ given by (6), (8), (11) and (13), respectively becomes a slow sequence and $\lambda(\omega)\to\infty$ then by Theorems 1 and 3 we have that $(X_i-\lambda(\omega))/\sqrt{\lambda(\omega)} \rightsquigarrow \mathscr{N}(0, 1)$. Also, by symmetry, replacing in (5), (7), (9) and (12) p with $q=1-p$ one can obtain suitable distributions of the random variable X_{n-i}. All these results agree with our intuition that for every $i=1, 2, \ldots, n-1$ the number of vertices of degree i increases first and then decreases in the process of the evolution of a random n-cube $Q_{n,p}$.

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BOUNDARY VALUE PROBLEMS FOR AN ABSTRACT DIFFERENTIAL EQUATION

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Abstract

In the paper there is considered an abstract differential equation

$$S^2 x = f$$

with conditions

$$s_q x = x_q,$$

$$Ax = x_A.$$

1. Introduction

Operational calculus is a set $(L^0, L^1, S, T_q, s_q, Q)$, where $L^1 \subset L^0$, L^1, L^0 are linear spaces. Operation $S \in \mathscr{L}(L^1, L^0)$ called a derivative is a surjection. The set Q is a set of indices q for the operation $T_q \in \mathscr{L}(L^0, L^1)$ and for the operation $s_q \in \in \mathscr{L}(L^1, \operatorname{Ker} S)$ such that

$$ST_q f = f \quad \text{for} \quad f \in L^1, \quad q \in Q,$$

$$T_q Sx = x - s_q x \quad \text{for} \quad x \in L^1, \quad q \in Q.$$

Operation T_q is called an integral. Operation s_q is called a limit condition (see [2, 3, 4], [8, 10, 11]).

Let be given a family $\{T_q\}_{q \in Q}$ of integrals and a family $\{s_q\}_{q \in Q}$ of limit conditions. It is easy to see that fixed derivative S and its corresponding family of integrals $\{T_q\}_{q \in Q}$ induces a family of limit conditions $\{s_q\}_{q \in Q}$.

2. The abstract differential equation

DEFINITION (see [4, 3]). We denote

$$L^2 \overset{\text{df}}{=} \{x \in L^1 \colon Sx \in L^1\}.$$

Let $A\colon L^2 \to \operatorname{Ker} S$ be a linear operation.

THEOREM 1. *The abstract differential equation*

(1) $$S^2 x = f, \quad x \in L^2, \quad f \in L^0$$

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with conditions

$$s_q x = x_q, \tag{2}$$

$$Ax = x_A \tag{3}$$

(a) *has at least one solution if operation* $AT_q|_{\mathrm{Ker}\,S}$ *is a surjection onto* Ker S;
(b) *has at the outmost one solution if operation* $AT_q|_{\mathrm{Ker}\,S}$ *is an injection*;
(c) *has only one solution if operation* $AT_q|_{\mathrm{Ker}\,S}$ *is a bijection onto* Ker S.

Proof. Operating on the equation (1) with the operation T_q^2 and applying the axioms of the operational calculus and condition (2) we obtain

$$x = T_q^2 f + T_q c + x_q, \quad c \in \mathrm{Ker}\, S.$$

From condition (3) we have

$$Ax = AT_q^2 + AT_q c + Ax_q = x_A.$$

We have obtained an equation

$$AT_q c = g, \quad x_A - Ax_q - AT_q^2 f = g \in \mathrm{Ker}\, S \tag{4}$$

with an unknown c.

The statement of the theorem follows directly from equation (4).

Corollary. *If operation* $AT_q|_{\mathrm{Ker}\,S}$ *is a bijection onto* Ker S *then the abstract differential equation* (1) *with conditions* (2), (3) *has only one solution given by formula*

$$x = T_q^2 f + T_q (AT_q|_{\mathrm{Ker}\,S})^{-1} g + x_q \tag{5}$$

where $g = x_A - Ax_q - AT_q^2 f$.

Theorem 2. *If the following assumptions are satisfied*

(i) L^2, L^1, L^0 *are commutative algebras with unity* $e \in \mathrm{Ker}\, S$ *on the field* Γ;

(ii) $S(cx) = c(Sx)$, $T_q(cf) = c(T_q f)$, $s_q(cx) = c(s_q x)$ *where* $c \in \mathrm{Ker}\, S$, $x \in L^1$, $f \in L^0$;

(iii) $A(T_q e) = e$;

(iv) $A(cf) = c(Af)$, $f \in L^2$, $c \in \mathrm{Ker}\, S$, *then three operations* $\tilde{S}, \tilde{T}_q, \tilde{s}_q$ *defined by the formulas*

$$\tilde{S}u \overset{\mathrm{df}}{=} S^2 u, \quad u \in L^2, \tag{6}$$

$$\tilde{T}_q f \overset{\mathrm{df}}{=} T_q^2 f - [A(T_q^2 f)] T_q e, \quad f \in L^0 \tag{7}$$

$$\tilde{s}_q u \overset{\mathrm{df}}{=} s_q u - [(s_q u)(Ae)] T_q e + (Au) T_q e = s_q u + [Au - (s_q u)(Ae)] T_q e, \quad u \in L^2 \tag{8}$$

satisfy the axioms of operational calculus. Operation $\tilde{S}$ *is a derivative, operation* $\tilde{T}_q$ *is an integral, operation* $\tilde{s}_q$ *is a limit condition.*

Proof. Operations $\tilde{S}, \tilde{T}_q, \tilde{s}_q$ are linear operations. We must show that

$$\tilde{S}\tilde{T}_q f = f \quad \text{and} \quad \tilde{T}_q \tilde{S} u = u - \tilde{s}_q u, \quad f \in L^0, \quad u \in L^2.$$

From the fact that operations S, T_q, s_q satisfy the axioms of the operational calculus and from the assumptions we have

$$\tilde{S}\tilde{T}_q f = S^2T_q^2f - S^2\{[A(T_q^2f)]T_qe\} = f - [A(T_q^2f)](S^2T_qe) = f.$$

Further we have that

$$\tilde{T}_q\tilde{S}u = T_q^2S^2u - [A(T_q^2S^2u)]T_qe = u - s_qu - T_qs_qSu - [A(u - s_qu - T_qs_qSu)]T_qe =$$
$$= u - s_qu - (s_qSu)T_qe - (Au)T_qe + [(s_qu)(Ae)]T_qe + (s_qSu)[A(T_qe)]T_qe =$$
$$= u - s_qu - (Au)T_qe + [(s_qu)(Ae)]T_qe = u - \tilde{s}_qu.$$

Operations $\tilde{S}$, $\tilde{T}_q$, $\tilde{s}_q$ satisfy the axioms of the operational calculus.

THEOREM 3. *If the assumptions of Theorem 2 are satisfied then the abstract differential equation* (1) *with conditions* (2), (3) *has only one solution given by the formula*

(9) $$x = T_q^2f - [A(T_q^2f)]T_qe + x_q - [x_q(Ae)]T_qe + x_AT_qe.$$

PROOF. The proof of Theorem 3 follows directly from Theorem 2 and from the theorems of the operational calculus.

REMARK 1. If $A \overset{\mathrm{df}}{=} s_qS$ then we obtain initial value problem. In the case of Theorem 2 for $A \overset{\mathrm{df}}{=} s_qS$ we have $Ae = 0$.

REMARK 2. If $A \overset{\mathrm{df}}{=} s_{q_1}$ then we obtain two-point boundary value problem considered in [9] $(n=2)$.

The operational calculus obtained in Theorem 2 enables for instance to solve abstract differential equations of the type

(10) $$\sum_{i=0}^{n} R_iS^{2i}u = f$$

with conditions

(11) $$s_qS^{2i}u = u_{iq}, \quad A(S^{2i}u) = u_{iA} \quad \text{for} \quad i = 0, 1, \ldots, n-1$$

applying the methods presented in [1—4, 8, 10, 11]. Coefficients of equation (10) can be scalars (numbers), commutative or non-commutative operations with derivative S and integral T_q and operation A.

3. Examples

1. The differential equation

$$\left(t\frac{d}{dt}\right)^2 x = \{f(t)\}$$

with conditions

$$x(1) = x_1, \quad x(a) + x(b) = x_{ab}$$

where $x \in C^2(\langle 1, \infty), R)$, $f \in C^0(\langle 1, \infty), R)$, $a, b \in \langle 1, \infty)$ has only one solution.

The derivative S, the integral $T_q=T_{t_0}$, the limit condition $s_q=s_{t_0}$ appearing above are defined by the formulas

$$S \overset{\mathrm{df}}{=} t\frac{d}{dt},$$

$$T_{t_0} f \overset{\mathrm{df}}{=} \left\{\int_{t_0}^{t} \frac{f(\tau)}{\tau}\, d\tau\right\}, \quad \text{(see [4])}$$

$$s_{t_0} u \overset{\mathrm{df}}{=} \{u(t_0)\}, \quad f\in C^0(\langle 1, \infty), R), \quad u\in C^2(\langle 1, \infty), R),$$

$t_0=1$ and operation $AT_{t_0}\big|_{\mathrm{Ker}\left(t\frac{d}{dt}\right)}$ is a bijection onto $\mathrm{Ker}\left(t\frac{d}{dt}\right)$, where $Ax \overset{\mathrm{df}}{=} \{x(a)+x(b)\}$, $x\in C^2(\langle 1, \infty), R)$.

2. The partial differential equation

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2 u = 0 \quad \text{(see Example 5)}$$

with conditions

$$u(x, 0) = 0,$$

$$\iint\limits_{D=\langle 0,1\rangle\times\left\langle 0,\frac{3}{4}\right\rangle} u(x, y)\, dx\, dy = 0$$

where $u\in C^3(R\times\langle y_1, y_2\rangle, R)$, $0, \frac{3}{4}, 1\in\langle y_1, y_2\rangle$ apart from a zero solution has the solution $u=\alpha(xy-y^2)$, $\alpha\in R$.

Operation $AT_q c \overset{\mathrm{df}}{=} \iint\limits_{D=\langle 0,1\rangle\times\left\langle 0,\frac{3}{4}\right\rangle} y\varphi(x-y)\, dx\, dy$ is not an injection, $c\in\mathrm{Ker}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)$, $\varphi\in C^1(R, R)$ and $c=\varphi(x-y)$.

3. In the case when $A \overset{\mathrm{df}}{=} s_{q_1}$ the examples are given in [9].

4. There are many examples of operational calculus such that operations S, T_q, s_q satisfy the assumptions of Theorem 2. Applying operation $\frac{d}{dt}$ instead of operation S and linear functional on $C^1(\langle 0, 1\rangle)$ instead of A in Theorem 2, we will get the operational calculus considered in [7, 8].

In [8] there are given operational culculi for the derivative $\tilde{S}=S^2=\frac{d^2}{dt^2}$ given a fixed A.

5. In the case of operational calculus with the derivative

$$S\{u(x_1, x_2, \dots, x_n)\} \overset{\text{df}}{=} \left\{\sum_{i=1}^{n} b_i \frac{\partial u(x_1, x_2, \dots, x_n)}{\partial x_i}\right\}$$

the integral

$$T_{x_n^0}\{f(x_1, x_2, \dots, x_n)\} \overset{\text{df}}{=} \left\{\frac{1}{b_n} \int_{x_n^0}^{x_n} f\left(x_1 - \frac{b_1}{b_n}(x_n-\tau), \dots, x_{n-1} - \frac{b_{n-1}}{b_n}(x_n-\tau), \tau\right) d\tau\right\}$$

and the limit condition

$$s_{x_n^0}\{u(x_1, x_2, \dots, x_n)\} \overset{\text{df}}{=} \left\{u\left(x_1 - \frac{b_1}{b_n}(x_n - x_n^0), \dots, x_{n-1} - \frac{b_{n-1}}{b_n}(x_n - x_n^0), x_n^0\right)\right\}$$

where

$$u \in L^1 \overset{\text{df}}{=} C^2(R^{n-1} \times \langle x_n^1, x_n^2 \rangle, R),$$

$$f \in L^0 \overset{\text{df}}{=} C^1(R^{n-1} \times \langle x_n^1, x_n^2 \rangle, R), \quad x_n^0 \in \langle x_n^1, x_n^2 \rangle, \quad b_i \in R$$

for $i = 1, 2, \dots, n$, $b_n \neq 0$ (see [6]) the derivative $\tilde{S}$, the integral $\tilde{T}_{x_n^0}$, the limit condition $\tilde{s}_{x_n^0}$ are defined by formulas

$$\tilde{S} \overset{\text{df}}{=} \left(\sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}\right)^2,$$

$$\tilde{T}_{x_n^0}\{f(x_1, x_2, \dots, x_n)\} \overset{\text{df}}{=}$$

$$\overset{\text{df}}{=} \left\{\frac{1}{b_n^2} \int_{x_n^0}^{x_n} (x_n - \tau) f\left(x_1 - \frac{b_1}{b_n}(x_n-\tau), \dots, x_{n-1} - \frac{b_{n-1}}{b_n}(x_n-\tau), \tau\right) d\tau -\right.$$

$$\left. - \frac{x_n - x_n^0}{b_n} A\left(\frac{1}{b_n^2} \int_{x_n^0}^{x_n} (x_n-\tau) f\left(x_1 - \frac{b_1}{b_n}(x_n-\tau), \dots, x_{n-1} - \frac{b_{n-1}}{b_n}(x_n-\tau), \tau\right) d\tau\right)\right\},$$

$$s_{x_n^0}\{u(x_1, x_2, \dots, x_n)\} \overset{\text{df}}{=} \left\{u\left(x_1 - \frac{b_1}{b_n}(x_n - x_n^0), \dots, x_{n-1} - \frac{b_{n-1}}{b_n}(x_n - x_n^0), x_n^0\right) + \right.$$

$$\left. + \frac{x_n - x_n^0}{b_n}\left[Au - \left\{u\left(x_1 - \frac{b_1}{b_n}(x_n - x_n^0), \dots, x_{n-1} - \frac{b_{n-1}}{b_n}(x_n - x_n^0), x_n^0\right)\right\}(A\{1\})\right]\right\}$$

(where $u \in L^1$, $f \in L^0$) if operation $A: L^2 \to \operatorname{Ker}\left(\sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}\right)$ satisfies assumptions (iii), (iv) of Theorem 2.

Partial differential equation

$$\left(\sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}\right)^2 u = f$$

with conditions

$$\{u(x_1, x_2, \ldots, x_{n-1}, x_n^0)\} = \{\varphi(x_1, x_2, \ldots, x_{n-1})\},$$

$$\left\{\int_{x_n^1}^{x_n^2} u\left(x_1-\frac{b_1}{b_n}(x_n-\tau), \ldots, x_{n-1}-\frac{b_{n-1}}{b_n}(x_n-\tau), \tau\right) d\tau\right\} = \psi,$$

$$u\in L^2, \quad f\in L^0, \quad \varphi\in C^3(R, R), \quad \psi\in \operatorname{Ker}\left(\sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}\right),$$

has only one solution given by formula

$$\{u(x_1, x_2, \ldots, x_n)\} =$$

$$= \left\{\frac{1}{b_n^2}\int_{x_n^0}^{x_n} (x_n-\tau) f\left(x_1-\frac{b_1}{b_n}(x_n-\tau), \ldots, x_{n-1}-\frac{b_{n-1}}{b_n}(x_n-\tau), \tau\right) d\tau +\right.$$

$$+\varphi\left(x_1-\frac{b_1}{b_n}(x_n-x_n^0), x_2-\frac{b_2}{b_n}(x_n-x_n^0), \ldots, x_{n-1}-\frac{b_{n-1}}{b_n}(x_n-x_n^0)\right) -$$

$$-\frac{2(x_n-x_n^0)}{(x_n^2)^2-(x_n^1)^2-2x_n^0x_n^2+2x_n^0x_n^1}\left[(x_n^2-x_n^1)\varphi\left(x_1-\frac{b_1}{b_n}(x_n-x_n^0), \ldots\right.\right.$$

$$\left.\left.\ldots, x_{n-1}-\frac{b_{n-1}}{b_n}(x_n-x_n^0)\right)-\psi\right] - \frac{(x_n-x_n^0)}{b_n^3} \times$$

$$\left.\times A\left[\int_{x_n^0}^{x_n} (x_n-\tau) f\left(x_1-\frac{b_1}{b_n}(x_n-\tau), \ldots, x_{n-1}-\frac{b_{n-1}}{b_n}(x_n-\tau), \tau\right) d\tau\right]\right\}.$$

In this case

$$Au \overset{\mathrm{df}}{=} \left\{\frac{2b_n}{(x_n^2)^2-(x_n^1)^2-2x_n^0x_n^2+2x_n^0x_n^1}\times\right.$$

$$\left.\times \int_{x_n^1}^{x_n^2} u\left(x_1-\frac{b_1}{b_n}(x_n-\tau), \ldots, x_{n-1}-\frac{b_{n-1}}{b_n}(x_n-\tau), \tau\right) d\tau\right\}$$

so A satisfies the assumptions of Theorem 2.

6. Into the space $C(N)$ of real sequences $x=\{x_k\}$, $k=0, 1, 2, \ldots$ let us introduce the derivative $S=\Delta$ according to the formula

$$\Delta\{x_k\} \overset{\mathrm{df}}{=} \{x_{k+1}-x_k\}.$$

The limit condition s_{k_0} corresponding to the derivative Δ has the form

$$s_{k_0}\{x_k\} \overset{\mathrm{df}}{=} \{x_{k_0}\}.$$

The integral T_{k_0} corresponding to the derivative Δ and the limit condition s_{k_0} has the form

$$T_{k_0}\{x_k\} \overset{\mathrm{df}}{=} \begin{cases} 0 & \text{for } k = k_0 \\ \sum_{i=k_0}^{k-1} x_i & \text{for } k_0 < k \\ -\sum_{i=k}^{k_0-1} x_i & \text{for } k_0 > k \end{cases} \qquad \text{(see [5]).}$$

If operation $A: C(N) \to \operatorname{Ker} \Delta$ satisfies assumptions (iii), (iv) of Theorem 2 then on the basis of Theorem 2 we can define the derivative $\tilde{S} = \Delta^2$, the integral $\tilde{T}_{k_0}$ and the limit condition $\tilde{s}_{k_0}$. Operations $\tilde{S}$, $\tilde{T}_{k_0}$, $\tilde{s}_{k_0}$ are defined by the formulas (6), (7), (8).

The difference equation

$$\Delta^2\{x_k\} = \{f_k\}$$

with conditions

$$x_{k_0} = \alpha,$$

$$\sum_{k=k_1}^{k_2} x_k = \beta, \quad k_0 < k_1 < k_2$$

has only one solution.

The operation

$$A\{x_k\} \overset{\mathrm{df}}{=} \left\{\frac{1}{\varrho} \sum_{k=k_1}^{k_2} x_k\right\}$$

where

$$\varrho = \frac{1}{(k_2-k_1+1)(k_1-k_0) + \dfrac{(k_2-k_1)(k_2-k_1+1)}{2}}$$

satisfies the assumptions (iii), (iv) of Theorem 2.

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CIRCLE PACKING WITH MAXIMUM TOTAL PERIMETER

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We consider a system of circles drawn on the plane such that their interiors are pairwise disjoint and their radii belong to the fixed interval $[r, R]$. We will prove the following

THEOREM. *The maximum perimeter density of the circle packing is attained when all the circles have radius r, and they form a honeycomb system (i.e. all the circles are as small as possible, and each one is tangent to 6 others, like the cells of the honeycomb).*

Roughly speaking, by *perimeter density* or for short by *density*, we mean the sum of the perimeters of the circles of the packing in a very large circle, divided by the area of this circle. More precisely, let p_i denote the perimeter of circle C_i of the packing, and D_ϱ the circle of radius ϱ centered at some fixed point O, then the perimeter density is defined by

$$\liminf_{\varrho\to\infty} \frac{\sum_{i:\, C_i\subset D_\varrho} p_i}{\varrho^2\pi}.$$

A similar theorem was proved for the usual density by A. Florian in [3].

PROOF. To prove our theorem, we start with the reduction procedure of L. Fejes Tóth and J. Molnár (see [1]):

First we make the circle system saturated by adding new circles to it until no room remains for any other one. This obviously does not decrease the density.

Second we construct the hyperbola cells around the circles, where each cell consists of the points closest to the given circle. It is proved in [1], that the dual of this cellulation consists of triangles with the following properties:

1. The vertices of these triangles are centers of some circles of our system;
2. These circles do not intersect the opposite sides of the triangles to which they belong.
3. Only the circles centered around the vertices of a certain triangle can have points inside that triangle.

Since these cells tile the plane, the maximum density relative to the triangles

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gives an upper bound for the density of the whole circle packing. Naturally, the (perimeter) density relative to a triangle means the sum of the length of the three circular arcs inside the triangle, divided by the area of the triangle.

Third step. It is proved in [2], that in a triangle the maximum weighted density occurs when the circles centered at the vertices touch each other, while we move the circles, but do not change their size. In particular, this theorem applies to the perimeter density as well, if the circles are weighted by their perimeter. Thus we reduce our problem to consider the density in triangles of the type showed by Figure 1, where unity is chosen equal to the longest radius and so $a \leqq x \leqq 1$.

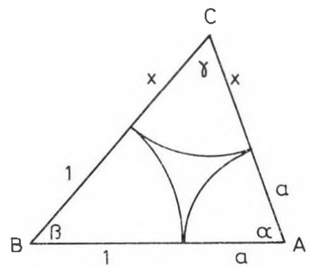

Fig. 1

Fourth step. We consider the density in this triangle when sidelength x varies. It will turn out, that the density is maximal, when $x=a$.

Fifth step. We fix the two congruent circles, choose the unity equal to their radius, and let the third circle vary. We will show, that the density increases as the largest radius decreases. This proves that the density is maximal if all the three radii are equal.

To complete the proof we note that decreasing the three equal radii increases the density. Therefore three pairwise touching circles of radius r yield the only "best" cell, where the density is maximum. Fortunately, the honeycomb circle system can be cut into such cells, thus it is of maximum perimeter density.

Proof of Step 4. Using the notations of Figure 1 the density to be considered is given by

$$S(x) = \frac{2(a\alpha+\beta+x\gamma)}{(a+1)(x+1)\sin\beta}. \tag{1}$$

Disregarding the constant factor $\frac{2}{a+1}$, later on we use

$$S_1(x) = \frac{a\alpha+\beta+x\gamma}{(x+1)\sin\beta}. \tag{2}$$

From the cosine theorem in the triangle of Figure 1 we get

$$
\begin{aligned}
&\cos\gamma = 1-\frac{2a}{(x+1)(x+a)}, \quad \sin\gamma = \frac{2\sqrt{ax(x+a+1)}}{(x+1)(x+a)},\\
(3)\qquad &\cos\beta = 1-\frac{2ax}{(x+1)(a+1)}, \quad \sin\beta = \frac{2\sqrt{ax(x+a+1)}}{(x+1)(a+1)},\\
&\cos\alpha = 1-\frac{2x}{(x+a)(a+1)}, \quad \sin\alpha = \frac{2\sqrt{ax(x+a+1)}}{(x+a)(a+1)}.
\end{aligned}
$$

Here α, β and γ are functions of x. Their derivatives are

$$
\begin{aligned}
&\gamma' = -\frac{(\cos\gamma)'}{\sin\gamma} = -\frac{a(2x+a+1)}{(x+1)(x+a)}\cdot\frac{1}{\sqrt{ax(x+a+1)}},\\
(4)\qquad &\beta' = \frac{a}{x+1}\cdot\frac{1}{\sqrt{ax(x+a+1)}},\\
&\alpha' = \frac{a}{x+a}\cdot\frac{1}{\sqrt{ax(x+a+1)}}.
\end{aligned}
$$

We have

$$\operatorname{sgn} S_1'(x) = \operatorname{sgn}\frac{S_2(x)}{(x+1)^2\sin^2\beta} = \operatorname{sgn} S_2(x),$$

where

(5) $S_2(x) = (a\alpha'+\beta'+\gamma+x\gamma')(x+1)\sin\beta-(a\alpha+\beta+x\gamma)[\sin\beta+(x+1)\cos\beta\cdot\beta']$.

Applying (3) and (4) the term above in square brackets can be written in the form

$$\frac{2a(2x+a+1)}{\sin\gamma(x+1)(x+a)(a+1)}.$$

It is positive, therefore we can divide S_2 by it without changing the sign. Let us denote the quotient by S_3,

$$
\begin{aligned}
(6)\qquad S_3(x) = {}&\gamma\frac{2x(x+a+1)}{2x+a+1}-x(\gamma+\sin\gamma)+\\
&+\left(\frac{a+1}{2x+a+1}\sin\beta-\beta\right)+a\left(\frac{a+1}{2x+a+1}\sin\alpha-\alpha\right).
\end{aligned}
$$

We need the derivative of $S_3(x)$, too. For this we calculate

$$
\begin{aligned}
(7)\qquad &\left[\gamma\frac{2x(x+a+1)}{2x+a+1}-x(\gamma+\sin\gamma)\right]' = \gamma\left[\frac{2(2x+a+1)^2-4x(x+a+1)}{(2x+a+1)^2}-1\right]+\\
&+\gamma'\left[\frac{2x(x+a+1)}{2x+a+1}-x(1+\cos\gamma)\right]-\sin\gamma,
\end{aligned}
$$

(8) $$\left[\frac{a+1}{2x+a+1}\sin\beta-\beta\right]'=-\frac{2(4x+a+3)ax(x+a+1)}{(2x+a+1)^2(x+1)^2\sqrt{ax(x+a+1)}},$$

(9) $$\left[\frac{a+1}{2x+a+1}\sin\alpha-\alpha\right]'=-\frac{2(4x+3a+1)ax(x+a+1)}{(2x+a+1)^2(x+a)^2\sqrt{ax(x+a+1)}}.$$

Putting these together we get

(10) $$S_3'(x)=\gamma\left(\frac{a+1}{2x+a+1}\right)^2-\frac{2\sqrt{ax(x+a+1)}}{(2x+a+1)^2(x+1)^2(x+a)^2}\times$$
$$\times[8(a+1)x^3+(7a^2+34a+7)x^2+(a^3+23a^2+23a+1)x+(3a^3+10a^2+3a)].$$

Now we show that $S_3'(x)$ is negative. For this we need the following

LEMMA.

(11) $$g(x)=\left[1+\left(\frac{\pi}{2}-1\right)(1-\cos x)\right]\sin x\geq x \quad \textit{if} \quad 0\leq x\leq\frac{\pi}{2}.$$

PROOF of the Lemma. Differentiating twice $g(x)$ we get

$$g''(x)=[2(\pi-2)\cos x-\pi/2]\sin x.$$

It is positive if $0<x<x_0=0.812\ldots$, $\left(\cos x_0=\frac{\pi}{4\pi-8}\approx 0.688\right)$, and negative if $x_0<x<\pi/2$. Thus $g(x)$ is convex in $(0, x_0)$ and concave in $(x_0, \pi/2)$. The line $y=x$ is the tangent of $g(x)$ at $x=0$, and it is therefore under the graph of $g(x)$ in $(0, x_0)$. The chord of $g(x)$ between x_0 and $\pi/2$ is above the line $y=x$ and under the graph of $g(x)$, proving our Lemma.

According to the Lemma, the expression in (10) increases of we replace γ by

$$g(\gamma)=\left[1+\left(\frac{\pi}{2}-1\right)\frac{2a}{(x+1)(x+a)}\right]\frac{2\sqrt{ax(x+a+1)}}{(x+1)(x+a)},$$

and then we get

(12) $$S_4(x)=-\frac{2\sqrt{ax(x+a+1)}}{(2x+a+1)^2(x+1)^2(x+a)^2}\{8(a+1)x^3+2(3a^2+16a+3)x^2+$$
$$+20a(a+1)x+[a^3(4-\pi)+2a^2(6-\pi)+a(4-\pi)]\}.$$

$S_4(x)$ is negative for any $x>0$, so is $S_3'(x)$. Consequently, $S_3(x)$ is a decreasing function of x.

Now we show, that $S_3(a)<0$. If $x=a$ then $\gamma=\alpha$, and

(13) $$S_3(a)=\frac{2a(2a+1)}{3a+1}-a(\alpha+\sin\alpha)+\frac{a+1}{3a+1}\sin\beta-\beta+$$
$$+a\left(\frac{a+1}{3a+1}\sin\alpha-\alpha\right)=-\frac{2a^2}{3a+1}\alpha-\frac{2a^2}{3a+1}\sin\alpha-\beta+\frac{a+1}{3a+1}\sin\beta.$$

Referring to triangle ABC we get $\beta=\pi-2\alpha$, and from the sine theorem $\sin\beta=$ $=\frac{2a}{a+1}\sin\alpha$. Applying these equalities and (13) we get

(14) $$S_3(a)=2\frac{-a^2+3a+1}{3a+1}\alpha+2a\frac{1-a}{3a+1}\sin\alpha-\pi.$$

Since $x\leqq\pi/2-\cos x$ for $0\leqq x\leqq\pi/2$, we have $\alpha\leqq\pi/2-\cos\alpha=\pi/2-1+\frac{1}{a+1}$. Substituting it to (14) we get

(15) $$S_3(a)\leqq\frac{-a^2+3a+1}{3a+1}\left[(\pi-2)+\frac{2}{a+1}\right]+2a\frac{1-a}{3a+1}\sin\alpha-\pi.$$

Since $\sin\alpha\leqq 1$, we need only to show, that

$$(-a^2+3a+1)[(\pi-2)a+\pi]+2a(1-a)(a+1)-\pi(3a+1)(a+1)<0.$$

This is obvious if we carry out the indicated multiplications, resulting

$$-\pi a^3-(\pi+4)a^2-2a<0.$$

Because $S_3(x)$ is decreasing and $S_3(a)<0$, $S_3(x)<0$ for $a\leqq x\leqq 1$. $S_3(x)$ has the same sign as $S'(x)$, and it completes our proof of Step 4, showing that the maximum density occurs when x is the smallest possible, $x=a$.

PROOF of Step 5.

Using the notations of Figure 2, we have to show, that the density

(16) $$T(x)=\frac{x\delta+\pi/2-\delta}{\sqrt{x^2+2x}}$$

is decreasing for $x\geqq 1$. Now

(17) $$\sin\delta=\frac{1}{x+1},\quad \delta'=\frac{-1}{(x+1)\sqrt{x^2+2x}}$$

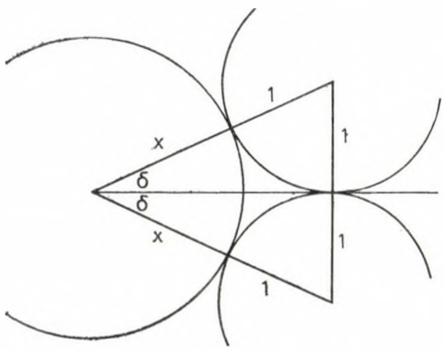

Fig. 2

and

$$T'(x) = \frac{[\delta+(x-1)\delta']\sqrt{x^2+2x}-[\pi/2+(x-1)\delta]\dfrac{x+1}{\sqrt{x^2+2x}}}{x^2+2x} =$$

(18)

$$= \delta\frac{2x+1}{(x^2+2x)^{3/2}} - \frac{(x-1)\sqrt{x^2+2x}+\pi/2(x+1)^2}{(x+1)(x^2+2x)^{3/2}}.$$

Let T_1 denote T' divided by the (positive) coefficient of δ in (18):

(19) $$T_1(x) = \delta - \frac{(x-1)\sqrt{x^2+2x}+\pi/2(x+1)^2}{(x+1)(2x+1)}.$$

We need its derivative, too:

$$T_1'(x) = \frac{-1}{(x+1)\sqrt{x^2+2x}} - \frac{\left[\sqrt{x^2+2x}+\dfrac{x^2-1}{\sqrt{x^2+2x}}+\pi x+\pi\right](x+1)(2x+1)}{(x+1)^2(2x+1)^2} -$$

(20)

$$- \frac{\left[(x-1)\sqrt{x^2+2x}+\dfrac{\pi}{2}(x+1)^2\right](4x+3)}{(x+1)^2(2x+1)^2}.$$

Let us multiply $T_1'(x)$ by the expression

$$(x+1)^2(2x+1)^2\sqrt{x^2+2x}.$$

The result is

(21) $$T_2(x) = -(x^2+2x)(7x+5)+\frac{\pi}{2}(x+1)^2\sqrt{x^2+2x}.$$

Its sign does not change if we replace both terms by their squares (i.e. multiplying T_2 by the sum of the two positive terms), and then divide the result by x^2+2x. We get

(22) $$T_3(x) = \left(\frac{\pi^2}{4}-49\right)x^4+(\pi-168)x^3+\left(\frac{3}{2}\pi-165\right)x^2+(\pi^2-50)x+\frac{\pi^2}{4}.$$

This is a decreasing function for positive values of x, because all the coefficients of the powers of x are negative. For $x=1$, $T_3(1)<0$, therefore $T_3(x)$ is negative for $x \geq 1$. So T_1' is also negative and T_1 is decreasing. In (19) we see that

$$T_1(1) = -\frac{\pi}{6} < 0,$$

and so T_1 is always negative. These prove that $T(x)$ is a decreasing function of x for $x \geq 1$.

Our proof is complete.

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NOTES ON EXTENSIONS OF QUASI-UNIFORMITIES FOR PRESCRIBED TOPOLOGIES

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Abstract

This paper answers two questions (raised by Császár [1, 2]) concerning extensions of a quasi-uniformity for a prescribed extension of the induced topology.

Császár [1] raised the following problem: let $(X, \mathcal{U})$ be a quasi-uniform space, $\mathcal{T}$ the topology induced by $\mathcal{U}$, and $(Y, \mathcal{S})$ an extension of $(X, \mathcal{T})$; find out as much as possible about the class of those quasi-uniform extensions of $\mathcal{U}$ which are compatible with the topology $\mathcal{S}$. He showed (among other results) that[1]

a) ([1] 1.1) if there is an extension of $\mathcal{U}$ compatible with $\mathcal{S}$ then each trace filter is round;

b) ([1] 2.2) if $\mathcal{S}$ is a loose extension of $\mathcal{T}$, and each trace filter is round then there is an extension of $\mathcal{U}$ compatible with $\mathcal{S}$;

c) ([1] 3.1) if there is at least one extension of $\mathcal{U}$ compatible with $\mathcal{S}$ then there exists a finest one among all such extensions.

§ 1. On coarsest extensions

The problem of the existence of a *coarsest* compatible extension was left open (see [1] 9.3). We are going to give an example showing that there is in general no coarsest extension of $\mathcal{U}$ compatible with $\mathcal{S}$, even if $\mathcal{S}$ is supposed to be a loose *and* strict extension of $\mathcal{T}$. There is, however, a coarsest one among those extensions of $\mathcal{U}$ compatible with a loose extension of $\mathcal{T}$ which satisfy an additional condition (see the theorem below); this extension will be of use in the counterexample.

THEOREM. *Let $(X, \mathcal{U})$ be a quasi-uniform space, $\mathcal{T}$ the topology induced by $\mathcal{U}$, and $(Y, \mathcal{S})$ a loose extension of $(X, \mathcal{T})$. Assume that $Y \neq X$, and the trace filters $\mathfrak{s}(p)$ $(p \in Y \setminus X)$ are round. For $p \in Y \setminus X$, $S \in \mathfrak{s}(p)$ and $U \in \mathcal{U}$, define $V = V(p, S, U) \subset$*

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[1] See the introduction of [1] for terminology and notations.

$\subset Y\times Y$ *by*

(1) $$(x, a)\in V \text{ iff } (x, a)\in U \quad (\text{for } x\in X, a\in Y);$$

(2) $$(p, a)\in V \text{ iff } a=p \text{ or } a\in U(S) \quad (\text{for } a\in Y);$$

(3) $$(q, a)\in V \text{ whenever } q\in Y\setminus X,\ q\neq p \text{ and } a\in Y.$$

Then

(4) $$\{V(p, S, U)\colon p\in Y\setminus X,\ S\in \mathfrak{s}(p),\ U\in\mathcal{U}\}$$

is a subbase for a quasi-uniformity $\mathcal{V}$ *on* Y. $\mathcal{V}$ *is the coarsest one among those extensions of* $\mathcal{U}$ *compatible with* $\mathcal{S}$ *for which* X *is far from* $Y\setminus X$.[2]

PROOF. Let us first show that if $V=V(p, S, U)$ and $V_0=V(p, S, U_0)$ where $U_0^2\subset U$ then $V_0^2\subset V$. Assume $(b, c)\in V_0$, $(c, d)\in V_0$. If $b\in Y\setminus X$, $b\neq p$ then $(b, d)\in V$ follows from (3). If $b=p$ then, by (2), either $c=p$ or $c\in U_0(S)$; in the first case, again by (2), $d=p$ or $d\in U_0(S)\subset U(S)$, hence $(b, d)\in V$; in the second case $c\in X$, thus (1) implies $(c, d)\in U_0$, i.e. $d\in U(S)$, and $(b, d)\in V$ again. Finally, if $b\in X$ then $(b, d)\in V$ is clear from (1). It is evident that each $V(p, S, U)$ contains the diagonal of $Y\times Y$, so (4) is indeed a subbase for a quasi-uniformity, which will be denoted by $\mathcal{V}$.

According to (1), $\mathcal{U}$ is the trace on X of a subbase for $\mathcal{V}$ [each $U\in\mathcal{U}$ can be obtained from at least one $V(p, S, U)$], therefore $\mathcal{V}|X=\mathcal{U}$. (1) implies that $V(X)=X$ for any element of the subbase (4), so X is far from $Y\setminus X$.

By (1), the $\mathcal{U}$-neighbourhoods of a point $x\in X$ form a base for the neighbourhood filter of x in the topology induced by $\mathcal{V}$, while (2) and (3) imply that the sets

(5) $$\{p\}\cup U(S) \qquad (U\in\mathcal{U},\ S\in\mathfrak{s}(p))$$

form a neighbourhood subbase (in fact, a neighbourhood base) of $p\in Y\setminus X$ in the topology induced by $\mathcal{V}$. As $\mathfrak{s}(p)$ is round, the sets in (5) coincide with those of the form $\{p\}\cup T$ $(T\in\mathfrak{s}(p))$, thus $\mathcal{V}$ is compatible with the loose extension.

Take now another extension $\mathcal{W}$ of $\mathcal{U}$ compatible with $\mathcal{S}$ such that there exists a $W_0\in\mathcal{W}$ with $W_0(X)=X$. It will complete the proof if we show that $\mathcal{V}\subset\mathcal{W}$, i.e. that each element of the subbase (4) for $\mathcal{V}$ belongs to $\mathcal{W}$.

Take a $V=V(p, S, U)$. Since $\mathcal{W}$ is an extension of $\mathcal{U}$, there is a $W_1\in\mathcal{W}$ with $W_1|X=U$. $\mathcal{W}$ is compatible with the loose extension, thus there is a $W_2\in\mathcal{W}$ with $W_2(p)=\{p\}\cup S$. Put $W=W_0\cap W_1\cap W_2$; we claim that $W\subset V$ (whence $V\in\mathcal{W}$).

Indeed: assume $(b, c)\in W$; if $b\in Y\setminus X$, $b\neq p$ then $(b, c)\in V$ by (3); if $b=p$ then $(b, c)\in W_2$ and (2) imply $(b, c)\in V$; finally, if $b\in X$ then it follows from $(b, c)\in W_0$ and $W_0(X)=X$ that $c\in X$, so we have $(b, c)\in U$ form $(b, c)\in W_1$ and $W_1|X=U$, i.e. $(b, c)\in V$ by (1).

EXAMPLE. Let $\mathbf{N}$ denote the set of the positive integers,

$$Y=\bigcup\{[2n, 2n+1]\colon n\in\mathbf{N}\},\quad X=\bigcup\{]2n, 2n+1]\colon n\in\mathbf{N}\},$$

$$\tilde{U}_n=\{(a, b)\in Y\times Y\colon |a-b|<1/n\},\quad U_n=\tilde{U}_n|X.$$

---

[2] This means that $V(X)=X$ for some $V\in\mathcal{V}$.

Let $\tilde{\mathcal{U}}$ and $\mathcal{U}$ be the uniformities on Y, respectively on X, inherited from the Euclidean uniformity of $\mathbf{R}$ (i.e. $\{\tilde{U}_n : n \in \mathbf{N}\}$ and $\{U_n : n \in \mathbf{N}\}$ are bases for $\tilde{\mathcal{U}}$, respectively for $\mathcal{U}$). Now if $\mathcal{T}$ is the topology of $\mathcal{U}$ and $\mathcal{S}$ denotes the Euclidean topology on Y then $\tilde{\mathcal{U}}$ is an extension of $\mathcal{U}$ compatible with $\mathcal{S}$. It is easy to check that $\mathcal{S}$ is both loose and strict extension of $\mathcal{T}$, with the trace filters generated by the filter bases

$$\{]2n, 2n+t[: 0 < t < 1\} \quad (n \in \mathbf{N}).$$

Let $\mathcal{V}$ denote the extension constructed in the theorem. We shall prove that $\mathcal{U}$ has no extension coarser than both $\tilde{\mathcal{U}}$ and $\mathcal{V}$; consequently, there is no coarsest extension of $\mathcal{U}$ for the topology $\mathcal{T}$.

Assume the contrary, i.e. let $\mathcal{W}$ be a quasi-uniformity on Y such that $\mathcal{W} \subset \tilde{\mathcal{U}} \cap \mathcal{V}$ and $\mathcal{W}|X = \mathcal{U}$ (we could also assume that $\mathcal{W}$ is compatible with $\mathcal{T}$, but this is not needed in the reasoning below). Take a $W \in \mathcal{W}$ with $W|X = U_1$, and pick a $W_0 \in \mathcal{W}$ with $W_0^2 \subset W$. Now $W_0 \in \tilde{\mathcal{U}}$, so there is a $k \in \mathbf{N}$ such that $\tilde{U}_k \subset W_0$. On the other hand, $W_0 \in \mathcal{V}$ implies that W_0 contains a finite intersection of entourages of the form (4), i. e.

$$\bigcap_{i=1}^{j} V(2n_i, S_i, Z_i) \subset W_0 \tag{6}$$

with suitable $j \in \mathbf{N}$, $n_i \in \mathbf{N}$, $S_i \in \mathfrak{s}(2n_i)$, $Z_i \in \mathcal{U}$ $(1 \leq i \leq j)$. Choose an $n \in \mathbf{N}$ different from each n_i, and put $x_0 = 2n + 1/2k$. We have $(x_0, 2n) \in \tilde{U}_k$, so $(x_0, 2n) \in W_0$; for an arbitrary $x \in X$, $(2n, x) \in W_0$ by (3) and (6); hence $(x_0, x) \in W$. This means that $(x_0, x) \in U_1$ holds for each $x \in X$, a contradiction.

§ 2. On regular extensions

A quasi-uniformity $\mathcal{U}$ on X is called *regular* ([1] § 8) if for any $U \in \mathcal{U}$ there is a $U_0 \in \mathcal{U}$ such that $\overline{U_0(x)} \subset U(x)$ whenever $x \in X$. If there is a regular extension of the quasi-uniformity $\mathcal{U}$ such that it is compatible with the extension $(Y, \mathcal{S})$ of the topological space $(X, \mathcal{T}(\mathcal{U}))$ then

a) ([1] 8.5) $\mathcal{U}$ is regular;

b) ([1] 8.1) $\mathcal{S}$ is regular;

c) ([1] 1.1) for each $p \in Y \setminus X$, $\mathfrak{s}(p)$ is round;

d) ([2] 2.1) for each $p \in Y \setminus X$, $\mathfrak{s}(p)$ is almost Cauchy. (According to [2] § 2, a filter $\mathfrak{s}$ is *almost Cauchy* if for any $U \in \mathcal{U}$ there are a $U_0 \in \mathcal{U}$ and an $S \in \mathfrak{s}$ such that $S \subset U(x)$ whenever $x \in X$ and each member of $\mathfrak{s}$ intersects $U_0(x)$.)

By [2] 3.1. the conditions a) to d) are sufficient for the existence of a regular extension of $\mathcal{U}$ assuming that $Y \setminus X$ is finite. [2] 5.1 asks whether these conditions are also sufficient in the general case; the example below shows that they are not.

EXAMPLE. Let

$$X = \bigcup_{n=2}^{\infty}]n, n+1/n[, \quad Y = X \cup \mathbf{N}.$$

Let $\mathcal{U}$ be the quasi-uniformity on X induced by the quasi-metric

$$d(x, y) = \begin{cases} y-x & \text{if } x \leqq y, \quad [x] = [y], \\ 1/[x] & \text{if } y \leqq x, \quad [x] = [y], \\ 1 & \text{if } [x] \neq [y] \end{cases}$$

where $[x]$ denotes the integral part of x. If

$$U_k = \{(x, y)\colon d(x, y) < 1/k\}$$

then $\{U_k : k \in \mathbf{N}\}$ is a base for $\mathcal{U}$. $\mathcal{T}(\mathcal{U})$ is the trace on X of the Sorgenfrey topology of $\mathbf{R}$. $\mathcal{U}$ is regular, since for each $k \in \mathbf{N}$ and $x \in X$, we have $\overline{U_k(x)} \subset U_k(x)$. Let now

$$\{]2p, 2p+t[\cup]2p+1, 2p+1+t[\colon 0 < t < 1/(2p+1)\}$$

be a base for the trace filter $\mathfrak{s}(p)$ $(p \in \mathbf{N})$. These filters are clearly round. The strict (in fact, the only) extension $\mathcal{S}$ of $\mathcal{T}(\mathcal{U})$ belonging to these trace filters is regular. Thus a), b) and c) are satisfied.

To prove that the trace filters are almost Cauchy, let $p \in \mathbf{N}$ be fixed, and take $U \in \mathcal{U}$; now $U_0 = U_{2p+1}$ and an arbitrary $S \in \mathfrak{s}(p)$ will do: indeed, there is no $x \in X$ such that each member of $\mathfrak{s}(p)$ meets $U_0(x)$.

Assume that there is a regular extension $\mathcal{V}$ of $\mathcal{U}$ compatible with $\mathcal{S}$. Choose a $V \in \mathcal{V}$ such that $V|X = U_1$. Take a $V_0 \in \mathcal{V}$ with $V_0^2 \subset V$, then a $W \in \mathcal{V}$ such that $\overline{W(a)} \subset V_0(a)$ $(a \in Y)$. Now $W|X \supset U_p$ for some $p \in \mathbf{N}$. For $0 < t < 1/2p$, $2p + t \in U_p(x_0)$ where $x_0 = 2p + 1/3p$, i.e. $2p + t \in W(x_0)$ and $p \in \overline{W(x_0)} \subset V_0(x_0)$. On the other hand, there is a $t_0 > 0$ such that $y_0 = 2p + 1 + t_0 \in V_0(p)$, hence $y_0 \in V(x_0)$, in other words, $y_0 \in U_1(x_0)$, a contradiction, since $d(x_0, y_0) = 1$.

REMARK. [2] Lemma 5.2 could have been used in proving that there is no regular extension. As a matter of fact, this lemma was of assistance in finding the example.

ADDED IN PROOF. The author prefers now the following terminology: *uniformly regular* for regular and *regularly tame* for almost Cauchy, cf. [3].

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A SUPPLEMENT TO THE GENERALIZED MARTINGALE FEFFERMAN INEQUALITY

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Abstract

S. Ishak and J. Mogyoródi [4], [5] and [6] have generalized the famous Fefferman inequality [2] in martingale language. If (Φ, Ψ) is a pair of complementary Young functions and Φ has finite power then the limit of $\mathsf{E}(X_n Y_n)$ exists and is finite, where $(X_n, \mathscr{F}_n)$ and $(Y_n, \mathscr{F}_n)$ are martingales belonging to the $\mathscr{H}_\Phi$- and $\mathscr{K}_\Psi$-spaces, respectively. In the present note we identify this limit. Namely, we prove that under the additional condition that Ψ has finite power, it can be expressed as a Lebesgue integral. Our result is that $\lim_{n\to+\infty} \mathsf{E}(X_n Y_n)$ is equal to

$$\mathsf{E}\Big(\sum_{i=1}^{\infty}(X_i-X_{i-1})(Y_i-Y_{i-1})\Big)$$

and can be considered as a generalization of Garsia's one (cf. [3], Remark I.5.1), which was proved in the case when $\Phi(x)=\dfrac{x^p}{p}$, $\Psi(x)=\dfrac{x^q}{q}$ and $p^{-1}+q^{-1}=1$, where $1<p\leqq 2$.

1. Basic notions and definitions

Let $(\Omega, \mathscr{A}, \mathsf{P})$ be a probability space and let $X, Y\in L_1(\Omega, \mathscr{A}, \mathsf{P})$ be random variables. Let $\mathscr{F}_0\subset\mathscr{F}_1\subset\dots$ be a sequence of σ-fields of events. Consider the martingales

$$X_n = \mathsf{E}(X|\mathscr{F}_n), \quad Y_n = \mathsf{E}(Y|\mathscr{F}_n), \quad n\geqq 0$$

where for the sake of simplicity we suppose that $\mathscr{F}_\infty=\sigma\big(\bigcup_{n=0}^{\infty}\mathscr{F}_n\big)=\mathscr{A}$ and $X_0=Y_0=0$ a.s. We denote by $d_0=0, d_1, d_2, \dots$ and $d_0'=0, d_1', d_2', \dots$ the corresponding martingale differences.

Recall that (Φ, Ψ) is called a pair of complementary Young functions if $\Phi(x)= \int_0^x \varphi(t)\,dt$ and $\varphi(t)$ is a right continuous increasing function with $\varphi(0)=0$, $\lim_{t\to\infty}\varphi(t)=\infty$ and $\Psi(x)=\int_0^x \psi(t)\,dt$, where $\psi(t)$ is the generalized inverse of $\varphi(t)$ and $\Psi(t)$ has the same properties as $\varphi(t)$. The power p of Φ is defined by

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the formula $p=\sup\limits_{x>0}\frac{x\varphi(x)}{\Phi(x)}$. Concerning the details of the theory of Young functions we refer to [7] and [8].

Further, let (Φ, Ψ) be a pair of complementary Young functions. We say that the random variable X belong to the Hardy space $\mathscr{H}_\Phi$ if its quadratic variation

$$S = \Big(\sum_{i=1}^{\infty} d_i^2\Big)^{1/2}$$

is an element of the Orlicz space $L^\Phi(\Omega, \mathscr{A}, \mathsf{P})$. A random variable Z is said to belong to $L^\Phi(\Omega, \mathscr{A}, \mathsf{P})$ if for some constant $a>0$ we have

$$\mathsf{E}\big(\Phi(a^{-1}|Z|)\big) \leqq 1$$

and in this case we define

$$\|Z\|_\Phi = \inf\big(a>0\colon\ \mathsf{E}\big(\Phi(a^{-1}|Z|)\big) \leqq 1\big).$$

It can be easily seen that $\|\cdot\|_\Phi$ is a norm.

If $X\in\mathscr{H}_\Phi$ then the $\mathscr{H}_\Phi$-norm of X is defined as follows

$$\|X\|_{\mathscr{H}_\Phi} = \|S\|_\Phi.$$

Suppose that $X\in\mathscr{H}_\Phi$ and for any fixed n consider the random variable X_n. The martingale which corresponds to X_n (with respect to the sequence $\{\mathscr{F}_n\}$ of σ-fields) is $X_0=0,\ X_1, \ldots, X_{n-1}, X_n, X_n, \ldots$. Therefore, the differences of this martingale are $d_0=0,\ d_1, \ldots, d_n, 0, 0, \ldots$ and the corresponding quadratic variation is

$$S_n = \Big(\sum_{i=1}^{n} d_i^2\Big)^{1/2}.$$

Consequently, if $X\in\mathscr{H}_\Phi$ then so does X_n and

$$\|X_n\|_{\mathscr{H}_\Phi} \leqq \|X\|_{\mathscr{H}_\Phi}.$$

The Hardy space $\mathscr{H}_1$ is defined separately, since the class of the Young functions Φ does not contain the identity function $\Phi(x)=x$. The random variable X belongs to the Hardy space $\mathscr{H}_1$ if $S\in L_1$ and in this case we define

$$\|X\|_{\mathscr{H}_1} = \|S\|_1.$$

It can be easily seen that $\|\cdot\|_{\mathscr{H}_1}$ is a norm. Also, if $X\in\mathscr{H}_1$ then $X_n\in\mathscr{H}_1$ and

$$\|X_n\|_{\mathscr{H}_1} \leqq \|X\|_{\mathscr{H}_1}.$$

We say that the random variable Y belongs to the space $\mathscr{K}_\Psi$ if there exists a random variable $\gamma\in L^\Psi(\Omega, \mathscr{A}, \mathsf{P})$ such that the inequality

$$\mathsf{E}\big(|Y-Y_{n-1}|\,\big|\mathscr{F}_n\big) \leqq \mathsf{E}(\gamma|\mathscr{F}_n)$$

holds a.s. for every $n\geqq 1$. Let

$$\Gamma_Y^{(\Psi)} = \big\{\gamma\colon \gamma\in L^\Psi,\ \mathsf{E}\big(|Y-Y_{n-1}|\,\big|\mathscr{F}_n\big) \leqq \mathsf{E}(\gamma|\mathscr{F}_n)\quad \text{a.s.},\quad \forall n \geqq 1\big\}.$$

If $Y\in\mathscr{K}_\Psi$ then we define its $\mathscr{K}_\Psi$-norm by the formula

$$\|Y\|_{\mathscr{K}_\Psi} = \inf_{\gamma\in\Gamma_Y^{(\Psi)}} \|\gamma\|_\Psi .$$

It can be seen that $\|\cdot\|_{\mathscr{K}_\Psi}$ is a norm. For arbitrary $k\geqq n\geqq 1$ we can show that (cf. [1])

$$\mathsf{E}(|Y_k-Y_{n-1}||\mathscr{F}_n) \leqq \mathsf{E}(|Y-Y_{n-1}||\mathscr{F}_n) \quad \text{a.s.},$$

and consequently, we see that if $Y\in\mathscr{K}_\Psi$ then so does Y_k for every k and

$$\|Y_k\|_{\mathscr{K}_\Psi} \leqq \|Y\|_{\mathscr{K}_\Psi}.$$

Especially, with $k=n$ for arbitrary $\gamma\in\Gamma_Y^{(\Psi)}$ we have

$$\mathsf{E}(|d_n'||\mathscr{F}_n) = |d_n'| \leqq \mathsf{E}(\gamma|\mathscr{F}_n) \quad \text{where} \quad d_n' = Y_n-Y_{n-1}$$

and

$$\|d_n'\|_{\mathscr{K}_\Psi} \leqq \|Y\|_{\mathscr{K}_\Psi}.$$

We also define here the notion of the BMO-space (bounded mean oscillation). Let $Y\in L_1(\Omega, \mathscr{A}, \mathsf{P})$ and consider the corresponding martingale $(Y_n, \mathscr{F}_n)$. We say that $Y\in$BMO if the set of random variables

$$\Gamma_Y^{(\infty)} = \{\gamma\colon \gamma\in L_\infty,\ \mathsf{E}(|Y-Y_{n-1}||\mathscr{F}_n) \leqq \mathsf{E}(\gamma|\mathscr{F}_n) \quad \text{a.s.,} \quad \forall n\geqq 1\}$$

is not empty. In this case we define

$$\|Y\|_{\mathrm{BMO}} = \inf_{\gamma\in\Gamma_Y^{(\infty)}} \|\gamma\|_\infty .$$

It can be easily seen that $\|\cdot\|_{\mathrm{BMO}}$ is a norm. An equivalent definition of $\|Y\|_{\mathrm{BMO}}$ is the following

$$\|Y\|_{\mathrm{BMO}} = \Big\|\sup_{n\geqq 1} \mathsf{E}(|Y-Y_{n-1}||\mathscr{F}_n)\Big\|_\infty .$$

2. The results

In [4], [5] and [6] the following assertion has been proved. We formulate it as

LEMMA 1. *Let (Φ, Ψ) be a pair of complementary Young functions and suppose that the power*

$$p = \sup_{x>0} \frac{x\varphi(x)}{\Phi(x)}$$

of the Young function Φ is finite. Here $\varphi(x)$ stands for the right-hand side derivative of Φ. Then for every $n\geqq 1$ we have

$$|\mathsf{E}(X_nY_n)| \leqq c_\Phi \|X_n\|_{\mathscr{H}_\Phi} \|Y_n\|_{\mathscr{K}_\Psi}$$

and the limit

$$\lim_{n\to+\infty} \mathsf{E}(X_nY_n)$$

exists and is finite. Moreover,

$$\left|\lim_{n\to+\infty} \mathsf{E}(X_n Y_n)\right| \leqq c_\Phi \|X\|_{\mathscr{H}_\Phi} \|Y\|_{\mathscr{K}_\Psi},$$

where c_Φ is a constant depending only on Φ.

Further, if $X\in\mathscr{H}_1$ and $Y\in BMO$ then

$$|\mathsf{E}(X_n Y_n)| \leqq c\|X_n\|_{\mathscr{H}_1}\|Y_n\|_{\mathrm{BMO}}.$$

Also, the limit

$$\lim_{n\to+\infty} \mathsf{E}(X_n Y_n)$$

exists and is finite and we have

$$\left|\lim_{n\to+\infty} \mathsf{E}(X_n Y_n)\right| \leqq c\|X\|_{\mathscr{H}_1}\|Y\|_{\mathrm{BMO}}.$$

Here $c>0$ is a constant.

In the papers [4], [5] and [6] the authors did not show that under the conditions of Lemma 1 the limit $\lim_{n\to+\infty} \mathsf{E}(X_n Y_n)$ is of the form of a Lebesgue integral of some random variable. Garsia in [3] has proved this fact in the case of $X\in\mathscr{H}_1$ and $Y\in$ BMO and also for $X\in\mathscr{H}_\Phi$ and $Y\in\mathscr{K}_\Psi$, where $\Phi(x)=\frac{x^p}{p}$ and $\Psi(x)=\frac{x^q}{q}$ with $p^{-1}+q^{-1}=1$ and $1<p\leqq 2$ (see, [3] Remark I.5.1).

Namely, he proved that in these cases $\sum_{i=1}^{\infty} |d_i d_i'|\in L_1$ and

$$\lim_{n\to+\infty} \mathsf{E}(X_n Y_n) = \mathsf{E}\Big(\sum_{i=1}^{\infty} d_i d_i'\Big).$$

Our aim is to show this result in the case of an arbitrary pair power (Φ, Ψ) of Young functions such that both Φ and Ψ have finite power. In this way we identify the limit $\lim_{n\to+\infty} \mathsf{E}(X_n Y_n)$ in all the possible cases and we show that it can be represented in the form of a Lebesgue integral.

First we prove the following

THEOREM 1. *Let (Φ, Ψ) be a pair of complementary Young functions both having finite power p and q, respectively. If $X\in\mathscr{H}_\Phi$ and $Y\in\mathscr{K}_\Psi$ then*

$$\mathsf{E}\Big(\sum_{i=1}^{\infty} |d_i d_i'|\Big) \leqq c_\Phi \|X\|_{\mathscr{H}_\Phi}\|Y\|_{\mathscr{K}_\Psi}$$

and

$$\lim_{n\to+\infty} \mathsf{E}(X_n Y_n) = \mathsf{E}\Big(\sum_{i=1}^{\infty} d_i d_i'\Big),$$

where c_Φ is a constant depending only on Φ.

PROOF. First we prove that for all $n\geqq 1$ we have

$$\mathsf{E}(X_n Y_n) = \sum_{i=1}^{n} \mathsf{E}(d_i d_i').$$

In fact, $X\in\mathscr{H}_\Phi$ implies that $d_i\in L^\Phi(\Omega,\mathscr{A},\mathsf{P})$ since

$$|X_i-X_{i-1}| = |d_i| \leqq S\in L^\Phi.$$

Also, from $Y\in\mathscr{K}_\Psi$ we deduce that

$$|Y_i-Y_{i-1}| = |d_i'| \leqq \mathsf{E}(\gamma|\mathscr{F}_i),$$

where $\gamma\in\Gamma_Y^{(\Psi)}$ is arbitrary. Consequently, $d_i'\in L^\Psi(\Omega,\mathscr{A},\mathsf{P})$. From these

$$\mathsf{E}(X_nY_n) = \sum_{i=1}^n \mathsf{E}(d_id_i') = \mathsf{E}(\sum_{i=1}^n d_id_i'),$$

since for $i\neq j$ we have $\mathsf{E}(d_id_j')=0$ given that by the Hölder inequality $\mathsf{E}(|d_id_j'|)\leqq\leqq 2\|d_i\|_\Phi\|d_j'\|_\Psi<+\infty$.

From this

$$|\mathsf{E}(X_nY_n)| \leqq \mathsf{E}(\sum_{i=1}^n |d_id_i'|) \leqq \mathsf{E}(S_nS_n') \leqq \mathsf{E}(SS'),$$

where $n=1,2,\dots$ is arbitrary and $S_n'^2=\sum_{i=1}^n d_i'^2$. By Hölder's inequality $\mathsf{E}(SS')\leqq\leqq 2\|S\|_\Phi\|S'\|_\Psi$.

Now $\|S'\|_\Psi$ is finite since $Y\in\mathscr{K}_\Psi$ and if both Φ and Ψ have finite power then Theorem 2 of [1] implies that $Y\in\mathscr{H}_\Psi$ and the norms $\|Y\|_{\mathscr{K}_\Psi}$ and $\|Y\|_{\mathscr{H}_\Psi}$ are equivalent. Consequently, $\|S'\|_\Psi=\|Y\|_{\mathscr{H}_\Psi}<+\infty$. Turning to the preceding inequality we get

$$|\mathsf{E}(X_nY_n)| \leqq \mathsf{E}(\sum_{i=1}^n |d_id_i'|) \leqq c_\Phi'\|X\|_{\mathscr{H}_\Phi}\|Y\|_{\mathscr{H}_\Psi}$$

for every n. Letting $n\to+\infty$ this implies that

$$\sum_{i=1}^\infty |d_id_i'|\in L_1,$$

i.e. the series

$$\sum_{i=1}^\infty d_id_i'$$

is absolutely integrable.

Since by Lemma 1 the limit

$$\lim_{n\to+\infty} \mathsf{E}(X_nY_n) = \lim_{n\to+\infty} \mathsf{E}(\sum_{i=1}^n d_id_i')$$

exists, which implies that

$$\lim_{n\to+\infty} \mathsf{E}(X_nY_n) = \mathsf{E}(\sum_{i=1}^\infty d_id_i').$$

This proves the assertion,

Especially, this assertion contains the case $\Phi(x)=\dfrac{x^p}{p}$, $\Psi(x)=\dfrac{x^q}{q}$,

$p^{-1}+q^{-1}=1$, where $p>1$ is arbitrary. It does not contain, however, the case $p=1$, $q=+\infty$ which corresponds to $X\in\mathscr{H}_1$ and $Y\in$BMO. Although the result corresponding to this case is well-known (see, [3] Remark I.5.1), for the sake of completeness we formulate it.

THEOREM 2. *Let $X\in\mathscr{H}_1$ and $Y\in BMO$. Then*

$$\left|\mathsf{E}\left(\sum_{i=1}^{\infty} d_i d_i'\right)\right| \leqq \mathsf{E}\left(\sum_{i=1}^{\infty} |d_i d_i'|\right) \leqq c\|X\|_{\mathscr{H}_1}\|Y\|_{\mathrm{BMO}},$$

and

$$\lim_{n\to+\infty} \mathsf{E}(X_n Y_n) = \mathsf{E}\left(\sum_{i=1}^{\infty} d_i d_i'\right).$$

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BIMEROTOPIES I

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Abstract

Asymmetric generalizations of such kinds of topological structures will be investigated whose systems of axioms do not contain an explicit "Axiom of Symmetry". The first steps in this direction were taken by Gantner and Steinlage [19].

In 1931, Wilson [63] and Niemytzki [49] brought the idea of asymmetry into topology: they dropped the Axiom of Symmetry from the definition of a metric. Wilson used the expression *quasi-metric*[1]. Nowadays there are asymmetric topological structures in abundance; to name only a few : quasi-uniformities, quasi-proximities, syntopogenous structures.

Concerning topologies deduced from different kinds of structures, there are two lines of investigation in the case of asymmetry: (i) one can keep (as far as possible) the construction used for symmetric structures, and assign to a structure a single topology; (ii) one can apply the same construction to the dual of the structure, too (assuming that dual structures do exist), and assign to a structure a *bitopology*, i.e. a pair of topologies. (It was already mentioned by Wilson that a quasi-metric induces two topologies.)

We are interested in the second problem; this means that a certain symmetry in the asymmetry has to be required, namely that the set of axioms for a kind of structure should be symmetrical, so that a dual structure satisfying the same axioms could be defined. To illustrate this vaguely described requirement, let us take the proximities as an example: δ is a *proximity* on the set X if it is a relation between subsets of X such that (with $\bar{\delta}$ meaning non-δ)

P1. $\emptyset \,\bar{\delta}\, X$;

P2. if $A \,\bar{\delta}\, B$ then $A \cap B = \emptyset$;

P3. if $A \,\bar{\delta}\, B$ and $B' \subset B$ then $A \,\bar{\delta}\, B'$;

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[1] "Quasi"- is the most usual prefix for denoting asymmetry, although the present usage cannot be traced back to [63].

P4. if $A \bar{\delta} B$ and $A \bar{\delta} C$ then $A \bar{\delta} (B \cup C)$;

PS. if $A \bar{\delta} B$ then $B \bar{\delta} A$;

P$\triangle$. if $A \bar{\delta} B$ then there is a C such that $A \bar{\delta} C$ and $(X \setminus C) \bar{\delta} B$.

Now if we drop Axiom PS (the Axiom of Symmetry) then δ^{-1} will only satisfy P2 and P$\triangle$, thus the duals of P1, P3 and P4 have to be added if δ^{-1} is expected to satisfy the same set of axioms:

P1^{-1}. $X \bar{\delta} \emptyset$;

P3^{-1}. if $A \bar{\delta} B$ and $A' \subset A$ then $A' \bar{\delta} B$;

P4^{-1}. if $A \bar{\delta} C$ and $B \bar{\delta} C$ then $(A \cup B) \bar{\delta} C$.

In fact, P1 to P4, P1^{-1} to P4^{-1} and P$\triangle$ make up the usual set of axioms for a *quasi-proximity*.

Consequently, several kinds of asymmetric structures are outside the scope of this paper, their sets of axioms being asymmetrical; see e.g. the proximities in the sense of Hušek [26, 27], Matolcsy's local syntopogenous structures [41], Leader's topological *d*-spaces [38]; see also [17, 21, 22, 50].

Let us consider now the covering uniformities (i.e. uniformities in the sense of Tukey [59]); the definition of an asymmetric version of this notion is problematic, since we have now no Axiom of Symmetry. The reader might raise here the objection that this is only a seeming problem, for covering uniformities are in a one-to-one correspondence with uniformities (defined in terms of entourages), and the asymmetric generalizations of uniformities, namely the quasi-uniformities, are well-known.

This scheme is, however, unworkable for *generalizations* of covering uniformities, as they cannot be completely described by generalizations of uniformities, see e.g. [11]. Gantner and Steinlage [19] solved the problem for covering uniformities: they defined a *covering quasi-uniformity* on X as a family of binary relations between subsets of X satisfying certain axioms. (Covering uniformities can be regarded as special covering quasi-uniformities by identifying a covering $\mathscr{C}$ with the relation $\{(C, C)\colon C \in \mathscr{C}\}$.) The aim of the present paper is to modify their method, and obtain asymmetric versions of certain generalizations of covering uniformities.[2]

The notion of a bimerotopy/biuniformity will be defined in § 2. § 3 deals with the question of symmetry; symmetric bimerotopies will turn out to be more general than merotopies. In § 4, we seek to characterize those bimerotopies/biuniformities that can be obtained from quasi-(semi)uniformities through some natural constructions. Bimerotopies defined on the same set will be compared in § 5. We introduce the notion of a binearness in § 6, then explain why a plausible definition of biuniformities is inadequate.

In the second part of this series, the definitions given in terms of bicoverings will be transcribed for micromeric relations, respectively near relations; then we shall introduce bimerotopic continuity.

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[2] Biperfect syntopogenous structures [3—6] also describe quasi-uniformities in terms of families of relations between subsets of X, but no generalization of them seems to be suitable for treating generalizations of covering uniformities.

List of symbols

Axioms

B1—B3 1.1
B4 1.1, 2.6
B5 1.1, 1.8
$B5_0$ 2.3
B$\triangle$ 1.1, 2.8
B$\triangle'$ 4.4
B$\triangle''$ 4.8
B$\triangle_1$, B$\triangle_2$ 4.11
B$\triangle_1'$ 4.18
B$\triangle_a$, B$\triangle_b$ 4.10
BN 6.2
BS 3.4
BS′, BS″ 3.9
C... 0.2
M1—M4, M$\triangle$, M$\triangle'$, MN 0.5
M$\triangle_1$, M$\triangle_2$ 0.12
P... Introduction
U... 0.4.

Operations

^{b} 1.3, 2.18, 2.28
^{b_0} 1.3
^{c} 2.18, 2.28
^{d} 1.8
^{f} 2.18, 2.28
^{m}, ^{m_0} 0.7
^{p} 0.2, 0.3
^{t} 0.4
^{u} 0.5, 1.3, 2.18, 2.28
^{1} 0.3
$^{-1}$ 0.3, 0.4, 2.5
c 0.3, 0.4, 0.5, 2.28
i 0.2, 6.3
cl, int, $\mathscr{P}$ 0.1
st 0.5, 1.1
$(\cap)$, $(\cup)$, $^-$ 0.1.

Relations

m, n, Δ 0.1
$<$ 0.1, 1.1, 1.8
$<^*$ 0.5, 2.11.

§ 0. Preliminaries

A. Notations, terminology, basic definitions

0.1 Let a set X be fixed. $\mathscr{P}(X)$ denotes the power set of X; $\Delta=\Delta_X$ is the diagonal of $X\times X$. For a set A, $R[A]=\{y: \exists x\in A,\ x\,R\,y\}$; for $x\in X$, $Rx=R[\{x\}]$ (where R is a binary relation on X). If p is a binary relation on $\mathscr{P}(X)$ then $\bar{p}=\mathscr{P}(X)\times\mathscr{P}(X)\setminus p$. If $*$ is a binary operation from $\mathscr{P}(X)$ into $\mathscr{P}(X)$, $\mathscr{A},\mathscr{B}\subset\mathscr{P}(X)$, $p,q\subset\mathscr{P}(X)\times\mathscr{P}(X)$ then

$$\mathscr{A}(*)\mathscr{B}=\{A*B: A\in\mathscr{A},\ B\in\mathscr{B}\},\quad p(*)q=\{(A*B, C*D): A\,p\,C,\ B\,q\,D\}.$$

(This notation will be used for $*=\cup$ or $\cap$.) $\mathrm{cl}_{\mathscr{T}}$ and $\mathrm{int}_{\mathscr{T}}$ denotes the closure, respectively the interior in the topology $\mathscr{T}$. For $\mathscr{A},\mathscr{B}\subset\mathscr{P}(X)$, $\mathscr{A}$ *refines* $\mathscr{B}$ ($\mathscr{A}$ is a refinement of $\mathscr{B}$, $\mathscr{A}<\mathscr{B}$) if for any $A\in\mathscr{A}$, there is a $B\in\mathscr{B}$ containing A. The relations $m, n\subset\mathscr{P}(X)\times\mathscr{P}(X)$ are defined as follows: $A\,m\,B$ means that $A\cap B\neq\emptyset$ or $A=B=X$; $A\,n\,B$ means that $A\neq\emptyset\neq B$ or $A=B=X$ (the conditions $X\,m\,X$ and $X\,n\,X$ are of course superfluous if $X\neq\emptyset$).

Now we are going to list the definitions of different kinds of structures; no one of them is new, some are known under other names, too. Selected references: [2—12, 14—16, 23—25, 28, 30—32, 36, 37, 43, 44, 46, 47, 51—53, 56, 58, 59, 62].

0.2 The function $\mathsf{c}\colon \mathscr{P}(X)\to\mathscr{P}(X)$ is a *closure* (on X) if

C1. $\mathsf{c}\emptyset=\emptyset$;

C2. $\mathsf{c}A\supset A \quad (A\subset X)$;

C3. $\mathsf{c}(A\cup B)=\mathsf{c}A\cup\mathsf{c}B \quad (A, B\subset X)$.

The function $\mathsf{i}\colon\mathscr{P}(X)\to\mathscr{P}(X)$ defined by $\mathsf{i}A=X\setminus\mathsf{c}(X\setminus A)$ $(A\subset X)$ is the *interior* associated with c. The interior associated with the closure c_α (α an arbitrary index) will always be denoted by i_α. F is a closed set in the topology c^p induced by the closure c iff $\mathsf{c}F=F$. (G is open in c^p iff $\mathsf{i}G=G$.)[3] A closure c is *topological* if

CT. $\mathsf{c}(\mathsf{c}A)=\mathsf{c}A \quad (A\subset X)$,

i.e. iff $\mathsf{c}=\mathrm{cl}_{\mathscr{T}}$ (equivalently, iff $\mathsf{i}=\mathrm{int}_{\mathscr{T}}$), where $\mathscr{T}=\mathsf{c}^p$. A *bitopology* is an ordered pair of topologies on the same set.

0.3 $\delta\subset\mathscr{P}(X)\times\mathscr{P}(X)$ is a *quasi-semiproximity* if it satisfies Axioms P1 to P4 and $\mathrm{P1}^{-1}$ to $\mathrm{P4}^{-1}$ (see in the introduction). The closure c_δ induced by δ is defined by $\mathsf{c}_\delta A=\{x\colon \{x\}\,\delta\, A\}$. δ^p denotes[4] the topology c_δ^p. If δ is a quasi-semiproximity then so is δ^{-1}. We shall write δ^{-p} for $\delta^{-1p}=(\delta^{-1})^p$ and δ^1 for δ; $\mathsf{c}_\delta^i=\mathsf{c}_{\delta^i}$, $\mathsf{i}_\delta^i=\mathsf{i}_{\delta^i}$ $(i=\pm 1)$; similar conventions will be used for other structures and other operations, too. (δ^{-p}, δ^p) is the bitopology induced by δ. A quasi-semiproximity δ is a *quasi-proximity* if it satisfies P$\triangle$. If δ is a quasi-proximity then (i) so is δ^{-1}; (ii) c_δ is topological. A quasi-(semi)proximity is a *(semi)proximity* if PS is fulfilled.

0.4 A *quasi-semiuniformity* $\mathscr{U}$ is a subset of $\mathscr{P}(X\times X)$ such that

U1. $\mathscr{U}\neq\emptyset$;

U2. if $U\in\mathscr{U}$ then $\Delta\subset\mathscr{U}$;

U3. if $U\in\mathscr{U}$ and $U\subset V\subset X\times X$ then $V\in\mathscr{U}$;

U4. if $U\in\mathscr{U}$ and $V\in\mathscr{U}$ then $U\cap V\in\mathscr{U}$.

As $\mathscr{U}$ is a filter on $X\times X$, one can speak about *(sub)bases* for $\mathscr{U}$, in the sense of a filter (sub)base. If $\mathscr{U}$ is a quasi-semiuniformity then so is $\mathscr{U}^{-1}=\{U^{-1}\colon U\in\mathscr{U}\}$. $\mathscr{U}$ induces a quasi-semiproximity $\mathscr{U}^t$ such that $A\,\mathscr{U}^t\,B$ iff for each $U\in\mathscr{U}$ there are $x\in A$ and $y\in B$ with $x\,U\,y$. We shall write $\mathsf{c}_{\mathscr{U}}^i$ for $\mathsf{c}_{\mathscr{U}^t}^i$ $(i=\pm 1)$. $x\in\mathsf{i}_{\mathscr{U}}A$ iff there is a $U\in\mathscr{U}$ with $Ux\subset A$. $\mathscr{U}^{tp}$ is the topology, $(\mathscr{U}^{-tp}, \mathscr{U}^{tp})$ the bitopology induced by $\mathscr{U}$. A quasi-semiuniformity $\mathscr{U}$ is a *quasi-uniformity* if

U$\triangle$. for any $U\in\mathscr{U}$ there is a $V\in\mathscr{U}$ with $V^2\subset U$.

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[3] Following [5], if a letter denoting an operation on structures is used as a superscript when applied to a structure then it will be printed as a superior letter even when it stands by itself: thus the operation defined here will be referred to as "the operation ^{p}", and p can denote something else without risking confusion. (Similarly, we shall later introduce an operation ^{m}, although a relation m has already been defined.)

[4] The notations c^p and δ^p are not in conflict, since it follows from C1 that a closure cannot be at the same time a quasi-semiproximity.

A quasi-(semi)uniformity $\mathscr{U}$ is a *(semi)uniformity* if

US. $U^{-1}\in\mathscr{U}$ whenever $U\in\mathscr{U}$,

i.e. iff $\mathscr{U}^{-1}=\mathscr{U}$. If $\mathscr{U}$ is a uniformity (quasi-uniformity, semiuniformity) then $\mathscr{U}^{t}$ is a proximity (quasi-proximity, semiproximity).

0.5 A *merotopy* $\mathfrak{M}$ is a subset of $\mathscr{P}(\mathscr{P}(X))$ such that

M1. $\mathfrak{M}\neq\emptyset$, $\emptyset\notin\mathfrak{M}$;

M2. each $\mathscr{C}\in\mathfrak{M}$ is a covering of X;

M3. if $\mathscr{C}\in\mathfrak{M}$ and $\mathscr{C}<\mathscr{D}$ then $\mathscr{D}\in\mathfrak{M}$;

M4. if $\mathscr{C},\mathscr{D}\in\mathfrak{M}$ then there is an $\mathscr{E}\in\mathfrak{M}$ with $\mathscr{E}<\mathscr{C}$, $\mathscr{E}<\mathscr{D}$.

(The second part of M1 is needed if $X=\emptyset$; the conclusion in M4 can be replaced by "$\mathscr{C}(\cap)\mathscr{D}\in\mathfrak{M}$".) $\mathfrak{B}\subset\mathfrak{M}$ is a *base* for $\mathfrak{M}$ if each element of $\mathfrak{M}$ is refined by some element of $\mathfrak{B}$. $\mathfrak{S}$ is a *subbase* for $\mathfrak{M}$ if the finite intersections of elements of $\mathfrak{M}$, in the sense $(\cap)$, form a base for $\mathfrak{M}$.[5] If $\mathfrak{M}$ is a merotopy then a base $\mathscr{B}$ for $\mathfrak{M}^u$ is defined by

$$\mathscr{B}=\{\mathscr{C}^u\colon \mathscr{C}\in\mathfrak{M}\},\quad \mathscr{C}^u=\bigcup\{C\times C\colon C\in\mathscr{C}\}.$$

$\mathfrak{M}^u$ is the semiuniformity, $\mathfrak{M}^{ut}$ the semiproximity, $c_{\mathfrak{M}^u}=c_{\mathfrak{M}}$ the closure and $\mathfrak{M}^{utp}$ the topology induced by $\mathfrak{M}$. For $A\subset X$ and $\mathscr{C}\subset\mathscr{P}(X)$, put $\mathrm{st}_{\mathscr{C}}A=\bigcup\{C\in\mathscr{C}\colon A\,m\,C\}$. $x\in i_{\mathfrak{M}}A$ iff there is a $\mathscr{C}\in\mathfrak{M}$ with $\mathrm{st}_{\mathscr{C}}\{x\}\subset A$. A merotopy $\mathfrak{M}$ is a *nearness* if

MN. $\mathscr{C}\in\mathfrak{M}$ implies $\{i_{\mathfrak{M}}C\colon C\in\mathscr{C}\}\in\mathfrak{M}$.

A merotopy $\mathfrak{M}$ is a *covering uniformity* if

M$\triangle$. for any $\mathscr{C}\in\mathfrak{M}$ there is a $\mathscr{D}\in\mathfrak{M}$ with $\mathscr{D}<^*\mathscr{C}$,

where $\mathscr{D}<^*\mathscr{C}$ ($\mathscr{D}$ is a *star-refinement* of $\mathscr{C}$) means that for each $D\in\mathscr{D}$ there is a $C\in\mathscr{C}$ with $\mathrm{st}_{\mathscr{D}}D\subset C$. M$\triangle$ is equivalent to

M$\triangle'$. if $\mathscr{C}\in\mathfrak{M}$ then there is a $\mathscr{D}\in\mathfrak{M}$ such that for each $x\in X$, $\mathrm{st}_{\mathscr{D}}\{x\}$ is contained by some $C\in\mathscr{C}$.

Each covering uniformity is a nearness. If $\mathfrak{M}$ is a covering uniformity then $\mathfrak{M}^u$ is a uniformity; the operation ^{u} establishes a one-to-one correspondence between covering uniformities and uniformities.

0.6 Remark. Several authors consider very general structures, where the axioms for quasi-semiproximities, quasi-semiuniformities, merotopies, or closure are further relaxed (this usually means dropping the "Union Axiom" ["Intersection Axiom"], i.e. P4+P4^{-1}, U4, M4, or the $\subset$ part of C3); see e.g. [8—11, 13, 20, 24, 33, 35, 39, 42, 48, 55, 60]. We shall not investigate which of our statements hold for such more general structures; the bimerotopies to be introduced will always be supposed to satisfy axioms corresponding to M1—M4.

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[5] The words base and subbase are often used in a different sense, see [31, 49].

B. The connexion between merotopies and semiuniformities

0.7 The usual way of giving the inverse of ^{u} restricted to the covering uniformities is through

$$\text{(0.8)} \qquad \{U^{m_0}\colon U\in\mathcal{U}\} \text{ is a base for } \mathcal{U}^m,$$

where $\mathcal{U}$ is a uniformity and

$$\text{(0.9)} \qquad U^{m_0}=\{Ux\colon x\in X\}.$$

Then $\mathcal{U}^{mu}=\mathcal{U}$ ($\mathcal{U}$ a uniformity) and $\mathfrak{M}^{um}=\mathfrak{M}$ ($\mathfrak{M}$ a covering uniformity). Unfortunately, neither of these equalities holds for semiuniformities and merotopies; to save at least one of them, (0.8) and (0.9) have to be modified:

$$\text{(0.10)} \qquad \{U^m\colon U\in\mathcal{U}\} \text{ is a base for } \mathcal{U}^m,$$

$$\text{(0.11)} \qquad U^m=\{A\colon A\times A\subset U\},$$

where $\mathcal{U}$ is a semiuniformity. The notation $\mathcal{U}^m$ is justified by the fact that (0.8) and (0.10) define the same covering uniformity if $\mathcal{U}$ is a uniformity. (This form of the definition is less usual, but still well-known, see e.g. [11].) For any entourage U, we have $U^{mu}=U$; if $\mathcal{U}$ is a semiuniformity then $\mathcal{U}^m$ is a merotopy and $\mathcal{U}^{mu}=\mathcal{U}$.

0.12 Császár [11] determined, based on an idea of Sandberg's [54], which merotopies are of the form $\mathcal{U}^m$ where $\mathcal{U}$ is a semiuniformity. His result was given in terms of micromeric systems; transcribed for coverings, we have: the merotopy $\mathfrak{M}$ can be induced by a semiuniformity iff

M$\triangle_1$. for any $\mathcal{C}\in\mathfrak{M}$ there is a $\mathcal{D}\in\mathfrak{M}$ such that if $S\subset X$ and for any pair of points in S there is a $D\in\mathcal{D}$ containing both points then there is a $C\in\mathcal{C}$ with $S\subset C$.

According to Sandberg [54], a merotopy $\mathfrak{M}$ is a covering uniformity iff it satisfies M$\triangle_1$ and

M$\triangle_2$. for any $\mathcal{C}\in\mathfrak{M}$ there is a $\mathcal{D}\in\mathfrak{M}$ such that if $A, B\in\mathcal{D}$ and $A\,m\,B$ then there is a $C\in\mathcal{C}$ with $A\cup B\subset C$.

(Sandberg's result was also written in terms of micromeric systems.) In other words, M$\triangle$=M$\triangle_1$+M$\triangle_2$. Consequently, a merotopy of the form $\mathcal{U}^m$ is a covering uniformity iff $\mathcal{U}$ is a uniformity iff $\mathcal{U}^m$ satisfies M$\triangle_2$.

§ 1. Covering quasi-uniformities

1.1 Let $\mathcal{U}$ be a quasi-uniformity on X. The coverings that can be refined by a covering of the form U^{m_0} (defined for $U\in\mathcal{U}$ according to (0.8)) are called quasi-uniform. Knowing the quasi-uniform coverings is not enough for reconstructing $\mathcal{U}$; not even the $\mathcal{U}$-quasi-uniform and the $\mathcal{U}^{-1}$-quasi-uniform coverings together contain sufficient information on $\mathcal{U}$. Therefore, following Gantner and Stenilage [19], coverings will be replaced by certain relations on X: $p\subset\mathcal{P}(X)\times\mathcal{P}(X)$ is a *bicover-*

ing (of X) ("conjugate pair of covers" in [19], "dual cover" in [1]) if for any $x \in X$, there is a pair $(A, B) \in p$ with $x \in A \cap B$. The bicovering p is *strong* if $p \subset m$. For bicoverings p and q, q *refines*[6] p (q is a refinement of p, $q < p$) if $A\, q\, B$ implies the existence of an $(A', B') \in p$ such that $A \subset A'$ and $B \subset B'$.

A subset $\mathfrak{M}$ of $\mathscr{P}(\mathscr{P}(X) \times \mathscr{P}(X))$ is a *covering quasi-uniformity* [19] if the following axioms are satisfied:

B1. $\mathfrak{M} \neq \emptyset$; $\emptyset \notin \mathfrak{M}$;

B2. the elements of $\mathfrak{M}$ are bicoverings;

B3. if $p \in \mathfrak{M}$ and $p < q$ then $q \in \mathfrak{M}$;

B4. if $p, q \in \mathfrak{M}$ then there is an $r \in \mathfrak{M}$ refining both p and q;

B5. if $p \in \mathfrak{M}$ then there is a $q \in \mathfrak{M}$ such that q is a strong bicovering and $q < p$;

B$\triangle$. if $p \in \mathfrak{M}$ then there is a $q \in \mathfrak{M}$ such that $A\, q\, B$ implies the existence of a $(C, D) \in p$ with $\mathrm{st}_q^{-1} A \subset C$ and $\mathrm{st}_q^{1} B \subset D$;

where

(1.2) $$\mathrm{st}_q^i S = \bigcup \{H \colon \exists G,\ S\, m\, G\, q^i\, H\} \quad (i = \pm 1).$$

$\mathfrak{B} \subset \mathfrak{M}$ is a *base* for $\mathfrak{M}$ if for any $p \in \mathfrak{M}$ there is a $q \in \mathfrak{B}$ refining p.

1.3 A one-to-one correspondence between quasi-uniformities and covering quasi-uniformities can be defined as follows [19]:

(1.4) $$\mathfrak{M}^u = \{p^u \colon p \in \mathfrak{M}\}$$

where $\mathfrak{M}$ is a covering quasi-uniformity and

(1.5) $$p^u = \bigcup \{A \times B \colon A\, p\, B\};$$

(1.6) $$\{U^{b_0} \colon U \in \mathscr{U}\} \text{ is a base for } \mathscr{U}^b$$

where $\mathscr{U}$ is a quasi-uniformity and

(1.7) $$U^{b_0} = \{(U^{-1}x, Ux) \colon x \in X\}.$$

Now $\mathfrak{M}^u$ is a quasi-uniformity, $\mathscr{U}^b$ is a covering quasi-uniformity, $\mathfrak{M}^{ub} = \mathfrak{M}$, $\mathscr{U}^{bu} = \mathscr{U}$. (1.6) and (1.7) are, similarly to (0.8) and (0.9), unsuitable for generalization, so they will have to be modified.

1.8 Let us first modify the definition of refinement. For a bicovering p, define a bicovering p^d by

(1.9) $$A\, p^d\, B \Leftrightarrow \exists (A', B') \in p, \quad A \times B \subset A' \times B'.$$

We shall say that q *refines* p (q is a refinement of p, $q < p$) if $q \subset p^d$; in other words: $q < p$ iff $q \cap n$ refines $p \cap n$ in the original definition. In particular, if $p \cap n = q \cap n$ then $p < q < p$.

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[6] This definition is only provisional, the final form will be given in 1.8.

One can easily verify, using Axiom B5, which is equivalent, with either definition of refinement, to

B5. if $p \in \mathfrak{M}$ then $p \cap m \in \mathfrak{M}$,

that this modification does not change the definition of a covering quasi-uniformity.

From now on, $q < p$ is always to be understood in the new definition.

§ 2. Bimerotopies

2.1 Definition. A set $\mathfrak{M} \subset \mathscr{P}(\mathscr{P}(X) \times \mathscr{P}(X))$ is a *bimerotopy* if it satisfies Axioms B1 to B4 (see in 1.1; do not forget 1.8). A bimerotopy $\mathfrak{M}$ is a *biuniformity* if it satisfies B$\triangle$. A bimerotopy $\mathfrak{M}$ is *strong* provided that B5 is fulfilled. (In particular, covering quasi-uniformities will be called strong biuniformitics.) A *base* for $\mathfrak{M}$ is defined just like in 1.1.

2.2 Lemma. *For a bimerotopy $\mathfrak{M}$, $p \in \mathfrak{M}$ iff $p \cap n \in \mathfrak{M}$.* □

2.3 Remark. The above lemma means that it would be enough to consider only bicoverings contained by n (just as it would be enough to consider only strong bicoverings in the definition of a covering quasi-uniformity); alternatively, if we had kept the original meaning of refinement then the following weaker version of B5 ought to have been added to the set of axioms for a bimerotopy:

B5_0. If $p \in \mathfrak{M}$ then there is a $q \in \mathfrak{M}$ such that $q \subset n$ and $q < p$

(in other words: if $p \in \mathfrak{M}$ then $p \cap n \in \mathfrak{M}$). Although our approach may seem to be the most inconvenient one of the three possibilities, it will later have some advantages, too.

2.4 Remark. The simplest (non-equivalent) way of defining a bimerotopy would be to use the original definition of refinement and take Axioms B1 to B4 (without adding B5_0); such a definition would have unwelcome consequences, which will be pointed out later.

2.5 Proposition. *If $\mathfrak{M}$ is a (strong) bimerotopy/biuniformity then so is $\mathfrak{M}^{-1} = \{p^{-1} : p \in \mathfrak{M}\}$.* □

2.6 Lemma. *In the definition of a bimerotopy,* B4 *can be replaced by the following condition: if $p, q \in \mathfrak{M}$ then $p (\cap) q \in \mathfrak{M}$.* □

2.7 Definition. $\mathfrak{S}$ is a *subbase* for $\mathfrak{M}$ if $\{p_1 (\cap) \ldots (\cap) p_k : k \geqq 2,\ p_i \in \mathfrak{S}\ (1 \leqq i \leqq k)\}$ is a base for $\mathfrak{M}$.

2.8 Lemma. *For a bimerotopy $\mathfrak{M}$,* B$\triangle$ *can be written in the following equivalent form:*

B$\triangle$. *if $p \in \mathfrak{M}$ then there is a $q \in \mathfrak{M}$ such that $A\, q\, B$ implies* $(\mathrm{st}_q^{-1} A)\, p^d\, (\mathrm{st}_q^1 B)$.

Proof. It is evident that the original axiom implies the new one; the converse follows from 2.2. □

2.9 Remark. As $p<p^d<p$ and $p^{dd}=p^d$ are evident from the definition of ^{d}, bimerotopies could be defined as families of bicoverings satisfying the additional condition $p=p^d$ (just as merotopies could be, but usually are not, regarded as systems of coverings $\mathscr{C}$ satisfying the condition that $B\subset C\in\mathscr{C}$ implies $B\in\mathscr{C}$). Such a modification of the definition would reduce $<$ to $\subset$ and simplify some axioms, but its drawbacks outweigh the advantages.

2.10 Lemma. *If $p=p^d$ and $q=q^d$ then $p(\cap)q=p\cap q$.* □

2.11 Definition. If q satisfies the condition in 2.8, we shall say that q is a *star-refinement* of p $(q<^*p)$.

2.12 Lemma. *Let p and q be bicoverings, $S, T\subset X$, $i=\pm 1$.*
a) $\mathrm{st}_p^i\, S\supset S$.
b) *If $T\subset S$ and $p<q$ then $\mathrm{st}_p^i\, T\subset \mathrm{st}_q^i\, S$.* □

2.13 Lemma. a) *$q<^*p$ implies $q<p$.*
b) *If $r<^*q<p$ or $r<q<^*p$ then $r<^*p$.* □

2.14 Remark. Observe that it is enough to assume B$\triangle$ for p taken from a base for $\mathfrak{M}$; in this case, q can also be required to belong to the base. The first part of this statement holds for subbases, too.

A bimerotopy is strong iff it has a base consisting of strong bicoverings; on the other hand, the next example shows that a bimerotopy with a subbase consisting of strong bicoverings is not necessarily strong.

2.15 Example. Let $X=\{1,2\}$, $p=\{(\{1\},X),(\{2\},\{2\})\}$, $q=\{(\{1\},\{1\}),(X,\{2\})\}$. Denote by $\mathfrak{M}$ the bimerotopy for which $\{p,q\}$ is a subbase. Now p and q are strong, but $\mathfrak{M}$ is not strong.

2.16 Lemma. *If $\mathscr{U}$ is a quasi-uniformity then $\{U^b\colon U\in\mathscr{U}\}$ is a base for the strong bimerotopy*[7] *$\mathscr{U}^b$ defined in* (1.7), *where*

(2.17) $$U^b=\{(A,B)\colon A\times B\subset U,\ A\, m\, B\}.$$

Proof. $U^b<U^{b_0}$, $U^{b_0}<(U^2)^b$. □

2.18 Definition. For an entourage U, let

(2.19) $$U^c=\{(A,B)\colon A\times B\subset U\},\quad U^b=U^c\cap m,$$

(i.e. U^b is defined by (2.17)), and

(2.20) $$U^f=\{(\{x\},\{y\})\colon x\,U\,y\}\cup\{(\emptyset,\emptyset)\}.$$

For a bicovering p, let p^u be defined by (1.5).

2.21 Lemma. *If U is an entourage then U^c, U^b and U^f are bicoverings. If p is a bicovering then p^u is an entourage. Furthermore, $U^{cu}=U^{bu}=U^{fu}=U$.*

[7] In fact, as we know from [19], $\mathscr{U}^b$ is a biuniformity, but, for the sake of completeness, we shall not use the results of [19]. (On the other hand, it is quite obvious that $\mathscr{U}^b$ is a strong bimerotopy.)

PROOF of $U^{bu}=U$. If $x\,U\,y$ then $\{x\}\,U^b\,\{x, y\}$, hence $x\,U^{bu}\,y$; conversely, if $x\,U^{bu}\,y$ then there are $A\ni x$ and $B\ni y$ with $A\,U^b\,B$, thus $A\times B\subset U$, and therefore $x\,U\,y$. □

2.22 LEMMA. a) *Let U and V be entourages and $^a={}^c$, ^{b} or ^{f}. Then $(U\cap V)^a=U^a\cap V^a$, $U^c(\cap)V^c=(U\cap V)^c$, $U^f(\cap)V^f<(U\cap V)^f$. If $U\subset V$ then $U^a\subset V^a$.*
b) *Let p and q be bicoverings. Then $(p\,(\cap)\,q)^u=p^u\cap q^u$. If $p<q$ then $p^u\subset q^u$.* □

2.23 EXAMPLE. If $A=\{1, 2\}$, $B=\{3, 4\}$, $X=A\cup B$, $U=\Delta\cup(A\times X)$, $V=\Delta\cup(X\times B)$, then $U^b(\cap)V^b \not< (U\cap V)^b$.

2.24 LEMMA. *Let p be a bicovering,*
a) *For $x, y\in X$, $\{x\}\,p^d\,\{y\}$ iff $x\,p^u\,y$.*
b) *For $x\in X$ and $i=\pm 1$, $\mathrm{st}^i_p\,\{x\}=(p^u)^i x$.* □

2.25 LEMMA. *If $\mathcal{U}$ is a quasi-semiuniformity then*

$$\{U^c\colon U\in\mathcal{U}\},\quad \{U^b\colon U\in\mathcal{U}\},\quad \{U^f\colon U\in\mathcal{U}\} \tag{2.26}$$

are bases for bimerotopies. If $\mathfrak{M}$ is a bimerotopy then

$$\{p^u\colon p\in\mathfrak{M}\} \tag{2.27}$$

is a quasi-semiuniformity. □

2.28 DEFINITION. If $\mathcal{U}$ is a quasi-semiuniformity then let $\mathcal{U}^c$, $\mathcal{U}^b$ and $\mathcal{U}^f$ denote the bimerotopies for which the systems in (2.26) are bases. If $\mathfrak{M}$ is a bimerotopy then denote the quasi-semiuniformity (2.27) by $\mathfrak{M}^u$. If $\mathcal{U}=\mathfrak{M}^u$ then we shall say that $\mathfrak{M}$ induces $\mathcal{U}$ ($\mathcal{U}$ is compatible with $\mathfrak{M}$). The quasi-semiproximity, the closure, the topology and the bitopology induced by $\mathfrak{M}^u$ will be said to be induced by $\mathfrak{M}$ (in other words, they are compatible with $\mathfrak{M}$). $c^i_{\mathfrak{M}^u}$ will also be written as $c^i_{\mathfrak{M}}$ $(i=\pm 1)$.

2.29 THEOREM. *If $\mathcal{U}$ is a quasi-semiuniformity then $\mathcal{U}^b$ is a strong bimerotopy. If $\mathcal{U}$ is a quasi-uniformity then $\mathcal{U}^c$ and $\mathcal{U}^b$ are biuniformities. If $\mathfrak{M}$ is a biuniformity then $\mathfrak{M}^u$ is a quasi-uniformity.*

PROOF. 1° The first statement is evident.
2° Suppose that $\mathcal{U}$ is a quasi-uniformity. Let $U\in\mathcal{U}$. Pick a $V\in\mathcal{U}$ with $V^3\subset U$. To prove that $\mathcal{U}^c$ is a biuniformity, it is enough, according to 2.14, to check that $V^c<^*U^c$. So assume $A\,V^c\,B$. If $x\in\mathrm{st}^{-1}_{V^c}A$ and $y\in\mathrm{st}^{1}_{V^c}B$ then there are A_1, A_2, B_2, B_1 such that $x\in A_1\,V^c\,A_2\,m\,A$ and $B\,m\,B_2\,V^c\,B_1\ni y$. Choosing $x'\in A_2\cap A$ and $y'\in B\cap B_2$, we have $x\,V\,x'\,V\,y'\,V\,y$, hence $x\,U\,y$; this implies $(\mathrm{st}^{-1}_{V^c}A)\times(\mathrm{st}^{1}_{V^c}B)\subset U$, i.e. $(\mathrm{st}^{-1}_{V^c}A)\;U^c\;(\mathrm{st}^{1}_{V^c}B)$, showing that $V^c<^*U^c$ holds indeed.
3° The proof for $\mathcal{U}^b$ is just the same; to conclude $(\mathrm{st}^{-1}_{V^b}A)\;U^b\;(\mathrm{st}^{1}_{V^b}B)$ at the end of the proof, observe that $A\,V^b\,B$ implies $A\,m\,B$, thus we have $(\mathrm{st}^{-1}_{V^b}A)\;m\;(\mathrm{st}^{1}_{V^b}B)$ by 2.12 a).
4° We skip here checking that $\mathfrak{M}^u$ is a quasi-uniformity, since a somewhat stronger statement will be proved in 4.12 b). □

2.30 THEOREM. *If $\mathcal{U}$ is a quasi-semiuniformity then $\mathcal{U}^{cu}=\mathcal{U}^{bu}=\mathcal{U}^{fu}=\mathcal{U}$.* □

2.31 EXAMPLE. Let $X=\{1, 2, 3, 4\}$ and let $\{U\}$ be a base for the quasi-uniformity $\mathscr{U}$ where $U=\Delta\cup(\{1, 2\}\times\{3, 4\})$. Then $\mathscr{U}^c$ and $\mathscr{U}^b$ are different biuniformities, while the bimerotopy $\mathscr{U}^f$ is not a biuniformity. (In order to see $\mathscr{U}^c\neq\mathscr{U}^b$, check that U^c does not refine U^b, since $\{1, 2\}\ U^c\ \{3, 4\}$, but $\{1, 2\}\ \overline{U^{bd}}\ \{3, 4\}$.)

2.32 REMARK. $\mathscr{U}^f$ is not a biuniformity even when $\mathscr{U}$ is the discrete uniformity on a two-point set. Nevertheless, this ill-behaved bimerotopy cannot be completely disregarded, since it has a certain extremal property among all the bimerotopies compatible with a given quasi-semiuniformity, cf. § 5.

2.33 REMARK. If $\mathscr{B}$ is a base for the quasi-semiuniformity $\mathscr{U}$ and $^a={}^c$, ^{b} or ^{f} then $\{U^a\colon\ U\in\mathscr{B}\}$ is a base for $\mathscr{U}^a$; this follows from the last statement in 2.22 a). Moreover, if $\mathscr{S}$ is a subbase for $\mathscr{U}$ and $^a={}^c$ or ^{f} then $\{U^a\colon\ U\in\mathscr{S}\}$ is a subbase for $\mathscr{U}^a$ [apply 2.22 a), using $U^f\cap V^f\subset U^f\,(\cap)\,V^f$]. The analogous statement for ^{b} is false: take X, U and V from 2.23, and let $\mathscr{S}=\{U, V\}$.

Conversely, 2.22 b) implies that if $\mathfrak{S}$ is a (sub)base for the bimerotopy $\mathfrak{M}$ then $\{p^u\colon u\in\mathfrak{S}\}$ is a (sub)base for $\mathfrak{M}^u$.

2.34 LEMMA. *If p is a bicovering, $S\subset X$ and $i=\pm 1$ then*

$$\mathrm{st}_p^i\, S = \bigcup_{x\in X} \mathrm{st}_p^i\,\{x\} = (p^u)^i[S] = p^{iu}[S].\quad\square$$

2.35 PROPOSITION. *Let $\mathfrak{M}$ be a bimerotopy, $A, B\subset X$, $x\in X$, $i=\pm 1$.*

a) *$A\ \mathfrak{M}^{ut}\ B$ iff $(\mathrm{st}_p^1 A)\ m\ B$ $(p\in\mathfrak{M})$; iff $A\ m\ (\mathrm{st}_p^{-1} B)$ $(p\in\mathfrak{M})$; iff for any $p\in\mathfrak{M}$, there are C and D with $A\ m\ C\ p\ D\ m\ B$.*

b) *$x\in \mathrm{i}_{\mathfrak{M}}^i A$ iff there is a $p\in\mathfrak{M}$ with $\mathrm{st}_p^i\{x\}\subset A$.*

c) *If $\mathfrak{M}$ is a biuniformity then $\{\mathrm{st}_p^i\,\{x\}\colon p\in\mathfrak{M}\}$ is the neighbourhood filter at x in the topology $\mathfrak{M}^{iut\sigma}$. If $\mathfrak{S}$ is a (sub)base for $\mathfrak{M}$ then $\{\mathrm{st}_p^i\,\{x\}\colon p\in\mathfrak{S}\}$ is a neighbourhood (sub)base.* $\square$

2.36 REMARK. It is possible to assign to each bicovering p a collection $\tilde{p}\subset\subset\mathscr{P}(X\times X)$ by $\tilde{p}=\{A\times B\colon\ A\,p\,B\}$. Taking the projections of the elements of $\tilde{p}$, the bicovering $p\cap n$ or $(p\cap n)\cup\{(\emptyset, \emptyset)\}$ can be reconstructed from $\tilde{p}$, which is enough to describe bimerotopies (cf. 2.3); therefore bimerotopies could also be axiomatized as subsets of $\mathscr{P}(\mathscr{P}(X\times X))$.

It will, however, turn out to be more important that the bimerotopies on X can be identified with certain merotopies on $X\times\{1, 2\}$. (See in Part II.)

§ 3. The question of symmetry

3.1 Let $\mathscr{C}$ be a covering of X. Then

$$p_{\mathscr{C}} = \{(C, C)\colon\ C\in\mathscr{C}\} \tag{3.2}$$

is a bicovering. If $\mathfrak{M}$ is a merotopy then

$$\{p_{\mathscr{C}}\colon\ \mathscr{C}\in\mathfrak{M}\} \tag{3.3}$$

is a base for a bimerotopy; $\mathscr{C}<\mathscr{D}$ iff $p_{\mathscr{C}}<p_{\mathscr{D}}$, therefore $\mathfrak{M}$ can be recovered from this bimerotopy by taking those coverings $\mathscr{C}$ for which $p_{\mathscr{C}}$ belongs to the bimerotopy.

Thus we shall identify the covering $\mathscr{C}$ with the bicovering $p_{\mathscr{C}}$ and the merotopy $\mathfrak{M}$ with the bimerotopy for which (3.3) is a base, i.e. merotopies will be regarded as special kinds of bimerotopies.

3.4 Lemma. a) *A bimerotopy $\mathfrak{M}$ is a merotopy iff*

BS. *for each $p\in\mathfrak{M}$ there is a $q\in\mathfrak{M}$ such that $q<p$, and $A=B$ whenever $A\,q\,B$.*

b) *Each merotopy is a strong bimerotopy.*
c) *The operations ^{u} defined in* 0.5 *and* 2.18 *coincide for merotopies.* □

3.5 Lemma. *A merotopy is a covering uniformity iff it is a biuniformity.*

Proof. Observe that $\mathscr{C}<^*\mathscr{D}$ iff $p_{\mathscr{C}}<^*p_{\mathscr{D}}$, then apply 2.14. □

3.6 Example. A semiuniformity $\mathscr{U}$ for which all the bimerotopies $\mathscr{U}^m$, $\mathscr{U}^c$, $\mathscr{U}^b$ and $\mathscr{U}^f$ are different. Let $X=\{1, 2, 3, 4\}$, and let $\{U\}$ be a base for $\mathscr{U}$ where

$$U = \Delta\cup(\{1, 2\}\times\{3, 4\})\cup(\{3, 4\}\times\{1, 2\}).$$

It is enough to check that if $p_1=U^c$, $p_2=U^b$, $p_3=U^m$, $p_4=U^f$ then p_i is not a refinement of p_j whenever $i<j$.

3.7 Lemma. *If $\mathscr{U}$ is a semiuniformity then $\mathscr{U}^b\subset\mathscr{U}^m$.* □

3.8 Proposition. *If $\mathscr{U}$ is a uniformity then $\mathscr{U}^m=\mathscr{U}^b=\mathscr{U}^c$.*

Proof. $\mathscr{U}^c\subset\mathscr{U}^b$ is obvious, so we have only to prove that $\mathscr{U}^m\subset\mathscr{U}^c$. For this purpose, it is enough to show that if $V^2\subset\mathscr{U}$ and $V\in\mathscr{U}$ is symmetrical then $V^c<U^m$.
If $A\,V^c\,B$ then $A\times B\subset V$ and, by the symmetry of V, $B\times A\subset V$, thus[8] $(A\cup B)\times(A\cup B)\subset U$, i.e. $(A\cup B)\,U^m\,(A\cup B)$. □

3.9 If one starts from the definition of a bimerotopy and—forgetting the merotopies—tries to introduce an Axiom of Symmetry then not BS but either of the following axioms seems to be the natural choice:

BS′. $\mathfrak{M}$ has a base consisting of symmetric bicoverings;

BS″. If $p\in\mathfrak{M}$ then $p^{-1}\in\mathfrak{M}$.

It is easy to check that BS′ and BS″ are equivalent (to prove BS″$\Rightarrow$BS′, apply 2.6).

3.10 Definition. We shall call a bimerotopy *symmetric* if it satisfies BS′.

3.11 Remark. Merotopies are symmetric, but a symmetric bimerotopy is not necessarily a merotopy (take $\mathscr{U}^c$, $\mathscr{U}^b$ or $\mathscr{U}^f$ from 3.6).

3.12 Proposition. *A biuniformity is symmetric iff it is a uniformity.*

Proof. Let $\mathfrak{M}$ be a symmetric biuniformity, and $p\in\mathfrak{M}$. Take $q<^*p$. By BS′, 2.13 b) and 2.2, we may assume that q is symmetric and $q\subset n$. Now if

[8] Here we use $A\,n\,B$, which can of course be assumed, e.g. by 2.2. In similar situations, $A\,n\,B$ will always be assumed implicitly.

$A\,q\,B$ then $A \subset \mathrm{st}_q^1 B$ and $B \subset \mathrm{st}_q^{-1} A$ follow from $B\,q\,A$, thus, by 2.12 a), $q < \{(A \cup B, A \cup B)\colon A\,q\,B\} <^* p$. □

3.13 Remark. 3.12 could also be deduced from 3.8, using 3.14, 2.29 and 5.14.

3.14 Proposition. *If $\mathfrak{M}$ is a symmetric bimerotopy then $\mathfrak{M}^u$ is a semiuniformity.* □

3.15 Example. Take X and p_2 from 3.6. Let $\{p\}$ be a base for the bimerotopy $\mathfrak{M}$, where $p = p_2 \cup \{(\{1, 2\}, \{3, 4\})\}$. Now $\mathfrak{M}^u$ is symmetric, although $\mathfrak{M}$ is not. (And $\mathfrak{M}$ is a biuniformity, too.)

§ 4. Around the Triangle Axiom

4.1 The main purpose of this section is to characterize those bimerotopies (biuniformities) which are of the form $\mathscr{U}^c$ or $\mathscr{U}^b$ for a suitable $\mathscr{U}$. As one can guess from 0.12, certain modifications of the Triangle Axiom B△ will play an important role in the solution of this problem. We shall also consider two other modifications of B△ that have nothing to do with the above mentioned question.

4.2 Definition. The bimerotopy $\mathfrak{M}$ is *coarse, basic,* respectively *fine* if there is a quasi-semiuniformity $\mathscr{U}$ such that $\mathfrak{M} = \mathscr{U}^c$, $\mathfrak{M} = \mathscr{U}^b$, respectively $\mathfrak{M} = \mathscr{U}^f$.

4.3 Remark. By 2.30, $\mathfrak{M}$ is coarse (basic, fine) iff $\mathfrak{M} = \mathfrak{M}^{uc}$ ($\mathfrak{M} = \mathfrak{M}^{ub}$, $\mathfrak{M} = \mathfrak{M}^{uf}$).

4.4 On the model of M△′, consider the following axiom:

B△′. if $p \in \mathfrak{M}$ then there is a $q \in \mathfrak{M}$ such that for any $x \in X$, $(\mathrm{st}_q^{-1}\{x\})\ p^d\ (\mathrm{st}_q^1\{x\})$.

If B△′ holds for p taken from a base then it holds for any $p \in \mathfrak{M}$. A bicovering q satisfying the condition in B△′ need not be a refinement of p. (Example: $p = U^b$, $q = U^c$ with U from 2.31.) All the same, it can always be assumed without changing B△′ that $q < p$. (Proof: replace q by $q(\cap)p$.) In contrast to the equivalence of M△ and M△′, we have here only:

4.5 Proposition. B△ *implies* B△′.

Proof. B2 and 2.12 b). □

4.6 Example. A bimerotopy satisfying B△′, but not B△. Let Z denote the set of the integers, $X = Z \times \{1, 2\}$. Let $\{r\}$ be a base for $\mathfrak{M}$ where

$$r = \{(A_{ki}, B_{ki})\colon k \in Z,\ i = 1, 2\} \cup \{(E, F)\},$$

$$A_{ki} = (\{h \in Z\colon h < k\} \times \{1, 2\}) \cup \{(k, i)\}, \quad B_{ki} = (\{h \in Z\colon h > k\} \times \{1, 2\}) \cup \{(k, i)\},$$

$$E = \{(1, 1), (1, 2)\}, \quad F = \{(2, 1), (2, 2)\}.$$

For $x = (k, i)$ we have $\mathrm{st}_r^{-1}\{x\} = A_{ki}$ and $\mathrm{st}_r^1\{x\} = B_{ki}$, thus B△′ is satisfied for $p = r$ with $q = r$. On the other hand, the pair (E, F) thwarts B△.

4.7 Lemma. B△ *and* B△′ *are equivalent for strong bimerotopies.*

PROOF. Assume B$\triangle'$, take $p\in\mathfrak{M}$, and choose q for p, then r for q according to B$\triangle'$. We may also assume that $r<q$. As $\mathfrak{M}$ is strong, $s=r\cap m\in\mathfrak{M}$. We claim that $s<^*p$.

Let $A\,s\,B$, and pick $x\in A\cap B$. It is enough to show that $\mathrm{st}_s^{-1}A\subset\mathrm{st}_q^{-1}\{x\}$ and $\mathrm{st}_s^1 B\subset\mathrm{st}_q^1\{x\}$; let us prove e.g. the second formula. If $y\in\mathrm{st}_s^1 B$ then there are C and D with $B\,m\,C\,s\,D\ni y$; choosing a point $z\in C\cap D$, we have $A\subset\mathrm{st}_s^{-1}\{z\}\subset\subset\mathrm{st}_r^{-1}\{z\}$ and $D\subset\mathrm{st}_s^1\{z\}\subset\mathrm{st}_r^1\{z\}$, thus $A\,q^d\,D$; hence $A\,q^d\{y\}$. Now $y\in\mathrm{st}_q^1\{x\}$ follows from $x\in A$. □

4.8 Since a pair of points in a bimerotopic space may well be considered a natural counterpart of a single point in a merotopic space, the following axiom is also an analogue of M$\triangle'$:

B$\triangle''$. if $p\in\mathfrak{M}$ then there is a $q\in\mathfrak{M}$ such that $\{x\}\,q^d\{y\}$ implies $(\mathrm{st}_q^{-1}\{x\})\,p^d\,(\mathrm{st}_q^1\{y\})$.

According to 2.24 a), it would be shorter here (and in other axioms to come) to write $x\,q^u\,y$ for $\{x\}\,q^d\{y\}$, but it seems to be incongruous to introduce entourages into an axiom concerning bicoverings.

4.9 PROPOSITION. B$\triangle'$ *and* B$\triangle''$ *are equivalent.*

PROOF. 1° B$\triangle''\Rightarrow$B$\triangle'$. Use B2 and apply B$\triangle''$ to $x=y$.

2° B$\triangle'\Rightarrow$B$\triangle''$. For $p\in\mathfrak{M}$ take q, and then for q take $r<q$ according to B$\triangle'$. We are going to show that B$\triangle''$ holds with r.

If $\{x\}\,r^d\{y\}$ then evidently $y\in\mathrm{st}_r^1\{x\}$, and (by the choice of r) $(\mathrm{st}_r^{-1}\{x\})\,q^d$ $q^d(\mathrm{st}_r^1\{x\})$; therefore $(\mathrm{st}_r^{-1}\{x\})\,q^d\{y\}$, i.e. $\mathrm{st}_r^{-1}\{x\}\subset\mathrm{st}_q^{-1}\{y\}$. From $r<q$ we have $\mathrm{st}_r^1\{y\}\subset\mathrm{st}_q^1\{y\}$, thus $(\mathrm{st}_q^{-1}\{y\})\,p^d\,(\mathrm{st}_q^1\{y\})$ implies $(\mathrm{st}_r^{-1}\{x\})\,p^d\,(\mathrm{st}_r^1\{y\})$. □

4.10 Before turning to the analogues of M$\triangle_1$ and M$\triangle_2$, let us decompose B$\triangle$ in a different way:

B$\triangle_a$. for each $p\in\mathfrak{M}$ there is a $q\in\mathfrak{M}$ such that $A\,q\,B$ implies $(\mathrm{st}_q^{-1}A)\,p^d\,B$;

B$\triangle_b$. for each $p\in\mathfrak{M}$ there is a $q\in\mathfrak{M}$ such that $A\,q\,B$ implies $A\,p^d\,(\mathrm{st}_q^1 B)$.

It can be easily verified (using 2.12) that B$\triangle$=B$\triangle_a$+B$\triangle_b$. One of these axioms alone is not enough for $\mathfrak{M}$ to be a biuniformity.

4.11 On the analogy of M$\triangle_1$ and M$\triangle_2$, let us introduce the following axioms:

B$\triangle_1$. for any $p\in\mathfrak{M}$ there is a $q\in\mathfrak{M}$ such that if $A, B\subset X$ and $\{x\}\,q^d\{y\}$ holds for each $x\in A$ and $y\in B$ then $A\,p^d\,B$;

B$\triangle_2$. for any $p\in\mathfrak{M}$ there is a $q\in\mathfrak{M}$ such that $A\,q\,B\,m\,C\,q\,D$ implies $A\,p^d\,D$.

(In B$\triangle_1$, we could write $A\times B\subset q^u$, cf. 4.8.) It is again enough to assume the axioms for p taken from a base. In B$\triangle_1$ q is, in B$\triangle_2$ q can be supposed to be, a refinement of p.

Unfortunately, B$\triangle_1$+B$\triangle_2$ is not equivalent to B$\triangle$: the biuniformity $\mathscr{U}^b$ in 2.31 does not satisfy B$\triangle_1$. (Nevertheless, these axioms will turn out to be of service in characterizing coarse or basic bimerotopies.) We have only the following implications, none of which can be reversed:

4.12 Lemma. a) *Any of the axioms* $B\triangle_a$, $B\triangle_b$ *and* $B\triangle'$ *implies* $B\triangle_2$.
b) *If* $\mathfrak{M}$ *satisfies* $B\triangle_2$ *then* $\mathfrak{M}^u$ *is a quasi-uniformity.*
c) *If* $\mathfrak{M}$ *satisfies* $B\triangle_1$ *and* $\mathfrak{M}^u$ *is a quasi-uniformity then* $\mathfrak{M}$ *is a biuniformity.*

Proof. a) $B\triangle_a \Rightarrow B\triangle_2$ and $B\triangle_b \Rightarrow B\triangle_2$: evident. $B\triangle' \Rightarrow B\triangle_2$: if $A\,q\,B\,m\,C\,q\,D$ and $x\in B\cap C$ then $A\subset \mathrm{st}_q^{-1}\{x\}$, $B\subset \mathrm{st}_q^1\{x\}$.

b) Choose q for $p\in\mathfrak{M}$ according to $B\triangle_2$. If $x\,q^u\,y\,q^u\,z$ then there are A, B, C, D such that $x\in A$, $y\in B\cap C$, $z\in D$ and $A\,q\,B$, $C\,q\,D$. As now $B\,m\,C$, we have $A\,p^d\,D$ from $B\triangle_2$, thus $x\,p^u\,z$. Hence $(q^u)^2\subset p^u$.

c) Let $p\in\mathfrak{M}$ and choose q according to $B\triangle_1$. Since $\mathfrak{M}^u$ is a quasi-uniformity, there is an $r\in\mathfrak{M}$ such that $(r^u)^2\subset q^u$. Assume $A\,r\,B$. By 2.34, $\mathrm{st}_r^1 B = r^u[B]$, thus for any $x\in A$ and $z\in \mathrm{st}_r^1 B$ we have $x\,r^u\,y\,r^u\,z$ with a suitable $y\in B$, therefore $x\,q^u\,z$, i.e. $\{x\}\,q^d\,\{z\}$. Now $A\,p^d\,(\mathrm{st}_r^1 B)$ follows from the choice of q; this proves $B\triangle_b$. □

4.13 Theorem. *The following conditions are equivalent for a coarse or basic bimerotopy* $\mathfrak{M}$:
a) $\mathfrak{M}$ *is a biuniformity;*
b) $\mathfrak{M}$ *satisfies* $B\triangle'$;
c) $\mathfrak{M}$ *satisfies* $B\triangle_2$;
d) $\mathfrak{M}^u$ *is a quasi-uniformity.* □

4.14 Lemma. a) *If* p *is a bicovering then* $p^{uf} < p < p^{uc}$.
b) *If* p *is a strong bicovering then* $p < p^{ub}$.

Proof. a) 1° If $A\,p^{uf}\,B$ then $|A|=|B|=1$ (or $A=B=\emptyset$), say $A=\{x\}$ and $B=\{y\}$, and $x\,p^u\,y$, i.e. $\{x\}\,p^d\,\{y\}$.
2° If $A\,p\,B$ then $A\times B\subset p^u$, hence $A\,p^{uc}\,B$.
b) The same argument. □

4.15 Theorem. *A bimerotopy* $\mathfrak{M}$ *is coarse iff it satisfies* $B\triangle_1$.

Proof. 1° Assume $\mathfrak{M}=\mathcal{U}^c$, $U\in\mathcal{U}$. If $\{x\}\,U^{cd}\,\{y\}$ for any $x\in A$ and $y\in B$ then $A\times B\subset U$, thus $A\,U^c\,B$; this means that $B\triangle_1$ holds for a base (with $q=p$).
2° Conversely, assume now that $B\triangle_1$ is satisfied; we have to prove that $\mathfrak{M}=\mathfrak{M}^{uc}$.
If $p\in\mathfrak{M}^{uc}$ then there is a $U\in\mathfrak{M}^u$ with $U^c<p$, i.e. there is a $q\in\mathfrak{M}$ with $q^{uc}<p$, thus $p\in\mathfrak{M}$ follows from $q<q^{uc}$ (4.14a)).
If $p\in\mathfrak{M}$ then choose a q according to $B\triangle_1$. Now $q^{uc}\in\mathfrak{M}^{uc}$, thus $p\in\mathfrak{M}^{uc}$ will follow from $q^{uc}<p$. To prove $q^{uc}<p$, let $A\,q^{uc}\,B$; this means $A\times B\subset q^u$, which is equivalent to the condition imposed on q in $B\triangle_1$; hence $A\,p^d\,B$. □

4.16 Theorem. *The following conditions are equivalent for a bimerotopy* $\mathfrak{M}$:
a) $\mathfrak{M}$ *is a coarse biuniformity;*
b) $\mathfrak{M}$ *is a biuniformity satisfying* $B\triangle_1$;
c) $\mathfrak{M}$ *satisfies* $B\triangle'$ *and* $B\triangle_1$;
d) $\mathfrak{M}$ *satisfies* $B\triangle_2$ *and* $B\triangle_1$. □

4.17 Remark. These results call the definition of a biuniformity in question: perhaps $B\triangle'$ or $B\triangle_2$ could be chosen instead of $B\triangle$; in fact 5.11 will provide a strong argument for $B\triangle'$. We shall try to justify our choice in § 6.

4.18 The following modification of $B\triangle_1$ will be needed for the characterization of basic bimerotopies:

$B\triangle_1'$ for any $p\in\mathfrak{M}$ there is a $q\in\mathfrak{M}$ such that if $A\,m\,B$ and $\{x\}\,q^d\,\{y\}$ holds for each $x\in A$ and $y\in B$ then $A\,p^d\,B$.

It is again enough to take p from a base; $q<p$ can be supposed. (It seems at first sight equally natural to write $\{x\}\,(q\cap m)^d\,\{y\}$ instead of, or besides, assuming $A\,m\,B$; such an axiom would be, however, too weak: it is satisfied by any fine bimerotopy.)

4.19 LEMMA. a) $B\triangle_1$ *implies* $B\triangle_1'$.
b) $B\triangle'=B\triangle_1'+B\triangle_2$.
c) *If* $B\triangle_1'$ *holds for* $\mathfrak{M}$ *and* $\mathfrak{M}^u$ *is a quasi-uniformity then* $B\triangle'$ *holds, too.*

PROOF. b) $B\triangle'\Rightarrow B\triangle_2$. 4.12 a).

$B\triangle'\Rightarrow B\triangle_1'$. A q chosen according to $B\triangle'$ will also be good in $B\triangle_1'$. Indeed, if $A\,m\,B$ and $\{x\}\,q^d\,\{y\}$ for each $x\in A$ and $y\in B$ then, with a point $z\in A\cap B$ we have $A\subset\mathrm{st}_q^{-1}\{z\}$ and $B\subset\mathrm{st}_q^1\{z\}$, thus $A\,p^d\,B$ follows from $B\triangle'$.

$B\triangle_1'+B\triangle_2\Rightarrow B\triangle'$. By 4.12 b), $\mathfrak{M}^u$ is a quasi-uniformity, thus this implication will be a consequence of c).

c) Let $p\in\mathfrak{M}$ and choose q according to $B\triangle_1'$. Take $r\in\mathfrak{M}$ with $(r^u)^3\subset q^u$; fix $z\in X$. Now $((r^u)^{-1}z)\times(r^u z)\subset q^u$, i.e. $\{x\}\,q^d\,\{y\}$ whenever $x\in(r^u)^{-1}z$ and $y\in r^u z$; furthermore, $((r^u)^{-1}z)\,m\,(r^u z)$, thus from $B\triangle_1'$ we have $((r^u)^{-1}z)\,p^d\,(r^u z)$. $B\triangle'$ follows now from 2.24 b). □

4.20 THEOREM. *A bimerotopy is basic iff it is strong and satisfies* $B\triangle_1'$.

PROOF. 1° If $\mathfrak{M}$ is basic then it is strong (2.29). For $p=U^b$, $B\triangle_1'$ is satisfied with $q=p$.

2° Assume conversely that $\mathfrak{M}$ is strong and satisfies $B\triangle_1'$; we have to show that $\mathfrak{M}=\mathfrak{M}^{ub}$.

If $p\in\mathfrak{M}^{ub}$ then there is a $U\in\mathfrak{M}^u$ with $U^b<p$, i.e. there is a $q\in\mathfrak{M}$ such that q is strong and $q^{ub}<p$; now 4.14 b) implies $q<p$, showing that $p\in\mathfrak{M}$.

$\mathfrak{M}\subset\mathfrak{M}^{ub}$ can be proved using a slight modification of the argument in the last paragraph of the proof of 4.15. □

4.21 REMARK. Observe that the strongness of $\mathfrak{M}$ is not used in the proof of $\mathfrak{M}\subset\subset\mathfrak{M}^{ub}$, while $B\triangle_1'$ is not needed when showing $\mathfrak{M}^{ub}\subset M$.

4.22 THEOREM. *The following conditions are equivalent for a bimerotopy* $\mathfrak{M}$:
a) $\mathfrak{M}$ *is a basic biuniformity;*
b) $\mathfrak{M}$ *is a strong biuniformity;*
c) $\mathfrak{M}$ *is strong and satisfies* $B\triangle'$;
d) $\mathfrak{M}$ *is strong and it satisfies* $B\triangle_2$ *and* $B\triangle_1'$. □

4.23 REMARK. The analogy between 4.16 and 4.22 becomes clear if we add the redundant condition $B\triangle_1'$ to 4.22 b) and c): $B\triangle_1$ of 4.16 is replaced by strong$+B\triangle_1'$ throughout 4.22. (Compare also 4.15 with 4.20.) None of the three conditions in 4.22 d) is superfluous: strongness is needed since each coarse biuniformity satisfies

B $\triangle_2$ and B $\triangle_1'$; 4.20 shows that B $\triangle_2$ cannot be dropped either; concerning B $\triangle_1'$, see the example below.

4.24 EXAMPLE. Let $|X|=3$, and $A\,p\,B$ iff $|A|=2=|B|$. Let $\{p\}$ be a base for $\mathfrak{M}$. Then $\mathfrak{M}$ is strong, it satisfies B $\triangle_2$, but it is not a biuniformity.

§ 5. Comparison of bimerotopies

5.1 DEFINITION. If $\mathfrak{M}$ and $\mathfrak{N}$ are bimerotopies and $\mathfrak{M}\subset\mathfrak{N}$, we shall say that $\mathfrak{N}$ is *finer than* $\mathfrak{M}$ (or $\mathfrak{M}$ is *coarser than* $\mathfrak{N}$).

5.2 PROPOSITION. *For a quasi-semiuniformity $\mathscr{U}$, $\mathscr{U}^c$ is the coarsest one, while $\mathscr{U}^f$ is the finest one among all the bimerotopies compatible with $\mathscr{U}$.*

PROOF. In other words, we have to show that for any bimerotopy $\mathfrak{M}$, $\mathfrak{M}^{uc}\subset\subset\mathfrak{M}\subset\mathfrak{M}^{uf}$.

1° If $p\in\mathfrak{M}^{uc}$ then there is a $q\in\mathfrak{M}$ with $q^{uc}<p$; now $q<q^{uc}$ (4.14 a)) implies $p\in\mathfrak{M}$.

2° Assume $p\in\mathfrak{M}$. Then $p^{uf}<p$ (4.14 a)) implies $p\in\mathfrak{M}^{uf}$. □

5.3 PROPOSITION. a) *If* $\mathfrak{M}\subset\mathfrak{N}$ *then* $\mathfrak{M}^u\subset\mathfrak{N}^u$.

b) *If* $\mathscr{U}\subset\mathscr{V}$ *then* $\mathscr{U}^c\subset\mathscr{V}^c$, $\mathscr{U}^b\subset\mathscr{V}^b$, $\mathscr{U}^f\subset\mathscr{V}^f$. □

5.4 THEOREM. *The bimerotopy $\mathfrak{M}$ is compatible with the quasi-semiuniformity $\mathscr{U}$ iff $\mathscr{U}^c\subset\mathfrak{M}\subset\mathscr{U}^f$.*

PROOF. 5.2 and 5.3 a). □

5.5 THEOREM. *The bimerotopy $\mathfrak{M}$ is basic iff it is the coarsest one among the strong bimerotopies compatible with $\mathfrak{M}^u$.*

PROOF. 1° Let $\mathfrak{M}$ be basic, i.e. $\mathfrak{M}=\mathfrak{M}^{ub}$. Then $\mathfrak{M}$ is strong by 4.20. If $\mathfrak{N}$ is another strong bimerotopy compatible with $\mathfrak{M}^u$ and $p\in\mathfrak{M}$ then there are a $q\in\mathfrak{M}$ such that $q^{ub}<p$ and an $r\in\mathfrak{N}$ with $r\subset m$, $r^u\subset q^u$. Hence $r^{ub}<q^{ub}<p$, thus $r<p$ by 4.14 b), i.e. $p\in\mathfrak{N}$.

2° The converse follows from 4.21. □

5.6 EXAMPLE. There is no finest one among the strong bimerotopies compatible with the indiscrete uniformity on $X=\{1,2\}$: take $p_1=\{(\{1\},X),(\{2\},X)\}$, $p_2=\{(X,\{1\}),(X,\{2\})\}$ and let $\{p_i\}$ be a base for $\mathfrak{M}_i$; if $\mathfrak{M}\supset\mathfrak{M}_1\cup\mathfrak{M}_2$ and $\mathfrak{M}$ is strong then $\{(\{1\},\{1\}),(\{2\},\{2\})\}=m\cap(p_1(\cap)p_2)\in\mathfrak{M}$, thus $\mathfrak{M}^u$ is the discrete uniformity, although $\mathfrak{M}_1^u=\mathfrak{M}_2^u$ is the indiscrete one.

5.7 REMARK. A modification of the above argument shows that if there is a finest one among the strong bimerotopies compatible with a quasi-semiuniformity $\mathscr{U}$ then $\mathscr{U}$ is necessarily discrete.

5.8 DEFINITION. A bimerotopy $\mathfrak{M}$ is *proper* if $\mathfrak{M}\subset\mathfrak{M}^{ub}$.

5.9 PROPOSITION. *A bimerotopy is basic iff it is strong and proper.* □

5.10 THEOREM. *A bimerotopy is proper iff it satisfies* $\mathrm{B}\triangle_1'$.

PROOF. 1° Let $\mathfrak{M}$ be proper, and $p\in\mathfrak{M}$. Then $p\in\mathfrak{M}^{ub}$, thus there is a $q\in\mathfrak{M}^{ub}$ with which $\mathrm{B}\triangle_1'$ is satisfied (4.20). We may assume that $q=r^{ub}$ where $r\in\mathfrak{M}$. Now $r^{uc}\in\mathfrak{M}^{uc}\subset\mathfrak{M}$, and $\mathrm{B}\triangle_1'$ is also satisfied with r^{uc} instead of r^{ub}. [Indeed: if ${}^a={}^b$ or ^{c} then $\{x\}\, r^{uad}\,\{y\}$ is equivalent to $x\, r^{uau}\, y$ (4.8), i.e. to $x\, r^u\, y$ (2.21).]

2° The converse follows from 4.21. □

5.11 THEOREM. *The bimerotopy* $\mathfrak{M}$ *satisfies* $\mathrm{B}\triangle'$ *iff* $\mathfrak{M}$ *is proper and* $\mathfrak{M}^u$ *is a quasi-uniformity.*

PROOF. 1° $\mathrm{B}\triangle'$ implies $\mathrm{B}\triangle_1'$ and $\mathrm{B}\triangle_2$ (4.19 b)), thus $\mathfrak{M}$ is proper by 5.10, and $\mathfrak{M}^u$ is a quasi-uniformity by 4.12 b).

2° If $\mathfrak{M}$ is proper and $\mathfrak{M}^u$ is a quasi-uniformity then $\mathfrak{M}$ satisfies $\mathrm{B}\triangle_1'$ by 5.10, thus $\mathrm{B}\triangle'$ follows from 4.19 c). □

5.12 REMARK. Neither $\mathrm{B}\triangle$ nor $\mathrm{B}\triangle_2$ could replace $\mathrm{B}\triangle'$ in 5.11. Examples: 4.6, respectively $\mathcal{U}^f$ for any non-discrete quasi-uniformity $\mathcal{U}$.

5.13 COROLLARY. *Biuniformities are proper.* □

5.14 REMARK. If $\mathcal{U}$ is a uniformity then there is exactly one biuniformity compatible with $\mathcal{U}$ (3.8, 5.13). This statement would not hold, had we defined bimerotopies as outlined in 2.4.

§ 6. Binearness

6.1 After introducing the notion of a binearness, we shall only prove one simple theorem, which will give a reason for our preferring $\mathrm{B}\triangle$ to $\mathrm{B}\triangle'$ in the definition of a biuniformity. In subsequent parts of this series, we shall concentrate on bimerotopies and biuniformities, and do not intend to deal with binearness spaces.

6.2 DEFINITION. The bimerotopy $\mathfrak{M}$ is a *binearness* if it satisfies the following axiom:

BN. if $p\in M$ then $\{(\mathrm{i}_{\mathfrak{M}}^{-1}A,\, \mathrm{i}_{\mathfrak{M}}^{1}B)\colon A\, p\, B\}\in\mathfrak{M}$.

6.3 NOTATION. The relation on the right-hand side of BN will be denoted by $\mathrm{i}_{\mathfrak{M}}\, p$.

6.4 PROPOSITION. *A merotopy is a binearness iff it is a nearness.* □

6.5 THEOREM. *Each biuniformity is a binearness.*

PROOF. Let $\mathfrak{M}$ be a biuniformity, and $p\in\mathfrak{M}$. We are going to show that if $q<^*p$ then $q<\mathrm{i}_{\mathfrak{M}}\, p$.

Assume $C\, q\, D$, and choose A and B such that $\mathrm{st}_q^{-1}C\subset A$, $\mathrm{st}_q^{1}D\subset B$, $A\, p\, B$. For any $x\in D$, $\mathrm{st}_q^1\{x\}\subset\mathrm{st}_q^1 D\subset B$, thus $x\in\mathrm{i}_{\mathfrak{M}}^1 B$ (2.35 b)), i.e. $D\subset\mathrm{i}_{\mathfrak{M}}^1 B$. Similarly, $C\subset\mathrm{i}_{\mathfrak{M}}^{-1}A$; hence $C\,(\mathrm{i}_{\mathfrak{M}}\, p)^d\, D$. □

6.6 This natural generalization of the theorem stating that each uniformity is a nearness would not be true if a biuniformity had been defined as a bimerotopy satisfying B△′. Example: if $\mathfrak{M}$ is from 4.6 then, with the notations used there, $i_{\mathfrak{M}} r = (r \setminus \{(E, F)\}) \cup \{(\emptyset, \emptyset)\} \notin \mathfrak{M}$.

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LOCALIZATION OF ASSOCIATED AND WEAKLY ASSOCIATED PRIME ELEMENTS AND SUPPORTS OF LATTICE MODULES OF FINITE LENGTH

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§ 0. Introduction

R. P. Dilworth [3] introduced the concept of a Noether lattice as an abstraction of the concept of the lattice of ideals of a Noetherian ring. He showed that many of the important theorems of classical ideal theory held in them. E. W. Johnson and J. A. Johnson [4, 5] introduced and studied Noetherian modules over multiplicative lattices, and hence many of Dilworth's ideas have been extended. Then D. D. Anderson [1] and H. M. Nakkar [7] studied multiplicative lattices and modules over them without chain condition. H. M. Nakkar and D. D. Anderson [11] introduced and studied the set $\mathrm{Ass}\,(M)$ of associated prime elements and the set $\underset{f}{\mathrm{Ass}}\,(M)$ of weakly associated prime elements for a module M over a multiplicative lattice L.

In this paper we study the localization of associated and weakly associated prime elements. We show (Theorem 2.2) that a prime element $[p]\in S^{-1}L$ belongs to $\underset{f}{\mathrm{Ass}}\,(S^{-1}M)$ if and only if $p\in\mathrm{Ass}\,(M)$ and $p\nleq s$ for any s of a multiplicatively closed set S. We prove (Theorem 2.5) that the element B of M maximal with respect to $S(B)=S(0)$ satisfies the conditions $\underset{f}{\mathrm{Ass}}\,(B)=\theta$ and $\underset{f}{\mathrm{Ass}}\,([0,B])=\underset{f}{\mathrm{Ass}}\,(M)-\theta$, where θ is the set of elements p of $\underset{f}{\mathrm{Ass}}\,(M)$ for which $s\nleq p$ for any $s\in S$. Section 3 is concerned with the introduction of the support of a module M, denoted $\mathrm{Supp}\,(M)$. We prove (Proposition 3.7) that if I_M is a finite join of weak principal elements, then $\underset{f}{\mathrm{Ass}}\,((0\colon I_M))\subseteq\underset{f}{\mathrm{Ass}}\,(M)\subseteq\mathrm{Supp}\,(M)=V((0\colon I_M))$ and that these sets have the same minimal elements. We show (Proposition 3.8) that a compact element $h\in L$ is M-nilpotent if and only if $h\leq p$ for every $p\in\mathrm{Supp}\,(M)$. In Section 4 we introduce relatively prime elements, and we prove (Theorem 4.5) that if $P_1, \dots, P_n$ are relatively prime in pairs in L and if M is modular, then the L-submodule $[(\bigwedge_{i=1}^{n} P_i)\,I_M,\ I_M]$ of M is isomorphic to the L-module $\prod_{i=1}^{n}[P_iI_M,\ I_M]$. In Section 5 we consider modules of finite length. We characterize these modules in terms of elements of $\mathrm{Ass}\,(M)$ and elements of $\mathrm{Supp}\,(M)$ (Theorem 5.4). We also show (Theorem 5.8) that if M is a Noetherian lattice module of finite length, then the element $0\in M$ has a unique primary decomposition $0=\bigwedge_{P\in\mathrm{Ass}(M)} Q(P)$, where $Q(P)$ is P-primary in M, and that the canonical mapping of M into $\prod_{P\in\mathrm{Ass}(M)}[Q(P), I_M]$ is bijective. In this case we have

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$\operatorname{Long}_L(M)=\sum_{P\in \operatorname{Ass}(M)} \operatorname{Long}_{L_P}(M_P)$. Finally we show that a Noether lattice L is of finite length if and only if L is isomorphic to $\prod_{i=1}^{m} [P_i^n, I]$, where n is an integer and $P_1, P_2, \ldots, P_m$ are maximal in L.

For this study we use K-lattices given by Nakkar in [7] for establishing the theory of localization with respect to multiplicatively closed sets.

§ 1. Preliminaries

Definition 1.1. A lattice module M over the multiplicative lattice L is called a CG-lattice (PG-lattice) if every element of M is a join of compact (principal) elements. Recall that M is a K-lattice ([7]), Definition 12) if it is a CG-lattice and for any compact element $h\in L$ and any compact element $H\in M$, the element hH is compact.

Definition 1.1. A nonempty subset S of L is called multiplicatively closed if it is closed under multiplication and every element of S is compact in L.

Definition 1.3. Let P be a prime element of L. Then we say P is associated (weakly associated) with the module M if there is a nonzero compact element $H\in M$ such that $P=(0\colon H)$ (P is minimal in $V((0\colon H))$). The set of prime elements associated (weakly associated) with M is denoted by $\operatorname{Ass}_L(M)(\operatorname{Ass}_{fL}(M))$ or simply by $\operatorname{Ass}(M)(\operatorname{Ass}_f(M))$.

In general we adopt the lattice terminology of [4], [5], [7] and [11] and the module (over a ring) terminology of [2] and [6].

Throughout this paper, L will denote a multiplicative lattice, M will denote a lattice module over L and S will denote a multiplicatively closed subset of L.

§ 2. Localization of associated and weakly associated prime elements

First we give the construction of the localization with respect to multiplicatively closed sets, established in [7] for K-lattices. Let M and L be K-lattices. For every element B of M we define $S(B)=\bigvee_{s\in S}(B\colon s)$. Clearly $B\sim D \Leftrightarrow S(B)=S(D)$ ($B, D\in M$) is an equivalence relation. Let $[B]$ be the equivalence class of B and let $S^{-1}(M)=\{[B]:B\in M\}$. The quotient lattice $S^{-1}L$ is a multiplicative lattice and the quotient module $S^{-1}M$ is an $S^{-1}L$-module. They are again K-lattices. If P is a prime element of L and $S_P=\{s\in L\colon s \text{ is compact and } s\not\leq P\}$, then we write $L_P(M_P)$ instead of $S_P^{-1}L(S_P^{-1}M)$. For more details the interested reader may refer to [7].

Proposition 2.1. *Let L and M be K-lattices and let B and D be elements of M such that $D\leq B$. The $S^{-1}L$-module $S^{-1}([D, B])$ is isomorphic to the submodule $[[D],[B]]$ of the $S^{-1}L$-module $S^{-1}M$.*

Proof. Let X be an element of $[D, B]$. Let $\bar{S}(X)=\bigvee_{s\in S}(X\colon s)$ $\left(S(X)=\bigvee_{s\in S}(X\colon s)\right)$

and let $[X]^*$ ($[X]$) denote the equivalence class of X regarded as an element of $[D, B]$ (of M), i.e., $[X]^* \in S^{-1}([D, B])$ and $[X] \in S^{-1}M$. Then we have $\bar{S}(X) = S(X) \wedge B$. Now let X be an element of M such that $S(X) \leqq S(B)$, then by Lemma 2.2 in [7] we have $\bar{S}(X) = S(S(X)) \wedge S(B) = S(S(X) \wedge B)$.

Define $\theta\colon S^{-1}([D, B]) \to [[D], [B]]$ by $\theta([X]^*) = [X]$. Then θ will clearly be an $S^{-1}L$-module homomorphism provided that it is well-defined. Suppose then that $[X]^*$ and $[Y]^*$ are any elements such that $[X]^* = [Y]^*$, then $\bar{S}(X) = \bar{S}(Y)$. This implies that $S(X) = S(Y)$ and hence $[X] = [Y]$. Clearly θ is injective. Now let $[X]$ be an arbitrary element of $[[D], [B]]$. Then by Lemma 2.2 in [7] we have $S(D) \leqq S(X) \leqq S(B)$ and hence $S(X) \wedge B \in [D, B]$. It follows that $\theta([S(X) \wedge B]^*) = [S(X) \wedge B] = [X]$.

THEOREM 2.2. *Let L and M be K-lattices, and let θ be the set of all prime elements p of L for which $s \not\leqq p$ for any $s \in S$. Then the mapping $p \mapsto [p]$ is a bijection of* $\operatorname{Ass}_f(M) \cap \theta$ *onto* $\operatorname{Ass}_f(S^{-1}M)$.

PROOF. The mapping $p \mapsto [p]$ is a one-to-one correspondence between the elements of θ and the prime elements of $S^{-1}L$ (Theorem 2.3 in [7]). Let p be an element of $\operatorname{Ass}_f(M) \cap \theta$, then there is a compact element $H \neq 0$ in M such that p is minimal in $V((0\colon H))$. By Lemma 9.2 in [7] we have $[(0\colon H)] = ([0]\colon [H])$. By Theorem 2.3 in [7] we get that $[p]$ is minimal in $V(([0]\colon [H]))$, and hence $[p] \in \operatorname{Ass}_f(S^{-1}M)$. Now let $[p] \in \operatorname{Ass}_f(S^{-1}M)$, then there exists a compact element $[H]$ of $S^{-1}M$ such that $[p]$ is minimal in $V(([0]\colon [H]))$. We can assume that H is compact in M and p is prime in L (Lemma 11.2 and Theorem 2.3 in [7]). Since $V(([0]\colon [H])) = V([(0\colon H)])$ then p is minimal in $V((0\colon H))$ and hence $p \in \operatorname{Ass}_f(M) \cap \theta$.

PROPOSITION 2.3. *Let L and M be K-lattices and let θ be the set of all prime elements $P \in L$ for which $s \not\leqq P$ for any $s \in S$. Then*

(i) *The mapping $P \mapsto [P]$ is a bijection of* $\operatorname{Ass}(M) \cap \theta$ *onto a subset of* $\operatorname{Ass}(S^{-1}M)$.

(ii) *If $P \in \theta$ is compact in L and $[P] \in \operatorname{Ass}(S^{-1}M)$, then $P \in \operatorname{Ass}(M)$.*

PROOF. If $P = (0\colon H)$, where H is a compact element of M, then we have $[P] = ([0]\colon [H])$ in $S^{-1}L$. Therefore (i) follows immediately from Theorem 2.2. Now let $[P] \in \operatorname{Ass}(S^{-1}M)$. Then there is a compact element H of M such that $[P] = ([0]\colon [H])$, therefore $[P][H] = [0]$ and hence $PH \leqq S(0)$. Since PH is compact in M, there exists an element $s \in S$ such that $sPH = 0$ (Lemma 1.2 in [7]). This implies that $P \leqq (0\colon sH)$. Since $P \in \theta$ we get $P = (0\colon sH)$ and hence $P \in \operatorname{Ass}(M)$.

COROLLARY 2.4. *Let L be a K-lattice, let M be a Noetherian L-module and let θ be the set of all prime elements p of L for which $s \not\leqq p$ for every $s \in S$. Then the mapping $p \mapsto [p]$ is a bijection of* $\operatorname{Ass}(M) \cap \theta$ *onto* $\operatorname{Ass}(S^{-1}M)$.

THEOREM 2.5. *Let L and M be K-lattices, let θ be the set of elements p of* $\operatorname{Ass}_f(M)$ *for which $s \not\leqq p$ for every $s \in S$, and let ψ be the set of all elements D of M such that*

$S(D)=S(0)$. *Then ψ has a maximum element and this maximum element B satisfies the following conditions:*

$$\underset{f}{\mathrm{Ass}}(B)=\theta \quad \textit{and} \quad \underset{f}{\mathrm{Ass}}([0,B])=\underset{f}{\mathrm{Ass}}(M)-\theta.$$

Moreover, if the greatest element I of L is compact, then B is the unique element of M which satisfies these conditions.

Proof. By Lemma 2.2 and Lemma 3.2 in [7], we have $S(\bigvee_{D\in\psi} D)=S(\bigvee_{D\in\psi} S(D))= =S(S(0))=S(0)$. This implies the existence of the maximum element B of ψ. Since $S(B)\in\psi$, we have $S(B)=B$. Now let $p\in L$ be an element of $\underset{f}{\mathrm{Ass}}([0,B])$. Then there exists a compact element H in $[0,B]$ such that p is minimal in $V((0:H))$. Since M is a CG-lattice, the element H is compact in M, thus $p\in\underset{f}{\mathrm{Ass}}(M)$. On the other hand, by Lemma 9.2 in [7], we get $S(p)\geqq S((0:H))=(S(0):S(H))= =(S(0):S(0))=I$. Since $p\neq I$ there exists an element $s\in S$ such that $s\leqq p$. This implies that $\underset{f}{\mathrm{Ass}}([0,B])\subseteqq\underset{f}{\mathrm{Ass}}(M)-\theta$. Let $p\in\underset{f}{\mathrm{Ass}}(B)$ and let H be a compact element of M such that $H\nleqq B$ and p is minimal in $V((B:H))$. If s is an element of S such that $s\leqq p$, then by Lemma 1.2 in [8] there exists an integer $n\geqq 1$ and a compact element $t\in L$ such that $t\nleqq p$ and $ts^n\leqq(B:H)$. It follows that $tH\leqq S(B)=B$, and hence $t\leqq(B:H)\leqq p$, a contradiction. Since $(B:H)=S((B:H))=(S(B):S(H))= =(S(0):S(H))=S((0:H))$, then p is a minimal element in $V((0:H))$ and hence $p\in\underset{f}{\mathrm{Ass}}(M)$. This implies that $\underset{f}{\mathrm{Ass}}(B)\subseteqq\theta$. By Proposition 2.5 in [11] we get that $\underset{f}{\mathrm{Ass}}(B)=\theta$ and $\underset{f}{\mathrm{Ass}}([0,B])=\underset{f}{\mathrm{Ass}}(M)-\theta$.

Finally, let D be an element of M such that $\underset{f}{\mathrm{Ass}}(D)=\theta$ and $\underset{f}{\mathrm{Ass}}([0,D])= =\underset{f}{\mathrm{Ass}}(M)-\theta$. By Theorem 2.2 and Proposition 2.1 we get $\underset{f}{\mathrm{Ass}}(S^{-1}([0,D]))= =\underset{f}{\mathrm{Ass}}([[0],[D]])=\phi$. By Proposition 2.10 in [11] we have $S(0)=S(D)$. Therefore $D\leqq B$. If $D\nleqq B$, then $[D,B]\neq\{0\}$ and hence $\underset{f}{\mathrm{Ass}}([D,B])\neq\phi$. Since $\underset{f}{\mathrm{Ass}}([D,B])\subseteqq \subseteqq\underset{f}{\mathrm{Ass}}([D,I])=\underset{f}{\mathrm{Ass}}(D)=\theta$, then $\phi\neq\underset{f}{\mathrm{Ass}}(S^{-1}([D,B]))=\underset{f}{\mathrm{Ass}}([[D],[B]])$ and hence $[D]\neq[B]$. This means that $S(D)\neq S(0)$, a contradiction.

Corollary 2.6. *Let L and M be K-lattices and let θ be an arbitrary finite subset of $\underset{f}{\mathrm{Ass}}(M)$ such that every element of $\underset{f}{\mathrm{Ass}}(M)-\theta$ is not contained in an element of θ. Then there exists an element B of M such that:*

$$\underset{f}{\mathrm{Ass}}(B)=\theta \quad \textit{and} \quad \underset{f}{\mathrm{Ass}}([0,B])=\underset{f}{\mathrm{Ass}}(M)-\theta.$$

Proof. This follows immediately if we take $S=\{h\in L: h$ is a compact element such that $h\nleqq p$ for any $p\in\theta\}$.

Corollary 2.7. *Let L and M be K-lattices and let P be an element of $\underset{f}{\mathrm{Ass}}(M)$. If P is minimal in $\underset{f}{\mathrm{Ass}}(M)$, then there exists an element Q of M such that $\underset{f}{\mathrm{Ass}}(Q)= =\{P\}$ and $\underset{f}{\mathrm{Ass}}([0,Q])=\underset{f}{\mathrm{Ass}}(M)-\{P\}$. Moreover, if the greatest element I is compact, then Q is P-primary in M.*

PROOF. This follows from Proposition 4.2 in [11].

PROPOSITION 2.8. *Let L be a K-lattice in which the greatest element I is compact, let M be a K-lattice and let B be an element of M such that every element of $\underset{f}{\operatorname{Ass}}(B)$ is minimal in $\underset{f}{\operatorname{Ass}}(B)$. Then the following conditions are equivalent:*

(i) *The set $\underset{f}{\operatorname{Ass}}(B)$ is finite.*

(ii) *The element B has a primary decomposition in M.*

PROOF. (i)$\Rightarrow$(ii). By Corollary 2.7 there exists for each $p \in \underset{f}{\operatorname{Ass}}(B)$ an element $Q(p)$ of $[B, I_M]$ such that $\underset{f}{\operatorname{Ass}}(Q(p)) = \{p\}$ and $\underset{f}{\operatorname{Ass}}([B, Q(p)]) = \underset{f}{\operatorname{Ass}}(B) - \{p\}$. Let us write $D = \bigwedge_{p \in \underset{f}{\operatorname{Ass}}(B)} Q(p)$. We shall show that $D = B$. By Proposition 2.5 in [11] we have for each $p \in \underset{f}{\operatorname{Ass}}(B)$:

$$\underset{f}{\operatorname{Ass}}([B, D]) \subseteq \underset{f}{\operatorname{Ass}}([B, Q(p)]) \subseteq \underset{f}{\operatorname{Ass}}(B) - \{p\}.$$

It follows that $\underset{f}{\operatorname{Ass}}([B, D]) = \phi$ and hence by Proposition 2.10 in [11] we get $D = B$. (ii)$\Rightarrow$(i). This follows from Proposition 2.6 and Proposition 4.2 in [11].

§ 3. The support of a module

DEFINITION 3.1. Let M be a K-lattice. The set of prime elements p of L such that $M_p \neq \{0\}$ is called the support of M and is denoted by $\operatorname{Supp}(M)$.

PROPOSITION 3.1. *Let M be a K-lattice.*

(i) *If B is an element of M, then* $\operatorname{Supp}(M) = \operatorname{Supp}([0, B]) \cup \operatorname{Supp}([B, I_M])$.

(ii) *If $I_M = \bigvee_{\alpha \in J} B_\alpha$ for a family $\{B_\alpha\}_{\alpha \in J}$ of elements of M, then* $\operatorname{Supp}(M) = \bigcup_{\alpha \in J} \operatorname{Supp}([0, B_\alpha])$.

PROOF. (i) Let p be a prime element of L. Clearly, $S_p(0) = S_p(I_M)$ if and only if $S_p(0) = S_p(B)$ and $S_p(B) = S_p(I_M)$. This means that $M_p = 0$ if and only if $[[0], [B]] = \{0\}$ and $[[B], [I_M]] = \{0\}$, and hence if and only if $([0, B])_p = \{0\}$ and $([B, I_M])_p = \{0\}$. (ii) Let $p \in \operatorname{Supp}(M)$. Since $[I_M] = \bigvee_{\alpha \in J} [B_\alpha]$ in M_p and $[I_M] \neq [0]$, then there exists an element B_α such that $[B_\alpha] \neq [0]$ in M_p. It follows that $p \in \operatorname{Supp}([0, B_\alpha])$. By (i) we get $\operatorname{Supp}(M) \supseteq \bigcup_{\alpha \in J} \operatorname{Supp}([0, B_\alpha])$.

PROPOSITION 3.2. *Let L and M be K-lattices and let p be a prime element of L. Then $p \in \operatorname{Supp}(M)$ if and only if $p \geqq q$ for some $q \in \underset{f}{\operatorname{Ass}}(M)$.*

PROOF. Let $p \in \operatorname{Supp}(M)$. Then there is a compact element H of M such that $[H] \neq [0]$ in M_p. Since $[I] \neq ([0] : [H]) = [(0 : H)]$ (Lemma 9.2 in [7]) we get $p \geqq (0 : H)$. By Zorn's Lemma there exists an element q of $\underset{f}{\operatorname{Ass}}(M)$ such that $q \leqq p$. Now let

$q \in \underset{f}{\mathrm{Ass}}(M)$ and let p be a prime element of L such that $p \geqq q$. Since $q \not\leqq s$ for any $s \in S_p$ then by Theorem 2.2 we get $[q] \in \underset{f}{\mathrm{Ass}}(M_p)$. Therefore $M_p \neq \{0\}$ and hence $p \in \mathrm{Supp}(M)$.

COROLLARY 3.3. *Let L and M be K-lattices and let the greatest element I of L be compact, then $M = \{0\}$ if and only if* $\mathrm{Supp}(M) = \phi$.

PROOF. This follows from Proposition 2.10 in [11].

For any element b of L, $V(b)$ will denote the set of all prime elements of L containing b.

LEMMA 3.4. *Let L and M be K-lattices. Let b be an element of L and let A be a weak principal element of M. Then*

(i) $\mathrm{Supp}([b, I]) = V(b)$, *and*

(ii) $\mathrm{Supp}([0, A]) = V((0\colon A))$.

PROOF. (i) Let p be a prime element of L. We note that $p \in V(b)$ if and only if $[b] \neq [I]$ in L_p and hence if and only if $([b, I])_p \neq \{0\}$. (ii) Since the L-submodule $[0, A]$ of M is isomorphic to the L-submodule $[(0\colon A), I]$ of L we have $\mathrm{Supp}([0, A]) = \mathrm{Supp}([0\colon A), I]) = V((0\colon A))$.

COROLLARY 3.5. *Let L and M be K-lattices. Let $\{A_\alpha\}_{\alpha \in J}$ be a collection of weak principal elements of M such that $I_M = \bigvee_{\alpha \in J} A_\alpha$. Then* $\mathrm{Supp}(M) = \bigcup_{\alpha \in J} V((0\colon A_\alpha))$.

PROOF. By Proposition 3.1 we get

$$\mathrm{Supp}(M) = \bigcup_{\alpha \in J} \mathrm{Supp}([0, A_\alpha]) = \bigcup_{\alpha \in J} V((0\colon A_\alpha)).$$

LEMMA 3.6. *Let M be a CG-lattice. For any weak principal element A of M and any compact element h of L, the element hA is compact in M.*

PROOF. This follows from Theorem 2.1 in [10].

PROPOSITION 3.7. *Let L and M be K-lattices and let $I_M = A_1 \vee \ldots \vee A_n$, where $A_1, \ldots, A_n$ are weak principal in M. Then* $\underset{f}{\mathrm{Ass}}((0\colon I_M)) \subseteq \underset{f}{\mathrm{Ass}}(M) \subseteq \mathrm{Supp}(M) = V((0\colon I_M))$. *Moreover, these sets have the same minimal elements.*

PROOF. Clearly $V((0\colon I_M)) = V((0\colon \bigvee_{i=1}^{n} A_i)) = V(\bigwedge_{i=1}^{n} (0\colon A_i)) = \bigcup_{i=1}^{n} V((0\colon A_i)) = \mathrm{Supp}(M)$. Now let $p \in \underset{f}{\mathrm{Ass}}((0\colon I_M))$ and let h be a compact element of L such that p is minimal in $V(((0\colon I_M)\colon h))$. It follows that $p \geqq \bigwedge_{i=1}^{n} (0\colon hA_i)$. By Lemma 3.6 hA_i $(i = 1, \ldots, n)$ is compact. Therefore $p \in \underset{f}{\mathrm{Ass}}(M)$. Finally, let p be a minimal element of $V((0\colon I_M))$. If Q is a compact element of L such that $Q \not\leqq p$, then $p \geqq ((0\colon I_M)\colon Q)$ and hence $p \in \underset{f}{\mathrm{Ass}}((0\colon I_M))$.

DEFINITION 3.2. Let b be an element of L. b is called M-nilpotent if there exists an integer $n \geqq 1$ such that $b^n I_M = 0$.

PROPOSITION 3.8. *Let L and M be K-lattices. Let I_M be a finite join of weak principal elements of M and let h be a compact element of L. Then h is M-nilpotent if and only if $h \leqq p$ for every $p \in \operatorname{Supp}(M)$.*

PROOF. Since $h^n \leqq (0: I_M)$ and $V((0:I_M)) = \operatorname{Supp}(M)$, then $h \leqq p$ for every $p \in \operatorname{Supp}(M)$. Now suppose $h^n \not\leqq (0:I_M)$ for all integers $n \geqq 1$. Then by Lemma 3.1 in [11] there exists a prime element p of L such that $p \geqq (0: I_M)$ and $p \not\geqq h$. Therefore $p \in \operatorname{Supp}(M)$ and $p \not\geqq h$.

§ 4. Relatively prime elements

DEFINITION 4.1. Let a, b be any elements of L. Then a and b are called relatively prime if $a \vee b = I$.

The following proposition, whose proof will be omitted, shows some of the properties of relatively prime elements.

PROPOSITION 4.1. *Let $a, b, d_1, \ldots, d_n$ be arbitrary elements of L.*
- (i) *If a, b are relatively prime, then a^s, b^t are relatively prime for any positive integers s and t.*
- (ii) *If a is relatively prime to each of the d_i $(1 \leqq i \leqq n)$, then it is relatively prime to $d_1 d_2 \ldots d_n$.*
- (iii) *If $d_1, d_2, \ldots, d_n$ are relatively prime in pairs, then $d_1 \wedge d_2 \wedge \ldots \wedge d_n = d_1 d_2 \ldots \ldots d_n$.*

PROPOSITION 4.2. *If $P_1, P_2, \ldots, P_n \in L$ are relatively prime in pairs, then*

$$P_1 I_M \wedge \ldots \wedge P_n I_M = (P_1 \ldots P_n) I_M = (P_1 \wedge \ldots \wedge P_n) I_M.$$

PROOF. It suffices to prove the first equation by induction by n. For $n=2$ we have $P_1 P_2 I_M \leqq P_1 I_M \wedge P_2 I_M = (P_1 \vee P_2)(P_1 I_M \wedge P_2 I_M) \leqq P_1 P_2 I_M$. Assume that $P_1 I_M \wedge \ldots \ldots \wedge P_{n-1} I_M = (P_1 \ldots P_{n-1}) I_M$. Since $P_1 \ldots P_{n-1}$, P_n are relatively prime, then $P_1 I_M \wedge \ldots \wedge P_{n-1} I_M \wedge P_n I_M = (P_1 \ldots P_{n-1}) I_M \wedge P_n I_M = (P_1 \ldots P_{n-1} P_n) I_M$.

PROPOSITION 4.3. *Let M be a modular L-module and let $P_1, \ldots, P_n \in L$ be relatively prime in pairs. Then for any element B of M we have*

$$\bigwedge_{i=1}^{n} (B \vee P_i I_M) = B \vee \left(\bigwedge_{i=1}^{n} P_i I_M \right).$$

PROOF. We shall prove the result by induction on n. For $n=2$ we have

$$(B \vee P_1 I_M) \wedge (B \vee P_2 I_M) = B \vee [(B \vee P_2 I_M) \wedge P_1 I_M] = B \vee [((P_1 \vee P_2) B \vee P_2 I_M) \wedge P_1 I_M] =$$

$$= B \vee [(P_1 B \vee P_2 I_M) \wedge P_1 I_M] = B \vee [P_1 B \vee (P_1 I_M \wedge P_2 I_M)] = B \vee (P_1 I_M \wedge P_2 I_M).$$

Assume that $\bigwedge_{i=1}^{n-1}(B\vee P_iI_M)=B\vee(\bigwedge_{i=1}^{n-1}P_iI_M)$, then

$$\bigwedge_{i=1}^{n}(B\vee P_iI_M)=\bigwedge_{i=1}^{n-1}(B\vee P_iI_M)\wedge(B\vee P_nI_M)=[B\vee(\bigwedge_{i=1}^{n-1}P_iI_M)]\wedge(B\vee P_nI_M)=$$

$$=(B\vee(P_1\dots P_{n-1})I_M)\wedge(B\vee P_nI_M)=B\vee[P_1\dots P_{n-1}I_M\wedge P_nI_M]=B\vee(\bigwedge_{i=1}^{n}P_iI_M).$$

PROPOSITION 4.4. *Let $P_1, P_2, \dots, P_n\in L$ be relatively prime in pairs. Then for any element $B_1, \dots, B_n$ of M there exists an element B of M such that $B\vee P_iI_M=B_i\vee\vee P_iI_M$ for any $i=1, 2, \dots, n$.*

PROOF. Let $B=\bigvee_{j=1}^{n}P_1\dots P_{j-1}P_{j+1}\dots P_nB_j$. *Since* $P_1\dots P_{i-1}P_{i+1}\dots P_n\vee P_i=I$ we have

$$B\vee P_iI_M=\bigvee_{j=1}^{n}P_1\dots P_{j-1}P_{j+1}\dots P_nB_j\vee P_iI_M=P_1\dots P_{i-1}P_{i+1}\dots P_nB_i\vee P_iI_M=$$

$$=(P_1\dots P_{i-1}P_{i+1}\dots P_n\vee P_i)B_i\vee P_iI_M=B_i\vee P_iI_M.$$

THEOREM 4.5. *Let M be a modular L-module and let $P_1, \dots, P_n\in L$ be relatively prime in pairs, then the canonical mapping $f\colon B\mapsto(B\vee P_1I_M, \dots, B\vee P_nI_M)$ is an isomorphism of the L-submodule $[(\bigwedge_{i=1}^{n}P_i)I_M, I_M]$ of M onto the L-module $\prod_{i=1}^{n}[P_iI_M, I_M]$.*

PROOF. It is clear that f is an L-module homomorphism, and by Proposition 4.3 and Proposition 4.4 we get that f is injective and surjective.

§ 5. Modules of finite length

REMARK 5.1. Let L be a multiplicative lattice in which 0 is a prime element. If L satisfies the D.C.C. and if every nonzero element of L contains a nonzero weak join principal element, then $L=\{0, 1\}$.

THEOREM 5.2. *Let L be a CG-lattice and let M be a Noetherian L-module. Then there exists a chain of elements $\{B_0, B_1, \dots, B_n\}$ of M such that:*

(i) $B_0=0_M$, $B_n=I_M$, $B_i=B_{i-1}\vee A_i$, *where A_i is a principal element of M for $i=1, 2, \dots, n$.*

(ii) *For any $i\in\{1, 2, \dots, n\}$ there exists a prime element $P_i\in L$ such that the L-submodule $[B_{i-1}, B_i]$ of M is isomorphic to the L-submodule $[P_i, I]$ of L.*

PROOF. See Theorem 2.11 in [11].

PROPOSITION 5.3. *Let L be a K-lattice, let M be a Noetherian lattice module and let $\{B_0, B_1, \dots, B_n\}$, $\{P_1, P_2, \dots, P_n\}$ be chains satisfying the conditions of Theorem 5.2. Then* Ass $(M)\subseteq\{P_1, \dots, P_n\}\subseteq$ Supp (M).

Proof. By Corollary 2.12 in [11] we have $\mathrm{Ass}(M) \subseteq \{P_1, \ldots, P_n\}$. Now let $P_i \in \{P_1, \ldots, P_n\}$, then by Lemma 3.4 we get $P_i \in V(P_i) = \mathrm{Supp}([P_i, I]) = \mathrm{Supp}([B_{i-1}, B_i]) \subseteq \mathrm{Supp}([0, B_i]) \subseteq \mathrm{Supp}(M)$.

Theorem 5.4. *Let L be a K-lattice and let M be a Noetherian lattice module. Then the following conditions are equivalent:*

(i) *M is of finite length.*

(ii) *Every element of* $\mathrm{Ass}(M)$ *is a maximal element of L.*

(iii) *Every element of* $\underset{f}{\mathrm{Ass}}(M)$ *is a maximal element of L.*

(iv) *Every element of* $\mathrm{Supp}(M)$ *is a maximal element of L.*

Proof. (i)⇒(ii). Let $\{B_0, \ldots, B_n\}$ and $\{P_1, \ldots, P_n\}$ be chains satisfying the conditions of Theorem 5.2. For $1 \leqq i \leqq n$ the L-submodule $[B_{i-1}, B_i]$ of M is isomorphic to the L-submodule $[P_i, I]$ of L. Since M is of finite length, then $[P_i, I]$ satisfies the D.C.C. By Remark 5.1 we get that P_i is maximal. Proposition 5.3 implies (ii). (ii)⇒(iii). This follows from Theorem 2.15 in [11]. (iii)⇒(iv). This follows from Proposition 3.2. (iv)⇒(i). If all the elements of $\mathrm{Supp}(M)$ are maximal, so are the P_i (Proposition 5.3). Hence the $[P_i, I]$ are simple L-modules. This implies that every $[B_{i-1}, B_i]$ is simple and therefore M is of finite length.

Corollary 5.5. *Let L be a K-lattice and let M be a Noetherian module. If M is of finite length, then* $\mathrm{Ass}(M) = \mathrm{Supp}(M)$.

Proof. This follows from Theorem 2.5 in [11] and Proposition 3.7.

Corollary 5.6. *Let L be a K-lattice, let M be a Noetherian module and let p be a prime element of L. For* M_p *to be a nonzero* L_p*-module of finite length, it is necessary and sufficient that p be a minimal element of* $\mathrm{Ass}(M)$.

Proof. Let p be a minimal element of $\mathrm{Ass}(M)$. Since $p \in \mathrm{Supp}(M)$ (Proposition 3.2) we have $M_p \neq \{0\}$ as an L_p-module. By Corollary 2.4 we get $\mathrm{Ass}(M_p) = \{[p]\}$. Since M_p is a Noetherian L_p-module (Theorem 1.3 in [7]) and since $[p]$ is a maximal element of L_p, then M_p is of finite length (Theorem 5.4). Now let M_p be a nonzero L_p-module of finite length. Then $p \in \mathrm{Supp}(M)$. By Proposition 3.2 p contains an element of $\underset{f}{\mathrm{Ass}}(M) = \mathrm{Ass}(M)$ (Theorem 2.15 in [11]). Let q be a minimal element of $\mathrm{Ass}(M)$ such that $q \leqq p$. By Corollary 2.5 we get $[p], [q] \in \mathrm{Ass}(M_p)$. By Theorem 5.4 we get $[p] = [q]$ and hence $p = q$ (Corollary 4.3 in [7]).

Remark 5.7. Let L be a K-lattice in which the greatest element I is compact, let M be a K-lattice, let n be a positive integer and let p be a maximal element of L. If $p^n I_M \neq I_M$ then $p^n I_M$ is p-primary in M.

Theorem 5.8. *Let L be a K-lattice in which the greatest element I is compact and let M be a Noetherian lattice module of finite length.*

(i) *The element* $0 \in M$ *only has a unique primary decomposition on M indexed by* $\mathrm{Ass}(M)$ *(necessarily reduced); let* $0 = \bigwedge_{p \in \mathrm{Ass}(M)} Q(p)$ *be this decomposition, where* $Q(p)$ *is p-primary in M.*

(ii) *There exists an integer* n_0 *such that for all* $n \geqq n_0$ *and all* $p \in \mathrm{Ass}(M)$, $Q(p) = p^n I_M$.

(iii) *For all* $p \in \mathrm{Ass}(M)$, *the canonical mapping of* M *to* M_p *is surjective and* $Q(p)$ *is the maximal element of* M *such that* $S_p(Q(p)) = S_p(0)$.

(iv) *The canonical mapping of* M *into* $\prod_{p \in \mathrm{Ass}(M)} [Q(p), I_M]$ *is bijective.*

PROOF. As every element $p \in \mathrm{Ass}(M)$ is minimal in $\mathrm{Ass}(M)$, then (i) follows from Corollary 4.9 and Theorem 4.11 in [11]. By Corollary 2.3 in [9] there exists an integer n_0 such that $p^n I_M \leqq Q(p)$ for all $p \in \mathrm{Ass}\, M)$ and all $n \geqq n_0$. But as p is a maximal element, $p^n I_M$ is p-primary in M, and as $\bigwedge_{p \in \mathrm{Ass}(M)} p^n I_M = 0$, it follows from (i) that necessarily $p^n I_M = Q(p)$ for all $p \in \mathrm{Ass}(M)$; whence (ii). (iii) This follows from Theorem 2.5. As the p^n, for $p \in \mathrm{Ass}(M)$, are relatively prime in pairs, then (iv) follows from Theorem 4.5.

COROLLARY 5.9. *Let* L *be a* K*-lattice in which the greatest element* I *is compact, and let* M *be a Noetherian lattice module of finite length. Then*

$$\mathrm{Long}_L(M) = \sum_{p \in \mathrm{Ass}(M)} \mathrm{Long}_{L_p}(M_p).$$

PROOF. This will follow from Theorem 5.8 if we prove that $\mathrm{long}_{L_p}(M_p) = \mathrm{Long}_L([Q(p), I_M])$. In fact the mapping $\varphi\colon X \mapsto [X]$ is an isomorphism of $[Q(p), I_M]$ onto M_p. Let $[B]$ be an element of M_p, then $B \vee Q(p) \in [Q(p), I_M]$ and $S_p \to (B \vee Q(p)) = S_p(S_p(B) \vee S_p(Q(p))) = S_p(S_p(B) \vee S_p(0)) = S_p(B)$. This means that φ is surjective. Let $X, Y \in [Q(p), I_M]$ such that $S_p(X) = S_p(Y)$. Since $X \leqq S_p(X) = S_p(Y)$ there exists an element $s \in S_p$ such that $sX \leqq Y$ (Lemma 1.2 in [7]). Let n be an integer such that $p^n I_M \leqq Q(p)$. Then $X = (s \vee p^n)X = sX \vee P^n X \leqq sX \vee Q(p) \leqq Y$. Similarly it follows that $Y \leqq X$ and therefore φ is injective.

REMARK 5.10. Since every modular module, over a Noether lattice, in which the greatest element is a finite join of principal elements is a Noetherian module (Proposition 2.11 in [9]), most of our results hold for them.

THEOREM 5.11. *Let* L *be a Noether lattice. The following conditions are equivalent·*

(i) *L satisfies the D.C.C.*

(ii) *All the prime elements of L are maximal elements.*

(iii) *All the elements of* $\mathrm{Ass}(L)$ *are maximal elements.*

If these conditions are fulfilled, L has only a finite number of prime elements, all of which are maximal and associated with the L-module L. Further, L is a semi-local lattice and its Jacobson radical $J(L)$ is nilpotent.

PROOF. (i)⇒(ii). Let P be a prime element of L. Then $[P, I]$ is a Noether lattice satisfying the D.C.C. By Remark 5.1 we get that P is maximal in L. Clearly, (ii) implies (iii). (iii)⇒(i) By Theorem 5.4 we get that L is an L-module of finite length and hence L satisfies the D.C.C. Now suppose that any of the three equivalent conditions holds. Since every prime element of L belongs to $\mathrm{Supp}(L)$ and $\mathrm{Ass}(L) = \mathrm{Supp}(M)$ (Corollary 5.5), then $\mathrm{Ass}(L)$ is the set of all prime elements of L, and all of them are maximal and associated with the L-module L. Since $\sqrt{0} = J(L)$, then $J(L)$ is nilpotent.

COROLLARY 5.12. *Let* L *be a Noether lattice. Then* L *is of finite length if and only* *if L is isomorphic to* $\prod_{i=1}^{m}[P_i^n, I]$, *where n is an integer and* $P_1, \ldots, P_m$ *are maximal in L.*

Proof. Let L be of finite length and let $\{P_1, \ldots, P_m\} = \mathrm{Ass}(L)$, then there exists an integer n such that $(J(L))^n = 0$. This means that $(\bigwedge_{i=1}^{m} P_i)^n = 0$. Since the elements $\{P_1^n, \ldots, P_m^n\}$ are relatively prime in pairs, then $\bigwedge_{i=0}^{m} P_i^n = 0$. By Theorem 4.5 we get that $L \approx \prod_{i=1}^{n}[P_i^n, I]$. Now since $[P_i^n, I]$ is of finite length (Theorem 5.4), then L is of finite length.

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LINEARIZATION NEAR THE SUMMABLE MANIFOLD FOR DISCRETE SYSTEMS

GARYFALOS PAPASCHINOPOULOS

Introduction

Consider the system of difference equations on $Z=\{, \dots, -1, 0, 1, \dots\}$

$$\begin{aligned} x(n+1) &= f(n, x(n), y(n)) \\ y(n+1) &= A(n)y(n)+g(n, x(n), y(n)) \end{aligned} \tag{1}$$

where $f(n, x, y)$ is a function of $Z\times R^l\times R^k$ into R^l, $g(n, x, y)$ is a function of $Z\times R^l\times R^k$ into R^k and $A(n)$ is an invertible matrix for all $n\in Z$ with bounded inverse. Consider also that $f(n, x, y)-x$, $g(n, x, y)$ are small Lipschitzian in x and y and the system

$$y(n+1) = A(n)y(n) \tag{2}$$

has an exponential dichotomy.

In [7] we have proved that there exists a summable manifold $v(n,x)$ for (1). Now for each using this result we prove that there exists a homeomorphism of the (x, y) space $n\in Z$ sending the solutions of (1) into the solutions of the linearized system

$$\begin{aligned} x(n+1) &= f(n, x(n), v(n, x(n))) \\ y(n+1) &= A(n)y(n). \end{aligned} \tag{3}$$

The results obtained in this paper are the discrete analogous of those of Palmer [3]. In the continuous case related results in this direction are included in [4], [5], [6].

In subsequent papers we are going to prove analogous results for the discrete case.

System (2) is said to possess an exponential dichotomy on Z if there exist a projection $P\,(P^2=P)$ and constants $K>0$, $0<p<1$ such that

$$\begin{aligned} |X(n)PX^{-1}(m)| &\leq Kp^{2(n-m)}, \quad n \geq m \\ |X(n)(I-P)X^{-1}(m)| &\leq Kp^{2(m-n)}, \quad m \geq n \end{aligned} \tag{4}$$

where $X(n)$ is a fundamental matrix of (2).

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A bounded function $v(n, x)$ of $Z\times R^l$ into R^l determines a summable manifold for (1) if for any solution $x(n)$ of the equation

$$x(n+1) = f(n, x(n), v(n, x(n)))$$

we have that $x(n), v(n, x(n))$ is a solution of (1) on Z. Moreover if $x(n), y(n)$ is a solution of (1) such that $\sup\{|y(n)|, n\in Z\}<\infty$ then $y(n)=v(n, x(n))$ for all $n\in Z$.

Main results

First we state our proposition.

PROPOSITION. *Suppose $f(n, x, y)$ is a function of $Z\times R^l\times R^k$ into R^l such that for all $x, \bar{x}\in R^l$, $y, \bar{y}\in R^k$, $n\in Z$ we have*

$$|f(n, x, y)-f(n, \bar{x}, \bar{y})-x+\bar{x}| \leqq q_1|x-\bar{x}|+N|y-\bar{y}|. \tag{5}$$

Let $A(n)$ be an invertible matrix for all $n\in Z$ such that

$$|A^{-1}(n)| \leqq M, \qquad M>0 \tag{6}$$

and (2) has a fundamental matrix solution $X(n)$ which satisfies (4). Suppose also that $g(n, x, y)$ is a function of $Z\times R^l\times R^k$ into R^k such that for all $x, \bar{x}\in R^l$, $y, \bar{y}\in R^k$, $n\in Z$ we have

$$|g(n, x, y)-g(n, \bar{x}, \bar{y})| \leqq q_2(|x-\bar{x}|+|y-\bar{y}|). \tag{7}$$

Then if

$$p<e^{-1}, \quad q_1<\frac{1}{4}, \quad q_2<\min\left\{\frac{p^2(1-p)}{4K}, \frac{1}{2M}\right\},$$

$$N<\min\left\{\frac{p(1-pe)}{32Kq_2}, \frac{p^3}{8Kq_2}, \frac{1}{2}\right\}$$

where p, K are the constants given in (4) and e is the base of the natural logarithms, there exists a function

$$H(n, x, y) = (H_1(n, x, y), H_2(n, x, y))$$

of $Z\times R^l\times R^k$ into $R^l\times R^k$ such that

(i) *for each fixed $n\in Z$, $H_n(x, y)=H(n, x, y)$ is a homeomorphism of $R^l\times R^k$,*

(ii) *if $x(n), y(n)$ is a solution of (1) then $H_1(n, x(n), y(n))$, $H_2(n, x(n), y(n))$ is a solution of (3),*

(iii) *if*

$$L(n, x, y) = (L_1(n, x, y), L_2(n, x, y)) = H_n^{-1}(x, y)$$

and $z(n), w(n)$ is a solution of (3) then $L_1(n, z(n), w(n))$, $L_2(n, z(n), w(n))$ is a solution of (1).

For the proof of the above proposition we give some lemmas.

LEMMA 1. *Let $A(n)$ be an invertible matrix for $n\in Z$ satisfying* (6). *Let also the functions f, g satisfy* (5) *and* (7) *correspondingly, and the constant q_1, N, q_2 satisfy*

$$0<q_1<\frac{1}{2},\quad 0<N<\frac{1}{2},\quad 0<q_2M<\frac{1}{2}.$$

Then for any (m, ξ, τ) $m\in Z$, $\xi\in R^l$, $\tau\in R^k$ there exists a unique solution $x(n)=x(n, m, \xi, \tau)$, $y(n)=y(n, m, \xi, \tau)$ of (1) *such that $x(m)=\xi$, $y(m)=\tau$. Moreover, the functions $x(n), y(n)$ are continuous in (ξ, τ) for each $n, m\in Z$.*

PROOF. Suppose m is fixed. Let $n\geqq m$. Obviously, the solution $x(n), y(n)$ of (1) such that $x(m)=\xi$, $y(m)=\tau$ exists and it is unique. We also have that the functions $x(n), y(n)$ are continuous in (ξ, τ) for $n\geqq m$ since the functions $f(n, x, y), g(n, x, y)$ are continuous in (x, y) for every $n\in Z$.

Let now $n<m$. Consider the set $I=\{u, u+1, \dots, m\}$ for fixed $u<m$ and a constant q such that

$$q<\min\{1-2q_1,\ 1-2N,\ M^{-1}-2q_2\}. \tag{8}$$

We choose any bounded sets $B_1\subset R^l$, $B_2\subset R^k$. Let S_1 be the space of all functions $x\colon I\times B_1\times B_2\to R^l$ which are continuous in $B_1\times B_2$ and

$$|x|=\sup\{|x(n, \xi, \tau)|\,q^{m-n},\ n\in I,\ \xi\in B_1,\ \tau\in B_2\}<\infty,$$

S_2 be the space of all functions $y\colon I\times B_1\times B_2\to R^k$ which are continuous in (ξ, τ) and

$$|y|=\sup\{|y(n, \xi, \tau)|\,q^{m-n},\ n\in I,\ \xi\in B_1,\ \tau\in B_2\}<\infty$$

and S be the space of all functions $z\colon I\times B_1\times B_2\to R^l\times R^k$ such that $z(n, \xi, \tau)=\operatorname{col}(x(n, \xi, \tau), y(n, \xi, \tau))$ where $x\in S_1$, $y\in S_2$. Consider $|z|=\max\{|x|, |y|\}$. Obviously, S is a Banach space. Let T_1, T_2 be the operators which are defined on S_1, S_2 correspondingly as follows:

$$T_1x(n, \xi, \tau)=\xi-\sum_{s=n}^{m-1} f(s,\ x(s, \xi, \tau),\ y(s, \xi, \tau))-x(s, \xi, \tau) \tag{9}$$

$$T_2y(n, \xi, \tau)=X(n)X^{-1}(m)\tau-\sum_{s=n}^{m-1} X(n)X^{-1}(s+1)g(s,\ x(s, \xi, \tau),\ y(s, \xi, \tau)). \tag{10}$$

Let $z_1, z_2\in S$ and $z_1(n, \xi, \tau)=\operatorname{col}(x_1(n, \xi, \tau),\ y_1(n, \xi, \tau))$, $z_2(n, \xi, \tau)=\operatorname{col}(x_2(n, \xi, \tau), y_2(n, \xi, \tau))$. From (5) and (9) we have

$$|T_1x_1(n, \xi, \tau)-T_1x_2(n, \xi, \tau)|\leqq$$

$$\leqq\sum_{s=n}^{m-1} q_1|x_1(s, \xi, \tau)-x_2(s, \xi, \tau)|+N|y_1(s, \xi, \tau)-y_2(s, \xi, \tau)|.$$

So

$$q^{m-n}|T_1x_1(n, \xi, \tau)-T_1x_2(n, \xi, \tau)|\leqq\sum_{s=n}^{m-1} q^{s-n}(q_1|x_1-x_2|+N|y_1-y_2|).$$

Therefore we get

$$(11)\qquad |T_1x_1-T_1x_2| \leqq \frac{1}{1-q}(q_1|x_1-x_2|+N|y_1-y_2|) < \frac{q_1+N}{1-q}|z_1-z_2|.$$

From (7), (10) and since from (6) $|X(n)X^{-1}(s+1)|\leqq M^{s-n+1}$, $s>n$ we get

$$q^{m-n}|T_2y_1(n,\xi,\tau)-T_2y_2(n,\xi,\tau)| \leqq \sum_{s=n}^{m-1}(Mq)^{s-n}Mq_2(|x_1-x_2|+|y_1-y_2|).$$

Therefore we have

$$(12)\qquad |T_2y_1-T_2y_2| \leqq \frac{Mq_2}{1-Mq}(|x_1-x_2|+|y_1-y_2|) \leqq \frac{2Mq_2}{1-Mq}|z_1-z_2|.$$

Let T be the operator on S defined by $Tz(n,\xi,\tau)=\mathrm{col}\,(T_1x(n,\xi,\tau),\ T_2y(n,\xi,\tau))$. So from (8), (11), (12) we have that T is a contraction on S. Hence there exists a unique $z\in S$ such that $Tz=z$. From (9) and (10) if $z(n,\xi,\tau)=\mathrm{col}\,(x(n,\xi,\tau),\ y(n,\xi,\tau))$ we have that $x(n)=x(n,\xi,\tau)$, $y(n)=y(n,\xi,\tau)$ is a solution of (1) on I such that $x(m)=\xi$, $y(m)=\tau$. Let now $\bar{x}(n)$, $\bar{y}(n)$ be a solution of (1) such that $\bar{x}(m)=\xi$, $\bar{y}(m)=\tau$. Then we can easily prove that

$$\bar{x}(n) = \xi-\sum_{s=n}^{m-1} f(s,\bar{x}(s),\bar{y}(s))-\bar{x}(s).$$

Using the variation of constants formula [2, p. 11—12] we get

$$\bar{y}(n) = X(n)X^{-1}(m)\tau-\sum_{s=n}^{m-1}X(n)X^{-1}(s+1)g(s,\bar{x}(s),\bar{y}(s)).$$

Hence from the last two relations we have $x(n)=\bar{x}(n)$, $y(n)=\bar{y}(n)$ for all $n\in I$. So the solution $x(n)$, $y(n)$ of (1) such that $x(m)=\xi$, $y(m)=\tau$ exists and it is unique for $n\in I$. We also have that $x(n)$, $y(n)$ are continuous functions in $B_1\times B_2$ for $n\in I$.

Now using the above argument we can easily prove that the solution $x(n)$, $y(n)$ of (1) such that $x(m)=\xi$, $y(m)=\tau$, $\xi\in B_1$, $\tau\in B_2$ exists and it is unique on every set $I=\{v, v+1, \dots, u\}$, $v\in Z$, $v\leqq u$, so exists and it is unique for all $n\leqq m$. We also have that $x(n)$, $y(n)$ are continuous in $B_1\times B_2$ for all $n\leqq m$. Since m is arbitrary and B_1 (B_2) is any bounded subset of R^l (R^k) the proof of the lemma is completed.

The following lemma is a type of the discrete Gronwall lemma [1, p. 68].

LEMMA 2. *Let $c(n)$, $k(n)$ be functions of Z into R^+ such that* $\frac{c(n+1)}{c(n)}\leqq p$, $0<p<1$, $c(n)\leqq wp^n$, $w>0$, $\frac{k(n+1)}{k(n)}\leqq 1$, $k(n)\leqq\mu$, $p(1+\mu)<1$ *for all* $n\in Z$. *Then for any positive solution $y(n)$ of the inequality*

$$(13)\qquad y(n)\leqq c(n)+\sum_{s=n+1}^{\infty}k(s)y(s)$$

such that

$$\|y\| = \sup\{y(n)p^{-n},\ n\in Z\}<\infty$$

we have

$$\|y\| \leqq \frac{(1-p)^2}{(1-p-p\mu)^2} w.$$

PROOF. Consider the equation

(14) $$x(n) = c(n) + \sum_{s=n+1}^{m} k(s)x(s), \quad n \in Z, \quad m \in Z, \quad m > n.$$

From the proof of Lemma 1 [7, p. 458—459] we have that

(15) $$x(n) = c(n) + \sum_{s=n+1}^{m} c(s)k(s) \prod_{v=n+1}^{s-1} (1+k(v))$$

is a solution of (14). The sum

$$\sum_{s=n+1}^{\infty} c(s)k(s) \prod_{v=n+1}^{s-1} (1+k(v))$$

converges for any $n \in Z$ since

$$\frac{c(s+1)k(s+1)}{c(s)k(s)} (1+k(s)) \leqq p(1+\mu) < 1.$$

Therefore from (15) if $m \to \infty$ we have that the function

(16) $$x(n) = c(n) + \sum_{s=n+1}^{\infty} c(s)k(s) \prod_{v=n+1}^{s-1} (1+k(v))$$

is a solution of the equation

$$x(n) = c(n) + \sum_{s=n+1}^{\infty} k(s)x(s).$$

From hypothesis and (16) we have

$$x(n) \leqq wp^n + w\mu(1+\mu)^{-n-1} \sum_{s=n+1}^{\infty} p^s(1+\mu)^s = \frac{wp^n(1-p)}{1-p(1+\mu)}.$$

So we have

(17) $$\|x\| \leqq \frac{w(1-p)}{1-p(1+\mu)}.$$

Consider a positive function $y(n)$ which satisfies (13) and $\|y\| < \infty$. Then we have

$$y(n) - x(n) \leqq \sum_{s=n+1}^{\infty} k(s)(y(s) - x(s)) \leqq \sum_{s=n+1}^{\infty} k(s)y(s).$$

So we obtain

$$y(n)p^{-n} \leqq x(n)p^{-n} + \sum_{s=n+1}^{\infty} \mu p^{s-n} p^{-s} y(s).$$

Therefore we have

$$\|y\| \leqq \frac{1-p}{1-p(1+\mu)} \|x\|.$$

So from (17) the proof of the lemma is completed.

Let E be a vector space and h be a function of Z into E. In the following we use the norms

$$\|h\| = \sup \{|h(n)| p^{-n}, \ n \in Z\}, \quad 0 < p < 1$$

$$\|h\|^{+} = \sup \{|h(n)| p^{-n}, \ n \geqq 0\}, \quad \|h\|^{-} = \sup \{|h(n)| p^{n}, \ n \leqq 0\}.$$

LEMMA 3. *Consider the set* $N=\{0, 1, \ldots\}$. *Let* $f(n, x)$, $k(n, x)$ *be functions of* $Z \times R^l$ *into* R^l *such that for* $n \in Z$

(18) $$|k(n, x_1)-k(n, x_2)-x_1+x_2| \leqq v|x_1-x_2|, \quad v > 0, \quad x_1, x_2 \in R^l.$$

Suppose that there exists a solution $x(n)$ *of the equation* $x(n+1)=f(n, x(n))$ *such that*

$$\sup \{|f(n, x(n))-k(n, x(n))| p^{-n}, \ n \in N\} < \infty, \quad 0 < p < 1.$$

Then if $v < 1-p$ *there exists a unique solution* $z(n)$ *of*

(19) $$x(n+1) = k(n, x(n))$$

such that

$$\sup \{|z(n)-x(n)| p^{-n}, \ n \in N\} < \infty.$$

Moreover if $k(n, x)=k(n, x, \zeta)$, $x(n)=x(n, \zeta)$, $f(n, x)=f(n, x, \zeta)$ *are continuous with respect to* ζ *for all* $n \in Z$, ζ *belongs to an Euclidean space S such that the constant* v *is independent of* ζ *and*

$$\sup \{|f(n, x(n, \zeta), \zeta)-k(n, x(n, \zeta), \zeta)| p^{-n}, \ n \geqq 0\} < \infty$$

uniformly with respect to ζ *then the solution* $z(n)=z(n, \zeta)$ *is continuous with respect to* ζ *for all* $n \in Z$.

PROOF. Let

$$M = \sup \{|f(n, x(n))-k(n, x(n))| p^{-n}, n \in N\}.$$

Consider the space E of all functions $z\colon N \to R^l$ such that

$$\sup \{|z(n)-x(n)| p^{-n}, \ n \in N\} < \infty$$

and the operator T on E given by

(20) $$Tz(n) = x(n) - \sum_{s=n}^{\infty} k(s, z(s)) - z(s) - f(s, x(s)) + x(s).$$

So from (18) we have

$$|Tz(n)-x(n)| \leqq \sum_{s=n}^{\infty} |k(s, z(s))-z(s)-k(s, x(s))+x(s)+k(s, x(s))-f(s, x(s))| \leqq$$

$$\leqq \sum_{s=n}^{\infty} v|z(s)-x(s)| + Mp^s \leqq \frac{(v\|z-x\|^{+}+M)}{1-p} p^n, \quad n \geqq 0.$$

So T is in E. We also have from (18) and (20)

$$\|Tz_1-Tz_2\|^+ \leqq \frac{v}{1-p}\|z_1-z_2\|^+.$$

Therefore T is a contraction on E which is a complete metric space by $d(z_1, z_2)=$ $=\|z_1-z_2\|^+$. So there exists a unique $z\in E$ such that $Tz=z$. Obviously, $z(n)$ is a solution of (19) such that $\|z-x\|^+<\infty$.

Now we prove that this solution is unique. Let $z_1(n)$ be another solution of (19) such that $\|z_1-x\|^+<\infty$. Consider $m=2k+1$, $k\in N$. From the proof of Lemma 2 [7, p. 459—461] we get for $m>n$

$$z_1(n) = z_1(m)+z(n)-z(m)+\sum_{s=n}^{m-1} z_1(s)-k(s, z_1(s))-z(s)+k(s, z(s)). \tag{21}$$

From the relation

$$\sup\{|z(m)-z_1(m)|\,p^{-m},\ m=2k+1\} \leqq \sup\{|z(n)-z_1(n)|\,p^{-n},\ n\in N\} = \|z-z_1\|^+$$

we have

$$|z(m)-z_1(m)| \leqq p^m\|z-z_1\|^+,\quad m=2k+1,\quad k\geqq 0.$$

So for $m\to\infty$ we have that $|z(m)-z_1(m)|$ tends to zero. Therefore from (21) we get

$$z_1(n) = z(n)+\sum_{s=n}^{\infty} z_1(s)-k(s, z_1(s))-z(s)+k(s, z(s)).$$

Then from (18) we obtain

$$|z_1(n)-z(n)| \leqq \frac{vp^n}{1-p}\|z_1-z\|^+,\quad \text{so}\quad \|z_1-z\|^+ \leqq \frac{v}{1-p}\|z_1-z\|^+.$$

Therefore $z_1(n)=z(n)$ for all $n\in N$ since $v<1-p$. So the proof of the first part of the lemma is completed.

Suppose now $k(n, x, \zeta)$, $x(n, x, \zeta)$, $f(n, x, \zeta)$ are continuous with respect to ζ for all $n\in N$. We replace E by the set of all continuous in S functions $z\colon N\times S\to R^l$ such that

$$\sup\{|z(n,\zeta)-x(n,\zeta)|\,p^{-n},\ n\in N\}<\infty. \tag{22}$$

Using the same argument as in the first part of the proof of the lemma we can easily prove that the operator T defined by

$$Tz(n,\zeta) = x(n,\zeta)-\sum_{s=n}^{\infty} k(s,\ z(s,\zeta),\ \zeta)-z(s,\zeta)-f(s,\ x(s,\zeta),\ \zeta)+x(s,\zeta)$$

has a unique fixed point z. We also have that $z(n,\zeta)$ is continuous with respect to ζ for all $n\in N$ and it is the unique solution of the equation $x(n+1)=k(n, x(n), \zeta)$ such that (22) holds. Using the same argument to prove the continuity in Lemma 1 and by an easy generalization of Lemma 2 [7, p. 459] the solution $z(n,\zeta)$ can be defined for all $n\in Z$, $\zeta\in S$ and it is continuous with respect to $\zeta\in S$ for all $n\in Z$. Thus the proof of the lemma is completed.

The following lemma is a generalization of Theorem 5 [8, p. 344].

LEMMA 4. *Let $A(n)$ be an invertible matrix for $n \geqq 0$ such that (2) satisfies (4) on N. Consider $h\colon N \to R^k$ such that $\|h\|^+ = \sup\{|h(n)|p^{-n},\ n\in N\} < \infty$ and $F\colon N\times \times R^k \to R^k$ such that*

$$(23)\qquad \begin{aligned} |F(n,x)| &\leqq q|x| \quad \text{for all} \quad n\in N,\quad x\in R^k \\ |F(n,x)-F(n,\bar{x})| &\leqq q|x-\bar{x}|, \quad n\in N, \quad x,\bar{x}\in R^K, \end{aligned}$$

where $q < \dfrac{1-p}{4Kp^{-1}}$. *Then if $x(n)$ is a bounded solution of the equation*

$$(24)\qquad x(n+1) = A(n)x(n)+h(n)+F(n,x(n))$$

such that $x(0)=\xi\in R^k$ we have

$$(25)\qquad |x(n)| \leqq 2\left(K|x(0)|+\frac{2Kp^{-1}}{1-p}\|h\|^+\right)p^n, \quad n\in N.$$

PROOF. Let E be the set of all functions $x\colon N\to R^k$ such that

$$|x(n)| \leqq 2\left(K|\xi|+\frac{2Kp^{-1}}{1-p}\|h\|^+\right)p^n, \quad n\in N.$$

Consider

$$z(n) = X(n)PX^{-1}(0)\xi+\sum_{s=0}^{n-1} X(n)PX^{-1}(s+1)h(s)-\sum_{s=n}^{\infty} X(n)(I-P)X^{-1}(s+1)h(s).$$

Since $\|h\|^+<\infty$ it is easy to show that

$$(26)\qquad |z(n)| \leqq \left(K|\xi|+\frac{2Kp^{-1}}{1-p}\|h\|^+\right)p^n, \quad n\in N.$$

Define the operator T on E as follows:

$$Tx(n) = z(n)+\sum_{s=0}^{n-1} X(n)PX^{-1}(s+1)F(s,x(s))-\sum_{s=n}^{\infty} X(n)(I-P)X^{-1}(s+1)F(s,x(s)).$$

From (23), (26) and since $\dfrac{2Kp^{-1}q}{1-p}<\dfrac{1}{2}$ we can easily prove that T is in E.

Using (23) if $x_1, x_2\in E$ we obtain

$$|Tx_1(n)-Tx_2(n)| \leqq \frac{2Kp^{-1}q}{1-p}\|x_1-x_2\|^+p^n, \quad n\geqq 0.$$

So T is a contraction on the Banach space E. Therefore there exists a unique $\bar{x}\in E$ such that $T\bar{x}=\bar{x}$. We have that $\bar{x}(n)$ satisfies (25) and it is easy to show that $\bar{x}(n)$ is a solution of (24). Now if we replace E by the set of all bounded functions on N and using the above argument we can easily prove that there exists a unique bounded function $\bar{\bar{x}}(n)$ on N such that $T\bar{\bar{x}}=\bar{\bar{x}}$. So, $\bar{x}(n)=\bar{\bar{x}}(n)$, $n\in N$. Using the proof of Theorem 2 [8, p. 340] we have that the bounded solution $x(n)$ of (24) such that $x(0)= =\xi$ is a fixed point for the operator T. Hence $\bar{x}(n)=\bar{\bar{x}}(n)=x(n)$, $n\in N$ and the proof of the lemma is completed.

LEMMA 5. *Suppose* (2) *satisfies* (4) *on* Z. *Consider a function* $f(n)$ *of* Z *into* R^k *such that* $|f(n)|\leqq\mu$ *for all* $n\in Z$. *Then the system*

$$y(n+1) = A(n)y(n)+f(n)$$

has a unique bounded solution $y(n)$, *such that*

$$|y(n)| \leqq \frac{2K\mu}{1-p^2}, \quad n\in Z.$$

We omit the proof because this lemma is a special case of Lemma 3 [7, p. 461].

LEMMA 6. *Consider system* (1) *where* $A(n)$ *is an invertible matrix such that* (6) *holds and* (2) *has a fundamental matrix satisfying* (4), f *and* g *are functions satisfying* (5) *and* (7) *correspondingly. Consider also that* $|g(n,x,y)|\leqq\mu$, $\mu>0$, $n\in Z$, $x\in R^l$, $y\in R^k$.

Let $x(n)=x(n,m,\xi,\tau)$, $y(n)=y(n,m,\xi,\tau)$ *be the solution of* (1) *such that* $x(m)=\xi$, $y(m)=\tau$ *and let* $q(n)$ *be a solution of* (2). *Suppose that*

$$p<\frac{1}{2}, \quad q_1<\min\left\{1-2p, \frac{1}{2}\right\}, \quad q_2<\min\left\{\frac{p^2(1-p)}{4K}, \frac{1}{2M}\right\},$$

$$N<\min\left\{\frac{p^3}{8Kq_2}, \frac{1}{2}\right\}.$$

Then for each (n,m,ξ,τ) *such that* $y(n,m,\xi,\tau)-q(n)$ *is bounded for* $n\geqq 0$ *there exists a unique solution* $\hat{x}(n)=\hat{x}(n,m,\xi,\tau)$, $\hat{y}=\hat{y}(n,m,\xi,\tau)$ *of* (1) *such that*

$$\|\hat{x}-x\|^+<\infty \quad \textit{and} \quad \sup\{|\hat{y}(n)-q(n)|,\ n\in Z\}<\infty.$$

Moreover, $\hat{x}(n)$, $\hat{y}(n)$ *are continuous with respect to* ξ,τ *for each* $n,m\in Z$ *and satisfy the inequalities*

$$\begin{aligned}&|\hat{x}(n)-x(n)| \leqq \frac{4NK}{p}|\hat{y}(m)-y(m)|p^{n-m}, \quad n\geqq m\\ &|\hat{y}(n)-y(n)| \leqq 4K|\hat{y}(m)-y(m)|p^{n-m}, \quad n\geqq m.\end{aligned} \tag{27}$$

PROOF. Consider the set E of all functions w such that $|w(n)-q(n)|\leqq\frac{2K\mu}{1-p^2}$, $n\in Z$, $\|w-y\|^+<\infty$. Let $w\in E$. Consider also the equations

$$\begin{aligned}x(n+1) &= f(n, x(n), y(n))\\ x(n+1) &= f(n, x(n), w(n)).\end{aligned} \tag{28}$$

It holds

$$\sup\{|f(n,x(n),y(n))-f(n,x(n),w(n))|p^{-n},\ n\geqq 0\}\leqq N\|w-y\|^+.$$

So from Lemma 3 there exists a unique solution $z(n)$ of (28) such that $\|z-x\|^+<\infty$. From the proof of Lemma 3 we have for $n\geqq 0$

$$z(n) = x(n)-\sum_{s=n}^{\infty} f(s,z(s),w(s))-z(s)-f(s,x(s),y(s))+x(s). \tag{29}$$

From Lemma 5 equation

$$v(n+1) = A(n)v(n)+g(n, z(n), w(n)), \quad n\in Z$$

has a unique bounded solution $v(n)$ on Z such that $|v(n)| \leq \frac{2K\mu}{1-p^2}$. Take

$$W(n) = q(n)+v(n).$$

We have

(30) $$W(n+1) = A(n)W(n)+g(n, z(n), w(n)).$$

We prove that W is in E. We get

$$|W(n)-q(n)| \leq \frac{2K\mu}{1-p^2}, \quad n\in Z.$$

Consider the function $u(n)=W(n)-y(n)=W(n)-q(n)+q(n)-y(n)$. So the function $u(n)$ is bounded for $n\geq 0$. We also have that $u(n)$ is a bounded solution of

$$u(n+1) = A(n)u(n)+h(n),$$

$$h(n) = g(n, z(n), w(n))-g(n, x(n), y(n)), \quad n\geq 0.$$

From the proof of Theorem 2 [8, p. 340] we obtain

$$u(n) = X(n)PX^{-1}(0)u(0)+$$

$$+\sum_{s=0}^{n-1} X(n)PX^{-1}(s+1)h(s)-\sum_{s=n}^{\infty} X(n)(I-P)X^{-1}(s+1)h(s).$$

Since

$$|h(n)| \leq q_2(|z(n)-x(n)|+|w(n)-y(n)|)$$

we have $\|h\|^+<\infty$. So

(31) $$|u(n)| \leq Kp^n|u(0)|+\frac{2K\|h\|^+p^{-1}p^n}{1-p}, \quad n\geq 0.$$

Therefore $\|u\|^+<\infty$. Hence W is in E.

The space E is a complete metric space by giving the metric $d(w_1, w_2)=\|w_1-w_2\|$, $w_1, w_2\in E$. We prove now that W is a contraction in E. Consider z_1, z_2, W_1, W_2 which correspond to w_1, w_2. From (30) and Lemma 5 the function $u_1(n)=W_1(n)-W_2(n)$ is the unique bounded solution of the equation

$$x(n+1) = A(n)x(n)+h_1(n), \quad n\in Z$$

where $h_1(n)=g(n, z_1(n), w_1(n))-g(n, z_2(n), w_2(n))$. It holds

(32) $$u_1(n) = \sum_{s=-\infty}^{n-1} X(n)PX^{-1}(s+1)h_1(s)-\sum_{s=n}^{\infty} X(n)(I-P)X^{-1}(s+1)h_1(s).$$

From (5) and (29) we obtain

$$|z_1(n)-z_2(n)| \leq \sum_{s=n}^{\infty} q_1|z_1(s)-z_2(s)|+N|w_1(s)-w_2(s)|.$$

So we have

$$|z_1(n)-z_2(n)| \leq \frac{q_1}{1-q_1} \sum_{s=n+1}^{\infty} |z_1(s)-z_2(s)| + \frac{Np^n}{(1-q_1)(1-p)} \|w_1-w_2\|, \quad n\in Z.$$

Therefore by Lemma 2 we get

(33) $$\|z_1-z_2\| \leq \frac{(1-p)N}{(1-p-pc)^2(1-q_1)} \|w_1-w_2\|, \quad c = \frac{q_1}{1-q_1}.$$

From (4), (7), (32) and (33) we take

(34) $$\|W_1-W_2\| \leq \frac{2Kp^{-1}q_2}{1-p} \left(\frac{(1-p)N}{(1-p-pc)^2(1-q_1)} + 1 \right) \|w_1-w_2\|.$$

From $p<\frac{1}{2}$ and $q_2<\frac{p^2(1-p)}{4K}$ we have $\frac{2Kp^{-1}q_2}{1-p}<\frac{1}{4}$. We also have

$$\frac{1}{(1-p-pc)^2} \cdot \frac{1}{1-q_1} \leq \frac{1}{p^2}$$

since $q_1<1-2p$. Therefore from $N\leq\frac{p^3}{8Kq_2}$ we get

$$\frac{2Kp^{-1}q_2 N}{(1-p-pc)^2(1-q_1)} \leq \frac{1}{4}.$$

So from (34) W is a contraction on E. Hence there exists a unique $\hat{y}\in E$ such that $W(\hat{y})=\hat{y}$. From (30) we have that $\hat{y}(n)$ is a solution of

$$\hat{y}(n+1) = A(n)\hat{y}(n)+g(n, \hat{x}(n)\hat{y}(n))$$

such that $|\hat{y}(n)-q(n)|\leq\frac{2K\mu}{1-p^2}$, $n\in Z$ and $\hat{x}(n)$ is a solution of the equation

$$\hat{x}(n+1) = f(n, \hat{x}(n), \hat{y}(n))$$

such that $\|x-\hat{x}\|^+<\infty$.

Now we prove that the solution $\hat{x}(n), \hat{y}(n)$ is unique. Let $x_1(n)$, $y_1(n)$ be another solution of (1) such that $|y_1(n)-q(n)|\leq\frac{2K\mu}{1-p^2}$, $n\in Z$ and $\|x_1-x\|^+<\infty$. So we have

$$\sup\{|y_1(n)-\hat{y}(n)|, \ n\in Z\} <\infty, \quad \|x_1-\hat{x}\|^+ <\infty.$$

We also have that $y_1(n)-\hat{y}(n)$ is a bounded solution for $n\geq 0$ of the equation

(35) $$u(n+1) = A(n)u(n)+h_2(n)+F(n, u(n))$$

where

$$h_2(n) = g(n, x_1(n), y_1(n))-g(n, \hat{x}(n), y_1(n)),$$

$$F(n, u) = g(n, \hat{x}(n), \hat{y}(n)+u)-g(n, \hat{x}(n), \hat{y}(n)).$$

From (7) and since $\|x_1-\hat{x}\|^+<\infty$ we have

$$\|h_2\|^+ \leqq q_2\|x_1-\hat{x}\|^+ <\infty.$$

From (7) for $n, u, \bar{u}\in R^k$ we get

$$|F(n, u)| \leqq q_2|u|, \quad |F(n, u)-F(n, \bar{u})| \leqq q_2|u-\bar{u}|, \quad n\in N.$$

Therefore from Lemma 4 if $u(n)=y_1(n)-\hat{y}(n)$ we get

$$|u(n)| \leqq 2\left(K|u(0)|+\frac{2Kp^{-1}}{1-p}\|h_2\|^+\right)p^n, \quad n\in N. \tag{36}$$

From the proof of the uniqueness of Lemma 3 we obtain

$$x_1(n)-\hat{x}(n) = -\sum_{s=n}^{\infty} f(s, x_1(s), y_1(s))-x_1(s)-f(s, \hat{x}(s), \hat{y}(s))+\hat{x}(s).$$

By (36) we get $\|u\|<\infty$ and using the same argument to prove (33) we have

$$\|x_1-\hat{x}\| \leqq \frac{(1-p)N}{(1-p-pc)^2(1-q_1)}\|y_1-\hat{y}\|, \quad c=\frac{q_1}{1-q_1}. \tag{37}$$

We also have that $y_1(n)-\hat{y}(n)$ is the unique bounded solution on Z of the equation

$$u(n+1) = A(n)u(n)+g(n, x_1(n), y_1(n))-g(n, \hat{x}(n), \hat{y}(n)) = A(n)u(n)+\bar{h}(n)$$

which is

$$y_1(n)-\hat{y}(n) = u(n) = \sum_{s=-\infty}^{n-1} X(n)PX^{-1}(s+1)\bar{h}(s)-\sum_{s=n}^{\infty} X(n)(I-P)X^{-1}(s+1)\bar{h}(s).$$

So from (7) and since $\|y_1-\hat{y}\|<\infty$ and $\|x_1-\hat{x}\|<\infty$ we have

$$\|u\| \leqq \frac{2Kq_2p^{-1}}{1-p}(\|\hat{x}-x_1\|+\|u\|).$$

So from (37) and using the same argument to prove that W is a contraction we have that $\|u\|=0$. Hence $y_1(n)=\hat{y}(n)$, $x_1(n)=\hat{x}(n)$, $n\in Z$ and the proof of the uniqueness is completed.

We prove now inequalities (27). Consider (35) where

$$h_2(n) = g(n, x(n), y(n))-g(n, \hat{x}(n), y(n)),$$

$$F(n, u) = g(n, \hat{x}(n), \hat{y}(n)+u)-g(n, \hat{x}(n), \hat{y}(n)).$$

Let

$$\|h_2\|_m = \sup\{|h_2(n)|p^{-(n-m)}, \ n\geqq m\}$$

for fixed $m\in Z$. Since $\|h_2\|_m\leqq q_2\|x-\hat{x}\|_m$ from Lemma 4 we get

$$|\hat{y}(n)-y(n)| \leqq 2K\left(|\hat{y}(m)-y(m)|+\frac{2p^{-1}q_2}{1-p}\|x-\hat{x}\|_m\right)p^{n-m}. \tag{38}$$

From the relation

$$\hat{x}(n) = x(n) - \sum_{s=n}^{\infty} f(s, \hat{x}(s), \hat{y}(s)) - \hat{x}(s) - f(s, x(s), y(s)) + x(s)$$

we have

$$p^{-(n-m)}|\hat{x}(n) - x(n)| \leq \sum_{s=n}^{\infty} q_1 p^{s-n} \|x - \hat{x}\|_m + \sum_{s=n}^{\infty} N p^{s-n} \|y - \hat{y}\|_m.$$

Therefore since $p_1 < 1 - 2p$ we get

$$\|\hat{x} - x\|_m \leq \frac{N}{1 - p - q_1} \|\hat{y} - y\|_m \leq \frac{N}{p} \|\hat{y} - y\|_m. \tag{39}$$

So from (38) and since $p < \frac{1}{2}$ and $N < \frac{p^3}{8Kq_2}$ we have

$$\|\hat{y} - y\|_m \leq 2K|\hat{y}(m) - y(m)| + \frac{1}{2} \|\hat{y} - y\|_m.$$

Therefore we obtain

$$\|\hat{y} - y\|_m \leq 4K|\hat{y}(m) - y(m)|. \tag{40}$$

So from (39) and (40) the inequalities (27) are hold.

It remains to prove the continuity of $\hat{x}(n)$, $\hat{y}(n)$ in (ξ, τ) for all $n, m \in Z$. Let m is fixed and B be a bounded subset of the (ξ, τ) space. Consider the space E of all functions $w(n, \xi, \tau)$ which are continuous in (ξ, τ) and satisfy

$$|w(n, \xi, \tau) - q(n)| \leq \frac{2K\mu}{1 - p^2}, \quad n \in Z,$$

$$\sup_B \sup_{n \geq 0} \{|w(n, \xi, \tau) - y(n, \xi, \tau)| p^{-n}\} < \infty.$$

According to Lemma 3 if $w \in E$ there exists a unique solution $z(n) = z(n, \xi, \tau)$ of

$$x(n+1) = f(n, x(n), w(n, \xi, \tau))$$

which is continuous in B for all $n \in Z$ and satisfies

$$\sup \{|z(n) - x(n, \xi, \tau)| p^{-n}, \ n \in N\} < \infty.$$

From Lemma 5 we have that the equation

$$v(n+1) = A(n)v(n) + g(n, z(n, \xi, \tau), w(n, \xi, \tau))$$

has a unique bounded solution $v(n, \xi, \tau)$. Using the same argument as in Lemma 1 we can easily prove that $v(n, \xi, \tau)$ is continuous in (ξ, τ) for all $n \in Z$. Put

$$W(n, \xi, \tau) = q(n) + v(n, \xi, \tau).$$

Since B is bounded we have $q(0) - y(0, \xi, \tau)$ is bounded in B. Therefore (31) holds and we can prove that W is in E. We also have that W is a contraction on E using the metric

$$d(w_1, w_2) = \sup \{|w_1(n, \xi, \tau) - w_2(n, \xi, \tau)| p^{-n}, \ n \in Z, \ (\xi, \tau) \in B\}.$$

So there exists a unique $\hat{y}\in E$ such that $W(\hat{y})=\hat{y}$. Using the same argument as in the first part of the lemma we find the solution $\hat{x}(n), \hat{y}(n)$ of (1) which is continuous in B for $n\in Z$. Since B is an arbitrary bounded subset of the (ξ, τ) space and m is also arbitrary we have that $\hat{x}(n)$, $\hat{y}(n)$ are continuous in $R^l\times R^k$ for any $n, m\in Z$. Thus the proof of the lemma is completed.

LEMMA 7. *Consider that all conditions of Lemma 6 are hold. Then if* $x(n)=x(n, m, \xi, \tau)$, $y(n)=y(n, m, \xi, \tau)$ *is a solution of* (1) *such that* $x(m)=\xi$, $y(m)=\tau$ *there exists a unique solution* $\hat{x}(n)=\hat{x}(n, m, \xi, \tau)$, $\hat{y}(n)=\hat{y}(n, m, \xi, \tau)$ *of* (1) *such that* $\sup\{|\hat{y}(n)|, n\in N\}<\infty$,

$$\sup\{|\hat{x}(n)-x(n)|p^n,\ n\leqq 0\}<\infty \quad \textit{and} \quad \sup\{|\hat{y}(n)-y(n)|,\ n\leqq 0\}<\infty.$$

We also have for $m\geqq n$

$$|\hat{x}(n)-x(n)|\leqq \frac{4NK}{p}|\hat{y}(m)-y(m)|p^{m-n}$$

$$|\hat{y}(n)-y(n)|\leqq 4K|\hat{y}(m)-y(m)|p^{m-n}$$

and the functions $x(n), y(n)$ *are continuous in* (ξ, τ) *for all* $n, m\in Z$.

Using Lemma 6 and the same argument as in the Remark [3, p. 252] we can easily prove the lemma.

Now we prove the main result of this paper.

PROOF of the proposition. Let $x(n), y(n)$ be a solution of (1) such that $x(m)=\xi$, $y(m)=\tau$. By Lemma 1 this solution is defined and it is unique. By Lemma 7 there exists a unique solution $\hat{x}(n)$, $\hat{y}(n)$ of (1) such that

$$\|x-\hat{x}\|^-<\infty,\ \sup\{|y(n)-\hat{y}(n)|, n\leqq 0\}<\infty \ \text{ and } \ \sup\{|\hat{y}(n)|,\ n\in N\}<\infty.$$

We have $\frac{1}{4}<1-2p$ since $p<e^{-1}$. Hence we have $q_1<1-2p$ since $q_1<\frac{1}{4}$. So from Lemma 6 if we take $q(n)=0$ there exists a unique solution $z(n)$, $w_1(n)$ of (1) such that $\|z-\hat{x}\|^+<\infty$, $\sup\{|w_1(n)|, n\in Z\}<\infty$.By Lemma 5 the equation

$$v(n+1)=A(n)v(n)+g(n, x(n), y(n))$$

has a unique bounded solution $v(n)$ such that $|v(n)|\leqq\frac{2K\mu}{1-p^2}$, $n\in Z$. Therefore the function $w(n)=y(n)-v(n)$ is the unique solution of (2) such that

$$|w(n)-y(n)|\leqq\frac{2K\mu}{1-p^2},\quad n\in Z. \tag{41}$$

Take $H_1(m, \xi, \tau)=z(m)$, $H_2(m, \xi, \tau)=w(m)$. From Lemma 6 we have that $z(m)$ is a continuous function of $(\hat{x}(m), \hat{y}(m))$ and $\hat{x}(m), \hat{y}(m)$ are continuous functions of (ξ, τ). So $z(m)$ is continuous in (ξ, τ). We also have that $w(m)$ is continuous in (ξ, τ) and from (41) we get

$$|H_2(m, \xi, \tau)-\tau|\leqq\frac{2K\mu}{1-p^2}.$$

It holds

$$H_1(n, x(n), y(n)) = z(n), \quad H_2(n, x(n), y(n)) = w(n), \quad n\in Z.$$

Let now $z(n)$, $w(n)$ be a solution of (3) such that $z(m)=\xi$, $w(m)=\tau$. (The relation $2p+\frac{1}{p^2}>e^2$ holds since $p<e^{-1}$. So from $e^2>2+e$ we get $2p+\frac{1}{p^2}>2+e$, which implies that $1>p^2(2+e-2p)$. Furthermore we have $\frac{1-p^2e}{2}>p^2(1-p)$ and, since $q_2<\frac{p^2(1-p)}{4K}$, we find $q_2<\frac{1-p^2e}{8K}$. Therefore from Proposition 1 [7, p. 462] there exists a summable manifold $v(n, x)$ for (1)). We apply Lemma 6 for the solution $z(n)$, $v(n, z(n))$ of (1) taking $q(n)=X(n)PX^{-1}(n)w(n)$ and we find the unique solution $\bar{z}(n)$, $\bar{w}(n)$ of (1) such that

$$\|\bar{z}-z\|^+ <\infty, \quad \sup\{|\bar{w}(n)-X(n)PX^{-1}(n)w(n)|, \ n\in Z\} <\infty.$$

Now using the same argument as in Lemma 6 for the solution $\bar{z}(n), \bar{w}(n)$ taking $q(n)=w(n)$ we find the unique solution $x(n)$, $y(n)$ of (1) such that

$$\|x-\bar{z}\|^- <\infty, \quad \sup\{|y(n)-w(n)|, \ n\in Z\} <\infty.$$

Take $L_1(m, \xi, \tau)=x(m)$, $L_2(m, \xi, \tau)=y(m)$. Then we obtain

$$L_1(n, z(n), w(n)) = x(n), \quad L_2(n, z(n), w(n)) = y(n).$$

Using the same argument as in [3, p. 254—255] we can easily prove that $L_n\circ H_n = H_n\circ L_n =$ the identity for any $n\in Z$. Thus the proof of the proposition is completed.

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СЕМЕЙСТВА ЭКСПОНЕНТ И УПРАВЛЯЕМОСТЬ ПРЯМОУГОЛЬНОЙ МЕМБРАНЫ

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Рассмотрена задача управления колебаниями однородной прямоугольной мембраны. Управление входит в граничные условия типа Неймана. Изучен вопрос и существовании, единственности и гладкости решения; при этом указаны пространства Соболева, в которых полученный результат неулучшаем. Далее доказано, что для одномерных управлений[1] эта система не является приближенно упрявляемой ни для какого конечного времени. Методом исследования является сведение задачи управления к проблеме моментов относительно семейства экспонент. Доказаны новые свойства этих семейств.

1. Пусть $a, b, T\in \mathbf{R}_+$, $\Omega=(0, a)\times(0, b)$, $\{x, y\}\in\Omega$, $t\in[0, T]$, $\Gamma=\partial\Omega$, $\Gamma_1=(0, a)\times\{0\}$, $\Gamma_0=\Gamma\setminus\Gamma_1$, $Q=\Omega\times(0, T)$, $\Sigma=\Gamma\times(0, T)$, $\Sigma_1=\Gamma_1\times(0, T)$, $\Sigma_0=\Gamma_0\times(0, T)$.

Рассматриваемая система описывается следующей начально-краевой задачей

$$\frac{\partial^2 z}{\partial t^2}=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2} \text{ в } Q \tag{1}$$
$$\left.\frac{\partial z}{\partial \nu}\right|_{\Sigma_1}=u, \qquad z|_{\Sigma_0}\equiv 0$$

здесь ν—внешняя нормаль к Γ,

$$u\in\mathscr{U}:=L^2(\Sigma_1)=L^2(0, T;\ L^2(\Gamma_1)). \tag{1'}$$

Ниже мы рассмотрим также случай (1″):

$$u(s, t)=v(t)\delta(s-s_0),\quad v\in L^2(0, T),\quad s_0\in(0, a). \tag{1''}$$

Начальные условия возьмем нулевым (для гиперболических систем в силу обратимости времени это не ограничивает общности решения проблемы упрявляемости):

$$z=z_t=0 \quad \text{при} \quad t=0. \tag{2}$$

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[1] Распределенных на границе или точечных.

2. С целью построения решения начально-краевой задачи (1), (2) и исследования управляемости рассмотрим следующую задачу на собственные значения:

(3) $$-\Delta\varphi = \lambda\varphi \text{ в } \Omega, \qquad \varphi|_{\Gamma_0} = 0, \qquad \left.\frac{\partial\varphi}{\partial\nu}\right|_{\Gamma_1} = 0.$$

Собственные значения этой задачи

(4) $$\lambda_{mn} = \left(\frac{\pi}{a}m\right)^2 + \left(\frac{\pi}{b}\left(n-\frac{1}{2}\right)\right)^2; \quad m, n\in\mathbf{N},$$

а собственные функции

(5) $$\varphi_{mn}(x, y) = \frac{2}{\sqrt{ab}}\sin\frac{\pi}{a}mx\cos\frac{\pi}{b}\left(n-\frac{1}{2}\right)y$$

образуют ортонормированный базис в $L^2(\Omega)$.

Для дальнейших построений удобно ввести следующие пространства. Положим

$$\mathbf{K} := \mathbf{Z}\diagdown\{0\}, \quad \omega_{mk} := (\operatorname{sgn} k)\sqrt{\lambda_{m|k|}}, \quad (m\in\mathbf{N},\ k\in\mathbf{K}),$$

$$l_r^2 := \left\{c = \{c_{mk}\}\colon c_{mk}\in\mathbf{C},\ m\in\mathbf{N},\ k\in\mathbf{K},\ \|c\|_r^2 := \sum_{m,k}|c_{mk}|^2|\omega_{mk}|^{2r} < \infty\right\} \quad (r\in\mathbf{R}),$$

$$W_r := \left\{f\,|\,f(x, y) = \sum_{m,n} f_{mn}\varphi_{mn}(x, y);\ m, n\in\mathbf{N},\ \|f\|_{W_r}^2 := \sum_{m,n}|f_{mn}|^2|\omega_{mn}|^{2r} < \infty\right\}.$$

При $r\geq 0$ ряд $\sum\limits_{m,n} f_{mn}\varphi_{mn}$ понимается в смысле сходимости в $L^2(\Omega)$, при $r<0$ — как обобщенная функция. Справедливы соотнощения (связь W_r с пространствами Соболева)

(6) $$\mathring{H}^r(\Omega)\subset W_r\subset H^r(\Omega), \qquad r\geq 0,$$

(7) $$W_r = H^r(\Omega), \qquad -\frac{1}{2} < r < \frac{1}{2}.$$

Действительно, для четных положительных r (6) следует из равенства $\operatorname{Dom}(-\Delta)^{r/2} = W_r$ $(r=2p,\ p\in\mathbf{N})$. Теперь для произвольных $r\geq 0$ (6) получается с помощью интерполяции [1, гл. 1]. Поскольку $H^r(\Omega) = \mathring{H}^r(\Omega)$ при $0\leq r<\frac{1}{2}$ [1, гл. 1] и $W_{-r} = (W_r)'$ ($(W_r)'$-двойственное к W_r), то верно и (7).

Теорема 1. *Пусть*

$$f = \sum_{m,n} f_{mn}\varphi_{mn}, \quad g = \sum_{m,n} g_{mn}\varphi_{mn}, \quad c_{mk} = i\omega_{mk}f_{m|k|} + g_{m|k|} \quad m, n\in\mathbf{N},\ k\in\mathbf{K}.$$

Тогда $\{f, g\}\in W_{r+1}\oplus W_r =: \mathscr{H}_r$ *тогда и только тогда* $c = \{c_{mk}\}\in l_r^2$. *При этом отображение* $\{f, g\}\to c$ *есть изоморфизм пространств* $\mathscr{H}_r$ *и* l_r^2.

Доказательство непосредственно следует из определения пространств $\mathscr{H}_r$ и l_r^2, ортонормированной базисности $\{\varphi_{mn}\}$ в $L^2(\Omega)$ и соотношений

$$f_{mn} = \frac{c_{mn} - c_{m,-n}}{2i\omega_{mn}}, \quad g_{mn} = \frac{c_{mn} + c_{m,-n}}{2}.$$

Аналогичное этой теореме утверждение доказано в [7] и в [8].

3. Теперь мы дадим корректное определение решения начально-краевой задачи (1), (2) и докажем теоремы существования, единственности и регулярности для случая (1′) и (1″). В духе метода транспонирования [1] возьмем $\psi \in L^2(0, T; W_r)$ (r достаточно большое, фиксированное) и рассмотрим вспомогательную начально-краевую задачу

$$\begin{cases} w_{tt} - \Delta w = \psi \quad \text{в } Q \\ w|_{\Sigma_0} = 0, \quad \left.\dfrac{\partial w}{\partial \nu}\right|_{\Sigma_1} = 0 \\ w = w_t = 0 \quad \text{при} \quad t = T. \end{cases} \tag{8}$$

Отображение $\psi \to \int\limits_{\Sigma_1} uw\, ds\, dt$ ($u \in \mathscr{U}$ фиксированное) при достаточно большом r является, очевидно, линейным непрерывным функционалом над $L^2(0, T; W_r)$. Следовательно, существует единственный $z \in (L^2(0, T; W_r))' = L^2(0, T; W_{-r})$ такой, что

$$\int\limits_Q z\psi\, dx\, dy\, dt = \int\limits_{\Sigma_1} uw\, ds\, dt \qquad (s \in \Gamma_1). \tag{9}$$

Поэтому в качестве решения задачи (1), (2) мы принимаем такой $z \in L^2(0, T; W_{-r})$, который удовлетворяет интегральному тождеству

$$\int\limits_Q z(w_{tt} - \Delta w)\, dx\, dy\, dt = \int\limits_{\Sigma_1} uw\, ds\, dt \tag{10}$$

для любого $\psi \in L^2(0, T; W_r)$ (w определяется по ψ с помощью (8)).

Для случая (1″): $u(s, t) = v(t)\delta(s - s_0)$ все предыдущие рассуждения остаются в силе (r надо взять больше, чем раньше); интегральное тождество примет вид

$$\int\limits_Q z(w_{tt} - \Delta w)\, dx\, dy\, dt = \int\limits_0^T v(t) w(s_0, t)\, dt. \tag{11}$$

Известно [3, гл. 4], что для случая (1′), (10) можно взять $r = 0$, т. е. существует единственное решение $z \in L^2(Q)$. Можно также указать наименьшее r для случая (1″), (11). Ниже мы докажем точные теоремы, которые интересны самостоятельно и важны для исследования управляемости.

Будем искать решение задачи (1), (1′), (2) (или что то же самое (10)) в виде

$$z(x, t) = \sum_{m,n} z_{mn}(t)\varphi_{mn}(x, y) \qquad (m, n \in \mathbf{N}). \tag{12}$$

Для определения коэффициентов $z_{mn}(t)$ положим в тождестве (10) $w(x, y, t) = \varphi_{mn}(x, y)f(t)$, где f — произвольная функция из $C^2[0, T]$ такая, что $f(T) = \dot{f}(T) = 0$. Получим

$$\int_0^T (z_{mn}\ddot{f} + \lambda_{mn} z_{mn} f)\, dt = \int_0^T u_m f\, dt,$$

где

$$u_m(t) = \int_{\Gamma_1} u(s, t)\varphi_{mn}(s)\, ds = \frac{2}{\sqrt{ab}} \int_0^a u(x, t) \sin\frac{\pi}{a} mx\, dx.$$

Используя произвольности функции f имеем

$$z_{mn}(t) = \int_0^t \frac{\sin\sqrt{\lambda_{mn}}(t-\tau)}{\sqrt{\lambda_{mn}}} u_m(\tau)\, d\tau,$$

(13)

$$\dot{z}_{mn}(t) = \int_0^t \cos\sqrt{\lambda_{mn}}(t-\tau) u_m(\tau)\, d\tau \qquad (m, n \in \mathbf{N},\ t \in [0, T]).$$

Положим

(14) $$\zeta_{mk}(t) := i\omega_{mk} z_{m,|k|}(t) + \dot{z}_{m,|k|}(t) \qquad (m \in \mathbf{N},\ k \in \mathbf{K}).$$

Равенства (13) можно записать в виде

(15) $$\zeta_{mk}(t) = \int_0^T e^{i\omega_{mk}(t-\tau)} u_m(\tau)\, d\tau \qquad (m \in \mathbf{N},\ k \in \mathbf{K}).$$

По Теореме 1

(15′) $$\|\{z(\cdot, t), z_t(\cdot, t)\}\|_{\mathscr{H}_r} \asymp \|\zeta(t)\|_{l_r^2}.$$

Теорема 2. (a) *Для любой* $u \in \mathscr{U} = L^2(0, T; L^2(\Gamma_1))$ *существует единственное решение* z *задачи* (1), (1′), (2) *такое, что*

$$\{z, \dot{z}_t\} \in C([0, T]; \mathscr{H}), \quad \mathscr{H} := H^{3/4}(\Omega) \oplus H^{-1/4}(\Omega).$$

(b) *Для любой* $p > -1/4$ *существует* $u \in \mathscr{U}$ *такое, что*

$$\{z(\cdot, T), \dot{z}_t(\cdot, T)\} \notin \mathscr{H}_p,$$

при всех $T > 0$.

Теорема 3. *Для любой* $v \in L^2(0, T)$ *существует единственное решение задачи* (1), (1″), (2) *такое, что*

$$\{z, z_t\} \in C([0, T]; \mathscr{H}_r) \quad \text{при} \quad r < -3/4.$$

Доказательству Теорем 2, 3 предпошлем ряд лемм.

4. Лемма 1. *Константа Карлесона* δ_m *множества* $\{v_{mk}\colon k \in \mathbf{K}\}$ *(m-фиксировано),* $v_{mk} := \omega_{mk} + i/2$, *удовлетворяет оценке* $\log\frac{1}{\delta_m} \prec \sqrt{m}$.

Напомним, (см. напр. [5]) что константой Карлесона счетного множества $\sigma \subset \mathbf{C}_+ := \{\nu | \operatorname{Im} \nu > 0\}$ называется величина

$$\delta(\sigma) := \inf_{\nu \in \sigma} \prod_{\substack{\mu \in \sigma \\ \mu \neq \nu}} \left| \frac{\nu - \mu}{\nu - \bar{\mu}} \right|.$$

Доказательство Леммы 1 приведено в конце работы (Приложение).

Лемма 2. *Для любых* T_1, T_2 $(0 \le T_1 < T_2 < \infty)$ *и любой* $f \in L^2(T_1, T_2)$

$$\sum_{k \in \mathbf{K}} |(f, e^{i\omega_{mk}t})_{L^2(T_1, T_2)}|^2 \prec \sqrt{m}\, \|f\|^2_{L^2(T_1, T_2)}, \quad (m \in \mathbf{N}). \tag{16}$$

Доказательство. Известно [5], что для счетного множества $\sigma \subset \mathbf{C}_+$ и любой $f \in L^2(0, \infty)$ справедлива оценка

$$\sum_{\mu \in \sigma} \left|(f, e^{i\mu t})_{L^2(0, \infty)} \sqrt{2 \operatorname{Im} \mu}\right|^2 \le \left(1 + 64 \log \frac{1}{\delta(\sigma)}\right) \|f\|^2_{L^2(0, \infty)}.$$

Применим этот результат к $\sigma = \{\nu_{mk} : m \in \mathbf{N},\ k \in \mathbf{K}\}$ и используем Лемму 1. Продолжая f нулем с $[T_1, T_2]$ на $(0, \infty)$, получим

$$\sum_{k \in \mathbf{K}} |(f, e^{i\nu_{mk}t})_{L^2(T_1, T_2)}|^2 \prec \sqrt{m}\, \|f\|^2_{L^2(T_1, T_2)}.$$

Так как отображение $f(t) \to f(t) e^{ct}$ является изоморфизмом $L^2(T_1, T_2)$, то отсюда верно и (16).

Доказательство Теоремы 2, пункт (a). В силу (15) $|\zeta_{mk}(t)| = |(u_m, e^{i\omega_{mk}\tau})_{L^2(0,t)}|$. Поэтому

$$\sum_{m,k} |\omega_{mk}|^{-1/2} |\zeta_{mk}(t)|^2 = \left(\|\zeta(t)\|^2_{l^2_{-1/4}}\right) = \sum_{m=1}^{\infty} \sum_{k \in \mathbf{K}} |\omega_{mk}|^{-1/2} |(u_m, e^{i\omega_{mk}\tau})_{L^2(0,t)}|^2.$$

Из очевидного неравенства $|\omega_{mk}|^{-1/2} \prec 1/\sqrt{m}$ и Леммы 2 имеем

$$\|\zeta(t)\|^2_{-1/4} \prec \sum_{m=1}^{\infty} \|u_m\|^2_{L^2(0,t)} = \frac{2}{b} \|u\|^2_{\mathscr{U}}.$$

Поэтому в силу (15′) для любого $t \ge 0$

$$\{z(\cdot, t),\ z_t(\cdot, t)\} \in \mathscr{H}_{-1/4} \subset \mathscr{H}$$

(в силу (6), (7)). Покажем непрерывность $\{z(\cdot, t), z_t(\cdot, t)\}$ по t в норме $\mathscr{H}_{-1/4}$ или, что то же самое, непрерывность $\zeta(t)$ в норме $l^2_{-1/4}$:

$$\|\zeta(t+h) - \zeta(t)\|^2_{-1/4} = \sum_{m,k} |\omega_{mk}|^{-1/2} \left| \int_0^{t+h} e^{i\omega_{mk}(t+h-\tau)} u_m(\tau)\, d\tau - \int_0^t e^{i\omega_{mk}(t-\tau)} u_m(\tau)\, d\tau \right|^2.$$

Далее,

$$\int_0^{t+h} \ldots - \int_0^t \ldots = (e^{i\omega_{mk}h} - 1) e^{i\omega_{mk}t} (u_m, e^{i\omega_{mk}\tau})_{L^2(0,t)} + e^{i\omega_{mk}(t+h)} (u_m, e^{i\omega_{mk}\tau})_{L^2(t, t+h)}.$$

Следовательно

$$\|\zeta(t+h)-\zeta(t)\|^2 \leqq$$

$$\leqq 2\sum_{m,k}|\omega_{mk}|^{-1/2}\{|(u_m, e^{i\omega_{mk}\tau})_{L^2(0,t)}|^2|e^{i\omega_{mk}h}-1|^2+|(u_m, e^{i\omega_{mk}\tau})_{L^2(t,t+h)}|^2\}.$$

Снова используя Лемму 2, аналогично доказанному выше имеем

$$\sum_{m,k}|\omega_{mk}|^{-1/2}|(u_m, e^{i\omega_{mk}\tau})_{L^2(t,t+h)}|^2 \prec \|u\|^2_{L^2(t,t+h;\, L^2(\Gamma_1))}.$$

$$\sum_{m,k}|\omega_{mk}|^{-1/2}|(u_m, e^{i\omega_{mk}\tau})_{L^2(0,t)}|^2\cdot|e^{i\omega_{mk}h}-1|^2=\sum_{m=1}^{N}\sum_{|k|\leqq N}\ldots+\sum_{m=N+1}^{\infty}\sum_{|k|>N}\ldots,$$

где $N\in\mathbf{N}$ выбрано так, что

$$\sum_{m=N+1}^{\infty}\sum_{|k|>N}|\omega_{mk}|^{-1/2}|(u_m, e^{i\omega_{mk}\tau})_{L^2(0,t)}|^2<\varepsilon/12$$

($\varepsilon>0$ произвольное).

Далее выберем h таким, что

$$\begin{cases}\displaystyle\sum_{m=1}^{N}\sum_{|k|\leqq N}|\omega_{mk}|^{-1/2}|(u_m, e^{-i\omega_{mk}\tau})_{L^2(0,t)}|^2|e^{i\omega_{mk}h}-1|^2<\frac{\varepsilon}{3},\\ \|u\|^2_{L^2(t,t+h;\, L^2(\Gamma_1))}<\dfrac{\varepsilon}{3}.\end{cases}$$

Тогда $\|\zeta(t+h)-\zeta(t)\|^2_{-1/4}<\varepsilon$. Тем самым утверждение (a) Теоремы 2 доказано.

Для доказательства пункта (b) нам понадобится следующая

Лемма 3. *Для любого $T>0$ найдутся положительные постоянные α, β зависящие только от a, b и T такие что*

$$\sum_{0<n<\alpha\sqrt{m}}|(e^{i(\pi/a)mt}, e^{i\omega_{mn}t})_{L^2(0,T)}|^2\geqq\beta\sqrt{m},\quad (m\in\mathbf{N}).$$

Доказательство дано в конце работы (Приложение).

Доказательство пункта (b) Теоремы 2. Выберем u таким, что

$$u_m(t)=\alpha_m e^{i(\pi/a)mt},\qquad \sum_{m=1}^{\infty}|\alpha_m|^2<\infty.$$

Тогда, очевидно, $u\in\mathscr{U}$. При этом

$$\|z(\cdot,T),\, z_t(\cdot,T)\|^2_{\mathscr{H}_p}\asymp\|\zeta(T)\|_p^2=\sum_{m,k}|\omega_{mk}|^{2p}|\zeta_{mk}(T)|^2=$$

$$=\sum_{m=1}^{\infty}\sum_{k\in\mathbf{K}}|\omega_{mk}|^{2p}|\alpha_m|^2|(e^{i(\pi/a)mt}, e^{i\omega_{mk}t})_{L^2(0,T)}|^2.$$

Пусть $p\in\left(-\frac{1}{4},\ 0\right]$. Тогда $|\omega_{mk}|^{2p}\succ m^{2p}$ и в силу Леммы 3

$$\|\zeta(T)\|_p^2\succ\sum_{m=1}^{\infty}\sum_{0<n<\alpha\sqrt{m}}|\omega_{mn}|^{2p}|\alpha_m|^2|(e^{i(\pi/a)mt},\ e^{i\omega_{mn}t})|^2\succ\sum_{m=1}^{\infty}m^{2p}|\alpha_m|^2\sqrt{m}.$$

Числа α_m можно выбрать так, что $\sum_{m=1}^{\infty}|\alpha_m|^2m^{1/2+2p}=\infty$. Тогда получим: $\{z(\cdot,T),\ z_t(\cdot,T)\}\notin\mathscr{H}_p$. Так как с ростом p пространства $\mathscr{H}_p$ сужаются, утверждение верно при всех $p>-\frac{1}{4}$. Теорема 2 доказана.

Замечание 1. Теорема 2 справедливо и для случая, когда $\Gamma_1=\Gamma$, $\Gamma_0=\Phi$ — управление на всей границе. Для доказательства надо разбить задачу на сумму $4-x$, в каждой из которых управление действует на одной стороне. Незначительное изменение вида $\varphi_{mn}(x)$ и λ_{mn} не влияет на ход доказательства и справедливость полученных результатов.

Замечание 2. Эта теорема усиливает результат [4], где доказано, что $\forall T>0$, $\forall\varepsilon>0$: $z\in H^{3/4-\varepsilon}(\Omega)$.

Доказательство Теоремы 3. Если $u(x,t)=v(t)\delta(x-x_0)$, $0<x_0<a$, то

$$u_m(t)=\frac{2}{\sqrt{ab}}\sin\frac{\pi m}{a}x_0v(t)=:\gamma_mv(t),$$

$$\zeta_{mk}(t)=\gamma_m\int_0^te^{i\omega_{mk}(t-\tau)}v(\tau)\,d\tau.$$

Поэтому

$$|\zeta_{mk}(t)|\prec|(v,\ e^{i\omega_{mk}\tau})_{L^2(0,t)}|.$$

Следовательно

$$\sum_{m,k}|\omega_{mk}|^{2r}|\zeta_{mk}|^2\prec\sum_{m=1}^{\infty}\sum_{k\in\mathbf{K}}|\omega_{mk}|^{2r}|(v,\ e^{i\omega_{mk}\tau})_{L^2(0,t)}|^2.$$

Пусть $r\leqq0$. Тогда $|\omega_{mk}|^{2r}\prec m^{2r}$ и воспользовавшись Леммой 2, получаем $\|\zeta(t)\|_{l_r^2}^2\prec\sum_{m=1}^{\infty}m^{2r}\sqrt{m}\,\|v\|_{L^2(0,t)}^2$, т.е. $\|\zeta(t)\|_r^2\prec\|v\|_{L^2(0,t)}$ при $2r+1/2<-1$ (т. е. $r<-3/4$). Значит для любого $t\leqq0$

$$\{z(\cdot,t),\ z_t(\cdot,t)\}\in\mathscr{H}_r=W_{r+1}\oplus W_r\quad\text{при}\quad r<-3/4.$$

Доказательство непрерывности в норме $\mathscr{H}_r$ по t проводится так же, как и в Теореме 2. Теорема 3 доказана.

5. Перейдем к изучению проблемы управляемости. Мы здесь ограничимся рассмотрением системы (1) при одномерных управлениях вида

(17) $$u(x,t)=b(x)v(t),\quad b\in L^2(0,a)\text{—фиксированный}$$

и

(18) $$u(x, t) =: = \delta(x-x_0)v(t), \quad x_0 \in (0, a);$$

управление $v \in L^2(0, T)$.

Введем множество достижимости: множество конечных состояний системы (1), (2), когда управление v пробегает все $L^2(0, T)$:

$$R(T) := \{\{z(\cdot, T), z_t(\cdot, T)\}: v \in L^2(0, T)\}.$$

Мы доказали, что для случая (17) $R(T) \subset \mathscr{H}_{-1/4}$, для случая (18) $R(T) \subset \mathscr{H}_r$, $r < -3/4$.

Систему (1), (17) назовем приближенно управляемой за время T, если $\overline{R(T)} = \mathscr{H}_{-1/4}$ (замыкание по норме $\mathscr{H}_{-1/4}$).

Аналогично для системы (1), (18): $\overline{R(T)} = \mathscr{H}_r$, $r < -3/4$ (замыкание по норме $\mathscr{H}_r$). Эти определения согласуются с общепринятыми, и корректны в силу Теорем 2, 3.

Обозначим через $\hat{R}(T)$ множество векторов $\zeta(T)$, соответствующее $R(T)$. В силу Теоремы 1 приближенную управляемость можно переписать в виде

$$\overline{\hat{R}(T)} = l^2_{-1/4}$$

для (1), (17);

$$\overline{\hat{R}(T)} = l^2_r, \qquad r < -\frac{3}{4},$$

для (1), (18).

Коэффиценты $\zeta_{mk}(T)$ определяются формулами (см. (15)):

(19) $$\zeta_{mk}(T) = \beta_m \int_0^T e^{i\omega_{mk}(T-t)} v(t)\, dt, \quad \beta_m := \frac{2}{\sqrt{ab}} \int_0^a b(x) \sin \frac{\pi}{a} mx\, dx$$

для (1), (17);

(20) $$\zeta_{mk}(T) = \gamma_m \int_0^T e^{i\omega_{mk}(T-t)} v(t)\, dt$$

для (1), (18). Чтобы избежать повторений мы будем проводить дальнейшие рассмотрения, в основном, для случая (1), (17). Положим

$$c_{mk} := |\omega_{mk}|^{-1/4} \zeta_{mk}(T) e^{-i\omega_{mk}T}$$

и запишем равенства (19) в виде

(21) $$c_{mk} = (v, e_{mk})_{L^2(0,T)}, \quad m \in \mathbf{N}, \quad k \in \mathbf{K},$$

где

$$e_{mk}(t) := \beta_m e^{i\omega_{mk}t} |\omega_{mk}|^{-1/4}.$$

Множество векторов $c = \{c_{mk}\}$, соответствующее $\hat{R}(T)$, обозначим $\tilde{R}(T)$. Очевидно, что $\tilde{R}(T) \subset l^2$ и $\overline{\hat{R}(T)} = l^2_{-1/4}$ тогда и только тогда $\overline{\tilde{R}(T)} = l^2$. Семейство векторов $\{e_{mk}\}$ ($m \in \mathbf{N}$, $k \in \mathbf{K}$) обозначим через $\mathscr{E}$. Будем говорить, что семейство $\mathscr{E}$ слабо ω-линейно независимо в $L^2(0, T)$ и писать $\mathscr{E} \in (w-\omega)$, если из условий

$$a=\{a_{mk}\}\in l^2,\quad \sum_{m=1}^{R}\sum_{|k|\leq R}(f, a_{mk}e_{mk})_{L^2(0,T)}\to 0$$

при $R\to\infty$ для любой $f\in L^2(0, T)$ следует, что $a=0$.

Лемма 4. *Для системы* (1), (17) $\overline{R(T)}=\mathscr{H}_{-1/4}$ *тогда и только тогда, когда* $\mathscr{E}\in(w-\omega)$.

Доказательство. Как было сказано выше $\overline{R(T)}=\mathscr{H}_{-1/4}\Leftrightarrow\overline{\tilde{R}(T)}=l^2$. Пусть $\mathscr{E}\notin(w-\omega)$, т. е. существует $a\in l^2$, $a\not\equiv 0$ такой, что

$$\sum_{m=1}^{R}\sum_{|k|\leq R}(f, a_{mk}e_{mk})_{L^2(0,T)}\to 0\quad\text{при}\quad R\to\infty\quad(\forall f\in L^2(0,T)),$$

т. е.

$$\sum_{m=1}^{R}\sum_{|k|\leq R}\bar{a}_{mk}(f, e_{mk})_{L^2(0,T)}\to 0\qquad(R\to\infty).$$

Поскольку $\{(f, e_{mk})\}\in l^2$ $\forall f\in L^2(0, T)$, то имеем $\sum_{m,k}\bar{a}_{mk}(f, e_{mk})=0$, т. е. $(c, a)_{l^2}=0$ $\forall f\in L^2(0, T)$. Иными словами $a\perp\tilde{R}(T)$. Приведенные утверждения легко обратимы. Лемма 4 доказана.

Сейчас мы займемся исследованием свойства $(w-\omega)$ семейства $\mathscr{E}$. Точнее говоря мы покажем, что для любого $T\in(0,\infty)$, $\mathscr{E}\notin(w-\omega)$ и поэтому $\overline{R(T)}\neq\neq\mathscr{H}_{-1/4}$. Для этого нам понадобится обобщение результата Д. Л. Русселя [6].

6. Предложение 1 [6]. *Пусть семейство* $\{e^{i\mu t}\}$, $\mu\in\sigma\subset\mathbf{C}$, *образует базис Рисса в* $L^2(0, T)$; $\sigma_s=\{\lambda_j\}_{j=1}^s\subset\mathbf{C}$, $\lambda_j\neq\lambda_k$, *при* $j\neq k$, $\sigma_s\cap\sigma=\emptyset$. *Тогда семейство* $\{\alpha_\mu e^{i\mu t}\}$, $\mu\in\sigma\cup\sigma_s$ *образует базис Рисса в* $H^s(0, T)$. *Здесь* α_μ — *нормировочные константы, чтобы сделать семейство почти нормированным в* $H^s(0, T)$; *можно взять* $\alpha_\mu=(1+|\mu|^2)^{-s/2}$.

Предложение 2 [7], [8]. *Пусть* Y_n, $n\in\mathbf{N}$*-семейство* N_n*-мерных подпространств* $L^2(0, T)$ *вида*

$$Y_n=\bigvee_{m=1}^{N_n}\exp(i\mu_{mn}t),\qquad N_n<\infty$$

($\vee:=$ *замыкание линейной оболочки*); Y_0-s*-мерное подпространство* $Y_0=\bigvee_{j=1}^{s}\exp(i\mu_j t)$, *причем*

$$\{\mu_j\}_{j=1}^s\cap\{\mu_{mn}\}_{n=1,m=1}^{\infty\ \ N_n}=\emptyset,\quad \mu_j\neq\mu_k\quad(j\neq k).$$

Если семейство $\{Y_n\}_{n=1}^\infty$ *образует базис Рисса из подпространств в* $L^2(0, T)$, *то* $Y_0\cup\{Y_n\}_{n=1}^\infty$ — *образует базис Рисса в* $H^s(0, T)$.

Напомним, что базисом Рисса из подпространств называется полное семейство подпространств, изоморфное ортонормированному.

Докажем теперь факт о зависимости семейства $\mathscr{E}$.

Теорема 4. *Для любого* $T>0$ *и любого* $N\in\mathbf{N}$ *существует* $\{c_{mk}\}_{m=1,\,k\in\mathbf{K}}^{\infty}$ *такой, что*

1) $\sum\limits_{m,k} c_{mk}e^{i\omega_{mk}t}=0$ *в* $L^2(0,T)$.

2) $\sum\limits_{m,k} |c_{mk}|^2|\omega_{mk}|^{2N}<\infty$.

3) $c_{mk}=0$ *при* $m\geqq m_0$.

Доказательство. Будем считать, что среди $\{\omega_{mk}\}$ нет общих точек, иначе теорема тривиальна. Это имеет место при $a^2/b^2\notin Q$. Фиксируем $N>0$, $T>0$ и выберем $M\in\mathbf{N}$ таким, что $T\leqq 2bM$. Утверждение теоремы достаточно проверить для $T=2bM$. Вместо ω_{mk} рассмотрим $v_{mk}=\omega_{mk}+i/2$. Если теорема верна для $\{v_{mk}\}$, то она, очевидно, верна и для $\{\omega_{mk}\}$. Положим $\sigma_k:=\{v_{mk}\}_{m=1}^{M}$, $k\in\mathbf{K}$; $\sigma_0:=\{v_{M+1,k}\}_{k=1}^{s}$, где s достаточно велико, например, $s>N+2M$; $Y_k=\bigvee\limits_{v\in\sigma_k} e^{ivt}$ $(k\in\mathbf{Z})$. Через δ_k обозначим константу Карлесона множества σ_k. Нам нужна

Лемма 5. (a) *Семейство подпространств* $\{Y_k\}$, $k\in\mathbf{K}$ *образует базис Рисса в замыкании своей линейной оболочки в* $L^2(0,\infty)$.

(b) *Ортопроектор* P_T *из* $\bigvee\limits_{L_2(0,\infty)}\{Y_k\}_{k\in\mathbf{K}}$ *на* $L^2(0,T)$ *есть изоморфизм.*

(c) *Существует* $c>0$ *такое, что для всех* $k\in\mathbf{Z}$ *и любого* $v\in\sigma_k$ *выполнена оценка* $\delta_k\geqq c|v|^{-M}$.

Лемму 5 мы докажем ниже, а сейчас закончим доказательство теоремы 4.

Из (a) и (b) Леммы 5 следует, что $\{Y_k\}_{k\in\mathbf{K}}$-базис Рисса в $L^2(0,T)$. Тогда в силу предложение 2 $\{Y_k\}_{k\in\mathbf{Z}}$ — базис Рисса в $H^s(0,T)$.

Разложим элемент e^{iv_0t}, $v_0=v_{M+2,1}$, не входящий в $\{Y_k\}_{k\in\mathbf{K}}$ по этому базису:

$$e^{iv_0t}=\sum_{k\in\mathbf{Z}}h_k,\qquad h_k=\sum_{v\in\sigma_k}b_ve^{ivt}.$$

По свойству базиса Рисса [5]

$$\sum_{k\in\mathbf{Z}}\left\|\left(\frac{d}{dt}\right)^s h_k\right\|^2<\infty,\quad\text{где}\quad\left(\frac{d}{dt}\right)^s h_k=\sum_{v\in\sigma_k}(iv)^sb_ve^{ivt}.$$

Известна оценка [5]:

$$\Big\|\sum_{\lambda\in\sigma}c_\lambda\sqrt{\operatorname{Im}\lambda}\,e^{i\lambda t}\Big\|^2_{L^2(0,\infty)}\geqq\frac{\delta^2}{1+64\log\dfrac{1}{\delta}}\sum_{\lambda\in\sigma}|c_\lambda|^2,$$

где δ — константа Карлесона счетного множества σ. Поэтому из (b), (c) Леммы 5 имеем

$$\infty>\sum_{k\in\mathbf{Z}}\Big\|\sum_{v\in\sigma_k}(iv)^sb_ve^{ivt}\Big\|^2_{L^2(0,T)}\asymp\sum_{k\in\mathbf{Z}}\Big\|\sum_{v\in\sigma_k}(iv)^sb_ve^{ivt}\Big\|^2_{L^2(0,\infty)}\geqq$$

$$\geqq\sum_{k\in\mathbf{Z}}\frac{\delta_k^2}{1+64\log\dfrac{1}{\delta_k}}\sum_{v\in\sigma_k}|b_v|^2|v|^{2s}\succ\sum_{k\in\mathbf{Z}}\sum_{v\in\sigma_k}|b_v|^2|v|^{2s-3M}>\sum_{k\in\mathbf{Z}}\sum_{v\in\sigma_k}|b_v|^2|v|^{2N}.$$

Мы видим, что ряд $\sum_{k\in\mathbf{Z}} h_k - e^{iv_0t}$ удовлетворяет всем требованиям Теоремы 4:

1) $\sum_{k\in\mathbf{Z}} h_k - e^{iv_0t} = 0$,

2) коэффициенты разложения по $e^{iv_{mk}t}$ сходятся с весом $|v_{mk}|^{2N}$,

3) в разложении встречаются элементы лишь первых $M+2$ серий, т. е. $m_0 = M+2$.

Осталось проверить Лемму 4.

(a) Положим $S_m := \{v_{mk}\}_{k\in\mathbf{K}}$; $m=1, 2, \ldots, M$. Числа v_{mk} лежат на прямой $\operatorname{Im} z = 1/2$ и отделимы. Поэтому константа Карлесона множеств S_m положительна [5]. Ясно также, что диам $\sigma_k \to 0$ при $|k|\to\infty$. Поэтому (a) следует из теоремы В. И. Васюнина [5].

(b) Известен критерий того, что P_T — изоморфизм [5]. Это условие Макенхоупта:

$$\sup_{I=[a,b]} \frac{1}{|I|}\int_I w\,dx\, \frac{1}{|I|}\int_I w^{-1}\,dx < \infty,$$

где

$$w = |F|^2, \quad F = \text{v.p.} \prod_{k\in\mathbf{K}} \prod_{v\in\sigma_k} (1 - z/v),$$

F — целая функция экспоненциального типа с индикаторой диаграммой ширины $2bM = T$.

Введем, следуя [9] функцию $F_\mu(z) = \cos\sqrt{(zb)^2 - \mu^2}$. При $\mu = \mu_m := \pi m/a$ нули F_μ совпадают с $\{\omega_{mk}\}$, $k\in\mathbf{K}$, и $|F(z-i/2)| \asymp 1$ при $z\in\mathbf{R}$. Поскольку $F(z) = F_{\mu_1}(z-i/2)\ldots F_{\mu_M}(z-i/2)$, то $|F(z)| \asymp 1$ при $z\in\mathbf{R}$ и условие Макенхоупта выполнено.

(c) Из явного вида ω_{mk} имеем $|\omega_{mk} - \omega_{jk}| \succ 1/|k|$. Тогда

$$\left|\frac{v_{mk} - v_{jk}}{v_{mk} - \bar{v}_{jk}}\right|^2 = \frac{|\omega_{mk} - \omega_{jk}|^2}{1 + |\omega_{mk} - \omega_{jk}|^2} \succ \frac{1}{|k|^2} \qquad (m \neq j;\ m, j = 1, 2, \ldots, M).$$

Отсюда

$$\delta_k := \inf_j \prod_{\substack{m=1,\ldots,M \\ m\neq j}} \left|\frac{v_{mk} - v_{jk}}{v_{mk} - \bar{v}_{jk}}\right| \succ |k|^{-M+1} \succ |k|^{-M}.$$

Для одной группы σ_k: $|v| \asymp |k|$ $(v\in\sigma_k)$ поэтому (c) доказано.

7. Применим полученные результаты к исследованию управляемости системы (1)+(17) или (18).

Теорема 5. *Для любого* $T>0$ *система* (1), (17) *и система* (1), (18) *не являются приближенно управляемыми.*

Доказательство. (a) Рассмотрим систему (1), (17). Покажем, что $\mathscr{E} \notin (w-\omega)$. Это очевидно, если хотя бы один коэффициент $\beta_m = 0$. Пусть все $\beta_m \neq 0$. Возьмем произвольное $T>0$ и построим последовательность $\{c_{mk}\}$, удовлетворяющую 1) — 3) Теоремы 4 при $N=1$. Положим $a_{mk} = c_{mk}|\omega_{mk}|^{1/4}/\beta_m$. В силу 2), 3) Теоремы 4 $\{a_{mk}\}\in l^2$, а в силу 1)

$$\sum_{m,k} a_{mk} e_{mk} = \sum_{m,k} c_{mk} |\omega_{mk}|^{1/4} \beta_m^{-1} \beta_m |\omega_{mk}|^{-1/4} e^{i\omega_{mk}t} = 0 \quad \text{в} \quad L^2(0, T).$$

Отсюда следует, что семейство $\mathscr{E}$ $w-\omega$-линейно зависимо (даже ω-линейно зависимо — определение см., например в [12]).

(б) Для системы (1), (18) надо вместо $\mathscr{E}$ рассмотреть семейство $\tilde{\mathscr{E}}=\gamma_m |\omega_{mk}|^r e^{i\omega_{mk}t}$, $r<-3/4$. Далее по $\{c_{mk}\}$, построенной по теореме 4 взять $a_{mk}=c_{mk}|\omega_{mk}|^{-r}\gamma_m^{-1}$ (предполагаем, что все $\gamma_m\neq 0$, т. е. $\sin \pi m\dfrac{x_0}{a}\neq 0$ $\forall m$, иначе $\tilde{\mathscr{E}}$, очевидно, линейно зависима). Получаем ω-линейную зависимость семейства $\tilde{\mathscr{E}}$. Теорема 5 доказана.

Управляемость системы (1) при $u\in\mathscr{U}=L^2(0, T;\ L^2(\Gamma_1))$ $(\Gamma_1=\{0, a\}\times\{0\})$ исследовалась в [9]. Результаты об управляемости системы (1) при разных управлениях есть в [10].

Можно доказать, что при $\beta_m\neq 0$ и отсутствии кратных точек в спектре $\{\lambda_{mn}\}$ для системы (1), (17) $\overline{\bigcup_{T>0} R(T)}=\mathscr{H}_{-1/4}$ (и аналогичный результат верна для системы (1), (18)). Результат такого типа справедлив для гиперболических систем общего вида в произвольных областях.

8. Приложение. Доказательство Леммы 2.

Проверку утверждение будем производить для $\tilde{\omega}_{mk}=\operatorname{sgn} k\sqrt{m^2+k^2}$ и $a=\pi$. Это сделано в целях простоты выкладок и доказательство леммы для ω_{mn} требует незначительных изменений. Положим $f_m(t)=e^{imt}$, $c_k=(f_m, e^{i\tilde{\omega}_{mk}t})_{L^2(0\ T)}$. Тогда, введя $\alpha_{mk}=\tilde{\omega}_{mk}-m$ имеем

$$c_k = \int_0^T e^{i(m-\tilde{\omega}_{mk})t}\,dt = \frac{1}{-i\alpha_{mk}}(1-e^{-i\alpha_{mk}T}) = -e^{-\alpha_{mk}T/2}\cdot T\cdot \sin\alpha_{mk}T/2\big/\alpha_{mk}T/2.$$

Покажем, что при $k<\sqrt{\pi m/T}$ аргумент синуса $\alpha_{mk}T/2$ меньше $\pi/2$. Действительно, пользуясь элементарным неравенством $\sqrt{1+x}\leqq 1+x$ $(x>0)$, получаем

$$\frac{\alpha_{mk}T}{2} = \frac{T}{2} m\left[\sqrt{1+\frac{k^2}{m^2}}-1\right] < \frac{T}{2}\frac{k^2}{m},$$

и при $k<\sqrt{\pi m/T}$:

$$\frac{\alpha_{mk}T}{2} < \frac{\pi}{2}.$$

Поэтому для таких k из элементарного неравенства $\sin x/x>2/\pi$, $x\in(0, \pi/2)$ имеем

$$|c_k|^2 = T^2\left|\frac{\sin\alpha_{mk}T/2}{\alpha_{mk}T/2}\right|^2 \geqq T^2\left(\frac{2}{\pi}\right)^2 =:\gamma.$$

Теперь оценим сумму квадратов коэффициентов

$$\sum_{0<k<\sqrt{\pi m/T}} |c_k|^2 \geqq \gamma \left[\sqrt{\frac{\pi m}{T}} - 1\right] \geqq \beta \sqrt{m}.$$

Лемма 2 доказана.

9. Доказательство леммы 1. Для простоты выкладок будем считать, что $v_{mn} = \frac{i}{2} + \operatorname{sgn} n \sqrt{m^2+n^2}$, где m—фиксированное число из $\mathbf{N}$, $n \in \mathbf{Z} \setminus \{0\} = \mathbf{K}$. Для случая

$$v_{mn} = \frac{i}{2} + \operatorname{sgn} n \sqrt{\left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi}{b}\left[|n| - \frac{1}{2}\right]\right)^2}$$

доказательство требует лишь небольших очевидных изменений.

Постоянную Карлесона δ_m множества $\{v_{mn}\}$, $n \in \mathbf{Z} \setminus \{0\}$, запишем в виде

$$1/\delta_m = \sup_n \prod_{k \neq n} \left|\frac{\bar{v}_{mk} - v_{mn}}{v_{mk} - v_{mn}}\right|.$$

Откуда

$$\log \delta_m^{-2} = \sup_n \sum_{k \neq n} \log \left|\frac{\bar{v}_{mk} - v_{mn}}{v_{mk} - v_{mn}}\right|^2 =$$

$$= \sup_{n \in \mathbf{Z} \setminus \{0\}} \sum_{k \neq n} \log \left[1 + \frac{1}{(\operatorname{sgn} k \sqrt{m^2+k^2} - \operatorname{sgn} k \sqrt{m^2+n^2})^2}\right].$$

Будем для определенности считать, что $n > 0$ и оценим $S_n = \sum_{k \neq n} \log [\ ]$ (очевидно $S_{-n} = S_n$). Представим S_n в виде

$$S_n = \left(\sum_{k=-\infty}^{-1} + \sum_{k=1}^{n-1} + \sum_{k=n+1}^{\infty}\right) \log [\ ] = S_n^1 + S_n^2 + S_n^3,$$

и оценим каждую сумму в правой части отдельно.

1) $S_n^1 < \sum_{k=1}^{\infty} \log\left(1 + \frac{1}{k^2}\right) < \infty$.

2) Случай $n < m$.

а) Тогда запишем S_n^2 в виде

$$S_n^2 = \sum_{p=1}^{n-1} \log \left[1 + \frac{1}{(\sqrt{m^2+n^2} - \sqrt{m^2+(n-p)^2})^2}\right]$$

и проверим, что для некоторого $c > 0$

$$\sqrt{m^2+n^2} - \sqrt{m^2+(n-p)^2} \geqq \frac{cp^2}{m} \qquad (1 \leqq p < n < m).$$

Действительно:

$$\sqrt{m^2+n^2} \geqq \frac{cp^2}{m} + \sqrt{m^2+(n-p)^2} \Leftrightarrow$$

$$\Leftrightarrow m^2+n^2 \geqq \frac{c^2p^4}{m^2} + m^2+n^2-2np+p^2+2\frac{cp^2}{m}\sqrt{m^2+(n-p)^2} \Leftrightarrow$$

$$\Leftrightarrow 2n \geqq \frac{c^2p^3}{m^2}+p+\frac{2cp}{m}\sqrt{m^2+(n-p)^2} \Leftarrow 2n \geqq c^2n+n+2cn\sqrt{2} \Leftrightarrow 1 \geqq c^2+2c\sqrt{2}.$$

Поэтому

$$\sum_{p=1}^{n-1} \log\left[1+\frac{1}{(\sqrt{m^2+n^2}-\sqrt{m^2+(n-p)^2})^2}\right] \leqq \sum_{p=1}^{n-1} \log\left(1+\frac{m^2}{cp^4}\right) <$$

$$< \sum_{p=1}^{\infty} \log\left(1+\frac{m^2}{cp^4}\right) = \int_1^{\infty} \log\left(1+\frac{m^2}{cx^4}\right) dx + O(\log m) \leqq c_1\sqrt{m} \quad (\exists c_1 \in (0,\infty)).$$

Мы заменили сумму интегралом, так как из за монотонности функции $\log x$ имеем

$$\sum_{p=1}^{\infty} \log\left(1+\frac{m^2}{cp^4}\right) - \int_0^{\infty} \log\left(1+\frac{m^2}{cx^4}\right) dx \leqq \log\left(1+\frac{m^2}{c\cdot 1}\right) = O(\log m).$$

3а) Теперь оценим $S_n^3 = \left(\sum_{k=n+1}^{m} + \sum_{k=m+1}^{\infty}\right) \log[\ ] = \tilde{S}_n^3 + \tilde{\tilde{S}}_n^3$.

$$\sum_{k=n+1}^{m} \log\left[1+\frac{1}{(\sqrt{m^2+k^2}-\sqrt{m^2+n^2})^2}\right] =$$

$$= \sum_{p=1}^{m-n} \log\left[1+\frac{1}{(\sqrt{m^2+(n+p)^2}-\sqrt{m^2+n^2})^2}\right] \leqq \tilde{c}_1\sqrt{m},$$

поскольку, аналогично 2а)

$$\sqrt{m^2+(n+p)^2}-\sqrt{m^2+n^2} \geqq \frac{cp^2}{m} \Leftrightarrow$$

$$\Leftrightarrow m^2+n^2+2np+p^2 \geqq \frac{c^2p^4}{m^2}+m^2+n^2+2\frac{cp^2}{m}\sqrt{m^2+n^2} \Leftrightarrow$$

$$\Leftrightarrow 2n+p \geqq \frac{c^2p^3}{m^2}+2cp\sqrt{1+n^2/m^2} \Leftarrow 3 \geqq c^2+2c\sqrt{2}.$$

Покажем теперь, что

$$\sum_{k=m+1}^{\infty} \log\left[1+\frac{1}{(\sqrt{m^2+k^2}-\sqrt{m^2+n^2})^2}\right] \asymp 1.$$

Действительно, запишем эту сумму в виде

$$\sum_{p=1}^{\infty} \log\left[1+\frac{1}{(\sqrt{m^2+(m+p)^2}-\sqrt{m^2+n^2})^2}\right]$$

и проверим, что

$$\exists c>0:\ \sqrt{m^2+(m+p)^2}-\sqrt{m^2+n^2} \geqq cp$$

$$m^2+m^2+2mp+p^2 \geqq c^2p^2+m^2+n^2+2cp\sqrt{m^2+n^2} \Leftarrow$$

$$\Leftarrow 2m+p \geqq c^2p+2c\sqrt{m^2+n^2} \Leftarrow c \leqq 1, \quad c\sqrt{2} \leqq 1 \Leftrightarrow c\in\left(0, \frac{1}{\sqrt{2}}\right).$$

Таким образом при $n<m$ мы получили оценку $S_n \leqq c_2\sqrt{m}$. Покажем, что при $n \geqq m$: $S_n \leqq c_3$. Тем самым лемма 1 будет доказана.

2b)
$$\sum_{x=1}^{n-1} \log\left[1+\frac{1}{(\sqrt{m^2+n^2}-\sqrt{m^2+x^2})^2}\right]=$$

$$=\sum_{p=1}^{n-1} \log\left[1+\frac{1}{(\sqrt{m^2+n^2}-\sqrt{m^2+(n-p)^2})^2}\right].$$

Проверим, что

$$\sqrt{m^2+n^2}-\sqrt{m^2+(n-p)^2} \geqq cp \quad (1 \leqq p \leqq n-1,\ m \leqq n) \Leftrightarrow$$

$$\Leftrightarrow m^2+n^2 \geqq c^2p^2+m^2+n^2-2np+p^2+2cp\sqrt{m^2+(n-p)^2} \Leftrightarrow$$

$$\Leftrightarrow 2n \geqq c^2p+p+2c\sqrt{m^2+(n-p)^2} \Leftarrow 1 \geqq c^2+c\cdot 2\sqrt{2}.$$

3b)
$$\sum_{k=n+1}^{\infty} \log\left[1+\frac{1}{(\sqrt{m^2+k^2}-\sqrt{m^2+n^2})^2}\right]=$$

$$=\sum_{p=1}^{\infty} \log\left[1+\frac{1}{(\sqrt{m^2+(n+p)^2}-\sqrt{m^2+n^2})^2}\right] < \text{const}.$$

так как

$$\sqrt{m^2+(n+p)^2}-\sqrt{m^2+n^2} \geqq cp \quad (p \geqq 1,\ m \leqq n) \Leftrightarrow$$

$$\Leftrightarrow m^2+n^2+p^2+2np \geqq c^2p^2+m^2+n^2+2cp\sqrt{m^2+n^2} \Leftrightarrow$$

$$\Leftrightarrow p+2n \geqq c^2p+2c\sqrt{m^2+n^2} \Leftarrow 1 \geqq c^2 \quad \text{и} \quad 1 \geqq c\sqrt{2}.$$

Лемма 1 доказана.

В данной работе мы использовали идеи работ [7], [8], [10], [14]. Аналогичными проблемами связаны интересные работы В. Коморник ([13]). Он применяет другой подход.

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BIMEROTOPIES II

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Abstract

After introducing the notion of micromeric relations and near relations in a bimerotopic space, we prove a theorem on extending a biuniformity to an extension of the induced bitopology.

Continuing the investigations of [5] (where §§ 0 to 6 and a list of symbols can be found), we deal with the following topics:

§ 7. Micromeric relations, analogous to the notion of micromeric systems, on which the original definition of a merotopy was based [12].

§ 8. Near relations, corresponding to near systems [11].

§ 9. Nearness of one collection of sets to another. (Leads to several open questions not explicitly stated.)

§ 10. Definition of bimerotopic continuity.

§ 11. Under what conditions can a biuniformity be extended to an extension of the induced bitopology? (We obtain only partial results; the complete answer is unknown even for quasi-uniformities, cf. [6—8].)

§ 7. Micromeric relations

7.1 Notations. For $\mathscr{A}\subset\mathscr{P}(X)$, $p\subset\mathscr{P}(X)\times\mathscr{P}(X)$ and $\mathfrak{A}\subset\mathfrak{P}$ where $\mathfrak{P}=\mathscr{P}(\mathscr{P}(X))$ or $\mathfrak{P}=\mathscr{P}(\mathscr{P}(X)\times\mathscr{P}(X))$, let

$$\sec\mathscr{A}=\{S\subset X:\ \forall A\in\mathscr{A},\ A\cap S\neq 0\},$$

$$\sec p=\big\{(A,B):\ A,B\subset X,\ (C\,p\,D\Rightarrow(A\cap C)\cup(B\cap D)\neq 0)\big\},$$

$$\sec\mathfrak{A}=\{\beta\in\mathfrak{P}:\ \forall\alpha\in\mathfrak{A},\ \alpha\cap\beta\neq 0\}.$$

For $\mathscr{A},\mathscr{B}\subset\mathscr{P}(X)$, we shall write $\mathscr{A}\sqsubset\mathscr{B}$ ($\mathscr{A}$ is *coarser* than $\mathscr{B}$, $\mathscr{B}$ is *finer* than $\mathscr{A}$) if any element of $\mathscr{A}$ contains an element of $\mathscr{B}$.

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7.2 It is well-known (see e.g. [11]) that merotopies can be described not only with coverings, but also in terms of micromeric systems (i.e. systems containing arbitrarily small sets) or of near systems. If $\mathfrak{M}$ is a merotopy on X, and we denote by $\mu\mathfrak{M}\subset\subset\mathscr{P}(\mathscr{P}(X))$ the collection of all the $\mathfrak{M}$-micromeric systems, respectively by $\nu\mathfrak{M}$ the collection of all the $\mathfrak{M}$-near systems then the equivalents of the System of Axioms M1—M4 and the relations between $\mathfrak{M}$, $\mu\mathfrak{M}$ and $\nu\mathfrak{M}$ can be given as follows (cf. [11] § 3):

μM1. $\emptyset\notin\mu\mathfrak{M}$, $\mu\mathfrak{M}\neq\emptyset$;

μM2. for any $x\in X$, $\{\{x\}\}\in\mu\mathfrak{M}$;

μM3. if $\mathscr{B}\in\mu\mathfrak{M}$ and $\mathscr{B}\sqsubset\mathscr{A}$ then $\mathscr{A}\in\mu\mathfrak{M}$;

μM4. if $\mathscr{A}\cup\mathscr{B}\in\mu\mathfrak{M}$ then $\mathscr{A}\in\mu\mathfrak{M}$ or $\mathscr{B}\in\mu\mathfrak{M}$.

νM1. $\{\emptyset\}\notin\nu\mathfrak{M}$, $\nu\mathfrak{M}\neq\emptyset$;

νM2. if $\bigcap\mathscr{A}\neq\emptyset$ then $\mathscr{A}\in\nu\mathfrak{M}$;

νM3. if $\mathscr{B}\in\nu\mathfrak{M}$ and $\mathscr{A}\sqsubset\mathscr{B}$ then $\mathscr{A}\in\nu\mathfrak{M}$;

νM4. if $\mathscr{A}(\cup)\mathscr{B}\in\nu\mathfrak{M}$ then $\mathscr{A}\in\nu\mathfrak{M}$ or $\mathscr{B}\in\nu\mathfrak{M}$.

$\mathfrak{B}\subset\mu\mathfrak{M}$ is a *base* for $\mu\mathfrak{M}$ if for any $\mathscr{A}\in\mu\mathfrak{M}$ there is a $\mathscr{B}\in\mathfrak{B}$ with $\mathscr{B}\sqsubset\mathscr{A}$. The notion of a base for $\mathfrak{M}$ was defined in 0.5.

(7.3) $$\sec\mathfrak{M}\ \text{is a base for}\ \mu\mathfrak{M};$$

(7.4) $$\sec\mu\mathfrak{M}\ \text{is a base for}\ \mathfrak{M};$$

(7.5) $$\mathscr{A}\in\nu\mathfrak{M}\ \text{iff}\ \{X\setminus A\colon A\in\mathscr{A}\}\notin\mathfrak{M};$$

(7.6) $$\mathscr{C}\in\mathfrak{M}\ \text{iff}\ \{X\setminus C\colon C\in\mathscr{C}\}\notin\nu\mathfrak{M};$$

(7.7) $$\mathscr{A}\in\mu\mathfrak{M}\ \text{iff}\ \sec\mathscr{A}\in\nu\mathfrak{M};$$

(7.8) $$\mathscr{A}\in\nu\mathfrak{M}\ \text{iff}\ \sec\mathscr{A}\in\mu\mathfrak{M}.$$

7.9 Given a *bi*merotopy $\mathfrak{M}$, we intend to introduce the notion of $\mathfrak{M}$-*micromeric relations*, respectively of $\mathfrak{M}$-*near relations* (the collection of all such relations will be denoted by $\mu\mathfrak{M}$, respectively by $\nu\mathfrak{M}$) in such a way that Axioms B1—B4 of a bimerotopy could be transcribed analogously to μM1—μM4 and νM1—νM4. We shall use a roundabout method, which makes it possible to *deduce* the definition of micromeric/near relations from the definition of micromeric/near collections, instead of just taking formal analogues of (7.3)—(7.6) and checking that they yield reasonable consequences.

Let $\hat{X}=X\times\{-1,1\}$. To any relation $p\subset\mathscr{P}(X)\times\mathscr{P}(X)$ we assign a system $\hat{p}\subset\mathscr{P}(\hat{X})$ as follows:

$$\hat{p}=\{(A\times\{-1\})\cup(B\times\{1\})\colon\ A\,p\,B\}.$$

For any $\mathscr{A}\subset\mathscr{P}(\hat{X})$, there is exactly one p with $\mathscr{A}=\hat{p}$. If p is a bicovering then $\hat{p}$

is a covering of $\hat{X}$ (but not conversely). Moreover,

(7.10) $$\hat{p} < \hat{q} \Rightarrow p < q,$$

(7.11) $$p < q, \quad p \subset n \cup \{(\emptyset, \emptyset)\} \Rightarrow \hat{p} < \hat{q}.$$

Given an $\mathfrak{A} \subset \mathscr{P}(\mathscr{P}(X) \times \mathscr{P}(X))$, put

$$\hat{\mathfrak{A}} = \{\hat{p}: p \in \mathfrak{A}\}.$$

In this way we obtain a one-to-one correspondence between $\mathscr{P}(\mathscr{P}(\mathscr{P}(X) \times \mathscr{P}(X)))$ and $\mathscr{P}(\mathscr{P}(\mathscr{P}(\hat{X})))$.

7.12 Lemma. *If $\mathfrak{M}$ is a bimerotopy on X then $\hat{\mathfrak{M}}$ is a merotopy on $\hat{X}$.*

Proof. M1 is evident from B1. We have already observed that M2 follows from B2. (7.10) and B3 imply M3.

In order to check M4, take $\hat{p}, \hat{q} \in \hat{\mathfrak{M}}$. Then $p, q \in \mathfrak{M}$, so by B4 there is an $r \in \mathfrak{M}$ such that $r < p$ and $r < q$; we may assume that $r \subset n$ (cf. e.g. the Footnote to the proof of 3.8). Now $\hat{r} < \hat{p}$ and $\hat{r} < \hat{q}$ follow from (7.11). □

7.13 Remark. It is not true that $\hat{\mathfrak{M}}$ is a covering uniformity whenever $\mathfrak{M}$ is a biuniformity (or even a strong biuniformity):

Let $\mathscr{U}_{so}$ denote the *Sorgenfrey quasi-uniformity* on $X = \mathbf{R}$, i.e. let

$$\{U_\varepsilon = \{(x, y): x \leqq y < x + \varepsilon\}: \varepsilon > 0\}$$

be a base for $\mathscr{U}_{so}$. Now $\mathfrak{M} = \mathscr{U}^b_{so} = \mathscr{U}^c_{so}$ is a strong biuniformity, but the merotopy $\hat{\mathfrak{M}}$ is not a covering uniformity.

7.14 Lemma. *Let $\mathfrak{M}$ be a bimerotopy. If $\hat{\mathfrak{B}}$ is a base for $\hat{\mathfrak{M}}$ then $\mathfrak{B}$ is a base for $\mathfrak{M}$; conversely, if $\mathfrak{B}$ is a base for $\mathfrak{M}$ and $\cup \mathfrak{B} \subset n$ then $\hat{\mathfrak{B}}$ is a base for $\hat{\mathfrak{M}}$.*

Proof. (7.10) and (7.11). □

7.15 Lemma. *A merotopy $\mathfrak{N}$ on $\hat{X}$ is of the form $\hat{\mathfrak{M}}$ with some bimerotopy $\mathfrak{M}$ on X iff it satisfies the following conditions:*

(i) *if $x \in X$ and $\mathscr{C} \in \mathfrak{N}$ then there is a $C \in \mathscr{C}$ such that $(x, i) \in C$ $(i = \pm 1)$;*

(ii) *there exists a base $\mathfrak{B}$ for $\mathfrak{N}$ such that for any $\emptyset \neq C \in \mathscr{C} \in \mathfrak{B}$, there are $x_i \in X$ $(i = \pm 1)$ with $(x_i, i) \in C$ $(i = \pm 1)$.*

Proof. Define $\mathfrak{M}$ by $\mathfrak{N} = \hat{\mathfrak{M}}$.

1° If $\mathfrak{M}$ is a bimerotopy then (i) follows from B2, (ii) from 2.2 and 7.14.

2° Conversely, assume that $\mathfrak{N}$ is a merotopy satisfying (i) and (ii); we have to check that $\mathfrak{M}$ is a bimerotopy. B1 is evident. B2 follows from (i), B4 from M4 and (7.10).

To prove B3, let $p \in \mathfrak{M}$ and $p < q$. Now $\hat{p} \in \mathfrak{N}$, and, according to (ii), there is a $\mathscr{C} \in \mathfrak{N}$ such that $\mathscr{C} < \hat{p}$, and for any $\emptyset \neq C \in \mathscr{C}$ there are $x_i \in X$ with $(x_i, i) \in C$. Taking r satisfying $\hat{r} = \mathscr{C}$, we have $r \in \mathfrak{M}$, and the above condition on $\mathscr{C}$ means $r \subset n \cup \cup \{(\emptyset, \emptyset)\}$. Now $\hat{r} = \mathscr{C} < \hat{p}$ implies, according to (7.10), that $r < p$, i.e. from $r < p < q$ we have $r < q$, and then $\hat{r} < \hat{q}$ follows from (7.11). $\hat{q} \in \mathfrak{N}$ by M3, hence $q \in \mathfrak{M}$. □

7.16 Lemma. *A merotopy $\mathfrak{N}$ on $\hat{X}$ is of the form $\hat{\mathfrak{M}}$ with some bimerotopy $\mathfrak{M}$ on X iff it satisfies the following conditions:*

(i)′ *if* $x \in X$ *then* $\{\{(x,-1),(x,1)\}\} \in \mu\mathfrak{N}$;

(ii)′ *there exists a base $\mathfrak{B}$ for $\mu\mathfrak{N}$ such that for any $\emptyset \neq A \in \mathscr{A} \in \mathfrak{B}$, there are $x_i \in X$ $(i=\pm 1)$ with $(x_i, i) \in A$ $(i=\pm 1)$.*

Proof. According to 7.15, it is enough to show that (i)⇔(i)′ and (ii)⇔(ii)′, which follow easily from (7.3) and (7.4) applied to $\mathfrak{N}$. [For the proof of (ii)⇔(ii)′, observe that it is enough to take a base on the left-hand side of (7.3) and (7.4).] □

7.17 Remark. The condition $\emptyset \neq C$ $(\emptyset \neq A)$ can be clearly dropped from (ii) and (ii)′ if $X \neq \emptyset$.

7.18 According to 7.12, 7.15 and 7.16, the bimerotopies on X can be identified with merotopies on $\hat{X}$ satisfying certain conditions. Given a merotopy $\mathfrak{M}$ on X, we can take first $\hat{\mathfrak{M}}$, then $\mu\hat{\mathfrak{M}}$, finally the system of relations $\mu\mathfrak{M}$ for which $\widehat{\mu\mathfrak{M}} = \mu\hat{\mathfrak{M}}$ holds. The elements of this $\mu\mathfrak{M}$ will be the $\mathfrak{M}$-*micromeric relations* we were looking for.

For u, $v \subset \mathscr{P}(X) \times \mathscr{P}(X)$, let us write $u \sqsubset v$ (u is *coarser* than v, v is *finer* than u) if $A\,u\,B$ implies the existence of an $(A', B') \in v$ with $A \supset A'$ and $B \supset B'$. (In other words, $u \sqsubset v$ iff $\hat{u} \sqsubset \hat{v}$.) The notion of a base for $\mu\mathfrak{M}$ is defined just like in 7.2: $\mathfrak{B}$ is a *base* for $\mu\mathfrak{M}$ (where $\mathfrak{M}$ is a bimerotopy) if $\mathfrak{B} \subset \mu\mathfrak{M}$, and for any $v \in \mu\mathfrak{M}$ there is a $u \in \mathfrak{B}$ with $u \sqsubset v$. Evidently, $\mathfrak{B}$ is a base for $\mu\mathfrak{M}$ iff $\hat{\mathfrak{B}}$ is a base for $\mu\hat{\mathfrak{M}}$, so it follows from (7.3), (7.4) (applied to $\hat{\mathfrak{M}}$) and 7.14 that (7.3) and (7.4) remain valid for bimerotopies. In fact, we have:

7.19 Proposition. *If $\mathfrak{B}$ is a base for the bimerotopy $\mathfrak{M}$, $\bigcup\mathfrak{B} \subset n$ and $\mathfrak{B} \subset \mathfrak{B}' \subset \mathfrak{M}$ then* sec $\mathfrak{B}'$ *is a base for $\mu\mathfrak{M}$. Conversely, if $\mathfrak{B}$ is a base for $\mu\mathfrak{M}$ then* sec $\mathfrak{B}$ *is a base for $\mathfrak{M}$.* □

7.20 Theorem. $\mathfrak{M} \subset \mathscr{P}(\mathscr{P}(X) \times \mathscr{P}(X))$ *is a bimerotopy iff $\mu\mathfrak{M}$ satisfies the following axioms:*

μB1. $\emptyset \notin \mu\mathfrak{M}$, $\mu\mathfrak{M} \neq \emptyset$;

μB2. *for any* $x \in X$, $\{(\{x\}, \{x\})\} \in \mu\mathfrak{M}$;

μB3. *if* $u \in \mu\mathfrak{M}$ *and* $u \sqsubset v$ *then* $v \in \mu\mathfrak{M}$;

μB4. *if* $u \cup v \in \mu\mathfrak{M}$ *then* $u \in \mu\mathfrak{M}$ *or* $v \in \mu\mathfrak{M}$;

μB5$_0$. *if* $u \in \mu\mathfrak{M}$ *then there is a* $v \in \mu\mathfrak{M}$ *such that* $v \subset n$ *and* $v \sqsubset u$.

Proof. Straightforward from 7.12, 7.16, 7.17 and $u \sqsubset v \Leftrightarrow \hat{u} \sqsubset \hat{v}$. ($\mu$B2 is equivalent to (i)′, and μB5$_0$ to (ii)′.) □

7.21 Remarks. a) Compare μB3 and μB5$_0$ with 2.3.

b) μB5$_0$ means that, similarly to the case of bicoverings, it would be enough to consider micromeric relations contained by n.

c) $\{(\emptyset, \emptyset)\} \in \mu\mathfrak{M}$.

d) The second part of μB1 is needed if $X = \emptyset$; it could be replaced by c).

7.22 Let $\mathfrak{M}_0$ be a merotopy on X. In 3.1, we identified $\mathfrak{M}_0$ with a bimerotopy on X satisfying BS. In order to make the present argument clear, we shall use a different symbol, namely $\mathfrak{M}$, for denoting this bimerotopy. We can take now $\mu\mathfrak{M}_0 \subset \mathscr{P}(\mathscr{P}(X))$, respectively $\mu\mathfrak{M} \subset \mathscr{P}(\mathscr{P}(X) \times \mathscr{P}(X))$. It is easy to deduce from 7.19 that $\{p_{\mathscr{A}} : \mathscr{A} \in \mu\mathfrak{M}_0\}$ (with $p_{\mathscr{A}}$ defined by (3.2)) is a base for $\mu\mathfrak{M}$.

§ 8. Near relations

8.1 Similarly to the method used in 7.18, we define $v\mathfrak{M}$ (the system of the $\mathfrak{M}$-*near relations*) through $\widehat{v\mathfrak{M}} = v\hat{\mathfrak{M}}$. $v\mathfrak{M}$ is connected with $\mathfrak{M}$ and $\mu\mathfrak{M}$ by the formulas below, which follow immediately from (7.5)—(7,8):

(8.2) $$s \in v\mathfrak{M} \quad \text{iff} \quad \{(X \setminus A, X \setminus B) : A\, s\, B\} \notin \mathfrak{M};$$

(8.3) $$p \in \mathfrak{M} \quad \text{iff} \quad \{(X \setminus A, X \setminus B) : A\, p\, B\} \notin v\mathfrak{M};$$

(8.4) $$u \in \mu\mathfrak{M} \quad \text{iff} \quad \sec u \in v\mathfrak{M};$$

(8.5) $$s \in v\mathfrak{M} \quad \text{iff} \quad \sec s \in \mu\mathfrak{M}.$$

$\mathfrak{B} \subset v\mathfrak{M}$ will be called a *base* for $v\mathfrak{M}$ if for any $t \in v\mathfrak{M}$ there is an $s \in \mathfrak{B}$ with $t \sqsubset s$.

8.6 Theorem. $\mathfrak{M} \subset \mathscr{P}(\mathscr{P}(X) \times \mathscr{P}(X))$ *is a bimerotopy iff* $v\mathfrak{M}$ *satisfies the following axioms:*[1]

vB1. $\{(\emptyset, \emptyset)\} \notin v\mathfrak{M}$, $v\mathfrak{M} \neq \emptyset$;

vB2. *if* $\bigcap\{A \cup B : A\, s\, B\} \neq \emptyset$ *then* $s \in v\mathfrak{M}$;

vB3. *if* $s \in v\mathfrak{M}$ *and* $t \sqsubset s$ *then* $t \in v\mathfrak{M}$;

vB4. *if* $s(\cup)t \in v\mathfrak{M}$ *then* $s \in v\mathfrak{M}$ *or* $t \in v\mathfrak{M}$;

vB5$_0$. *if* $X \neq \emptyset$ *and* $s \in v\mathfrak{M}$ *then* $s \cup \{(\emptyset, X), (X, \emptyset)\} \in v\mathfrak{M}$.

Proof. Starting from 7.15, proceed on the analogy of 7.16—7.20. (We do not go into details, because the argument is even simpler than for $\mu\mathfrak{M}$.) □

8.7 Remark. It follows from vB3 [μB3] that a base for $v\mathfrak{M}$ [$\mu\mathfrak{M}$] uniquely determines $\mathfrak{M}$.

8.8 Proposition. *If* $\mathfrak{B}$ *is a base for* $\mu\mathfrak{M}$ *then* $\{\sec u : u \in \mathfrak{B}\}$ *is a base for* $v\mathfrak{M}$. *Conversely, if* $\mathfrak{B}$ *is a base for* $v\mathfrak{M}$ *then* $\{\sec s : s \in \mathfrak{B}\}$ *is a base for* $\mu\mathfrak{M}$.

Proof. Using (8.4) and (8.5), show first that the statement holds for $\mathfrak{B} = \mu\mathfrak{M}$, respectively for $\mathfrak{B} = v\mathfrak{M}$. □

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[1] Recall that $s(\cup)t$ was defined to mean $\{(A \cup B, C \cup D) : A\, s\, C,\ B\, t\, D\}$.

8.9 It is possible to transcribe the axioms for special kinds of bimerotopies; e.g. B5 (strongness), BS (merotopy) and $BS'=BS''$ (symmetry) can be written as follows:

μB5. if $u\in\mu\mathfrak{M}$ then there is a $v\in\mu\mathfrak{M}$ such that $v\subset m$ and $v\sqsubset u$;

νB5. if $X\neq\emptyset$ and $s\in\nu\mathfrak{M}$ then $s\cup\{(A, X\setminus A)\colon A\subset X\}\in\nu\mathfrak{M}$;

μBS. for each $u\in\mu\mathfrak{M}$ there is a $v\in\mu\mathfrak{M}$ such that $v\sqsubset u$, and $A=B$ whenever $A\, v\, B$;

μBS″ if $u\in\mu\mathfrak{M}$ then $u^{-1}\in\mu\mathfrak{M}$;

νBS″ if $s\in\nu\mathfrak{M}$ then $s^{-1}\in\nu\mathfrak{M}$.

The condition "for each $s\in\nu\mathfrak{M}$ there is a $t\in\nu\mathfrak{M}$ such that $s\sqsubset t$, and $A=B$ whenever $A\, t\, B$" will not do as νBS, see 8.13. The analogoues of BS′, namely the conditions "there is a base for $\mu\mathfrak{M}$ [$\nu\mathfrak{M}$] consisting of symmetric relations" are equivalent to each other, but not to BS′. (Example: $\mathcal{U}^c$ from 3.6 is symmetric, but it does not satisfy the above conditions.)

8.10 PROPOSITION. *For a bimerotopy* $\mathfrak{M}$, $A\,\mathfrak{M}^{ut}\,B$ *iff* $\{(A,\emptyset), (\emptyset, B)\}\in\nu\mathfrak{M}$.

PROOF. According to (8.2), it is sufficient to show that

$$p=\{(X\setminus A, X), (X, X\setminus B)\}\in\mathfrak{M} \tag{8.11}$$

ff $A\,\bar{\delta}\,B$ where $\delta=\mathfrak{M}^{ut}$. As $A\,\bar{\delta}\,B$ is equivalent to

$$U=X\times X\setminus A\times B\in\mathfrak{M}^{u}, \tag{8.12}$$

we have only to prove the equivalence of (8.11) and (8.12).

As $U=p^u$, $p\in\mathfrak{M}$ clearly implies $U\in\mathfrak{M}^u$.

Conversely, assume that $U\in\mathfrak{M}^u$. Now $U^c<p$ can be easily checked, thus $p\in\mathfrak{M}^{uc}$ follows from $U^c\in\mathfrak{M}^{uc}$. Hence $p\in\mathfrak{M}$ by 5.2. □

8.13 Near relations seem to give a less satisfactory way of describing bimerotopies than either bicoverings or micromeric relations. To support this statement, let us make two observations:

a) According to 2.3 (respectively $\mu B5_0$), a bimerotopy can be completely described by bicoverings (respectively micromeric relations) contained by n. The analogous statement for near relations is false, even in the case of strong biuniformities: take all the non-discrete quasi-uniformities $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$ on a two-point set; now the $\mathcal{U}_i^b$-near $=\mathcal{U}_i^c$-near relations contained by n are the same for $i=1, 2, 3$.

b) Let $\mathfrak{M}_0$ and $\mathfrak{M}$ be as in 7.22. Then $\{p_{\mathcal{A}}\colon \mathcal{A}\in\nu\mathfrak{M}_0\}$ will *never* be a base for $\nu\mathfrak{M}$ (except when $X=\emptyset$).

§ 9. Nearness of two collections

9.1 Let $\mathfrak{M}$ be a bimerotopy on X, and $\mathscr{A}, \mathscr{B} \subset \mathscr{P}(X)$. We shall write $\mathscr{A}\, \mathfrak{M}^n\, \mathscr{B}$, and say that $\mathscr{A}$ is *near* ($\mathfrak{M}$-near) to $\mathscr{B}$ if for any $p \in \mathfrak{M}$, there are $C \in \sec \mathscr{A}$ and $D \in \sec \mathscr{B}$ such that $C\, p\, D$. ($\mathfrak{M}$ can be replaced here by a base.) We have:

$$\mathscr{A}\, \mathfrak{M}^n\, \mathscr{B} \quad \text{iff} \quad \sec \mathscr{A} \times \sec \mathscr{B} \in \mu\mathfrak{M}; \tag{9.2}$$

$$\mathscr{A}\, \mathfrak{M}^n\, \mathscr{B} \quad \text{iff} \quad \{(A, \emptyset)\colon A \in \mathscr{A}\} \cup \{(\emptyset, B)\colon B \in \mathscr{B}\} \in \nu\mathfrak{M}; \tag{9.3}$$

$$A\, \mathfrak{M}^{ut}\, B \quad \text{iff} \quad \{A\}\, \mathfrak{M}^n\, \{B\}. \tag{9.4}$$

By (9.4), the relation $\mathfrak{M}^n$ determines $\mathfrak{M}^{ut}$. Moreover, if $\mathscr{U}$ is a uniformity then $\mathscr{A}\, \mathscr{U}^{cn}\, \mathscr{B}$ iff $\mathscr{A} \cup \mathscr{B}$ is a near collection in the sense of 7.2, so if $\mathscr{U}_1$ and $\mathscr{U}_2$ are both uniformities such that $\mathscr{U}_1^{cn} = \mathscr{U}_2^{cn}$ then $\mathscr{U}_1 = \mathscr{U}_2$; this means that, under suitable restrictions, $\mathfrak{M}^n$ can determine $\mathfrak{M}$ itself. One may wonder whether $\mathfrak{M}^n$ contains in general enough information to recover $\mathfrak{M}$, or at least $\mathfrak{M}^u$. By (9.2) and (9.3), knowing $\mathfrak{M}^n$ only means that we know those near (respectively, micromeric) relations that satisfy an additional condition.

If $\mathfrak{M}$ is a fine bimerotopy (see 4.2) on $X \neq \emptyset$ then $\mathscr{A}\, \mathfrak{M}^n\, \mathscr{B}$ iff $(\bigcap \mathscr{A})\, \mathfrak{M}^{ut}\, (\bigcap \mathscr{B})$ (with the convention $\bigcap \emptyset = X$), so $\mathfrak{M}^n$ conveys now exactly as much information as $\mathfrak{M}^{ut}$, thus not even $\mathfrak{M}^u$ can be recovered from it. $\mathfrak{M}^n$ is in fact insufficient even for determining much better bimerotopies:

9.5 Example. On $X = \mathbf{R}$, let

$$\{\{(x, y)\colon y < x + \varepsilon\}\colon \varepsilon > 0\}$$

be a base for the quasi-uniformity $\mathscr{U}$. With $\mathfrak{M} = \mathscr{U}^c = \mathscr{U}^b$, $\mathscr{A}\, \mathfrak{M}^n\, \mathscr{B}$ iff

$$-\infty < \inf_{A \in \mathscr{A}} \sup A \geqq \sup_{B \in \mathscr{B}} \inf B < +\infty. \tag{9.6}$$

Let $\mathscr{U}_1$ be the inverse of $\mathscr{U}$ under the mapping $x \mapsto x^3$. Now $\mathscr{U}$ and $\mathscr{U}_1$ are clearly different, but, with $\mathfrak{M}_1 = \mathscr{U}_1^c = \mathscr{U}_1^b$, $\mathscr{A}\, \mathfrak{M}_1^n\, \mathscr{B}$ is also equivalent to (9.6).

9.7 The above example shows that one can put a non-trivial intermediate notion between quasi-uniformity and quasi-proximity, namely $\mathscr{U}^{cn}$ (or perhaps $\mathscr{U}^{bn}$). Such a definition gives rise to several natural questions, which are yet to be investigated.

§ 10. Continuity

10.1 Definition. Let $(X, \mathfrak{M})$ and $(Y, \mathfrak{N})$ be bimerotopic spaces, and $f\colon X \to Y$. We say that f is $(\mathfrak{M}, \mathfrak{N})$-*continuous* if

$$\hat{f} = f \times \Delta_{\{-1,1\}}\colon \hat{X} \to \hat{Y}$$

is $(\hat{\mathfrak{M}}, \hat{\mathfrak{N}})$-continuous.

10.2 Notations. For $A, B\subset X$, $C, D\subset Y$, $p\subset\mathcal{P}(X)\times\mathcal{P}(X)$, $q\subset\mathcal{P}(Y)\times\mathcal{P}(Y)$ and $f\colon X\to Y$, let

$$f(A, B) = (f[A], f[B]), \quad f^{-1}(C, D) = (f^{-1}[C], f^{-1}[D]),$$

$$fp = \{f(A, B)\colon\ A\ p\ B\}, \quad f^{-1}q = \{f^{-1}(C, D)\colon\ C\ q\ D\}.$$

10.3 Proposition. *f is $(\mathfrak{M}, \mathfrak{N})$-continuous iff any of the following conditions holds:*
(i) *if* $q\in\mathfrak{N}$ *then* $f^{-1}q\in\mathfrak{M}$;
(ii) *if* $u\in\mu\mathfrak{M}$ *then* $fu\in\mu\mathfrak{N}$;
(iii) *if* $s\in\nu\mathfrak{M}$ *then* $fs\in\nu\mathfrak{N}$. □

10.4 Having defined continuity, one can consider the category of bimerotopies, its subcategories (biuniformities, coarse/basic/strong/etc. bimerotopies) and their relations to other categories (bitopologies, quasi-uniformities, merotopies, etc.). This will be the topic of the third part of this series[2]. It will be sufficient here to make some simple observations:

a) The relation finer/coarser introduced in 5.1 tallies with our definition of continuity. $\mathfrak{N}$ is finer than $\mathfrak{M}$ (i.e. $\mathfrak{M}\subset\mathfrak{N}$) iff $\mu\mathfrak{M}\supset\mu\mathfrak{N}$ iff $\nu\mathfrak{M}\supset\nu\mathfrak{N}$.

b) The restriction $\mathfrak{N}|X$ of the bimerotopy $\mathfrak{N}$ on Y to $X\subset Y$ can be given as follows:

$$\mathfrak{N}|X = \{p|X\colon\ p\in\mathfrak{N}\} \tag{10.5}$$

where

$$p|X = \{(A\cap X, B\cap X)\colon\ A\ p\ B\}.$$

Moreover, we have

$$\mu(\mathfrak{N}|X) = \{u|X\colon\ u\in\mu\mathfrak{N}\} = \mu\mathfrak{N}\cap\mathcal{P}(\mathcal{P}(X)\times\mathcal{P}(X)); \tag{10.6}$$

$$\nu(\mathfrak{N}|X) = \nu\mathfrak{N}\cap\mathcal{P}(\mathcal{P}(X)\times\mathcal{P}(X)); \tag{10.7}$$

$$(\mathfrak{N}|X)^u = \mathfrak{N}^u|X. \tag{10.8}$$

Using the notation of (10.5) for arbitrary collections of relations, $\mathfrak{B}|X$ is a base for $\mathfrak{N}|X$ whenever $\mathfrak{B}$ is a base for $\mathfrak{N}$; similarly, if $\mathfrak{B}$ is a base for $\mu\mathfrak{N}$ then $\mathfrak{B}|X$ is a base for $\mu(\mathfrak{N}|X)$.

§ 11. Extending a biuniformity

11.1 Given a biuniform space $(X, \mathfrak{M})$ and an extension $(Y;\ \mathcal{T}^{-1}, \mathcal{T}^{1})$ of the induced bitopological space $(X;\ \mathfrak{M}^{-utp}, \mathfrak{M}^{utp})$ [i.e. $(X;\ \mathfrak{M}^{-utp}, \mathfrak{M}^{utp})$ is a subspace of $(Y;\ \mathcal{T}^{-1}, \mathcal{T}^{1})$, and X is $\mathcal{T}^{-1}$-dense as well as $\mathcal{T}^{1}$-dense], we are looking for an extension $\mathfrak{N}$ of $\mathfrak{M}$ compatible with $(\mathcal{T}^{-1}, \mathcal{T}^{1})$ [i.e. $\mathfrak{N}|X=\mathfrak{M}$, $\mathfrak{N}^{iutp}=\mathcal{T}^{i}$ $(i=\pm 1)$]. We investigated the analogous problem for quasi-uniformities in [6, 7]; the results

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[2] *Added in proof.* The third part will probably never be written; instead, we shall discuss the category of bimerotopies in [16], where it will not be considered *per se*, but as a device for solving a problem in categorical topology.

obtained there cannot be applied to the present situation, since the existence of an extension of $\mathfrak{M}^u$ does not guarantee that there is an extension of $\mathfrak{M}$, too (see 11.45).

Let $\mathcal{N}^i(a)$ denote the $\mathcal{T}^i$-neighbourhood filter of the point $a \in Y$, and

$$\mathcal{F}^i(a) = \mathcal{N}^i(a)|X = \{S \cap X: S \in \mathcal{N}^i(a)\}.$$

$(\mathcal{F}^{-1}(a), \mathcal{F}^1(a))$ is called the *trace filter pair* of *a*.

A bitopology is *completely regular* if it can be induced by a quasi-uniformity (equivalently: by a quasi-proximity or by a biuniformity), cf. [9, 15, 10, 3]. $(\mathcal{T}^{-1}, \mathcal{T}^1)$ is *regular* [13] if any $\mathcal{T}^i$-neighbourhood of a point contains a $\mathcal{T}^{-i}$-closed $\mathcal{T}^i$-neighbourhood ($i = \pm 1$). Completely regular bitopologies are regular.

11.2 Definition. In a biuniform space $(X, \mathfrak{M})$, the filter pair $(\mathcal{F}^{-1}, \mathcal{F}^1)$ is round ($\mathfrak{M}$-round) if for any $S \in \mathcal{F}^i$ and $i \in \{-1, 1\}$ there are an $S_0 \in \mathcal{F}^i$ and a $p \in \mathfrak{M}$ such that $\mathrm{st}_p^i\, S_0 \subset S$.

11.3 Remark. $(\mathcal{F}^{-1}, \mathcal{F}^1)$ is $\mathfrak{M}$-round iff it is $\mathfrak{M}^u$-round ([6] Definition 1.1 a)), i.e. iff $\mathcal{F}^i$ is $\mathfrak{M}^{ui}$-round ($\mathfrak{M}^{uti}$-round) in the usual sense for $i = \pm 1$, cf. [14, 1, 2, 4, 6].

11.4 Proposition. *If $(X, \mathfrak{M})$ is a biuniform space, and $\mathfrak{M}$ can be extended to the extension $(Y; \mathcal{T}^{-1}, \mathcal{T}^1)$ of $(X; \mathfrak{M}^{-utp}, \mathfrak{M}^{utp})$ then*

(i) $(\mathcal{T}^{-1}, \mathcal{T}^1)$ *is completely regular;*
(ii) $(\mathcal{F}^{-1}(a), \mathcal{F}^1(a))$ *is round* $(a \in Y)$;
(iii) $\mathcal{F}^{-1}(a) \times \mathcal{F}^1(a) \in \mu\mathfrak{M}$ $(a \in Y)$.

Proof. (ii) By (10.8) and 11.3, [6] Theorem 1.1 b) can be applied. (But one can also give a straightforward direct proof.)

(iii) Denote by $\mathfrak{N}$ an extension compatible with $(\mathcal{T}^{-1}, \mathcal{T}^1)$, and let a point $a \in Y$ be fixed. For $q \in \mathfrak{N}$ take $r \in \mathfrak{N}$ with $r <^* q$. Then $\mathrm{st}_r^{-1}\{a\}\; q^d\; \mathrm{st}_r^1\{a\}$, so 2.35 c) implies that $\mathcal{N}^{-1}(a) \times \mathcal{N}^1(a) \in \mu\mathfrak{N}$ (7.19 is to be applied here to $\mathfrak{B}' = \{p^d: p \in \mathfrak{N}\}$). Now $\mathcal{F}^{-1}(a) \times \mathcal{F}^1(a) \in \mu\mathfrak{M}$ follows from the first part of (10.6). □

11.5 The conditions in 11.4 will turn out to be sufficient in a special case, namely for fine regular extensions. Recall from [6] that if we are given round Cauchy trace filter pairs $(\mathcal{F}^{-1}(a), \mathcal{F}^1(a))$ $(a \in Y)$ in a quasi-uniform space $(X, \mathcal{U})$ ($\mathcal{U}$-Cauchy means that the product of the filters is $\mathcal{U}^c$-micromeric) then there do exist regular bitopological extensions associated with these trace filter pairs; there is a finest one among these extensions, called therefore *fine regular*, which is in fact completely regular (cf. [6] Lemma 2.3, Theorems 2.1 and 3.1).

A base for the neighbourhood filter $\mathcal{N}^i(a)$ in a fine regular extension can be given as follows:

$$\{N_S^i(a): S \in \mathcal{B}^i(a)\} \tag{11.6}$$

where $\mathcal{B}^i(a)$ is an arbitrary base for $\mathcal{F}^i(a)$, and

$$N_S^i(a) = \{a\} \cup \{b: S \in \sec \mathcal{F}^{-i}(b)\} \tag{11.7}$$

([6] 2.1 (4) and (5)).

11.8 THEOREM. *Let $(X, \mathfrak{M})$ be a biuniform space, and $(Y; \mathcal{T}^{-1}, \mathcal{T}^{1})$ a fine regular extension of $(X; \mathfrak{M}^{-utp}, \mathfrak{M}^{utp})$. Then $\mathfrak{M}$ has a biuniform extension compatible with $(\mathcal{T}^{-1}, \mathcal{T}^{1})$ iff the trace filter pairs satisfy* 11.4 (ii) *and* (iii).

11.9 REMARKS. a) It follows from 11.4 applied to the special case $Y=X$ that neighbourhood filter pairs always satisfy (ii) and (iii), so it is enough to assume these conditions for the points of $Y \setminus X$.

b) [6] Theorem 3.1 can be deduced from 11.8 (applied to $\mathcal{U}^c$), but not conversely.

11.10. PROOF of 11.8. *Necessity.* 11.4.

Sufficiency. We may assume that $X \neq \emptyset$. Let Φ^i denote the family of all those functions $f\colon Y \to \mathcal{P}(X)$ for which

$$f(a) \in \mathcal{F}^i(a) \quad (a \in Y). \tag{11.11}$$

Consider the system

$$\mathfrak{B} = \{q(f^{-1}, f^1, p)\colon f^i \in \Phi^i \quad (i = \pm 1),\ p \in \mathfrak{M}\}, \tag{11.12}$$

where, with q standing for $q(f^{-1}, f^1, p)$,

$$\begin{aligned} A\, q\, B \quad \text{iff} \quad & \exists Z \subset Y,\ \ |Z| \leq 1,\ \ \exists (A^\circ, B^\circ) \in p^d \cap n, \\ & (e \in Z \Rightarrow A^\circ \subset f^{-1}(e),\ B^\circ \subset f^1(e)), \\ & A \subset Z \cup s^{-1}(A^\circ), \quad B \subset Z \cup s^1(B^\circ), \end{aligned} \tag{11.13}$$

and

$$s^i(E) = s^i(f^{-i}, p;\ E) = \{b \in Y\colon (\mathrm{st}^i_p E)\, m\, f^{-i}(b)\}, \tag{11.14}$$

in other words,

$$b \in s^i(E) \Leftrightarrow \exists F, G, \quad E\, m\, F\, p^i\, G\, m\, f^{-i}(b). \tag{11.15}$$

For an arbitrary index α, let $q_\alpha = q(f_\alpha^{-1}, f_\alpha^1, p_\alpha)$ and $s_\alpha^i(E) = s^i(f_\alpha^{-i}, p_\alpha;\ E)$.

1° *$\mathfrak{B}$ is a base for a bimerotopy.* Each $q \in \mathfrak{B}$ is a bicovering of Y, since for any $a \in Y$ there are, according to 11.4 (iii), sets $g^i(a) \in \mathcal{F}^i(a)$ with $g^{-1}(a)\, p^d\, g^1(a)$, and then $\{a\}\, q^d\, \{a\}$ is shown by taking $Z = \{a\}$, $A^\circ = f^{-1}(a) \cap g^{-1}(a)$ and $B^\circ = f^1(a) \cap \cap g^1(a)$. ($A^\circ\, n\, B^\circ$ follows from (11.11).)

If $p < p_1$ and

$$f^i(a) \subset f_1^i(a) \quad (a \in Y,\ i = \pm 1) \tag{11.16}$$

then $q < q_1$. (Indeed, assuming $A\, q\, B$, take Z, A° and B° according to (11.13). Now $A\, q_1\, B$ will follow from the fact that the conditions in (11.13) are satisfied with the same sets Z, A° and B° for q_1, too: $A^\circ\, (p_1^d \cap n)\, B^\circ$ follows from $p < p_1$; (11.16) implies that $A^\circ \subset f_1^{-1}(e)$ and $B^\circ \subset f_1^1(e)$ if $e \in Z$; finally, we have $A \subset Z \cup s_1^{-1}(A^\circ)$ and $B \subset Z \cup s_1^1(B^\circ)$ from $s^i(E) \subset s_1^i(E)$, which is an immediate consequence of (11.14), $\mathrm{st}^i_p E \subset \mathrm{st}^i_{p_1} E$ and (11.16).) Hence $\mathfrak{B}$ satisfies B4, because $\mathfrak{M}$ does so, and the sets $f^i(a)$ are taken from filters.

Let $\mathfrak{N}$ denote the bimerotopy generated by $\mathfrak{B}$.

2° $\mathfrak{N}$ *is a biuniformity*. For $q \in \mathfrak{B}$ fixed, take p_0, p_1 and f_0^i such that

$$p_0 \in \mathfrak{M}, \quad p_0 <^* p_1 <^* p, \tag{11.17}$$

$$f_0^i \in \Phi^i \quad (i = \pm 1), \quad f_0^{-1}(a)\, p_0^d\, f_0^1(a) \quad (a \in Y), \tag{11.18}$$

$$f_0^i(a) \subset f^i(a) \quad (a \in Y,\ i = \pm 1) \tag{11.19}$$

(the conditions on f_0^i can be fulfilled, since 11.4 (iii) was assumed). $q_0 \in \mathfrak{B}$ is obvious; we claim that $q_0 <^* q$.

To show this, take $A\, q_0\, B$; then what we have to prove is

$$(\mathrm{st}_{q_0}^{-1} A)\, q^d\, (\mathrm{st}_{q_0}^1 B). \tag{\textit{11.20}[3]}$$

As $A\, q_0\, B$, we can take Z, A_0° and B_0° such that

$$\emptyset \neq A_0^\circ\, p_0^d\, B_0^\circ \neq \emptyset, \tag{11.21}$$

$$e \in Z \Rightarrow A_0^\circ \subset f_0^{-1}(e), \quad B_0^\circ \subset f_0^1(e), \tag{11.22}$$

$$A \subset Z \cup s_0^{-1}(A_0^\circ), \quad B \subset Z \cup s_0^1(B_0^\circ). \tag{11.23}$$

We are going to show that the conditions for (11.20) are satisfied with the sets Z and

$$A^\circ = \begin{cases} A_0^\circ & \text{if} \quad Z = \emptyset, \\ f_0^{-1}(e) & \text{if} \quad Z = \{e\}, \end{cases} \qquad B^\circ = \begin{cases} B_0^\circ & \text{if} \quad Z = \emptyset, \\ f_0^1(e) & \text{if} \quad Z = \{e\}, \end{cases} \tag{11.24}$$

i.e. the following four statements have to be proved:

$$\emptyset \neq A^\circ\, p^d\, B^\circ \neq \emptyset, \tag{11.25}$$

$$e \in Z \Rightarrow A^\circ \subset f^{-1}(e), \quad B^\circ \subset f^1(e) \tag{11.26}$$

[(11.25) follows either from (11.24), (11.21) and (11.17), or from (11.24), (11.18) and (11.17), according as $Z = \emptyset$ or not; (11.26) is evident from (11.24) and (11.19)], and

$$\mathrm{st}_{q_0}^{-1} A \subset Z \cup s^{-1}(A^\circ), \tag{\textit{11.27}}$$

$$\mathrm{st}_{q_0}^1 B \subset Z \cup s^1(B^\circ). \tag{\textit{11.28}}$$

We shall only deal with (11.28), since, for reasons of symmetry, (11.27) can be proved in the same way.

Take now a point

$$d \in \mathrm{st}_{q_0}^1 B. \tag{11.29}$$

The proof of 2° will be complete if we show that

$$d \in Z \cup s^1(B^\circ). \tag{\textit{11.30}}$$

According to (11.29), there are sets C, D and a point c such that

$$c \in B \cap C, \quad C\, q_0\, D, \quad d \in D. \tag{11.31}$$

[3] In order to make the proof easier to read, numbers in italics will be assigned to formulas not yet known to be true.

We can now choose Z', C° and D° satisfying

(11.32) $$\emptyset \neq C^\circ\ p_0^d\ D^\circ \neq \emptyset,$$

(11.33) $$e \in Z' \Rightarrow C^\circ = f_0^{-1}(e), \quad D^\circ = f_0^1(e),$$

(11.34) $$C \subset Z' \cup s_0^{-1}(C^\circ), \quad D \subset Z' \cup s_0^1(D^\circ).$$

[The equalities in (11.33) can be achieved using the method of (11.24); (11.32) remains valid in consequence of (11.18).] Moreover, (11.22) to (11.24) imply:

(11.35) $$e \in Z \Rightarrow B^\circ = f_0^1(e),$$

(11.36) $$B \subset Z \cup s_0^1(B^\circ).$$

If $c \in Z \cap Z'$ then, by (11.33) and (11.35), $B^\circ = D^\circ$, thus (11.24), (11.36) and $Z = Z'$ (which follows from $|Z|, |Z'| \leqq 1$) imply $D \subset Z \cup s_0^1(B^\circ)$, i.e. (11.30) holds now in consequence of (11.31) and $s_0^1(B^\circ) \subset s^1(B^\circ)$ [where the last formula is clear from (11.17) and (11.19)]. Therefore we have only to deal with the following three possibilities [cf. (11.33) to (11.36)]:

(i) $c \in s_0^1(B^\circ) \cap s_0^{-1}(C^\circ)$;

(ii) $c \in s_0^1(B^\circ), \quad f_0^{-1}(c) = C^\circ$,

(iii) $f_0^1(c) = B^\circ, \quad c \in s_0^{-1}(C^\circ)$.

In the first case, (11.15) gives us sets F, G, F' and G' such that, recalling also (11.18),

$$B^\circ\ m\ F\ p_0\ G\ m\ f_0^{-1}(c)\ p_0^d\ f_0^1(c)\ m\ F'\ p_0\ G'\ m\ C^\circ.$$

By (11.17), this means $F\ p_1^d\ G'$ (cf. the definition of $<^*$), hence we have

(11.37) $$\exists F'', G'', \quad B^\circ\ m\ F''\ p_1\ G''\ m\ C^\circ,$$

which clearly holds in the remaining cases, too (even with p_0 instead of p_1).

If $d \in Z'$ then, by (11.33), $C^\circ = f_0^{-1}(d)$, thus (11.30) can be obtained from (11.37), (11.19) and (11.15).

On the other hand, if $d \notin Z'$ then (11.31), (11.34) and (11.15) give sets F''' and G''' such that

$$D^\circ\ m\ F'''\ p_0\ G'''\ m\ f_0^{-1}(d).$$

Now (11.37), (11.32) and (11.17) imply $F''\ p^d\ G'''$, so we have (11.30), just like before.

3° $\mathfrak{M}$ *is the trace of* $\mathfrak{N}$. We are going to show that, with q_0 chosen as in 2°,

(11.38) $$q_0|X < p < q|X.$$

a) Let us first prove $p < q|X$. Assume $A\,(p \cap n)\,B$. Then the conditions in (11.13) hold with $Z = \emptyset$, $A^\circ = A$ and $B^\circ = B$. (Recall that $x \in f^i(x)$ whenever $x \in X$.) Thus $p \cap n \subset q$, and therefore $p < q|X$ (cf. the definition of $q|X$).

b) To prove the first part of (11.38), assume $A'(q_0|X)B'$. Then there are A and B such that $A'=A\cap X$, $B'=B\cap X$ and $A\,q_0\,B$. Take A° and B° as in 2°. If $x\in B'$ then either $x\in s_0^1(B^\circ)$, implying that (with suitable sets F and G)

$$B^\circ\, m\, F\, p_0\, G\, m\, f_0^{-1}(x)\, p_0^d\, f_0^1(x)\ni x,$$

and so

$$x\in \mathrm{st}_{p_1}^1 B^\circ, \tag{11.39}$$

or $B^\circ=f_0^1(x)\ni x$, i.e. (11.39) holds in any case. Now (11.39) and its counterpart for A' mean

$$A'\subset \mathrm{st}_{p_1}^{-1} A^\circ,\quad B'\subset \mathrm{st}_{p_1}^1 B^\circ,$$

hence $A'\,p^d\,B'$ [since (11.24), (11.21), (11.18) and (11.17) imply $A^\circ\,p_1^d\,B^\circ$].

4° $\mathfrak{N}$ *is compatible with* $(\mathscr{T}^{-1},\mathscr{T}^1)$. Instead of giving a direct proof, it will be easier to recall a construction from [6]. The condition 11.4 (iii) implies that $\mathscr{F}^{-1}(a)\times\times\mathscr{F}^1(a)\in\mu\mathscr{U}^c$, where $\mathscr{U}=\mathfrak{M}^u$ [cf. 5.2 and 11.2 a)], so the trace filter pairs are $\mathscr{U}$-Cauchy; they are also $\mathscr{U}$-round by 11.3, thus [6] Theorem 3.1 can be applied, and according to [6] 3.1 (1)—(3),

$$\{V(f^{-1},f^1,U)\colon f^i\in\Phi^i\quad (i=\pm 1),\ U\in\mathscr{U}\}$$

is a base for a quasi-uniformity $\mathscr{V}$ compatible with $(\mathscr{T}^{-1},\mathscr{T}^1)$, where

$$a\,V(f^{-1},f^1,U)\,b\quad\text{iff}\quad\text{either}\quad a=b \tag{11.40}$$
$$\text{or}\quad \exists x\in f^1(a),\quad \exists y\in f^{-1}(b),\quad x\,U\,y.$$

Now it is enough to prove that $\mathfrak{N}^u=\mathscr{V}$. This follows from

$$q_0^u\subset V=V(f^{-1},f^1,p^u)\subset q^u, \tag{11.41}$$

where $q_0=q(f^{-1},f^1,p_0)$ and $p_0<^*p$.

a) To prove $V\subset q^u$, assume $a\,V\,b$ and $a\neq b$. Then there are $x\in f^1(a)$ and $y\in f^{-1}(b)$ such that $x\,p^u\,y$, i.e. that $\{x\}\,p^d\,\{y\}$. Taking $Z=\emptyset$, $A^\circ=\{x\}$ and $B^\circ=\{y\}$, we see from (11.13) that $\{a\}\,q\,\{b\}$, so $a\,q^u\,b$. Hence $V\subset q^u$, indeed.

b) Assume now $a\,q_0^u\,b$, $a\neq b$. Then $\{a\}\,q_0^d\,\{b\}$, so we can take Z, A° and B° according to (11.13).

If $a\notin Z$ and $b\notin Z$ then there are F, G, F', G' such that

$$f^1(a)\,m\,F\,p_0\,G\,m\,A^\circ\,p_0^d\,B^\circ\,m\,F'\,p_0\,G'\,m\,f^{-1}(b),$$

so $F\,p^d\,G'$, therefore the right-hand side of (11.40) holds for $U=p^u$ with $x\in f^1(a)\cap F$ and $y\in G'\cap f^{-1}(b)$.

If $a\in Z$ and $b\notin Z$ then we have

$$f^1(a)\supset B^\circ\,m\,F'\,p_0\,G'\,m\,f^{-1}(b),$$

and now we can take $x\in f^1(a)\cap F'$ and $y\in G'\cap f^{-1}(b)$. The remaining case $a\notin Z$, $b\in Z$ being analogous, this completes the proof of (11.41). □

11.42 Proposition. *Under the assumptions of* 11.8, *the biuniformity constructed in* (11.12)—(11.14) *is the finest biuniform extension of* $\mathfrak{M}$ *compatible with* $(\mathscr{T}^{-1},\mathscr{T}^1)$.

PROOF. Let $\mathfrak{N}$ be the extension from 11.8, and $\breve{\mathfrak{N}}$ another extension of $\mathfrak{M}$. Given a $\breve{q}\in\breve{\mathfrak{N}}$, we need a $q\in\mathfrak{N}$ such that $q<\breve{q}$.

Take $\breve{q}_0$ and $\breve{q}_1$ from $\breve{\mathfrak{N}}$ such that $\breve{q}_0<^*\breve{q}_1<^*\breve{q}$. Now $q<\breve{q}$ will hold with

$$p=\breve{q}_0|X,\quad f^i(a)=\mathrm{st}^i_{\breve{q}_0}\{a\}\cap X. \tag{11.43}$$

Indeed, assume $A\,q\,B$, and choose Z, A° and B° according to (11.13).

a) If $Z=\emptyset$ then $B\subset s^1(B^\circ)$, so for each $b\in B$,

$$\mathrm{st}^1_{\breve{q}_0}B^\circ\supset\mathrm{st}^1_p B^\circ\ m\ f^{-1}(b)\subset\mathrm{st}^{-1}_{\breve{q}_0}\{b\};$$

hence $B\subset\mathrm{st}^1_{\breve{q}_1}B^\circ$, and analogously $A\subset\mathrm{st}^{-1}_{\breve{q}_1}A^\circ$. Now $A^\circ\,p^d\,B^\circ$ implies $A^\circ\,\breve{q}_0^d\,B^\circ$, so $A\,\breve{q}^d\,B$.

b) If $Z=\{e\}$ then $A^\circ\subset f^{-1}(e)$ and $B^\circ\subset f^1(e)$, so by (11.43),

$$\{e\}\cup A^\circ\subset\mathrm{st}^{-1}_{\breve{q}_0}\{e\},\quad \{e\}\cup B^\circ\subset\mathrm{st}^1_{\breve{q}_0}\{e\},$$

implying (cf. B$\triangle'$) that

$$(\{e\}\cup A^\circ)\,\breve{q}_1^d\,(\{e\}\cup B^\circ). \tag{11.44}$$

Now just like in a), $B\setminus\{e\}\subset s^1(B^\circ)\subset\mathrm{st}^1_{\breve{q}_1}B^\circ$, and an analogous statement holds for A, so

$$A\subset\mathrm{st}^{-1}_{\breve{q}_1}(\{e\}\cup A^\circ),\quad B\subset\mathrm{st}^1_{\breve{q}_1}(\{e\}\cup B^\circ),$$

therefore $A\,q^d\,B$ follows from (11.44). □

11.45 EXAMPLE. Let $X=\mathbf{R}\setminus\{0\}$, $Y=\mathbf{R}$, $\mathcal{U}=\mathcal{U}_{so}|X$ (with $\mathcal{U}_{so}$ from 7.13), $(\mathcal{T}^{-1},\mathcal{T}^1)=(\mathcal{U}_{so}^{-tp},\mathcal{U}_{so}^{tp})$. Now $(\mathcal{T}^{-1},\mathcal{T}^1)$ is a fine regular extension of $(\mathcal{U}^{-tp},\mathcal{U}^{tp})$, the trace filter pair $(\mathcal{F}^{-1}(0),\mathcal{F}^1(0))$ is round, $\mathcal{F}^{-1}(0)\times\mathcal{F}^1(0)\in\mu\mathcal{U}^c$, but $\notin\mu\mathcal{U}^c$, therefore $\mathcal{U}^c$ can be extended to $(\mathcal{T}^{-1},\mathcal{T}^1)$ (in fact, $\mathcal{U}_{so}^c=\mathcal{U}_{so}^b$ is the only possible extension), but $\mathcal{U}^b$ cannot be.

11.46 REMARK. It is most likely that several other results from [6—8] can be generalized to biuniformities. We do not intend to carry through such a programme.

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DIGITAL MODELLING OF CHAOTIC MOTION

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Summary

The effect of digitalization on the modelling of chaotic (ergodic) motion is investigated. The quality Q of the digital model is defined as a non-negative scalar function of the computation precision C (number of cells on unit length). The arbitrary upper bound $Q(C)=1$ (corresponding to an ergodic digital model) means, that the strict monotonic increase of the function $Q(C)$ is a sufficient condition for the convergence of the digitalization procedure. Computational formulas for $Q(C)$ are derived and illustrated. On the basis of the probabilistic approximation of the digital model the mixed precision method is introduced for practical applications.

0. Introduction

In the research field of chaotic motion the examination of specific dynamical ystems is still of considerable interest. This systems display ergodic behaviour in certain regions of the phase space. The examination is usually carried out by simulation, i.e. by the application of numerical models running on digital computers. Despite the fact, that many results in engineering mechanics are based on data produced oy such digital simulation, it has not been investigated yet, in which sense are the ergodic properties of the dynamical system preserved in the digital model. It was registrated only, that the simulation results display extreme sensitivity to the change of pomputation precision.

The aim of this paper is to define a scalar quantity, which serves as a measure of resemblance between the ergodic dynamical system and the digital simulation. This scalar will be denote by Q and called the model quality.

In Sections 1 and 2 computational formulas for the model quality are derived and illustrated. In Sections 3 and 4 a discrete random approach is made to the digital simulation in order to separate the effects of digitalization. In Section 5 the mixed precision method is introduced for the practical applications in digital simulation. Section 6 discusses strange phenomena arising in two dimensions which can be explained on the basis of the previous results.

The applied method and the derived formulas permit us to follow the challenging transformation of a deterministic process (digital simulation) into a random one (ergodic dynamical system). The amount of information necessary to describe the deterministic process increases rapidly with the computation precision. An outside observer, unaware of this information, might call the process random. Above a cer-

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tain computation precision we are compelled to be such "outside observers" because the necessary information exceeds any computer memory. Quite surprisingly the formulas derived from the discrete random approach (which represents the outside observer's point of view) fit the model fairly well. This illustrates, that the origine of chance is (at least partly) our lack of knowledge.

1. Basic notations

DEFINITION 1. The vector function $\mathbf{x}(t)$ describing the continuous, differentiable motion of a material point in the n-dimensional Euclidean space $\mathbf{R}^n$ is wanted in engineering mechanics usually as the solution trajectory of the first order differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) \tag{1}$$

where t denotes time, dot denotes differentiation with respect to t.

DEFINITION 2. The $2n$-dimensional phase space $\mathbf{R}^{2n}$ of the trajectory $\mathbf{a}(t)$ is spanned by the components of the vectors $\mathbf{x}$ and $\dot{\mathbf{x}}$.

DEFINITION 3. The trajectory $\mathbf{x}(t)$ is called chaotic if it remains whithin a bounded region of the phase space without converging to any periodic trajectory.

The efficiency of the methods for the numerical solution of the differential equation (1) were studied in [1]. Chaotic motion in the presence of random external noise is discussed in [2] and [3].

DEFINITION 4. The intersection of a subspace $\mathbf{R}^{2n-1}$ of the phase space with the trajectory $\mathbf{x}(t)$ is called the Poincaré map of the trajectory. The Poincaré map is identical with a vector series in $\mathbf{R}^{2n-1}$ defined by the recursive formula $\mathbf{a}_i = F(\mathbf{a}_{i-1})$, which can be interpreted as an iteration procedure, as well.

For the sake of simplicity we will treat cases with scalar $(n=1)$ mappings F of the one-dimensional unit interval onto itself:

$$a \to F(a), \quad a, F(a) \in [0, 1]. \tag{2}$$

We do not restrict the generality severely with this simplification, because Poincaré mappings of type (2) have been used for the analysis of a wide range of physical problems in higher dimensions, see [4], [5], [6] and [7]. The definition of the model quality will be valid for arbitrary dimensions.

According to Definition 4 mapping (2) defines the iteration procedure

$$a_i = F(a_{i-1}) \tag{3}$$

which is the subject of the forthcoming investigations.

DEFINITION 5. The computation precision C is equal to the number of cells on the unit interval. A cell is a semi-closed interval (containing its starting point) of length $1/C$. The starting point of the k-th cell is at $(k-1)/C$.

DEFINITION 6. Iteration (3) is called ergodic according to [8], if for any integrable function $h(a)$ the equation

$$\lim \frac{1}{n} \sum_{i=0}^{n} h(a_i) = \int_0^1 h(a)\, d\mu(a) \tag{4}$$

holds for μ-almost all values of the initial value a_0. The measure $\mu(a)$ which is preserved by the mapping M is a probability distribution.

Ergodicity means in this case, that the position of the members a_i is a random variable with stationary distribution. Ergodicity is characteristic for iterations corresponding to chaotic motion.

DEFINITION 7. The digitalization a^D of the unit interval $a \in [0, 1]$ is a (one-way) unique mapping:

$$a^D = k(a)$$

where

$$k(a) = (\text{int}\,(Ca))/C + 1.$$

The digitalization I^D of a part-interval I is carri. d out by the digitalization of its endpoints:

$$I = [a^1, a^2], \quad I^D = [k(a^1), k(a^2)].$$

The digitalization $F^D(a)$ of the mapping $F(a)$ is called the digital model of the continuous motion $\mathbf{x}(t)$ and is given by

$$F^D(a) = k(F(k(a))).$$

The digitalized mapping $F^D(a)$ defines the digitalized series a_i^D by

$$a_i^D = F^D(a_{i-1}^D).$$

REMARK. The a's with the upper indices (a^1, a^2, etc.) have nothing to do with the series a_i or a_i^D. This notation is used for constants in the unit interval.

The aim of this paper is to investigate the convergence of digitalization, i.e. to analyse, whether by increasing C the digital model becomes any better or not.

This question is independent of the problem in [1] and can be regarded as a special case of the problem in [2] and [3], since the digitalization can be interpreted as a special random perturbation.

In order to carry out our investigations we will attempt to define the quality Q of the digital model as a non-negative scalar function of the computation precision C. The arbitrary upper bound $Q(C) \leq 1$ means, that the strict monotonic growth of the function $Q(C)$ is a sufficient condition of its convergence. The analysis of the function $Q(C)$ will lead to some hints for the practical applications.

DEFINITION 8. An event A is the visiting of an interval I by the series a_i.

DEFINITION 9. The digitalized event A^D is the visiting of the digitalized interval I^D by the digitalized series a_i^D.

Events defined above will be used to compare the digital model with the original one.

DEFINITION 10. The model quality Q is defined as

$$Q = \min\left(\frac{\mathsf{P}(A^D)}{\mathsf{P}(A)}, \frac{\mathsf{P}(A)}{\mathsf{P}(A^D)}\right)$$

where

$A =$ control event, defined arbitrarily in the sense of Definition 8.
$\mathsf{P}(A) =$ the probability assigned to the event A by the measure $\mu(a)$.
$\mathsf{P}(A^D) =$ the probability of the digitalized control event A^D in the digital model.

Definition 10 is a simplified version of the definition in [10].

Although the von Mises definition [9] of randomness is generally rejected [17], the concept of sample space (Kollektiv) introduced by him in the same book is of fundamental importance. In our case several different sample spaces may be defined in which we are looking for the occurrence of the control event. Depending on which definition we accept, the resulting probabilities will be quite different. We present two alternatives:

DEFINITION 11a. The sample space is the set of infinite iteration sequences, the initial cells of which are selected randomly with equal probability from among the C possibilities in the digital model.

REMARK. The stationary probability distribution $\mu(a)$ guarantees, that during an infinite time-interval the series a_i visits any part-interval of the unit interval, therefore $\mathsf{P}(A)=1$, $Q=\mathsf{P}(A^D)$.

DEFINITION 11b. The sample space is the set of single iteration steps during infinite iteration sequences, the initial cells of which are selected as in Definition 11a in the digital model.

REMARK. In the case of Definition 11b not only the existence, but the exact form of $\mu(a)$ is needed to calculate the model quality.

All forthcoming investigations will be based on Definition 11a.

Definition 10 demonstrates, that the model quality Q depends not only on the computation precision C, but on the control event A, as well. Since A is defined arbitrarily, this may leave some doubts as to the reliability of our definition. To answer this doubts we mention two arguments:

For the comparison of a set of digital models only the relative value of $Q(C)$ is interesting, therefore we may compare them on the basis of any constant control event.

In the following examples we will consider a very general control event A^*. The values $Q^*(C)$ computed on this basis might be regarded as absolute measures of model quality.

DEFINITION 12. The symbolical dynamics is an oriented graph, uniquely determined by the mapping (2) and the computation precision C. The graph is indicating all possible transitions between the cells in the digital model during the iteration (3).

The construction principles of the symbolical dynamics will be illustrated in the first example (Fig. 1). The model quality Q may be directly computed by using the measure $\mu(a)$ and the symbolical dynamics .

2. Example I: The linearized Lorenz-model

We will examine the so-called diadic mapping

$$F(a) = (2 \cdot a) \bmod 1. \tag{5}$$

According to [6], this mapping is a linearized form of the Lorenz model. It is shown in [11] that mapping (5) induces the uniform distribution

$$\mu(a) = a. \tag{6}$$

DEFINITION 13. The random control event A^* is defined by letting A^* occur whenever a member of the infinite series a_i lies within the random interval I. The length s and starting point $\bar{a}$ of this interval I are random variables, defined by

$$\mathsf{P}(s \in [a^1, a^2]) = a^2 - a^1, \quad a^1, a^2, s \in [0, 1], \quad a^2 \geqq a^1 \tag{7}$$

$$\mathsf{P}(\bar{a} \in [a^1, a^2]) = \frac{a^2 - a^1}{1 - s}, \quad a^1, a^2, \bar{a} \in [0, 1-s], \quad a^2 \geqq a^1. \tag{8}$$

REMARK. The above choice of the random interval I is equivalent to

(7*) choosing its length uniformly in the [0, 1] interval, then

(8*) depending on s, choosing the starting point $\bar{a}$ uniformly in the interval $[0, 1-s]$.

The computation of Q (C) based on A^* is easier to understand if we first investigate a simplified version A_1, illustrate the computation procedure in detail, and than generalize the control event in two steps. The other reason for this stepwise introduction is, that computational formulas for A_1 and A_2 are simpler and can be applied in practical cases, as well.

DEFINITION 14. The random control event A_1 is defined by letting A_1 occur, whenever a member of the infinite series a_i lies within the random interval I. The length s and starting point $\bar{a}$ of this interval are defined as

s is a constant with $s \leqq 1/C$

$\bar{a}$ is defined by (8), i.e. uniformly distributed over $[0, 1-s]$.

The computation will be specialized for $C = 3$. The construction principles of the symbolical dynamics are illustrated in Fig. 1 for this case.

DEFINITION 15. If a cell is contained in the digitalized interval I^D (see Definition 7) then it will be called "activated by I".

If there is a property of the mapping $F(a)$ which holds in each point of the interval I, then this property will hold over the full activated cell in the digitalized mapping

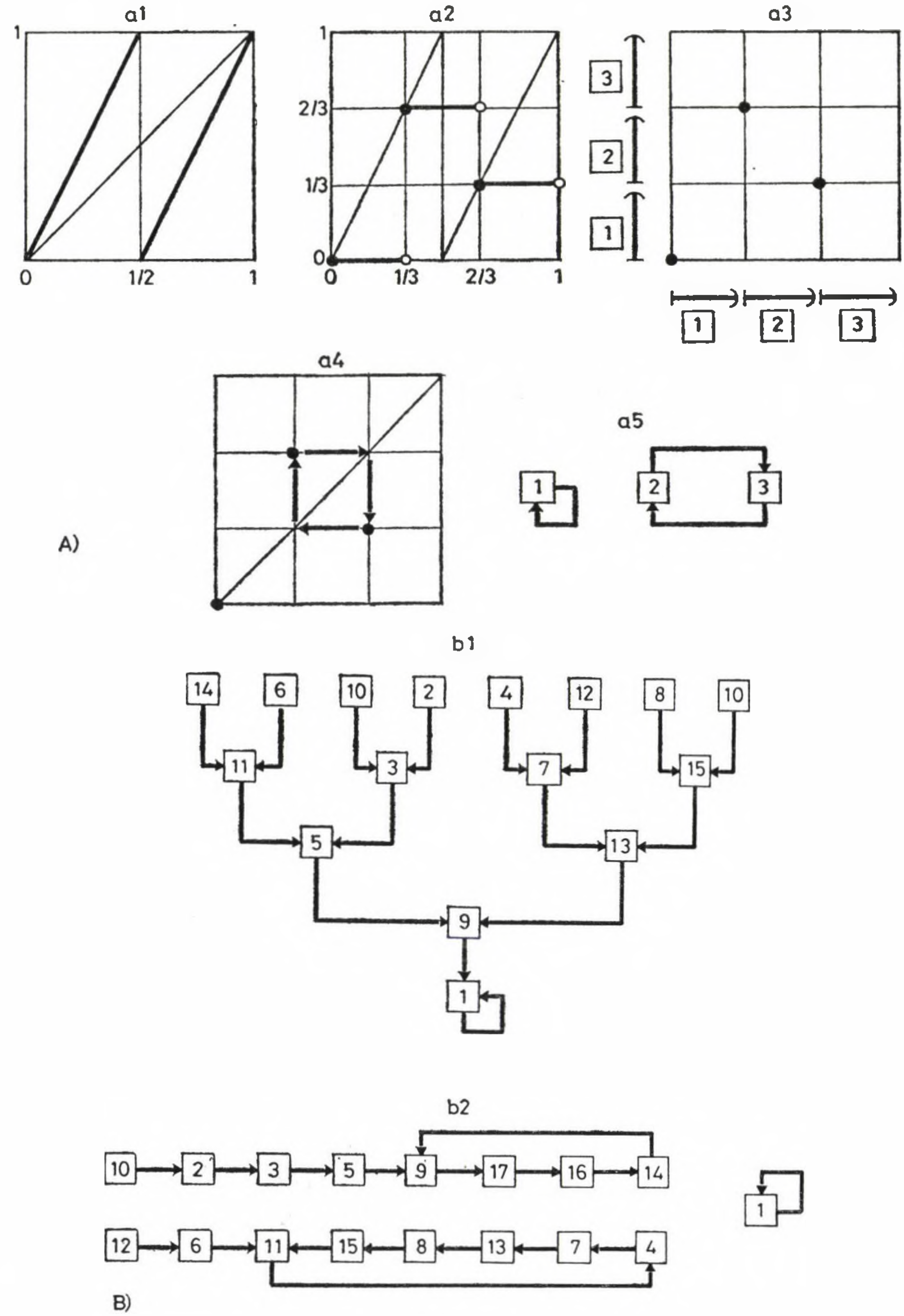

Fig. 1. Symbolical dynamics. Continuous mapping (a1), Discretization $C=3$ (a2), Cell mapping (a3), Iteration (a4), Graph (a5). Examples for $C=16$ (b1) and for $C=17$ (b2)

(see Definition 7). E. g., if I represents the escape gate from the strange attractor in transient chaos discussed in [5] and [12], then this gate will be extended for the full activated cell.

We will use the following shorthand notations for some events:

$$V = \text{The series } a_i \text{ visits } I;$$

$$V_i = \text{The series } a_i^D \text{ visits the } i\text{-th cell};$$

(9) $$E_i = \text{The } i\text{-th cell is activated by } I;$$

$$N(a^1, a^2) = \bar{a} \in (a^1, a^2), \quad a^1 \leq a^2;$$

$$T(a^1, a^2) = s \in (a^1, a^2), \quad a^1 \leq a^2.$$

Using this notations:

(10) $$A_1 = V \cap N[0, 1-s]$$

with probability

(11) $$\mathsf{P}(A_1) = \mathsf{P}(V)\mathsf{P}(N[0, 1-s]) = 1 \cdot 1 = 1.$$

This is in full accordance with our previous statement, that the control event has always probability 1 in the continuous model if we accept Definition 11a. The digitalized control event A_1^D can be written according to Definition 9 and notation (9):

(12) $$A_1^D = \bigcup_{i=1}^{3} (E_i \cap V_i).$$

Since the events E_i are mutually exclusive and independent of the events V_i the probability of A_1^D can be expressed as

(13) $$\mathsf{P}(A_1^D) = \sum_{i=1}^{3} \mathsf{P}(E_i)\mathsf{P}(V_i).$$

Using the definition of the activated cells (Definition 14) and the notation (9) we get:

$$E_1 = N[0, 0],$$

(14) $$E_2 = N\left[\frac{1}{3} - s, \frac{1}{3}\right],$$

$$E_3 = N\left[\frac{2}{3} - s, \frac{2}{3}\right].$$

The probabilities for these are:

$$\mathsf{P}(E_1) = 0,$$

(15) $$\mathsf{P}(E_2) = \frac{s}{1-s},$$

$$\mathsf{P}(E_3) = \frac{s}{1-s}.$$

The probabilities of the V_i events may be computed by using the symbolical dynamics in Fig. 1/a5. We mention again, that a uniform choice of the initial cell is assumed (Definition 11a) therefore the relative frequency computed on the basis of three distinguishable trials is equal to the probability:

$$\mathrm{P}(V_1) = \frac{1}{3},\qquad \mathrm{P}(V_2) = \frac{2}{3},\qquad \mathrm{P}(V_3) = \frac{2}{3}. \tag{16}$$

Substituting the equations (15) and (16) into (13) yields

$$\mathrm{P}(A_1^D) = 0\cdot\frac{1}{3}+\frac{s}{1-s}\cdot\frac{2}{3}+\frac{s}{1-s}\cdot\frac{2}{3} = \frac{4}{3}\cdot\frac{s}{1-s}. \tag{17}$$

Remembering Definition 10 and the remark after Definition 11a, we arrive at

COROLLARY 1. *For the linearized Lorenz model we have:*

$$Q(3) = \frac{4}{3}\cdot\frac{s}{1-s}.$$

This formula is easily generalized for an arbitrary value of C, under the restriction $s \leqq 1/C$, yielding

THEOREM 1. *In the case of the control event A_1 we have*

$$Q(C) = \frac{s}{1-s}\sum_{i=2}^{C} \mathrm{P}(V_i). \tag{18}$$

REMARK. The above formula and all further equations in this section are valid for any mapping of type (2) satisfying (4). The figures illustrate results connected with the diadic mapping.

The function $Q(C)$ was computed on the basis of (18) for (5) and is illustrated with continuous $s=0.05$ line in Fig. 2. Formula (18) confirmes that the sign of the difference function

$$\Delta Q(C) = Q(C)-Q(C-1) \tag{19}$$

does not depend on s, provided $s \leqq 1/C$.

It may be of interest to note that in the investigated range of s the maximum of $Q(C)$ is a linear function of C:

$$Q^{\max}(C) = (C-1)\frac{s}{1-s}. \tag{20}$$

This function is marked with dotted line in Fig. 2. The fact, that the model quality cannot reach the value 1 under this circumstances is explained by considering that

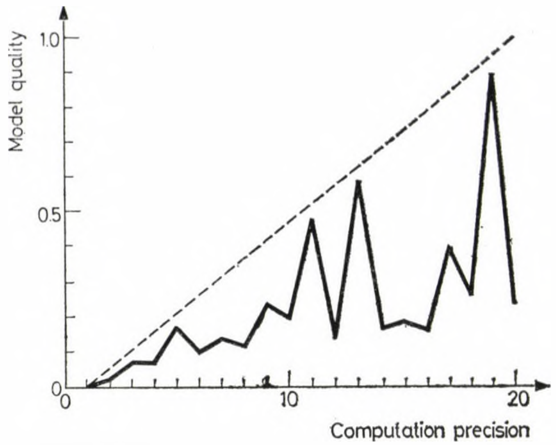

Fig. 2. Model quality (s=const. lines) computed on the basis of the A_1 control event

there is a certain probability that the interval I does not activate any cell of the digital model.

If the value of s exceeds $1/C$ then this probability equals zero. This case will be investigated by using control event A_2.

DEFINITION 16. The random control event A_2 is defined by letting A_2 occur, whenever a member of the infinite series a_i lies within the random interval I. The length s and starting point $\bar{a}$ of this interval are defined as

s is a constant with $s \leqq 1$,

$\bar{a}$ is defined by (8), i.e. uniformly distributed over $[0, 1-s]$.

In this case there is an integer M with

(21) $$\frac{M}{C} < s \leqq \frac{M+1}{C}, \quad 0 \leqq M \leqq C-1.$$

Our former shorthand notations (9) need to be supplemented:

$$s_1 = s - \frac{M}{C},$$

$$s_2 = \frac{M+1}{C} - s,$$

(22) $$V_i^j = \bigcup_{k=0}^{j-1} V_{i+k}, \quad \text{if} \quad j > 0,$$

$$E_i^j = \bigcap_{k=0}^{j-1} E_{i+k}, \quad \text{if} \quad j > 0,$$

$$V_i^j = E_i^j = \{0\}, \quad \text{if} \quad j = 0,$$

$\{0\}$ stands for the empty set.

The brief verbal explanation of these symbols is:

Symbol V_i^j means, that the series a_i visits any one of the cells of j subsequent cells, starting with the i-th cell. Symbol E_i^j means the simultaneous activation of the above mentioned cell group by I. (I covers the starting points of these j cells.)

The digitalized control event A_2^D is described by the generalized version of (12):

$$(23)\qquad A_2^D = \Big[\bigcup_{i=2}^{C-M+1} (E_i^M \cap V_i^M)\Big] \cup \Big[\bigcup_{i=2}^{C-M} (E_i^{M+1} \cap V_i^{M+1})\Big].$$

Due to the uniform choice of the random interval I, the activation of cell 1 has probability zero, therefore this term $(i=1)$ is excluded from our formulas.

In order to compute the model quality, the components of the control event and their probabilities have to be determined. Using (22), these can be expressed similarly to the equation (14):

$$(24)\qquad \begin{aligned} E_i^M &= N\left(\frac{i-2}{C}, \frac{i-2}{C}+s_2\right), \\ E_i^{M+1} &= N\left[\frac{i-1}{C}-s_1, \frac{i-1}{C}\right]. \end{aligned}$$

We get the analogons of (15) by (8):

$$(25)\qquad \begin{aligned} \mathsf{P}(E_i^M) &= \frac{s_2}{1-s}, \\ \mathsf{P}(E_i^{M+1}) &= \frac{s_1}{1-s}. \end{aligned}$$

This results in a form for the model quality as follows:

THEOREM 2. *In the case of the control event A_2 we have:*

$$(26)\qquad Q(C) = \mathsf{P}(A_2^D) = \frac{s_2}{1-s} \sum_{i=2}^{C-M+1} \mathsf{P}(V_i^M) + \frac{s_1}{1-s} \sum_{i=2}^{C-M} \mathsf{P}(V_i^{M+1}).$$

Setting $\mathsf{P}(V_i^j)=1$ for all probabilities in (26) the maximum of the model quality may be computed:

$$(27)\qquad Q^{\max}(C) = 1$$

which confirms our formulas.

The curves for $s=$constant, computed on the basis of (26), are demonstrated in Fig. 3 $(s=i\cdot 0.05,\ i=1, 2, \ldots, 20)$. The upper limit $Q(C)=1-\frac{1}{C}$ is due to the unstable fixed point at 0 of the dyadic mapping.

Now we return to the original, general control event A^*. Using the notations (9) we can generalize (10):

$$(28)\qquad A^* = V \cap T[0, 1) \cap N[0, 1-s].$$

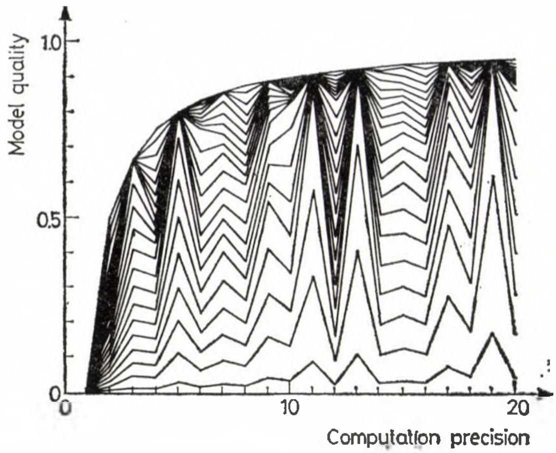

Fig. 3. Model quality (s = const. lines) computed on the basis of the A_2 control event

Naturally:

$$P(A^*) = 1, \quad Q^*(C) = P(A^{*D}). \tag{29}$$

The digitalized control event A^{*D} is described by

$$A^{*D} = \bigcup_{M=0}^{C-1} \{[\bigcup_{i=2}^{C-M+1} (E_i^M \cap V_i^M)] \cup [\bigcup_{i=2}^{C-M} (E_i^{M+1} \cap V_i^{M+1})]\} \tag{30}$$

with components

$$E_i^M = N\left(\frac{i-1}{C}, \frac{i-1}{C}+s_2\right) \cap T\left[\frac{M}{C}, \frac{M+1}{C}\right),$$
$$E_i^{M+1} = N\left[\frac{i}{C}-s_1, \frac{i}{C}\right] \cap T\left[\frac{M}{C}, \frac{M+1}{C}\right]. \tag{31}$$

Since the events N depend on the value of s, the probabilities $P(E_i^j)$ can be computed by integration:

$$P(E_i^M) = \int_{\frac{M}{C}}^{\frac{M+1}{C}} \frac{s_2}{1-s} ds,$$
$$P(E_i^{M+1}) = \int_{\frac{M}{C}}^{\frac{M+1}{C}} \frac{s_1}{1-s} ds. \tag{32}$$

The value ds denotes the density function of s, i.e. the probability $P(T[s, s+ds])$ according to (7). The model quality $Q^*(C)$ on the basis of A^* can be expressed now by considering (29):

(33)
$$Q^*(C)=\sum_{M=0}^{C-1}\left[\int_{\frac{M}{C}}^{\frac{M+1}{C}}\frac{s_2}{1-s}\,ds\sum_{i=2}^{C-M+1}\mathrm{P}(V_i^M)\right]+\sum_{M=0}^{C-2}\left[\int_{\frac{M}{C}}^{\frac{M+1}{C}}\frac{s_1}{1-s}\,ds\sum_{i=2}^{C-M}\mathrm{P}(V_i^{M+1})\right].$$

Since the conditional probability $\mathrm{P}(A^{*D}|s=s_0)$ is equal to $\mathrm{P}(A_2^D)$, and the latter is expressed in (26), the formula (33) can be interpreted as the integration of (26) according to the uniform distribution of s. After carrying out the integration, substituting $K=C-M$ and remembering (29), we can arrive at

THEOREM 3. *In the case of the control event A^* we have*

(34)
$$Q^*(C)=\frac{1}{C}\sum_{K=2}^{C-1}\left[\ln\left(\frac{(K-1)^{(K-1)}(K+1)^{(K+1)}}{K^{2K}}\right)\sum_{i=2}^{K+1}\mathrm{P}(V_i^{C-K})\right]+\frac{2(C-1)}{C^2}\ln 2.$$

The surprising last member is due to the fact, that $\mathrm{P}(s=1)=0$, therefore the last member $(M=C-1)$ had to be drawn out of the summation and computed extra. The function $Q^*(C)$ solely depends on mapping (2) and the computation precision. It is illustrated in Fig. 4 for the mapping (5). The function $Q^*(C)$ is readily seen to be non-monotonic.

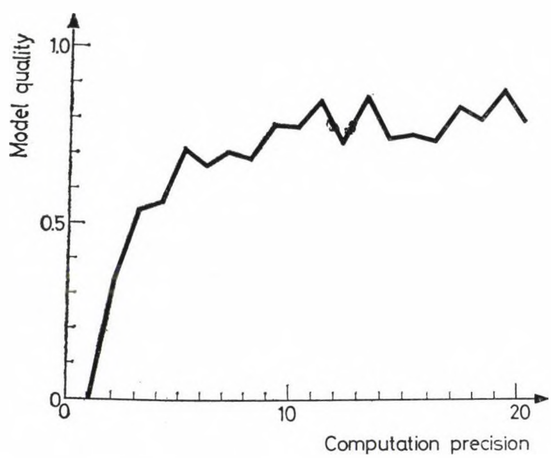

Fig. 4. Model quality Q^*, computed on the basis of the A^* control event

3. The probabilistic approach

Although the direct computation of the model quality $Q(C)$ in the previous section provided much information about the effects of the digitalization procedure, a theoretical and a practical question remained unanswered:

(a) Are the fluctuations in the $Q(C)$ function solely due the to irregular development of the symbolical dynamics?

(b) When running a model on a computer, with maximal computation precision, what is the safest way of improving the model quality?

The clue to both questions is the probabilistic approximation (not to be confused with stochastic approximation) of the digital model. This may sound as a contradiction, because the digital model is a deterministic approximation of the continuous (random) motion, and seemingly we got back to the original problem. However, this is not the case, because we will apply a discrete random approach. This approach enables us to exclude all effects of the symbolical dynamics (question (a)) and to give analytical estimates about the behaviour of the $Q(C)$ function for high values of C (question (b)).

The basis of the probabilistic model is the discretization of the probability distribution $\mu(a)$ defined in (3). We will regard $\mu(a)$ as the distribution of a continuous random variable to which the discrete random variable α is associated in the following way:

$$\text{(35)} \qquad \mathsf{P}_1^i = \mathsf{P}\left(\alpha = \frac{i}{C}\right) = \mu\left(\frac{i+1}{C}\right) - \mu\left(\frac{i}{C}\right) \qquad (i = 1, 2, \ldots, C).$$

The iteration procedure (3) will be substituted by a sequence of drawings distinguishable balls B^i from an urn with replacement, where the probability of drawing the i-th ball is P_1^i. We carry on doing this as long as we are drawing different balls. If a ball appears the second time, we stop. Such sequences form the sample space. There are infinite independent sequences, therefore this sample space is different from the one defined in Definition 11a.

In the probabilistic approach we will treat the simplest control event A_1, defined in Definition 14. The probabilities of the events E_i are (similarly to (15)):

$$\text{(36)} \qquad \mathsf{P}(E_i) = \frac{s}{1-s}, \quad s \leqq 1/C, \quad i = 2, 3, \ldots, C.$$

The probabilities $\mathsf{P}(V_i)$ cannot be computed on the basis of the symbolical dynamics, since the form of the mapping is not known. (It was our aim to exclude the effect of the symbolical dynamics.) The length t of the above mentioned sequences (elements of the sample space) will be denoted with the discrete random variable:

(37)

$$\mathsf{P}(\beta = t) = \mathsf{P}_2^t = \sum_{i_1=1}^{C} \Big[\mathsf{P}_1^{i_1} \cdot \sum_{\substack{i_2=1 \\ i_2 \neq i_1}}^{C} \Big[\mathsf{P}_1^{i_2} \ldots \sum_{\substack{i_{t-1}=1 \\ i_{t-1} \neq i_1, i_2, \ldots, i_{t-2}}}^{C} \mathsf{P}_1^{i_{t-1}}(\mathsf{P}_1^{i_1} + \mathsf{P}_1^{i_2} + \ldots + \mathsf{P}_1^{i_{t-1}})\Big] \ldots\Big].$$

If our sequence ends before we saw the ball B^i, then its logical value is 0, since we are interested in the probability $\mathsf{P}(V_i)$ of the sequences where B^i was observed. Formula (37) describes the probability of the sequences of length t. If we exclude the sequences not containing B^i, and sum up their probabilities from $t=2$ to $C+1$, then we will get $(1-\mathsf{P}(V_i))$. Since formula (37) is a product of probabilities, excluding the sequences containing B^i may be done simply by setting

$$\text{(38)} \qquad \mathsf{P}_1^i = 0$$

into (37). According to this considerations:

(39) $$\mathsf{P}(V_i) = 1 - \sum_{t=2}^{C+1} \mathsf{P}_2^t \qquad (\mathsf{P}_1^i = 0).$$

Now we can substitute into the equation (18)

(40) $$Q(C) = \frac{s}{1-s} \sum_{i=2}^{C} \left(1 - \sum_{t=2}^{C+1} \mathsf{P}_2^t\right) \qquad (\mathsf{P}_1^i = 0).$$

To compute (40) we have to know the values P_1^i, which can be computed using (35) for any specific $\mu(a)$.

4. Example II: Resonance with periodic distribution

We will investigate the periodic distribution

(41) $$\mu(a) = a - \frac{1}{8\pi}\cos(8\pi a) + 1.$$

The function $Q(C)$ is illustrated in Fig. 5 for $s=0.01$. Now the difference function (19) does not depend on s.

Fig. 5 demonstrates, that the non-monotonity of the function is not only due to the irregular change of the symbolical dynamics. The valley in the function is due to a special interaction between the probability distribution and the computation precision. We could call this phenomenon resonance, because the valley is at double frequency of the density function. Similar phenomena may occur in practical computations, because the investigated density functions are often non-smooth, non-continuous. This answers now question (a) of Section 3.

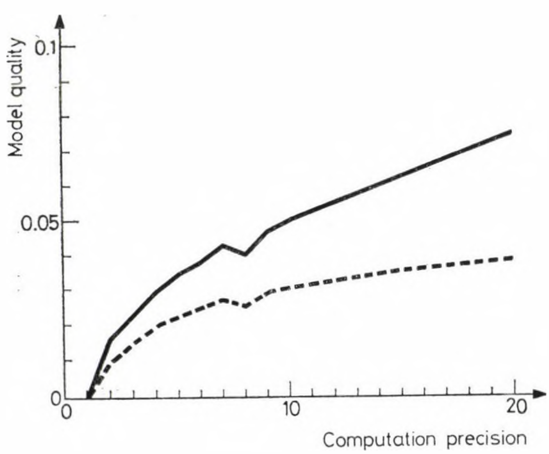

Fig. 5. Model quality and Shannon entropy (dotted line, divided by 100), computed on the basis of the A_1 control event, using the random approach

It may be of interest, that the diagram of the Shannon-entropy H ([15]) of the variable α as a function of C (dotted line in Fig. 5) very closely resembles the diagram of $Q(C)$. This fact is explained if we take into account, that the first reoccurrence time may be regarded as a special case $k=1$ of the first reoccurrence of a k-tuple in a random series. This explanation is based on a conjecture of T. Nemetz.

5. The method of mixed precision

We will now proceed to answer question (b) at the beginning of Section 3.

For the uniform distribution it is shown in [3], that the expected value $\mathsf{E}(\beta)$ of the first reoccurrence converges to

$$(42) \qquad \mathsf{E}(\beta) \underset{C\to\infty}{=} \sqrt{0.5\pi C}.$$

DEFINITION 16. The cycle length γ is equal to the number of steps between the first and the second observation of the same ball B^r.

REMARK. The second observation means, that the sequence ends. This integer was denoted by t.

Using the formulas for conditional probability we can derive from (42), that the expected value of the cycle length γ is:

$$(43) \qquad \mathsf{E}(\gamma) \underset{C\to\infty}{=} 0.5\mathsf{E}(\beta).$$

Both results are in very fair accordance with the empirical data collected during over 10^5 trials, thus the probabilistic approach is empirically justified, and the results derived from this approach may be regarded as explanation to the behaviour of the digital model.

Equation (43) may be the clue to the extraordinary stability of the cycles in the digital model. Regarding the symbolical dynamics as a random graph, the equation (43) indicates that the order of magnitude of cells visited by one cycle is with high probability equal to C. This stability makes it impossible to improve the model quality significantly by small random perturbations of the iteration.

The best way to improve the model quality is to vary the computation precision during the iteration. If we change the value of C, then the iteration will jump to a totally different symbolical dynamics, and the model quality will rapidly converge to 1. However, it is necessary to know when to change C. It is advisable to wait until the iteration falls into a cycle. To notice this we have to know the approximate time, when the iteration enters the cycle. This approximation is given by (4), and may be multiplicated by a small safety factor in practical applications. If the cycle is completed the value of C is to be changed.

Since a slight change of C is impossible in practical cases (high precision), we imitate this by applying numerous constant local perturbations to the symbolical dynamics. A slight increase of C is imitated, for example, by using the perturbed discrete mapping $F(a)$ defined by

$$(44)\qquad F(a)=\begin{cases} F(a) & \text{if} \quad a<0.5 \quad \text{or} \quad a=1 \\ F\left(a+\dfrac{1}{C}\right) & \text{if} \quad a\geqq 0.5 \end{cases} \qquad \left(a=\frac{k}{C},\ k=0,1,2,\dots,C\right).$$

Many similar mappings (containing only local perturbations) can be constructed and used cyclically during the iteration.

Since this perturbed mappings can be regarded as imitations of different computation precisions, the described method can be called the mixed precision method and answers question (b) of Section 3.

The mixed precision method presents an alternative for generating long binary sequences without cycles. The problem is usually discussed in connection with so-called shift-registers [16].

The mixed precision method improves $Q(C)$ nearly to the maximum.

6. Examples in two dimensions

In this section we try to illustrate only the problems and strange phenomena arising during digital experiments in two dimensions, without presenting further theoretical results.

First we investigate the iteration procedure consisting of two independent one-dimensional mappings:

$$(45)\qquad \begin{aligned} F(a) &= 4a(1-a), \\ y_{i+1} &= F(y_i), \\ x_{i+1} &= F(x_i). \end{aligned}$$

This mapping is the well-known logistic mapping, the ergodicity of which is discussed in [3] and [18]. The values x_i and y_i are therefore independent random variables, and if we plot the points $P_i(x_i, y_i)$ on the unit square the figure should look like Fig. 6/a where the pairs (x_i, y_i) are completely independent. However, from a number of starting points P_0 the picture is totally different (Fig. 6/b). The computations were carried out at $C=2\cdot 10^5$ precision. The explanation of the appearance of the strange functions lies in the symbolical dynamics and the equation (53). This equation indicates, that even at very high precision only very few, huge cycles coexist in the symbolical dynamics. The probability that both x and y will be started from the attraction territory of such a big cycle is not negligible. In this case, after entering the cycle, there will be a constant functional relation

$$(46)\qquad \begin{aligned} y &= F^n(x), \\ x &= F^{-n}(y) \end{aligned}$$

between the x and y values, n denoting their distance in the cycle. The line in Fig. 6/b is the $y=M^2(x)$ line.

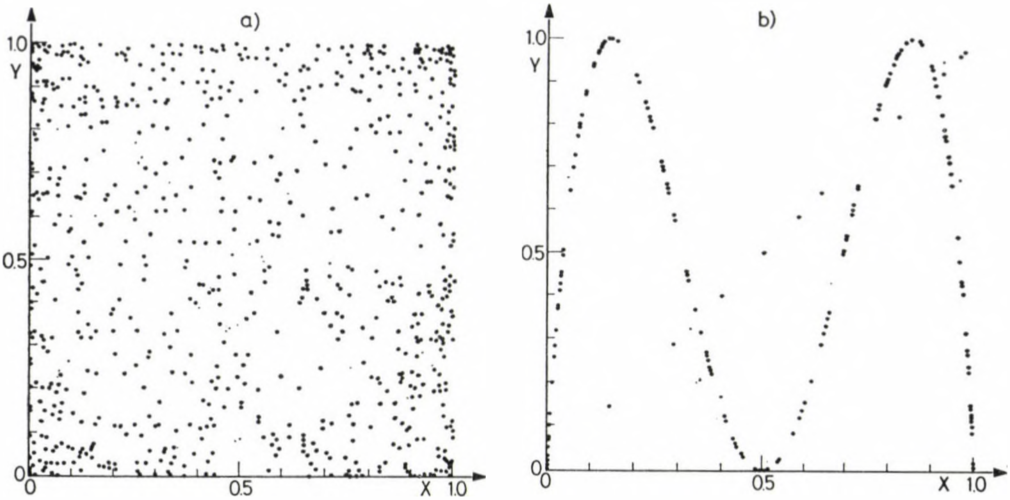

Fig. 6. Asymmetric trap of two-dimensional motion

Even more interesting is the case of the

$$x_{i+1} = F(y_i), \qquad (47)$$
$$y_{i+1} = F(x_i)$$

mapping. Fig. 7 illustrates the picture on the unit square. (This is not typical, but occurs at a number of starting points.) If the two starting points enter the same cycle,

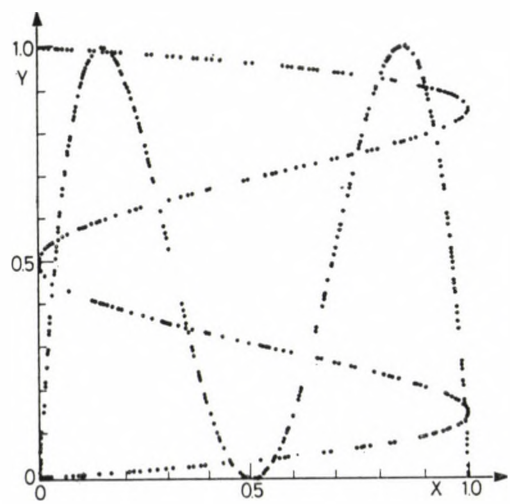

Fig. 7. Symmetric trap of two-dimensional motion

then the

$$x = F^n(y), \quad y = F^n(x) \tag{48}$$

symmetric relation holds. Fig. 7 illustrates the case $n=2$.

This phenomena, although strange and interesting, are disturbing when investigating a real model, because the functional relations (46) and (48) do not hold in the continuous model. They may be very disturbing when examining simultaneously the trajectories of a dynamical system to detect the global behaviour [14], because eq. (47) may be regarded as two simultaneous (independent) solutions of a one-dimensional problem. To get a clear picture of the dynamical system we have to destroy this phenomena, and this can be done efficiently by the mixed precision method. In higher dimensions, or, when following simultaneously more trajectories it is essential that the precisions applied for the computation of the different trajectories (or dimensions) should be different. This is a supplement to the former description of the mixed precision method.

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UNIFORM AND PROXIMAL EXTENSIONS WITH CARDINALITY LIMITATIONS

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Abstract

Theorems of the following type will be proved: if a compatible/continuous uniformity/proximity given on a subspace of a topological space can be extended to a compatible/continuous uniformity/proximity on the whole space then, under suitable conditions, it has an extension satisfying some additional requirements.

Throughout this paper, $X=(X, \mathcal{T})$ *is a completely regular (not necessarily* T_0*) topological space, and S a subspace of it.*

Generalizing results due to Gantner [10] and Úry [22], we shall give necessary and sufficient conditions for the existence of a *compatible* extension to X of a uniformity on S such that the weight, the covering character and the point character of the extension do not exceed prescribed cardinals; one of the conditions will require that there should exist an extension at all (Theorem 1.8). We shall also prove a similar result for *continuous* extensions of uniformities (Theorem 1.11).

§ 2 contains analogous statements (deduced from the results of § 1) for extensions of proximities (Theorems 2.1 and 2.3).

Notations and terminology. Uniformities will be described in terms of coverings. Lower case letters u and v will be used for denoting coverings. $<$ means refinement, $<^*$ star refinement in the stronger sense, $<^{(*)}$ star refinement in the weaker sense, i.e.

$$u <^* v \quad \text{iff} \quad \mathrm{St}(u, u) < v,$$

$$u <^{(*)} v \quad \text{iff} \quad \{\mathrm{St}(x, u)\colon x \in X\} < v,$$

where (with $A \subset X$):

$$\mathrm{St}(x, u) = \bigcup \{B\colon x \in B \in u\}, \quad \mathrm{St}(A, u) = \bigcup_{x \in A} \mathrm{St}(x, u),$$

$$\mathrm{St}(u_0, u) = \{\mathrm{St}(A, u)\colon A \in u_0\}$$

(the last notation will also be used for an arbitrary family u_0 of subsets in X). Moreover,

$$u(\cap)v = \{A \cap B\colon A \in u,\ B \in v\}.$$

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$u|S$ is the restriction of the covering u to S. Analogous notation will be used for restrictions of uniformities, and of filters.

Let $\mathfrak{f}$ be a filter in X, and $\mathcal{U}$ a uniformity on X. $\mathfrak{f}$ is called *$\mathcal{U}$-round* [12] if for any $F\in\mathfrak{f}$ there are $F_0\in\mathfrak{f}$ and $u\in\mathcal{U}$ such that $\mathrm{St}(F_0, u)\subset F$.

A uniformity, proximity, or pseudometric on X is *continuous* [1] if it induces a topology coarser than $\mathcal{T}$. S is *P-embedded in X* [19] in case every continuous pseudometric on S can be extended to a continuous pseudometric on X.

§ 1. Extensions of uniformities

1.1 The *weight* [21] of the uniformity $\mathcal{U}$, denoted by $w(\mathcal{U})$, is the smallest infinite cardinal $\varkappa$ for which $\mathcal{U}$ has a base (equivalently: a subbase) of cardinality $\leqq\varkappa$. Úry [22] proved:

THEOREM. *If $\mathcal{U}$ is a compatible uniformity on the closed subset S, $w(\mathcal{U})\leqq\varkappa$, there exists a compatible uniformity $\mathcal{V}$ on X such that $w(\mathcal{V})\leqq\varkappa$, and S is P-embedding in X then $\mathcal{U}$ has a compatible extension $\tilde{\mathcal{U}}$ to X with $w(\tilde{\mathcal{U}})\leqq\varkappa$.*

If S is P-embedded then each compatible uniformity has a continuous extension [1] (induce the uniformity by a family of pseudometrics; extend this family; take the induced uniformity). Therefore it will be a generalization of the above theorem if we only assume that the uniformity $\mathcal{U}$ itself has a continuous extension (Corollary 1.8).

1.2 The *covering character* [11] of a uniformity $\mathcal{U}$, denoted by $\mathrm{cc}(\mathcal{U})$, is the smallest infinite cardinal $\varkappa$ for which there is a base (equivalently: a subbase) $\mathcal{B}$ for $\mathcal{U}$ such that $|u|<\varkappa$ whenever $u\in\mathcal{B}$. According to Gantner [10], the remark after 3.22, we have:

THEOREM. *If GCH holds, $\varkappa$ is an infinite successor cardinal, $\mathcal{U}$ is a compatible uniformity on S, $\mathrm{cc}(\mathcal{U})\leqq\varkappa$, and $\mathcal{U}$ has a compatible extension to X, then it has a compatible extension $\tilde{\mathcal{U}}$ with $\mathrm{cc}(\tilde{\mathcal{U}})\leqq\varkappa$.*

A slight modification of the proof shows that this result holds for arbitrary infinite cardinals, i.e. the conclusion can be written as $\mathrm{cc}(\mathcal{U})=\mathrm{cc}(\tilde{\mathcal{U}})$. We shall see (Corollary 1.9) that GCH can also be dropped. GCH is, however, essential in the original proof, since it makes use of the following lemma, which is not true without GCH (cf. [15] Theorem 2).

LEMMA ([9] 3.10; see also [13, 14]). *If GCH holds, $\mathcal{U}$ is a uniformity, $\varkappa$ is an infinite cardinal, $u\in\mathcal{U}$ and $|u|<\varkappa$ then there is a $v\in\mathcal{U}$ such that $v<^* u$ and $|v|<\varkappa$.*

Let us also remark that the following weaker form of our Corollary 1.9 is essentially contained by [10] § 2:

PROPOSITION. *If $\varkappa$ is an infinite successor cardinal, and each compatible uniformity $\mathcal{V}$ on S satisfying $\mathrm{cc}(\mathcal{V})\leqq\varkappa$ can be extended to a compatible uniformity on X then each uniformity $\mathcal{U}$ of this type has a compatible extension $\tilde{\mathcal{U}}$ with $\mathrm{cc}(\tilde{\mathcal{U}})\leqq\varkappa$.*

Thus, instead of GCH, it has to be assumed here that several uniformities other than $\mathcal{U}$ can be extended, too; in this respect, the proposition is very similar to Theorem 1.1.

1.3 We shall consider one more cardinal invariant of uniformities. For a covering u of X, let ord u denote the smallest (possibly finite) cardinality λ for which no point of X is contained by λ different elements of u. Now the *point character* [8] of the uniformity $\mathcal{U}$, denoted by pc $(\mathcal{U})$, is the smallest (possibly finite) cardinal $\varkappa$ for which there exists a base $\mathcal{B}$ for $\mathcal{U}$ such that ord $u<\varkappa$ whenever $u\in\mathcal{B}$.

LEMMA. a) *For* $\varkappa\geqq\omega$, $\mathrm{pc}(\mathcal{U})\leqq\varkappa$ *iff there is a subbase* $\mathcal{S}$ *for* $\mathcal{U}$ *such that* ord $u<\varkappa$ *whenever* $u\in\mathcal{S}$.

b) *If* $\mathcal{U}=\sup\{\mathcal{U}_1,\mathcal{U}_2\}$ *then* $w(\mathcal{U})\leqq w(\mathcal{U}_1)w(\mathcal{U}_2)$, $\mathrm{cc}(\mathcal{U})\leqq\mathrm{cc}(\mathcal{U}_1)\,\mathrm{cc}(\mathcal{U}_2)$, $\mathrm{pc}(\mathcal{U})\leqq\mathrm{pc}(\mathcal{U}_1)\,\mathrm{pc}(\mathcal{U}_2)$.

c) *If* $\mathcal{U}=\mathcal{V}|S$ *then* $w(\mathcal{U})\leqq w(\mathcal{V})$, $\mathrm{cc}(\mathcal{U})\leqq\mathrm{cc}(\mathcal{V})$, $\mathrm{pc}(\mathcal{U})\leqq\mathrm{pc}(\mathcal{V})$. □

REMARKS. a) In the earlier definitions of the point character [15, 16], one writes ord $u\leqq\varkappa$ instead of ord $u<\varkappa$ and/or "contained by more than λ" instead of "contained by λ"; these definitions give a less fine classification of uniformities than the one we have taken from [8].

b) It is not at all evident that there exist uniformities with arbitrarily large point character, see [20, 15].

c) For $-1\leqq n<\omega$, $\Delta d\mathcal{U}=n$ iff $\mathrm{pc}(\mathcal{U})=n+3$ (where Δd is the large dimension from [11]).

1.4 We shall need a construction that assigns to a uniformity on X and to another uniformity on S a third one on X. For two systems of sets u and v in X, let

$$v+u = v\cup \mathrm{St}(u, v). \tag{1}$$

If $\mathcal{U}$ is a uniformity on S and $\mathcal{V}$ a uniformity on X then define

$$\mathcal{V}+\mathcal{U} = \{w\colon \exists v\in\mathcal{V},\ \exists u\in\mathcal{U},\ v+u < w\}. \tag{2}$$

Under suitable conditions, $\mathcal{V}+\mathcal{U}$ will be a uniformity on X. Observe that if $v_1<v$ and $u_1<u$ then $v_1+u_1<v+u$, therefore it is enough to take v and u in (2) from bases

LEMMA. *Assume that* $\mathcal{V}$ *is a continuous uniformity on* X, $\mathcal{U}$ *is a uniformity on* S, *and*

$$\mathcal{U}\subset\mathcal{V}|S. \tag{3}$$

Then

a) $\mathcal{V}+\mathcal{U}$ *is a continuous uniformity, and it is an extension of* $\mathcal{U}$;

b) *if* S *is closed,* $\mathcal{V}$ *and* $\mathcal{U}$ *are compatible then so is* $\mathcal{V}+\mathcal{U}$;

c) *if* $\mathcal{V}$ *and* $\mathcal{U}$ *are complete, and the conditions of* b) *are fulfilled then* $\mathcal{V}+\mathcal{U}$ *is complete, too;*

d) $w(\mathcal{V}+\mathcal{U})\leqq w(\mathcal{V})w(\mathcal{U})$;

e) $\mathrm{cc}(\mathcal{V}+\mathcal{U})\leqq\mathrm{cc}(\mathcal{V})\mathrm{cc}(\mathcal{U})$;

f) $\mathrm{pc}(\mathcal{V}+\mathcal{U})\leqq\mathrm{pc}(\mathcal{V})+\mathrm{pc}(\mathcal{U})$.

REMARKS. a) It follows from (3) and the continuity of $\mathscr{V}$ that $\mathscr{U}$ is continuous, too.

b) A similar construction for metrics was given by Bing [3]. In [6], we defined $\mathscr{V}+\mathscr{U}$ in terms of entourages, and proved lemmas more general than a) and b) of the above lemma. For the convenience of the reader, we shall give complete proofs instead of relying on [6]. (Let us note that the proof of the Triangle Axiom will be more cumbersome than with the technique of entourages.)[1]

c) The inequality in f) is not sharp if $\mathrm{pc}(\mathscr{V})$ and $\mathrm{pc}(\mathscr{U})$ are finite. (The same holds for Lemma 1.3 b)). Instead of e), $\mathrm{cc}(\mathscr{V}+\mathscr{U}) \leqq \mathrm{cc}(\mathscr{V})$ is also true.

PROOF. a) 1° Given $v \in \mathscr{V}$ and $u \in \mathscr{U}$, choose $u_1, u_0 \in \mathscr{U}$ such that $u_0 <^* u_1 <^* u$. By (3) there is a $v_1 \in \mathscr{V}$ with $v_1|S = u_0$. Pick now a $v_0 \in \mathscr{V}$ such that $v_0 <^* v_1$ and $v_0 <^* v$. We claim that

$$(4) \qquad v_0 + u_0 <^* v + u.$$

Therefore $\mathscr{V}+\mathscr{U}$ is a uniformity (since the other axioms are evidently satisfied).

To prove (4), take an $A \in v_0 + u_0$; we have to find an element of $v+u$ containing $\mathrm{St}(A, v_0+u_0)$.

If $\mathrm{St}(A, v_0+u_0) \cap S = \emptyset$ then $\mathrm{St}(A, v_0+u_0) = \mathrm{St}(A, v_0)$ and $A \in v_0$, so $v_0 <^* v$ implies that $\mathrm{St}(A, v_0+u_0)$ is contained by an element of $v \subset v+u$.

Otherwise, pick a $B \in v_0 + u_0$ such that $B \cap A \neq \emptyset \neq B \cap S$; in case $A \cap S \neq \emptyset$, let $B = A$. Now

$$(5) \qquad B \subset \mathrm{St}(H_0, v_0)$$

with a suitable $H_0 \in u_0$ (since either B is of the form $\mathrm{St}(H_0, v_0)$, or $B \in v_0$, and then we can take any $H_0 \in u_0$ meeting B). As $u_0 <^* u_1 <^* u$, there is a $H \in u$ such that

$$(6) \qquad \mathrm{St}(\mathrm{St}(H_0, u_0), u_0) \subset H.$$

It is now enough to check that

$$(7) \qquad \mathrm{St}(A, v_0+u_0) \subset \mathrm{St}(H, v),$$

since the right-hand side of (7) belongs to $\mathrm{St}(u, v) \subset v+u$.

In order to prove (7), let us first observe that if

$$(8) \qquad T = \mathrm{St}(\mathrm{St}(H_0, v_0), v_0)$$

then $A \subset T$ [if $B = A$ then A is already in $\mathrm{St}(H_0, v_0)$ by (5); if $B \neq A$ then $A \cap S = \emptyset$, hence $A \in v_0$, and $A \subset T$ follows from (5) and $A \cap B \neq \emptyset$]. Thus (7) will be proved if we show that $C \in v_0 + u_0$ and $C \cap T \neq \emptyset$ imply $C \subset \mathrm{St}(H, v)$.

If $C \in v_0$ then (8) and $v_0 <^* v$ give $C \subset \mathrm{St}(H_0, v) \subset \mathrm{St}(H, v)$.

On the other hand, if $C \in \mathrm{St}(u_0, v_0)$ then

$$C \subset \mathrm{St}(\mathrm{St}(\mathrm{St}(T, v_0) \cap S, u_0), v_0).$$

[1] ADDED IN PROOF. The two constructions will be compared in [23] 2.4.

From $v_0 <^* v_1$ and (8) we have $\mathrm{St}(T, v_0) \subset \mathrm{St}(H_0, v_1)$, so $v_1|S = u_0$ gives

$$C \subset \mathrm{St}(\mathrm{St}(\mathrm{St}(H_0, u_0), u_0), v_0),$$

and then (6) implies $C \subset \mathrm{St}(H, v_0)$, i.e. $C \subset \mathrm{St}(H, v)$ again.

2° $\mathscr{V}+\mathscr{U}$ is continuous because we have $\mathscr{V}+\mathscr{U} \subset \mathscr{V}$ from $v+u \supset v$.

3° $u < v+u|S$, hence $\mathscr{V}+\mathscr{U}|S \subset \mathscr{U}$. On the other hand, if v_0 and u_0 are chosen as in 1° then $v_0+u_0|S < u$, thus $\mathscr{V}+\mathscr{U}|S \supset \mathscr{U}$. Therefore $\mathscr{V}+\mathscr{U}$ is indeed an extension of $\mathscr{U}$.

b) Let S be closed, $\mathscr{V}$ and $\mathscr{U}$ compatible. As $\mathscr{V}+\mathscr{U}$ is continuous, the compatibility will follow if we show that for any $a \in X$ and $v \in \mathscr{V}$ there are $u_0 \in \mathscr{U}$ and $v_0 \in \mathscr{V}$ such that

$$\mathrm{St}(a, v_0+u_0) \subset \mathrm{St}(a, v).$$

If $a \notin S$ then take $v_0 \in \mathscr{V}$ such that $v_0 < v$ and $\mathrm{St}(a, v_0) \cap S = \emptyset$. Now an arbitrary u_0 will do.

If $a \in S$ then choose $v_1 \in \mathscr{V}$, $u_1, u_0 \in \mathscr{U}$ and finally $v_0 \in \mathscr{V}$ with the following properties: $v_1 <^* v$, $\mathrm{St}(a, u_1) \subset \mathrm{St}(a, v_1)$, $u_0 <^* u_1$, $v_0|S < u_0$, $v_0 < v_1$.

c) Assume now that $\mathscr{V}$ and $\mathscr{U}$ are complete. To prove that $\mathscr{V}+\mathscr{U}$ is complete, too, it is enough to show that each $\mathscr{V}+\mathscr{U}$-round $\mathscr{V}+\mathscr{U}$-Cauchy filter has a cluster point. Let $\mathfrak{f}$ be such a filter.

If some element of $\mathfrak{f}$ does not meet S then, as $\mathfrak{f}$ is round, there are $F \in \mathfrak{f}$, $v_1 \in \mathscr{V}$ and $u_1 \in \mathscr{U}$ such that

$$\mathrm{St}(F, v_1+u_1) \cap S = \emptyset. \tag{9}$$

This implies that $\mathfrak{f}$ is $\mathscr{V}$-Cauchy. Indeed, if $v \in \mathscr{V}$ then, as $\mathfrak{f}$ is $\mathscr{V}+\mathscr{U}$-Cauchy, there is an $F_0 \in \mathfrak{f} \cap (v_0+u_1)$ where $v_0 = v(\cap)v_1$; by (9), $F_0 \cap S = \emptyset$ (because $F_0 \cap F \neq \emptyset$ and $v_0+u_1 < v_1+u_1$); therefore (1) implies that $F_0 \in \mathfrak{f} \cap v_0$. Now $\mathfrak{f}$ is convergent because $\mathscr{V}$ is complete (recall that, by b), $\mathscr{V}$ and $\mathscr{V}+\mathscr{U}$ induce the same topology).

On the other hand, if each element of $\mathfrak{f}$ meets S then $\mathfrak{f}|S$ is a filter, which is $\mathscr{V}+\mathscr{U}|S$-Cauchy, and so $\mathscr{U}$-Cauchy by a). Since $\mathscr{U}$ is complete, $\mathfrak{f}|S$ has a cluster point; hence $\mathfrak{f}$ has a cluster point, too.

d) and e). Evident.

f) Given $u \in \mathscr{U}$ and $v \in \mathscr{V}$, we need $u_0 \in \mathscr{U}$ and $v_0 \in \mathscr{V}$ such that $u_0 < u$, $v_0 < v$, and

$$\mathrm{ord}(v_0+u_0) < \mathrm{pc}(\mathscr{V}) + \mathrm{pc}(\mathscr{U}). \tag{10}$$

First choose $u_1 \in \mathscr{U}$ with $u_1 < u$ and $\mathrm{ord}\, u_1 < \mathrm{pc}(\mathscr{U})$. Now we can take a uniform strict shrinking of u_1 (cf. [11] IV. 19), i.e. there are $u_0, u_2 \in \mathscr{U}$ and a surjection $\varphi: u_1 \to u_0$ such that

$$\mathrm{St}(\varphi(A), u_2) \subset A \qquad (A \in u_1). \tag{11}$$

Clearly, $u_0 < u$. According to (3), there is a $v_2 \in \mathscr{V}$ with $v_2|S = u_2$. Choose $v_0 \in \mathscr{V}$ such that $v_0 < v$, $v_0 <^* v_2$, and $\mathrm{ord}\, v_0 < \mathrm{pc}(\mathscr{V})$. To prove (10), it is enough to show that

$$\text{(12)} \qquad \operatorname{ord} \operatorname{St}(u_0, v_0) \leqq \operatorname{ord} u_1$$

(cf. (1) and the inequalities for $\operatorname{ord} v_0$ and $\operatorname{ord} u_1$).

Let $b \in X$ be fixed. If $\operatorname{St}(b, v_0) \cap S = \emptyset$ then no element of $\operatorname{St}(u_0, v_0)$ contains b. Otherwise, we may choose a point $c \in \operatorname{St}(b, v_0) \cap S$. Now if $b \in \operatorname{St}(\varphi(A), v_0)$ for some $A \in u_1$ then $c \in \operatorname{St}(\varphi(A), v_2)$, hence $c \in \operatorname{St}(\varphi(A), u_2) \subset A$. This means that if b is contained by $\varkappa$ different elements of $\operatorname{St}(u_0, v_0)$ then c is contained by at least $\varkappa$ different elements of u_1, so (12) holds indeed. □

1.5 LEMMA ([10]) § 2, weaker results in [2, 19]). *Let $\mathscr{U}$ be the fine uniformity of the topology $\mathscr{T}$, and $\varkappa \geqq \omega$. Then*

$$\mathscr{B}_\varkappa(\mathscr{T}) = \{u \in \mathscr{U}\colon |u| < \varkappa\} \quad \textit{and} \quad \mathscr{B}^{lf}_\varkappa(\mathscr{T}) = \{u \in \mathscr{B}_\varkappa(\mathscr{T})\colon u \text{ is locally finite}\}$$

are bases for the same uniformity, which is compatible with $\mathscr{T}$.

NOTATION. The uniformity furnished by this lemma will be denoted by $\mathscr{U}_\varkappa(\mathscr{T})$.

PROOF. If $u_1, u_2 \in \mathscr{B}_\varkappa(\mathscr{T})$ then $u_1 (\cap) u_2 \in \mathscr{B}_\varkappa(\mathscr{T})$, and similarly for $\mathscr{B}^{lf}_\varkappa(\mathscr{T})$. We are going to show that for any $u \in \mathscr{B}_\varkappa(\mathscr{T})$ there is a $u_0 \in \mathscr{B}^{lf}_\varkappa(\mathscr{T})$ with $u_0 <^{(*)} u$. This implies that $\mathscr{B}_\varkappa(\mathscr{T})$ and $\mathscr{B}^{lf}_\varkappa(\mathscr{T})$ are bases for the same uniformity.

As $u \in \mathscr{U}$, and $\mathscr{U}$ is fine, there are locally finite coverings $u_1, u_2 \in \mathscr{U}$ such that $u_2 <^* u_1 < u$ ([11] VII. 4). For each $A \in u_1$, choose a $\varphi(A) \in u$ with $A \subset \varphi(A)$. Assign to each point x a $\psi(x) \in u_1$ such that $\operatorname{St}(x, u_2) \subset \psi(x)$. If $B \in u_2$ then $\psi[B]$ is finite, because $B \subset \psi(x)$ $(x \in B)$ and u_1 is locally finite. Let

$$u_0 = \big\{\bigcup \{B \in u_2\colon \varphi[\psi[B]] = u'\}\colon u' \subset u,\ |u'| < \omega\big\}.$$

Now u_0 is clearly locally finite; $u_2 < u_0$ implies that $u_0 \in \mathscr{U}$; $|u_0| < \varkappa$ follows from $|u| < \varkappa$. Hence $u_0 \in \mathscr{B}^{lf}_\varkappa(\mathscr{T})$.

If $y \in \operatorname{St}(x, u_0)$ then there are $B, C \in u_2$ such that $x \in B$, $y \in C$, $\varphi[\psi[B]] = \varphi[\psi[C]]$. Thus $\varphi(\psi(x)) = \varphi(\psi(z))$ for some $z \in C$, and therefore

$$y \in C \subset \operatorname{St}(z, u_2) \subset \psi(z) \subset \varphi(\psi(z)) = \varphi(\psi(x)).$$

This means that $\operatorname{St}(x, u_0) \subset \varphi(\psi(x)) \in u$, therefore $u_0 <^{(*)} u$.

$\mathscr{U}_\varkappa(\mathscr{T})$ is compatible with $\mathscr{T}$, because $\mathscr{U}_\omega(\mathscr{T}) \subset \mathscr{U}_\varkappa(\mathscr{T}) \subset \mathscr{U}$, and $\mathscr{U}_\omega(T)$ is the precompact reflexion of $\mathscr{U}$. □

1.6 We shall need only a weaker form of the following lemma, whose proof is straightforward:

LEMMA. *For any uniformity $\mathscr{U}$, there is a base $\mathscr{B}$ consisting of open coverings such that $|\mathscr{B}| \leqq w(\mathscr{U})$, $|u| < \operatorname{cc}(\mathscr{U})$ $(u \in \mathscr{B})$ and $\operatorname{ord} u < \operatorname{pc}(\mathscr{U})$ $(u \in \mathscr{B})$.* □

1.7 LEMMA. *If $\mathscr{V}$ is a continuous uniformity on X, S is dense, and $\mathscr{U} = \mathscr{V}|S$ then $w(\mathscr{U}) = w(\mathscr{V})$, $\operatorname{cc}(\mathscr{U}) = \operatorname{cc}(\mathscr{V})$ and $\operatorname{pc}(\mathscr{U}) = \operatorname{pc}(\mathscr{V})$.*

PROOF. S being dense in the original topology, it is also dense in the coarser topology induced by $\mathscr{V}$, thus we may assume without loss of generality that $\mathscr{V}$ is compatible. By Lemma 1.3 c), it is enough to prove that $w(\mathscr{U}) \geqq w(\mathscr{V})$, etc.

Recall that $\mathscr{V}$ can be obtained from $\mathscr{U}$ in the following way[2]: Denote by $\mathfrak{f}(a)$ the trace on S of the neighbourhood filter of $a\in X$. ($\mathfrak{f}(a)$ is a Cauchy filter.) Take an arbitrary base $\mathscr{B}$ for $\mathscr{U}$. Let

$$s(A)=\{b\in X:\ A\in\mathfrak{f}(b)\}\quad (A\subset S),$$

$$u^\circ=\{s(A):\ A\in u\}\quad (u\in\mathscr{U}),\quad \mathscr{B}^\circ=\{u^\circ:\ u\in\mathscr{B}\}.$$

Now $\mathscr{B}^\circ$ is a base for $\mathscr{V}$. Starting from a suitable $\mathscr{B}$, one can immediately see that $w(\mathscr{U})\geqq w(\mathscr{V})$ and $\operatorname{cc}(\mathscr{U})\geqq\operatorname{cc}(\mathscr{V})$.

To prove $\operatorname{pc}(\mathscr{U})\geqq\operatorname{pc}(\mathscr{V})$, it is enough to show that if $u_1\in\mathscr{U}$ and u_0 is a uniform strict shrinking of u_1 then

$$\text{(1)}\qquad \operatorname{ord} u_0^\circ\leqq\operatorname{ord} u_1.$$

Let $b\in X$ be fixed. Using the notations of the proof of Lemma 1.4 f), let us take an $F\in\mathfrak{f}(b)\cap u_2$ (there is such an F, since $\mathfrak{f}(b)$ is Cauchy), and pick a $c\in F$. Now if $b\in s(\varphi(A))$ for some $A\in u_1$ then $F\cap\varphi(A)\neq\emptyset$, and so 1.4 (11) implies $c\in A$, showing that (1) holds. □

1.8 LEMMA. *If $\mathscr{U}$ is a uniformity on S, and it has a continuous extension to X then there exists a continuous uniformity $\mathscr{V}_1$ on X such that $w(\mathscr{V}_1)\leqq w(\mathscr{U})$, $\operatorname{cc}(\mathscr{V}_1)\leqq\operatorname{cc}(\mathscr{U})$, $\operatorname{pc}(\mathscr{V}_1)\leqq\omega_1$ and $\mathscr{U}\subset\mathscr{V}_1|S$.*

PROOF. Let $\mathscr{V}_0$ denote the fine uniformity of $\mathscr{T}$. Take a base $\mathscr{B}$ for $\mathscr{U}$ such that $|\mathscr{B}|\leqq w(\mathscr{U})$ and $|u|<\varkappa$ $(u\in\mathscr{B})$ where $\varkappa=\operatorname{cc}(\mathscr{U})$ (Lemma 1.6). As $\mathscr{U}$ has a continuous extension, which is necessarily coarser than $\mathscr{V}_0$, there is for each $u\in\mathscr{B}$ a $u'\in\mathscr{V}_0$ such that $u=u'|S$. Now

$$u'<u_0''=\{A\cup(X\setminus S):\ A\in u\}\in\mathscr{V}_0,$$

and $u_0''|S=u$, $|u_0''|<\varkappa$ $(u\in\mathscr{B})$. According to Lemma 1.5, there are $u_n''\in\mathscr{U}_\varkappa(\mathscr{T})$ such that $u_n''<^*u_{n-1}''$, $|u_n''|<\varkappa$ and $\operatorname{ord} u_n''<\omega_1$ $(u\in\mathscr{B},\ 1\leqq n<\omega)$. Let

$$\{u_n'':\ u\in\mathscr{B},\ 1\leqq n<\omega\}$$

be a subbase for the uniformity $\mathscr{V}_1$. $\mathscr{U}\subset\mathscr{V}_1|S$ is clear. $\mathscr{V}_1$ is continuous on X because $\mathscr{V}_1\subset\mathscr{U}_\varkappa(\mathscr{T})$. Moreover, $w(\mathscr{V}_1)\leqq\omega|\mathscr{B}|\leqq w(\mathscr{U})$, $\operatorname{cc}(\mathscr{V}_1)\leqq\varkappa=\operatorname{cc}(\mathscr{U})$ (as one can take subbases in the definition of w and cc), and $\operatorname{pc}(\mathscr{V}_1)\leqq\omega_1$ (Lemma 1.3 a)). □

THEOREM. *Assume that $\varkappa_1\geqq\omega$, $\varkappa_2\geqq\omega$, $\varkappa_3\geqq\omega_1$, and $\mathscr{U}$ is a compatible uniformity on S. Then $\mathscr{U}$ has a [complete] compatible extension $\bar{\mathscr{U}}$ to X such that $w(\bar{\mathscr{U}})\leqq\varkappa_1$, $\operatorname{cc}(\bar{\mathscr{U}})\leqq\varkappa_2$ and $\operatorname{pc}(\bar{\mathscr{U}})\leqq\varkappa_3$ iff the following conditions hold:*

(i) $w(\mathscr{U})\leqq\varkappa_1$, $\operatorname{cc}(\mathscr{U})\leqq\varkappa_2$, $\operatorname{pc}(\mathscr{U})\leqq\varkappa_3$;

(ii) *there exists a [complete] compatible uniformity $\mathscr{W}$ on X such that $w(\mathscr{W})\leqq\varkappa_1$, $\operatorname{cc}(\mathscr{W})\leqq\varkappa_2$;*

(iii) *$\mathscr{U}$ has a compatible extension to X;*

[(iv) *the compatible extension of $\mathscr{U}$ to* cl S *is complete*].

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[2] See [12] or [5], where this construction is described with entourages, or [18], where coverings are used, but only the special case of the completion is considered.

If S is closed then (iii) *and* (iv) *can be replaced by*

(iii)′ *$\mathcal{U}$ has a continuous extension to X;*

[(iv)′ *$\mathcal{U}$ is complete*].

REMARK. It is correct to speak of *the* extension of $\mathcal{U}$ to cl S in (iv), since there cannot be more than one compatible extension, and the existence of an extension is guaranteed by (iii).

PROOF. The necessity of the conditions is evident.

1° Assume first that S is closed, and (i), (ii), (iii)′ [and (iv)′] hold. Let $\mathcal{V}_1$ be the uniformity furnished by Lemma 1.8. Using Lemma 1.5 one can obtain a compatible uniformity $\mathcal{W}_1 \supset \mathcal{W}$ with $w(\mathcal{W}_1) \leqq \varkappa_1$, $\mathrm{cc}(\mathcal{W}_1) \leqq \varkappa_2$ and $\mathrm{pc}(\mathcal{W}_1) \leqq \omega_1$. Taking $\mathcal{V} = \sup \{\mathcal{V}_1, \mathcal{W}_1\}$, we have $w(\mathcal{V}) \leqq \varkappa_1$, $\mathrm{cc}(\mathcal{V}) \leqq \varkappa_2$ and $\mathrm{pc}(\mathcal{V}) \leqq \omega_1$ from Lemma 1.3 b). Moreover, $\mathcal{V}$ is compatible, $\mathcal{U} \subset \mathcal{V}|S$ [and $\mathcal{V}$ is complete, because it is finer than $\mathcal{W}$, and they induce the same topology]. Lemma 1.4 can now be applied to $\mathcal{V}$ and $\mathcal{U}$.

2° If S is an arbitrary subspace then apply Lemma 1.7 and 1°. □

COROLLARY. *If S is closed, $\mathcal{U}$ is a compatible uniformity on S, $w(\mathcal{U}) \leqq \varkappa \geqq \omega$, $\mathcal{U}$ has a continuous extension to X, and there is on X a compatible uniformity of weight $\leqq \varkappa$ then $\mathcal{U}$ has a compatible extension of weight $\leqq \varkappa$.* □

1.9 COROLLARY. *If the uniformity $\mathcal{U}$ on S has a compatible extension to X then it has a compatible extension $\tilde{\mathcal{U}}$ with* $\mathrm{cc}(\tilde{\mathcal{U}}) = \mathrm{cc}(\mathcal{U})$. □

1.10 It follows from Theorem 1.8 that if a compatible uniformity on a closed subspace has a continuous extension to the whole space then it has a compatible extension, too. (Instead of using Theorem 1.8, it is enough to take $\mathcal{V}_0 + \mathcal{U}$, where $\mathcal{V}_0$ is the fine uniformity of $\mathcal{T}$.)

THEOREM. *For a closed subspace, the following conditions are equivalent:*

(i) *each compatible uniformity has a compatible extension;*

(ii) *each compatible uniformity has a continuous extension;*

(iii) *each continuous uniformity has a continuous extension.*

PROOF. (iii)⇒(ii). Evident.

(ii)⇒(i). See the foregoing observation.

(i)⇒(iii). Let $\mathcal{U}$ be a continuous uniformity, and $\mathcal{U}_0$ the fine uniformity on S. Take a compatible extension $\mathcal{V}$ of $\mathcal{U}_0$. Then $\mathcal{U} \subset \mathcal{V}|S$, so $\mathcal{V} + \mathcal{U}$ is a continuous extension of $\mathcal{U}$ to X by Lemma 1.4 a). □

REMARK. (ii)⇒(i) could also be deduced from [10] 3.6 combined with [1] Theorem 2. (Or from [22] 2.5—2.6.)

1.11 The analogue of Theorem 1.8 for continuous uniformities is much simpler:

THEOREM. *If the continuous uniformity $\mathcal{U}$ has a continuous extension then it has also a continuous extension $\tilde{\mathcal{U}}$ such that $w(\tilde{\mathcal{U}}) = w(\mathcal{U})$, $\mathrm{cc}(\tilde{\mathcal{U}}) = \mathrm{cc}(\mathcal{U})$ and $\mathrm{pc}(\tilde{\mathcal{U}}) \leqq \leqq \omega_1 \mathrm{pc}(\mathcal{U})$.*

PROOF. Lemmas 1.8 and 1.4 a), d), e), f). □

REMARK. A somewhat weaker statement can be easily deduced from [9] 3.4.

§ 2. Extensions of proximities

2.1 A collection $\mathscr{P}$ of subsets of X is a *base* [17] for the proximity δ on X if for any pair of far sets A and B there is a $P\in\mathscr{P}$ such that $A\subset P$ and $P\cap B=\emptyset$. The *weight* [17] of δ, denoted by $w(\delta)$, is the smallest infinite cardinal $\varkappa$ for which there exists a base of cardinality $\leqq\varkappa$. If $\mathscr{U}$ is the precompact uniformity associated with δ then $w(\mathscr{U})=w(\delta)$ [17]. Let $w(\mathscr{T})$ denote the *weight* of the topology $\mathscr{T}$. If δ is compatible with $\mathscr{T}$ then $w(\mathscr{T})\leqq w(\delta)$; $\mathscr{T}$ can be induced by a proximity δ with $w(\mathscr{T})=$ $=w(\delta)$ [17].

THEOREM. *The compatible proximity δ on S has a compatible extension ε to X such that $w(\varepsilon)\leqq\varkappa$ iff $w(\delta)\leqq\varkappa$, $w(\mathscr{T})\leqq\varkappa$, and δ has a compatible extension.*

PROOF. Apply Theorem 1.8 to the precompact uniformity associated with δ, taking $\varkappa_1=\varkappa$, $\varkappa_2=\omega$ and $\varkappa_3=\omega_1$; use the foregoing observations. □

2.2 If a compatible proximity given on a closed subspace has a continuous extension then it has a compatible extension, too. (Apply the observation made in the first paragraph of 1.10.)

THEOREM. *For a closed subspace, the following conditions are equivalent:*
(i) *each compatible proximity has a compatible extension;*
(ii) *each compatible proximity has a continuous extension;*
(iii) *each continuous proximity has a continuous extension.*

PROOF. (iii)⇒(ii). Evident.
(ii)⇒(i). See above.
(i)⇒(iii). Let δ be a continuous proximity on S, $\mathscr{U}$ the precompact uniformity associated with δ, and $\mathscr{U}_1$ the precompact reflexion of the fine uniformity on S. Clearly, $\mathscr{U}\subset\mathscr{U}_1$. By (i), there is a (precompact) compatible extension $\mathscr{V}$ of $\mathscr{U}_1$. Consider now $\mathscr{V}+\mathscr{U}$, apply Lemma 1.4 a), and take the proximity induced by $\mathscr{V}+\mathscr{U}$. □

REMARKS. a) Given a proximity δ on S and a proximity ε on X, one can define a proximity $\varepsilon+\delta$ on X in an obvious way, through the associated precompact uniformities. A direct construction of $\varepsilon+\delta$ will be given in [7], as a special case of a construction for syntopogenous structures.

b) The statement that *if X is normal and S is closed then any compatible proximity on S has a compatible extension* [10, 4] can be deduced from (ii) ⇒(i) of the above theorem. This question will also be discussed in [7].

2.3 THEOREM. *If the continuous proximity δ has a continuous extension then it has also a continuous extension ε with $w(\varepsilon)=w(\delta)$.*

PROOF. Theorem 1.11. □

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ON THE DISTRIBUTIONS OF THE SUPREMUM OF WEIGHTED QUANTILE PROCESSES

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Summary

The asymptotic distributions of the supremum of weighted uniform and general quantile processes are studied. We show that these two processes are near to each other in weighted metrics and that the limiting distributions of the supremum of these weighted quantile processes are the same when their asymptotic distributions are determined by Gaussian processes.

1. Introduction

Let $X_1, X_2, \ldots$ be independent identically distributed random variables (i.i.d.rv) with an absolutely continuous distribution function F, having a positive density function f on the open support of F. We denote the continuous inverse of F by Q, the quantile function of F. Let $X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n,n}$ be the order statistics of X_i $(i=1, \ldots, n)$, and define the empirical quantile function

$$Q_n(s) = X_{k,n}, \quad (k-1)/n < s \leq k/n \quad (k = 1, \ldots, n). \tag{1.1}$$

The quantile process ϱ_n is defined by

$$\varrho_n(s) = n^{1/2} f(Q(s))(Q(s)-Q_n(s)), \quad 0 < s < 1. \tag{1.2}$$

Let $U_1=F(X_1)$, $U_2=F(X_2), \ldots$. Then $U_1, \ldots, U_n$ are independent uniform-(0,1) rv with their corresponding order statistics $U_{1,n} \leq U_{2,n} \leq \ldots \leq U_{n,n}$ and empirical quantile function

$$U_n(s) = U_{k,n}, \quad (k-1)/n < s \leq k/n \quad (k = 1, \ldots, n). \tag{1.3}$$

The uniform quantile process u_n is then

$$u_n(s) = n^{1/2}(s-U_n(s)), \quad 0 < s \leq 1. \tag{1.4}$$

The idea of studying the quantile process ϱ_n via computing its supremum distance from u_n was introduced by Csörgő and Révész (1978). Assuming the condition

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C1 (i) *F is twice differentiable on its open support,*

(ii) $f(Q(s))>0, \quad 0<s<1,$

(iii) $\sup_{0<s<1} s(1-s)\dfrac{|f'(Q(s))|}{f^2(Q(s))}<\infty,$

they proved a fast enough rate of convergence to zero for

$$\sup_{1/(n+1)\le s\le n(n+1)} |\varrho_n(s)-u_n(s)|$$

so that appropriate Gaussian approximations for u_n became also valid for ϱ_n.

There have been many papers studying also the asymptotic distribution of weighted empirical and quantile processes. For a review we may for example refer to Chapter 5 of Csörgő (1983). In our Section 2 we will study the asymptotic behaviour of weighted uniform quantile processes, while in Section 3 we will do the same for ϱ_n. We will point out that the weighted supremum norm behaviour of u_n and that of ϱ_n can be the same or it can be also different, depending on the weights used and also on the part of the unit interval where the supremum is taken. We determine also the exact order in probability of weighted differences of ϱ_n and u_n.

Without loss of generality we are assuming that the underlying probability space $(\Omega, \mathcal{A}, \mathrm{P})$ is so rich that it accommodates all the rv and processes introduced so far and also later on.

For the sake of simplicity our results will be proved only on the intervals $[1/(n+1), 1/2], [1/(n+1), k_n/n]$ and $[k_n/n, 1/2]$. Their counterparts on the other halves $[1/2, n/(n+1)], [1-m_n/n, n/(n+1)], [1/2, 1-m_n/n]$ can be easily formulated, and hence also on $[1/(n+1), n/(n+1)]$ and $[k_n/n, 1-m_n/n]$. The latter results will be discussed in remarks after the appropriate theorems.

We will assume throughout that $\{k_n\}$ and $\{m_n\}$ are sequences of positive numbers such that

(1.5) $$1\le k_n\le n \quad (n\ge 1), \quad k_n\to\infty \quad \text{and} \quad k_n/n\to 0 \quad (n\to\infty),$$

and

(1.6) $$1\le m_n\le n \quad (n\ge 1), \quad m_n\to\infty \quad \text{and} \quad m_n/n\to 0 \quad (n\to\infty).$$

2. Uniform quantile process

Generalizing a method and earlier results of Csörgő and Révész [8], [9], Csörgő et al. [4] obtained the following approximation.

THEOREM 2.1 ([4]). *We can define a sequence of Brownian bridges $\{B_n(s); 0\le s\le 1\}$ such that*

(2.1) $$\sup_{1/(n+1)\le s\le n/(n+1)} n^{\tau}|u_n(s)-B_n(s)|/(s(1-s))^{1/2-\tau}=O_{\mathrm{P}}(1), \quad n\to\infty,$$

for every $0\le\tau<1/2$.

A new direct proof of Theorem 2.1 can be found in [6]. Introduce the integral

$$I(q, c) = \int_0^{1/2} s^{-1} \exp\{-cq^2(s)/s\}\, ds.$$

Whenever $I(q, c)<\infty$ for some $c>0$ and q is a positive nondecreasing function in a neighbourhood of zero, then

$$q(s)/s^{1/2} \to \infty \quad (s\downarrow 0). \tag{2.2}$$

Csörgő et al. [4] proved that Theorem 2.1 implied the following results.

THEOREM 2.2 ([4]). *Let q be a positive nondecreasing function in a neighbourhood of zero. Then, as $n\to\infty$,*

$$\sup_{1/(n+1)\le s\le 1/2} |u_n(s)|/q(s) \xrightarrow{\mathcal{D}} \sup_{0\le s\le 1/2} |B(s)|/q(s)$$

if and only if $I(q, c)<\infty$ for some $c>0$, where B is a Brownian bridge.

THEOREM 2.3 ([4]). *Let q be a positive nondecreasing function in a neighbourhood of zero, and let $\{k_n\}$ be as in (1.5). Then, as $n\to\infty$,*

$$\sup_{k_n/n\le s\le 1/2} |u_n(s)|/q(s) \xrightarrow{\mathcal{D}} \sup_{0\le s\le 1/2} |B(s)|/q(s),$$

if and only if $I(q, c)<\infty$ for some $c>0$.

THEOREM 2.4 ([4]). *Let q be a positive nondecreasing function in a neighbourhood of zero, and let $\{k_n\}$ be as in (1.5). Then, as $n\to\infty$,*

$$\sup_{1/(n+1)\le s\le k_n/n} |u_n(s)|/q(s) \xrightarrow{P} c_0,$$

where c_0 is a nonnegative constant, if and only if $I(q, c)<\infty$ for some $c>0$.

In [4] we called the functions q in Theorems 2.2, 2.3 and 2.4 E-F-K-P upper class functions. For a discussion of these functions and their relationship to other integral tests we refer to the latter paper.

In this paper we will work with weight functions of the form $s^\nu L(s)$ $(-\infty<\nu<\infty)$, on $(0, 1/2]$, where $L(s)$ is slowly varying at zero, i.e., it is measurable, positive and

$$\lim_{s\downarrow 0} \frac{L(\lambda s)}{L(s)} = 1 \quad \text{for all} \quad \lambda > 0. \tag{2.3}$$

For the sake of further motivation we mention that Theorems 2.2, 2.3 and 2.4 remain true if q is a regularly varying function at zero $(q(t)=t^\nu L(t)$, where L is slowly varying at zero). Thus, in this case, we can drop the monotonicity required of q (cf. Propositions 2.1, 2.2, 2.3 in [5]).

On can easily verify that finiteness of $I(s^\nu L(s), c)$ for some $c>0$ implies

$$s^\nu L(s)/s^{1/2} \to \infty, \quad s\downarrow 0. \tag{2.4}$$

The latter in turn implies that $-\infty<\nu\leqq 1/2$. When $-\infty<\nu<1/2$, then c_0 in Theorem 2.4 is equal to zero for all slowly varying functions L. Hence on multiplying the rv

$$\sup_{1/(n+1)\leqq s\leqq k_n/n} |u_n(s)|/(s^\nu L(s))$$

by an appropriate sequence of constants tending to ∞ as $n\to\infty$, we can hope for a non-degenerate limit distribution.

In order to give the next result we define

$$Y_\nu = \sup_{0\leqq s\leqq 1} |W(s)|/s^\nu, \quad -\infty<\nu<1/2, \tag{2.5}$$

where $\{W(s);\ 0\leqq s\leqq 1\}$ is a standard Wiener process.

THEOREM 2.5. *Let L be as in* (2.3) *and* $\{k_n\}$ *as in* (1.5). *Then for any* $-\infty<\nu<1/2$

$$(k_n/n)^{\nu-1/2}L(k_n/n) \sup_{1/(n+1)\leqq s\leqq k_n/n} |u_n(s)|/(s^\nu L(s)) \xrightarrow{\mathcal{D}} Y_\nu, \tag{2.6}$$

as $n\to\infty$.

The latter theorem was proved by Csörgő and Mason [7] in the special case of $L(s)=1$ and $0\leqq\nu<1/2$.

REMARK 2.1. By symmetry of u_n obvious counterparts of Theorems 2.2, 2.3 and 2.4 hold true over the subintervals $[1/2, n/(n+1)]$ and $[1/2, 1-m_n/n]$. In case of Theorem 2.5 we have

$$(m_n/n)^{\nu-1/2}L(m_n/n) \sup_{1-m_n/n\leqq s\leqq n/(n+1)} |u_n(s)|/((1-s)^\nu L(1-s)) \to Y_\nu, \tag{2.7}$$

where $\{m_n\}$ is as in (1.6). By Satz 4 in Rossberg (1967) the left rv of (2.6) and (2.7) are asymptotically independent.

We note also that

(2.8)
$$\sup_{k_n/n\leqq s<1-m_n/n} |u_n(s)|/((s(1-s))^\nu L(s(1-s))) \xrightarrow{\mathcal{D}} \sup_{0<s<1} |B(s)|/((s(1-s))^\nu L(s(1-s))),$$

provided $I(s^\nu L(s), c)<\infty$ for some $c>0$, and $\{k_n\}$, $\{m_n\}$ respectively are as in (1.5) and (1.6). It is easy to see that we can also replace k_n/n by $1/(n+1)$ and $1-m_n/n$ by $n/(n+1)$ in (2.8).

The standard deviation of a Brownian bridge $\{B(s);\ 0\leqq s\leqq 1\}$ is $(s(1-s))^{1/2}$ and the thus standardized uniform quantile process is $\{u_n(s)/(s(1-s))^{1/2};\ 1/(n+1)\leqq s\leqq n/(n+1)\}$. The latter is then a natural enough process to be studied on its own. Eicker [12] and Jaeschke [14] initiated such studies, and further variations will also be given here.

Let

$$a(x) = (2\log x)^{1/2} \tag{2.9}$$

$$b(x) = 2\log x + (1/2)\log\log x - (1/2)\log \pi \tag{2.10}$$

and

$$c(x) = \log((1-x)/x). \tag{2.11}$$

Define V to be the maximum of two independent rv whose distribution function is $\exp(-\exp(-x))$, $-\infty<x<\infty$.

THEOREM 2.6. *Let* $\{k_n\}$ *be as in* (1.5). *Then, as* $n\to\infty$,

(2.12) $$a((1/2)\log n)\sup_{1/(n+1)\le s\le 1/2}|u_n(s)|/(s(1-s))^{1/2}-b((1/2)\log n)\xrightarrow{\mathcal{D}}V,$$

(2.13) $$a((1/2)\log k_n)\sup_{1/(n+1)\le s\le k_n/n}|u_n(s)|/(s(1-s))^{1/2}-b((1/2)\log k_n)\xrightarrow{\mathcal{D}}V,$$

and

(2.14) $$a((1/2)c(k_n/n))\sup_{k_n/n\le s\le 1/2}|u_n(s)|/(s(1-s))^{1/2}-b((1/2)c(k_n/n))\xrightarrow{\mathcal{D}}V.$$

REMARK 2.2. A general two-sided version of Theorem 2.6 can be formulated as follows: Let $1/(n+1)\le\varepsilon_1(n)$, $\varepsilon_2(n)\le n/(n+1)$, $\varepsilon_1(n)<1-\varepsilon_2(n)$, and assume that $(1-\varepsilon_1(n))(1-\varepsilon_2(n))/(\varepsilon_1(n)\varepsilon_2(n))\to\infty$. Then, as $n\to\infty$,

(2.15) $$a((1/2)(c(\varepsilon_1(n))+c(\varepsilon_2(n))))\sup_{\varepsilon_1(n)\le s\le 1-\varepsilon_2(n)}|u_n(s)|/(s(1-s))^{1/2}-$$
$$-b((1/2)(c(\varepsilon_1(n))+c(\varepsilon_2(n))))\xrightarrow{\mathcal{D}}V.$$

A proof of (2.15) is along the lines of that of Theorem 2.6. On choosing special forms for $\varepsilon_1(n)$ and $\varepsilon_2(n)$ the normalizing sequences take up their familiar forms. For example if $\varepsilon_1(n)=\varepsilon_2(n)=1/(n+1)$, then we can choose $a(\log n)$ and $b(\log n)$ as normalizing sequences in (2.15) (cf. Theorem 5.5.1 in [10]).

REMARK 2.3. A natural question is to ask for Theorem 2.6 like results for $|u_n(s)|/(s^{1/2}L(s))$, where L satisfies (2.3). This, however, is impossible in general, due to Theorem 2.4. For example, let $L(s)=(\log\log(1/s))^{1/2}$, then Theorem 2.4 holds with $c_0=2^{1/2}$.

Considering now the special case of $L(s)=1$ and using the observation that

$$|u_n(s)|/s^{1/2}=\left|u_n(s)/(s(1-s))^{1/2}-u_n(s)/(s^{-1/2}\tilde{L}(s))\right|,$$

where $\tilde{L}(s)\to 2$ as $s\downarrow 0$, we obtain the following results: if

(2.16) $$(k_n/n)(\log\log k_n)^{1/2}\to 0,$$

then

$$a((1/2)\log k_n)\sup_{1/(n+1)\le s\le k_n/n}|u_n(s)|/s^{1/2}-b((1/2)\log k_n)\xrightarrow{\mathcal{D}}V,$$

and if

(2.17) $$(k_n/n)(\log\log k_n)^{1/2}\to\infty,$$

then

$$2(n/k_n)\sup_{1/(n+1)\le s\le k_n/n}|u_n(s)|/s^{1/2}\xrightarrow{\mathcal{D}}Y_{-1/2}.$$

These statements follow immediately from Theorems 2.5 and 2.6. An example for (2.16) is $k_n=n^\alpha$, $0<\alpha<1$, and with $k_n=n/\log\log\log n$ we have (2.17).

REMARK 2.4. In general the suitably normalized rv

$$\sup_{1/(n+1)\leqq s\leqq k_n/n} |u_n(s)|/(s(1-s))^{1/2}, \qquad \sup_{1-m_n/n\leqq s\leqq n/(n+1)} |u_n(s)|/(s(1-s))^{1/2}$$

are not independent for all sequences $\{k_n\}$ and $\{m_n\}$ satisfying (1.5) and (1.6). For a detailed discussion we refer to Csörgő and Mason (1985).

Let now

$$Z_\nu = |S_1-1| \vee \sup_{1\leqq i<\infty} \sup_{i<u\leqq i+1} |S_{i+1}-u|/u^\nu, \qquad 1/2<\nu<\infty, \tag{2.18}$$

where the $\{S_i;\ i\geqq 1\}$ are partial sums of i.i.d. exponential rv with expectation 1. If $1/2<\nu\leqq 1$, then

$$Z_\nu = |S_1-1| \vee \sup_{1\leqq i<\infty} \left(|S_{i+1}-i|/i^\nu \vee |S_{i+1}-(i+1)|/(i+1)^\nu\right).$$

The next theorems are generalizations of earlier results of Csáki [1] and Mason [15] [16].

THEOREM 2.7. *Let L be as in* (2.3) *and* $1/2<\nu<\infty$. *Then, as* $n\to\infty$,

$$n^{1/2-\nu}L(1/n) \sup_{1/(n+1)\leqq s\leqq 1/2} |u_n(s)|/(s^\nu L(s)) \xrightarrow{\mathscr{D}} Z_\nu.$$

THEOREM 2.8. *Let L be as in* (2.3) *and* $\{k_n\}$ *as in* (1.5). *Then, for any* $1/2<\nu<\infty$, *as* $n\to\infty$,

$$(k_n/n)^{\nu-1/2}L(k_n/n) \sup_{k_n/n\leqq s\leqq 1/2} |u_n(s)|/(s^\nu L(s)) \xrightarrow{\mathscr{D}} Y_{1-\nu}.$$

THEOREM 2.9. *Let L be as in* (2.3) *and* $\{k_n\}$ *as in* (1.5). *Then, for any* $1/2<\nu<\infty$, *as* $n\to\infty$,

$$n^{1/2-\nu}L(1/n) \sup_{1/(n+1)\leqq s\leqq k_n/n} |u_n(s)|/(s^\nu L(s)) \xrightarrow{\mathscr{D}} Z_\nu. \tag{2.19}$$

REMARK 2.5. By symmetry of u_n obvious counterparts of Theorems 2.7 and 2.8 hold true over the subintervals $[1/2, n/(n+1)]$ and $[1/2, 1-m_n/n]$. In case of Theorem 2.9 we have

$$n^{1/2-\nu}L(1/n) \sup_{1-m_n/n\leqq s\leqq n/(n+1)} |u_n(s)|/((1-s)^\nu L(1-s)) \xrightarrow{\mathscr{D}} Z_\nu, \tag{2.20}$$

where $\{m_n\}$ is as in (1.6). By Satz 4 in Rossberg (1967) the left rv of (2.19) and (2.20) are asymptotically independent.

REMARK 2.6. Let $I_A(\cdot)$ be indicator function, and define the following Poisson process

$$N(t) = \sum_{i=1}^{\infty} i I_{(S_i, S_{i+1}]}(t). \tag{2.21}$$

Then it is immediate that for Z_ν of (2.18) we have

$$Z_\nu = \sup_{S_1 < t < \infty} |N(t)-t|/(N(t))^\nu, \quad 1/2 < \nu \leqq 1.$$

REMARK 2.7. All the results of this section have immediate counterparts when replacing $|u_n|$ by u_n or $-u_n$.

REMARK 2.8. Using the methods of this section one can obtain similar results for the empirical process.

3. General quantile process

In this section we first study the quantile process ϱ_n via computing its weighted supremum distance from u_n. We want to see how much ϱ_n can look like u_n the weighted way.

THEOREM 3.1. *Let F satisfy the condition* C1, *and let L be as in* (2.3), $\{k_n\}$ *as in* (1.5). *Then, as* $n \to \infty$

(3.1)
$$\sup_{1/(n+1) \leqq s \leqq 1/2} |\varrho_n(s)-u_n(s)| = O_P(n^{-1/2}\log\log n),$$

(3.2)
$$\sup_{k_n/n \leqq s \leqq 1/2} |\varrho_n(s)-u_n(s)| = O_P\big(n^{-1/2}\log\log(n/k_n)\big).$$

With $0 < \nu < \infty$

(3.3)
$$\sup_{1/(n+1) \leqq s \leqq 1/2} |\varrho_n(s)-u_n(s)|/\big(s^\nu L(s)\big) = O_P\big(n^{\nu-1/2}/L(1/n)\big)$$

and

(3.4)
$$\sup_{k_n/n \leqq s \leqq 1/2} |\varrho_n(s)-u_n(s)|/\big(s^\nu L(s)\big) = O_P\big(k_n^{-\nu} n^{\nu-1/2}/L(k_n/n)\big) \quad as \quad n \to \infty.$$

We show in Theorem 3.7 that the rates in Theorem 3.1 are essentially optimal.

Theorem 3.1 immediately implies the following results.

THEOREM 3.2. *Let F satisfy the condition* C1, *let* $\{k_n\}$ *be as in* (1.5), *and let q be a positive non-decreasing function in a neighbourhood of zero. If* $I(q,c) < \infty$ *for some* $c > 0$, *then, as* $n \to \infty$,

(3.5)
$$\sup_{1/(n+1) \leqq s \leqq 1/2} |\varrho_n(s)|/q(s) \xrightarrow{\mathcal{D}} \sup_{0 \leqq s \leqq 1/2} |B(s)|/q(s),$$

(3.6)
$$\sup_{k_n/n \leqq s \leqq 1/2} |\varrho_n(s)|/q(s) \xrightarrow{\mathcal{D}} \sup_{0 \leqq s \leqq 1/2} |B(s)|/q(s),$$

and

$$\sup_{1/(n+1) \leqq s \leqq k_n/n} |\varrho_n(s)|/q(s) \xrightarrow{P} c_0,$$

where c_0 *is a nonnegative constant.*

The problem of

$$\sup_{1/(n+1)\le s\le 1/2} |\varrho_n(s)-B_n(s)|/q(s) = o_P(1), \quad n\to\infty, \tag{3.7}$$

where B_n are the Brownian bridges of Theorem 2.1, has been studied in a number of works under various conditions on F. An answer essentially amounts to saying that (3.7) holds true with any q for which we have $I(q, c)<\infty$ for all $c>0$. For a discussion of these results we refer to Csörgő [3].

The results in Theorem 3.2 remain true if q is a regularly varying function at zero and in this case we can drop the monotonicity required of q (cf. Theorem 3.3 in Csörgő and Horváth [5]).

The next theorem is a parallel of Theorem 2.5.

THEOREM 3.3. *Let F satisfy the condition* C1, *let $\{k_n\}$ be as in* (1.5), *and let the function L be as in* (2.3). *Then, as $n\to\infty$, for any $-\infty<\nu<1/2$*

$$(k_n/n)^{\nu-1/2}L(k_n/n) \sup_{1/(n+1)\le s\le k_n/n} |\varrho_n(s)|/(s^\nu L(s)) \xrightarrow{\mathcal{D}} Y_\nu .$$

REMARK 3.1. It is easy to see that in Remark 1 u_n can be replaced by ϱ_n under the conditions of the present section.

The following theorem contains Eicker—Jaeschke type results for ϱ_n.

THEOREM 3.4. *Let F satisfy the condition* C1 *and let $\{k_n\}$ be as in* (1.2). *Then, as $n\to\infty$,*

$$a((1/2)\log n) \sup_{1/(n+1)\le s\le 1/2} |\varrho_n(s)|/(s(1-s))^{1/2} - b((1/2)\log n) \xrightarrow{\mathcal{D}} V, \tag{3.8}$$

$$a((1/2)\log k_n) \sup_{1/(n+1)\le s\le k_n/n} |\varrho_n(s)|/(s(1-s))^{1/2} - b((1/2)\log k_n) \xrightarrow{\mathcal{D}} V, \tag{3.9}$$

and

$$a((1/2)c(k_n/n)) \sup_{k_n/n\le s\le 1/2} |\varrho_n(s)|/(s(1-s))^{1/2} - b((1/2)c(k_n/n)) \xrightarrow{\mathcal{D}} V. \tag{3.10}$$

REMARK 3.2. The comments we made about u_n in Remarks 2.2, 2.3 and 2.4 remain valid also for ϱ_n.

Turning now to heavy weights for ϱ_n, first we deal with its behaviour on $[k_n/n, 1/2]$.

THEOREM 3.5. *Let F satisfy the condition* C1, *let $1/2<\nu<\infty$, $\{k_n\}$ be as in* (1.5), *and let the function L be as in* (2.3). *Then, as $n\to\infty$,*

$$(k_n/n)^{\nu-1/2}L(k_n/n) \sup_{k_n/n\le s\le 1/2} |\varrho_n(s)|/(s^\nu L(s)) \xrightarrow{\mathcal{D}} Y_{1-\nu} .$$

We have seen so far that the weighted sup-norm behaviour of u_n was inherited by ϱ_n, given only the condition C1. On the other hand (3.3) and (3.4) give only

$$n^{1/2-\nu}L(1/n) \sup_{1/(n+1)\le s\le k_n/n} |\varrho_n(s)-u_n(s)|/(s^\nu L(s)) = O_P(1). \tag{3.11}$$

Hence our next question should be whether $O_P(1)$ of (3.11) could possibly be replaced by $o_P(1)$. Obviously, the latter will be sorted out by the nature of the tail behaviour of Q at zero. We introduce the following condition:

C2 $$\lim_{s\downarrow 0} s(1-s)\frac{|f'(Q(s))|}{f^2(Q(s))} = \gamma.$$

THEOREM 3.6. *Let F satisfy the conditions* C1 *and* C2, *let* $1/2<\nu<\infty$, $\{k_n\}$ *be as in* (1.5), *and let the function L be as in* (2.3). *If* $\gamma=0$ *in condition* C2, *then, as* $n\to\infty$,

(3.12) $$n^{1/2-\nu}L(1/n) \sup_{1/(n+1)\le s\le k_n/n} |\varrho_n(s)-u_n(s)|/(s^\nu L(s)) = o_P(1).$$

Combining now Theorem 3.5, (3.12) of Theorem 3.6 and Theorem 2.9, we get

COROLLARY 3.1. *Let F satisfy* C1 *and* C2 *with* $\gamma=0$, *let* $1/2<\nu<\infty$, $\{k_n\}$ *be as in* (1.5), *and let L be as in* (2.3). *Then, as* $n\to\infty$,

$$n^{1/2-\nu}L(1/n) \sup_{1/(n+1)\le s\le k_n/n} |\varrho_n(s)|/(s^\nu L(s)) \xrightarrow{\mathscr{D}} Z_\nu,$$

and

$$n^{1/2-\nu}L(1/n) \sup_{1/(n+1)\le s\le 1/2} |\varrho_n(s)|/(s^\nu L(s)) \xrightarrow{\mathscr{D}} Z_\nu.$$

Next we consider the case when γ of condition C2 is not zero.

THEOREM 3.7. *Let F satisfy the conditions* C1, *and* C2 *with* $\gamma>0$. *Let* $1/2<\nu<\infty$, $\{k_n\}$ *be as in* (1.5), *and let the function L be as in* (2.3). *Assume also that* $f'(Q(t))$ *is a regularly varying function at zero and* $t(1-t)f'(Q(t))/f^2(Q(t))$ *is continuous on* $[0,1]$. *Then, as* $n\to\infty$, *we have*

(3.13) $$\frac{n^{1/2}}{\log\log(n/k_n)} \sup_{k_n/n\le s\le 1/2} |\varrho_n(s)-u_n(s)| \xrightarrow{P} \gamma,$$

(3.14) $$k_n^\nu n^{1/2-\nu}L(k_n/n) \sup_{k_n/n\le s\le 1/2} |\varrho_n(s)-u_n(s)|/(s^\nu L(s)) \xrightarrow{\mathscr{D}} \frac{\gamma}{2}Y^2_{1/2-\nu/2},$$

and for any $\delta>0$ *there exist* $\varepsilon>0$ *and* n_0 *such that we have*

(3.15) $$\mathrm{P}\left\{\frac{n^{1/2}}{\log\log n} \sup_{1/(n+1)\le s\le 1/2} |\varrho_n(s)-u_n(s)| > \varepsilon\right\} \ge 1-\delta,$$

as well as

(3.16) $$\mathrm{P}\{n^{1/2-\nu}L(1/n) \sup_{1/(n+1)\le s\le 1/2} |\varrho_n(s)-u_n(s)|/(s^\nu L(s)) > \varepsilon\} \ge 1-\delta,$$

whenever $n\ge n_0$.

It follows immediately from (3.16) that the asymptotic behaviour of the rv

(3.17) $$\sup_{1/(n+1)\le s\le 1/2} |\varrho_n(s)|/(s^\nu L(s))$$

can be completely different from the uniform case (cf. Csörgő and Horváth [5]). Properly normalized, the limit distribution of the rv in (3.17) depends on the three domains of attraction of the extreme value distributions. For details we refer to Horváth [13].

4. Proofs of results of section 2

The first result here is 7 of Corollary 1.2.1 of De Haan [11]. It will be used frequently throughout.

LEMMA 4.1. *Let* $\varepsilon \neq 0$, *and* L *be a slowly varying function at zero and consider* $s^{\varepsilon}L(s)$. *Then there exists another slowly varying function at zero,* L^*, *such that* $s^{\varepsilon}L^*(s)$ *is a strictly monotone function and*

$$\lim_{s\downarrow 0}\frac{L^*(s)}{L(s)} = 1. \tag{4.1}$$

Using Karamata's representation, De Haan [11] in his Corollary 1.2.1 proved also

LEMMA 4.2. *Let* L *be a slowly varying function at zero. Then*

$$\lim_{s\downarrow 0}\sup_{a\le\lambda\le b}\left|\frac{L(s)}{L(\lambda s)}-1\right| = 0, \quad 0 < a \le b < \infty.$$

PROOF of Theorem 2.5. First we show that

$$\sup_{1/(n+1)\le s\le k_n/n} L(k_n/n)/L(s) = O(k_n^{\varepsilon}) \quad \text{for all} \quad \varepsilon > 0, \quad n\to\infty. \tag{4.2}$$

Applying Lemma 4.1 to $s^{-\varepsilon}L(s)$, $\varepsilon>0$, we get that $s^{-\varepsilon}L^*(s)$ is a monotone decreasing function. Hence

$$\frac{(k_n/n)^{-\varepsilon}L^*(k_n/n)}{s^{-\varepsilon}L^*(s)}\,\frac{s^{-\varepsilon}}{(k_n/n)^{-\varepsilon}} \le 2k_n^{\varepsilon}, \qquad 1/(n+1) \le s \le k_n/n,$$

and the latter with (4.1) implies (4.2).

Now using Theorem 2.1 with $\tau=1/2-\nu$ and (4.2) with $\varepsilon=(1/2-\nu)/2$ we get

$$\begin{aligned}(k_n/n)^{\nu-1/2}L(k_n/n)&\sup_{1/(n+1)\le s\le k_n/n}|u_n(s)-B_n(s)|/(s^{\nu}L(s)) = \\ &= O_P\big(k_n^{\nu-1/2}\sup_{1/(n+1)\le s\le k_n/n}L(k_n/n)/L(s)\big) = O_P(k_n^{(\nu-1/2)/2}) = o_P(1),\end{aligned} \tag{4.3}$$

if $0<\nu<1/2$. When $\nu\le 0$, we let $0<\delta<1/2$, and note that

$$(k_n/n)^{\nu-1/2} \le (k_n/n)^{\delta-1/2}/s^{\delta-\nu}, \quad 1/(n+1) \le s \le k_n/n.$$

Hence

$$(k_n/n)^{\nu-1/2}L(k_n/n)\sup_{1/(n+1)\le s\le k_n/n}|u_n(s)-B_n(s)|/(s^{\nu}L(s)) \le$$

$$\le (k_n/n)^{\delta-1/2}L(k_n/n)\sup_{1/(n+1)\le s\le k_n/n}|u_n(s)-B_n(s)|/(s^{\delta}L(s)) = o_P(1), \quad n\to\infty,$$

by (4.3).

In order to complete the proof of Theorem 2.5, it suffices to show that

(4.4) $$(k_n/n)^{\nu-1/2}L(k_n/n)\sup_{1/(n+1)\le s\le k_n/n}|B(s)|/(s^\nu L(s))\xrightarrow{\mathcal{D}} Y_\nu.$$

Let $0<\delta<1/2$. Applying Lemma 4.1 to $s^{-\varepsilon}L(s)$, $\varepsilon<0$, we get that $s^{-\varepsilon}L^*(s)$ is a monotone decreasing function, and therefore

$$\sup_{1/k_n\le t\le\delta} L^*(k_n/n)/\big(t^{-\varepsilon}L^*(tk_n/n)\big)=L^*(k_n/n)/\big(\delta^{-\varepsilon}L^*(\delta k_n/n)\big)\to\delta^\varepsilon,\quad n\to\infty.$$

Since by (4.1)

$$\sup_{1/k_n\le t\le\delta} L^*(tk_n/n)/L(tk_n/n)=\sup_{1/n\le u\le\delta k_n/n} L^*(u)/L(u)\to 1,\quad n\to\infty,$$

therefore

(4.5) $$\sup_{1/k_n\le t\le\delta} t^\varepsilon L(k_n/n)/L(tk_n/n)=O(1),\quad n\to\infty.$$

Using Doob's transformation $\{W(t);\ t\ge 0\}\overset{\mathcal{D}}{=}\{(t+1)B(t/(t+1));\ t\ge 0\}$ we get

$$(k_n/n)^{\nu-1/2}L(k_n/n)\sup_{1/(n+1)\le s\le k_n/n}|B(s)|/(s^\nu L(s))=(k_n/n)^{\nu-1/2}L(k_n/n)\times$$

$$\times\sup_{(1/k_n)(1-k_n/n)\le t\le 1}\frac{\left|B\left(t\frac{k_n}{n-k_n}\Big/\left(t\frac{k_n}{n-k_n}+1\right)\right)\right|}{\left(t\frac{k_n}{n-k_n}\Big/\left(t\frac{k_n}{n-k_n}+1\right)\right)^\nu L\left(t\frac{k_n}{n-k_n}\Big/\left(t\frac{k_n}{n-k_n}+1\right)\right)}\overset{\mathcal{D}}{=}$$

$$\overset{\mathcal{D}}{=}(k_n/n)^{\nu-1/2}L(k_n/n)\times$$

$$\times\sup_{(1/k_n)(1-k_n/n)\le t\le 1}\frac{\left|W\left(t\frac{k_n}{n-k_n}\right)\right|}{\left(t\frac{k_n}{n-k_n}+1\right)^{1-\nu}\left(t\frac{k_n}{n-k_n}\right)^\nu L\left(t\frac{k_n}{n-k_n}\Big/\left(t\frac{k_n}{n-k_n}+1\right)\right)}=$$

$$=(k_n/n)^{\nu-1/2}L(k_n/n)\sup_{1/n\le s\le k_n/(n-k_n)}\frac{|W(s)|}{s^\nu(1+s)^{1-\nu}L(s/(s+1))}=$$

$$=L(k_n/n)\sup_{1/k_n\le t\le n/(n-k_n)}\frac{\left(\frac{k_n}{n}\right)^{-1/2}\left|W\left(t\frac{k_n}{n}\right)\right|}{t^\nu\left(\frac{k_n}{n}t+1\right)^{1-\nu}L\left(\frac{tk_n}{tk_n+n}\right)}\overset{\mathcal{D}}{=}$$

$$\overset{\mathcal{D}}{=}\sup_{1/k_n\le t\le n/(n-k_n)}\frac{|W(t)|}{t^\nu}\,\frac{L(k_n/n)}{\left(\frac{k_n}{n}t+1\right)^{1-\nu}L\left(\frac{tk_n}{tk_n+n}\right)}.$$

Next by Lemma 4.2

$$\sup_{1/k_n\le t\le n/(n-k_n)}\left|\frac{L\left(t\frac{k_n}{n}\right)}{L\left(t\frac{k_n}{n}\frac{1}{tk_n/n+1}\right)}-1\right| = \sup_{1/n\le u\le k_n/(n-k_n)}\left|\frac{L(u)}{L(u/(u+1))}-1\right|\le$$

$$\le \sup_{1/n\le u\le k_n/(n-k_n)}\ \sup_{1/2\le\lambda\le 1}\left|\frac{L(u)}{L(\lambda u)}-1\right| = o(1),\quad n\to\infty. \tag{4.6}$$

Also, it is easily verified that

$$\sup_{1/k_n<t<n/(n-k_n)}\left|\left(\frac{k_n}{n}t+1\right)^{\nu-1}-1\right| = o(1),\quad n\to\infty. \tag{4.7}$$

Hence, by (4.6) and (4.7) it is enough to show that

$$\sup_{1/k_n\le t\le n/(n-k_n)}\frac{|W(t)|}{t^\nu}\frac{L(k_n/n)}{L(t(k_n/n))}\xrightarrow{\mathcal{D}} Y_\nu,\quad n\to\infty. \tag{4.8}$$

By Lemma 4.2 we have with $0<\delta<1/2$

$$\sup_{\delta\le t\le n/(n-k_n)}\left|\frac{L(k_n/n)}{L(t(k_n/n))}-1\right| = o(1),\quad n\to\infty,$$

and so, as $n\to\infty$,

$$\sup_{\delta\le t\le n/(n-k_n)}\frac{|W(t)|}{t^\nu}\frac{L(k_n/n)}{L(t(k_n/n))}\xrightarrow{\mathcal{D}}\sup_{\delta\le t\le 1}\frac{|W(t)|}{t^\nu}. \tag{4.9}$$

Now in (4.5) we let $\varepsilon=1/2((1/2)-\nu)$ and get

$$\sup_{1/k_n\le t\le\delta}\frac{|W(t)|}{t^\nu}\frac{L(k_n/n)}{L(t(k_n/n))} = O\left(\sup_{0\le t\le\delta}\frac{|W(t)|}{t^{\nu+\varepsilon}}\right). \tag{4.10}$$

By the law of the iterated logarithm for the Wiener process we have

$$\lim_{\delta\downarrow 0}\sup_{0\le t\le\delta}|W(t)|/t^{\nu+\varepsilon}=0,\quad \text{a.s.},$$

and hence by (4.9) and (4.10) we get (4.8).

For the proof of Theorem 2.6 the following lemma is needed.

LEMMA 4.3. *Let* $0<\varepsilon_1(n)$, $\varepsilon_2(n)<1$, $\varepsilon_1(n)<1-\varepsilon_2(n)$ *and assume that* $(1-\varepsilon_1(n))(1-\varepsilon_2(n))/(\varepsilon_1(n)\varepsilon_2(n))\to\infty$ *as* $n\to\infty$. *Then*

$$a((1/2)(c(\varepsilon_1(n))+c(\varepsilon_2(n))))\sup_{\varepsilon_1(n)\le s\le 1-\varepsilon_2(n)}|B(s)|/(s(1-s))^{1/2}-$$

$$-b((1/2)(c(\varepsilon_1(n))+c(\varepsilon_2(n))))\xrightarrow{\mathcal{D}} V.$$

PROOF. The proof of this lemma is along the lines of Corollary 1.9.1 in Csörgő and Révész [10]. There is, however, a misprint in their proof. We have to use the transformation $e^{2t}=s/(1-s)$ in order to get the Ornstein—Uhlenbeck process.

PROOF of Theorem 2.6. Choose any $0<\tau<1/2$ in Theorem 2.1. Let $\varepsilon(n)=(\log n)^3/n$. Then

$$\begin{aligned} &a((1/2)\log n) \sup_{\varepsilon(n)\leq s\leq 1/2} |u_n(s)-B_n(s)|/(s(1-s))^{1/2} \leq \\ (4.11)\qquad &\leq \frac{2a((1/2)\log n)}{(\log n)^{3\tau}} n^\tau \sup_{1/(n+1)\leq s\leq n/(n+1)} |u_n(s)-B_n(s)|/(s(1-s))^{1/2-\tau} = o_P(1), \quad n\to\infty. \end{aligned}$$

Next consider

$$\begin{aligned} &\Big|\sup_{1/(n+1)\leq s\leq \varepsilon(n)} |u_n(s)|/(s(1-s))^{1/2} - \sup_{1/(n+1)\leq s\leq \varepsilon(n)} |B_n(s)|/(s(1-s))^{1/2}\Big| / a((1/2)\log(n\varepsilon(n))) \leq \\ (4.12)\qquad &\leq \sup_{1/(n+1)\leq s\leq n/(n+1)} |u_n(s)-B_n(s)|/((s(1-s))^{1/2} a((1/2)\log(n\varepsilon(n)))) = o_P(1) \end{aligned}$$

by Theorem 2.1. Let now $\varepsilon_1(n)=1/(n+1)$, $\varepsilon_2(n)=1-\varepsilon(n)$ in Lemma 4.3. We obtain

$$(4.13)\qquad \sup_{1/(n+1)\leq s\leq \varepsilon(n)} |B_n(s)|/((s(1-s))^{1/2} a((1/2)\log(n\varepsilon(n)))) \xrightarrow{P} 1,$$

and therefore by (4.12) we get also

$$(4.14)\qquad \sup_{1/(n+1)\leq s\leq \varepsilon(n)} |u_n(s)|/((s(1-s))^{1/2} a((1/2)\log(n\varepsilon(n)))) \xrightarrow{P} 1.$$

Consequently, by (4.13)

$$(4.15)\qquad a((1/2)\log n) \sup_{1/(n+1)\leq s\leq \varepsilon(n)} |B_n(s)|/(s(1-s))^{1/2} - b((1/2)\log n) \xrightarrow{P} -\infty,$$

and by (4.14)

$$(4.16)\qquad a((1/2)\log n) \sup_{1/(n+1)\leq s\leq \varepsilon(n)} |u_n(s)|/(s(1-s))^{1/2} - b((1/2)\log n) \xrightarrow{P} -\infty.$$

Hence, in order to prove (2.12), by (4.11), (4.15) and (4.16) it suffices to show that

$$(4.17)\qquad a((1/2)\log n) \sup_{\varepsilon(n)\leq s\leq 1/2} |B_n(s)|/(s(1-s))^{1/2} - b((1/2)\log n) \xrightarrow{\mathcal{D}} V.$$

Let $\varepsilon_1(n)=\varepsilon(n)$ and $\varepsilon_2(n)=1/2$. Then Lemma 4.3 gives (4.17), on account of $c(\varepsilon_2(n))=0$, and

$$b((1/2)\log n) - b((1/2)c((\log n)^3/n)) \to 0,$$

$$\left|\frac{a((1/2)\log n)}{a((1/2)c((\log n)^3/n))} - 1\right| \frac{b((1/2)\log n)}{a((1/2)c((\log n)^3/n))} \to 0.$$

Now the proof of (2.12) is complete.

In order to prove (2.13), first by Theorem 2.1 with $0<\tau<1/2$ we get

$$a((1/2)\log k_n) \sup_{(\log k_n)/n\le s\le k_n/n} |B_n(s)-u_n(s)|/(s(1-s))^{1/2} =$$

$$(4.18)\quad = O\big(a((1/2)\log k_n)(n/\log k_n)^{\tau} \sup_{1/(n+1)\le s\le 1/2} |B_n(s)-u_n(s)|/(s(1-s))^{1/2-\tau}\big) =$$

$$= O_{\mathrm{P}}\big(a((1/2)\log k_n)(n/\log k_n)^{\tau_n-\tau}\big) = o_{\mathrm{P}}(1).$$

Again by Theorem 2.1 with $\tau=0$ we get

$$(4.19)\quad \begin{aligned} &\sup_{1/(n+1)\le s\le(\log k_n)/n} |B_n(s)-u_n(s)|/\big((s(1-s))^{1/2}a((1/2)\log\log k_n)\big) = \\ &= \big(1/a((1/2)\log\log k_n)\big)O_{\mathrm{P}}(1) = o_{\mathrm{P}}(1). \end{aligned}$$

By Lemma 4.3 we have also

$$\sup_{1/(n+1)\le s\le(\log k_n)/n} |B(s)|/\big((s(1-s))^{1/2}a((1/2)\log(nk_n/(n-\log k_n)))\big) \xrightarrow{\mathrm{P}} 1,$$

and since by Lemma 4.2

$$\lim_{n\to\infty} \frac{a((1/2)\log(nk_n/(n-\log k_n)))}{a((1/2)\log\log k_n)} = 1,$$

we get

$$(4.20)\quad \sup_{1/(n+1)\le s\le(\log k_n)/n} |B(s)|/\big((s(1-s))^{1/2}a((1/2)\log\log k_n)\big) \xrightarrow{\mathrm{P}} 1.$$

Hence by (4.19) and (4.20) we get

$$(4.21)\quad \sup_{1/(n+1)\le s\le(\log k_n)/n} |u_n(s)|/\big((s(1-s))^{1/2}a((1/2)\log\log k_n)\big) \xrightarrow{\mathrm{P}} 1.$$

Since with any $K>0$ we have

$$Ka((1/2)\log k_n)a((1/2)\log\log k_n) - b((1/2)\log k_n) \to -\infty,$$

by (4.20) and (4.21) we get

$$(4.22)\quad a((1/2)\log k_n) \sup_{1/(n+1)\le s\le(\log k_n)/n} |B(s)|/(s(1-s))^{1/2} - b((1/2)\log k_n) \xrightarrow{\mathrm{P}} -\infty$$

and

$$(4.23)\quad a((1/2)\log k_n) \sup_{1/(n+1)\le s\le(\log k_n)/n} |u_n(s)|/(s(1-s))^{1/2} - b((1/2)\log k_n) \xrightarrow{\mathrm{P}} -\infty.$$

Using now (4.18), for the sake of verifying (2.13), by (4.22) and (4.23) it suffices to show that

$$(4.24)\quad a((1/2)\log k_n) \sup_{(\log k_n)/n\le s\le k_n/n} |B(s)|/(s(1-s))^{1/2} - b((1/2)\log k_n) \xrightarrow{\mathcal{D}} V.$$

On account of Lemma 4.3 we have

$$a\left((1/2)\log\frac{k_n}{n-k_n}\,\frac{n-\log k_n}{\log k_n}\right)\sup_{(\log k_n)/n\le s\le k_n/n}|B(s)|/(s(1-s))^{1/2}-$$
$$-b\left((1/2)\log\frac{k_n}{n-k_n}\,\frac{n-\log k_n}{\log k_n}\right)\xrightarrow{\mathcal{D}} V,$$

and because

$$a\left((1/2)\log\frac{k_n}{n-k_n}\,\frac{n-\log k_n}{\log k_n}\right)\left|\frac{a((1/2)\log k_n)}{a\left((1/2)\log\frac{k_n}{n-k_n}\,\frac{n-\log k_n}{\log k_n}\right)}-1\right|\to 0,$$

and

$$b\left((1/2)\log\frac{k_n}{n-k_n}\,\frac{n-\log k_n}{\log k_n}\right)-b((1/2)\log k_n)\to 0,$$

we also obtain (4.24).

Now we prove (2.14). First assume that for some $0<\tau<1/2$ we have

$$a((1/2)c(k_n/n))/k_n^{\tau}\to 0. \tag{4.25}$$

Then for the same $0<\tau<1/2$ we have

$$\begin{aligned}&a((1/2)c(k_n/n))\sup_{k_n/n\le s\le 1/2}|u_n(s)-B_n(s)|/(s(1-s))^{1/2}\le\\ &\le 2\,\frac{a((1/2)c(k_n/n))}{k_n^{\tau}}\,n^{\tau}\sup_{1/(n+1)\le s\le 1/2}|u_n(s)-B_n(s)|/(s(1-s))^{1/2-\tau}=o_P(1)\end{aligned} \tag{4.26}$$

by (4.25) and Theorem 2.1. The latter implies (2.14) by Lemma 4.3, provided (4.25) holds true.

If (4.25) does not hold, then assume that for any $0<\tau<1/2$

$$\limsup_{n\to\infty} a((1/2)c(k_n/n))/k_n^{\tau}>0.$$

The latter combined with (1.5) implies

$$\limsup_{n\to\infty}(\log\log n)^{1/2}/k_n^{\tau}>0.$$

Hence it suffices to show that for any subsequence $\{n_m\}$ such that

$$\liminf_{m\to\infty}(\log\log n_m)^{1/2}/k_{n_m}^{\tau}>0 \tag{4.27}$$

we have

$$a((1/2)c(k_{n_m}/n_m))\sup_{k_{n_m}/n_m\le s\le 1/2}|u_{n_m}(s)|/(s(1-s))^{1/2}-b((1/2)c(k_{n_m}/n_m))\xrightarrow{\mathcal{D}} V. \tag{4.28}$$

By (4.27) we get

$$(\log n_m)/k_{n_m}\to\infty. \tag{4.29}$$

Obviously then we have

$$\frac{a((1/2)\log k_{n_m})}{(\log\log\log n_m)^{1/2}} = O(1),$$

and hence by (2.13)

$$\sup_{1/(n_m+1)\le s\le k_{n_m}/n_m} |u_{n_m}(s)|/(s(1-s))^{1/2} = O_P((\log\log\log n_m)^{1/2}).$$

Consequently by (2.12) we get

$$(4.30)\quad a((1/2)\log n_m) \sup_{k_{n_m}/n_m\le s\le 1/2} |u_{n_m}(s)|/(s(1-s))^{1/2} - b((1/2)\log n_m) \xrightarrow{\mathcal{D}} V.$$

Elementary calculations show that

$$a((1/2)\log n_m)\left|\frac{a((1/2)c(k_{n_m}/n_m))}{a((1/2)\log n_m)} - 1\right| \to 0,$$

$$b((1/2)\log n_m) - b((1/2)c(k_{n_m}/n_m)) \to 0,$$

and hence (4.30) implies (4.28). This also completes the proof of (2.14).

Proof of Theorem 2.7. First we show that

$$(4.31)\qquad \max_{1\le i\le k_n} i^{-\varepsilon}\frac{L(1/n)}{L(i/n)} = O(1).$$

We apply Lemma 4.1 to $s^\varepsilon L(s)$, $\varepsilon>0$, and get that $s^\varepsilon L^*(s)$ is a monotone increasing function. Hence

$$\frac{(1/n)L^*(1/n)}{(i/n)^\varepsilon L^*(i/n)} \le 1, \qquad 1\le i\le n,$$

and by (4.1) we have also

$$\max_{1\le i\le k_n}\frac{L^*(i/n)}{L(i/n)} \le \sup_{1/n\le s\le k_n/n}\frac{L^*(s)}{L(s)} \to 1, \qquad n\to\infty.$$

This also completes proof of (4.31). Next consider

$$\frac{n^{1/2-\nu}L(1/n)}{(k_n/n)^{\nu-1/2}L(k_n/n)} = k_n^{1/2-\nu}\frac{L(1/n)}{L(k_n/n)} = O(k_n^{1/2-\nu+\varepsilon}) = o(1), \quad n\to\infty,$$

where the latter $O(\cdot)$ term is by (4.31). Consequently, Theorems 2.8 and 2.9 together imply Theorem 2.7.

Proof of Theorem 2.8. We observe that

$$(4.32)\qquad (k_n/n)^\varepsilon \sup_{k_n/n\le s\le 1/2}\frac{L(k_n/n)}{s^\varepsilon L(s)} = O(1)$$

because

$$(k_n/n)^\varepsilon L^*(k_n/n) \le s^\varepsilon L^*(s), \quad k_n/n\le s\le 1/2,$$

and

$$\sup_{k_n/n \leq s \leq 1/2} L^*(s)/L(s) = O(1), \quad n \to \infty,$$

by Lemma 4.1. By (4.32)

$$(k_n/n)^{\nu-1/2} L(k_n/n) \sup_{k_n/n \leq s \leq 1/2} \frac{|u_n(s) - B_n(s)|}{s^\nu L(s)} =$$

$$= O\left((k_n/n)^{\nu-1/2-\varepsilon} \sup_{k_n/n \leq s \leq 1/2} \frac{|u_n(s) - B_n(s)|}{s^{\nu-\varepsilon}}\right),$$

where $0 < \varepsilon < 1/2$. Let now $\tau = \varepsilon$ in Theorem 2.1. Then

$$(k_n/n)^{\nu-1/2-\varepsilon} \sup_{k_n/n \leq s \leq 1/2} \frac{|u_n(s) - B_n(s)|}{s^{\nu-\varepsilon}} \leq$$

$$\leq (k_n/n)^{-\varepsilon} \sup_{1/(n+1) \leq s \leq 1/2} \frac{|u_n(s) - B_n(s)|}{s^{1/2-\varepsilon}} = O_p(k_n^{-\varepsilon}) = o_P(1).$$

In order to prove the statement of Theorem 2.8, it suffices to verify

$$(4.33) \qquad (k_n/n)^{\nu-1/2} L(k_n/n) \sup_{k_n/n \leq s \leq 1/2} |B(s)|/(s^\nu L(s)) \xrightarrow{\mathscr{D}} Y_{1-\nu}, \quad n \to \infty.$$

The latter in turn is proved along the lines of (4.4).

PROOF of Theorem 2.9. With $\Lambda > 0$ we have

$$(4.34) \qquad \begin{aligned} \sup_{1/(n+1) \leq s \leq k_n/n} (ns)^{-2\varepsilon} \left| \frac{L(1/n)}{L(s)} - 1 \right| &\leq \sup_{n/(n+1) \leq t \leq \Lambda} \left| \frac{L(1/n)}{L(t/n)} - 1 \right| + \\ + \Lambda^{-\varepsilon} \sup_{\Lambda \leq t \leq k_n} t^{-\varepsilon} \left| \frac{L(1/n)}{L(t/n)} - 1 \right| &= o(1) + \Lambda^{-\varepsilon} O(1) = o(1), \quad n \to \infty, \end{aligned}$$

by Lemma 4.2 and (4.31).

By (4.34) we have

$$n^{1/2-\nu} L(1/n) \sup_{1/(n+1) \leq s \leq k_n/n} |u_n(s)|/(s^\nu L(s)) =$$

$$= n^{1/2-\nu} \sup_{1/(n+1) \leq s \leq k_n/n} |u_n(s)|/s^\nu + n^{1/2-\nu} \sup_{1/(n+1) \leq s \leq k_n/n} (|u_n(s)|/s^\nu) \left(\frac{L(1/n)}{L(s)} - 1\right) =$$

$$= n^{1/2-\nu} \sup_{1/(n+1) \leq s \leq k_n/n} |u_n(s)|/s^\nu + o(1) n^{1/2-\nu+\varepsilon} \sup_{1/(n+1) \leq s \leq k_n/n} |u_n(s)|/s^{\nu-\varepsilon}.$$

Hence Theorem 2.9 will be proved if we can show that

$$(4.35) \qquad n^{1/2-\nu} \sup_{1/(n+1) \leq s \leq k_n/n} |u_n(s)|/s^\nu \xrightarrow{\mathscr{D}} Z_\nu.$$

We observe

$$\left|\frac{n}{S_{n+1}}-1\right| \max_{1\le i\le k_n} \frac{S_i}{i^\nu} \le \left|\frac{n}{S_{n+1}}-1\right| \max_{1\le i\le k_n} \frac{|S_i-i|}{i^\nu}+ \tag{4.36}$$

$$+\left|\frac{n}{S_{n+1}}-1\right| n^{1-\nu}(k_n/n)^{1-\nu} = o_P(1), \quad n\to\infty,$$

where $\{S_i;\ i\ge 1\}$ are partial sums as in (2.18). By the law of the iterated logarithm for partial sums

$$\sup_{k_n<i<\infty} |S_i-i|/i^\nu = o_P(1), \quad n\to\infty. \tag{4.37}$$

It is easy to see that

$$n^{1/2-\nu} \sup_{1/(n+1)\le s\le k_n/n} |u_n(s)|/s^\nu = n^{1-\nu}\{ \sup_{1/(n+1)\le s\le 1/n} |U_{1,n}-s|/s^\nu \vee$$

$$\vee \max_{1\le i\le [k_n]-1} \sup_{i<u\le i+1} |U_{i+1,n}-u|/u^\nu \vee \sup_{[k_n]-1<u\le [k_n]} |U_{[k_n]+1}-u|/u^\nu\}.$$

Since for each n

$$\{S_i/S_{n+1};\ 1\le i\le n\} \overset{\mathcal{D}}{=} \{U_{i,n};\ 1\le i\le n\}, \tag{4.38}$$

and by (4.36)

$$\sup_{1\le i\le k_n+1} \left| \frac{n^{1-\nu}(S_i/S_{n+1}-i/n)}{(i/n)^\nu} - \frac{S_i-i}{i^\nu}\right| = o_P(1),$$

we obtain (4.35) by (4.37).

5. Proofs of results of Section 3

PROOF of Theorem 3.1. By a two-term Taylor expansion we get

$$\varrho_n(s) = u_n(s)-(1/2)n^{-1/2}\frac{u_n^2(s)}{s(1-s)}\left\{\frac{s(1-s)}{\theta_n(s)(1-\theta_n(s))}\right\}\times \tag{5.1}$$

$$\times\left\{\theta_n(s)(1-\theta_n(s))\frac{f'(Q(\theta_n(s)))}{f^2(Q(\theta_n(s)))}\right\}\left\{\frac{f(Q(s))}{f(Q(\theta_n(s)))}\right\},$$

where

$$U_n(s)\wedge s < \theta_n(s) < U_n(s)\vee s, \qquad 0<s<1.$$

By Wellner [20]

$$\sup_{1/(n+1)\le s\le n/(n+1)} \frac{s(1-s)}{\theta_n(s)(1-\theta_n(s))} = O_P(1), \quad n\to\infty, \tag{5.2}$$

and by C1 (iii) and Lemma 4.5.2 of Csörgő and Révész [10]

$$\sup_{1/(n+1)\le s\le n/(n+1)} \frac{f(Q(s))}{f(Q(\theta_n(s)))} = O_P(1), \quad n\to\infty. \tag{5.3}$$

By Theorem 2.6 we have

(5.4) $$\sup_{1/(n+1)\le s\le 1/2} u_n^2(s)/s = O_P(\log\log n), \quad n\to\infty.$$

Now (3.1) follows from the above lines.

By Theorem 2.6 again

(5.5) $$\sup_{k_n/n\le s\le 1/2} u_n^2(s)/s = O_P\left(\frac{b(c(k_n/n))}{a(c(k_n/n))}\right) = O_P(\log\log(n/k_n)).$$

Thus (3.2) follows by (5.1), (5.2), (5.3) and (5.5).

Since we have (5.2), (5.3) and condition C1, by (5.1) we have that for the proof of (3.3) and (3.4) we only have to verify

(5.6) $$\sup_{1/(n+1)\le s\le 1/2} u_n^2(s)/(s^{1+\nu}L(s)) = O_P(n^\nu/L(1/n))$$

and

(5.7) $$\sup_{k_n/n\le s\le 1/2} u_n^2(s)/(s^{1+\nu}L(s)) = O_P((k_n/n)^{-\nu}/L(k_n/n)),$$

respectively. Theorem 2.7 implies (5.6), and (5.7) follows from Theorem 2.8.

Proof of Theorem 3.2. By (3.1) we have for any $\delta\in(0, 1/2)$ that

(5.8) $$\sup_{\delta\le s\le 1/2} |\varrho_n(s)-u_n(s)|/q(s) = o_P(1), \quad n\to\infty.$$

By (3.3) with $\nu=1/2$ and $L(s)=1$ we get, as $n\to\infty$,

(5.9) $$\sup_{1/(n+1)\le s\le \delta} |\varrho_n(s)-u_n(s)|/q(s) = 1\Big/\inf_{1/(n+1)\le s\le\delta}(s^{-1/2}q(s))O_P(1).$$

Using now (2.2) via choosing δ small enough, by (5.8) and (5.9) we get

(5.10) $$\sup_{1/(n+1)\le s\le 1/2} |\varrho_n(s)-u_n(s)|/q(s) = o_P(1), \quad n\to\infty.$$

Hence by Theorems 2.2 and 2.3 we obtain (3.5) and (3.6).

Proof of Theorem 3.3. By (3.3), in the case of $0<\nu<1/2$,

$$(k_n/n)^{\nu-1/2}L(k_n/n)\sup_{1/(n+1)\le s\le k_n/n} |\varrho_n(s)-u_n(s)|/(s^\nu L(s)) =$$
$$= O_P((k_n/n)^{\nu-1/2}n^{\nu-1/2}L(k_n/n)/L(1/n)).$$

Using now (4.2) and the latter statement we get

(5.11) $$(k_n/n)^{\nu-1/2}L(k_n/n)\sup_{1/(n+1)\le s\le k_n/n} |\varrho_n(s)-u_n(s)|/(s^\nu L(s)) = o_P(1).$$

We note that for $0<\delta<1/2$

$$(k_n/n)^{\nu-1/2} \le (k_n/n)^{\delta-1/2}/s^{\delta-\nu}, \qquad 1/(n+1)\le s\le k_n/n.$$

Hence by (5.11) with $\nu \leqq 0$ we get

$$(k_n/n)^{\nu-1/2} L(k_n/n) \sup_{1/(n+1) \leqq s \leqq k_n/n} |\varrho_n(s)-u_n(s)|/(s^\nu L(s)) \leqq$$
$$\leqq (k_n/n)^{\delta-1/2} L(k_n/n) \sup_{1/(n+1) \leqq s \leqq k_n/n} |\varrho_n(s)-u_n(s)|/(s^\delta L(s)) = o_P(1).$$

Consequently, (5.11) is now proved for $-\infty<\nu<1/2$, and Theorem 3.3 follows by Theorem 2.5.

PROOF of Theorem 3.4. The proof of this theorem is similar to that of Theorem 2.6. Hence we will give its short outline only.

We start with proving (3.8). Let $\varepsilon(n)=(\log n)^3/n$. Using Theorems 2.1 and 3.1 we get

$$a((1/2)\log n) \sup_{\varepsilon(n) \leqq s \leqq 1/2} |\varrho_n(s)-B_n(s)|/(s(1-s))^{1/2} = o_P(1) \tag{5.12}$$

and

$$\Big| \sup_{1/(n+1) \leqq s \leqq \varepsilon(n)} |\varrho_n(s)|/(s(1-s))^{1/2} - \sup_{1/(n+1) \leqq s \leqq \varepsilon(n)} |B_n(s)|/(s(1-s))^{1/2}\Big| / a((1/2)\log(n\varepsilon(n))) = o_P(1). \tag{5.13}$$

By (5.13) and (4.15) we have

$$a((1/2)\log n) \sup_{1/(n+1) \leqq s \leqq \varepsilon(n)} |\varrho_n(s)|/(s(1-s))^{1/2} - b((1/2)\log n) \xrightarrow{P} -\infty. \tag{5.14}$$

Consequently (4.15), (5.14), (5.12) and (4.17) give (3.8).

In order to prove (3.9), we first note that by Theorems 2.1 and 3.1 we get

$$a((1/2)\log k_n) \sup_{(\log k_n)/n \leqq s \leqq k_n/n} |B_n(s)-\varrho_n(s)|/(s(1-s))^{1/2} = o_P(1) \tag{5.15}$$

and

$$\sup_{1/(n+1) \leqq s \leqq (\log k_n)/n} |B_n(s)-\varrho_n(s)|/((s(1-s))^{1/2} a((1/2)\log\log k_n)) = o_P(1). \tag{5.16}$$

On using now (5.16) and (4.22) we obtain

$$a((1/2)\log k_n) \sup_{1/(n+1) \leqq s \leqq (\log k_n)/n} |\varrho_n(s)|/(s(1-s))^{1/2} - b((1/2)\log k_n) \xrightarrow{P} -\infty. \tag{5.17}$$

Hence (4.22), (5.17), (5.15) and (4.24) yield (3.9).

Finally we prove (3.10). First we assume that for some $0<\tau<1/2$ we have (4.25). In this case by Theorems 2.1 and 3.1 we conclude

$$a((1/2)c(k_n/n)) \sup_{k_n/n \leqq s \leqq 1/2} |\varrho_n(s)-B_n(s)| = o_P(1), \tag{5.18}$$

and hence Lemma 4.3 results in (3.10). If (4.25) does not hold, then the same argument as was used in the proof of (2.14) gives that (3.8) and (3.9) imply (3.10).

PROOF of Theorem 3.5. From (3.4) we get

$$(k_n/n)^{\nu-1/2}L(k_n/n) \sup_{k_n/n\le s\le 1/2} |\varrho_n(s)-u_n(s)|/(s^\nu L(s)) = O_P(k_n^{-1/2}) = o_P(1)$$

and therefore Theorem 2.8 yields the result.

PROOF of Theorem 3.6. When $\gamma=0$, then in (5.1) we have

$$\sup_{1/(n+1)\le s\le k_n/n} \theta_n(s)(1-\theta_n(s))\frac{f'(Q(\theta_n(s)))}{f^2(Q(\theta_n(s)))} = o_P(1) \tag{5.19}$$

and by (5.2) and (5.3) we get

$$n^{1/2-\nu}L(1/n) \sup_{1/(n+1)\le s\le k_n/n} |\varrho_n(s)-u_n(s)|/(s^\nu L(s)) =$$

$$= o_P\big(n^{-\nu}L(1/n) \sup_{1/(n+1)\le s\le k_n/n} u_n^2(s)/(s^{1+\nu}L(s))\big).$$

Thus Theorem 2.9 implies (3.12).

PROOF of Theorem 3.7. We use (5.1). By Smirnov (1949) we have, as $n\to\infty$,

$$\sup_{k_n/n\le s\le 1/2} |\theta_n(s)/s-1| = o_P(1). \tag{5.20}$$

Since $f'(Q(s))$ is assumed to be regularly varying at zero, so is also $f(Q(s))$. Hence by (5.20)

$$\sup_{k_n/n\le s\le 1/2} \big|f(Q(s))/f(Q(\theta_n(s)))-1\big| = o_P(1) \tag{5.21}$$

and

$$\sup_{k_n/n\le s\le 1/2}\left|\theta_n(s)(1-\theta_n(s))\frac{f'(Q(\theta_n(s)))}{f^2(Q(s))}-s(1-s)\frac{f'(Q(s))}{f^2(Q(s))}\right| = o_P(1). \tag{5.22}$$

By Theorems 2.1, 2.6 and Lemma 4.3 we obtain

$$\sup_{k/n\le s\le 1/2}\frac{|B_n^2(s)-u_n^2(s)|}{s(1-s)} =$$

$$= \sup_{k_n/n\le s\le 1/2}\frac{|B_n(s)-u_n(s)|}{(s(1-s))^{1/2}}\,\frac{|B_n(s)+u_n(s)|}{(s(1-s))^{1/2}} = O_P\big((\log\log(n/k_n))^{1/2}\big). \tag{5.23}$$

For any $\varepsilon>0$ we have, as $n\to\infty$,

$$\sup_{\varepsilon\le s\le 1/2} |\varrho_n(s)-u_n(s)| = O_P(n^{-1/2}). \tag{5.24}$$

For any $\varepsilon>0$ there exists a $\delta>0$ such that

$$\sup_{0\le s\le\varepsilon}\left|\gamma-s(1-s)\frac{f'(Q(s))}{f^2(Q(s))}\right| < \delta. \tag{5.25}$$

Hence by (5.23), (5.24) and (5.25) it suffices to consider

$$(1/2)n^{-1/2}\gamma \sup_{k_n/n\leqq s\leqq\varepsilon} B_n^2(s)/(s(1-s)).$$

Now Lemma 4.3 implies (3.13).

Using again (5.1), we prove (3.14). We have for all $\varepsilon>0$

$$(1/2)n^{-1/2} \sup_{\varepsilon\leqq s\leqq 1/2} |\varrho_n(s)-u_n(s)|/(s^\nu L(s)) = O_P(n^{-1/2})$$

and

$$\sup_{\varepsilon\leqq s\leqq 1/2} u_n^2(s)/(s^{1+\nu}L(s)) = O_P(1).$$

Hence it is enough to consider

$$(1/2)\gamma n^{-1/2} \sup_{k_n/n\leqq s\leqq 1/2} u_n^2(s)/(s^{1+\nu}L(s)). \tag{5.26}$$

By Theorems 2.1 and 2.8 we have

$$\left(\frac{k_n}{n}\right)^\nu L(k_n/n) \sup_{k_n/n\leqq t\leqq 1/2} \frac{|u_n(t)-B_n(t)|}{t^{1/2-\alpha}} \frac{|u_n(t)+B_n(t)|}{t^{1/2+\nu+\alpha}L(t)} = O_P(k_n^{-\alpha}), \tag{5.27}$$

where $0<\alpha<1/2$. Now, similarly to the proof of (4.33), it is easy to see that we have

$$\begin{aligned}(1/2)\gamma\left(\frac{k_n}{n}\right)^\nu L(k_n/n) \sup_{k_n/n\leqq s\leqq 1/2} B_n^2(s)/(s^{1+\nu}L(s)) &\overset{\text{a.s.}}{=} \\ \overset{\text{a.s.}}{=} (1/2)\gamma\left(\frac{k_n}{n}\right)^\nu L(k_n/n)\Big(\sup_{k_n/n\leqq s\leqq 1/2} |B(s)|/(s^{1/2+\nu/2}L^{1/2}(s))\Big)^2 &\xrightarrow{\mathcal{D}} (\gamma/2)Y_{1/2-\nu/2}^2.\end{aligned} \tag{5.28}$$

By (5.27) and (5.28) we get (3.14).

In order to prove (3.15) we again use (5.1). First we note

$$\sup_{1/(n+1)\leqq s\leqq k_n/n} \theta_n(s)(1-\theta_n(s))\frac{f'(Q(\theta_n(s)))}{f^2(Q(\theta_n(s)))} \xrightarrow{P} \gamma. \tag{5.29}$$

By Wellner [20] and the regular variation of $f(Q(s))$ at zero we have that for any $\delta>0$ there exist $\varepsilon>0$ and n_0 such that

$$P\left\{\inf_{1/(n+1)\leqq s\leqq k_n/n} \frac{s(1-s)}{\theta_n(s)(1-\theta_n(s))}\frac{f(Q(s))}{f(Q(\theta_n(s)))} > \varepsilon\right\} \geqq 1-\delta \tag{5.30}$$

if $n\geqq n_0$. By Theorem 2.6 we have

$$\sup_{1/(n+1)\leqq s\leqq k_n/n} u_n^2(s)/(2s(1-s)\log\log n) \xrightarrow{P} 1. \tag{5.31}$$

Now (3.15) follows from (5.1) and (5.29)—(5.31).

The proof of (3.16) is similar to that of (3.15) and hence it is omitted.

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MONOTONICITY PROPERTIES OF THE ZEROS OF DERIVATIVE OF BESSEL FUNCTIONS

ÁRPÁD ELBERT, PÁL KOSIK and ANDREA LAFORGIA

Summary

For $\nu \geqq 0$ let $c_{\nu k}$ be the kth positive zero of the general cylinder function

$$C_\nu(x) = \cos \alpha J_\nu(x) - \sin \alpha Y_\nu(x), \quad 0 \leqq \alpha < \pi$$

where $J_\nu(x)$ and $Y_\nu(x)$ denote the Bessel funtions of the first and second kind, respectively. Since the notation $c_{\nu k}$ does not reflect the dependence on the values of α it is useful to define the function $j_{\nu\varkappa}$ as in [3]. The sequence $j_{\nu 1}\, j_{\nu 2}, \ldots$ is used to denote the sequence of the zeros of $J_\nu(x)$ corresponding to $\alpha=0$. Now for any $\varkappa \in (k-1, k)$, where k is some natural number, let $j_{\nu\varkappa}=c_{\nu k}$ with $\alpha=(k-\varkappa)\pi$. The correspondence between $j_{\nu\varkappa}$ and $c_{\nu k}$ is one to one.

In this paper we define the new function $j'_{\nu\varkappa}$ as the zeros of the derivative $(d/dx)C_\nu(x)= =C'_\nu(x)$ and we investigate the behaviour of this function establishing new properties of the functions $j'_{\nu\varkappa}$ and $j_{\nu\varkappa}$.

1. Introduction

For $\nu \geqq 0$ we use $c_{\nu k}$ to denote the kth positive zero of the cylinder function

$$C_\nu(x) = J_\nu(x) \cos \alpha - Y_\nu(x) \sin \alpha, \qquad 0 \leqq \alpha < \pi$$

where $J_\nu(x)$ and $Y_\nu(x)$ are the Bessel functions of the first and second kind, respectively.

In [2], [3] we introduced the notation $j_{\nu\varkappa}$ to denote the function $c_{\nu k}$ as follows: let $\varkappa = k - \alpha/\pi$, then $j_{\nu\varkappa} = c_{\nu k}$ for $k=1, 2, \ldots$.

We know that the function $j_{\nu\varkappa}$ is increasing with respect to both variables. For fixed $\varkappa$ this follows from the Watson formula [9, p. 508]

$$\text{(1.1)} \qquad \frac{d}{d\nu} j_{\nu\varkappa} = 2 j_{\nu\varkappa} \int_0^\infty K_0(2 j_{\nu\varkappa} \sinh t) e^{-2\nu t}\, dt,$$

where $K_0(u)$ is the modified Bessel function of order zero, which is positive on $(0, \infty)$. On the other hand for fixed ν we have proved in [2] that $j_{\nu\varkappa}$ is strictly increasing with respect to $\varkappa$. For later reference we express this property in the form

$$\text{(1.2)} \qquad j_{\nu\varkappa'} > j_{\nu\varkappa}, \quad \varkappa' > \varkappa > 0, \quad \nu > -\varkappa.$$

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In a similar but more complicated way, we shall introduce the function $j'_{\nu\varkappa}$ for the positive zeros $c'_{\nu k}$ of the function $C'_\nu(x)=(d/dx)C_\nu(x)$.

Concerning the sequence $\{j'_{\nu,\varkappa+k+1}-j'_{\nu,\varkappa+k}\}_{k=1}^{\infty}$ L. Lorch, M. E. Muldoon and P. Szego in [6] and J. Vosmansky in [8] have proved, that for fixed $\varkappa>0$ and $\nu\geqq 0$, this sequence decreases. This result suggests that $j'_{\nu\varkappa}$ is concave in $\varkappa$ for fixed $\nu\geqq 0$. However, we are able to prove this result only in the particular case $\nu=1/2$.

Finally we investigate the determinant

$$A = A(\delta) = \begin{vmatrix} j'_{\nu\varkappa} & j'_{\nu+\delta,\varkappa} \\ j_{\nu\varkappa'} & j_{\nu+\delta,\varkappa'} \end{vmatrix} \tag{1.3}$$

and we show that under some restrictions $A<0$.

This result gives some information on the connection between the zeros of $C_\nu(x)$ and $C'_\nu(x)$. We mention that the last author has studied similar properties for $c'_{\nu,k}$ [5].

2. The function $j'_{\nu\varkappa}$

In [4] we have introduced and studied the function $j'_{\nu\varkappa}$ as the zeros of the function $C'_\nu(x)$. There we assumed that $j'_{\nu\varkappa}>|\nu|$. Now we are interested only in the case $\nu\geqq 0$ and we will give a *modified* definition of $j'_{\nu\varkappa}$ more suitable for our investigations. This will be made in three steps.

Step A. For $\alpha=0$ we have $C_\nu(x)=J_\nu(x)$. If $\nu>0$ the zeros of $J_\nu(x)$ are denoted by $0, j_{\nu 1}, j_{\nu 2}, \ldots$. By Rolle's theorem we have that the elements of the sequence $j'_{\nu 1}$, $j'_{\nu 2}, \ldots$ of the zeros of $J'_\nu(x)$ satisfy the chain of inequalities $0<j'_{\nu 1}<j_{\nu 1}<j'_{\nu 2}<j_{\nu 2}<\ldots$. Moreover, by [9, p. 486] it follows the more stringent inequality $\nu<j'_{\nu 1}<j_{\nu 1}$.

Step B. Now, let $0<\alpha<\pi$. Since $\lim_{x\to 0^+} Y_\nu(x)=-\infty$, we get $\lim_{x\to 0^+} C_\nu(x)= +\infty$ and, consequently $C_\nu(x)>0$ on some right neighbourhood of $x=0$. Let us suppose that the equation $C'_\nu(x)=0$ has a solution $c'_{\nu 1}$ on the interval $0<x<\nu$ (see Fig. 1). Then by a result of Muldoon and Spigler [7] this may happen only in the case $0<\alpha<\pi/6$ (see Fig. 1). Another result of these authors in this paper states that

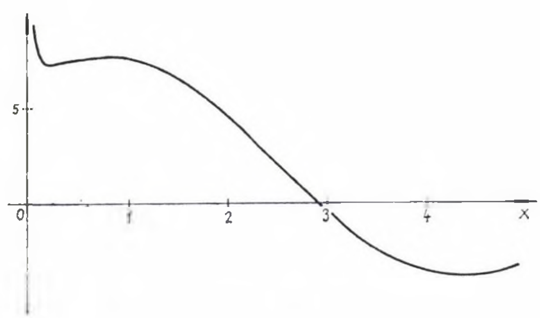

Fig. 1. The function $C_{1/2}(x)$ with small α

the function $C_\nu(x)$ may have a zero on $0<x\leqq\nu$ only if $5\pi/6<\alpha<\pi$. Therefore if $0<c'_{\nu 1}\leqq\nu$, then $C_\nu(x)>0$ on the interval $0<x\leqq\nu$ and for the first zero of $C_\nu(x)$ we have $c_{\nu 1}>\nu$ or, with our notation $j_{\nu\varkappa}>\nu$, where $\varkappa=1-\alpha/\pi$.

We recall that the function $C_\nu(x)$ is a solution of the differential equation

$$x^2y''+xy'+(x^2-\nu^2)y=0. \tag{2.1}$$

Hence if $c'_{\nu 1}=\nu$ then $C''_\nu(c'_{\nu 1})=0$ and $c'_{\nu 1}=\nu$ is a double zero of $C'_\nu(x)$. If $0<c'_{\nu 1}<\nu$ then from (2.1) we have $C''_\nu(c'_{\nu 1})>0$, that is $C_\nu(x)$ has a local minimum at $x=c'_{\nu 1}$. On the other hand the differential equation (2.1) can be rewritten in the form

$$x(xy')'=(\nu^2-x^2)y \tag{2.1'}$$

implying that $xC'_\nu(x)$ is an increasing function of x on $0<x\leqq\nu$ because $C_\nu(x)>0$ on $(0,\nu]$ in our case. Consequently we have $C'_\nu(\nu)>0$, hence the function $C'_\nu(x)$ must vanish again on $(\nu, c_{\nu 1})$. But it vanishes exactly once because by (2.1′) the function $xC'_\nu(x)$ is strictly decreasing on $(\nu, c_{\nu 1})$. Let $c'_{\nu 2}$ be this second zero of $C'_\nu(x)$. We denote $c'_{\nu 2}$ by $j'_{\nu\varkappa}$, so it is clear that $\nu<j'_{\nu\varkappa}<j_{\nu\varkappa}$ with $\varkappa=1-\alpha/\pi$. By similar argumentation we have that between the consecutive zeros $j_{\nu\varkappa}$, $j_{\nu,\varkappa+1}$ of the function $C_\nu(x)$ there is exactly one zero of $C'_\nu(x)$ denoted by $j'_{\nu,\varkappa+1}$ which satisfies the inequalities $j_{\nu\varkappa}<j'_{\nu,\varkappa+1}<j_{\nu,\varkappa+1}$ where $\varkappa=k-\alpha/\pi$, $k=1,2,\ldots$.

Step C. Let us consider now such function $C_\nu(x)$ which decreasing on $(0, c_{\nu 1}]$. This is for example the case when $\alpha=\pi/2$, i.e. $C_\nu(x)=-Y_\nu(x)$. Then the zeros of $C'_\nu(x)$ are $c'_{\nu 1}, c'_{\nu 2}, \ldots$ and they satisfy the chain of inequalities $c_{\nu 1}<c'_{\nu 1}<c_{\nu 2}<c'_{\nu 2}<\ldots$. Now we have $c'_{\nu 1}>\nu$. This is clearly true if $c_{\nu 1}>\nu$ and by the above mentioned results of Muldoon and Spigler the case $c_{\nu 1}\leqq\nu$ is possible only if $\alpha\in(5\pi/6,\pi)$, hence $c'_{\nu 1}$ cannot be on the interval $(0,\nu]$ because then it would be $\alpha\in(0,\pi/6)$. Let $j'_{\nu\varkappa}$ be defined by the relations $j'_{\nu\varkappa}=c'_{\nu k}$, $\varkappa=k+1-\alpha/\pi$, $k=1,2,\ldots$.

Summarizing the properties of the function $j'_{\nu\varkappa}$ just defined we have

$$j'_{\nu\varkappa}>\nu \tag{2.2}$$

$$j_{\nu,\varkappa-1}<j'_{\nu\varkappa}<j_{\nu\varkappa}, \quad \text{provided that} \quad j_{\nu,\varkappa-1} \text{ exists.} \tag{2.3}$$

The advantage of this *new* definition of $j'_{\nu\varkappa}$ lies first of all in the validity of the relations (2.2) and especially in (2.3) because the second inequality in (2.3) does not hold always using the *old* definition (see Remark 2.4).

Now we have to deal with some properties of newly defined function $j'_{\nu\varkappa}$.

Lemma 2.1. *Let j' be a zero of the function $C'_\nu(x)=(d/dx)C_\nu(x)$, $(\nu\geqq 0)$ satisfying the inequality $j'>\nu$. Then there exists a $\varkappa>0$ such that $j'=j'_{\nu\varkappa}$.*

Proof. It is sufficient to show that during the definiton of $j'_{\nu\varkappa}$ above we have taken into consideration all the possibilities. Suppose the contrary.

Looking over the definition we find that a missing function $C_\nu(x)$ should be monotonic on $(0,\nu)$ and not monotonic on $(0, c_{\nu 1}]$, i.e. not monotonic on $[\nu, c_{\nu 1}]$. Then there would be at least one zero c' of $C'_\nu(x)$ such that $\nu<c'<c_{\nu 1}$. By (2.1′) the function $xC'_\nu(x)$ is decreasing on $[\nu, c_{\nu 1}]$ and $C'_\nu(\nu)>0$. By our assumptions the function $C_\nu(x)$ is monotonic on $(0,\nu]$ and $\operatorname{sign} C'_\nu(x)=\operatorname{sign} C'_\nu(\nu)=1$ there. Among the functions $C_\nu(x)$ with $0\leqq\alpha<\pi$ only $J_\nu(x)$ is increasing on $(0,\nu]$. But the zeros of $J'_\nu(x)$

have already been defined as $j'_{\nu 1}, j'_{\nu 2}, \ldots$. This contradiction shows that our definition takes into account every $C_\nu(x)$ and the proof is complete.

Let the function $\bar{\alpha}=\bar{\alpha}(\nu)$ be defined by

(2.4) $$\tan \bar{\alpha}(\nu) = \frac{J'_\nu(\nu)}{Y'_\nu(\nu)}.$$

In [7] the authors have shown that the function $J'_\nu(\nu)/Y'_\nu(\nu)$ increases from zero to $\sqrt{3}/3$ as ν varies from 0 to ∞. By this and (2.4) the function $\bar{\alpha}(\nu)$ increases, $\bar{\alpha}(0)=0$, $\lim_{\nu\to+\infty} \bar{\alpha}(\nu)=\pi/6$. Let the function $\bar{\varkappa}=\bar{\varkappa}(\nu)$ be defined by

(2.5) $$\bar{\varkappa} = \bar{\varkappa}(\nu) = 1-\frac{\bar{\alpha}(\nu)}{\pi},$$

hence $\bar{\varkappa}(\nu)$ decreases and the relations $\bar{\varkappa}(0)=1$, $\bar{\varkappa}(\nu)>\lim_{\nu\to+\infty} \bar{\varkappa}(\nu)=5/6$ hold.

Now we can formulate the next result.

LEMMA 2.2. *The function $j'_{\nu\varkappa}$ is defined if $\varkappa>\bar{\varkappa}(\nu)$ and $\nu\geqq 0$, where $\bar{\varkappa}$ is the same as in* (2.5).

PROOF. The function $j'_{\nu 1}$ was defined above for every $\nu>0$. In the case $\varkappa>1$ the values $j_{\nu,\varkappa-1}, j_{\nu\varkappa}$ are consecutive zeros of $C_\nu(x)$, hence by Rolle's theorem the value $j'_{\nu\varkappa}$ exists for all $\nu>0$. Thus we have to consider only the case $0<\varkappa<1$, i.e. $\varkappa=1-\alpha/\pi$. This case is dealt in Step B. Hence $j'_{\nu\varkappa}$ is the second zero of $C'_\nu(x)=\cos \alpha J'_\nu(x)-\sin \alpha Y'_\nu(x)$ and the relations $\nu<j'_{\nu\varkappa}<j_{\nu\varkappa}=c_{\nu 1}$ hold. By (2.1′) the function $xC'_\nu(x)$ is decreasing on $[\nu, c_{\nu 1}]$, so we have $C'_\nu(\nu)=\cos \alpha J'_\nu(\nu)-\sin \alpha Y'_\nu(\nu)>0$ and therefore

(2.6) $$\operatorname{cotan} \alpha > \frac{Y'_\nu(\nu)}{J'_\nu(\nu)}.$$

Comparing (2.6) with (2.4) we find $\alpha<\bar{\alpha}(\nu)$ and by (2.5) $\varkappa=1-\alpha/\pi>1-\bar{\alpha}(\nu)/\pi=\bar{\varkappa}(\nu)$ which was to be proved.

Let us consider the particular case $\nu=1/2$. Then $C_{1/2}(x)=\sqrt{2/\pi x}\sin(x+\alpha)$ and $j'_{1/2,\varkappa}$ is solution of the equation

(2.7) $$\sin(j'+\alpha)-2j'\cos(j'+\alpha) = 0.$$

By our notation $\varkappa=k-\alpha/\pi$, $k=1, 2, \ldots$, hence $\alpha=k\pi-\varkappa\pi$ and the equation (2.7) becomes

(2.8) $$\sin(\varkappa\pi-j')+2j'\cos(\varkappa\pi-j') = 0.$$

From [1, p. 468] $j'_{1/2,1}=1.1655\ldots$, therefore if we put

(2.9) $$\delta_\varkappa = \varkappa\pi-j'_{1/2,\varkappa}$$

we obtain $\pi/2<\delta_1<3\pi/4$ and from (2.8) it follows

$$\frac{\pi}{2} < \delta_\varkappa < \pi$$

and

$$\varkappa = -\frac{1}{2\pi}\tan\delta_\varkappa + \frac{1}{\pi}\delta_\varkappa. \tag{2.10}$$

The function on the right-hand side of (2.10) is convex on $\pi/2<\delta_\varkappa<\pi$ and it has the minimum at $\delta_\varkappa=3\pi/4$ where it takes on the value $\varkappa_0=3/4+1/2\pi$ which belongs to the interval (0, 1). Thus the function $\delta_\varkappa$ is defined for $\varkappa>\varkappa_0$ and the relations

$$\begin{gathered}\frac{\pi}{2}<\delta_\varkappa<\frac{3\pi}{4} \quad \text{for} \quad \varkappa>\varkappa_0,\\ \lim_{\varkappa\to\varkappa_0+0}\delta_\varkappa=\frac{3\pi}{4}, \qquad \lim_{\varkappa\to\infty}\delta_\varkappa=\frac{\pi}{2}\end{gathered} \tag{2.11}$$

hold. Now we collect the results concerning the function $j'_{1/2,\varkappa}$ in a lemma.

LEMMA 2.3. *The function* $j'_{1/2,\varkappa}$ *is defined for* $\varkappa>\varkappa_0=3/4+1/2\pi=0.90915\ldots$ *and satisfies the limit relations*

$$\lim_{\varkappa\to\varkappa_0+0} j'_{1/2,\varkappa}=1/2, \qquad \lim_{\varkappa\to\infty} j'_{1/2,\varkappa}=\infty.$$

Moreover, it is strictly increasing and concave.

PROOF. By (2.9) and (2.10) $j'_{1/2,\varkappa}=\varkappa\pi-\delta_\varkappa=1/2\tan\delta_\varkappa$ hence by (2.11) the limit relations for $j'_{1/2,\varkappa}$ are clearly true. On the other hand by (2.10) the function $\delta_\varkappa$ is decreasing convex function of $\varkappa$. Hence $j'_{1/2,\varkappa}$ is increasing convex function. This completes the proof of Lemma 2.3.

REMARK 2.1. We have seen above that the function $\delta_\varkappa=\varkappa\pi-j'_{1/2,\varkappa}$ is decreasing convex and $\lim_{\varkappa\to+\infty}\delta_\varkappa=\pi/2$. The following questions arise naturally. Let $\delta_{\nu\varkappa}=j_{\nu\varkappa}-j'_{\nu\varkappa}$. If $\nu\geqq 0$ is fixed, is the function $\delta_{\nu\varkappa}$ decreasing and convex?

REMARK 2.2. We conjecture that $j'_{\nu\varkappa}$ is concave with respect to $\varkappa$ for all positive ν. This conjecture is supported also by the property proved in [6] and [8] that the sequence $\{j'_{\nu,\varkappa+k+1}-j'_{\nu,\varkappa+k}\}_{k=0}^{\infty}$ is completely monotonic, i.e., among other things it is decreasing.

The next result also supports the *new* definition of $j'_{\nu\varkappa}$.

THEOREM 2.1. *The function* $j'_{\nu\varkappa}$ *is continuous with respect to both variables* ν *and* $\varkappa$ *with* $\nu>0$ *and* $\varkappa>\bar\varkappa(\nu)$ *where* $\bar\varkappa$ *is defined by* (2.5). *Moreover, it satisfies the inequality*

$$j'_{\nu\varkappa'}>j'_{\nu\varkappa} \quad \textit{for} \quad \varkappa'>\varkappa>\bar\varkappa(\nu), \quad \nu>0 \tag{2.12}$$

and the relation

$$\lim_{\varkappa\to\bar\varkappa(\nu)+0} j'_{\nu\varkappa}=\nu$$

holds.

PROOF. Let the value of $\varkappa$ be fixed. Then the function $j'_{\nu\varkappa}$ is a solution of the non-linear integro-differential equation [9, p. 510]

$$\frac{d}{d\nu} j' = 2j' \int_0^\infty \frac{j'^2 \cosh 2t - \nu^2}{j'^2 - \nu^2} K_0(2j' \sinh t) e^{-2\nu t}\, dt. \tag{2.13}$$

The right-hand side is Lipschitzian with respect to j' provided $j' \neq \nu$ and $j' > 0$. Here we are concerned in values $j'_{\nu\varkappa} > \nu > 0$. Hence the initial value problem for (2.13) has unique solution.

Let $\varkappa > \varkappa_0$, where $\varkappa_0 \in (0, 1)$ is the same as in Lemma 2.3. Then the solution of (2.13) with the initial condition $j'|_{\nu=1/2} = j'_{1/2,\varkappa}$ is the function $j'_{\nu\varkappa}$ provided $j'_{\nu\varkappa} > \nu > 0$.

In [4] we have proved that $(d/d\nu) j'_{\nu\varkappa} > 1$ if $j'_{\nu\varkappa} > \nu$, hence the condition $j'_{\nu\varkappa} > \nu$ is satisfied when $\nu \geqq 1/2$. The continuous dependence of the solutions on the initial conditions ensures that the function $j'_{\nu\varkappa}$ is continuous in both variables ν and $\varkappa$ for $\varkappa > \bar{\varkappa}(\nu) > \varkappa_0$. Moreover we claim that

$$j'_{\nu\varkappa'} > j'_{\nu\varkappa} \quad \text{for} \quad \varkappa' > \varkappa > \varkappa_0 \tag{2.14}$$

as long as $j'_{\nu\varkappa} > \nu$. To prove this relation we state first that $j'_{1/2,\varkappa'} > j'_{1/2,\varkappa} > 1/2$. This is true because by Lemma 2.3 the function $j'_{1/2,\varkappa}$ is strictly increasing as $\varkappa$ increases.

Thus the uniqueness of the above mentioned initial value problem posed at $\nu = 1/2$ shows that the equality $j'_{\bar{\nu}\varkappa'} = j'_{\bar{\nu}\varkappa}$ never occurs if $j'_{\bar{\nu}\varkappa} > \bar{\nu} > 0$ hence (2.14) is true.

Now we are going to prove that the value $\varkappa_0$ in (2.14) can be replaced by $\bar{\varkappa}(\nu)$. Since $j'_{\nu 1} > \nu$ (see Step A above), (2.14) holds for $\varkappa' > \varkappa \geqq 1$. Now we should consider the case $\varkappa, \varkappa' < 1$.

First we claim that the first zero $y'_{\nu 1}$ of $Y'_\nu(x)$ is $j'_{\nu,3/2}$. Since for $\alpha = \pi/2$ we have $C_\nu(x) = -Y_\nu(x)$, and the parameter $\varkappa$ may take on the values $1/2, 3/2, 5/2, \ldots$. The value $1/2$ is not possible because by Lemma 2.2 $\varkappa$ should be greater than $\bar{\varkappa}(\nu)$ and $\bar{\varkappa}$ varies between 1 and 5/6. Thus we find $y'_{\nu 1} = j'_{\nu,3/2}$. Then by (2.14) $j'_{\nu,3/2} > j'_{\nu,1}$ and the relations

$$J'_\nu(x) > 0, \quad Y'_\nu(x) > 0, \quad \text{for} \quad 0 < x < j'_{\nu 1} \tag{2.15}$$

hold.

For fixed $\nu > 0$ we suppose $\bar{\varkappa}(\nu) < \varkappa < 1$. During the procedure in the definition of $j'_{\nu\varkappa}$ we have seen that this is the case in Step B when there are two zeros of $C'_\nu(x)$ on the interval $(0, j_{\nu\varkappa})$, namely $c'_{\nu 1}$ and $c'_{\nu 2} = j'_{\nu\varkappa}$ satisfying the relations $0 < c'_{\nu 1} < \nu < j'_{\nu\varkappa} < j_{\nu\varkappa}$ and $C'_\nu(x) < 0$ on $(0, c'_{\nu 1}) \cup (j'_{\nu\varkappa}, j_{\nu\varkappa})$ and $C'_\nu(x) > 0$ on $(c'_{\nu 1}, j'_{\nu\varkappa})$. Since $C_\nu(x) = \cos \alpha J_\nu(x) - \sin \alpha Y_\nu(x)$ and $\alpha = \pi - \varkappa\pi \in \left(0, \frac{\pi}{6}\right)$, therefore by (2.15)

$$\frac{\partial}{\partial \varkappa} C'_\nu(x) = \pi[\sin \alpha J'_\nu(x) + \cos \alpha Y'_\nu(x)] > 0 \tag{2.16}$$

for $0 < x < j'_{\nu 1}$. Hence the function $C'_\nu(x)$ increases as $\varkappa$ increases and ν, x fixed. Consequently, the function $j'_{\nu\varkappa}$ increases and $c'_{\nu 1}$ decreases as $\varkappa$ increases. Taking into account also (2.14) we have found that (2.12) is true as long as $j'_{\nu\varkappa}$ exists.

By (2.15) $C'_\nu(x) < 0$ for $\alpha = \pi/2$ and $0 < x < j'_{\nu 1}$, therefore by (2.16) there is exactly one value $\alpha^* \in (0, \pi/2)$ or $\varkappa^* \in (1/2, 1)$ with $\alpha^* = \pi(1 - \varkappa^*)$ such that $x = j'_{\nu\varkappa^*}$

is a double zero for $C'_\nu(x)$ with $\alpha=\alpha^*$. Hence $C'_\nu(j'_{\nu\varkappa^*})=C''_\nu(j'_{\nu\varkappa^*})=0$ and by (2.1) we find $j'_{\nu\varkappa^*}=\nu$. Owing to (2.5), (2.6) this can happen only if $\alpha^*=\bar{\alpha}(\nu)$ or $\varkappa^*=\bar{\varkappa}(\nu)$ which proves the limit relation of Theorem 2.1. The proof is complete.

REMARK 2.3. Since the right-hand side of (2.12) is Lipschitzian with respect to j' for $j'\neq\nu$, $j'>0$ not only for $\nu>0$ but also for $\nu\leqq 0$, then the range of the validity of the relations above can be extended continuously to $\nu\leqq 0$ as well. The complete situation for $-2\leqq\nu\leqq 2$ and $0<j'\leqq 5$ displayed by Fig. 2 where the curves show the points which the value of $\varkappa$ is common for.

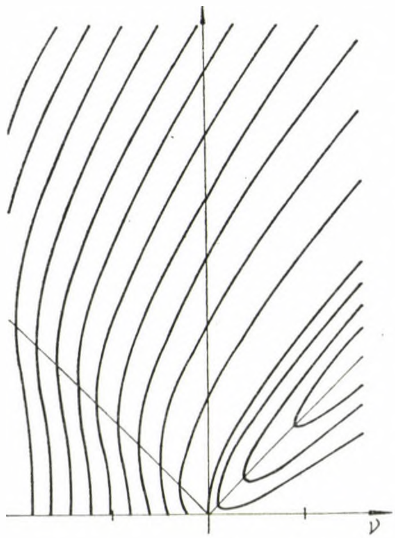

Fig. 2. The graphs of $j'_{\nu\varkappa}$ for $-2\leqq\nu\leqq 2$

REMARK 2.4. The relation $y'_{\nu 1}=j'_{\nu,3/2}$ is a good example to show the advantage of our *new* definition for $j'_{\nu\varkappa}$.

From [1] we know that $y'_{1/2,1}=2.975086$ and the two first zeros of $J'_{1/2}(x)$ are $j'_{1/2,1}=1.165561$, $j'_{1/2,2}=4.604217$. The monotonic behaviour of $j'_{\nu\varkappa}$ with respect to $\varkappa$ given by (2.12) would be broken if the first zero $y'_{1/2,1}$ of $Y'_{1/2}(x)$ were denoted by $j'_{1/2,1/2}$ in accordance with the *old* and seemingly *natural* definition.

3. A determinantal inequality

The result of this section is concerned with the monotonic character of the function $j_{\nu\varkappa'}/j'_{\nu\varkappa}$ with respect to ν and $\varkappa'>\varkappa$ fixed.

THEOREM 3.1. *Let* $\nu_0\geqq 0$ *and* $\varkappa'>\varkappa>\bar{\varkappa}(\nu_0)$. *Then the determinant* $A=A(\delta)$ *defined by* (1.3) *is negative if* $\delta>0$ *for all* $\nu>\nu_0$.

PROOF. First we observe that by Lemma 2.2 $j'_{\nu+\delta,\varkappa}$ is defined for all $\delta>0$ because the function $\bar{\varkappa}(\nu)$ is decreasing. Thus $\bar{\varkappa}(\nu+\delta)<\bar{\varkappa}(\nu)<\varkappa$ and by (2.2) $j'_{\nu+\delta,\varkappa}>\nu+\delta$. Making use of (1.1) and (2.13) we get

$$A'(\delta)=\frac{d}{d\delta}A(\delta)=2j'_{\nu\varkappa}j_{\nu+\delta,\varkappa'}\int_0^\infty K_0(2j_{\nu+\delta,\varkappa'}\sinh t)e^{-2(\nu+\delta)t}\,dt-$$

$$-2j_{\nu\varkappa'}j'_{\nu+\varkappa}\int_0^\infty Q(j'_{\nu+\delta,\varkappa},\nu+\delta,t)K_0(2j'_{\nu+\delta,\varkappa}\sinh t)e^{-2(\nu+\delta)t}\,dt$$

where

$$Q(x,\nu,t)=\frac{x^2\cosh 2t-\nu^2}{x^2-\nu^2}.$$

By (1.3) it follows that $A(0)=0$, moreover

$$A'(0)=2j_{\nu\varkappa'}j'_{\nu\varkappa}\int_0^\infty[K_0(2j_{\nu\varkappa'}\sinh t)-K_0(j'_{\nu\varkappa}\sinh t)Q(j'_{\nu\varkappa},\nu,t)]e^{-2\nu t}\,dt.$$

We claim that $A'(0)<0$. Since $\cosh 2t>1$ for $t>0$ and $j'_{\nu\varkappa}>\nu$ we have $Q(j'_{\nu\varkappa'},\nu,t)>1$. Recalling that $K_0(u)$ decreases with respect to u and using the inequalities (1.2) and (2.3) we obtain $j_{\nu\varkappa'}\geqq j_{\nu\varkappa}>j'_{\nu\varkappa}$, hence $K_0(2j_{\nu\varkappa'}\sinh t)<$ $<K_0(2j'_{\nu\varkappa}\sinh t)$ which yields $A'(0)<0$.

Thus the function $A(\delta)$ is negative on some right neighbourhood of $\delta=0$. We have to show that $A(\delta)<0$ for all $\delta>0$. Suppose the contrary. Then there exists δ_1 such that $A(\delta_1)\geqq0$. Let δ_0 be defined by $\delta_0=\min\limits_{\delta>0}\{\delta\colon A(\delta)\geqq0\}$. It is clear that $\delta_0>0$, $A(\delta_0)=0$, $A(\delta)<0$ for $0<\delta<\delta_0$. Hence $A'(\delta_0)\geqq0$ and by (1.3)

$$j_{\nu\varkappa'}j'_{\nu+\delta_0,\varkappa}=j'_{\nu\varkappa}j_{\nu+\delta_0,\varkappa'},$$

and therefore

$$A'(\delta_0)=2j_{\nu\varkappa'}j'_{\nu+\delta_0,\varkappa}\times$$

$$\times\int_0^\infty[K_0(2j_{\nu+\delta_0,\varkappa'}\sinh t)-K_0(2j'_{\nu+\delta_0,\varkappa}\sinh t)Q(j'_{\nu+\delta_0,\varkappa},\nu+\delta_0,t)]e^{-2(\nu+\delta_0)t}\,dt.$$

Now we have $j'_{\nu+\delta_0,\varkappa}>\nu+\delta_0$, hence $Q(j'_{\nu+\delta_0,\varkappa},\nu+\delta_0,t)>1$ for $t>0$ and by $j_{\nu+\delta_0,\varkappa'}>j'_{\nu+\delta_0,\varkappa}$ the inequality $K_0(2j_{\nu+\delta_0,\varkappa'}\sinh t)<K_0(2j'_{\nu+\delta_0,\varkappa}\sinh t)$ follows and consequently $A'(\delta_0)<0$. But this contradicts to the inequality $A'(\delta_0)\geqq0$ shown above. Therefore $A(\delta)<0$ for all $\delta>0$, which was to be proved.

An equivalent formulation of Theorem 3.1 is the following

COROLLARY 3.1. *Let* $\varkappa,\varkappa'$ *be the same as in Theorem 3.1. Then the function* $j_{\nu\varkappa'}/j'_{\nu\varkappa}$ *decreases as* ν *increases and* $\nu>\nu_0$.

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ON PROXIMITY-LIKE RELATIONS INTRODUCED BY F. RIESZ IN 1908

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Abstract

We investigate the connexion between *chainings* introduced by F. Riesz to axiomatize the notion of nearness of two sets and *proximities* in the wider sense (Čech proximities), and show how a theorem on extensions of chainings (Theorem 1.7), whose proof was sketched out by Riesz, can be precisely formulated and proved.

According to Riesz [10], a function d: $\exp X \to \exp X$ is an *accumulation ("Verdichtung")* on the set X if

A1 $\qquad A \subset B \subset X$ implies $d(A) \subset d(B)$;

A2 $\qquad d(A \cup B) \subset d(A) \cup d(B) \qquad (A, B \subset X)$;

A3 $\qquad d(\{x\}) = \emptyset \qquad (x \in X)$.

The elements of $d(A)$ are the *accumulation points* of A. A1 and A2 together can be replaced by $d(A \cup B) = d(A) \cup d(B)$.

A relation $\varrho \subset \exp X \times \exp X$ is a *chaining ("Verkettung")* if (with $\bar{\varrho}$ meaning $\exp X \times \exp X \setminus \varrho$)

Ch0 $\qquad \emptyset \, \bar{\varrho} \, X$; if $A \, \varrho \, B$ then $B \, \varrho \, A$;

Ch1 $\qquad$ if $A \, \varrho \, B$ and $B \subset C \subset X$ then $A \, \varrho \, C$;

Ch2 $\qquad$ if $A \, \varrho \, B \cup C$ then $A \, \varrho \, B$ or $A \, \varrho \, C$;

Ch3 $\qquad \{x\} \, \bar{\varrho} \, \{y\} \qquad (x, y \in X)$.

(The two conditions in Ch0 are not explicitly stated in [10], but it is clear from the context that they were meant to hold.) If $A \, \varrho \, B$ then A and B are said to be *chained*. Ch1+Ch2 can be replaced by the axiom $A \, \varrho \, B \cup C$ iff $A \, \varrho \, B$ or $A \, \varrho \, C$.

In current terminology, an accumulation is nothing but the usual derived set operation in a separated closure space (called semi-separated or T_1 closure in [2]),

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where a function c: $\exp X \to \exp X$ is a *separated closure* on X if

C0 $$c(\emptyset) = \emptyset;$$

C1 $$\text{if} \quad A \subset B \subset X \quad \text{then} \quad c(A) \subset c(B);$$

C2 $$c(A \cup B) \subset c(A) \cup c(B) \qquad (A, B \subset X);$$

C3 $$c(\{x\}) = \{x\} \qquad (x \in X).$$

[In the definition of a *closure*, C3 is replaced by the weaker condition $x \in c(\{x\})$ $(x \in X)$, or equivalently by $A \subset c(A)$ $(A \subset X)$.] A one-to-one correspondence between accumulations and separated closures can be obtained in the following way: if d is an accumulation and c a separated closure on X then a separated closure c_d and an accumulation d_c are defined by

$$c_d(A) = A \cup d(A) \quad (A \subset X);$$

$$x \in d_c(A) \quad \text{iff} \quad x \in c(A \setminus \{x\}) \quad (x \in X,\ A \subset X).$$

Now $c_{d_c} = c$ and $d_{c_d} = d$.

A chaining ϱ on X *induces* an accumulation d_ϱ as follows [10]:

$$x \in d_\varrho(A) \quad \text{iff} \quad \{x\}\ \varrho\ A \qquad (x \in X,\ A \subset X);$$

we shall also say that ϱ is *compatible* with d_ϱ. c_{d_ϱ} will be denoted by c_ϱ. For any accumulation d on X, there are chainings compatible with it (more than one, in general); ϱ_d^* is the finest one, where

$$A\ \varrho_d^*\ B \quad \text{iff} \quad A \cap d(B) \neq \emptyset \quad \text{or} \quad d(A) \cap B \neq \emptyset \qquad (A, B \subset X).$$

[The chaining ϱ_1 is *finer* than the chaining ϱ_2 (or ϱ_2 is *coarser* than ϱ_1) if $\varrho_1 \subset \varrho_2$.] There is also a coarsest one, namely

$$A\ \varrho_d^\circ\ B \quad \text{iff} \quad A\ \varrho_d^*\ B \quad \text{or} \quad A \quad \text{and} \quad B \quad \text{are infinite} \quad (A, B \subset X).$$

Chainings are sometimes cited in the literature as something very similar to proximities, but (as far as we know) the connexion of these two notions has never been worked out in detail: we shall deal with this problem in § 2 and § 3. In § 1, we shall solve a problem raised by Riesz (proximities will not be needed there). § 4 contains some additional material, mostly without proofs.

The topic of the present paper is mainly of historical interest; without the paper by Riesz, there would be no reason for introducing chainings, since the same idea (the nearness of two sets) is much more conveniently described by proximities.

NOTATIONS AND TERMINOLOGY. The zero filter $\exp X$ is regarded as one of the filters on X. $\operatorname{fil} \mathfrak{b} = \operatorname{fil}_X \mathfrak{b}$ is the filter on X generated by the filter base $\mathfrak{b}$. A filter $\mathfrak{f}$ is *free* or *fixed* according as $\bigcap \mathfrak{f}$ is empty or not; $\mathfrak{f}$ is *fixed at* the point x if $x \in \bigcap \mathfrak{f}$. $\dot{x}$ is the ultrafilter fixed at x (the fundamental set will be clear from the context). For $\mathfrak{a} \subset \exp X$, $\sec \mathfrak{a} = \sec_X \mathfrak{a}$ is the collection of all the subsets of X that meet each element of $\mathfrak{a}$. If $\mathfrak{a} \subset \exp X$ and $S \subset X$ then $\mathfrak{a}|S = \{A \cap S\colon A \in \mathfrak{a}\}$ is the *trace* of $\mathfrak{a}$ on S.

§ 1. Chainings obtainable as restrictions to subsets of special types of chainings

1.1 After defining the accumulation induced by a chaining, Riesz introduces one more axiom for chainings, and then restricts his attention to such chainings. We shall say that the chaining ϱ is *Riesz* if

ChR $$d_\varrho(A)\cap d_\varrho(B)\neq\emptyset \quad \text{implies} \quad A\ \varrho\ B.$$

The finest Riesz chaining compatible with a given accumulation d is ϱ_d, where [10]

(1) $$A\ \varrho_d\ B \quad \text{iff} \quad A\ \varrho_d^*\ B \quad \text{or} \quad d(A)\cap d(B)\neq\emptyset.$$

Riesz raised the following problem: which chainings can be obtained as a restriction of a chaining of the type ϱ_d? ($\varrho|X=\varrho\cap\exp X\times\exp X$ is the *restriction* of the chaining ϱ on Y to $X\subset Y$.) Let us observe that the answer to the analogous question for ϱ_d^* or ϱ_d° is a triviality, since $\varrho_d^*|X=\varrho_{d|X}^*$ and $\varrho_d^\circ|X=\varrho_{d|X}^\circ$, where $(d|X)(A)=d(A)\cap X$.

In order to solve the problem, we introduce one more axiom: a chaining ϱ is *saturated* if

ChS $$A\ \varrho\ B \quad \text{and} \quad A\cap d_\varrho(B)=\emptyset \quad \text{imply} \quad A\ \varrho\ A.$$

LEMMA. *If d is an accumulation on Y and $X\subset Y$ then $\varrho_d|X$ is a saturated Riesz chaining.*

PROOF. First let us show that ϱ_d is saturated. As ϱ_d is compatible with d, what we have to show is that

(2) $$A\ \varrho_d\ B \quad \text{and} \quad A\cap d(B)=\emptyset$$

imply $A\ \varrho_d\ A$. By (1), (2) and the definition of ϱ_d^*, $d(A)\cap B\neq\emptyset$ or $d(A)\cap d(B)\neq\emptyset$; in both cases, $d(A)\neq\emptyset$, thus $A\ \varrho_d\ A$ follows from (1). It is an immediate consequence of the definitions that $\varrho|X$ is saturated, too (observe that $d_{\varrho|X}=d_\varrho|X$).

The proof of the other part of the lemma is of similar simplicity. □

EXAMPLES. a) *A Riesz chaining that is not saturated.* Let X be the disjoint union of the infinite sets P and Q. Put $A\ \varrho\ B$ iff either $A\cap P$ and $B\cap Q$ or $A\cap Q$ and $B\cap P$ are infinite. ϱ is a Riesz chaining (it is Riesz, because $d_\varrho(A)=\emptyset$ for any $A\subset X$), but it is not saturated, for $P\ \varrho\ Q$, $P\cap d_\varrho(Q)=\emptyset$ and $P\ \bar\varrho\ P$.

b) *A saturated chaining that is not Riesz.* Let $X=\mathbf{R}$, and d the accumulation associated with the Euclidean closure. Put $A\ \varrho\ B$ iff either $A\ \varrho_d^*\ B$ or $A\cap B$ is infinite. ϱ is clearly a chaining compatible with d. It is saturated, since if $A\ \varrho\ B$ and $A\cap d(B)=$ $=\emptyset$ then A has to be infinite. It is shown by the sets $A=]-1,\ 0[$ and $B=]0,\ 1[$ that ϱ is not Riesz.

REMARKS. a) In the finest saturated chaining compatible with an accumulation d, A and B are chained iff $A\ \varrho_d^*\ B$ or $d(A\cap B)\neq\emptyset$. ϱ_d is the finest saturated Riesz chaining compatible with d. ϱ_d° is saturated and Riesz, so it is the coarsest saturated chaining as well as the coarsest Riesz chaining compatible with d.

b) For Riesz chainings (but not in general), ChS is equivalent to

ChS′ if $A\ \varrho\ B$ and $A\ \overline{\varrho^*_{d_\varrho}}\ B$ then $A\ \varrho\ A$.

It will be apparent from 1.3 and 4.2 that ChS is the better choice.

1.2 We shall see that the statement of Lemma 1.1 can be reversed, i.e. that the saturated Riesz chainings give the answer to Riesz's question. The proof will be essentially the same as the one sketched out by Riesz, but we shall use filters instead of the dual notion (which is currently called a grill, see later in § 3).

For $A, B\subset X$ and a filter $\mathfrak{f}$ on X, we write $(A, B)\in \operatorname{Sec}\mathfrak{f}$ iff $A, B\in \sec\mathfrak{f}$ and $\mathfrak{f}|A\cup B$ is not a fixed ultrafilter. The filter $\mathfrak{f}$ is *ϱ-compressed* if $\operatorname{Sec}\mathfrak{f}\subset\varrho$.

Lemma. *For an arbitrary filter* $\mathfrak{f}$, $\operatorname{Sec}\mathfrak{f}$ *satisfies* Ch0, Ch1 *and* Ch2.

Proof. Ch0 and Ch1 are evident. To prove Ch2, assume $(A, B\cup C)\in\operatorname{Sec}\mathfrak{f}$. Then either B or C belongs to $\sec\mathfrak{f}$, say $B\in\sec\mathfrak{f}$. As $A\in\sec\mathfrak{f}$, too, there is nothing to prove if $\mathfrak{f}|A\cup B$ is not a fixed ultrafilter. So we may assume that $\mathfrak{f}|A\cup B=\dot{x}$ with some $x\in A\cap B$. Since $\mathfrak{f}|A\cup B\cup C$ is not a fixed ultrafilter,

$$(1)\qquad \emptyset\notin\mathfrak{f}|C\setminus(A\cup B),$$

i.e. each element of $\mathfrak{f}$ meets $A\cup C$ in x as well as in some other point, $A\in\sec\mathfrak{f}$, and by (1) $C\in\sec\mathfrak{f}$, too, hence $(A, C)\in\operatorname{Sec}\mathfrak{f}$. □

1.3 The following lemma is the key to the problem. (Concerning the analogous result for proximities, see [11, 12, 4].)

Lemma. *If ϱ is a saturated chaining and $A\ \varrho\ B$ then there is a ϱ-compressed filter $\mathfrak{f}$ such that* $(A, B)\in\operatorname{Sec}\mathfrak{f}$.

Remark. ChS′ would not be enough here, see the example after the proof. The statement of the lemma can be reversed: if $A\ \varrho\ B$ implies the existence of a ϱ-compressed filter $\mathfrak{f}$ such that $(A, B)\in\operatorname{Sec}\mathfrak{f}$ then ϱ is saturated.

Proof. 1° As $\operatorname{Sec}\mathfrak{f}$ satisfies Ch1, it is enough to prove the lemma for smaller sets instead of A and B. So we may assume:

$$(1)\qquad \text{if}\quad A\cap d_\varrho(B)\neq\emptyset\quad\text{then}\quad |A|=1$$

[because A can be replaced by $\{x\}$ with some $x\in A\cap d_\varrho(B)$], and similarly, reducing the sets that already satisfy (1), we have also

$$(1')\qquad \text{if}\quad d_\varrho(A)\cap B\neq\emptyset\quad\text{then}\quad |B|=1.$$

2° According to Ch2, if $A\ \varrho\ B$ then either $A\ \varrho\ B\setminus A$ or $A\setminus B\ \varrho\ B$ or $A\cap B\ \varrho$ $\varrho\ A\cap B$. Hence we may also assume that

$$(2)\qquad \text{either}\quad A\cap B=\emptyset\quad\text{or}\quad A=B;$$

$$(3)\qquad \text{if}\quad A=B,\quad A_0, B_0\subset A,\quad A_0\ \varrho\ B_0\quad\text{then}\quad A_0\cap B_0\neq\emptyset$$

[(3) may be assumed, because if $A=B$ contains disjoint chained subsets then the first part of (2) holds with these subsets]. Observe that these further reductions of A and B do not upset (1) and (1′).

3° Consider those filters $\mathfrak{f}$ on X for which

$$A\cap S\ \varrho\ B\cap S \qquad (S\in\mathfrak{f}). \tag{4}$$

There are such filters, e.g. $\{X\}$. Using Zorn's lemma, we may take a filter $\mathfrak{f}$ maximal with respect to property (4). If $\mathfrak{f}$ satisfies (4) then so does $\mathrm{fil}_X(\mathfrak{f}|A\cup B)$, so the maximality of $\mathfrak{f}$ implies that $A\cup B\in\mathfrak{f}$.

4° $\mathfrak{f}|A$ is an ultrafilter. This is a consequence of the following statements, which we shall prove in 5° and 6°:

$$\text{if}\quad A=A_1\cup A_2\quad\text{then there is an } i \text{ such that}\quad A_i\cap S\ \varrho\ B\cap S\quad (S\in\mathfrak{f}); \tag{5}$$

$$\text{if}\quad E\subset A\quad\text{and}\quad E\cap S\ \varrho\ B\cap S\quad (S\in\mathfrak{f})\quad\text{then}\quad E\in\mathfrak{f}|A. \tag{6}$$

For reasons of symmetry, $\mathfrak{f}|B$ is an ultrafilter, too.

5° If (5) were not true then there would be sets $S_i\in\mathfrak{f}$ such that $A_i\cap S_i\ \bar{\varrho}\ B\cap S_i$ $(i=1,2)$. Now Ch2 implies $A\cap S_1\cap S_2\ \bar{\varrho}\ B\cap S_1\cap S_2$, a contradiction.

6° a) Assume first that $A\cap B=\emptyset$. If the premiss of (6) holds then $\mathrm{fil}_X(\mathfrak{f}|E\cup B)$ satisfies (4), so $E\cup B\in\mathfrak{f}$ by the maximality of $\mathfrak{f}$, and therefore $E\in\mathfrak{f}|A$.

b) On the other hand, if $A=B$ and (3) is fulfilled then, according to Ch2, $E\cap S\ \varrho\ A\cap S$ implies $E\cap S\ \varrho\ E\cap S$, since from (3) we have $E\cap S\ \bar{\varrho}\ (A\setminus E)\cap S$. Hence $\mathrm{fil}_X(\mathfrak{f}|E)$ satisfies (4), and so $E\in\mathfrak{f}$, again by the maximality of $\mathfrak{f}$.

7° $A, B\in\sec\mathfrak{f}$ is clear from (4). $\mathfrak{f}|A\cup B$ cannot be a fixed ultrafilter; this is evident if A and B are disjoint, and follows from (4) and Ch3 if $A=B$. Therefore $(A, B)\in\mathrm{Sec}\,\mathfrak{f}$.

8° It has to be proved that if $(C, D)\in\mathrm{Sec}\,\mathfrak{f}$ then $C\ \varrho\ D$. As $A\cup B\in\mathfrak{f}$, we have $(C\cap(A\cup B), D\cap(A\cup B))\in\mathrm{Sec}\,\mathfrak{f}$, thus it can be assumed that $C\cup D\subset A\cup B$. By Lemma 1.2 (applied twice), one of the pairs $(A\cap C, A\cap D)$, $(B\cap C, B\cup D)$, $(A\cap C, B\cap D)$ and $(B\cap C, A\cap D)$ belongs to $\mathrm{Sec}\,\mathfrak{f}$, so we may assume (for reasons of symmetry) that $C\subset A$ and either $D\subset A$ or $D\subset B$.

a) If $D\subset A$ then $(C, D)\in\mathrm{Sec}\,\mathfrak{f}$ implies that $\mathfrak{f}|A$ is not a fixed ultrafilter, so $|A|\neq 1$, and

$$(C\cap D)\cap d_\varrho(B)\subset A\cap d_\varrho(B)=\emptyset \tag{7}$$

by (1). As $\mathfrak{f}|A$ is an ultrafilter, $C, D\in\sec\mathfrak{f}$, i.e. $C, D\in\sec_A(\mathfrak{f}|A)$ means $C, D\in\mathfrak{f}|A$, so $C\cap D\in\mathfrak{f}|A$, and then (4) implies $C\cap D\ \varrho\ B$. ϱ is saturated, thus form (7) we have $C\cap D\ \varrho\ C\cap D$, therefore $C\ \varrho\ D$.

b) If $D\subset B$ then only the case $A\cap B=\emptyset$ has to be considered, since otherwise $B=A$, and so $D\subset A$. Just like above, $C\in\mathfrak{f}|A$ and $D\in\mathfrak{f}|B$, so there is an $S\in\mathfrak{f}$ such that $S\cap A\subset C$ and $S\cap B\subset D$, hence $C\ \varrho\ D$ follows from (4). □

EXAMPLE. Let X be an infinite set, and $z\in X$ fixed. Consider the accumulation d on X for which $d(A)=\{z\}$ whenever A is infinite, and put $\varrho=\varrho_d^*$. ϱ satisfies ChS', for $\varrho^*_{d_\varrho}=\varrho^*_d=\varrho$. If $A=X\setminus\{z\}$ and $B=\{z\}$ then $A\,\varrho\,B$, but there is no ϱ-compressed filter $\mathfrak{f}$ with $(A, B)\in\operatorname{Sec}\mathfrak{f}$, i.e. the statement of the lemma is not valid for this ϱ.

Assume indirectly that there is such an $\mathfrak{f}$. Then $\mathfrak{f}|A$ is a free filter, since $\mathfrak{f}$ is fixed at z, and a ϱ-compressed filter cannot be fixed at more than one point. Therefore $(A, A)\in\operatorname{Sec}\mathfrak{f}$, implying $A\,\varrho\,A$, a contradiction.

1.4 LEMMA. *If ϱ is a chaining then every ϱ-compressed filter contains a minimal ϱ-compressed filter.*

PROOF. Zorn's lemma can be applied if we show that $\mathfrak{f}=\bigcap_{\alpha\in I}\mathfrak{f}_\alpha$ is ϱ-compressed whenever $\{\mathfrak{f}_\alpha : \alpha\in I\}$ is a non-empty family of ϱ-compressed filters, linearly ordered by inclusion.

Assume that $(A, B)\in\operatorname{Sec}\mathfrak{f}$. Then $\mathfrak{f}|A\cup B$ is not a fixed ultrafilter, so there is an $\alpha_1\in I$ such that $\mathfrak{f}_{\alpha_1}|A\cup B$ is not a fixed ultrafilter either (otherwise $\mathfrak{f}_\alpha|A\cup B=\dot{x}$ would hold with x independent of α, implying $\mathfrak{f}|A\cup B=\dot{x}$). There is an $\alpha_2\in I$ with $A\in\operatorname{sec}\mathfrak{f}_{\alpha_2}$ (otherwise there would be sets $S_\alpha\in\mathfrak{f}_\alpha$ such that $A\cap S_\alpha=\emptyset$; then $S=\bigcup_{\alpha\in I}S_\alpha\in\mathfrak{f}$ and $S\cap A=\emptyset$, contradicting that $A\in\operatorname{sec}\mathfrak{f}$). Similarly, there is an $\alpha_3\in I$ with $B\in\operatorname{sec}\mathfrak{f}_{\alpha_3}$. Now if $\mathfrak{f}_\beta$ is the coarsest one of these three filters then $(A, B)\in\operatorname{Sec}\mathfrak{f}_\beta$, thus $A\,\varrho\,B$. □

1.5 In an accumulation space (X, d), the *d-neighbourhood* filter $\mathfrak{n}(x)$ of the point $x\in X$ (the same as the c_d-neighbourhood filter in the usual sense) is defined as follows:

$$A\in\mathfrak{n}(x) \quad\text{iff}\quad x\in A \quad\text{and}\quad x\notin d(X\setminus A).$$

$\mathfrak{n}(x)$ is fixed at x, and (by Ch3) only at x. Conversely, if a filter $\mathfrak{n}(x)$ fixed at x only is assigned to each point $x\in X$ then there is exactly one accumulation d for which these are the neighbourhood filters:

$$x\in d(A) \quad\text{iff}\quad A\setminus\{x\}\in\operatorname{sec}\mathfrak{n}(x) \quad\text{iff}\quad (\{x\}, A)\in\operatorname{Sec}\mathfrak{n}(x).$$

$A\in\operatorname{sec}\mathfrak{n}(x)$ iff either $x\in A$ or $x\in d(A)$.

LEMMA. $(A, B)\in\operatorname{Sec}\mathfrak{n}(x)$ *iff either* $x\in d(A)\cap d(B)$ *or* $x\in A\cap d(B)$, *or* $x\in d(A)\cap B$.

PROOF. If $(A, B)\in\operatorname{Sec}\mathfrak{n}(x)$ then, according to the foregoing observation, either one of the three possibilities in the lemma holds, or $x\in A\cap B$. In the last case, $(A\cup B)\setminus\{x\}\in\operatorname{sec}\mathfrak{n}(x)$, because $\mathfrak{n}(x)|A\cup B\neq\dot{x}$. Consequently, either $A\setminus\{x\}\in\operatorname{sec}\mathfrak{n}(x)$ or $B\setminus\{x\}\in\operatorname{sec}\mathfrak{n}(x)$, and so $x\in d(A)\cap B$ or $x\in d(B)\cap A$.

Conversely, if $x\in A\cap d(B)$ then from $(\{x\}, B)\in\operatorname{Sec}\mathfrak{n}(x)$ we have $(A, B)\in\operatorname{Sec}\mathfrak{n}(x)$. If $x\in d(A)\cap d(B)$ then $A\setminus\{x\}$, $B\setminus\{x\}\in\operatorname{sec}\mathfrak{n}(x)$, and $\mathfrak{n}(x)|(A\setminus\{x\})\cup(B\setminus\{x\})$ is not a fixed ultrafilter, because $\mathfrak{n}(x)$ is not fixed at any point different from x. Hence $(A, B)\in\operatorname{Sec}\mathfrak{n}(x)$ again. □

1.6 LEMMA. a) *If d is an accumulation on X then* $\varrho_d=\bigcup_{x\in X}\operatorname{Sec}\mathfrak{n}(x)$.

b) *A chaining ϱ is Riesz iff the d_ϱ-neighbourhood filters are ϱ-compressed.*

Proof. Lemma 1.5. □

1.7 Lemma. a) *If ϱ is a chaining and $\mathfrak{f}$ is a ϱ-compressed filter fixed at x then $\mathfrak{n}(x)\subset\mathfrak{f}$ [where $\mathfrak{n}(x)$ is the d_ϱ-neighbourhood filter of x].*

b) *If ϱ is a Riesz chaining then the fixed minimal ϱ-compressed filters are the same as the d_ϱ-neighbourhood filters.*

Proof. a) If $N\in\mathfrak{n}(x)$ then $x\notin d_\varrho(X\setminus N)$, i.e. $\{x\}\,\bar{\varrho}\,X\setminus N$, so $(\{x\}, X\setminus N)\notin \operatorname{Sec}\mathfrak{f}$, which implies that $X\setminus N\notin \sec\mathfrak{f}$ (since $\{x\}\in\sec\mathfrak{f}$ and $x\notin X\setminus N$), therefore $N\in\mathfrak{f}$.

b) Lemma 1.6 b). □

Theorem. *Let (X, ϱ) be a chaining space. Then there is an accumulation space (Y, d) such that $X\subset Y$ and $\varrho=\varrho_d|X$ iff ϱ is saturated and Riesz.*

Proof. Necessity: Lemma 1.1.

Sufficiency: Denote by $\mathfrak{n}_X(x)$ the d_ϱ-neighbourhood filter of x. Take an $Y\supset X$ and a bijection $\mathfrak{f}$ from Y onto the system of all the minimal ϱ-compressed filters such that $\mathfrak{f}(x)=\mathfrak{n}_X(x)$ $(x\in X)$; by Lemma 1.7 there are such Y and $\mathfrak{f}$, and for each $p\in Y\setminus X$, $\mathfrak{f}(p)$ is a free filter. Assign to each $a\in Y$ the neighbourhood filter

$$\mathfrak{n}(a)=\{S\subset Y\colon a\in S,\ S\cap X\in\mathfrak{f}(a)\}.$$

$\mathfrak{n}(a)$ is fixed at a and only at a, so we have defined in this way an accumulation d on Y. We claim that $\varrho=\varrho_d|X$.

If $A\,\varrho\,B$ then, by Lemmas 1.3 and 1.4, there is an $a\in Y$ such that $(A, B)\in\operatorname{Sec}_X\mathfrak{f}(a)$, and so $(A, B)\in\operatorname{Sec}_Y\mathfrak{n}(a)$, thus $A\,\varrho_d\,B$ by Lemma 1.6 a), i.e. $A\,(\varrho_d|X)\,B$.

Conversely, if $A\,\varrho_d\,B$ and $A, B\subset X$ then by Lemma 1.6 a) there is an $a\in Y$ with $(A, B)\in\operatorname{Sec}_Y\mathfrak{n}(a)$, i.e. $(A, B)\in\operatorname{Sec}_X\mathfrak{f}(a)$. $\mathfrak{f}(a)$ is ϱ-compressed, so $A\,\varrho\,B$. □

Remarks. a) X is dense in the accumulation space (Y, d) defined above (i.e. for each $a\in Y$, $\mathfrak{n}(a)|X$ is a proper filter).

b) We have much freedom in defining the neighbourhood filters $\mathfrak{n}(a)$ (cf. [6] § 2). What we have chosen is (in case of open filters in a topological space) the loose (or simple) extension associated with the given trace filter system.

c) In general, there is no compact accumulation space (Y, d) satisfying the conditions of the theorem. (d is *compact* [2] if any system $N_a\in\mathfrak{n}(a)$ $(a\in Y)$ contains a finite covering of Y.) Example: take the discrete chaining $\emptyset$ on an infinite set.

1.8 The following notion will enable us to add a supplement to Theorem 1.7: a chaining ϱ is *Lodato* (the name will be justified by Proposition 2.7) if

ChL $$A\,\varrho\,d_\varrho(B)\quad\text{implies}\quad A\,\varrho\,B.$$

Lemma. *For a chaining ϱ, each of the following conditions is equivalent to ChL:*

(1) $$A\,\varrho\,c_\varrho(B)\quad\text{implies}\quad A\,\varrho\,B;$$

(2) $$c_\varrho(A)\,\varrho\,c_\varrho(B)\quad\text{implies}\quad A\,\varrho\,B.$$

Proof. ChL⇒(1): If $A\,\varrho\,c_\varrho(B)$ then by Ch2 either $A\,\varrho\,B$ or $A\,\varrho\,d_\varrho(B)$; in the second case, $A\,\varrho\,B$ follows from ChL.

(1)⇒(2): Apply (1) twice.

(2)⇒ChL: If $A\,\varrho\,d_\varrho(B)$ then $c_\varrho(A)\,\varrho\,c_\varrho(B)$ by Ch1. □

PROPOSITION. *Each Lodato chaining is Riesz.*

PROOF. If there is an $x\in d_\varrho(A)\cap d_\varrho(B)$ then $\{x\}\ \varrho\ B$, $\{x\}\subset c_\varrho(A)$, $B\subset c_\varrho(B)$, so we have $A\ \varrho\ B$ from the lemma above. □

An accumulation d on X will be called *topological* if $d(d(A))\subset d(A)$ $(A\subset X)$; this is equivalent to the statement that c_d is topological in the usual sense, i.e. that $c_d(c_d(A))=c_d(A)$.

THEOREM. *Let (X, ϱ) be a chaining space. Then there is a topological accumulation space (Y, d) such that $X\subset Y$ and $\varrho=\varrho_d|X$ iff ϱ is saturated and Lodato.*

PROOF. Necessity: ϱ is saturated by Lemma 1.1. To prove that ϱ is Lodato, assume $A\ \varrho\ d_\varrho(B)$. This means that $A\ \varrho_d\ d(B)\cap X$, therefore $A\ \varrho_d\ d(B)$. According to the definition of ϱ_d, one of the following sets is non-empty: $A\cap d(d(B))$, $d(A)\cap d(B)$, $d(A)\cap d(d(B))$. Now $A\ \varrho_d\ B$ follows from the assumption $d(d(B))\subset d(B)$; thus $A\ \varrho\ B$.

Sufficiency: If ϱ is Lodato then d_ϱ is topological, since $x\in d_\varrho(d_\varrho(A))$ means $\{x\}\ \varrho\ d_\varrho(A)$, which implies $\{x\}\ \varrho\ A$, i.e. $x\in d_\varrho(A)$. By the preceding proposition, ϱ is Riesz, so ϱ satisfies the conditions in Theorem 1.7. It is enough to show now that the free minimal ϱ-compressed filters are open, since then the construction in the proof of Theorem 1.7 yields the neighbourhood filters for a topology (cf. Remark 1.7 b)).

Let $\mathfrak{f}$ be a free ϱ-compressed filter. We shall see that the filter $\mathfrak{g}$ generated by the open elements of $\mathfrak{f}$ is ϱ-compressed, too; consequently, if $\mathfrak{f}$ is minimal then it is open.

Assume $A\ \bar\varrho\ B$; we have to prove that $(A, B)\notin \operatorname{Sec}\mathfrak{g}$. Since ϱ is Lodato, $c_\varrho(A)\ \bar\varrho\ c_\varrho(B)$. $\mathfrak{f}$ is ϱ-compressed, so $(c_\varrho(A), c_\varrho(B))\notin \operatorname{Sec}\mathfrak{f}$. $\mathfrak{f}$ is free, so this means that either $c_\varrho(A)\notin \sec\mathfrak{f}$ or $c_\varrho(B)\notin\sec\mathfrak{f}$. A closed set belongs to $\sec\mathfrak{f}$ iff it belongs to $\sec\mathfrak{g}$, so $c_\varrho(A)\notin\sec\mathfrak{g}$ or $c_\varrho(B)\notin\sec\mathfrak{g}$; hence $(A, B)\notin\operatorname{Sec}\mathfrak{g}$. □

§ 2. The connexion between chainings and proximities

2.1 A relation $\delta\subset\exp X\times\exp X$ is a *separated proximity* on X if

P0 $\qquad 0\ \bar\delta\ X$; if $A\ \delta\ B$ then $B\ \delta\ A$;

P1 $\qquad$ if $A\ \delta\ B$ and $B\subset C\subset X$ then $A\ \delta\ C$;

P2 $\qquad$ if $A\ \delta\ B\cup C$ then $A\ \delta\ B$ or $A\ \delta\ C$;

P3 $\qquad \{x\}\ \delta\ \{y\}$ iff $x=y\in X$.

[In the definition of a *proximity* (in the sense of Čech [2]), P3 is replaced by the following weaker axiom: for any $x\in X$, $\{x\}\ \delta\ \{x\}$.]

PROPOSITION. *If ϱ is a chaining on X then*

$$A\ \delta_\varrho\ B \quad \textit{iff} \quad A\ \varrho\ B \quad \textit{or} \quad A\cap B\neq\emptyset \qquad (A, B\subset X)$$

defines a separated proximity on X. □

We shall say that ϱ *induces* δ_ϱ, or that ϱ is *compatible* with δ_ϱ. For each separated proximity, there are chainings compatible with it:

2.2 PROPOSITION. *If δ is a separated proximity on X then two chainings on X are defined by*

$$A\ \varrho_\delta^*\ B \text{ iff there are } A'\subset A \text{ and } B'\subset B \text{ such that } A'\ \delta\ B' \text{ and } A'\cap B' = \emptyset;$$

$$A\ \varrho_\delta^\circ\ B \text{ iff either } A\ \varrho_\delta^*\ B \text{ or } A\cap B \text{ is infinite.}$$

$\varrho_\delta^*\subset\varrho_\delta^\circ$, and a chaining ϱ is compatible with δ iff $\varrho_\delta^*\subset\varrho\subset\varrho_\delta^\circ$.

PROOF. 1° It is evident that ϱ_δ^* and ϱ_δ° are chainings, and $\varrho_\delta^*\subset\varrho_\delta^\circ$. Assume $\varrho_\delta^*\subset\varrho\subset\varrho_\delta^\circ$; we have to show that $\delta_\varrho=\delta$.

a) Assume $A\ \delta\ B$. If $A\cap B\neq\emptyset$ then clearly $A\ \delta_\varrho\ B$; if A and B are disjoint then $A\ \varrho_\delta^*\ B$, thus $A\ \varrho\ B$, and so $A\ \delta_\varrho\ B$.

b) Let $A\ \delta_\varrho\ B$; we may assume again that $A\cap B=\emptyset$. Now $A\ \varrho\ B$, and therefore $A\ \varrho_\delta^\circ\ B$. It follows from the definition of ϱ_δ° that $A\ \delta\ B$.

2° Assume conversely that $\delta_\varrho=\delta$; we have to show that $\varrho_\delta^*\subset\varrho\subset\varrho_\delta^\circ$.

a) If $A\ \varrho_\delta^*\ B$ then there are disjoint $A'\subset A$ and $B'\subset B$ such that $A'\ \delta\ B'$, i.e. $A'\ \delta_\varrho\ B'$; therefore $A'\ \varrho\ B'$ and $A\ \varrho\ B$.

b) If $A\ \varrho\ B$ then either $A\ \varrho\ B\setminus A$ or $A\setminus B\ \varrho\ B$ or $A\cap B\ \varrho\ A\cap B$. In the first case, $A\ \varrho\ B\setminus A$ implies $A\ \delta_\varrho\ B\setminus A$, i.e. $A\ \delta\ B\setminus A$, so $A\ \varrho_\delta^*\ B\setminus A$, and therefore $A\ \varrho_\delta^\circ\ B$. The second case is analogous. In the third case, $A\cap B$ is infinite by Ch3, thus $A\ \varrho_\delta^\circ\ B$ again. □

2.3 If δ is a (separated) proximity on X then a (separated) closure c_δ is defined by $x\in c_\delta(A)$ iff $\{x\}\ \delta\ A$. It is the closure *induced* by δ. In case δ is separated we shall write d_δ for d_{c_δ}.

PROPOSITION. For any chaining ϱ, $c_{\delta_\varrho}=c_\varrho$. □

2.4 LEMMA. *If ϱ_1 and ϱ_2 are chainings inducing the same proximity, $\varrho_1\subset\varrho_2$, and ϱ_1 is saturated then so is ϱ_2.*

PROOF. Let $A\ \varrho_2\ B$ and $A\cap d_{\varrho_2}(B)=\emptyset$. If $A\ \varrho_2\ A\cap B$ then evidently $A\ \varrho_2\ A$, so we may assume that $A\ \varrho_2\ B\setminus A$. Now $A\ \delta_{\varrho_2}\ B\setminus A$, i.e. $A\ \delta_{\varrho_1}\ B\setminus A$ and (the sets being disjoint) $A\ \varrho_1\ B\setminus A$, so $A\ \varrho_1\ B$. Moreover, $\delta_{\varrho_1}=\delta_{\varrho_2}$ implies $d_{\varrho_1}=d_{\varrho_2}$ (Proposition 2.3), thus $A\cap d_{\varrho_1}(B)=\emptyset$; hence $A\ \varrho_1\ A$ (since ϱ_1 is saturated), and so $A\ \varrho_2\ A$. □

PROPOSITION. *If δ is a separated proximity then a saturated chaining ϱ_δ compatible with δ is defined by*

$$A\ \varrho_\delta\ B \text{ iff } A\ \varrho_\delta^*\ B \text{ or there are } C\subset A\cap B \text{ and } D \text{ such that } C\ \varrho_\delta^*\ D \text{ and } C\cap d_\varrho(D) = \emptyset.$$

A chaining ϱ compatible with δ is saturated iff $\varrho_\delta\subset\varrho$. In particular, ϱ_δ° is always saturated.

PROOF. 1° ϱ_δ evidently satisfies Ch0 and Ch1. To prove Ch2, assume $A\ \varrho_\delta\ (B_1\cup B_2)$. We have two possibilities:

a) If $A\ \varrho_\delta^*\ (B_1\cup B_2)$ then $A\ \varrho_\delta^*\ B_i$ with $i=1$ or 2, implying $A\ \varrho_\delta\ B_i$.

b) If there are C and D as in the definition of ϱ_δ then taking $C_i=C\cap B_i$, $C_i\ \varrho_\delta^*\ D$ holds with $i=1$ or 2, moreover $C_i\subset A\cap B_i$ and $C_i\cap d_\delta(D)=\emptyset$. Hence the sets C_i and D show that $A\ \varrho_\delta\ B_i$ with $i=1$ or 2.

Ch3 follows from the observation that the set C in the definition of ϱ_δ has to be infinite: if $C\,\varrho_\delta^*\,D$ then there are $C'\subset C$ and $D'\subset D$ such that $C'\,\delta\,D'$, $C'\cap D'=\emptyset$ and (in consequence of $C\cap d_\varrho(D)=\emptyset$) $C'\cap d_\delta(D')=\emptyset$; C' and D' being disjoint, we have $C'\cap c_\delta(D')=\emptyset$, thus $\{x\}\,\bar\delta\,D'$ for each $x\in C'$, and then P2 would imply $C'\,\bar\delta\,D'$ if C' were finite.

2° $\varrho_\delta^*\subset\varrho_\delta$ is clear. If $A\,\varrho_\delta\,B$ then either $A\,\varrho_\delta^*\,B$ or, as we have just seen, $A\cap B$ is infinite. Thus $\varrho_\delta\subset\varrho_\delta^\circ$, and Proposition 2.2 gives that ϱ_δ is compatible with δ.

3° To prove that ϱ_δ is saturated, assume that $E\,\varrho_\delta\,F$ and $E\cap d_{\varrho_\delta}(F)=\emptyset$; we have to show that $E\,\varrho_\delta\,E$. First assume $E\,\varrho_\delta^*\,F$. ϱ_δ is compatible with δ, so $d_{\varrho_\delta}=d_\delta$, therefore $E\cap d_\delta(F)=\emptyset$. Now the conditions in the definition of ϱ_δ hold for $A=B=E$ with $C=E$ and $D=F$. On the other hand, if $E\,\overline{\varrho_\delta^*}\,F$ then the sets C and D for the pair E, F will do for the pair E, E as well.

4° If ϱ is compatible with δ, and $\varrho_\delta\subset\varrho$ then ϱ is saturated by the lemma.

Conversely assume that ϱ is a saturated chaining compatible with δ; we have to prove that $\varrho_\delta\subset\varrho$. Let $A\,\varrho_\delta\,B$. If $A\,\varrho_\delta^*\,B$ then $A\,\varrho\,B$ by Proposition 2.2; otherwise, take appropriate sets C and D. $C\,\varrho_\delta^*\,D$ implies $C\,\varrho\,D$; $C\cap d_\delta(B)=\emptyset$ means $C\cap d_\delta(B)=\emptyset$, thus $C\,\varrho\,C$ (because ϱ is saturated); hence $A\,\varrho\,B$ again. □

2.5 A proximity δ will be called *Riesz* if $c_\delta(A)\cap c_\delta(B)\neq\emptyset$ implies $A\,\delta\,B$, or equivalently if $d_\delta(A)\cap d_\delta(B)\neq\emptyset$ implies $A\,\delta\,B$. (Such proximities are known in the literature under different names, see [12, 4, 7, 6].) δ is *Lodato* (the terminology is established in this case) if one of the following equivalent conditions holds: $c_\delta(A)\,\delta\,c_\delta(B)$ implies $A\,\delta\,B$; $A\,\delta\,c_\delta(B)$ implies $A\,\delta\,B$; $A\,\delta\,d_\delta(B)$ implies $A\,\delta\,B$. If δ is Lodato then it is clearly Riesz.

PROPOSITION. a) *If ϱ is a Riesz chaining then δ_ϱ is Riesz, too.*

b) *A saturated chaining ϱ is Riesz iff δ_ϱ is Riesz.*

PROOF. a) Evident.

b) Assume that ϱ is saturated and δ_ϱ is Riesz. Let $x\in d_\varrho(A)\cap d_\varrho(B)$; we have to show that $A\,\varrho\,B$. Two cases will be considered:

If $x\in d_\varrho(B\setminus A)$ then we have $x\in d_{\delta_\varrho}(A)\cap d_{\delta_\varrho}(B\setminus A)$, and so ($\delta_\varrho$ being Riesz) $A\,\delta_\varrho\,B\setminus A$, consequently $A\,\varrho\,B\setminus A$ and $A\,\varrho\,B$.

If $x\in d_\varrho(A\cap B)$ then $A\cap B\,\varrho\,\{x\}$ and $A\cap B\cap d_\varrho(\{x\})\subset d_\varrho(\{x\})=\emptyset$; ϱ is saturated, thus $A\cap B\,\varrho\,A\cap B$, and $A\,\varrho\,B$ again. □

2.6 LEMMA. *If ϱ_1 and ϱ_2 are chainings, $\varrho_1\subset\varrho_2$, $d_{\varrho_1}=d_{\varrho_2}$ (in particular, if $\delta_{\varrho_1}=\delta_{\varrho_2}$) and ϱ_1 is Riesz then so is ϱ_2.* □

PROPOSITION. *Let δ be a separated Riesz proximity and ϱ a chaining compatible with δ. Then $\varrho_\delta^R=\varrho_\delta^*\cup\varrho_{d_\delta}$ is a chaining compatible with δ, and ϱ is Riesz iff $\varrho_\delta^R\subset\varrho$.*

PROOF. ϱ_δ^R is a chaining, since a union of chainings is again a chaining. $\varrho_{d_\delta}\subset\varrho_\delta^\circ$, because ϱ_δ° is a Riesz chaining compatible with d_δ [Propositions 2.4, 2.5 b) and 2.3], and ϱ_{d_δ} is the finest Riesz chaining with this property. Thus $\varrho_\delta^*\subset\varrho_\delta^R\subset\varrho_\delta^\circ$, and so ϱ_δ^R is compatible with δ by Proposition 2.2. It follows from the lemma that if $\varrho_\delta^R\subset\varrho$ and ϱ is compatible with δ then ϱ is Riesz. Conversely if ϱ is a Riesz chaining compatible with δ then ϱ is compatible with d_δ, thus $\varrho_{d_\delta}\subset\varrho$; on the other hand, $\varrho_\delta^*\subset\varrho$ by Proposition 2.2, therefore $\varrho_\delta^R\subset\varrho$. □

Remark. According to Proposition 2.5 b), if δ is a separated Riesz proximity then $\varrho_\delta^R\subset\varrho_\delta$.

2.7 Proposition. a) *A chaining ϱ is Lodato iff it is Riesz and δ_ϱ is Lodato.*
b) *A saturated chaining ϱ is Lodato iff δ_ϱ is Lodato.*

Proof. a) Necessity: By Proposition 1.8, ϱ is Riesz. To prove that δ_ϱ is Lodato, assume that $A\,\delta_\varrho\,d_{\delta_\varrho}(B)$. Then $A\,\delta_\varrho\,d_\varrho(B)$. If $A\cap d_\varrho(B)=\emptyset$ then $A\,\varrho\,d_\varrho(B)$, so $A\,\varrho\,B$ and $A\,\delta\,B$. Otherwise take an $x\in A\cap d_\varrho(B)$; now $\{x\}\,\varrho\,B$, therefore $A\,\delta\,B$ again.

Sufficiency: Assume that ϱ is Riesz and δ_ϱ is Lodato. If $A\,\varrho\,d_\varrho(B)$ then, by A2 and Ch2, either $A\,\varrho\,d_\varrho(B\cap A)$ or $A\,\varrho\,d_\varrho(B\setminus A)$. In the first case $d_\varrho(B\cap A)\neq\emptyset$, so we have $B\cap A\,\varrho\,B\cap A$ from the Riesz property, thus $A\,\varrho\,B$. In the second case $A\,\delta_\varrho\,d_{\delta_\varrho}(B\setminus A)$, so (since δ_ϱ is Lodato) $A\,\delta_\varrho\,B\setminus A$; as the two sets are disjoint, this implies $A\,\varrho\,B\setminus A$, thus $A\,\varrho\,B$ again. Hence ϱ is Lodato.

b) Necessity: Part a) above.

Sufficiency: If δ_ϱ is Lodato then it is Riesz, so ϱ is Riesz by Proposition 2.5 b), and Part a) can be applied again. □

Remark. If δ is a separated Lodato proximity then ϱ_δ^R is the finest Lodato chaining compatible with δ.

§ 3. A structure theorem for chainings

3.1 The aim of this section is to describe all the chainings compatible with a given separated proximity more precisely than we did in Proposition 2.2.

Recall [3, 12] that a *grill* $\mathfrak{g}$ on X is a family $\mathfrak{g}\subset\exp X\setminus\{\emptyset\}$ such that $A'\supset A\in\mathfrak{g}$ implies $A'\in\mathfrak{g}$ and $A\cup B\in\mathfrak{g}$ implies $A\in\mathfrak{g}$ or $B\in\mathfrak{g}$. $\mathfrak{g}$ is a grill iff $\sec\mathfrak{g}$ is a filter; $\mathfrak{f}$ is a filter iff $\sec\mathfrak{f}$ is a grill; if $\mathfrak{a}$ is a grill or a filter then $\sec\sec\mathfrak{a}=\mathfrak{a}$, so there is a one-to-one correspondence between grills and filters. The grill $\mathfrak{g}$ and the filter $\sec\mathfrak{g}$ will be said to be *associated.* The empty grill is associated with the zero filter. $\mathfrak{g}$ is a grill iff it is the union of a collection of ultrafilters (in general, there may be different ways of writing the same grill as a union of ultrafilters); the associated filter is the intersection of the same ultrafilters (with an appropriate convention for the intersection of an empty collection). A grill $\mathfrak{g}$ is called *free* if it satisfies one of the following equivalent conditions: (i) $\mathfrak{g}$ is the union of free ultrafilters; (ii) each element of $\mathfrak{g}$ is infinite; (iii) $\sec\mathfrak{g}$ is a free filter.

3.2 Let δ be a separated proximity on X. We shall call a set $A\subset X$ *δ-large* if there are disjoint sets $B, C\subset A$ such that $B\,\delta\,C$. (A δ-large set is infinite.) $A\subset X$ is *δ-small* if it is not δ-large.

Take all those grills that consist of δ-large sets; this collection is non-empty, because the empty grill belongs to it. The union of these grills is again a grill consisting of δ-large sets, so there exists a largest grill consisting of δ-large sets; it will be denoted by $\mathfrak{g}(\delta)$. In other words $\mathfrak{g}(\delta)$ is the union of all those (free) ultrafilters whose elements are

δ-large. $\mathfrak{g}(\delta)$ may be different from the system of all the δ-large sets, since the latter is not always a grill.

Let ϱ be a chaining on X. $A\subset X$ is δ_ϱ-large iff there are disjoint sets $B, C\subset A$ such that $B\ \varrho\ C$. Let $\mathfrak{g}(\varrho)$ denote the system of all those subsets of X which contain a δ_ϱ-small subset B such that $B\ \varrho\ B$. $\mathfrak{g}(\varrho)$ is a grill; it is the union of all those (free) ultrafilters $\mathfrak{u}$ which contain at least one δ_ϱ-small set (equivalently: which have a base consisting of δ_ϱ-small sets), and for which $A\ \varrho\ A$ whenever $A\in\mathfrak{u}$. Observe that $\mathfrak{g}(\varrho)\cap\mathfrak{g}(\delta_\varrho)$ does not contain any ultrafilter.

LEMMA. *Let ϱ be a chaining on X, and $A\subset X$. Then $A\in\mathfrak{g}(\varrho)\cup\mathfrak{g}(\delta_\varrho)$ iff A belongs to some (free) ultrafilter $\mathfrak{u}$ for which $B\ \varrho\ B$ whenever $B\in\mathfrak{u}$.* □

THEOREM. *If δ is a separated proximity and $\mathfrak{g}$ a free grill on X then a chaining $\varrho(\delta,\mathfrak{g})$ compatible with δ is defined by*

$$A\ \varrho(\delta,\mathfrak{g})B \quad \textit{iff} \quad A\ \varrho_\delta^*\ B \quad \textit{or} \quad A\cap B\in\mathfrak{g} \qquad (A, B\subset X).$$

Conversely, if ϱ is a chaining compatible with δ then there are grills $\mathfrak{g}$ such that

$$\varrho=\varrho(\delta,\mathfrak{g}). \tag{1}$$

A grill $\mathfrak{g}$ satisfies (1) *iff* $\mathfrak{g}(\varrho)\subset\mathfrak{g}\subset\mathfrak{g}(\varrho)\cup\mathfrak{g}(\delta)$.

PROOF. 1° $\varrho(\delta,\mathfrak{g})$ clearly satisfies Ch0, Ch1 and Ch3. Ch2 is also easy to check. It is compatible with δ, since $\varrho_\delta^*\subset\varrho(\delta,\mathfrak{g})\subset\varrho_\delta^\circ$, where the latter inclusion holds because the elements of $\mathfrak{g}$ are infinite.

2° $\varrho\subset\varrho(\delta,\mathfrak{g})$ *iff* $\mathfrak{g}(\varrho)\subset\mathfrak{g}$.

Indeed, assume first that $\varrho\subset\varrho(\delta,\mathfrak{g})$, and take an $A\in\mathfrak{g}(\varrho)$. Then there is a δ-small set $B\subset A$ with $B\ \varrho\ B$. We have now $B\ \varrho(\delta,\mathfrak{g})\ B$ and $B\ \overline{\varrho_\delta^*}\ B$ (since B is δ-small), thus $B\in\mathfrak{g}$, and therefore $A\in\mathfrak{g}$.

Conversely, assume that $\mathfrak{g}(\varrho)\subset\mathfrak{g}$ and $A\ \varrho\ B$. If $A\ \varrho_\delta^*\ B$ then clearly $A\ \varrho(\delta,\mathfrak{g})\ B$. Otherwise $A\cap B$ is a δ-small set such that $A\cap B\ \varrho\ A\cap B$. This means that $A\cap B\in\mathfrak{g}(\varrho)$, thus $A\cap B\in\mathfrak{g}$, and therefore $A\ \varrho(\delta,\mathfrak{g})\ B$.

3° $\varrho(\delta,\mathfrak{g})\subset\varrho$ *iff* $\mathfrak{g}\subset\mathfrak{g}(\varrho)\cup\mathfrak{g}(\delta)$.

Indeed, assume first that $\varrho(\delta,\mathfrak{g})\subset\varrho$, and take an $A\in\mathfrak{g}$. Then $A\in\mathfrak{u}$ with some ultrafilter $\mathfrak{u}\subset\mathfrak{g}$. Now $B\ \varrho(\delta,\mathfrak{g})\ B$ for each $B\in\mathfrak{u}$, thus $B\ \varrho\ B$ $(B\in\mathfrak{u})$. Hence $A\in\mathfrak{g}(\varrho)\cup\mathfrak{g}(\delta)$ by the lemma.

Conversely, assume that $\mathfrak{g}\subset\mathfrak{g}(\varrho)\cup\mathfrak{g}(\delta)$, and $A\ \varrho(\delta,\mathfrak{g})\ B$. If $A\ \varrho_\delta^*\ B$ then $A\ \varrho\ B$, because ϱ is compatible with δ. Otherwise $A\cap B\in\mathfrak{g}\subset\mathfrak{g}(\varrho)\cup\mathfrak{g}(\delta)$, thus $A\cap B\ \varrho\ A\cap B$ by the lemma. Hence $A\ \varrho\ B$. □

REMARKS. a) Let δ be a separated proximity. Then $\varrho\mapsto\mathfrak{g}(\varrho)$ is a bijection from the chainings compatible with δ onto the grills smaller than $\mathfrak{g}(\varrho_\delta^\circ)$, and $\varrho\mapsto\mathfrak{g}(\varrho)\cup\mathfrak{g}(\delta)$ is a bijection from the chainings compatible with δ onto the free grills larger than $\mathfrak{g}(\delta)$.

b) If $\delta_1\subset\delta_2$ and $\mathfrak{g}_1\subset\mathfrak{g}_2$ then $\varrho(\delta_1,\mathfrak{g}_1)\subset\varrho(\delta_2,\mathfrak{g}_2)$. Conversely, if $\varrho_1\subset\varrho_2$ then $\delta_{\varrho_1}\subset\delta_{\varrho_2}$ and $\mathfrak{g}(\varrho_1)\cup\mathfrak{g}(\delta_{\varrho_1})\subset\mathfrak{g}(\varrho_2)\cup\mathfrak{g}(\delta_{\varrho_2})$; if $\varrho_1\subset\varrho_2$ and $\delta_{\varrho_1}=\delta_{\varrho_2}$ then $\mathfrak{g}(\varrho_1)\subset\mathfrak{g}(\varrho_2)$ (it is not enough to assume that $d_{\varrho_1}=d_{\varrho_2}$).

§ 4. Miscellaneous

4.1 Continuity. Let (X, ϱ) and (Y, ϱ') be chaining spaces, and $f\colon X \to Y$. We would like to define the (ϱ, ϱ')-continuity of f, and introduce in this way the category CHAIN of chainings. The most obvious definition (call f (ϱ, ϱ')-continuous if $A \varrho B$ implies $f[A]\, \varrho'\, f[B]$) is not satisfactory, since in this case the constant functions would not be continuous. Instead, we choose the following definition: f is (ϱ, ϱ')-*continuous* if $A \varrho B$ implies that either $f[A]\, \varrho'\, f[B]$ or there is an infinite set $C \subset A \cup B$ such that $C \cap A \neq \emptyset \neq C \cap B$ and f is constant on C. Thus we obtain a category CHAIN; our definition tallies with the notions "finer" and "restriction" introduced earlier for chainings. Moreover, the category **SPROX** of separated proximities can be regarded as a full subcategory of CHAIN; for this purpose, identify δ with ϱ°_δ.

(We only prove that (δ, δ')-continuity implies (ϱ, ϱ')-continuity, where $\varrho = \varrho^\circ_\delta$ and $\varrho' = \varrho^\circ_{\delta'}$. Let f be (δ, δ')-continuous and $A \varrho B$; it has to be shown that either $f[A]\, \varrho'\, f[B]$ or there is a set C as above. If $D = f[A] \cap f[B]$ is infinite then $f[A]\, \varrho'\, f[B]$, so we may assume that D is finite. It is enough to consider two cases: $A = B$ or $A \cap B = \emptyset$. A is infinite in the first case, so there is an $x \in D$ such that $C = f^{-1}[\{x\}] \cap A$ is infinite, and then this set C has the required properties. In the second case, let $E = f^{-}[D]$; now one of the following relations holds: $A\, \delta\, B \setminus E$, $A \setminus E\, \delta\, B$, $A \cap E\, \delta\, B \cap E$. If $A\, \delta\, B \setminus E$ then the (δ, δ')-continuity of f implies that $f[A]\, \delta'\, f[B \setminus E]$; these sets are disjoint, so $f[A]\, \varrho'\, f[B \setminus E]$, and therefore $f[A]\, \varrho'\, f[B]$. The case $A \setminus E\, \delta\, B$ is analogous. If $x, y \in E$, $x \neq y$ then from $\{x\}\, \overline{\delta'}\, \{y\}$ we have $A \cap f^{-1}[\{x\}]\, \overline{\delta}\, B \cap f^{-1}[\{y\}]$, so $A \cap E\, \delta\, B \cap E$ implies that there is an $x \in E$ with $A \cap f^{-1}[\{x\}]\, \delta\, B \cap f^{-1}[\{x\}]$; now $C = f^{-1}[\{x\}] \cap (A \cup B)$ will do.)

CHAIN is not a topological category. (This is no wonder, since SPROX is not topological either.) To obtain a topological supercategory of CHAIN, Ch3 has to be replaced by $\{x\}\, \bar{\varrho}\, \{x\}$ $(x \in X)$.

4.2 Chainings and screens. (Consult [4, 5] for terminology, or skip to 4.3.) Let us call a screen $\mathfrak{S}$ *separated* if no $\mathfrak{s} \in \mathfrak{S}$ is fixed at more than one point. $\mathfrak{S}$ is separated iff $\delta(\mathfrak{S})$ is separated. If $\mathfrak{S}$ is a separated screen then $\varrho(\mathfrak{S}) = \bigcup_{\mathfrak{s} \in \mathfrak{S}} \operatorname{Sec} \mathfrak{s}$ is a saturated chaining. Conversely, if ϱ is a saturated chaining then ϱ can be induced by screens, e.g. $\varrho = \varrho(\mathfrak{S})$ where $\mathfrak{S}$ consists of all the minimal ϱ-compressed filters. For each separated screen $\mathfrak{S}$, $\delta_{\varrho(\mathfrak{S})} = \delta(\mathfrak{S})$. A saturated chaining is Riesz iff it can be induced by a unipunctual screen.

4.3 Extending a chaining. Let (X, d) be an accumulation space, $X \subset Y$, and ϱ a chaining compatible with $d|X$. Can ϱ be extended to a chaining compatible with d? The answer is yes, similarly to the case of proximities and closures [6]. An extension can be obtained in two steps: δ_ϱ can be extended to a proximity compatible with d [6], and then ϱ can be extended to a chaining compatible with this proximity. In fact, much more is true:

THEOREM. a) *If (Y, δ) is a separated proximity space, (X_i, ϱ_i) $(i \in I)$ is a family of chaining spaces such that $X_i \subset Y$, ϱ_i is compatible with $\delta|X_i$ $(i \in I)$, and $\varrho_i|X_i \cap X_j = \varrho_j|X_i \cap X_j$ $(i, j \in I)$ then there exists a chaining ϱ compatible with δ such that $\varrho|X_i = \varrho_i$ $(i \in I)$.*

b) (Generalizes [6] 5.1.) *If* (Y, c) *is a symmetrical closure space*[1], (X_i, δ_i) $(i\in I)$ *is a family of proximity spaces such that* $X_i\subset Y$, δ_i *is compatible with* $c|X_i$ $(i\in I)$, *and* $\delta_i|X_i\cap X_j=\delta_j|X_i\cap X_j$ $(i, j\in I)$ *then there exists a proximity* δ *compatible with* c *such that* $\delta|X_i=\delta_i$ $(i\in I)$.

Hint for the proof. a) For $A, B\subset Y$, put $A\,\varrho\,B$ iff $A\,\varrho_\delta^*\,B$ or there is an $i\in I$ such that $(A\cap X_i)\,\varrho_i\,(B\cap X_i)$.

b) A similar proof using the finest proximity δ_c^* compatible with c $(A\,\delta_c^*\,B$ iff $A\cap c(B)\neq\emptyset$ or $c(A)\cap B\neq\emptyset$, see e.g. [4]). □

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Added in proof. See [13] for a more detailed discussion of Theorem 4.3 b) and related problems.

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[1] A closure c is *symmetrical* ("semi-uniformizable" in [2]) if $x\in c(\{y\})$ implies $y\in c(\{x\})$. A closure admits a compatible proximity iff it is symmetrical. Separated closures are symmetrical.

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ON THE ZEROS OF JACOBI POLYNOMIALS

P. VÉRTESI[1]

Abstract

We prove an asymptotic formula for the zeros of Jacobi polynomials for arbitrary $\alpha, \beta \geqq -1$. Various applications are mentioned.

1. Introduction. Preliminary results

1.1 If $\alpha, \beta > -1$, the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ of degree exactly n are the orthogonal polynomials on $[-1, 1]$ with respect to the weight $w^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$, $n=0, 1, \ldots$. They satisfy

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}, \tag{1.1}$$

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x). \tag{1.2}$$

If $\alpha = -1$, $\beta \geqq -1$, let

(1.3)

$$nP_n^{(-1,\beta)}(x) = \frac{n+\beta}{2}(x-1)P_{n-1}^{(1,\beta)}(x) = \frac{n+\beta}{2}(x-1)(-1)^{n-1}P_{n-1}^{(\beta,1)}(-x), \quad \beta \geqq -1.$$

By (1.2) and (1.3) *we have defined* $P_n^{(\alpha,\beta)}(x)$ *for arbitrary* $\alpha, \beta \geqq -1$.

Let $x_{kn}^{(\alpha,\beta)} = \cos \theta_{kn}^{(\alpha,\beta)}$ be the n zeros of $P_n^{(\alpha,\beta)}(x)$, numbering such that

$$-1 \leqq x_{nn}^{(\alpha,\beta)} < x_{n-1,n}^{(\alpha,\beta)} < \ldots < x_{1n}^{(\alpha,\beta)} \leqq 1. \tag{1.4}$$

If $\alpha, \beta > -1$ they are in $(-1, 1)$ (cf. G. Szegő [7, § 2.4, 1, (4.1.1), (4.22.2) and Theorem 3.3.1]).

1.2 In many applications the location of the zeros is of fundamental importance. By [7, (8.9.1)],

$$\theta_{kn}^{(\alpha,\beta)} = \frac{k\pi + O(1)}{n}, \qquad \alpha, \beta > -1, \tag{1.5}$$

with $O(1)$ being uniformly bounded for all values of $k = 1, 2, \ldots, n$, $n = 1, 2, \ldots$. (Here and later, sometimes omitting the superfluous notation, the constant $c, c_1, \ldots$ and the symbols $O(1)$, $o(1)$ may vary with α and β.)

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Formula (1.5) can be applied, e.g. for Lagrange or Hermite—Fejér interpolation (cf. [6, 14.4 and 14.6]), however, in many cases we need sharper estimations. A typical asymptotic expansion is as follows:

$$(1.6)\quad \theta_{kn}^{(\alpha,\beta)} = \eta_{kn}^{(\alpha,\beta)} + \frac{1}{4N^2}\left[\left(\frac{1}{4}-\alpha^2\right)\cot\frac{\eta_{kn}^{(\alpha,\beta)}}{2} - \left(\frac{1}{4}-\beta^2\right)\tan\frac{\eta_{kn}^{(\alpha,\beta)}}{2}\right] + O\left(\frac{1}{n^4}\right)$$

supposing $-1/2 \leqq \alpha, \beta \leqq 1/2$ and $0<\varepsilon\leqq\theta_{kn}^{(\alpha,\beta)}\leqq\pi-\varepsilon$. Here $N=n+(\alpha+\beta+1)/2$ and $\eta_{kn}=(2k+\alpha-1/2)\pi/(2N)$ (cf. L. Gatteschi—G. Pittaluga [3, (4.2)]).

2. Results

2.1 Our first statement is

THEOREM 2.1. *Let* $\alpha, \beta \geqq -1$, $0<\varepsilon<1$ *be arbitrary fixed. Then, uniformly in* k *and* n,

$$(2.1)\qquad \theta_{kn}^{(\alpha,\beta)} = \eta_{kn}^{(\alpha,\beta)} + \varrho_{kn}^{(\alpha,\beta)} \quad \textit{with} \quad |\varrho_{kn}^{(\alpha,\beta)}| \leqq \frac{c}{kn} \quad \textit{and} \quad 1 \leqq k \leqq (1-\varepsilon)n$$

where, as before, $\eta_{kn}^{(\alpha,\beta)}=(2k+\alpha-1/2)\pi/(2N)$.

REMARKS 1. If we consider (1.6) it is easy to get that the order of $\varrho_{kn}^{(\alpha,\beta)}$ generally is the best possible.

2. By Theorem 2.1 we verify: *Let* $\alpha, \beta > -1$, $\theta_{0n}=0$, *and* $\theta_{n+1,n}=\pi$. *Then*

$$(2.2)\qquad \frac{c_1}{n} < \theta_{k+1,n}^{(\alpha,\beta)} - \theta_{kn}^{(\alpha,\beta)} < \frac{c_2}{n}, \qquad k=0,1,\dots,n, \quad n \geqq n_0.$$

Indeed, by (2.1) and (1.2), $\theta_{k+1}-\theta_k=\pi/N+O(1/(KN))$ where $K:=\min(k, n-k+1)$, whence we get (2.2) if $K_0\leqq k\leqq n-K_0$, $n\geqq n_0$. If $K<K_0$ we can apply the relation

$$(2.3)\qquad \lim_{n\to\infty} n\theta_{kn}^{(\alpha,\beta)} = j_k^{(\alpha)}, \quad k \text{ is fixed,}$$

(where $j_k^{(\alpha)}$ denote the k-th positive zeros of the Bessel function $J_\alpha(x)$) and (1.2) (cf. [7, (8.1.3)]). (2.2) was proved by G. I. Natanson [4] and G. Freud, A. Sharma [2, Lemma 2]. Our proof is different.

3. In my paper [8, 4.8] I sketched the proof of a weaker version of (2.1) (cf. (3.3)). Papers [1], [8], [9] and [10] show some other application of (2.1) (or (3.3)) where formula (1.6) could not do the job.

2.2 The previous theorem was based on a Hilb's type formula for $P_n^{(\alpha,\beta)}$ proved by G. Szegő (cf. (3.1)). However, using a finer estimation obtained by G. Szegő [6] (cf. (3.5)) we can improve (2.1). Namely

THEOREM 2.2. *By the notations of Theorem 2.1,*

$$\varrho_{kn}^{(\alpha,\beta)} = \frac{1-4\alpha^2}{8N^2\eta_{kn}^{(\alpha,\beta)}} + O(1)\left(\frac{1}{nk^3}+\frac{k}{n^3}\right), \quad 1 \leqq k \leqq (1-\varepsilon)n, \tag{2.4}$$

uniformly in k and n.

By (2.4) $\theta_k \approx \eta_k + (1-4\alpha^2)/(8N^2\eta_k)$ if $n\to\infty$, whenever $K_0 \leqq k \leqq \delta_n n$, where $\delta_n \to 0$ (cf. (1.6)).

3. Proofs

3.1 Proof of Theorem 2.1. Let $\alpha, \beta \geqq -1$. Then with $k(\theta) = \pi^{-1/2}(\sin\theta/2)^{-\alpha-1/2}(\cos\theta/2)^{-\beta-1/2}$, $\gamma = -(\alpha+1/2)\pi/2$ and $x = \cos\theta$, we will apply the following relations:

$$P_n^{(\alpha,\beta)}(\cos\theta) = \frac{k(\theta)}{\sqrt{n}}\left[\cos(N\theta+\gamma) + \frac{O(1)}{n\sin\theta}\right], \quad \frac{c}{n} \leqq \theta \leqq \pi - \frac{c}{n}, \tag{3.1}$$

$$\frac{d}{d\theta}[P_n^{(\alpha,\beta)}(\cos\theta)] = \sqrt{n}\,k(\theta)\left[-\sin(N\theta+\gamma) + \frac{O(1)}{n\sin\theta}\right], \quad \frac{c}{n} \leqq \theta \leqq \pi - \frac{c}{n}, \tag{3.2}$$

(cf. [7, (8.21.18), (8.8.1) and their proofs]).

By (1.5), it is enough to prove (2.1) for $k \geqq K_0$ and $n \geqq n_0$. Let $\varepsilon = 1/4$, say. The next formula is fundamental. We have

$$\theta_{kn}^{(\alpha,\beta)} = \eta_{kn}^{(\alpha,\beta)} + \frac{M_n(\alpha,\beta)}{N}\pi + \varrho_{kn}^{(\alpha,\beta)}, \quad K_0 \leqq k \leqq \frac{3n}{4}, \quad n \geqq n_0, \quad \alpha, \beta \geqq -1 \tag{3.3}$$

with proper integers $K_0 \geqq 1$, $n \geqq n_0$ and M_n. If $\alpha, \beta > -1$, the proof was sketched in P. Vértesi [8]. Here we give a detailed argument for any $\alpha, \beta \geqq -1$ (cf. the proof of [7, Theorem 8.9.1]). Let $\delta_k = A/k$ where $A > 0$ will be determined later. If $\varphi_k = \eta_k + \delta_k/N$ and $\psi_k = \eta_k - \delta_k/N$, then with a proper $K_0 = K_0(A)$, $\varphi_k < \psi_{k+1}$, and

$$\frac{6}{7}\frac{k}{N}\pi \leqq \varphi_k, \quad \psi_k \leqq \frac{7}{6}\frac{k}{N}\pi, \quad \text{if} \quad K_0 \leqq k \leqq \frac{3n}{4}, \quad n \geqq n_0.$$

Hence for the bracket [...] in (3.1) we get

$$\left.\begin{aligned} (-1)^k[\ldots]_{\theta=\varphi_k} &\geqq \sin\delta_k - \frac{c_1}{k} \geqq \frac{1}{2}\sin\delta_k \\ (-1)^{k+1}[\ldots]_{\theta=\psi_k} &\geqq \sin\delta_k - \frac{c_1}{k} \geqq \frac{1}{2}\sin\delta_k \end{aligned}\right\} \quad K_0 \leqq k \leqq \frac{3n}{4}, \quad n \geqq n_0, \tag{3.4}$$

if A is big enough, which we suppose. (Notice that c_1 does not depend on A.) So sign $P_n(\cos\varphi_k) = -$sign $P_n(\cos\psi_k)$ whence $P_n(\cos\theta)$ has at least one zero in (ψ_k, φ_k). Using similar argument it can be seen that $P_n(\cos\theta)$ has no zeros in the "complementary intervals" $[\varphi_{k-1}, \psi_k]$ and $[\varphi_k, \psi_{k+1}]$. Further, using the same argument for

(3.2), we get sign $d/d\theta[P_n(\cos\theta)]=(-1)^k$ if $\theta\in[\psi_k,\varphi_k]$, i.e. $P_n(\cos\theta)$ has only one zero in (ψ_k,φ_k). Denoting its index by $k-M_n$ we obtain (3.3).

Next we prove that

$$M_n^{(\alpha,\beta)}=0 \quad \text{if} \quad n\geqq n_0. \tag{3.5}$$

Indeed, by (1.5) and (2.3) for any fixed k, $\lim_{n\to\infty} N\theta_{kn}^{(\alpha,\beta)}=j_k^{(\alpha)}$, whence

$$N\theta_{kn}^{(\alpha,\beta)}+\varepsilon_{kn}^{(\alpha)}=j_k^{(\alpha)},\quad k \text{ is fixed},\quad \lim_{n\to\infty}\varepsilon_{nk}^{(\alpha)}=0. \tag{3.6}$$

Further, from McMahon's expansion

$$j_k^{(\alpha)}=N\eta_{kn}^{(\alpha,\beta)}+\chi_k^{(\alpha)} \quad \text{where} \quad \chi_k^{(\alpha)}=O(1/k) \quad \text{uniformly if} \quad k\geqq K_1 \tag{3.7}$$

(see F. W. J. Olwer [5, 7(6.03)]; notice that $N\eta_{kn}$ does not depend on n). By (3.3), (3.6) and (3.7)

$$N\eta_{kn}+M_n\pi+N\varrho_{kn}+\varepsilon_{kn}=N\eta_{kn}+\chi_k \quad \text{if} \quad k=K_2$$

where $K_2\geqq\max(K_0,K_1)$. By a proper K_2, $2|\chi_k-N\varrho_{kn}-\varepsilon_{kn}|\leqq\pi$ if $n\geqq n_0$, whence $M_n=0$, as it was stated. □

3.2 Proof of Theorem 2.2. The proof uses the estimation $\varrho_{kn}=O(1/kn)$ as an "initial value". Instead of (3.1) we apply

$$\begin{aligned}P_n^{(\alpha,\beta)}(x)=\frac{k(\theta)}{\sqrt{N}}&\left(1-\frac{\alpha\beta}{N}\sqrt{\frac{\tan\frac{\theta}{2}}{2\theta}}\right)\times\\ &\times\left\{\cos(N\theta+\gamma)\left[1+\frac{O(1)}{(n\theta)^2}\right]-\sin(N\theta+\gamma)\left[\frac{4\alpha^2-1}{8N\theta}+\frac{O(1)}{(n\theta)^3}\right]+O(1)\frac{\theta}{n}\right\},\end{aligned} \tag{3.8}$$

$c/n\leqq\theta\leqq\pi-\delta$, $x=\cos\theta$, which comes from

$$P_n^{(\alpha,\beta)}(x)=\sqrt{\frac{\pi}{2}\theta}\,k(\theta)\left(1-\frac{\alpha\beta}{N}\sqrt{\frac{\tan\frac{\theta}{2}}{2\theta}}\right)J_\alpha(N\theta)+\theta^{1/2-\alpha}O(n^{-3/2}),\quad \frac{c}{n}\leqq\theta\leqq\pi-\delta$$

(G. Szegő [6, § 15, (49)]) substituting $J_\alpha(N\theta)$ by the first two members of its asymptotic expression

$$\begin{aligned}J_\alpha(z)=\sqrt{\frac{2}{\pi z}}&\left\{\cos(z+\gamma)\left[\sum_{k=0}^{p-1}(-1)^k\frac{A_{2k}(\alpha)}{z^{2k}}+O(z^{-2p})\right]-\right.\\ &\left.-\sin(z+\gamma)\left[\sum_{k=0}^{p-1}(-1)^k\frac{A_{2k+1}(\alpha)}{z^{2k+1}}+O(z^{-2p-1})\right]\right\},\quad z\to\infty,\end{aligned} \tag{3.9}$$

where

$$A_0(\alpha)=1,\quad A_k(\alpha)=\frac{(4\alpha^2-1^2)(4\alpha^2-3^2)\dots[4\alpha^2-(2k-1)^2]}{k!\,8^k},\quad k=1,2,\dots$$

(see [6, § 2, (14), (15)], say).

Writing $\theta_k=\eta_k+\varrho_k$, $K_0\le k\le(1-\varepsilon)n$, $n\ge n_0$,

$$0=P_n(\cos\theta_k)=(-1)^k\frac{k(\theta_k)}{\sqrt{N}}\left(1-\frac{\alpha\beta}{N}\sqrt{\frac{\tan\frac{\theta_k}{2}}{2\theta_k}}\right)\times$$

$$\times\left\{\sin N\varrho_k\left[1+\frac{O(1)}{k^2}\right]+\cos N\varrho_k\left[\frac{4\alpha^2-1}{8N\theta_k}+\frac{O(1)}{k^3}\right]+O(1)\frac{k}{n^2}\right\}$$

whence $\{\ldots\}=0$. So dividing by $\cos N\varrho_k$ ($\ge 1/2$ if K_0 is big enough),

$$\tan N\varrho_k=\frac{1-4\alpha^2}{8N\theta_k}+O(1)\left(\frac{1}{k^3}+\frac{k}{n^2}\right).$$

If we use $\tan N\varrho_k=N\varrho_k+O(k^{-3})$ and $|\eta_k^{-1}-\theta_k^{-1}|=|\varrho_k/(\eta_k\theta_k)|=O(nk^{-3})$, we obtain $N\varrho_k=(1-4\alpha^2)/(8N\eta_k)+O(k^{-3}+kn^{-2})$, which gives (2.4), when $k\ge K_0$, $n\ge n_0$. If $1\le k<K_0$, $n\le n_0$, (2.4) is obvious. □

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ON AN OPERATIONAL DIFFERENTIAL EQUATION

T. FÉNYES

Introduction

In the papers [1], [2] we have discussed the algebraic differential equation

$$D^2(x)+2gD(x)+[g^2+D(g)-a]x = h \tag{1}$$

defined in the Mikusiński operator field based on the convolution product of continuous functions and on the Cauchy product of discrete functions.

In what follows D is the symbol of the well-known algebraic derivative, g, h are given locally integrable functions of [1], and discrete functions of [2], a is a given number in both papers and x is the unknown of the differential equation (1).

In this paper we shall give an operational treatment of the above differential equation, being defined in the special operator field M_D based on the number-theoretical Dirichlet product of discrete functions.

Papers [3], [4], [5], [6] contain the theory and applications of the discrete operational calculus based on the Dirichlet product. In Chapter 1 we briefly summarize the results of these papers, Chapter 2 contains the operational theory of (1). We motivate the discussion of (1) by the fact that (1) is the simplest second-order differential equation with non-constant coefficients in the sense that its invariant — if it exists — is the "simplest" operator, i.e. it is a number.

In what follows **Z** will denote the set of natural numbers.

§ 1. Discrete Mikusiński operators based on the Dirichlet product

Let $a=\{a(n)\}$, $b=\{b(n)\}$ be arbitrary complex-valued functions defined on **Z**. The symbols $a(n)$, $b(n)$ denote the values of these functions for arbitrary fixed n.

Let E denote the set of the discrete functions. If we introduce in E the following two operations

(i) $a+b := \{a(n)\}+\{b(n)\} = \{a(n)+b(n)\}$ (addition)

(ii) $ab\colon \{a(n)\}\{b(n)\} = \left\{\sum_{\nu|n} a(\nu)b\left(\frac{n}{\nu}\right)\right\}$ (multiplication),

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then E becomes a commutative ring without divisor of zero and can be extended to a quotient field.

This is called the discrete Mikusiński operator field and is denoted by M_D. The elements of M_D are called M_D-operators, or simply operators.

The definition and properties of the "discrete" Dirac function.

We define the discrete Dirac function by

$$\delta(N) = \{\delta(n, N)\},$$

where

$$\delta(n, N) = \begin{cases} 0, & \text{for} \quad n \neq N \\ 1, & \text{for} \quad n = N, \end{cases} \qquad N \in \mathbf{Z}.$$

For later purposes we enumerate some properties of the Dirac function.

PROPERTY 1. $\delta(N)\{a(n)\} = \{b(n)\}$,

(1.1)
$$b(n) = \begin{cases} a\left(\dfrac{n}{N}\right) & \text{for} \quad N|n \\ 0 & \text{otherwise.} \end{cases}$$

(1.2)
$$\delta(N_1)\delta(N_2) = \delta(N_1 N_2), \qquad N_1, N_2 \in \mathbf{Z}.$$

PROPERTY 2.

(1.3)
$$x = \frac{\{a(n)\}}{\delta(N)} \in E, \qquad N \in \mathbf{Z}$$

holds if and only if

(1.4)
$$a(n) = 0$$

for those values of n for which N is not a divisor of n. If (1.4) holds, then

(1.5)
$$x = \{a(nN)\}.$$

The field K of the complex numbers can be embedded isomorphically into the operator field M_D. The common unit element of K, E, M is the function $\delta(1)$ and we write

$$\delta(1) = 1.$$

Moreover,

$$c\delta(1) = c, \quad c\{a(n)\} = \{ca(n)\}; \quad c \in K, \quad a \in E.$$

Every operator of the form

$$x = \frac{\{a(n)\}}{\{b(n)\}}$$

is a function if $b(1) \neq 0$.

The operator function $\delta(\varepsilon)$

For arbitrary rational number $\varepsilon=\dfrac{N_1}{N_2}$ we define

$$\delta(\varepsilon)=\frac{\delta(N_1)}{\delta(N_2)}. \tag{1.6}$$

From this definition it follows that for $\varepsilon=N$ we have

$$\delta(\varepsilon)=\delta(N)=\{\delta(n,N)\}.$$

If

$$\frac{N_1}{N_2}=\frac{N_3}{N_4},$$

then

$$\delta\left(\frac{N_1}{N_2}\right)=\delta\left(\frac{N_3}{N_4}\right)$$

holds, so the definition (1.6) is correct.

PROPERTY 3. Let α, β be arbitrary positive rational numbers, then

$$\delta(\alpha)\delta(\beta)=\delta(\alpha\beta), \tag{1.7}$$

$$\delta\left(\frac{1}{\alpha}\right)=\frac{1}{\delta(\alpha)}. \tag{1.8}$$

The set of the operators $\delta(\varepsilon)$ is a subgroup of M_D.

The definition of the ring E^*

Let $E^*\subset M_D$ be the subset of M_D whose elements are of the form

$$x=\frac{\{a(n)\}}{\delta(N)},\quad N\in\mathbf{Z},\quad a\in E. \tag{1.9}$$

E^* is a ring and, by choosing $N=1$, we have

$$E\subset E^*.$$

PROPERTY 4.

$$x=\frac{\{a(n)\}}{\delta(\varepsilon)}\in E^*,\quad \varepsilon=\frac{N_1}{N_2},\quad (N_1, N_2 \text{ are relatively primes}).$$

Moreover, $x\in E$ if and only if

$$a(n)=0$$

for those values of n for which N_1 is not a divisor of n. If the condition is satisfied, we have

$$x(n)=\begin{cases} a\left(\dfrac{nN_1}{N_2}\right) & \text{for } N_2|n,\\ 0 & \text{otherwise.}\end{cases}$$

Definition of the convergence in the ring E

Let $\{a_k(n)\} \in E$, $(k=1, 2, \ldots)$ be an infinite sequence of functions. By definition

$$\lim_{k\to\infty} \{a_k(n)\} = \{a(n)\} \tag{1.10}$$

if for every fixed n

$$\lim_{k\to\infty} a_k(n) = a(n).$$

This convergence can be extended to infinite series of functions as usual. Let

$$f(z) = \sum_{k=0}^{\infty} \beta_k z^k, \qquad \beta_k \in K$$

be an arbitrary entire function of the complex variable z. Then

$$f(a) = \sum_{k=0}^{\infty} \beta_k \{a(n)\}^k, \quad a \in E, \quad a^0 = 1 \tag{1.11}$$

holds in the sense of the convergence defined above. We have

$$e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!},$$

having the property

$$e^a e^b = e^{a+b}, \quad a, b \in E,$$

moreover, if we write

$$e^a = \{e_a(n)\},$$

so

$$e_a(1) = e^{a(1)} \tag{1.12}$$

holds.

The algebraic derivation and integration

For the sake of easy reading we recapitulate some definitions and facts of the algebraic derivation and integration

$$\begin{gathered} D(a) = \{-\log n \cdot a(n)\}, \quad a \in E, \\ D\left(\frac{a}{b}\right) = \frac{bD(a) - aD(b)}{b^2}, \quad a, b \in E, \qquad \frac{a}{b} \in M_D. \end{gathered} \tag{1.13}$$

Property 5.

$$\begin{gathered} D\left[\frac{a}{\delta(\varepsilon)}\right] = \frac{\left\{-\log \frac{n}{\varepsilon} \cdot a(n)\right\}}{\delta(\varepsilon)} \in E^*, \quad a \in E, \qquad \varepsilon = \frac{N_1}{N_2}, \\ D[\delta(\varepsilon)] = -\log \varepsilon \cdot \delta(\varepsilon). \end{gathered} \tag{1.14}$$

PROPERTY 6.

$$D(e^a) = D(a)e^a. \tag{1.15}$$

If for a given $x \in M_D$ there exists a $y \in M_D$ such that

$$D(y) = x,$$

we say that x is algebraically integrable and we write

$$y = \int x.$$

PROPERTY 7. If $x \in M_D$ and

$$D(x) = 0$$

then x is an arbitrary number.

Two algebraic integrals of an operator may differ only by an arbitrary number.
The algebraic differentiation and integration is a linear operation over the field of the real (complex) numbers.

The algebraic integration

PROPERTY 8. The operator

$$x = \frac{a}{\delta(\varepsilon)}, \quad a \in E, \quad \varepsilon = \frac{N_1}{N_2} \tag{1.16}$$

is algebraically integrable in M_D if and only if

$$\text{either} \quad \varepsilon \neq N, \quad N \in \mathbf{Z},$$

$$\text{or} \quad \varepsilon = N \quad \text{and} \quad a(N) = 0$$

holds true. Every algebraic integral of (1.16) belonging to E^* is given by

$$\int \frac{a}{\delta(\varepsilon)} = \frac{\left\{-\dfrac{a(n)}{\log \dfrac{n}{\varepsilon}}\right\}}{\delta(\varepsilon)} + c, \qquad c \in K, \tag{1.17}$$

where in the case of $\varepsilon = N$ the symbol

$$\frac{a(N)}{\log \dfrac{N}{N}}$$

denotes an arbitrary real (complex) number. We shall choose this to be null.
For $\varepsilon = 1$ we have that a is integrable if and only if $a(1) = 0$, and

$$\int a = \left\{\frac{-a(n)}{\log n}\right\} + c.$$

The algebraic differential equation with constant coefficients

Let us consider the differential equation

(1.18) $D^m(x)+a_{m-1}D^{m-1}(x)+\ldots+a_0x=0, \quad m\in\mathbf{Z}, \quad a_i\in K, \quad i=0,1,\ldots,m.$

By Property 7 we can say that the elements of K are the constants of the field M_D and (1.18) is an algebraic differential equation with constant coefficients. We assume that the numbers a_i are real.

The algebraic equation

(1.19) $$u^m+a_{m-1}u^{m-1}+\ldots+a_0=0$$

is called the characteristic equation of (1.18).

PROPERTY 9. To every real root u_i of the characteristic equation (1.19) for which

$$\varrho_i=e^{-u_i}$$

is rational there corresponds the nontrivial solution $\delta(\varrho_i)$ of (1.18). The general solution in M_D is of the form

(1.20) $$x=\sum_i \gamma_i\delta(\varrho_i), \qquad \gamma_i\in K.$$

Let

$$\mu_i=\gamma_i, \quad \text{for} \quad \varrho_i\in\mathbf{Z},$$

$$\mu_i=0, \quad \text{for} \quad \varrho_i\notin\mathbf{Z}$$

so the general function solution in M_D has the form

(1.21) $$x=\sum_i \mu_i\delta(\varrho_i).$$

The general inhomogeneous second order algebraic differential equation

Let us consider the differential equation

(1.22) $$D^2(x)+p_1D(x)+p_2x=r, \qquad p_1,p_2,r\in M_D.$$

We assume that the corresponding homogeneous equation has two linearly independent solutions $x_1, x_2\in M_D$.

The operator

(1.23) $$w=x_1D(x_2)-x_2D(x_1)$$

is called the Wronski determinant. Obviously, $w\neq 0$. If $w=0$ held, then by

$$w=D\left(\frac{x_2}{x_1}\right)x_1^2$$

we would obtain

$$D\left(\frac{x_2}{x_1}\right)=0,$$

and by Property 7 we would have

$$x_2 = cx_1, \quad c \in K,$$

so x_1, x_2 would be linearly dependent.

PROPERTY 10. (1.22) has a solution $x_p \in M_D$ if and only if the algebraic integrals

$$\int \frac{x_1 r}{w}, \quad \int \frac{x_2 r}{w}$$

exist. If they exist, then a particular solution of (1.22) is of the form

$$x_p = x_2 \int \frac{x_1 r}{w} - x_1 \int \frac{x_2 r}{w},$$

the general solution is

(1.24) $$x = c_1 x_1 + c_2 x_2 + x_p, \quad c_1, c_2 \in K.$$

§ 2. On the differential equation (1)

In the sequel we shall deal with the linear second order algebraic differential equation

(2.1) $$D^2(x) + 2gD(x) + [g^2 + D(g) - a]x = h$$

where a, g, h are given. We assume that a is real and we motivate (2.1) as follows.

From the elements of the theory of the (classical) ordinary differential equations it is known that

(2.2) $$y''(t) + 2b(t)y'(t) + c(t)y(t) = 0, \qquad t \in R$$

can be reduced to the normal form

$$U''(t) + I(t)U(t) = 0$$

where

$$U(t) = y(t) \exp \int b(t)\,dt.$$

Here $I(t)$ is the invariant of (2.2):

(2.3) $$I(t) = c(t) - b^2(t) - b'(t).$$

We have the simplest normal form if $I(t)$ is the constant function. Denoting its value by $-\gamma$, we obtain

$$c(t) = b^2(t) + b'(t) - \gamma$$

and

(2.4) $$y''(t) + 2b(t)y'(t) + [b^2(t) + b'(t) - \gamma]y(t) = 0.$$

(2.4) can be regarded as the "simplest" second order differential equation with non-constant coefficients in the sense that its invariant is a constant.

Analogously, the algebraic differential equation (2.1) defined in M_D can be regarded as the "simplest" one, since the invariant of the corresponding homogeneous equation is a number. We have

$$D^2(v) - av = 0, \tag{2.5}$$

$$v = xe^{\int g}. \tag{2.6}$$

Here an interesting situation occurs. By Property 8, $\int g$ exists if and only if $g(1)=0$. So, if $g(1)\neq 0$, the substitution (2.6) looses its meaning and (2.1) has no normal form. However, instead of (2.6) we can apply the substitution

$$v = x \exp\left[\int (g - g(1))\right] \tag{2.7}$$

and an elementary calculation gives

$$D^2(v) + 2g(1)D(v) + [g(1)^2 - a]v = 0, \tag{2.8}$$

which is an algebraic differential equation with constant coefficients. For $g(1)\neq 0$ (2.1) can still be regarded as the "simplest" equation in the sense that the corresponding homogeneous differential equation can be reduced to an algebraic differential equation with constant coefficients given by (2.8).

First we discuss the homogeneous case ($h=0$). By (1.19), the characteristic equation of (2.1) is

$$u^2 + 2g(1)u + g(1)^2 - a = 0,$$

having the roots

$$u_{1,2} = -g(1) \pm \sqrt{a}.$$

(2.7) gives

$$x = v \exp\left[-\int (g - g(1))\right].$$

So (2.1) has a nontrivial operational solution if (2.8) has the same and it can be easily seen that

$$x \in E \leftrightarrow v \in E.$$

Property 9 shows that (2.1) has only the trivial solution if $a<0$.

Let $a \geqq 0$. We extend the definition of the δ functions for irrational arguments by

$$\delta(\varrho) = 0 \quad (\varrho \text{ is irrational}). \tag{2.9}$$

By the application of Property 9 we obtain the general solution of (2.1) in the form

$$x = [c_1\delta(e^{g(1)-\sqrt{a}}) + c_2\delta(e^{g(1)+\sqrt{a}})] \exp\left[-\int (g - g(1))\right], \quad c_1, c_2 \in K. \tag{2.10}$$

Introducing the coefficients

$$\begin{aligned} \overline{c_1} &= c_1, \quad \text{for} \quad e^{g(1)-\sqrt{a}} \in \mathbf{Z}, \\ \overline{c_1} &= 0, \quad \text{for} \quad e^{g(1)-\sqrt{a}} \notin \mathbf{Z}, \\ \overline{c_2} &= c_2, \quad \text{for} \quad e^{g(1)+\sqrt{a}} \in \mathbf{Z}, \\ \overline{c_2} &= 0, \quad \text{for} \quad e^{g(1)+\sqrt{a}} \notin \mathbf{Z} \end{aligned} \tag{2.11}$$

and replacing c_1, c_2 by $\overline{c_1}, \overline{c_2}$ in (2.10), the general function solution $x \in E$ will be obtained. (2.10), (2.11) give not only the explicit form of the solutions but also contain the existence criteria of the non-trivial operational (or function) solutions.

In the following we shall deal with the inhomogeneous equation ($h \neq 0$). First we assume that the corresponding homogeneous equation has two linearly independent solutions $x_1, x_2 \in M_D$. Then, as we have seen in the foregoing, $a > 0$. By Property 10 (2.1) has a solution $x_p \in M_D$ if and only if

$$\int \frac{x_1 h}{w}, \quad \int \frac{x_2 h}{w}$$

exist and

(2.12)
$$x_p = x_2 \int \frac{x_1 h}{w} - x_1 \int \frac{x_2 h}{w}.$$

Since

$$x_1 = \delta(e^{g(1)-\sqrt{a}}) e^{-\int (g-g(1))},$$

$$x_2 = \delta(e^{g(1)+\sqrt{a}}) e^{-\int (g-g(1))},$$

$$w = D\left(\frac{x_2}{x_1}\right) x_1^2 = D\left(\frac{\delta(e^{\sqrt{a}+g(1)})}{\delta(e^{g(1)-\sqrt{a}})}\right) \delta^2(e^{g(1)-\sqrt{a}}) \exp\left[2\int (g-g(1))\right]$$

is obtained. Applying Properties 3 and 5 we have

(2.13)
$$w = -2\sqrt{a}\, \delta(e^{2g(1)}) \exp\left[-2\int (g-g(1))\right].$$

By substituting (2.13) and x_1, x_2 into (2.12) and introducing the function

$$H = h e^{\int [g-g(1)]}$$

we get

(2.14)
$$x_p = \frac{1}{2\sqrt{a}} \exp\left[-\int (g-g(1))\right] \left[\delta(e^{g(1)-\sqrt{a}}) \int \frac{H}{\delta(e^{g(1)-\sqrt{a}})} - \delta(e^{g(1)+\sqrt{a}}) \int \frac{H}{\delta(e^{g(1)+\sqrt{a}})}\right].$$

If the rational arguments of the δ functions are not integer, then by Property 8 the occurring algebraic integrals exist in M_D and the application of (1.17) gives

$$x_p = \frac{1}{2\sqrt{a}} \exp\left[-\int (g-g(1))\right] \left\{\frac{H(n)}{\log \dfrac{n}{e^{g(1)+\sqrt{a}}}} - \frac{H(n)}{\log \dfrac{n}{e^{g(1)-\sqrt{a}}}}\right\}.$$

Since

$$\frac{1}{\log \frac{n}{e^{g(1)+\sqrt{a}}}} - \frac{1}{\log \frac{n}{e^{g(1)-\sqrt{a}}}} =$$

$$= \frac{1}{\log n - g(1) - \sqrt{a}} - \frac{1}{\log n - g(1) + \sqrt{a}} = \frac{2\sqrt{a}}{[\log n - g(1)]^2 - a},$$

(2.15)
$$x_p = \exp\left[-\int (g-g(1))\right]\left\{\frac{H(n)}{[\log n - g(1)]^2 - a}\right\}.$$

Obviously, $x_p \in E$ and it does not contain the δ functions explicitly.

If any of the δ functions has an integer argument, then by Property 8 and 10 we get that (2.1) has a solution if and only if $H(n)=0$ for those values of n, for which

(2.16)
$$(\log n - g(1))^2 - a = 0.$$

For such a value $n=N$ the symbol

(2.17)
$$\frac{H(N)}{(\log N - g(1))^2 - a} = \frac{0}{0}$$

denotes the number zero and formula (2.15) holds true.

If the homogeneous differential equation corresponding to (2.1) has not two linearly independent solutions, then (2.12) looses its meaning. Here we distinguish two cases.

I. The homogeneous equation has only the trivial solution. Then, as we have seen either $a<0$, or $a \geqq 0$ and the numbers $e^{g(1)+\sqrt{a}}$, $e^{g(1)-\sqrt{a}}$ are irrational.

It can be easily seen that (2.16) cannot hold for integer n, (2.15) has a meaning and an easy substitution shows that (2.15) is really a solution of (2.1).

II. The homogeneous equation has a nontrivial solution x_h, but has not two linearly independent solutions.

We may assume without loss of generality that $x_h = x_1$. If $x_1 \notin E$, then the situation is similar to the case I. (2.15) is a solution of (2.1). If $x_1 \in E$, then there is exactly one value $N \in \mathbf{Z}$ satisfying (2.16). If $H(N)=0$ the formula (2.15) is meaningful, when we retain (2.17). We obtain that (2.15) satisfies (2.1).

(2.15) is meaningless for $H(N) \neq 0$. We have to show that (2.1) has no solution in M_D at all.

Namely, by introducing the substitution

$$x = x_1 u, \quad u \in M_D$$

in (2.1), we have

$$D^2(u) + \left[\frac{2D(x_1)}{x_1} + 2g\right] D(u) = \frac{h}{x_1}.$$

Since

$$x_1 = \delta(e^{g(1)+\sqrt{a}}) \exp\left[-\int (g-g(1))\right], \qquad e^{g(1)+\sqrt{a}} = N \in \mathbf{Z},$$

an easy calculation gives

$$D^2(u)-2\sqrt{a}\,D(u)=\frac{he^{\int(g-g(1))}}{\delta(N)}=\frac{H}{\delta(N)}.$$

If $u_1 \in M_D$ were a solution of the latter equation, then by an algebraic integration we should have that u_1 would be a solution of

$$D(u)-2\sqrt{a}\,u=\int\frac{H}{\delta(N)}+c,$$

too, for certain $c\in K$. But this cannot hold since by Property 8

$$\int\frac{H}{\delta(N)}$$

does not exist for $H(N)\neq 0$. Consequently, (2.1) has no operational solution if $H(N)\neq 0$. Therefore the following theorem is valid.

THEOREM. *The homogeneous differential equation corresponding to* (2.1) *has only the trivial solution, if* $a<0$. *Let* $a\geqq 0$. *The general operational solution and function solution of the homogeneous equation are given by* (2.9), (2.10), (2.11).

If $\nexists n\in \mathbf{Z}$ *satisfying*

$$(\log n-g(1))^2-a=0, \tag{2.18}$$

then the inhomogeneous differential equation (2.1) *has a function solution of the form*

$$x_p=\exp\left[-\int(g-g(1))\right]\left\{\frac{H(n)}{(\log n-g(1))^2-a}\right\}, \tag{2.19}$$

where

$$H=he^{\int(g-g(1))}.$$

If $\exists n\in\mathbf{Z}$ *satisfying* (2.18) *then* (2.1) *has a solution in* M_D *if and only if*

$$H(n)=0$$

holds for those n that satisfy (2.18). *A solution is given by* (2.19) — *which is at the same time a function* — *where*

$$\frac{H(n)}{(\log n-g(1))^2-a}$$

denotes the number zero for those values of n that satisfy (2.18).

The general solution of (2.1) *is the sum of the general solution of the homogeneous equation and of* x_p. *Every solution of* (2.1) *belongs to the ring* E^*.

A simple consequence is the following

ALTERNATIVE THEOREM. *If the homogeneous differential equation has only the trivial solution in the ring E, then the inhomogeneous differential equation has for every*

$h \in E$ exactly one solution in E. If the homogeneous differential equation has a non-trivial solution in E, then the inhomogeneous equation has either no solution, or infinitely many solutions in E, according to $h \in E$.

REMARK. This Alternative Theorem does not hold in the field M_D.

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DISTRIBUTION OF VALUES AND PROXIMITY OF a-POINTS FOR QUOTIENTS OF BLASCHKE PRODUCTS WITH NEARBY ZEROS

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O. Frostman ([11]) constructed, for an arbitrary set $K\subset\{w:|w|<1\}$ of planar measure zero, a Blaschke product $B(z)$ that takes w finitely many times exactly when $w\in K$.

Thus Picard's theorem, let alone the fundamental theorems of value distribution according to which most values are taken equally often are not valid for Blaschke products.[1]

In this paper we show that the quotient of two Blaschke products with corresponding zeros close to each other already obeys these theorems and moreover, it has the "proximity property" (see [3]) i.e., quantitatively speaking, different values are taken at nearby points. It will be shown that the image by such quotients of some very small sets almost coincides with the image of the full unit disk and estimations of the derivative at simple points will also be derived.

Let us consider the quotient of two Blaschke products

$$B(z)=\frac{\prod_{n=1}^{\infty}\frac{\bar a_n}{|a_n|}\frac{a_n-z}{1-\bar a_n z}}{\prod_{n=1}^{\infty}\frac{\bar b_n}{|b_n|}\frac{b_n-z}{1-\bar b_n z}}$$

where $\sum_{n=1}^{\infty}(1-|a_n|)<+\infty$, $\sum_{n=1}^{\infty}(1-|b_n|)<+\infty$, $a_n\neq b_k$ for all integers n, k and

$$\sum_{n=1}^{\infty}\frac{|a_n-b_n|}{1-|a_n|}<+\infty. \tag{1}$$

We shall also consider products of the form

$$\Pi(z)=\prod_{n=1}^{\infty}\frac{z-a_n}{z-b_n},\quad |z|<1$$

where $|a_n|$, $|b_n|<1$, $a_n\neq b_k$ $(n,k\in N)$ and (1) is fulfilled. We observe that the latter condition implies the absolute and locally uniform convergence of this product for $|z|<1$.

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[1] We assume that the reader is familiar with the basic notions and results of the theory of value distribution (see [2]).

THEOREM 1. *Let* $(c_i)_{i=1}^q \in \bar{C}$, $c_i \neq c_j$ *for* $i \neq j$, $q > 2$. *Then we have for* $B(z)$

$$\sum_{i=1}^{q}\left(1-\frac{n(r_n, c_i)}{n(r_n, 0)}\right)+\sum_{i=1}^{q}\frac{n_1(r_n, c_i)}{n(r_n, 0)} \leqq 2+o(1) \quad (n\to\infty), \tag{2}$$

where r_n *is a suitable sequence of radii tending to* 1 *as* $n\to\infty$, $n_1(r, c)=n(r, c)-\bar{n}(r, c)$, $\bar{n}(r, c)$ *standing for the number of distinct roots of the equation* $B(z)=c$ *in the disc* $|z|\leqq r$.

It will be established in the proof that $n(r_n, 0)\sim A(r_n)$ as $n\to\infty$. Taking this into account (2) is equivalent to Ahlfors' form of the second fundamental theorem, (see [2]).

The latter is known under the assumption $\overline{\lim}_{r\to 1}(1-r)A(r)=\infty$. In our case this limit equals 0 and yet Ahlfors' theorem proves to be true in our situation.

THEOREM 2. *Let* $\varphi(r)$ $(0\leqq r<1)$ *be an increasing function tending to* $+\infty$ *as* $r\to 1$, $\varphi(r)<\log n(r, 0)$. *Then we can find for* $B(z)$ *a certain number (to be denoted by* $\Phi(r)$*) of pairwise disjoint regions* $E_i(r)$ $(i=1, \dots, \Phi(r))$ *in* $|z|<r$ *having the following properties.*

(i) $\Phi(r_n)\sim n(r_n, 0)$ $(n\to\infty)$ *where* r_n *is a suitable sequence of radii going to* 1 *as* $n\to\infty$.

(ii) *In each region* $E_i(r)$ *the function* $B(z)$ *is univalent (though it may have multiple points in the boundary); the closure of the image* $B(E_i(r))$ *coincides with the closed plane minus a finite number (to be denoted by* K_i*) of simply connected regions* Δ_j^i $(j=1, \dots, K_i)$.

(iii) $\varrho(\Delta_j^i)\leqq 1/\varphi(r_n)$ $(i=1, \dots, \Phi(r_n),\ j=1, \dots, K_i)$ *where* $\varrho(\ )$ *denotes the diameter with respect to the spherical metric of the region in question.*

(iv) $\sum_{i=1}^{\Phi(r_n)} K_i\leqq(2+o(1))n(r_n, 0)$ $\qquad (n\to\infty)$.

(3) (v) $d(E_i(r))\leqq K\varphi^s(r_n)(\log n(r_n, 0))/n(r_n, 0)$ $\qquad (n=1, 2, \dots)$

where $d(\ )$ *denotes ordinary euclidean diameter and* $K<+\infty$ *is a constant.*

Properties (i)—(v) signify that $E_i(r_n)$ are small and in most of these regions $B(z)$ takes all values $a\in C$ with the exception of a set of negligeable area and thus with very few exceptions the a-points and b-points belonging to $E_i(r_n)$ lie close to each other. This is the property of proximity referred to in the title.

Theorem 2 extends to $B(z)$ the statements of Theorems 1.1 of [3] except that $A(r)$ is replaced by $n(r, 0)$ and (v) is in a sharpened form here. Theorem 1.1 of [3] incorporates a number of improvements over the basic theorems of value distribution theory, L. Ahlfors' theory of covering surfaces and over the results on Julia directions and exhausting discs. The same way as in [3] these results can also be extended to $B(z)$; the present sharper form of (v) leads to a better result in the case of $B(z)$ concerning the discs of exhaustion.

THEOREM 3. *Let $c_\nu \in C$ $(\nu=1, \dots, q)$ be distinct, $\psi(r)$ $(r\in[0, 1))$ non-increasing and tending to 0 as $r\to 1$. For $B(z)$ there is then a sequence $r_n\to 1$ $(n\to\infty)$ such that in the disc $|z|\leqq r_n$ we can find $n_0(r_n, c_\nu)$ simple c_ν-points $z_{\nu,k}$ $(k=1, \dots, n_0(r_n, c_\nu))$ such that*

$$\sum_{\nu=1}^{q} n_0(r_n, c_\nu) \geqq (q-4)n(r_n, 0)+o(n(r_n, 0)) \quad (n\to\infty), \tag{4}$$

$$|w'(z_{\nu,k})| \geqq \text{const}\,\frac{\psi(r_n)n(r_n, 0)}{\log n(r_n, 0)} \quad (n = 1, 2, \dots). \tag{5}$$

As the bound in (4) is exactly the one guaranteed by the theory of value distribution for the number of simple c_ν-points, Theorem 3 concerns the majority of simple c_ν-points and (5) shows that the derivative is large in most of these points.

THEOREM 4. *Theorems 1, 2 and 3 hold for $\Pi(z)$ in place of $B(z)$ provided that the number of zeros $n(r, 0, \Pi)$ has finite lower order, i.e. $\lambda=\liminf(\log n(r, 0))/(-\log(1-r))<+\infty$.*

For the proofs of these theorems we first establish some lemmas. Here and in what follows K stands for constants not necessarily the same at different occurrences.

LEMMA 1. *We have for $B(z)$ and every $r'<r<1$*

$$\int_{1-2(1-r)}^{r}\int_{0}^{2\pi}\left|\frac{B'}{B}(te^{i\varphi})\right| t\,d\varphi\,dt \leqq K(1-r)\log n\left(1-\frac{1-r}{2}, 0\right), \tag{6}$$

$$\int_{0}^{r}\int_{0}^{2\pi}\left|\frac{B'}{B}(te^{i\varphi})\right| t\,d\varphi\,dt \leqq K\log n\left(1-\frac{1-r}{2}, 0\right). \tag{7}$$

PROOF. As

$$\frac{B'}{B}(z) = \sum_{n=1}^{\infty}\left(\frac{1}{z-a_n}-\frac{1}{z-b_n}\right)+\sum_{n=1}^{\infty}\left(\frac{\bar a_n}{1-\bar a_n z}-\frac{\bar b_n}{1-\bar b_n z}\right),$$

we have

$$\begin{aligned}\int_{1-2(1-r)}^{r}\int_{0}^{2\pi}\left|\frac{B'}{B}(te^{i\varphi})\right| t\,d\varphi\,dt \leqq \sum_{|a_n|\leqq r_0} + \sum_{r_0<|a_n|\leqq 1-3(1-r)} + \sum_{1-3(1-r)<|a_n|\leqq 1-(1-r)/2} + \\ + \sum_{1-(1-r)/2<|a_n|}\int_{1-2(1-r)}^{r}\int_{0}^{2\pi}\left|\frac{1}{te^{i\varphi}-a_n}-\frac{1}{te^{i\varphi}-b_n}\right| t\,d\varphi\,dt + \\ + \sum_{n=1}^{\infty}\int_{1-2(1-r)}^{r}\int_{0}^{2\pi}\left|\frac{\bar a_n}{1-\bar a_n te^{i\varphi}}-\frac{\bar b_n}{1-\bar b_n te^{i\varphi}}\right| t\,d\varphi\,dt = I_1+I_2+I_3+I_4+I_5 \quad (r_1\leqq r).\end{aligned} \tag{8}$$

Obviously, for $|a_n|\leqq r_0$ and $z\in D(r)=\{z\colon 1-2(1-r)\leqq|z|\leqq r\}$ we have $|z-a_n|\geqq K$, $|z-b_n|\geqq K$, $r_2\leqq r$, hence

$$I_1 \leqq K\sum_{|a_n|\leqq r_0}|a_n-b_n|\iint_{D(r)} d\sigma \leqq K(1-r) \tag{9}$$

where $d\sigma$ is the area element.

Fix r_0 such that

$$\sum_{r_0<|a_n|} \frac{|a_n-b_n|}{1-|a_n|} \leqq \frac{1}{100}.$$

For estimating I_2 we use the inequality $(|a|<r_1<r_2<+\infty)$

$$\begin{aligned}\int_{r_1}^{r_2}\int_0^{2\pi} \frac{t\,d\varphi\,dt}{|te^{i\varphi}-a|^2} &= \int_{r_1}^{r_2} t\left\{\frac{1}{t^2}\int_0^{2\pi} \sum_{k=0}^{\infty}\left(\frac{a}{te^{i\varphi}}\right)^k \sum_{k=0}^{\infty}\left(\frac{\bar a}{te^{i\varphi}}\right)^k\right\} d\varphi\,dt = \\ &= \int_{r_1}^{r_2} t\left\{\frac{2\pi}{t^2}\sum_{k=0}^{\infty}\left|\frac{a}{t}\right|^{2k}\right\} dt = 2\pi\int_{r_1}^{r_2}\frac{tdt}{t^2-|a|^2} \leqq 2\pi\frac{r_2}{r_1}\log\frac{r_2-|a|}{r_1-|a|}.\end{aligned} \tag{10}$$

We observe that for $r_0<|a_n|\leqq 1-3(1-r)$

$$|z-b_n| \geqq |z-a_n|-|a_n-b_n| > \frac{1}{2}|z-a_n|. \tag{11}$$

In fact, according to the choice of r_0, we have

$$2|a_n-b_n|+|a_n| \leqq \frac{2}{100}(1-|a_n|)+|a_n| \leqq \frac{2}{100}3(1-r)+1-3(1-r) < 1-2(1-r),$$

$$2|a_n-b_n| < 1-2(1-r)-|a_n| \leqq |z-a_n| \qquad (z\in D(r))$$

and (11) follows.

By the inequalities (10) and (11) we get

$$I_2 \leqq 2\sum_{r_0<|a_n|\leqq 1-3(1-r)} |a_n-b_n| \int_{1-2(1-r)}^{r}\int_0^{2\pi} \frac{td\varphi dt}{|te^{i\varphi}-a_n|^2} \leqq$$

$$\leqq 4\pi\frac{r}{1-2(1-r)}\sum_{r_0<|a_n|\leqq 1-3(1-r)} \frac{|a_n-b_n|}{1-|a_n|}(1-|a_n|)\log\frac{r-|a_n|}{1-2(1-r)-|a_n|} \quad (r_1\leqq r).$$

As the function

$$(1-x)\log\{(r-x)/(1-2(1-r)-x)\}$$

increases for $0<x<1-2(1-r)<1$, we have

$$I_2 \leqq K(1-r)\sum_{r_0<|a_n|\leqq 1-3(1-r)} \frac{|a_n-b_n|}{1-|a_n|} \leqq K(1-r) \quad (r_1\leqq r). \tag{12}$$

Let us estimate now I_4. The same way as (10) we easily establish the inequality

$$\int_{r_1}^{r_2}\int_0^{2\pi} \frac{t\,d\varphi\,dt}{|te^{i\varphi}-a|^2} \leqq 2\pi\frac{r_2}{r_1}\log\frac{|a|-r_1}{|a|-r_2} \qquad (0\leqq r_1<r_2<|a|). \tag{13}$$

Observe that for $1-(1-r)/2<|a_n|$

$$|a_n-b_n| < \frac{1-|a_n|}{100} < \frac{1}{2}|z-a_n| \qquad (z\in D(r)),$$

i.e.

$$|z-b_n| \geqq |z-a_n|-|a_n-b_n| > \frac{1}{2}|z-a_n| \qquad (z\in D(r)). \tag{14}$$

By (13), (14) and the fact that $(1-x)\log\{[x-(1-2(1-r))]/(x-r)\}$ decreases for $x>1-(1-r)/2$ we get

$$\begin{aligned} I_4 &\leqq 2\pi\frac{r}{1-2(1-r)}\sum_{1-(1-r)/2<|a_n|}\frac{|a_n-b_n|}{1-|a_n|}(1-|a_n|)\log\frac{|a_n|-(1-2(1-r))}{|a_n|-r} \leqq \\ &\leqq K(1-r)\sum_{1-(1-r)/2<|a_n|}\frac{|a_n-b_n|}{1-|a_n|}\leqq K(1-r) \qquad (r_1\leqq r). \end{aligned} \tag{15}$$

Let now, for $1-3(1-r)<|a_n|\leqq 1-(1-r)/2$,

$$\delta_n(r) = \max\left\{|a_n-b_n|,\ \frac{1-r}{n\left(1-\frac{1-r}{2},\ 0\right)}\right\},$$

$$D_n(r) = \left\{z\colon\ |z-z_n|\leqq\delta_n(r),\ z_n=\frac{a_n+b_n}{2}\right\},$$

$$D(a_n,r) = \left\{z\colon\ |z-a_n|\leqq\frac{1}{2}\delta_n(r)\right\}.$$

We have

$$\begin{aligned} I_3 &\leqq \sum_{1-3(1-r)<|a_n|\leqq 1-(1-r)/2}\left\{3|a_n-b_n|\iint\limits_{D(r)\setminus D_n(r)}\frac{d\sigma}{|z-a_n|^2}+\iint\limits_{D_n(r)}\frac{d\sigma}{|z-a_n|}+\iint\limits_{D_n(r)}\frac{d\sigma}{|z-b_n|}\right\}\leqq \\ &\leqq 6\pi\sum_{1-3(1-r)<|a_n|\leqq 1-(1-r)/2}\left\{|a_n-b_n|\iint\limits_{D(r)\setminus D(a_n,r)}\frac{d\sigma}{|z-a_n|^2}+\delta_n(r)\right\} \quad (r_1\leqq r). \end{aligned} \tag{16}$$

Suppose first that $r-(1-r)/2\leqq|a_n|\leqq 1-(1-r)/2$. Let us denote by $|l_n(R)|$ the arc length of $l_n(R)=\{z\colon\ |z-a_n|=R\}\cap D(r)$ for $\delta_n(r)/2\leqq R\leqq|a_n|+r$. From geometric considerations it is clear that

$$|l_n(R)|\leqq K(1-r),\quad |a_n|-(1-2(1-r))\leqq R\leqq\sqrt{|a_n|^2+(1-2(1-r))^2}\quad (r_3\leqq r).$$

Hence

$$\iint\limits_{D(r)\setminus D(a_n,r)} \frac{d\sigma}{|z-a_n|^2} \leqq 2\pi \int\limits_{1/2\delta_n(r)}^{|a_n|-(1-2(1-r))} \frac{dR}{R} + \int\limits_{|a_n|-(1-2(1-r))}^{\sqrt{|a_n|^2+(1-2(1-r))^2}} \frac{|l_n(R)|}{R^2}\,dR +$$

$$(17)\qquad + \int\limits_{\sqrt{|a_n|^2+(1-2(1-r))^2}}^{|a_n|+r} \frac{dR}{R} \leqq 2\pi \log\left(2\frac{|a_n|-(1-2(1-r))}{1-r}\, n\left(1-\frac{1-r}{2},0\right)\right) +$$

$$+K\frac{1-r}{|a_n|-(1-2(1-r))} + 2\pi\log\frac{|a_n|+r}{\sqrt{|a_n|^2+(1-2(1-r))^2}} \leqq$$

$$\leqq K\log n\left(1-\frac{1-r}{2},0\right) \qquad (r_4 \leqq r).$$

In the case $1-3(1-r)<|a_n|<r-(1-r)/2$ we obtain analogously the same bound.

From (16) and (17) it follows that

$$I_3 \leqq K(1-r) \sum_{1-3(1-r)<|a_n|\leqq 1-(1-r)/2} \times$$

$$(18)\qquad \times\left\{\frac{|a_n-b_n|}{1-|a_n|}\log n\left(1-\frac{1-r}{2},0\right) + \frac{|a_n-b_n|}{1-|a_n|} + \frac{1}{n\left(1-\frac{1-r}{2},0\right)}\right\} \leqq$$

$$\leqq K(1-r)\log n\left(1-\frac{1-r}{2},0\right) \qquad (r_4 \leqq r).$$

Using the inequality $|1-b_n z| \geqq K|1-a_n z|$ $(z\in D(r))$ we have

$$I_5 \leqq K\sum_{n=1}^{\infty}|a_n-b_n| \int\limits_{1-2(1-r)}^{r}\int\limits_0^{2\pi} \frac{t\,d\varphi\,dt}{|1-\bar a_n te^{i\varphi}|^2} \leqq$$

$$(19)\qquad \leqq K\sum_{n=1}^{\infty}|a_n-b_n| \int\limits_{1-2(1-r)}^{r} t\left\{\int\limits_0^{2\pi}\sum_{k=0}^{\infty}|a_n t|^{2k}\,d\varphi\right\}dt \leqq$$

$$\leqq K\sum_{n=1}^{\infty}|a_n-b_n| \int\limits_{1-2(1-r)}^{r} \frac{t\,dt}{1-|a_n t|^2} \leqq K(1-r)\sum_{n=1}^{\infty}\frac{|a_n-b_n|}{1-|a_n|} \leqq K(1-r) \quad (r_1 \leqq r).$$

By adding (8), (9), (12), (15), (18), (19) we get inequality (6) for $r_5<r<1$.

For $r_5<1-2(1-r)$ let us divide the interval $[0, r]$ into subintervals

$$[0,r] = [1-2(1-r), r]\cup\ldots\cup[1-2^n(1-r), 1-2^{n-1}(1-r)]\cup[0, 1-2^n(1-r)],$$

where n is chosen according to

$$1-2^{n+1}(1-r) < r_5 \leqq 1-2^n(1-r).$$

Using inequality (6) for each segment $[1-2^m(1-r),\ 1-2^{m-1}(1-r)]$ $(m=1, \dots, n)$ we get

$$\int_0^r \int_0^{2\pi} \left|\frac{B'}{B}(te^{i\varphi})\right| t\, d\varphi\, dt = \int_0^{1-2^n(1-r)} \int_0^{2\pi} + \sum_{m=1}^{n} \int_{1-2^m(1-r)}^{1-2^{m-1}(1-r)} \int_0^{2\pi} \left|\frac{B'}{B}(te^{i\varphi})\right| t\, d\varphi\, dt \leqq$$

$$\leqq K + K(1-r)\log n\left(1-\frac{1-r}{2},\ 0\right) \sum_{m=1}^{n} (2^m - 2^{m-1}) \leqq$$

$$\leqq K(1-r)2^n \log n\left(1-\frac{1-r}{2},\ 0\right) \leqq K \log n\left(1-\frac{1-r}{2},\ 0\right) \quad (r_6 \leqq r).$$

This proves Lemma 1 for $\max(r_5, r_6) = r' < r < 1$.

Remark. From the proof of Lemma 1, in particular from the bounds for the integrals I_1, I_2, I_3, I_4 it follows that the inequalities (6) and (7) also hold for the function $\Pi(z)$ since

$$\frac{\Pi'}{\Pi}(z) = \sum_{n=1}^{\infty} \left(\frac{1}{z-a_n} - \frac{1}{z-b_n}\right).$$

Lemma 2. *Let* $\varphi(r)$ $(r \in [0, 1))$ *be a non-decreasing function tending to* $+\infty$ *as* $r \to 1$. *If* $\varphi(r)$ *has finite lower order, i.e.*

$$\varliminf_{r \to 1} (\log \varphi(r))/\log \frac{1}{1-r} = \lambda < +\infty,$$

then for each $1 < c < \infty$ *one can exhibit a sequence* $r_n \to 1$ $(n \to \infty)$ *such that*

$$\varphi\left(1-\frac{1-r_n}{c}\right) \leqq c^{\lambda+1}\varphi(r_n) \qquad (n = 1, 2, \dots).$$

The proof of this well-known type of result follows e.g. from Lemma 1.3.1 of [4] applied to the function $\varphi((u-1)/u)$ $(u \geqq 1)$.

Because of its independent interest, we formulate the next auxiliary result as

Theorem 5. *With a suitable sequence* $r_n \to 1$ $(n \to \infty)$ *the function* $B(z)$ *satisfies the relations*

(20) $$A(r_n) \sim n(r_n, 0) \qquad (n \to \infty),$$

(21) $$L(r_n) = o(A(r_n)) \qquad (n \to \infty),$$

(22) $$\int_0^{r_n} L(t)\, dt \leqq K \log n(r_n, 0) \qquad (n = 1, 2, \dots).$$

Proof. According to Lemma 1.1 of [5] we have $(z = te^{i\varphi})$

$$n(t, 0) - A(t) = \frac{t}{2\pi} \int_{\Delta_1(t,0)} -\frac{\partial}{\partial t} \log |B(z)|\, d\varphi + hL(t)$$

where $\Delta_1(t, 0) = \{z\colon |z| = t,\ |B(z)| < 1\}$, $|h| < h(0) = \text{const} < \infty$.

From (6) it follows that

$$\int_{1-2(1-r)}^{r}\left\{\frac{t}{2\pi}\int_{\Delta_1(t,0)}\left|\frac{\partial}{\partial t}\log|B(z)|\right|d\varphi+|h|L(t)\right\}dt\leqq \tag{23}$$
$$\leqq K\int_{1-2(1-r)}^{r}\int_{0}^{2\pi}\left|\frac{B'}{B}(te^{i\varphi})\right|t\,d\varphi\,dt\leqq K(1-r)\log n\left(1-\frac{1-r}{2},0\right)\quad (r'<r).$$

Since the function $n(r,0)$ has finite lower order, in fact $(1-r)n(r,0)\to 0$ $(r\to 1)$ as is well-known, Lemma 2 applied to (23) yields

$$\int_{1-2(1-r_n')}^{r_n'}\left\{\frac{t}{2\pi}\int_{\Delta_1(t,0)}\left|\frac{\partial}{\partial t}\log|B(z)|\right|d\varphi+|h|L(t)\right\}dt\leqq$$
$$\leqq K\int_{1-2(1-r_n')}^{r_n'}\int_{0}^{2\pi}\left|\frac{B'}{B}(te^{i\varphi})\right|t\,d\varphi\,dt\leqq K(1-r_n')\log n(1-2(1-r_n'),0)\quad (n=1,2,\ldots)$$

where $r_n'\to 1$ $(n\to\infty)$.

By the mean value theorem we find an $r_n\in[1-2(1-r_n'),r_n']$ $(n=1,\ldots)$ such that

$$|n(r_n,0)-A(r_n)|\leqq\frac{r_n}{2\pi}\int_{\Delta_1(r_n,0)}\left|\frac{\partial}{\partial r}\log|B(z)|\right|d\varphi+|h|L(r_n)\leqq$$
$$\leqq K\log n(r_n,0)\qquad (n=1,2,\ldots).$$

Dividing by $n(r_n,0)$ we get (20) and (21), while (22) follows from

$$\int_0^{r_n}L(t)\,dt\leqq\int_0^{r_n'}L(t)\,dt\leqq K\log n(1-2(1-r_n'),0)\leqq K\log n(r_n,0)\quad (n=1,2,\ldots).$$

THEOREM 6. *Theorem 5 is valid for $\Pi(z)$ provided that $n(r,0,\Pi)$ has finite lower order.*

The proof only differs from that of Theorem 5 in that (23) follows from the remark made after the proof of Lemma 1.

PROOF of Theorem 1. By the second fundamental theorem of L. Ahlfors (see [2]) we have

$$\sum_{i=1}^{q}\left(1-\frac{n(r,c_i)}{A(r)}\right)+\sum_{i=1}^{q}\frac{n_1(r,c)}{A(r)}\leqq 2+h\frac{L(r)}{A(r)} \tag{24}$$

where $|h|=h(c_1,\ldots,c_q)=\text{const}<\infty$.

The inequality (2) of Theorem 1 follows from (20), (21) and (24).

PROOF of Theorem 2. This is based on the following proposition (see [3], pp. 414—415) concerning a meromorphic function $w(z)$, a region $D=\{z\colon |z|\leqq r\}$ and $n=[\varphi(r)]$.

Let $w(z)$ be meromorphic in $|z|<1$. In the disc $|z|\leqq r$ one can then find $\Phi(r)$ disjoint regions $E_i(r)$ $(i=1, \ldots, \Phi(r))$ satisfying (ii) of Theorem 2 and such that

$$|\Phi(r)-A(r)| \leqq K\frac{A(r)}{\varphi(r)}+K\varphi^{17}(r)L(r) \tag{25}$$

$$\varrho(\Delta_j^i) \leqq \frac{K}{\varphi(r)} \quad (i=1, \ldots, \Phi(r),\ j=1, \ldots, K_i),$$

$$\sum_{i=1}^{\Phi(r)} K_i \leqq 2A(r)+K\frac{A(r)}{\varphi(r)}+K\varphi^{17}(r)L(r) \tag{26}$$

$$\sum_{i=1}^{\Phi(r)} d(E_i(r)) \leqq K\varphi^{17}(r)\int_0^r L(t)\,dt+K\varphi(r)r. \tag{27}$$

Putting $w(z)=B(z)$ and applying (20), (21) and (22) to the inequalities (25), (26) and (27) we get the statements (i), (iv) and the inequality

$$\sum_{i=1}^{\Phi(r_n)} d(E_i(r_n)) \leqq K\varphi^7(r_n)\log n(r_n, 0) \quad (n=1, 2, \ldots). \tag{28}$$

Hence the number of regions $E_i(r)$ such that

$$d(E_i(r_n)) \geqq K\varphi^8(r_n)\frac{\log n(r_n, 0)}{n(r_n, 0)}$$

is $o(n(r_n, 0))$ $(n\to\infty)$ and dropping these regions we get (3) with an obvious change of notation. This proves Theorem 2.

Proof of Theorem 3. In the course of proof of Theorem 1′ in [6] it is shown that one can find $n_0(r, c_\nu)$ simple c_ν-points $z_{\nu,k}$ $(\nu=1, \ldots, q,\ k=1, \ldots, n_0(r, c_\nu))$ in $|z|\leqq r$ such that

$$\sum_{i=1}^{q} n_0(r, c_\nu) \geqq (q-4)A(r)-\text{const } L(r),$$

$$|w'(z_{\nu,k})| \geqq \frac{\text{const}}{d(E_{\nu,k})}.$$

(For the definition of $E_{\nu,k}$ see [6].)

The same way as in [6], using Theorem 5 we obtain (4) and (5).

Taking into account Theorem 6, Theorem 4 is proved analogously as Theorems 1, 2 and 3.

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WELL-ORDERINGS WHICH ARE TIGHT RELATIVE TO A PRESCRIBED DISTANCE FUNCTION

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Abstract

It is shown that under certain conditions (that can naturally be formulated using the concepts introduced in the elementary arithmetic of ordinal numbers) the following property holds: Given a well-ordering of the set of all two-element subsets of the set H, that is, a one-to-one and onto mapping $\varphi: \{\{x, y\}|x\neq y,\ x, y\in H\}\to\beta\setminus\{0\}$ for some ordinal $\beta(=\{\alpha|\alpha<\beta\})$, one can define another well-ordering $\psi: H\to\alpha$ (one-to-one and onto) such that, for arbitrary $x, y\in H$, we have $\psi(y)\leqq\psi(x)+\varphi(\{x, y\})$. To explain the title, it is suggested to think of φ as a distance function; then the last inequality expresses a certain tightness condition.

1. Introduction

Recently, considerable attention has been attracted by the problem, how to order the vertices of a graph in such a way that those vertices connected by an edge be as close as possible. In [2], starting from a number theoretic problem, we arrived at a related question and obtained the following result. If the edges of a graph are labelled with distinct positive integers, then there exists a labelling of the vertices of G with distinct, consecutive integers $\geqq 0$, such that, denoting the edge connecting the vertices x and y by xy and denoting the label associated with the edge a (respectively, vertex x) by $d(a)$ (respectively, $h(x)$), we have

$$h(x)-h(y)\leqq d(xy) \tag{1}$$

for all edges xy of G. Furthermore, the vertex labelled with 0 can be chosen arbitrarily.

In this note we generalize the above theorem in two directions. Firstly, this result can clearly be reformulated to a theorem about orderings of the vertices. On the other hand, its proof in [2] is based upon a cardinality argument. This provokes the question whether the theorem remains valid if we allow d and h to take their values from a set of ordinal numbers. In this note we give an affirmative answer (Theorem 1) under a certain regularity condition. It is an open question whether this condition can be dropped. The present proof essentially relies upon this condition, as it is shown by the counter-example of Section 3.

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Secondly observe that, if already one repetition in the labels of the edges is allowed, then the theorem is no longer true. In fact, let G consists of the vertices x, y, and z, and edges xy and yz with $d(xy)=d(yz)=1$. Then there is no labelling satisfying (1) and starting from y, that is, satisfying $h(y)=0$. This leads us to the question, how to modify the above result, if we want to allow repetitions in the labels of the edges. Theorem 1 is generalized to an answer to the second question in Section 4.

A few words about the set theoretic background are said in Section 5.

2. The main theorem

By a graph we mean an ordered pair $G=(V, E)$, where V is a set (the set of vertices), and E (the set of edges) is a subset of the set of all two element subsets of V. A path in a graph is a sequence of edges $c_1, c_2, \ldots, c_n$ such that the $c_i \cap c_{i+1}$, $i=1, 2, \ldots, n-1$ are pairwise disjoint one element subsets of V. For simplicity's sake, we denote the edge connecting the vertices x and y by xy, as before.

As usual, ω denotes the order type of the set of natural numbers. If α and β are ordinals such that $\alpha \leq \beta$, then there exists a unique ordinal ξ such that $\alpha+\xi=\beta$. This ordinal will be denoted by $-\alpha+\beta$. Another fundamental property is that Euclidean division can be carried out between two ordinal numbers, in the sense that, for any ordinals α and β, $\beta \neq 0$, there exist unique ordinals δ and ε such that $\alpha = \beta\delta+\varepsilon$ and $\varepsilon<\beta$. The answer to the problems posed will be based upon this fact. Let α and γ be ordinals, then there exist δ and ε with $\varepsilon<\omega^\gamma$ such that $\alpha=\omega^\gamma\delta+\varepsilon$. Define $[\alpha]_\gamma=\omega^\gamma \cdot \delta$, and, if α and β are two ordinal numbers, write $\alpha \prec_\gamma \beta$ for $[\alpha]_\gamma = [\beta]_\gamma$ and write $\alpha <_\gamma \beta$ for $[\alpha]_\gamma < [\beta]_\gamma$. Now we are ready to state our main result.

THEOREM 1. *Let $G=(V, E)$ be a graph, let $\mathcal{O}$ be a set of ordinals, $0 \notin \mathcal{O}$, and let $d\colon E \to \mathcal{O}$ be a one-to-one mapping. Assume that, for any ordinal γ, and for any edges $a, b \in E$ with $d(a) \sim_\gamma d(b)$, there exists a path $a=c_1, c_2, \ldots, c_n=$ such that $d(c_i) \prec_\gamma d(a)$ for all i. Then there exists a one-to-one mapping $h\colon V \to \mathcal{O}'$ to a set $\mathcal{O}'$ of ordinals such that, for any edge $xy \in E$ with $h(x)<h(y)$, we have*

$$-h(x)+h(y) \leq d(xy).$$

3. Proof of Theorem 1

Before proceeding to the proof, we need some preparations. For ordinals α, β $(\alpha<\beta)$ let

$$[\alpha, \beta) := \{\eta \mid \alpha \leq \eta < \beta\}.$$

The following lemma is proved by an easy transfinite induction over γ.

LEMMA 1. *Let α and γ be ordinals such that $\alpha<\omega^\gamma$. Then $[\alpha, \omega^\gamma)$ is similar to $[0, \omega^\gamma)$.*

Now we turn to defining h. At the same time we introduce some more notations which we shall need in the sequel. We may assume that G is connected, as from this particular case Theorem 1 easily follows in full generality. Choose an arbitrary vertex x and define $h(x)$ to be 0.

Now, by induction, let $\beta>0$, and assume that for all $\alpha<\beta$ we have chosen a vertex z with $h(z)=\alpha$. Define

$$V_\beta = \{z\in V \mid \exists\alpha(<\beta):\ h(z) = \alpha\}$$

$$\mathcal{O}_\beta = \{d(a) \mid a\in E,\quad a\cap V_\beta = \emptyset,\ a\setminus V_\beta \neq \emptyset\}$$

$$\lambda_\beta = \min \mathcal{O}_\beta .$$

Then there is a unique edge $x_0 y_0$, $x_0\in V_\beta$, $y_0\notin V_\beta$ such that

$$\lambda_\beta = d(x_0 y_0).$$

Define

$$h(y_0) = \beta.$$

The proof of Theorem 1 is divided into two parts. First we show (Lemma 2) that the occurrances of the ordinals in the $\mathcal{O}_\alpha$'s show a certain regularity. This will be exploited to prove Lemma 3, a modification of Theorem 1 which is more suitable for a proof by transfinite induction, than the theorem itself. For an ordinal β, call the set $[[\beta]_\gamma, [\beta]_\gamma+\omega^\gamma)$ the ω^γ-class of β and denote it by $\mathscr{C}_\gamma(\beta)$.

LEMMA 2. *Let α and γ be ordinals such that, for some ξ with $\alpha\leqq\xi<\alpha+\omega^\gamma$, $\mathcal{O}_\xi\neq\emptyset$. Let $\beta=\min \bigcup_{\alpha\leqq\xi<\alpha+\omega^\gamma} \mathcal{O}_\xi$. Then $\mathcal{O}\cap\mathscr{C}_\gamma(\beta)\subseteq \bigcup_{1\leqq\xi<\alpha+\omega^\gamma} \mathcal{O}_\xi$.*

PROOF. We proceed via transfinite induction over γ. For $\gamma=0$, the lemma is obvious.

Let $\gamma=\varkappa+1$. Let $\beta'\sim_\gamma\beta$, and $\beta'\in\mathcal{O}$. We have to show that $\beta'\in \bigcup_{1\leqq\xi<\alpha+\omega^\gamma} \mathcal{O}_\xi$. First of all, let α_0 be the smallest of those ordinals α' satisfying $\beta\in\mathcal{O}_{\alpha'}$, and $\alpha\leqq\alpha'$. Then there is a natural number n_0 with $\alpha_0\leqq\alpha+\omega^\varkappa\cdot n_0$. By the assumptions of Theorem 1, there is a path $e_1, e_2, \ldots, e_n$ of G with $d(e_1)=\beta$, $d(e_n)=\beta'$, and $d(e_i)\prec_\gamma\beta$. Choose $k_1\in\{1, 2, \ldots, n\}$ such that e_{k_1} has a point in V_{α_0}, but, for $i>k_1$, $e_i\cap V_{\alpha_0}=\emptyset$. Such a k_1 exists, or else $\beta'\in \bigcup_{1\leqq\xi<\alpha_0} \mathcal{O}_\xi\subseteq \bigcup_{1\leqq\xi<\alpha+\omega^\gamma} \mathcal{O}_\xi$ proving the lemma. By the choice of k_1, $d(e_{k_1})\in\mathcal{O}_{\alpha_0}$, thus $\beta\leqq d(e_{k_1})$, whence $\beta\sim_\gamma d(e_{k_1})$. By the induction hypothesis, $\mathscr{C}_\varkappa(\beta)\subseteq \bigcup_{1\leqq\xi<\alpha_0+\omega^\varkappa} \mathcal{O}_\xi$. The induction hypothesis yields also that $\mathscr{C}_\varkappa(\min \bigcup_{\alpha_0+\omega^\varkappa\leqq\xi<\alpha_0+\omega^\varkappa\cdot 2} \mathcal{O}_\xi)\subseteq \bigcup_{1\leqq\xi<\alpha_0+\omega^\varkappa\cdot 2} \mathcal{O}_\xi$. Clearly $\mathscr{C}_\varkappa(\min \bigcup_{\alpha_0+\omega^\varkappa\leqq\xi<\alpha_0+\omega^\varkappa\cdot 2} \mathcal{O}_\xi)=$ $=\mathscr{C}_\varkappa(\beta)$. Iterating this argument, we obtain that $\bigcup_{1\leqq\xi<\alpha_0+\omega^\varkappa\cdot n} \mathcal{O}_\xi$ contains n distinct $\omega^\varkappa$-classes. Now, if none of these n classes contains $d(e_{k_1})$, then all of them are in between $\mathscr{C}_\varkappa(\beta)$ and $\mathscr{C}_\varkappa(d(e_{k_1}))$ in the natural ordering of the $\omega^\varkappa$-classes. As there are only finitely many $\omega^\varkappa$-classes between $\mathscr{C}_\varkappa(\beta)$ and $\mathscr{C}_\varkappa(d(e_{k_1}))$, we reach $\mathscr{C}_\varkappa(d(e_{k_1}))$ after finitely many repetitions, whence we obtain that there exists a natural number n_{k_1} and an ordinal α_{k_1} such that $d(e_{k_1})\in\mathcal{O}_{\alpha_{k_1}}$ and $\alpha_{k_1}\leqq\alpha_0+\omega^\varkappa\cdot n_{k_1}$. Now, either $\beta'\in\bigcup_{1\leqq\xi<\alpha_{k_1}}\mathcal{O}_\xi$, which finishes the proof, or there exists a k_2 such that $e_{k_2}\cap V_{\alpha_{k_1}}\neq\emptyset$, but, for $i>k_2$, $e_i\cap V_{\alpha_{k_1}}=\emptyset$. Similarly as above, we obtain that there exist an ordinal α_{k_2} and natural number n_{k_2} with $\alpha_{k_2}\leqq\alpha_{k_1}+\omega^\varkappa\cdot n_{k_2}$ such that $d(e_{k_2})\in\mathcal{O}_{\alpha_{k_2}}$. Continuing this procedure, we obtain that there exist natural numbers $s, k_1, k_2, \dots, k_s, n_{k_1}, n_{k_2}, \dots, n_{k_s}$, and ordinals $\alpha_{k_1}, \alpha_{k_2}, \dots, \alpha_{k_s}$, such that $\beta'\in\mathcal{O}_{\alpha_{k_s}}$ and $\alpha_{k_1}\leqq\alpha_0+\omega^\varkappa\cdot n_{k_1}$, $\alpha_{k_2}\leqq\alpha_{k_1}+\omega^\varkappa\cdot n_{k_2}, \dots, \alpha_{k_s}\leqq\alpha_{k_{s-1}}+\omega^\varkappa\cdot n_{k_s}$ (whence $\alpha_{k_s}\leqq\alpha+\omega^\varkappa(n_0+n_{k_1}+\dots+n_{k_s})$), or else $\beta'\in\bigcup_{1\leqq\xi<\alpha_{k_i}}\mathcal{O}_\xi$ for some $i\leqq s-1$. In both cases $\beta'\in\bigcup_{1\leqq\xi<\alpha+\omega^\gamma}\mathcal{O}_\xi$ as claimed.

If γ is a limit-number, then the statement is obvious from the induction hypothesis. Now let $\{\beta\}_\gamma$ denote the remainder of the Euclidean division of β by ω^γ, that is, $\{\beta\}_\gamma=-[\beta]_\gamma+\beta$. Since $0\notin\mathcal{O}$, all elements of $\mathcal{O}$ can be represented in the form $1+\beta$.

LEMMA 3. *Let* $1+\beta\in\mathcal{O}_\alpha$. *Assume that*

$$\{1+\eta|0\leqq\eta<[\beta]_\gamma\}\cap\mathcal{O}_\xi=\emptyset$$

for all $\xi\in[\alpha,\alpha+\{\beta\}_\gamma+1)$. *Then* $1+\beta\notin\mathcal{O}_{\alpha+\{\beta\}_\gamma+1}$.

COROLLARY. *Let* $1+\beta\in\mathcal{O}_\alpha$. *Then* $1+\beta\notin\mathcal{O}_{\alpha+\beta+1}$.

Before proving Lemma 3 we have to mention another well-known fact about ordinals, namely, that any ordinal can be written in the normal form

$$\omega^{\gamma_1}\cdot n_1+\omega^{\gamma_2}\cdot n_2+\dots+\omega^{\gamma_k}\cdot n_k \tag{2}$$

where $k, n_1, n_2, \dots, n_k$ are natural numbers, $n_1, n_2, \dots, n_k\neq0$, and $\gamma_1, \gamma_2, \dots, \gamma_k$ are ordinals such that $\gamma_1>\gamma_2>\dots>\gamma_k$. This is an easy consequence of that Euclidean division can be extended to ordinals (see e.g. [1]).

PROOF of Lemma 3. For $\beta=0$, the Lemma is obvious.

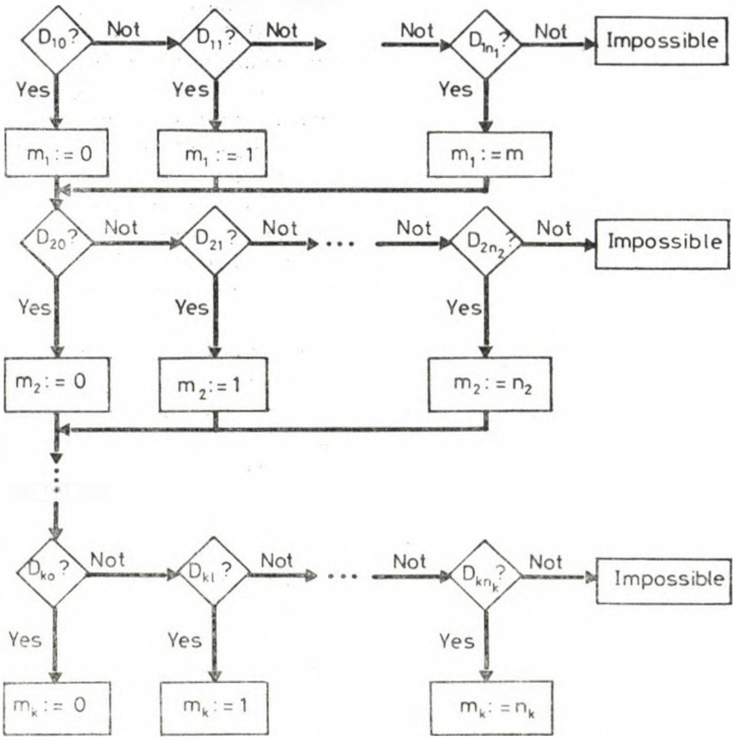
Let $\beta>0$, let γ be arbitrary, and let $\{\beta\}_\gamma$ have the normal form (2). Assume that Lemma 3 is valid for all $\beta'<\beta$. We now prove it for β. We may assume that $k\neq0$, otherwise the Lemma is trivial. Define m_i inductively, by the following scheme, where D_{ij} is the statement

$$\min(\cup\mathcal{O}_\xi|\,\alpha+\omega^{\gamma_1}\cdot m_1+\dots+\omega^{\gamma_{i-1}}\cdot m_{i-1}+\omega^{\gamma_i}\cdot j\leqq\xi<$$

$$<\alpha+\omega^{\gamma_1}\cdot m_1+\dots+\omega^{\gamma_{i-1}}\cdot m_{i-1}+\omega^{\gamma_i}(j+1))\geqq1+[\beta]_\gamma+\omega^{\gamma_1}\cdot n_1+\dots+\omega^{\gamma_i}\cdot n_i.$$

First of all we show that

$(*)$ $D_{i0}, D_{i1}, \dots, D_{in_i}$ *cannot be false at a time,*



as indicated on the diagram. This shows that the definition of the m_i's is correct. Suppose D_{10} is not true, that is,

$$\min(\cup \mathcal{O}_\xi | \alpha \leqq \xi < \alpha + \omega^{\gamma_1}) < 1 + [\beta]_\gamma + \omega^{\gamma_1} \cdot n_1 .$$

Then every element $\neq 0$ of the ω^{γ_1}-class of $\min(\cup \mathcal{O}_\xi | \alpha \leqq \xi < \alpha + \omega^{\gamma_1})$ is contained in some of the $\mathcal{O}_\xi$'s for $1 \leqq \xi < \alpha + \omega^{\gamma_1}$ by Lemma 2, that is, if $1 + \beta' \in \mathscr{C}_{\gamma_1}(\min(\cup \mathcal{O}_\xi | \alpha \leqq \leqq \xi < \alpha + \omega^{\gamma_1}))$, and ξ_0 is the smallest of those ordinals ξ satisfying $1 + \beta' \in \mathcal{O}_\xi$, then $\xi_0 < \alpha + \omega^{\gamma_1}$. Now we have

$$1 + \beta' < 1 + [\beta]_\gamma + \omega^{\gamma_1} \cdot n_1 \leqq 1 + [\beta]_\gamma + \{\beta\}_\gamma = 1 + \beta$$

thus the induction hypothesis can be applied for β' (with γ_1 in the place of γ). We obtain that $1 + \beta' \notin \mathcal{O}_{\xi_0 + \{\beta'\}_{\gamma_1} + 1}$. Hence $1 + \beta' \notin \mathcal{O}_{\xi_0 + \eta}$ for all $\eta \geqq \{\beta'\}_{\gamma_1} + 1$. Now

$$\xi_0 + \{\beta'\}_{\gamma_1} + 1 \leqq \xi_0 + \omega^{\gamma_1} \leqq \alpha + \omega^{\gamma_1}$$

(the latter inequality follows from Lemma 1). Thus $1 + \beta' \notin \mathcal{O}_{\alpha + \eta}$ for all $\eta \geqq \omega^{\gamma_1}$. That is, $\mathscr{C}_{\gamma_1}(\min(\cup \mathcal{O}_\xi | \alpha \leqq \xi < \alpha + \omega^{\gamma_1})) \cap \mathcal{O}_{\alpha + \eta} = \emptyset$ for all $\eta \geqq \omega^{\gamma_1}$. Therefore, $\min(\cup \mathcal{O}_\xi | \alpha + \omega^{\gamma_1} \leqq \xi < \alpha + \omega^{\gamma_1} 2))$ is in a different ω^{γ_1}-class, that is $\mathscr{C}_{\gamma_1}(\min \cup \mathcal{O}_\xi | \alpha \leqq \xi < \alpha + \omega^{\gamma_1})) \neq \mathscr{C}_{\gamma_1}(\min(\cup \mathcal{O}_\xi | \alpha + \omega^{\gamma_1} \leqq \xi < \alpha + \omega^{\gamma_1} \cdot 2))$. Similarly, we obtain, in general, that the ω^{γ_1}-classes of the elements $\min(\cup\mathcal{O}_\xi|\alpha+\omega^{\gamma_1}\cdot j\leqq\xi<\alpha+\omega^{\gamma_1}\cdot(j+1))$, $j=0, 1, \ldots, n_1$ are distinct, and, if all of D_{1j}, $j=0, 1, \ldots, n_1$ are false, then all the elements of these ω^{γ_1}-classes are majorated (in the strict sense) by $1+[\beta]_\gamma+\omega^{\gamma_1}\cdot n_1$. Since we assumed that $[1, 1+[\beta]_\gamma)\cap\mathcal{O}_\xi=\emptyset$ for all $\xi\in[\alpha, \alpha+\{\beta\}_\gamma+1)$, these ω^{γ_1}-classes are also minorated by $1+[\beta]_\gamma$. But this is a contradiction, for there are only n_1 distinct ω^{γ_1}-classes in the interval $[1+[\beta]_\gamma, 1+[\beta]_\gamma+\omega^{\gamma_1}\cdot n_1)$. Thus we proved the statement $(*)$ in the case $i=0$. The case $i=1$ differs in that, at the end of the proof, where we utilized that the ω^{γ_1}-classes in question are minorated by $1+[\beta]_\gamma$ (which was one of the assumption of Lemma 3), we now have to apply that the corresponding ω^{γ_2}-classes occurring in the case $i=1$ are minorated by $1+[\beta]_\gamma+\omega^{\gamma_1}\cdot n_1$, which is exactly the assertion of D_{1m_1}. The general case is now analogous, completing the proof of the statement $(*)$.

Now let $1+\mu=\min\mathcal{O}_{\alpha+\omega^{\gamma_1}m_1+\ldots+\omega^{\gamma_k}\cdot m_k}$. By D_{km_k}, $1+\mu\geqq 1+[\beta]_\gamma+\omega^{\gamma_1}\cdot n_1+\ldots$ $\ldots+\omega^{\gamma_k}\cdot n_k=1+\beta$ (the reverse inequality is also true, but it is of no relevance here). Thus $1+\beta=\min\mathcal{O}_\xi$ for some $\xi\leqq\alpha+\omega^{\gamma_1}\cdot m_1+\ldots+\omega^{\gamma_k}\cdot m_k\leqq\alpha+\omega^{\gamma_1}\cdot n_1+\ldots+\omega^{\gamma_k}\cdot n_k=$ $=\alpha+\{\beta\}_\gamma$. Thus $1+\beta$ has been chosen as λ_ξ for some $\xi\leqq\alpha+\{\beta\}_\gamma$, whence $1+\beta\notin\mathcal{O}_{\alpha+\{\beta\}_\gamma+1}$, completing the proof of Lemma 3. To get the Corollary just choose γ such that $\beta<\omega^\gamma$ and apply the Lemma.

Finally, we return to the proof of Theorem 1. Let $h(x)<h(y)$, $h(x)=\alpha$, $d(xy)=$ $=1+\beta$. It follows that $x\in V_{\alpha+1}$, $y\notin V_{\alpha+1}$. Hence $1+\beta=d(xy)\in\mathcal{O}_{\alpha+1}$. By the Corollary of Lemma 3, $1+\beta\notin\mathcal{O}_{(\alpha+1)+\beta+1}$, that is $1+\beta=\min\mathcal{O}_\xi$ for some $\xi\leqq(\alpha+1)+\beta$. Hence $h(y)=\xi\leqq(\alpha+1)+\beta=\alpha+(1+\beta)=h(x)+d(xy)$, equivalently, $-h(x)+h(y)\leqq$ $\leqq d(xy)$. This finishes the proof of Theorem 1.

Now we show that the present proof need not work for arbitrary one-to-one mapping d. Some another proof, however, still might provide the result without imposing the connectivity condition of Theorem 1. Consider the graph on the points $a_0, a_1, a_2, \ldots, a_\omega, a_{\omega+1}$, with the edges a_0a_α for $\alpha\neq 1$, $0\leqq\alpha<\omega+2$ and $a_\omega a_1$. Let $d(a_0a_\alpha)=\alpha$ whenever a_0a_α is an edge and let $d(a_\omega a_1)=1$. Set $h(a_0)=0$. Then the above definition of h yields $h(a_2)=1$, $h(a_3)=2, \ldots, h(a_\omega)=\omega$, $h(a_1)=\omega+1$ and $h(a_{\omega+1})=\omega+2$. Now $-h(a_0)+h(a_{\omega+1})=\omega+2>\omega+1=d(a_0a_{\omega+1})$, that is, the assertion of Theorem 1 does not hold for this mapping h. Unfortunately this is no counter-example to the assertion itself, as the mapping h_1 defined by $h_1(a_0)=0$, $h_1(a_{\omega+1})=1$, $h_1(a_\alpha)=\alpha$ for $2\leqq\alpha<\omega+1$, and $h_1(a_1)=\omega+1$ obviously satisfies $-h_1(x)+h_1(y)\leqq d(xy)$ as well as the assumption $h_1(a_0)=0$.

4. A generalization of Theorem 1

The following theorem gives also an answer to the second question posed in the Introduction.

THEOREM 2. *Let $G=(V, E)$ be a graph, let $\mathcal{O}$ be a set of ordinals, and let $d\colon E\to\mathcal{O}\setminus\{0\}$ be a mapping satisfying the following conditions.*

(i) *For any ordinal γ, and for any edges $a, b\in E$ with $d(a)\sim_\gamma d(b)$, there exists a path $a=c_1, c_2, \ldots, c_n=b$ such that $d(c_i)<_\gamma d(a)$ for all i.*

(ii) *For any ordinal γ, the subgraph of G, spanned by all those edges e of G with $d(e)=\gamma$, has at most $n+1$ points.*

Then there exists a mapping $h: V \to \mathcal{O}'$ *to a set* $\mathcal{O}'$ *of ordinals such that, for any edge* $xy \in E$ *with* $h(x) < h(y)$

$$-h(x)+h(y) \leqq d(xy) \tag{3}$$

holds, and, for all $\gamma \in \mathcal{O}'$ *the set of all* $x \in V$ *with* $h(x)=\gamma$ *has at most n points.*

PROOF (sketch). The proof is analogous with that of Theorem 1. Therefore, we do not go into details, we only define the mapping h, but do not prove the inequality $-h(x)+h(y) \leqq d(xy)$, except for the case of finite graphs. This special case is discussed in order to show how the condition (ii) is applied in the proof.

We first define a mapping h_0 as follows. Choose $x \in V$ with $h_0(x)=0$ arbitrarily. Now, by induction, let $\beta > 0$ and assume that, for all $\alpha (<\beta)$, we have chosen a vertex z with $h_0(z)=\alpha$. Let V_β be the set of all these vertices z. Choose an edge $x_0y_0 \in E$ with $x_0 \in V_\beta$, $y_0 \notin V_\beta$ such that

$$d(x_0y_0) = \min_{\substack{x \in V_\beta \\ y \notin V_\beta \\ xy \in E}} d(xy), \tag{4}$$

and define

$$h_0(y_0) = \beta.$$

Finally, let $h(x)$, $x \in V$ be the quotient of the Euclidean division of $h_0(x)$ by n.

Now we restrict our considerations to the finite case. We first show that, for any vertices x, y,

$$h_0(y)-h_0(x) \leqq n \cdot d(xy). \tag{5}$$

We may assume that $h_0(y) > h_0(x)$, that is $h_0(x)$ is defined by the above procedure earlier than $h_0(y)$. Let $d(xy)=k$. We have to show that $h(y)$ is defined not later than in the nk-th step after defining $h(x)$ (defining $h(x)$ is the 0-th step). Assume that this statement is not true, that is, after defining $h(x)$, in nk consecutive steps y is never chosen as the next element. Thus in this nk steps and in at least one more step $k=d(xy)$ is among the numbers for which the minimum (4) is formed. Hence the value of (4) is $\leqq k$ in $nk+1$ steps. Therefore one of the numbers $1, 2, \ldots, k$ must occur as the minimum (4) in at least $n+1$ steps, which is impossible by (ii). Thus (5) is proved, and (3) is a weakened form of (5).

It is worth mentioning that this theorem has a consequence in connection with the original number theoretic problem of [2]. Analogously to the main result of [2], it follows that, whenever $m_1, m_2, \ldots, m_r$ are natural numbers such that the pairwise greatest common divisors $d_{ij}=(m_i, m_j)$, $i \neq j$ are different from 1, then in order that there exist $a_1, a_2, \ldots, a_r$ such that for any n-element subset K of $\{1, 2, \ldots, r\}$

$$\bigcap_{i \in K} \{x \mid x \equiv a_i \pmod{m_i}\} = \emptyset$$

holds, it is sufficient that for any natural number a and for any subset $L \subseteq \{1, 2, \ldots, r\}$ satisfying

$$\forall i, j \in L\colon (m_i, m_j) = a$$

we have $|L| \leqq n+1$. We omit the proof. For a motivation of these number theoretic question the reader is referred to [2].

5. Open problems

PROBLEM 1. Is there any numbering of the edges for which the property formulated in Theorem 1 is not valid? Characterize these numberings.

We think, the occurrance of graphs in Theorem 1 and in Problem 1 is incidental and there is some more fundamental property of ordinal numbers lying behind Theorem 1 and Problem 1. Our next problem concerns this property.

Imagine the following game played by coins which are marked with distinct non-zero ordinal numbers. The game is played between us and a slot machine in transfinite time, such that, for any ordinal number α, in the α-th moment a pair of moves is made. Before the game all the coins are in the slot machine. The slot machine opens the game by throwing out a set of coins. Then, provided the set of coins at our disposal is not empty, we have to choose one of these coins and insert it in the slot machine. In reply, in the next move, the machine throws out another set of coins, whereafter we have to choose a coin from the union of this set and the set of coins remained after our previous move, etc. In every move, the coin inserted is swallowed by the slot machine once and for all and will never again occur in the set of coins thrown out by the machine. Our aim is to get rid of every coin in not more steps than the number on the coin.

PROBLEM 2. Can we win the game?

In this formulation the slot machine takes the place of the graph of Theorem 1. It should be the subject of a further study how exactly this problem is related to Problem 1. The present formulation even admits some ambiguity. To mention two of them, the answer might be different according to whether the moves of the slot machine only depend upon time or they also depend upon the coin just inserted and according to whether we know the function describing this dependence or not.

In case there are only finitely many coins (then we may assume that the game is played in finite time), the answer is much easier: we can always win the game, no matter how the machine works. One winning strategy is the following (this is the strategy that our present proof of Theorem 1 relies upon): In each move we get rid of the coin wearing the smallest number, no matter when we received it. Another winning strategy is always to insert the coin which is the most urgent to get rid of, that is, for which the sum of the number on the coin and the time when we received it is minimum. (It is a good exercise to prove that this strategy also works.)

PROBLEM 3. Find the general winning strategy in the finite case. Find at least a common generalization of the two strategies described above.

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UNIQUE FACTORIZATION IN QUADRATIC NUMBER FIELDS

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Abstract

Generalizing a method of Edit Gyarmati we give a necessary and sufficient condition of new type, which provides, for any given quadratic field (imaginary or real), the possibility to decide in a bounded number of steps whether the unique factorization is valid or not in the ring of the integers of the field.

Introduction

We denote by $I(d)$ the ring of algebraic integers of the extension of the rational field by $\sqrt{d}$, where d is an arbitrary squarefree rational integer.

For the imaginary case Heilbronn and Linfoot [3], [4] proved that there are at most ten $d<0$, for which $I(d)$ is a unique factorization domain (UFD).

Their proof is not elementary using ideal-theory. Nine of these values were known, namely

$$d = -1, -2, -3, -7, -11, -19, -43, -67, -163. \tag{1}$$

The problem of the existence of a tenth value resisted many efforts, till finally Stark [6] proved that no such tenth d exists.

For the real case, it is still undecided whether there are finitely or infinitely many $d>0$, for which $I(d)$ is UFD, hence the determination of all such values of d seems to be much more complicated. For detailed references see [11] p. 404.

Edit Gyarmati [1] gave an elementary proof of the fact that $I(d)$ is a UFD for the values enumerated in (1) (see also [2]). Her proof has also some interesting theoretical consequences, which may give a base for an eventual elementary proof of Stark's result (we also note that her paper appeared earlier than Stark obtained his result).

In our paper we shall generalize her method for both the imaginary and the real case and give necessary and sufficient condition of new type for the property of $I(d)$ to be a UFD. It is enough to check finitely many irreducible elements of $I(d)$, whether they are also primes in $I(d)$. Some related problems are discussed in [7] and [8].

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Notation and preliminaries

We shall denote the elements of $I(d)$ by small Greek letters, the rational integers by Latin letters. It is well-known that the elements of $I(d)$ can be represented in the form:

$$\alpha = a + b\vartheta$$

where

$$\vartheta = \begin{cases} \sqrt{d} & \text{if} \quad d \equiv 2 \text{ or } 3 \pmod 4, \\ \dfrac{1+\sqrt{d}}{2} & \text{if} \quad d \equiv 1 \pmod 4). \end{cases}$$

For α we put

$$\bar{\alpha} = \begin{cases} a - b\vartheta = a - b\sqrt{d} & \text{if} \quad d \equiv 2 \text{ or } 3 \pmod 4, \\ a + b\dfrac{1-\sqrt{d}}{2} = a + b - b\vartheta & \text{if} \quad d \equiv 1 \pmod 4, \end{cases}$$

and

$$N(\alpha) = |\alpha\bar{\alpha}| = \begin{cases} |a^2 - db^2| & \text{if} \quad d \equiv 2 \text{ or } 3 \pmod 4, \\ \left| a^2 + ab - \dfrac{d-1}{4} b^2 \right| & \text{if} \quad d \equiv 1 \pmod 4). \end{cases}$$

Let us recall that $\alpha \in I(d)$ different from 0 and units are called irreducible if $\alpha = \beta\gamma$ implies that either β or γ is a unit, and it is called prime if $\alpha | \beta\gamma$ implies $\alpha | \beta$ or $\alpha | \gamma$ (or both).

ε will always denote a unit, π, π_i and ϱ irreducible elements and p, p_i, q, q_i rational primes.

The main result

THEOREM. *Let T be as follows:*

$$T = \begin{cases} \dfrac{2}{\sqrt{3}}\sqrt{-d} & \text{if} \quad d < 0,\ d \equiv 2 \text{ or } 3 \pmod 4), \\ \dfrac{1}{\sqrt{3}}\sqrt{-d} & \text{if} \quad d < 0,\ d \equiv 1 \pmod 4), \\ \dfrac{2}{\sqrt{5}}\sqrt{d} & \text{if} \quad d > 0,\ d \equiv 2 \text{ or } 3 \pmod 4), \\ \dfrac{1}{\sqrt{5}}\sqrt{d} & \text{if} \quad d > 0,\ d \equiv 1 \pmod 4). \end{cases} \tag{2}$$

Then $I(d)$ is a UFD if and only if the following condition holds:

(3) *each p with $0 < p < T$ which is irreducible in $I(d)$, is also a prime in $I(d)$.*

PROOF. Trivially $I(d)$ is a UFD if and only if all irreducible elements are prime. This immediately shows that condition (3) is necessary.

To prove the converse we need some lemmata.

LEMMA 1. *If every rational prime is a prime in $I(d)$ or a product of $I(d)$-primes then $I(d)$ is a UFD.*

PROOF. The norm of an α different from 0 and from units has a factorization

$$N(\alpha) = p_1 p_2 \cdots p_n.$$

By assumption each p_i is prime or a product of primes in $I(d)$. This implies that the factorization of

$$N(\alpha) = |\alpha\bar{\alpha}| = \varepsilon\alpha\bar{\alpha} \qquad (\varepsilon = +1 \text{ or } -1)$$

is unique, hence the factorization of α is unique as well.

LEMMA 2. *If $p|N(\alpha)$ implies $p|\alpha$, then p is prime in $I(d)$.*

PROOF. By

$$p|\alpha\beta, \quad p^2|N(\alpha)\,N(\beta)$$

we have

$$p|N(\alpha) \quad \text{or} \quad p|N(\beta),$$

and hence by assumption of the lemma we obtain

$$p|\alpha \quad \text{or} \quad p|\beta.$$

The following lemma plays a key role in the proof of the theorem.

LEMMA 3. *If p is not a prime in $I(d)$ and T is the constant defined by* (2), *then there exists a $\beta \neq 0$ with*

$$N(\beta) = pn, \quad n \leqq T.$$

PROOF. For a rational prime p which is not a prime in $I(d)$ we can find by Lemma 2 an $\alpha \in I(d)$ for which

$$p|N(\alpha), \quad \text{but} \quad p \nmid \alpha.$$

(i) Let first be

$$d \equiv 2 \quad \text{or} \quad 3 \pmod 4.$$

Then d is, by the definition of the norm a quadratic residue (mod p), i.e.

$$z^2 - d = pn$$

for some z and n. Let us form the following lattice:

$$x = pu + zv,$$

$$y = v,$$

where u and v run through the rational integers independently. The area of the fundamental parallelogram is

$$\Delta = \begin{vmatrix} p & 1 \\ 0 & 1 \end{vmatrix} = p.$$

For any lattice point $(x; y)$ we have

$$x^2 - dy^2 = p^2u^2 + 2puzv + z^2v^2 - dv^2 = p(pu^2 + 2uzv) + v^2(z^2 - d),$$

hence

$$p \mid x^2 - dy^2.$$

In this case we have

$$p \mid z^2 - d.$$

(ii). If

$$d \equiv 1 \pmod 4,$$

and if 2 is prime or product of primes in $I(d)$ then it is trivial that

$$p \mid N(\alpha) \quad \text{but} \quad p \nmid \alpha$$

if and only if

$$p \mid N(2\alpha) \quad \text{but} \quad p \nmid 2\alpha.$$

Since

$$N(2\alpha) = 4x^2 + 4xy - 4\frac{d-1}{4}y^2 = 4x^2 + 4xy - dy^2 + y^2 = (2x+y)^2 - dy^2,$$

d must be a quadratic residue (mod p) i.e.

$$z_0^2 - d = pn$$

for some z_0 and n. Trivially there will be

$$(z_0 - p)^2 - d = pn_1$$

for some n_1, too, and since z_0 or $z_0 - p$ is odd (because $p \neq 2$), hence

$$z_0 = 2z + 1 \quad \text{or} \quad z_0 - p = 2z + 1$$

for some z, where z is a rational integer. Let us form now the following lattice:

$$x = pu - \frac{1}{2}pv + \frac{1}{2}zv,$$

$$y = \frac{1}{2}v,$$

where u, v run through the rational integers independently. The area of the fundamental parallelogram of this lattice is

$$\Delta' = \begin{vmatrix} p & -\frac{1}{2}p + \frac{1}{2}z \\ 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2}p = \frac{1}{2}\Delta.$$

For any lattice point $(x; y)$ we have:

$$(2x+y)^2-dy^2 = 4p^2u^2+p^2v^2-4p^2uv+4puvz+$$
$$+2puv-2pv^2z-pv^2+v^2z^2+v^2z+1/4v^2-(d/4)v^2 =$$
$$= pA+v^2\left(z^2+z-\frac{d-1}{4}\right),$$

and since

$$p\,|(2z+1)^2-d = 4z^2+4z+1-d = 4\left(z^2+z-\frac{d-1}{4}\right),$$

hence

$$p|(2x+y)^2-dy^2.$$

(iii) We want to find a lattice point $(x_0; y_0)$ for which

$$|x_0^2-dy_0^2| = c_0$$

is small but not 0.

(iii/a) Let first be $d<0$. Then

$$x^2-dy^2 = c \quad (c \text{ is a positive constant})$$

is a circle in the coordinates

$$\xi = x, \quad \eta = \sqrt{-d}\, y.$$

The points (ξ, η) for integers u, v form a lattice in the (ξ, η) plane in which the area Δ^* of the fundamental parallelograms is $\sqrt{-d}\,\Delta$ or $\sqrt{-d}\Delta'$, respectively. It is well-known (see e.g. [9] or [10]) that the circle

$$\xi^2+\eta^2 \leqq \frac{2}{\sqrt{3}}\Delta^*$$

always contains a lattice point (ξ_0, η_0) different from the origin. So we can choose $c_0 = \frac{2}{\sqrt{3}}\Delta^*$ which means that there is a β in $I(d)$ with

$$N(\beta) = x_0^2-dy_0^2 = pn \leqq \frac{2}{\sqrt{3}}p\sqrt{-d} \text{ or } \leqq \frac{2}{\sqrt{3}}\cdot\frac{1}{2}p\sqrt{-d},$$

i.e.

$$n \leqq T = \begin{cases} 2\sqrt{\dfrac{-d}{3}} & \text{if } d \equiv 2 \text{ or } 3 \pmod 4 \\ \sqrt{\dfrac{-d}{3}} & \text{if } d \equiv 1 \pmod 4). \end{cases}$$

(iii/b) For $d>0$

$$x^2-dy^2 = \pm c \quad (c \text{ is a positive constant})$$

is a pair of hyperbolas which can be written in the form

$$(x+\sqrt{d}y)(x-\sqrt{d}y) = \pm c.$$

The points

$$\xi = x+\sqrt{d}y, \quad \eta = x-\sqrt{d}y$$

in the (ξ, η)-plane form a lattice. The area Δ^* of the fundamental parallelograms can be calculated easily; it is $2\sqrt{d}\Delta$ or $2\sqrt{d}\Delta'$, respectively.

It is well-known (see again [9] or [10] e.g.) that there is always a lattice point (ξ_0, η_0) different from the origin for which

$$|\xi_0\eta_0| \leqq \frac{\Delta^*}{\sqrt{5}}.$$

This means that there is a β in $I(d)$ with

$$N(\beta) = |x_0^2-dy_0^2| = pn \leqq \frac{2p\sqrt{d}}{\sqrt{5}} \text{ or } \leqq \frac{p\sqrt{d}}{\sqrt{5}},$$

i.e.

$$n \leqq T = \begin{cases} 2\sqrt{\dfrac{d}{5}} & \text{if} \quad d \equiv 2 \text{ or } 3 \pmod 4) \\ \sqrt{\dfrac{d}{5}} & \text{if} \quad d \equiv 1 \pmod 4). \end{cases}$$

LEMMA 4. *$N(\alpha)=p$ implies that α is prime in $I(d)$.*

PROOF. We note that the irreducibility of α is trivial, the stress lies on the prime property. Lemma 4 can be easily proved by ideal theory. An elementary but complicated proof was given in [5]. Now we give a simple elementary proof of the statement.

It is well-known that $0, 1, 2, \ldots, p-1$ form a complete residue system (mod α) if $N(\alpha)=p$.

Thus we can find to any γ, δ an a and b such that

$$0 \leqq a, b \leqq p-1$$

and

$$\gamma = a \pmod{\alpha},$$

$$\delta = b \pmod{\alpha}.$$

Now if $\alpha|\gamma\delta$ then $\alpha|ab$ which implies

$$N(\alpha) = p|a^2 b^2$$

i.e.

$$p|a \quad \text{or} \quad p|b.$$

Since this is possible only if $a=0$ or $b=0$, we have

$$\alpha|\gamma \quad \text{or} \quad \alpha|\delta,$$

respectively, which shows that α is really a prime.

LEMMA 5. *Condition* (3) *implies that any* $0<p\leq T$ *is a prime or a product of primes in* $I(d)$.

PROOF. *If p is irreducible then it is prime by assumption. If p is reducible then only*

$$p = \varepsilon\pi\bar{\pi}$$

is possible, i.e.

$$N(\pi) = N(\bar{\pi}) = p,$$

but then, by Lemma 4, π *and* $\bar{\pi}$ *are primes.*

LEMMA 6. *If* $N(\gamma)=pt$, *where t is prime or a product of primes in* $I(d)$ *then also p is prime or a product of primes.*

PROOF. Let

$$t = \pi_1\pi_2\ldots q_1 q_2\ldots,$$

where π_i and q_j are primes in $I(d)$. Then

$$\gamma\bar{\gamma} = p\pi_1\bar{\pi}_1\pi_2\bar{\pi}_2\ldots q_1 q_2\ldots.$$

Since π_1 is prime,

$$\pi_1|\gamma \quad \text{or} \quad \pi_1|\bar{\gamma},$$

and hence

$$\bar{\pi}_1|\bar{\gamma} \quad \text{or} \quad \bar{\pi}_1|\gamma.$$

In any case we can reduce both sides by $\pi_1\bar{\pi}_1$.

We obtain similarly that e.g.

$$q_1|\gamma \quad \text{and} \quad q_1|\bar{\gamma},$$

hence either $p=q_1$ or we can reduce both sides by q_1^2.

Thus we have either $p=q_i$ for some i, which means that p is prime, or finally $\varepsilon\gamma'\bar{\gamma}'=p$, which shows, by Lemma 4, that p is a product of primes.

Proof of the theorem

Now we are ready to prove our theorem. By Lemma 1 it is enough to verify that the rational primes are primes or product of primes in $I(d)$.

If p is not prime (and $p=2$ if $d\equiv 1 \pmod 4$), then by the Lemma 3 there will exist a $\beta\in I(d)$, $\beta\neq 0$ with

$$N(\beta) = pn, \quad n\leq T.$$

For $d<-12$ or $d>20$ and $d\equiv 1 \pmod 4$ $2>T$, because we have to test only $p=2$, otherwise it is easy to show that 2 is prime or product of primes in $I(d)$ especially for

$$d = -11, -7, 3, 5, 13, 17$$

$I(d)$ is UFD. Trivially, T is irrational if $d \neq -3$ or 5, hence we can write

$$n < T \quad \text{instead of} \quad n \leqq T.$$

Factoring n among the rational integers and using Lemma 5 we obtain that n is prime or a product of primes. Finally, by Lemma 6, we conclude that p is prime or a product of primes.

Some applications

At present we do not see, though hope, that our theorem might give some information about the number of real quadratic fields which are UFD.

As an illustration we prove that $I(d)$ is a UFD for $d=-67$ and $d=93$.

We shall use the well-known fact, that an odd p is prime if and only if the $\left(\frac{d}{p}\right)$ Legendre-symbol is -1, and 2 is prime if and only if $d \equiv 5 \pmod 8$.

For $d=-67$

$$T = \sqrt{\frac{67}{3}} = \sqrt{22{,}3} < 5,$$

hence we have to check only $p=2$ and 3. Since

$$-67 \equiv 5 \pmod 8$$

and

$$\left(\frac{-67}{3}\right) = -1,$$

these numbers are primes in $I(-67)$, which verifies that $I(-67)$ is a UFD.

For $d=93$

$$T = \sqrt{\frac{93}{5}} = \sqrt{18{,}6} < 5,$$

hence we have to check again $p=2$ and 3. 2 is prime, while 3 is reducible, but

$$N(4+\vartheta) = |-3| = 3,$$

thus $I(93)$ is a UFD.

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MARKOVIAN MODELS OF URBAN TRAFFIC. AN APPLICATION OF THE FEYNMAN-KAC FORMULA

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Abstract

If the time-parameter of a second order stationary process is replaced by the integral of a real function of a stationary Markov process, then the spectral density of the resulting process is given by an integral transform, see [2]. The kernel of this integral operator is calculated by means of the Feynman—Kac formula for some explicitely solvable models. Asymptotics of the integral operator is investigated as the randomness of this rescaling of time diminishes. Our studies are motivated by some theoretical problems related to a probabilistic description of vibrations of vehicles in urban traffic.

0. Introduction

The following problem emerged from a mathematical study of certain vibrations of vehicles caused by the unevenness of the road, see [1, 2, 3] for a more detailed explanation. Let $U(s)$, $U\in \mathbf{R}$ denote the level of the road at a distance $s\in \mathbf{R}$ from the origin, and consider a vehicle moving with a random velocity, $V=V(t)$ at time $t\in \mathbf{R}$. Let $S(t) = \int_0^t V(u)\,du$ denote the distance covered in time t, then the road profile as seen by the driver is just the composed process $U(t)=U[S(t)]$. It is natural to assume that U is a second order stationary process of zero mean and covariance $K(s)=\mathbf{E}[U(r+s)U(r)]$, V is a nonnegative function of a stationary Markov process such that U and V are completely independent. It is easy to see that under such conditions U is again a second order stationary process of zero mean and covariance $\tilde{K}$,

$$\tilde{K}(t) = \int_0^\infty K(s)F_t(ds) \tag{0.1}$$

if $t\geq 0$, $K(-t)=K(t)$, where F_t denotes the distribution function of $S(t)$, i.e. $F_t(s)=\mathbf{P}[S(t)<s]$ for $t\geq 0$.

The empirical information we are a priori given is the spectral density, f, of U; thus f is a nonnegative even function such that

$$K(s) = \int_{-\infty}^{+\infty} e^{isx}f(x)\,dx. \tag{0.2}$$

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Therefore

$$\tilde{K}(t) = \int_{-\infty}^{+\infty} r(t, x) f(x)\, dx, \tag{0.3}$$

where $r(t, \cdot)$ is the characteristic function of F_t,

$$r(t, x) = \int_{0}^{+\infty} e^{isx} F_t(ds). \tag{0.4}$$

Suppose now that $\tilde{K}$ is integrable, then it is the Fourier transform of a symmetric real function $\tilde{f}$,

$$\tilde{f}(y) = \frac{1}{\pi} \operatorname{Re} \int_{0}^{+\infty} e^{-ity} \tilde{K}(t)\, dt, \tag{0.5}$$

consequently

$$\tilde{f}(y) = \frac{1}{\pi} \int_{-\infty}^{+\infty} R(x, y) f(x)\, dx, \tag{0.6}$$

where

$$R(x, y) = \operatorname{Re} \int_{0}^{+\infty} e^{-ity} r(t, x)\, dt = \operatorname{Re} \int_{0}^{+\infty} \int_{0}^{+\infty} \exp(isx - ity) F_t(ds)\, dt. \tag{0.7}$$

Since the equations of vibration are usually solved by methods of Fourier analysis, relevant information is contained in the spectral density of the road profile as seen from the vehicle. The distortion given by (0.6) does really depend on the structure of our velocity process, V, thus comprehensive models for the velocity process of buses in an urban traffic are of great interest. Unfortunately, precise models do not allow us to make explicite calculations. The main purpose of this paper is to explore the possibilities of a pragmatic approach to this problem.

In the next section we introduce a fairly general Markovian model of the velocity process V. The Feynman—Kac formula reduces the calculation of R to the Cauchy problem of an operator equation involving the generator of the underlying Markov process. This approach initiates a generalization of the results of [4] to a large class of models. In some regular cases we can apply Fourier transform to eliminate time from the operator equation mentioned above. The resulting equation can explicitely be solved even in some nontrivial cases. The kernel of (0.6) is analytically determined in the simplest non-trivial case, and the first approximation to the integral operator (0.6) is calculated as the variance of the velocity process V goes to zero.

1. Markovian models

We are going to investigate the following class of models. Let $\varphi = \varphi_t$, $t \geq 0$ denote a uniformly stochastically continuous Markov family in a complete separable metric space, Ω. The associated semigroup is defined as $\mathbf{P}_t g = \mathbf{P}_t g(u) = \mathbf{E}[g(\varphi_t) | \varphi_0 = u]$, $\mathbf{P}_t$ is a strongly continuous map of the Banach space of continuous and bounded complex valued functions on Ω into itself. Let $\mathbf{C}_b(\Omega)$ denote the space of such functions, and let $A: \mathbf{D}_A \to \mathbf{C}_b(\Omega)$ denote the infinitesimal generator of the transition semigroup introduced above. Stationary distributions, if any, are characterized by the stationary Kolmogorov equation:

(1.1) $$\int Ag\,d\mu = 0 \quad \text{for all} \quad g \in \mathbf{D}_A.$$

For definitions and elementary statements of this section see Chapter 10 of [5].

Suppose now that the initial distribution of φ_t is a stationary measure μ of $\mathbf{P}_t$, and consider $V(t)=c(\varphi_t)$ as the velocity process of our vehicle. Notice that the model is specified by Ω, A, μ and c; the static characterics f need not be explicitely given. Although it is not necessary everywhere, through this paper we assume that the a priori spectral density f is twice continuously differentiable with compact support.

Introduce now

(1.2) $$\omega(t, u, x) = \mathbf{E}\left[\exp\left(ix\int_0^t c(\varphi_s)\,ds\right)\middle|\,\varphi_0 = u\right],$$

the conditional characteristic function of $S(t)$, then $r=r(t, x)$ of (0.3)—(0.4) is represented as $r(t, x)=\int \omega(t, u, x)\mu(du)$. In view of the Theorem of Section 10.3 of [5] we can identify ω as the only exponentially bounded solution to the Cauchy problem $\omega(0, u, x)=1$ for

(1.3) $$\frac{\partial\omega(t, u, x)}{\partial t} = A_u\omega(t, u, x)+ix\,c(u)\omega(t, u, x);$$

the superscript u of A indicates that A is acting on ω as a function of $u\in\Omega$, while $x\in\mathbf{R}$ is fixed. Expectations like (1.2) for solving parabolic equations are usually referred to as Feynman—Kac formulae.

Of course, an equation like (1.3) cannot be solved in general. Nevertheless, the probabilistic interpretation of ω suggests one further step towards reducing the problem to an easier one. Indeed, if φ has some good mixing properties, then $S(t)= \int_0^t c(\varphi_s)ds$ is asymptotically normal, that is

$$\omega(t, u, x) \sim \exp\left(ivtx-\frac{t}{2}\sigma^2x^2\right) \quad \text{as} \quad t\to+\infty,$$

where v is the mean velocity, and $\sigma>0$ denotes the dispersion of S per unit time. Therefore we expect that ω is an integrable function of time if $x\neq 0$, thus

(1.4) $$R(x, y) = \operatorname{Re}\int w(x, y, u)\mu(du),$$

where

(1.5) $$w(x, y, u) = \int_0^{+\infty} \omega(t, u, x)e^{-ity}\,dt.$$

On the other hand, multiplying both sides of (1.3) by e^{-ity}, and integrating by parts, we obtain an equation of elliptic type for w, namely

(1.6) $$A_u w(x, y, u)+i[xc(u)-y]\,w(x, y, u)+1 = 0.$$

In the next section (1.6) will be used to calculate the kernel function in a concrete example.

To reduce (1.3) to (1.6) we need that ω is an integrable function of time. Since $r(t,x)=\int\omega(t,u,x)\mu(du)$, this question is closely related to the integrability of $\tilde{K}$.

LEMMA 1.7. *Suppose that*

(i) $$\int_0^{+\infty}[\sup_{s\geq r}|K(s)|]\,dr<+\infty,$$

and we have some $\delta>0$ *such that*

(ii) $$\int_0^{+\infty}F_t(\delta t)\,dt<+\infty,$$

then $\tilde{K}$ *is integrable.*

PROOF. Let $M(r)$ denote the integrand of (i), and observe that $|K(s)|\leq K(0)$ as $f\geq 0$. From (0.1) we get

$$|\tilde{K}(t)|\leq\int_0^{\delta t}|K(s)|\,F_t(ds)+\int_{\delta t}^{+\infty}|K(s)|\,F_t(ds)\leq K(0)F_t(\delta t)+M(\delta t),$$

which proves the statement.

Condition (i) is a little bit stronger than integrability of K. Since f'' is integrable by assumption, we have $|K(s)|\leq C/s^2$ implying (i). The second condition is related to the velocity process. Let $v=\int c(u)\mu(du)$ denote the mean velocity, assume that $v>0$. The mean value of $S(t)$ is just vt, therefore (ii) is quite natural if $\delta<v$. To ensure mixing of the underlying Markov process, we assume that it is geometrically ergodic in the following sense. There exists an $\alpha>0$ such that

(1.8) $$-\mathrm{Re}\int\bar{g}Ag\,d\mu\geq\alpha\int|g|^2\,d\mu\quad\text{if}\quad\int g\,d\mu=0,\quad g\in\mathbf{D}_A.$$

The presence of this spectral gap of size α implies not only the uniqueness of the stationary measure μ, but also the geometric ergodicity of φ_t in $\mathrm{L}^2(\Omega,\mu)$. If Ω is a finite set, then (1.8) is equivalent to saying that there is only one stationary measure.

Now we are in a position to formulate a general statement, the main result of this section.

PROPOSITION 1.9. *Suppose* (1.8) *and* $v>0$. *If f is twice continuously differentiable with compact support then $\tilde{K}$ is an integrable function given by* (0.3), *where* $r(t,x)=\int\omega(t,u,x)\mu(du)$, *and ω is the only exponentially bounded solution to* (1.3) *with initial condition* $\omega(0,u,x)=1$.

PROOF. It is sufficient to verify the second condition of Lemma 1. Since $v>0$, (ii) follows from a bound $\mathrm{E}[S(T)-vT]^4\leq KT$ by the Markov inequality. Let $b=b(u)=c(u)-v$, and observe that

(1.10) $$\mathrm{E}[S(T)-vT]^2=\frac{1}{24}\iiiint_W E(t_1,t_2,t_3,t_4)\,dt_1\,dt_2\,dt_3\,dt_4,$$

where W is a four-dimensional simplex defined by $0<t_1<t_2<t_3<t_4<T$, while

$$E(t_1, t_2, t_3, t_4) = \mathbf{E}\left[\prod_{k=1}^{4} b(\varphi_{t_k})\right] = \int d\mu\, b\mathbf{P}_{t_2-t_1}[b\mathbf{P}_{t_3-t_2}(b\mathbf{P}_{t_4-t_3}b)].$$

Now we are in a position to exploit the geometric ergodicity of φ_t. Let $g\in \mathbf{C}_b(\Omega)$, then $\mathbf{P}_t g\in \mathbf{D}_A$ satisfies $\partial z/\partial t = Az$ with $z(0,u)=g(u)$. It is plain that $\mathbf{P}_t$ is a contraction semigroup also in $\mathbf{L}^2(\Omega,\mu)$, and its generator is an extension of A, therefore (1.8) implies

$$\int |\mathbf{P}_t g|^2\, d\mu \leqq e^{-2\alpha t}\int |g|^2\, d\mu, \tag{1.11}$$

provided that $\int g\, d\mu = 0$. Therefore by the Schwarz inequality we obtain that

$$\left|\int d\mu\, b\mathbf{P}_t b\right| \leqq e^{-\alpha t}\int b^2\, d\mu.$$

Similarly, as $\int b\, d\mu = 0$, and $\mu\mathbf{P}_t=\mu$, we have

$$\int d\mu\, b\mathbf{P}_t(b\mathbf{P}_s b) = \int d\mu\, b\mathbf{P}_t\left[b\mathbf{P}_s b - \int d\mu\, b\mathbf{P}_s b\right],$$

whence

$$\left|\int d\mu\, b\mathbf{P}_t(b\mathbf{P}_s b)\right| \leqq K_1 \exp(-\alpha t - \alpha s)$$

with some K_1 depending also on the bound of c. Repeating this argument once more, we arrive at

$$|E(t_1, t_2, t_3, t_4)| \leqq K_2 \exp[-\alpha(t_4-t_1)],$$

which completes the proof by a direct calculation. QED.

Let us remark that equations (1.4) and (1.8) are not involved in this result. In concrete calculations of the next section (1.4) will be used to verify (1.5), the kernel function will be determined by means of (1.1) and (1.8).

The simplest model of this kind is certainly the case of a vehicle travelling with a constant velocity $v>0$, then $\Omega=\mathbf{R}$, $\varphi(t)=v$, $c(u)=u$, and $A=0$; thus $\omega(t,u,x)=$ $=e^{ixut}$, and μ is concentrated in the point v. This model is not asymptotically normal as $\sigma=0$. Nevertheless, calculation in the space of generalized functions yields

$$f(y) = \frac{1}{\pi}\,\mathrm{Re}\int_0^{+\infty}\int_{-\infty}^{+\infty} e^{ixvt-iyt} f(x)\, dt\, dx = \frac{1}{v} f\left(\frac{y}{v}\right). \tag{1.12}$$

Of course, (1.12) follows directly from (0.1), too.

Another principal model is $S(t)=vt+\sigma W(t)$, where v is a positive constant, and W is a standard Wiener process. This case is not compatible with our framework, because velocity here is a white noise process. Since F_t is a normal distribution of mean vt and variance $\sigma^2 t$ in this case, a direct calculation yields

$$R(x,y) = \frac{2\sigma^2 x^2}{\sigma^4 x^4 + 4(y-vx)^2}. \tag{1.13}$$

In the next section we are going to investigate a nontrivial Markovian model such that (1.1) and (1.8) admit explicit solutions. Although this model is degenerate in the sense that the velocity can take on only two different values, it demonstrates the main phenomena appearing in case of more complex models. In the last section an asymptotic expansion of the integral operator (0.4) in the neighbourhood of (1.12) will be given. Universality of this expansion is to be discussed in a forthcoming paper.

2. Calculation of the kernel of (0.4)

Let $\Omega=\{v_1, v_2\}$ be the set of allowed velocities, and put $c(u)=u$. It is very important that $v_1\neq v_2$, $v_1>0$, $v_2>0$. Then $\mathbf{C}_b(\Omega)$ is simply a two-dimensional space; if $g=(g_1, g_2)$ is an arbitrary element of $\mathbf{C}_b(\Omega)$, and λ_1, λ_2 are positive constants, then

$$Ag = \begin{bmatrix} -\lambda_1 g_1+\lambda_1 g_2 \\ \lambda_2 g_1-\lambda_2 g_2 \end{bmatrix} = \begin{bmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

is the most general form of the generator of a stochastically continuous Markov process in Ω, see [5, 7]. The intuitive content of (2.1) is the following. If $\varphi(t)=v_i$ at $t\geqq 0$, then $\varphi(t+s)=v_i$ remains in force for $s<\tau_i$, where τ_i is an exponential holding time of parameter λ_i. At $s=\tau_i$ the process changes its value, and the procedure continues in such a way that, given the value of φ, past and future are completely independent. It is plain that φ has a unique stationary distribution μ. Let $\mu(\{v_1\})=p$, $\mu(\{v_2\})=1-p$, and $\lambda=\lambda_1+\lambda_2$, then (1.1) turns into

$$\int_\Omega Ag\, d\mu = (-\lambda_1 g_1+\lambda_2 g_2)p+(\lambda_2 g_1-\lambda_2 g_2)(1-p) = (\lambda_2-\lambda p)g_1+(\lambda p-\lambda_2)g_2 = 0,$$

whence

(2.2) $$\mu(\{v_1\}) = \lambda_2/\lambda, \qquad \mu(\{v_2\}) = \lambda_1/\lambda.$$

Further important parameters are the mean velocity v and the variance σ^2 of velocity in the stationary state μ. An easy calculation yields

(2.3) $$v = \frac{\lambda_1 v_2+\lambda_2 v_1}{\lambda}, \qquad \sigma^2 = \frac{\lambda_1\lambda_2}{\lambda^2}(v_1-v_2)^2.$$

Our basic result is

THEOREM 2.4. *Consider the above Markovian model and suppose that U satisfies* (1.9). *Then all conclusion of Proposition 1.11 hold, i.e. U and $\tilde{U}$ admit continuous and bounded spectral densities f and $\tilde{f}$, respectively, furthermore*

$$\tilde{f}(y) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\lambda\sigma^2 x^2 f(x)\, dx}{(y-v_1 x)^2(y-v_2 x)^2+\lambda^2(y-vx)^2}.$$

PROOF. We have to verify conditions of Proposition 1.11. In this case (1.10) is trivial because $0<\delta<\min\{v_1, v_2\}$ implies $F_t(\delta t)=0$, verification of (1.5) is based on (1.4), which reads as

(2.5) $$\dot{\omega}(t, v_k, x) = (-\lambda_k+ixv_k)\omega(t, v_k, x)+\lambda_k\omega(t, v_{\bar{k}}, x), \quad k = 1, 2$$

with $\omega(0, v_1, x)=\omega(0, v_2, x)=1$; here $\dot{\omega}$ denotes temporal derivative of ω, and $\bar{k}=2$ if $k=1$, $\bar{k}=1$ if $k=2$. The characteristic equation of system (2.5) is just

(2.6) $$z^2+(\lambda-ixv_1-ixv_2)z-v_1v_2x^2-i\lambda vx = 0;$$

let $p+iq$ be one of the square roots of the discriminant of (2.6). We have to investigate the real parts of the roots of (2.6), it is sufficient to show that for each $\varepsilon>0$ there exists a $b>0$ such that $|p|-\lambda<-2b$ if $|x|>\varepsilon$. Since $p^2-q^2=\lambda^2-x^2(v_1-v_2)^2$, $pq=-x(\lambda_1-\lambda_2)(v_1-v_2)$, and $p\neq 0$ may be assumed, we obtain that

$$\lambda^2-p^2 = x^2(v_1-v_2)^2[1-p^{-2}(\lambda_1-\lambda_2)^2],$$

whence the statement follows directly as $|\lambda_1-\lambda_2|<\lambda$.

Therefore w can be determined as the unique solution of (1.8), and

(2.7) $$R(x, y) = \mathrm{Re}[w(x, y, v_1)\lambda_2/\lambda+w(x, y, v_2)\lambda_1/\lambda].$$

On the other hand, (1.8) reads as

(2.8) $$-\lambda_k w(x, y, v_{\bar{k}})+(\lambda_k-iv_k x+iy)w(x, y, v_k) = 1, \qquad k = 1, 2,$$

whence

(2.9) $$w(x, y, v_k) = \frac{\lambda+i(y-v_{\bar{k}}x)}{i\lambda(y-vx)-(y-v_1x)(y-v_2x)}, \qquad k = 1, 2.$$

Comparing (2.7) and (2.9) we obtain the explicit form of R by an easy calculation.

Explicit solutions can be obtained in some more complex cases when the velocity may take on several different values.

3. The first approximation to (2.9)

Let $\mathscr{R}$ denote the integral operator defined by (2.9), and set $\mathscr{R}_0 f=\frac{1}{v}f\left(\frac{y}{v}\right)$. Since $\mathscr{R}_0$ corresponds to the case of constant velocity, we expect that $\mathscr{R}$ approaches $\mathscr{R}_0$ as randomness of the velocity disappears from the model. To obtain a more intuitive picture on the nature of distortions of the spectral density resulting from randomness of the velocity, we investigate asymptotics of $\mathscr{R}$ near $\mathscr{R}_0$. More exactly, we calculate the first derivative of $\mathscr{R}f$ at $\mathscr{R}_0 f$ as a function of σ^2.

THEOREM 3.1. *Suppose that f is a twice continuously differentiable bounded function with bounded first and second derivatives, and let $\tilde{f}=\mathscr{R}f$. If λ_1, λ_2 and $v>0$ are fixed as σ goes to zero, then*

$$\tilde{f}(y) = \frac{1}{v}f\left(\frac{y}{v}\right)+\frac{\sigma^2}{v^3}\int_{-\infty}^{+\infty} g_\lambda(u)f\left(\frac{y+u}{v}\right)du+$$

$$+\frac{2y\sigma^2}{v^3}\int_{-\infty}^{+\infty} g_\lambda(u)\frac{1}{u}\left[f\left(\frac{y+u}{v}\right)-f\left(\frac{y}{v}\right)\right]du+$$

$$+\frac{y^2\sigma^2}{v^3}\int_{-\infty}^{+\infty} g_\lambda(u)u^{-2}\left[f\left(\frac{y+u}{v}\right)-f\left(\frac{y}{v}\right)-\frac{u}{v}f'\left(\frac{y}{v}\right)\right]du+o(\sigma^2)$$

for all y, where $g_\lambda(u)=(\lambda/\pi)(\lambda^2+u^2)^{-1}$ denotes the Cauchy density of parameter λ.

PROOF. Introducing $u=vx-y$ as a new variable, (2.9) turns into

$$\tilde{f}(y) = \frac{1}{\pi v} \int_{-\infty}^{+\infty} \lambda v^2 \sigma^2 (y+u)^2 |p_\sigma(u)|^{-2} f\left(\frac{y+u}{v}\right) du, \tag{3.2}$$

where

$$p_\sigma(u) = \left[v_1 u + \sigma y \sqrt{\frac{\lambda_1}{\lambda_2}}\right]\left[v_2 u - \sigma y \sqrt{\frac{\lambda_2}{\lambda_1}}\right] + i\lambda u v^2. \tag{3.3}$$

Observe that $(\lambda/\pi) u^2 v^4 |p_\sigma(u)|^{-2}$ converges to $g_\lambda(u)$ as σ goes to zero, an easy calculation shows that this convergence is a dominated one, consequently we have

$$\lim_{\sigma\to 0} \frac{v^4}{\pi} \int_{-\infty}^{+\infty} \lambda u^2 |p_\sigma(u)|^{-2} h(u)\, du = \int_{-\infty}^{+\infty} g_\lambda(u) h(u)\, du \tag{3.4}$$

for all bounded $h\colon \mathbf{R}\to\mathbf{R}$. On the other hand,

$$\begin{aligned}(u+y)^2 f\left(\frac{y+u}{v}\right) &= u^2 f\left(\frac{y+u}{v}\right) + (y^2+2yu) f\left(\frac{y}{v}\right) + \\ &+ 2yu^2 \frac{1}{u}\left[f\left(\frac{y+u}{v}\right) - f\left(\frac{y}{v}\right)\right] + y^2 \frac{u}{v} f'\left(\frac{y}{v}\right) + \\ &+ y^2 u^2 \frac{1}{u^2}\left[f\left(\frac{y+u}{v}\right) - f\left(\frac{y}{v}\right) - \frac{u}{v} f'\left(\frac{y}{v}\right)\right],\end{aligned} \tag{3.5}$$

whence the statement of (3.1) follows directly by (3.4) as

$$\int_{-\infty}^{+\infty} u |p_\sigma(u)|^{-2}\, du = 0, \tag{3.6}$$

and

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} \lambda v \sigma^2 (y^2+2yu) |p_\sigma(u)|^{-2}\, du = \frac{1}{v}. \tag{3.7}$$

Proofs of (3.6) and (3.7) are straightforward but long, we only indicate the main steps of the calculations. We need

LEMMA 3.8. *Let* u_1 *and* u_2 *denote the roots of* $u^2+Bu+C=0$, *where* $\operatorname{Im} C=0$, $\operatorname{Re} C\neq 0$. *If* $\operatorname{Im} u_1<0$ *and* $\operatorname{Im} u_2<0$, *then*

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{a+bu}{|u^2+Bu+C|^2}\, du = \frac{a}{C \operatorname{Im} B}.$$

PROOF of (3.8). Since

$$\lim_{r\to+\infty} \frac{1}{\pi} \int_{-r}^{+r} \frac{1}{u-z}\, du = i \operatorname{sign} \operatorname{Im} z \tag{3.9}$$

if $\operatorname{Im} z \neq 0$, we obtain that

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{du}{(u-z_1)(u-z_2)} = \frac{i}{z_1 - z_2} [\operatorname{sign} \operatorname{Im} z_1 - \operatorname{sign} \operatorname{Im} z_2], \tag{3.10}$$

provided that $\operatorname{Im} z_1 \neq 0$ and $\operatorname{Im} z_2 \neq 0$. On the other hand,

$$\frac{a+bu}{|u^2+Bu+C|^2} = \frac{a+bu}{|u_1-u_2|^2} \left[\frac{1}{u-u_1} - \frac{1}{u-u_2}\right]\left[\frac{1}{u-\bar{u}_1} - \frac{1}{u-\bar{u}_2}\right] =$$

$$= \frac{a+bu_1}{|u_1-u_2|^2} \left[\frac{1}{(u-u_1)(u-\bar{u}_1)} - \frac{1}{(u-u_1)(u-\bar{u}_2)}\right] +$$

$$= \frac{a+bu_2}{|u_1-u_2|^2} \left[\frac{1}{(u-u_2)(u-\bar{u}_2)} - \frac{1}{(u-u_2)(u-\bar{u}_1)}\right],$$

whence by (3.10) and $C = \bar{C}$ we obtain for the value I of the integral in (3.8) that

$$I = \frac{2i}{(u_1-u_2)} \left[\frac{a+bu_1}{(u_1-\bar{u}_1)(u_1-\bar{u}_2)} - \frac{a+bu_2}{(u_2-\bar{u}_1)(u_2-\bar{u}_2)}\right] =$$

$$= \frac{2i}{(u_1-u_2)} \left[\frac{a+bu_1}{(\bar{B}-B)u_1} - \frac{a+bu_2}{(\bar{B}-B)u_2}\right] = \frac{-1}{(u_1-u_2)\operatorname{Im} B} \left[\frac{a}{u_1} - \frac{a}{u_2}\right],$$

which completes the proof of (3.8). QED

To conclude (3.6) and (3.7) we have to verify that the roots u_1 and u_2 of p_σ satisfy the conditions $\operatorname{Im} u_1 < 0$ and $\operatorname{Im} u_2 < 0$ of (3.8). Letting σ go to zero in (3.3) we see that one of the roots, say u_1, goes to zero, while u_2 converges to $-i\lambda$ as $\sigma \to 0$. Since $u_1 u_2 = -\sigma^2 y^2 / v_1 v_2$, we obtain that $u_1 = -i\sigma^2 y^2/\lambda v^2 + o(\sigma^2)$, consequently $\operatorname{Im} u_1 < 0$ and $\operatorname{Im} u_2 < 0$ hold true, at least for small values of σ. Applying (3.8) to calculate (3.6) and (3.7) we obtain the statement of Theorem (3.1). QED.

Very similar methods work in the case of (1.13); we obtain

THEOREM 3.11. *Suppose that f is twice continuously differentiable and $f(x)=0$ for $|x|>\varrho$. If*

$$\tilde{f}(y) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{2\sigma^2 x^2 f(x)\,dx}{\sigma^4 x^4 + 4(y-vx)^2} \tag{3.12}$$

and $r > |y| + \varrho v$, then

$$\tilde{f}(y) = \frac{1}{v} f\left(\frac{y}{v}\right) - \frac{\sigma^2 y^2}{\pi r v^3} f\left(\frac{y}{v}\right) +$$

$$+ \frac{\sigma^2}{2\pi v^2} \int_{-\infty}^{+\infty} f(x)\,dx + \frac{\sigma^2 y}{\pi v^3} \int_{-r}^{r} \frac{1}{u}\left[f\left(\frac{y+u}{v}\right) - f\left(\frac{y}{v}\right)\right] du + \tag{3.13}$$

$$+ \frac{\sigma^2 y^2}{2\pi v^3} \int_{-r}^{r} u^{-2}\left[f\left(\frac{y+u}{v}\right) - f\left(\frac{y}{v}\right) - \frac{u}{v} f'\left(\frac{y}{v}\right)\right] du + o(\sigma^2)$$

as $\sigma \to 0$; the right-hand side does not depend on r.

PROOF. Substituting $vx-y=u$ we obtain that

$$\tilde{f}(y)=\frac{1}{\pi v}\int_{-r}^{r}\frac{2\sigma^2v^2(y+u)^2f\left(\frac{y+u}{v}\right)du}{\sigma^4(y+u)^4+4v^4u^2}. \tag{3.14}$$

Therefore, in view of (3.5) and

$$\frac{1}{\pi}\int_{-r}^{r}\frac{2\sigma^2v^2y^2\,du}{\sigma^4(y+u)^4+4v^4u^2}=1-\frac{\sigma^2y^2}{rv^3}+o(\sigma^2), \tag{3.15}$$

$$\int_{-r}^{r}\frac{u\,du}{\sigma^4(y+u)^4+4v^4u^2}=o(\sigma^2) \tag{3.16}$$

we obtain (3.13) by the Dominated Convergence Theorem. To prove (3.15) and (3.16) let $u=\sigma^2z$, differentiating the transformed integrals with respect to σ^2 at $\sigma^2=0$ we obtain the statements of (3.15) and (3.16). QED.

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ON BITOPOLOGICAL SPACES I

J. DEÁK

This is the first part of a series investigating various topics in the theory of bitopological spaces. In § 0, the elementary bitopological constructions will be lined up; some of the material will be used only in later parts of the series. § 1 is a survey of the most important bitopological separation axioms. § 2 and most of § 3 constitute a rather trivial digression into the field of very weak separation properties. § 4 establishes a connexion between bitopologies and pseudo-directional structures. We shall prove, among others, that a bitopological space is completely regular iff it admits a compatible orderly pseudo-directional structure. (For the convenience of the reader, all the necessary definitions will be furnished.) The results of this section will be extensively used in Parts II and III of this series.

In Part II, we shall continue the investigations initiated by Smithson [103] on bitopologies induced by multifunctions into topological spaces. Part III will deal with complete regularity and compactifications.

References will be given in a liberal quantity, even simple statements and obvious definitions will be credited to their authors; there are, however, some extremely trivial statements and counter-examples whose sources we could not or would not trace.

References made by numbers in italics indicate papers known to us only indirectly, through reviews and/or citations.

Added in 1987. The contents of Remarks 1.7 c) and 1.7 h) together with Proposition 1.7 can be found in:

Mrševič, M., On bitopological separation axioms, *Mat. Vesnik* **38** (1986), no. 3, 313—318.

§ 0. Preliminaries

A. Notations and terminology

0.0 *Ordering* means a linear ordering. Symbols for orderings will be used in the strict sense, while $\subset$ stands for not necessarily proper containing. **R** denotes the set of the *real numbers*, **J** the *interval* [0, 1]; $\mathscr{E}$ is the *Euclidean topology* on **R**. $|A|$ is the *cardinality* of the set A. dom r and ran r denote the *domain* and *range*, respectively, of the relation r.

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The word *topology* will be used in the sense "collection of the open sets". An *open subbase* is not required to cover the space. A *closed (sub)base* means a subbase for the collection of the closed sets. *Neighbourhoods* and *neighbourhood bases* are not necessarily open.

0.1 Given a system of sets $\mathscr{S}$, (i) the symbol $\mathscr{T} \triangleq \mathscr{S}$ means that $\mathscr{T}$ is the topology for which $\mathscr{S}$ is an open subbase, (ii)

$$\text{co-}\mathscr{S} = \{X \setminus S\colon\ S \in \mathscr{S}\};$$

in both cases, the fundamental set X has to be clear from the context.

$\text{cl}_{\mathscr{T}}$ and $\text{int}_{\mathscr{T}}$ denote the closure and the interior with respect to the topology $\mathscr{T}$. For typographical reasons, we shall occasionally write $\mathscr{T}$-cl and $\mathscr{T}$-int. $\sup\limits_{\alpha \in A} \mathscr{T}_\alpha$ and $\inf\limits_{\alpha \in A} \mathscr{T}_\alpha$ are the supremum and the infimum of the topologies $\mathscr{T}_\alpha$; for $|A|=2$, the symbols $\mathscr{P} \vee \mathscr{Q}$ and $\mathscr{P} \wedge \mathscr{Q}$ will be used. ($\mathscr{P} \wedge \mathscr{Q} = \mathscr{P} \cap \mathscr{Q}$.)

Concerning topological separation axioms, we shall follow the usage of Császár's book [26]: a topology is S_i if its T_0-identification is T_i; i.e. S_1 is Shanin's weak regularity [93] (usually called R_0 after Davis [29]), S_2 is the weak Hausdorff property of Banaschewski and Maranda [8] (R_1 of Davis [29]), S_3=regular, S_π= completely regular, $S_4 = S_1$+normal.

If r is a relation between subsets of a set X, we shall write, according to [23]:

$$r^c = \{(X \setminus B,\ X \setminus A)\colon\ ArB\},$$

$$r^q = \{(A, B)\colon\ A = \bigcup \mathscr{A},\ B = \bigcap \mathscr{B},\ |\mathscr{A}| < \omega,\ |\mathscr{B}| < \omega,\ CrD\ (C \in \mathscr{A},\ D \in \mathscr{B})\},$$

$$r^s = (r \cup r^c)^q.$$

0.2 Let Σ be a structure (e.g. a topology, a proximity, etc.) on the set X and take some $A \subset X$. Then the induced structure $\Sigma|A$ on A is obtained by intersecting the sets at the appropriate level in Σ with A (or with $A \times A$, etc., according to the nature of the structure), e.g. if Σ is a family of relations between subsets of X then

$$\Sigma|A = \{r|A\colon\ r \in \Sigma\}$$

where

(1) $$r|A = \{(B \cap A,\ C \cap A)\colon\ BrC\};$$

if $\Sigma = (\mathscr{P}, \mathscr{Q})$ is a pair of topologies then

$$(\mathscr{P}, \mathscr{Q})|A = (\mathscr{P}|A,\ \mathscr{Q}|A) = (\{G \cap A\colon\ G \in \mathscr{P}\},\ \{G \cap A\colon\ G \in \mathscr{Q}\}).$$

(If (1) is applied to a proximity, r is to mean "far", not "near".) The restriction of a relation r to a subset of its domain will be denoted by $r \restriction A$. [So if $r \subset X \times X$ then $r \restriction A = r \cap (A \times X)$, while $r|A = r \cap (A \times A)$.]

Let Σ and Θ be structures on X and Y, respectively; then a mapping $f\colon X \to Y$ will often be referred to as a "mapping from (X, Σ) into [onto] (Y, Θ)" and written as $f\colon (X, \Sigma) \to (Y, \Theta)$, especially if it has a property with respect to Σ and Θ (e.g. continuity). A similar convention will be used also if only one of the sets X and Y is equipped with a structure.

B. Basic constructions in bitopology

0.3 A *bitopology* is a pair of topologies on the same set. A *bitopological space* (or *bispace*) is a pair consisting of a fundamental set and a bitopology on it; we shall write $(X; \mathscr{P}, \mathscr{Q})$ for $(X, (\mathscr{P}, \mathscr{Q}))$. Bispaces were introduced by Kelly [51], although some other authors were much earlier on the brink of making this definition, e.g. Wilson [116], Albert [4], Nachbin [70], [71], Tamari [107], Weston [115].

Bispaces will be denoted by upright sans-serif capitals (X, Y, A, etc.). The symbol X will always mean

$$\mathsf{X} = (X; \mathscr{P}, \mathscr{Q}).$$

0.4 Let $(S, <)$ be an ordered set. Then

$$\mathsf{S}_< = (S; \mathscr{P}_<, \mathscr{Q}_<)$$

is the *order bitopology* of S (with respect to the ordering $<$), where

$$\mathscr{P}_< = \{]\leftarrow, x[: x\in S\},\ \mathscr{Q}_< = \{]x, \rightarrow[: x\in S\}.$$

(Order bispaces are not to be confused with bitopological ordered spaces, which are bispaces equipped with a partial order, see e.g. [99].)

$$\mathscr{T}_< = \mathscr{P}_< \vee \mathscr{Q}_<$$

is the usual order topology.

As the most important special case, take $S = \mathbf{R}$ with $<$ the natural ordering of $\mathbf{R}$; we shall denote this bispace by

$$\mathsf{R} = (\mathbf{R}; \overline{\mathscr{E}}, \underline{\mathscr{E}}).$$

In the usual terminology, $\overline{\mathscr{E}}$ and $\underline{\mathscr{E}}$ are the topologies of the upper, respectively lower semicontinuity. Several authors work with $(\mathbf{R}; \underline{\mathscr{E}}, \overline{\mathscr{E}})$ instead of R.[1]

0.5 For $A\subset X$, $(A, (\mathscr{P}, \mathscr{Q})|A)$ defined in 0.2 is a *sub-bispace* of X (cf. [52] and [10]). If there is no danger of misunderstanding, we shall use the following conventions: (i) given $A\subset X$, $(A, (\mathscr{P}, \mathscr{Q})|A)$ will be denoted by A; (ii) given a sub-bispace A of X, it will be understood that the fundamental set of A is A (and similarly for other letters). In particular, let

$$\mathsf{J} = (\mathbf{J}; \overline{\mathscr{E}}|\mathbf{J}, \underline{\mathscr{E}}|\mathbf{J}).$$

0.6 The product of a family of bispaces was defined by Bîrsan [9] as follows:

$$\prod_{\alpha\in A} (X_\alpha; \mathscr{P}_\alpha, \mathscr{Q}_\alpha) = \Big(\mathop{\times}_{\alpha\in A} X_\alpha;\ \prod_{\alpha\in A} (\mathscr{P}_\alpha, \mathscr{Q}_\alpha)\Big) = \Big(\mathop{\times}_{\alpha\in A} X_\alpha;\ \prod_{\alpha\in A} \mathscr{P}_\alpha,\ \prod_{\alpha\in A} \mathscr{Q}_\alpha\Big).$$

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[1] *Added in proof* (April 1991). Perhaps it would have been better to take the two topologies in reverse order (not only here, but also in 4.2, 5.1 and 9.0). Cf. 0.3 in: DEÁK, J., Extensions of quasi-uniformities for prescribed bitopologies I, *Studia Sci. Math. Hungar.* **25** (1990), 45—67.

0.7 A partial order between bitopologies on the same set can be defined by

$$(\mathscr{P}_1, \mathscr{Q}_1) \leqq (\mathscr{P}_2, \mathscr{Q}_2) \Leftrightarrow \mathscr{P}_1 \subset \mathscr{P}_2, \mathscr{Q}_1 \subset \mathscr{Q}_2$$

(Kim [54]); in this case $(\mathscr{P}_2, \mathscr{Q}_2)$ is called finer than $(\mathscr{P}_1, \mathscr{Q}_1)$. Supremum and infimum of bitopologies are taken with respect to this partial order, i.e.

$$\sup_{\alpha \in A} (\mathscr{P}_\alpha, \mathscr{Q}_\alpha) = (\sup_{\alpha \in A} \mathscr{P}_\alpha, \sup_{\alpha \in A} \mathscr{Q}_\alpha).$$

0.8 Let P be a topological property. We shall say that the bitopology $(\mathscr{P}, \mathscr{Q})$, or the bispace X, is *sup-P (inf-P)*, respectively *bi-P* if $\mathscr{P} \vee \mathscr{Q}$ $(\mathscr{P} \wedge \mathscr{Q})$ is P, respectively $\mathscr{P}$ and $\mathscr{Q}$ are both P, cf. [56]. (Instead of sup-, the prefix "semi-" or the words "doubly" and "jointly" are also used, see [27], [23], [25], [18]. Semi- in this sense can be misleading; see [61] for the generally accepted meaning of this prefix.)[2] The properties sup-P, inf-P, and bi-P are bitopological generalizations of the topological property P in the sense that they reduce to P if $\mathscr{P} = \mathscr{Q}$. Usually there are, of course, more interesting bitopological generalizations of P, which depend on how the open sets of the two topologies are situated with respect to each other; one can (and usually does) pick one of these generalizations and call it the bitopological property P; most authors use the expression "pairwise P", but we shall allocate this term to the sense defined in 0.9 below. The word "pairwise" will be dropped even from citations, i.e. if we say that certain bispaces are somewhere called "P", it may mean that they are called there "pairwise P".

Similar conventions apply to relative properties of sub-bispaces (e.g. density) and to properties of mappings (e.g. continuity). For example: a sub-bispace A of X is bi-P in X if $(A, \mathscr{P}|A)$ is P in $(X, \mathscr{P})$ as well as $(A, \mathscr{Q}|A)$ is P in $(X, \mathscr{Q})$; a mapping is bi-P if it is P both in the first and the second topologies. (When first defined, bicontinuous mappings were called continuous, see Pervin [79].)

0.9 The topology

$$\mathscr{T} \triangleq \{A \cup B \colon A \in \mathscr{P} \setminus \{\emptyset\}, B \in \mathscr{Q} \setminus \{\emptyset\}\} \tag{1}$$

is called the *pairwise topology* of X; the bispace X, or the bitopology $(\mathscr{P}, \mathscr{Q})$, is *pairwise P* if this topology $\mathscr{T}$ is P (Singal and Jain [95]).

0.10 A subset of X is *quasi-open* in X (Datta [27], see also *[20]* and *[59]*) if it is the union of a $\mathscr{P}$-open and a $\mathscr{Q}$-open set. Let us denote the system of the quasi-open sets by $\mathscr{S}$. $\mathscr{S}$ is closed for arbitrary unions, $\emptyset \in \mathscr{S}$, $X \in \mathscr{S}$, but $\mathscr{S}$ is not a topology, since it is not closed for finite intersections; $\mathscr{S}$ is only a pretopology in the sense of Garg and Naimpally [45] ("supratopology" in [63]). Topological properties can be easily extended to pretopologies, so the bispace X will be called *quasi-P* (P a topological property) if the pretopology $\mathscr{S}$ defined above is P (Datta [27]).

[2] *Added in proof.* The author prefers now the following terminology: *doubly P* for bi-P and *subspace* for sub-bispace. See the paper cited in Footnote 1.

Conditions equivalent in topological spaces may fail to remain equivalent in pretopological spaces, therefore the properties quasi-P are to be defined individually for each P considered. For *quasi-*T_i ($i=0, 1, 2$) and *quasi-*S_i ($i=1, 2$), use the definitions with open sets, e.g. a bispace is quasi-T_2 if two distinct points are always contained by disjoint quasi-open sets (Datta [27]). A bispace can be called *quasi-compact* if each quasi-open covering of it contains a finite subcovering; by the Alexander subbase theorem, this is equivalent to the sup-compactness. The subset A of X (the sub-bispace A of X) can be called *quasi-dense* in X if each non-empty quasi-open set of X meets A; this is evidently equivalent to the bi-density.

Two similar schemes of axioms are obtainable by considering the pretopology on the right-hand side of 0.9 (1), or the system of sets $\mathscr{P}\cup\mathscr{Q}$, which is not even a pretopology.

It is also possible to extend topological properties to bitopologies using the tools of category theory, see e.g. [19] and the references therein.

0.11 A *bi-continuous real function on* X is a bi-continuous mapping from X into R, i.e. a $\mathscr{P}$-upper and $\mathscr{Q}$-lower semicontinuous real function.

0.12 Similarly to the T_0-identification (T_0-reflexion) of topological spaces, we introduce the *w*T_0*-identification* of bispaces (the name will be explained by Definition 1.2). In the bispace X, let us call two points equivalent if they have the same neighbourhood filter in $\mathscr{P}\vee\mathscr{Q}$. Denote by ϱX the set of the equivalence classes, by $\varrho\mathscr{P}$ and $\varrho\mathscr{Q}$ the quotient topologies, and by ϱ_{X} the bi-continuous mapping from X onto $\varrho\mathsf{X}=(\varrho X;\ \varrho\mathscr{P}, \varrho\mathscr{Q})$ with $x\in\varrho_{\mathsf{X}}(x)$ $(x\in X)$. (Although it is not shown in the notations, ϱX, $\varrho\mathscr{P}$, and $\varrho\mathscr{Q}$ depend on $(\mathscr{P}, \mathscr{Q})$; $(\varrho\mathscr{P}, \varrho\mathscr{Q})$ is a *quotient-bitopology* in the sense of [9] and [2].) The wT_0-identification is a reflexion, i.e. if Y is a wT_0-bispace (=sup-T_0-bispace) and f: X→Y is bi-continuous then there is a unique bi-continuous mapping ϱf: ϱX→Y such that $f=\varrho f\circ\varrho_{\mathsf{X}}$.

§ 1. Reconsidering the bitopological separation axioms

1.0 In this section, we make an attempt at introducing a consistent terminology for the bitopological separation properties. As a by-product, we obtain some new axioms, which will be needed for filling in some gaps in the hierarchy of the separation axioms (see the diagram in 1.8). Perhaps more than a hundred bitopological separation axioms can be found in the literature, so we shall have to make some restrictions in order to avoid getting lost in the jungle.

a) Only the generalizations of the most important topological separation axioms listed in 0.1 will be considered. For axioms between T_0 and T_1, between T_1 and T_2, with semi-open sets, etc., see [5], [14], [37], [50], [61], [62], [74], [81], [94], [95], [98], [100], [114], *[47]*, *[60]*, *[64]*, *[75]*, *[101]*, *[108]*. For higher separation axioms such as complete, total or perfect normality, see [22], [52], [57], [77], [84], *[16]*, *[55]*.

b) The not really bitopological axioms bi-P, sup-P and inf-P (P a topological separation property) will not appear in the diagram in 1.8, although we shall state some facts about the relations of bi-P and sup-P to the real bitopological separation axioms. It should be noted, however, that some of the axioms whose definitions look bitopological enough turn out to be of the type bi-P or sup-P (they are: T_0, wT_0,

qT_0, sT_1, qT_1 and qS_1). The inf-P axioms will be completely disregarded; see [95] for such axioms.

c) Bitopological generalizations of hereditary topological properties can be reasonably expected to be hereditary, too; consequently, we do not consider separation axioms failing to satisfy this natural condition, e.g. the pairwise T_i axioms in the sense of 0.9, see [95].

1.1 DEFINITION. The bispace X is

a) *strongly* T_0 (sT_0) if for distinct points $x, y \in X$, there is either a $U \in \mathcal{P}$ with $x \in U$, $y \notin U$ or a $V \in \mathcal{Q}$ with $y \in V$, $x \notin V$;

b) *strongly* T_1 (sT_1) if for distinct points $x, y \in X$, there are $U \in \mathcal{P}$ and $V \in \mathcal{Q}$ such that $x \in U \setminus V$, $y \in V \setminus U$;

c) *strongly* T_2 (sT_2) or *strongly Hausdorff* if for distinct points $x, y \in X$, there are disjoint sets $U \in \mathcal{P}$ and $V \in \mathcal{Q}$ such that $x \in U$ and $y \in V$.

REMARKS. a) Axiom sT_0 was introduced by Fletcher, Hoyle and Patty [43] under the name T_0; such bispaces are called $T_{1/2}$ in [84].

b) Thampuran [110] and Reilly [84] defined sT_1, calling it T_1. Evidently, $sT_1 = $ bi-T_1, cf. [84].

c) Axiom sT_2 was already considered by Weston [115] when bispaces were still unknown; he used the expression "$\mathcal{P}$ and $\mathcal{Q}$ are consistent". In [51], sT_2-bispaces are called Hausdorff.

d) The terminology adopted in the above definition is used in [80] and [3].

e) A different property is called strongly T_2 in [114].

1.2 DEFINITION. The bispace X is *weakly* T_i (wT_i) ($i=0, 1, 2$) or *weakly Hausdorff* (for $i=2$) if for distinct points $x, y \in X$, the condition in the definition of sT_i holds either for x and y or for y and x.

REMARKS. a) Murdeshwar and Naimpally [69] introduced wT_0 under the name T_0; the term wT_0 was first used in [90].

b) Axiom wT_1 is due to Swart [106]; wT_1-bispaces are called T_1 in [3].

c) Kim [52] introduced wT_2-bispaces, calling them $T_{1^1/_2}$; they were renamed wT_2 in [11]. In [103] semi-Hausdorff, in [3] T_2 is used.

d) The idea of arranging the bitopological separation axioms into two sequences—strong ones and weak ones— is due to Saegrove [90]. (Although he calls the strong axioms strong, he does not put the word strong or the letter s into their names.)

1.3 NOTATION. We shall write qT_i for quasi-T_i ($i=0, 1, 2$).

REMARKS. a) $qT_0 = wT_0 = $ sup-T_0.

b) $qT_1 = $ sup-T_1; this axiom was introduced by Murdeshwar and Naimpally [69] under the name T_1.

c) Quasi-Hausdorff (qT_2) bispaces were first considered by Datta [27].

d) qT_i bispaces are called weakly T_i in [3] ($i=1, 2$).

e) It is clear that $wT_1 \Rightarrow qT_1$, but the converse is not true, see Example b) below.

f) $wT_2 \Rightarrow qT_2 \Rightarrow$ sup-T_2, but neither of these implications can be reversed, as shown by the examples below.

EXAMPLES. a) Let

$$X=\{1,2,3\},\quad \mathscr{P}\triangleq\{\{1,2\},\{2,3\}\},\quad \mathscr{Q}\triangleq\{\{1,3\}\}.$$

Then X is sup-T_2, but not qT_2.

b) A two-point set with the discrete and the indiscrete topologies is qT_2, but not wT_1.

1.4 REMARK. Bi-$T_2\Rightarrow qT_2$ (evidently), but $sT_2 \not\Rightarrow$ bi-$T_2 \not\Rightarrow wT_2$ ([56], [95]), see the next examples.

EXAMPLES. a) On $X=J$, let $\mathscr{Q}|]0,1[$ and $\mathscr{P}$ be Euclidean topologies, and let

$$\{\{0\}\cup]t,1[:\ t\in]0,1[\}\quad\text{and}\quad\{\{1\}\cup]0,t[:\ t\in]0,1[\}$$

be $\mathscr{Q}$-neighbourhood bases of 0 and 1, respectively. Now $\mathscr{P}$ and $\mathscr{Q}$ are T_2 (in fact, both are homeomorphic to a real interval with the Euclidean topology), but X is not wT_2. (Example 9.12 in [95] is similar, but seems to be wrong.)

b) ([56], [95]) Take the discrete and the co-finite topologies on an infinite set. This bispace is sT_2, but not bi-T_2.

1.5 DEFINITION. a) For topologies $\mathscr{P}$ and $\mathscr{Q}$ on a set X,

aa) $\mathscr{P}$ is S_1 *with respect to* $\mathscr{Q}$ if for $x,y\in X$ and $U\in\mathscr{P}$ with $x\in U$, $y\notin U$, there is a $V\in\mathscr{Q}$ with $y\in V$, $x\notin V$;

ab) $\mathscr{P}$ is S_2 *with respect to* $\mathscr{Q}$ if for $x,y\in X$ and $U\in\mathscr{P}$ with $x\in U$, $y\notin U$, there are disjoint $V\in\mathscr{Q}$ and $W\in\mathscr{P}$ such that $x\in W$ and $y\in V$.

b) The bispace X is S_i if $\mathscr{P}$ is S_i with respect to $\mathscr{Q}$ and vice versa $(i=1,2)$.

REMARKS. a) S_1-bispaces were introduced by Murdeshwar and Naimpally [69] under the name R_0; their property R_1 is different from S_2 and will be discussed later.

b) Part aa) of the above definition was given by Ćirić [22] and Žižović [117], with R_0 instead of S_1.

c) S_2-bispaces were introduced by Žižović [117] and Reilly [88], under the name R_1.

d) $S_2 \not\Rightarrow$ bi-$S_2 \not\Rightarrow S_2$ [88] and a similar statement holds for S_1, too; in fact, bi-$S_2 \not\Rightarrow S_1$ (see Example a) below) and $S_2 \not\Rightarrow$ bi-S_1 (take R).

e) ([87], [88]) We have $sT_i=S_i+sT_0$ $(i=1,2)$.

f) $wT_2 \not\Rightarrow S_1$, see Example b) below.

g) According to Raghavan and Reilly [83], $\mathscr{P}$ has property R with respect to $\mathscr{Q}$ ($\mathscr{P}$R$\mathscr{Q}$) if for each $x\in X$,

$$\text{(1)}\qquad \mathrm{cl}_{\mathscr{P}}\{x\}=\bigcap\{V\in\mathscr{Q}:\ x\in V\}=\bigcap\{\mathrm{cl}_{\mathscr{P}}V:\ x\in V\in\mathscr{Q}\}.$$

In spite of the asymmetrical formulation of the definition, $\mathscr{P}$R$\mathscr{Q}$ holds iff X is S_2. Indeed, (1) is equivalent to (2) and (3):

$$\text{(2)}\qquad \mathrm{cl}_{\mathscr{P}}\{x\}\subset\bigcap\{V\in\mathscr{Q}:\ x\in V\},$$

$$\text{(3)}\qquad \bigcap\{\mathrm{cl}_{\mathscr{P}}V:\ x\in V\in\mathscr{Q}\}\subset\mathrm{cl}_{\mathscr{P}}\{x\};$$

here (2) holds iff $\mathscr{Q}$ is S_1 with respect to $\mathscr{P}$, and (3) means that $\mathscr{P}$ is S_2 with respect to $\mathscr{Q}$; now one has only to observe that these conditions together are sufficient for S_2.

EXAMPLES. a) If

$$X = \{1, 2, 3\}, \quad \mathscr{P} \triangleq \{\{1, 2\}, \{3\}\}, \quad \mathscr{Q} \triangleq \{\{1\}, \{2, 3\}\}$$

then X is bi-S_2 but not S_1.

b) If $X=\{1, 2\}$, $\mathscr{P}$ is discrete and $\mathscr{Q} \triangleq \{\{1\}\}$ then X is wT_2 but not S_1.

1.6 DEFINITION. The bispace X is

a) *strongly S_i* or *sS_i* $(i=0, 1, 2)$, respectively *weakly S_i* or *wS_i* $(i=1, 2)$, if the requirements in the definition of sT_i, respectively wT_i, are satisfied for each pair of points x, y having different neighbourhood filters in $\mathscr{P}\vee\mathscr{Q}$;

b) *T_i* $(i=1, 2)$ or *Hausdorff* (for $i=2$) if it is S_i and wT_0.

NOTATION. We shall write qS_i for quasi-S_i $(i=1, 2)$.

REMARKS. a) sS_2 is equivalent to Axiom R_1 introduced by Murdeshwar and Naimpally [69]. Raghavan and Reilly [83] give several statements about bispaces X for which $\mathscr{P}$R$\mathscr{Q}$ holds (cf. Remark 1.5 g)) and $\mathscr{P}$ is an S_1-topology; one can readily check that these are just the sS_2-bispaces.

b) It is easy to see that the bispace X is qS_i iff the requirements of the definition of qT_i hold for pairs of points with different neighbourhood filters in $\mathscr{P}\vee\mathscr{Q}$ $(i=1, 2)$. A similar statement would be valid for S_i and T_i if we had given a direct definition for T_i instead of deriving it from S_i. For example, a definition for T_1 could run as follows: the bispace X is T_1 if for distinct points $x, y\in X$, the statements

(1) $$\exists U\in\mathscr{P},\ x\in U,\ y\notin U,$$

(2) $$\exists V\in\mathscr{Q},\ x\notin V,\ y\in V,$$

(3) $$\exists V\in\mathscr{Q},\ x\in V,\ y\notin V,$$

(4) $$\exists U\in\mathscr{P},\ x\notin U,\ y\in U$$

are either all true, or (1) and (2) are true, (3) and (4) false, or (3) and (4) are true, (1) and (2) false.

b) qS_1=sup-S_1, qS_2⇒sup-S_2, but sup-S_2=|⇒qS_2, see Example 1.3 a).

c) sS_1⇒bi-S_1 (evidently), but sS_2=|⇒bi-S_2 (Example 1.4 b)). It is immediate that bi-S_i⇒qS_i $(i=1, 2)$, but bi-S_2=|⇒wS_1 (Example 1.5 a)).

d) For relations between the bitopological separation properties defined so far, look at the bottom half of the diagram in 1.8.

1.7 DEFINITION. a) For topologies $\mathscr{P}$ and $\mathscr{Q}$ on a set X,

aa) (Kelly [51]) $\mathscr{P}$ is *regular with respect to* $\mathscr{Q}$ if given $x\in U\in\mathscr{P}$, there are a $V\in\mathscr{Q}$ and a $W\in\mathscr{P}$ with $x\in W\subset X\setminus V\subset U$;

ab) (Lane [57]) $\mathscr{P}$ is *completely regular with respect to* $\mathscr{Q}$ if given $x\in U\in\mathscr{P}$, there is a bi-continuous real function f on X such that $f(x)=0$ and $f(y)=1$ $(y\notin U)$.

b) "Regular" and "completely regular" in Part a) will be abbreviated by S_3 and S_π, respectively.

c) (Kelly [51]) The bispace X is *normal* if given disjoint sets F_1 and F_2 such that F_1 is $\mathscr{P}$-closed and F_2 is $\mathscr{Q}$-closed, there are disjoint $U\in\mathscr{P}$ and $V\in\mathscr{Q}$ with $F_1\subset V$ and $F_2\subset U$.

d) For topologies $\mathscr{P}$ and $\mathscr{Q}$ on a set X, $\mathscr{P}$ is S_4 *with respect to* $\mathscr{Q}$ if the bispace X is normal and $\mathscr{P}$ is S_1 with respect to $\mathscr{Q}$.

e) The bispace X is

ea) S_i ($i=3, \pi, 4$) or (Kelly [51]) *regular* (for $i=3$) or (Fletcher [42] and Lane [57]) *completely regular* (for $i=\pi$) if $\mathscr{P}$ is S_i with respect to $\mathscr{Q}$ and vice versa;

eb) T_i if it is S_i and wT_0 ($i=3, \pi, 4$);

ec) *strongly* S_i (sS_i) if it is S_i and sS_0 ($i=3, \pi, 4$);

ed) *strongly* T_i (sT_i) if it is S_i and sT_0 ($i=3, \pi, 4$).

Remarks. a) Axioms T_3 and T_π were introduced by Saegrove [90] under the name WT_3 and $WT_{3^1/_2}$, respectively.

b) Axioms sT_i ($i=3, \pi, 4$) were given by Reilly [84] under the name T_i.

c) Axiom R_2 introduced by Žižović [117] looks stronger than, but is equivalent to, regularity.

d) A special case of bitopological normality was already considered by Nachbin [70], [72].

e) M. K. Singal and A. R. Singal [98] seem to use T_3 in the sense regular + + sup-T_1; if so, their Theorem 5.4 (wT_0+regular$\Rightarrow T_3$) is correct, in spite of a statement made by Reilly [85]. Theorem 5.5 in [98] claiming that T_3 implies a condition stronger than sT_2 is, however, mistaken; the proof of it is certainly wrong, whatever T_3 may mean.

f) If $\mathscr{P}$ is S_4 with respect to $\mathscr{Q}$ then $\mathscr{P}$ is also S_π with respect to $\mathscr{Q}$; this follows from the bitopological generalization of Urysohn's Lemma, see Kelly [51]. Consequently, $S_4 \Rightarrow S_\pi$ (Mršević [67]).

Notation. In order to make some parallel statements more similar in form, we shall occasionally use the alternative name T_0 for $wT_0 = qT_0$.

Remarks. g) We have (in part by definition):

$$sS_i = S_i + sS_0, \quad T_i = S_i + T_0.$$
$$sT_i = sS_i + T_0 = T_i + sS_0 = S_i + sT_0$$

for $i=1, 2, 3, \pi, 4$ (the first and the second equalities in the second line also for $i=0$). Further,

$$P_4 = \text{normal} + P_1 \qquad (P = S, sS, T, sT).$$

h) Saegrove [90] defines WT_4=normal+wT_1 and claims that $WT_4 \Rightarrow WT_{3^1/_2}$, i.e. normal+$wT_1 \Rightarrow T_\pi$ in our terminology. This implication is not true: Example 1.5 b) is normal and wT_3, but not even S_1. Let us note, however, that:

Proposition. *Normal*+$wT_1 \Rightarrow wT_2$.

Proof. Let X be normal and wT_1. Take distinct points $x, y \in X$.

a) First assume $y \in \mathrm{cl}_{\mathscr{P}}\{x\}$. By wT_1, $x \notin \mathrm{cl}_{\mathscr{P}}\{y\}$. For any $z \in \mathrm{cl}_{\mathscr{Q}}\{x\} \cap \mathrm{cl}_{\mathscr{P}}\{y\}$, we have $z \in \mathrm{cl}_{\mathscr{Q}}\{x\} \cap \mathrm{cl}_{\mathscr{P}}\{x\}$, since $\mathrm{cl}_{\mathscr{P}}\{y\} \subset \mathrm{cl}_{\mathscr{P}}\{x\}$. By wT_1, this means $z=x$, contradicting $x \notin \mathrm{cl}_{\mathscr{P}}\{y\}$. Thus $\mathrm{cl}_{\mathscr{Q}}\{x\}$ and $\mathrm{cl}_{\mathscr{P}}\{y\}$ are disjoint. Consequently, normality gives disjoint sets $U \in \mathscr{P}$ and $V \in \mathscr{Q}$ such that $x \in U$, $y \in V$.

b) The case $x \in \mathrm{cl}_{\mathscr{Q}}\{y\}$ is analogous. So assume now $x \notin \mathrm{cl}_{\mathscr{Q}}\{y\}$. If $\mathrm{cl}_{\mathscr{P}}\{x\}$ and $\mathrm{cl}_{\mathscr{Q}}\{y\}$ are disjoint then normality gives disjoint neighbourhoods as required in the definition of wT_2. On the other hand, if there is a point $z \in \mathrm{cl}_{\mathscr{P}}\{x\} \cap \mathrm{cl}_{\mathscr{Q}}\{y\}$ then

Part a) of this proof can be applied to the points x and z (since $x \notin \mathrm{cl}_{\mathscr{Q}}\{y\}$ implies $x \neq z$), thus there are disjoint $U \in \mathscr{P}$ and $V \in \mathscr{Q}$ with $x \in U$ and $z \in V$. Now $y \in V$, because $z \in \mathrm{cl}_{\mathscr{Q}}\{y\}$.

1.8 The diagram below shows all the relations between the bitopological separation properties considered in this section. A bispace $(X, \mathscr{P}, \mathscr{P})$ satisfies one of the axioms iff the topology $\mathscr{P}$ satisfies the axiom shown in round brackets in the same line.

$$\begin{array}{lllllll}
(\mathrm{T}_4) & \mathrm{sT}_4 & \Rightarrow & \mathrm{T}_4 & & & \\
 & \Downarrow & & \Downarrow & & & \\
(\mathrm{S}_4) & \mathrm{sS}_4 & \Rightarrow & \mathrm{S}_4 & & & \\
 & & & \Downarrow & & & \\
(\text{normal}) & & & \text{normal} & & & \\
(\mathrm{T}_i) & \mathrm{sT}_i & \Rightarrow & \mathrm{T}_i & & & \\
 & \Downarrow & & \Downarrow & & & (i = \pi, 3) \\
(\mathrm{S}_i) & \mathrm{sS}_i & \Rightarrow & \mathrm{S}_i & & & \\
(\mathrm{T}_j) & \mathrm{sT}_j & \Rightarrow & \mathrm{T}_j & \Rightarrow \mathrm{wT}_j & \Rightarrow \mathrm{qT}_j & \\
 & \Downarrow & & \Downarrow & \quad\Downarrow & \quad\Downarrow & (j = 2, 1) \\
(\mathrm{S}_j) & \mathrm{sS}_j & \Rightarrow & \mathrm{S}_j & \Rightarrow \mathrm{wS}_j & \Rightarrow \mathrm{qS}_j & \\
(\mathrm{T}_0) & \mathrm{sT}_0 & \Rightarrow & \mathrm{T}_0 & = \mathrm{wT}_0 & = \mathrm{qT}_0 & \\
 & \Downarrow & & & & & \\
(\text{any space}) & \mathrm{sS}_0 & & & & &
\end{array}$$

$$\mathrm{P}_n \Rightarrow \mathrm{P}_k \quad (\mathrm{P} = \mathrm{sT}, \mathrm{T}, \mathrm{wT}, \mathrm{qT}, \mathrm{sS}, \mathrm{S}, \mathrm{wS}, \mathrm{qS};\ n > k).$$

In order to see that no other implications hold, it is enough to consider (i) the usual counterexamples for topological separation axioms, doubling their topologies; (ii) $\mathbb{R}$; (iii) Example 1.3 b); (iv) Example 1.5 b).

Remarks. a) Searching for equivalent characterizations of bitopological separation properties and for invariance theorems (under operations and at mappings) offers a practically unlimited scope for investigation, with some interesting and a lot of trivial results. It does not seem to be worth while to make a systematic study of such questions. Nevertheless, we shall deal with some problems of this type, mainly in connexion with complete regularity.

b) For relations between S_3, S_π, normality and the corresponding bi- and sup-properties, see Lal [56].

c) In addition to the papers cited elsewhere in this section, bitopological separation properties are also dealt with in [6], [27], [28], [36], [38], [39], [40], [48], [53], [65], [66], [97], [103], *[2]*, *[7]*, *[21]*, *[41]*, *[59]*, *[113]*.[3]

d) Observe that the bispace X is sS_i (respectively S_i, wS_i or qS_i) iff its wT_0-identification $\varrho\mathrm{X}$ is sT_i (respectively T_i, wT_i or qT_i) (for any possible choice of i).

[3] *Added in proof.* For recent publications, see: Ivanov, A. A., A bibliography of bitopological spaces, *Studies in topology VI, Zap. Nauchn. Sem. LOMI* **167** (1988), 63—78 (in Russian).

§ 2. Below sT_1

2.0 In this section, we consider separation axioms weaker than sT_1. After defining two natural weakenings of sT_1, we shall look for axioms below S_1 which are equivalent to the aforesaid axioms in wT_0-bispaces, thus yielding analogues to the statements $sT_i = S_i + sT_0$. This section will be concluded with defining another two separation axioms below sS_1 and indicating how one could define dozens of such axioms.

2.1 It is possible to insert two separation axioms between sT_0 and sT_2 such that they are independent of each other and one of them is a generalization of the topological axiom T_1, the other of T_0:

(a) for distinct points $x, y \in X$, there are $U, V \in \mathscr{P}$ or $U, V \in \mathscr{Q}$ such that $x \in U \setminus V$, $y \in V \setminus U$;

(b) for distinct points $x, y \in X$, there are $U \in \mathscr{P}$ and $V \in \mathscr{Q}$ such that one of the points is in $U \cap V$, the other is outside $U \cup V$.

Clearly, $sT_1 \Rightarrow (a) \Rightarrow sT_0$ and $sT_1 \Rightarrow (b) \Rightarrow sT_0$. Elementary examples show that (a) and (b) are different from sT_0, sT_1, and each other; even $(a)+(b) \not\Rightarrow sT_1$. Žižović [117] calls a bispace sT_0 if it satisfies condition (b), and proves that $sT_i = S_i + (b)$ $(i = 1, 2)$, which is a weaker form of Remark 1.5 e).

2.2 Similarly to (a) and (b), we can define two axioms more general than S_1:

(c) if $U \in \mathscr{P}$, $V \in \mathscr{Q}$, $x \in U \cap V$, $y \notin U \cup V$ then there is a $W \in \mathscr{P}$ or a $W \in \mathscr{Q}$ with $y \in W$, $x \notin W$;

(d) if $U, V \in \mathscr{P}$ [respectively, $U, V \in \mathscr{Q}$], $x \in U \setminus V$, $y \in V \setminus U$ then there is a $W \in \mathscr{Q}$ [respectively, a $W \in \mathscr{P}$] containing exactly one of the points x and y.

If $\mathscr{P} = \mathscr{Q}$ then (c) reduces to the topological axiom S_1, while (d) turns into an empty condition. Similarly to Remark 1.7 g), we have

$$(1) \qquad (a) = (c) + sT_0, \quad (b) = (d) + sT_0.$$

One could try looking for similar weakenings of some other separation axioms with index 1 and inserting all these axioms into the diagram in 1.8, but in this case the diagram would lose its appealing regularity, e.g. we have the unwished-for implications $qS_1 \Rightarrow (c)$ and $wS_1 \Rightarrow (d)$. If we want to retain (1), the only alternatives to (c) and (d) are the stronger conditions

(c*) ϱX is sup-T_1;

(d*) ϱX is bi-T_0.

(Here (c*) is given in this roundabout way in order to make it look similar to (d*), which has no simpler form.) From the point of view of the regularity of the diagram, (c*) is even worse than (c), since $(c^*) = qS_1$; furthermore, $wS_1 \Rightarrow (d^*)$.

2.3 Observe that (a), (b) and wT_1 form a triplet of separation axioms insomuch as (i) each of them requires that if the four conceivable kinds of neighbourhood containing exactly one of the given points are paired off in a certain way then both kinds of neighbourhood belonging to one of these pairs should in fact exist and (ii) a fourth axiom of this type cannot exist. In the same sense, qS_1, (d*) and sS_0 are triplets, too. We can also formulate an axiom belonging to (c) and (d) in a similar way:

(e) if $U \in \mathscr{P}$, $V \in \mathscr{Q}$, $x \in U \setminus V$, $y \in V \setminus U$ then there is a $W \in \mathscr{P}$ with $y \in W$, $x \notin W$ or there is a $W \in \mathscr{Q}$ with $x \in W$, $y \notin W$.

This axiom will be of some use in the next section. qS_1, (d*), sS_0 and (c), (d), (e) each states that the existence of some kind(s) of neighbourhood implies the existence of some other kind(s). The strongest possible of such axioms is clearly sS_1. S_1, wS_1 and the asymmetrical axioms "the first topology is S_1 with respect to the second" and "the second topology is S_1 with respect to the first" also belong to this category. It is a trivial but tedious exercise to collect all such axioms and draw a diagram showing the implications between them. According to our count, there are 19 symmetrical and 2×23 asymmetrical axioms of this kind (the correctness of these numbers is not guaranteed). Only one of the axioms not mentioned before seems worth formulating:

(f) for each $x\in X$, $\mathrm{cl}_{\mathscr{P}}\{x\} = \mathrm{cl}_{\mathscr{Q}}\{x\}$.

Observe that S_1, bi-S_1 and (f) form a triplet.

§ 3. More on S_i and sS_i

3.0 Let us consider the following problem: what additional condition makes an S_i-bispace satisfy the stronger axiom sS_i, too. (Clearly, there is no analogous problem for topological spaces.) We saw in § 1 that

(1) $$sS_i = S_i + A \qquad (i = 1, 2, 3, \pi, 4)$$

holds if $A = sS_0$ or $A = sS_1$, the latter being the strongest possible choice for A in (1). There are, however, axioms A weaker than sS_0 satisfying (1), e.g. (e) from 2.3. We can choose, of course, any axiom between sS_1 and (e), e.g. "$\mathscr{P}$ is S_1" or (f) from 2.3 (and several others referred to but not explicitly given in 2.3). Žižović [118] has essentially proved that (1) holds with $A=(g)$ where

(g) if $x, y\in X$, $\mathrm{cl}_{\mathscr{P}}\{x\} \cap \mathrm{cl}_{\mathscr{Q}}\{y\} = \emptyset$ then $\mathrm{cl}_{\mathscr{Q}}\{x\} \cap \mathrm{cl}_{\mathscr{P}}\{y\} = \emptyset$.

Axioms (e) and (g) are independent. Both are generalized by

(h) if $x, y\in X$, $\mathrm{cl}_{\mathscr{P}}\{x\} \cap \mathrm{cl}_{\mathscr{Q}}\{y\} = \emptyset$ then $y\notin \mathrm{cl}_{\mathscr{Q}}\{x\}$ or $x\notin \mathrm{cl}_{\mathscr{P}}\{y\}$,

which can also be chosen for A in (1). The easy proof is left to the reader.

3.1 (To Remark 1.5 g).) [83] contains several statements about bispaces X in which $\mathscr{P}$ is S_1 and $\mathscr{P}R\mathscr{Q}$; these are just the sS_2-bispaces, see above. [83] Lemma 1 claims that if $\mathscr{P}$ is S_3 with respect to $\mathscr{Q}$ then $\mathscr{P}R\mathscr{Q}$. This is, however, false, see Example 1.5 b).

3.2 The original definition of sS_2 in [69] (there: R_1) runs as follows: X is sS_2 if $x, y\in X$ and $\mathrm{cl}_{\mathscr{P}}\{x\}\neq \mathrm{cl}_{\mathscr{Q}}\{y\}$ imply that there are disjoint $U\in\mathscr{Q}$ and $V\in\mathscr{P}$ with $x\in U$ and $y\in V$. If we require this condition only for $x\neq y$, we get an axiom strictly weaker than sS_2 and strictly stronger then S_2. (Examples: R and a two-point sub-bispace of R.) An axiom between S_1 and sS_1 can be obtained in a similar way.

3.3 Some facts about completely regular bispaces will be needed later (see [11], [98], [106], [90], [48]).

a) $\mathscr{P}$ is S_π with respect to $\mathscr{Q}$ iff there is a subbase $\mathscr{S}$ for $\mathscr{P}$ such that given $x\in U\in\mathscr{S}$, there is a bi-continuous real function f on X with $f(x)=0$ and $f(y)=1$ $(y\notin U)$;

b) the supremum of a family of S_π-bitopologies is S_π, too;

c) the product of an arbitrary family of S_π-bispaces is S_π;

d) S_π is a hereditary property;

e) any T_π-bispace is bi-homeomorphic to a sub-bispace of some power of R.

§ 4. Bitopologies induced by pseudo-directional structures

4.0 There are several ways of defining bitopologies via other structures, e.g. pairs of a topology and a partial order (Nachbin [72], Adnađević [1]), quasi-uniformities (Nachbin [71], Fletcher [42], Lane [57]; see also [107], [69], [43], *[109]*), certain generalizations of quasi-uniformities (Thampuran [112], Brown [15], *[17]*, [18]), quasi-proximities (Lane [58], see also [25], *[92]*, [46], [44], [48], [49]), generalizations of quasi-proximities (Singal and Lal *[96]*, [97], Mršević [68]), quasi-(pseudo-)metrics (Wilson [116], Kelly [51], see also [77], [52], [105], [91],*[76]*, [89]), families of quasi-pseudo-metrics (Reilly [86]), generalized quasi-pseudo-metrics (Thampuran [110]) metric-like functions with values not from **R** (Pervin and Anton [80], Borşan [12], [13]), families of multifunctions into topological spaces (Smithson [103]), syntopogenous structures (Császár [25], Thampuran [111]). In this section, we aim at adding a new item to the above list.

4.1 Definition (E. Deák [30], [31], [33], [34]). a) A binary relation d between subsets of a set X is a *pseudo-direction on* X if

(i) $\emptyset d \emptyset$, XdX;

(ii) $GdF \Rightarrow G \subset F$;

(iii) $G_1 dF_1$, $G_2 dF_2$, $(G_1, F_1) \neq (G_2, F_2) \Rightarrow F_1 \subset G_2$ or $F_2 \subset G_1$.

d is a *direction* if, in addition,

(iv) $\mathscr{A} \subset \operatorname{dom} d \Rightarrow \bigcup \mathscr{A} \in \operatorname{dom} d$;

(v) $\mathscr{A} \subset \operatorname{ran} d \Rightarrow \bigcap \mathscr{A} \in \operatorname{ran} d$.

The pseudo-direction d is *orderly* if

(vi) $\bigcup \{F \setminus G\colon GdF\} = X$.

A*(n orderly) (pseudo-)directional structure* on X is a family of (orderly) (pseudo-) directions on X.

b) A pseudo-direction d and a pseudo-directional structure D *induce the topologies*

$$\mathscr{T}_d \cong \operatorname{dom} d \cup \text{co-ran}\, d,$$

$$\mathscr{T}_D = \sup_{d \in D} \mathscr{T}_d \cong \bigcup_{d \in D} (\operatorname{dom} d \cup \text{co-ran}\, d).$$

c) Let $(X, \mathscr{T})$ be a topological space. d is a *pseudo-direction of the space* if $\mathscr{T}$ is finer than $\mathscr{T}_d$. D is a *pseudo-directional structure of the space* if $\mathscr{T}$ is finer than $\mathscr{T}_D$. The pseudo-direction d, respectively the pseudo-directional structure D, is *compatible (with* $\mathscr{T}$*)* if $\mathscr{T} = \mathscr{T}_d$, respectively $\mathscr{T} = \mathscr{T}_D$.

4.2 By keeping dom d and co-ran d apart, we can get bitopologies instead of topologies:

Definition. a) A pseudo-direction d, respectively a pseudo-directional structure D, on a set X *induces the bitopology* $(\mathscr{P}_d, \mathscr{Q}_d)$, respectively $(\mathscr{P}_D, \mathscr{Q}_D)$, where

$$\mathscr{P}_d \cong \operatorname{dom} d, \quad \mathscr{Q}_d \cong \text{co-ran}\, d$$

$$\mathscr{P}_D = \sup \{\mathscr{P}_d\colon d \in D\}, \quad \mathscr{Q}_D = \sup \{\mathscr{Q}_d\colon d \in D\}.$$

The corresponding bispaces will be denoted by

$$\mathsf{X}_d = (X; \mathscr{P}_d, \mathscr{Q}_d), \quad \mathsf{X}_D = (X; \mathscr{P}_D, \mathscr{Q}_D).$$

b) $\boldsymbol{d}$ is a *pseudo-direction*, respectively $\boldsymbol{D}$ is a *pseudo-directional structure, of the bispace* X if $(\mathscr{P}, \mathscr{Q})$ is finer than $(\mathscr{P}_d, \mathscr{Q}_d)$, respectively $(\mathscr{P}_D, \mathscr{Q}_D)$. $\boldsymbol{d}$ is a *compatible pseudo-direction*, respectively $\boldsymbol{D}$ is a *compatible pseudo-directional structure, of the bispace* X [in other words: $\boldsymbol{d}$, respectively $\boldsymbol{D}$, is compatible with the bitopology $(\mathscr{P}, \mathscr{Q})$] if it induces $(\mathscr{P}, \mathscr{Q})$, i.e. if $(\mathscr{P}, \mathscr{Q})=(\mathscr{P}_d, \mathscr{Q}_d)$, respectively $(\mathscr{P}, \mathscr{Q})=(\mathscr{P}_D, \mathscr{Q}_D)$.

REMARK. A pseudo-direction $\boldsymbol{d}$ on X is a pseudo-direction of X iff G is $\mathscr{P}$-open and F is $\mathscr{Q}$-closed whenever $G\boldsymbol{d}F$. A pseudo-directional structure on X is a pseudo-directional structure of X iff it consists of pseudo-directions of X.

4.3 Some results about pseudo-directions will be needed; see E. Deák [30], [33], [34] for proofs:

a) Each pseudo-direction $\boldsymbol{d}$ has a natural ordering $<_d$ defined by

$$(G_1, F_1) <_d (G_2, F_2) \Leftrightarrow F_1 \subset G_2, G_1 \neq F_2.$$

In index, we shall write $<\boldsymbol{d}$ for $<_d$.

b) If $\boldsymbol{d}$ is an orderly pseudo-direction on X then for each $x \in X$, there is a unique $(G_d(x), F_d(x)) \in \boldsymbol{d}$ with $x \in F_d(x) \setminus G_d(x)$. Define $\chi_d \colon X \to \boldsymbol{d}$ by

$$\chi_d(x) = (G_d(x), F_d(x)) \quad (x \in X).$$

If $\boldsymbol{d}$ is an orderly pseudo-direction of the space $(X, \mathscr{T})$ then χ_d is a continuous mapping into the space $(\boldsymbol{d}, \mathscr{T}_{<d})$.

c) If $\boldsymbol{D}$ is a compatible orderly pseudo-directional structure of the T_0-space $(X, \mathscr{T})$ then

$$\chi_D \colon (X, \mathscr{T}) \to \prod_{d \in D} (\boldsymbol{d}, \mathscr{T}_{<d})$$

is a topological embedding if the $\boldsymbol{d}$th coordinate of $\chi_D(x)$ is defined as $\chi_d(x)$ $(\boldsymbol{d} \in \boldsymbol{D}, x \in X)$.

d) Any orderly pseudo-direction $\boldsymbol{d}$ on a set X is contained by an orderly direction $\boldsymbol{e}$ such that (in our terminology) $\mathsf{X}_d = \mathsf{X}_e$. There is a maximal $\boldsymbol{e}$ with this property.

e) If $\boldsymbol{d}$ is a direction then $\mathscr{T}_{<d}$ is compact.

f) A space is completely regular iff it has a compatible orderly (pseudo-)directional structure.

4.4 EXAMPLE. Let $(S, <)$ be an ordered set and put

$$\boldsymbol{p}(<) = \{(\emptyset, \emptyset)\} \cup \{(]\leftarrow, x[,]\leftarrow, x]) \colon x \in S\} \cup \{(S, S)\}.$$

Then $\boldsymbol{p}(<)$ is a compatible orderly pseudo-direction of the bispace $\mathsf{S}_<$, i.e. $\mathscr{P}_< = \mathscr{P}_{p(<)}$, $\mathscr{Q}_< = \mathscr{Q}_{p(<)}$.

NOTATIONS. Let $\boldsymbol{m}(<)$ denote the maximal orderly direction belonging to the above $\boldsymbol{p}(<)$ according to 4.3 d). For an arbitrary pseudo-direction $\boldsymbol{d}$, put $\bar{\boldsymbol{d}} = \boldsymbol{m}(<_d)$ and $\mathsf{d} = (\boldsymbol{d}; \mathscr{P}_{\bar{d}}, \mathscr{Q}_{\bar{d}})$, i.e. the bitopology of d is induced by the natural ordering of the pseudo-direction $\boldsymbol{d}$, or equivalently by the pseudo-direction $\boldsymbol{p}(<_d)$, or by the direction $\bar{\boldsymbol{d}}$. Analogous notations will be used for pseudo-directions denoted by other symbols, e.g. $\mathsf{d}_1 = (\boldsymbol{d}_1; \mathscr{P}_{\bar{d}_1}, \mathscr{Q}_{\bar{d}_1})$, $\mathsf{e} = (\boldsymbol{e}; \mathscr{P}_{\bar{e}}, \mathscr{Q}_{\bar{e}})$. If $\boldsymbol{D}$ is a pseudo-directional structure then put

$$\mathsf{D} = (\Pi \boldsymbol{D}; \mathscr{P}_{\bar{D}}, \mathscr{Q}_{\bar{D}}) = \prod_{d \in D} \mathsf{d}, \quad \mathscr{T}_{\bar{D}} = \prod_{d \in D} \mathscr{T}_d.$$

Although it is not necessary in the above definition to assign any meaning to the symbol $\overline{D}$, it is consistent with our notations to regard $\overline{D}$ as the compatible orderly directional structure of $\Pi D = \prod_{d \in D} d$ obtained by pulling up the directions $\overline{d}$ $(d \in D)$ to ΠD, i.e. replacing G and F in each $(G, F) \in \overline{d}$ by their inverse images at the projection onto the appropriate coordinate.

4.5 Proposition. *Each bispace has a compatible directional structure.*

Proof. Put

$$D = \{\{(\emptyset, \emptyset), (G, X), (X, X)\}: G \in \mathscr{P}\} \cup \{\{(\emptyset, \emptyset), (\emptyset, F), (X, X)\}: F \in \text{co-}\mathscr{Q}\}.$$

Remark. If we want to induce only one of the topologies of a bispace X, it can always be done by an *orderly* directional structure:

$$D^{\circ} = \{\{(\emptyset, \emptyset), (\emptyset, G), (G, X), (X, X)\}: G \in \mathscr{P}\}$$

induces a bitopology $(\mathscr{P}, \mathscr{Q}^{\circ})$ where $\mathscr{Q}^{\circ}$ is usually different from $\mathscr{Q}$. There is an obvious analogy with Pervin's quasi-uniformity [78] (see also [69], 1.19).

4.6 On the model of 4.3 b) and c), we have:

Theorem. a) *If d is an orderly pseudo-direction of the bispace X then χ_d is a bi-continuous mapping into* d.

b) *Let D be a compatible orderly pseudo-directional structure of the bispace* X. *Then the mapping χ_D:* X $\to$ D *is bi-continuous and $\varrho\chi_D$ is a bitopological embedding.*

Proof. The easy verification is left to the reader.

4.7 Now we are in a position to prove the main result of this section, the bitopological counterpart of 4.3 f).

Theorem. *For a bispace* X, *the following conditions are equivalent:*
(i) X *is completely regular;*
(ii) X *has a compatible orderly directional structure;*
(iii) X *has a compatible orderly pseudo-directional structure.*

Proof. (i)$\Rightarrow$(ii): Assume X is S_π. Then ϱX is T_π (Remark 1.8 d)), so it is bi-homeomorphic to a sub-bispace A of R^α for some cardinality α (3.3 e)). Let us denote this bi-homeomorphism by η. For $i \in \alpha$, let d_i be the orderly direction of A obtained by pulling up the direction $m(<_R)$ from the ith coordinate to R^α (4.4) and then restricting it to A. Now

$$\{(\eta \circ \varrho_X)^{-1} d_i: i \in \alpha\}$$

is a compatible orderly directional structure of X, where the inverse image of a pseudo-direction is defined as follows: if $\varphi: Y \to Z$ and d is a pseudo-direction on Z then

$$\varphi^{-1} d = \{(\varphi^{-1}[G], \varphi^{-1}[F]): G d F\}.$$

(ii)$\Rightarrow$(iii): Evident.

(iii)⇒(i): It is easy to see that any orderly pseudo-direction induces an S_π-bitopology (in fact, an S_4-bitopology), so only 3.3 b) has to be applied. (An alternative proof: use Theorem 4.6 b), 3.3 c) and 3.3 d).)

4.8 Now come some lemmas paving the way to another proof and a generalization of Theorem 4.7.

Lemma. *Let* X *be a bispace and* $A, B \subset X$. *Then the existence of a bi-continuous real function* f *on* X *with* $f(x)=0$ $(x\in A)$ *and* $f(x)=1$ $(x\in X\setminus B)$ *is equivalent to the existence of an orderly (pseudo-)direction* $\boldsymbol{d}$ *of* X *such that there are* $F\in \operatorname{ran} \boldsymbol{d}$ *and* $G\in \operatorname{dom} \boldsymbol{d}$ *with* $A\subset F\subset G\subset B$.

Remark. For the case of topological spaces, see E. Deák [34].

Proof. a) First assume the existence of an f. Then

$$\boldsymbol{d}=f^{-1}\boldsymbol{m}(<_{\mathbf{R}}),\quad F=f^{-1}[]\leftarrow,0]],\quad G=f^{-1}[]\leftarrow,1[]$$

will do. (Observe that $\boldsymbol{d}$ is a direction.)

b) Assume now that there is a pseudo-direction $\boldsymbol{d}$ as described in the lemma. If $F=G$ then put $f(x)=0$ if $x\in F$ and $f(x)=1$ if $x\in X\setminus F$; this function is clearly bi-continuous, since $F\in\mathscr{P}\cap\text{co-}\mathscr{Q}$. If $F\neq G$ then there are G^* and F^* with $G^*\boldsymbol{d}F$, $G\boldsymbol{d}F^*$ and $(G^*,F)<_d(G,F^*)$. d is normal, being an order-bispace, so for the $\mathscr{Q}_{\bar d}$-closed set $\boldsymbol{a}=]\leftarrow,(G^*,F)]$ and the $\mathscr{P}_{\bar d}$-closed set $\boldsymbol{b}=[(G,F^*),\rightarrow[$, there is a bi-continuous real function g with $g(y)=0$ $(y\in\boldsymbol{a})$ and $g(y)=1$ $(y\in\boldsymbol{b})$. Now $f=g\circ\chi_d$ is bi-continuous by Theorem 4.6 a). Further, if $x\in A$ then $x\in F$, i.e. $\chi_d(x)\in\boldsymbol{a}$, so $f(x)=0$. Similarly, if $x\in X\setminus B$ then $x\in X\setminus G$, i.e. $\chi_d(x)\in\boldsymbol{b}$, so $f(x)=1$.

Definition. The situation described in the lemma will be referred to by the expression f *separates* A *from* $X\setminus B$, respectively $\boldsymbol{d}$ *separates* A *from* $X\setminus B$. (Being separated in this sense is not a symmetric relation.)

4.9 Definition. A pseudo-direction $\boldsymbol{d}$ is *normal* if

$$(1)\qquad \begin{gathered} F\in\operatorname{ran}\boldsymbol{d},\ G\in\operatorname{dom}\boldsymbol{d},\ F\subset G\Rightarrow\\ \Rightarrow\exists G'\in\operatorname{dom}\boldsymbol{d},\ \exists F'\in\operatorname{ran}\boldsymbol{d},\ F\subset G'\subset F'\subset G.\end{gathered}$$

Remarks. a) It clearly does not change the definition if the condition $F\subset G$ in the premiss of (1) is replaced by $\emptyset\neq F\subsetneqq G\neq X$.

b) A pseudo-direction $\boldsymbol{d}$ is normal iff X_d is normal (equivalently: $\mathscr{T}_d$ is normal). This can be easily checked by observing that $\mathscr{P}_d\cup\text{co-}\mathscr{Q}_d$ is ordered by inclusion.

c) If the pairs $(G_0,F)<_d(G,F_0)$ are not neighbours in the ordering then (1) is clearly satisfied for this F and G; therefore the normality of $\boldsymbol{d}$ is also equivalent to the following condition: if (G_1,F_1) and (G_2,F_2) are neighbours in the ordering $<_d$ then at least two of the sets G_1,G_2,F_1,F_2 coincide.

d) Orderly pseudo-directions are evidently normal. In contrast to 4.3 d), a normal pseudo-direction is not always contained by a normal direction inducing the same bitopology.

e) Using normal pseudo-directions is an adaptation of the "$(\mathscr{H},\mathbf{R})$-method" of E. Deák and Hamburger [35] p. 132.

LEMMA. *If $\boldsymbol{d}$ is a normal pseudo-direction of a bispace X then there exists an orderly direction $\boldsymbol{e}$ of X such that sets separated by $\boldsymbol{d}$ are separated by $\boldsymbol{e}$, too.*

PROOF. Applying Zorn's Lemma, choose a maximal pseudo-direction $\boldsymbol{c} \supset \boldsymbol{d}$ of X. Put

$$A^* = \begin{cases} \bigcup \{F \in \operatorname{ran} \boldsymbol{c} \colon F \subset A\} & \text{if } A \in \operatorname{dom} \boldsymbol{c} \\ \bigcap \{G \in \operatorname{dom} \boldsymbol{c} \colon A \subset G\} & \text{if } A \in \operatorname{ran} \boldsymbol{c} \end{cases}$$

and

$$\boldsymbol{e} = \{(G^*, F^*) \colon G\boldsymbol{c}F\}.$$

It is left to the reader to verify that $\boldsymbol{e}$ fulfils the conditions in the conclusion of the lemma. (Or see in [34].)

4.10 LEMMA. *For a bispace X, the following conditions are equivalent:*
(i) *$\mathscr{P}$ is S_π with respect to $\mathscr{Q}$;*
(ii) *if $x \notin B \in$ co-$\mathscr{Q}$ then $\{x\}$ can be separated from B by an orderly direction of X;*
(iii) *if $x \notin B \in$ co-$\mathscr{Q}$ then $\{x\}$ can be separated from B by a normal pseudo-direction of X.*

PROOF. (i)⇔(ii): Lemma 4.8.
(ii)⇒(iii): Evident.
(iii)⇒(ii): Lemma 4.9.

REMARK. Compare this lemma with a result of Smirnov's (the formulation given here is from Császár [24], the original version in [102] is less general): if a relation $\boldsymbol{r}$ between subsets of a topological space $(X, \mathscr{T})$ satisfies the condition

$$(1) \qquad F\boldsymbol{r}G \Rightarrow F \in \text{co-}\mathscr{T},\ G \in \mathscr{T},\ F \subset G,\ \exists G',\ \exists F',\ F\boldsymbol{r}G' \subset F'\boldsymbol{r}G$$

then F and $X \setminus G$ can be separated by a continuous real function whenever $F\boldsymbol{r}G$.

4.11 LEMMA. *Let X be a bispace and n a positive integer.*
a) *If the sets A_i can be separated from the set B by normal pseudo-directions of X $(1 \leqq i \leqq n)$ then $\bigcup_{i=1}^{n} A_i$ can also be separated from B by a normal pseudo-direction of X.*
b) *If the set A can be separated from the sets B_i by normal pseudo-directions of X $(1 \leqq i \leqq n)$ then A can be separated also from $\bigcup_{i=1}^{n} B_i$ by a normal pseudo-direction of X.*

PROOF. a) Let $\boldsymbol{d}_i$ denote a normal pseudo-direction of X separating A_i from B $(1 \leqq i \leqq n)$. Choose $F_i^* \in \operatorname{ran} \boldsymbol{d}_i$ and $G_i^* \in \operatorname{dom} \boldsymbol{d}_i$ such that

$$(1) \qquad A_i \subset F^* \subset G_i^* \subset X \setminus B \quad (1 \leqq i \leqq n)$$

and put

$$\boldsymbol{d}_0 = \Big\{(\emptyset, \emptyset), \big(\emptyset, \bigcup_{i=1}^{n} F_i^*\big), \big(\bigcup_{i=1}^{n} G_i^*, X\big), (X, X)\Big\}.$$

Let $\boldsymbol{D}$ denote the system of those pseudo-directions $\boldsymbol{d}$ of X which contain $\boldsymbol{d}_0$ and satisfy the following condition:

$$(2)\qquad \begin{gathered} F\in \operatorname{ran} \boldsymbol{d},\ G\in \operatorname{dom} \boldsymbol{d},\ F\subset G \Rightarrow \\ \Rightarrow \exists F_i\in \operatorname{ran} \boldsymbol{d}_i,\ \exists G_i\in \operatorname{dom} \boldsymbol{d}_i,\ F_i\subset G_i \quad (1\leqq i\leqq n),\quad F\subset \bigcup_{i=1}^{n} F_i,\ \bigcup_{i=1}^{n} G_i\subset G. \end{gathered}$$

Clearly, $\boldsymbol{d}_0\in \boldsymbol{D}$. Let $\boldsymbol{e}$ be a maximal element of $\boldsymbol{D}$. $\boldsymbol{e}$ is a pseudo-direction of X and it separates $\bigcup_{i=1}^{n} A_i$ from B (since already $\boldsymbol{d}_0$ does so, by (1)). Assume $\boldsymbol{e}$ is not normal. Then there exist $(\Gamma, F) <_{\boldsymbol{e}} (G, \Phi)$ which are neighbours such that $\Gamma\neq F$ and $G\neq \Phi$. Choose sets G_i and F_i to this F and G according to (2) and then $F_i'\in \operatorname{ran} \boldsymbol{d}_i$ and $G_i'\in \operatorname{dom} \boldsymbol{d}_i$ with $F_i\subset G_i'\subset F_i'\subset G_i$ by the normality of $\boldsymbol{d}_i$ $(1\leqq i\leqq n)$. We have

$$F\subset G' = \bigcup_{i=1}^{n} G_i' \subset F' = \bigcup_{i=1}^{n} F_i' \subset G,$$

thus $(G', F')\notin \boldsymbol{e}$. Now $\boldsymbol{e}\cup\{(G', F')\}$ also satisfies (2), contradicting the maximality of $\boldsymbol{e}$.

b) Apply a) to $(X, \mathscr{Q}, \mathscr{P})$. [Observe that (i) $\boldsymbol{d}$ is a pseudo-direction of X iff $\boldsymbol{d}^c$ is a pseudo-direction of $(X, \mathscr{Q}, \mathscr{P})$; (ii) $\boldsymbol{d}$ is normal iff $\boldsymbol{d}^c$ is normal; (iii) $\boldsymbol{d}$ separates A from B iff $\boldsymbol{d}^c$ separates B from A.]

Remarks. a) The pseudo-direction $\boldsymbol{e}$ obtained in the above proof satisfies a condition somewhat stronger than normality: if (G_1, F_1) and (G_2, F_2) are neighbours in the ordering $<_{\boldsymbol{e}}$ then $G_1=F_1$ or $G_2=F_2$ (cf. Remark 4.9 c)). Equivalently: $\boldsymbol{e}$ satisfies 4.9 (1) with the stronger assumption $G'\boldsymbol{e}F'$ in the conclusion. Each normal pseudo-direction $\boldsymbol{d}$ can be enlarged to a pseudo-direction $\boldsymbol{e}$ satisfying this stronger condition such that $\operatorname{dom} \boldsymbol{d}=\operatorname{dom} \boldsymbol{e}$ and $\operatorname{ran} \boldsymbol{d}=\operatorname{ran} \boldsymbol{e}$ through saturating it in the sense of E. Deák [32] (0.4), i.e. putting

$$\boldsymbol{e} = \boldsymbol{d}\cup\{(A, A)\colon A\in \operatorname{dom} \boldsymbol{d}\cap \operatorname{ran} \boldsymbol{d}\};$$

therefore the choice between these two possible definitions of normality is only a matter of taste. Our choice has been motivated by Remark 4.9 d). Observe that if we substitute $F\subsetneqq G$ for $F\subset G$ in the premiss of (1) simultaneously with requiring $G'\boldsymbol{d}F'$ in the conclusion, we are back at normality.

b) A shorter proof of the lemma could be given by invoking Lemmas 4.8 and 4.9. It will be explained presently why we have decided for a direct proof. (The lemma can also be proved through a bitopological analogue of Smirnov's theorem mentioned in Remark 4.10, with co-$\mathscr{T}$ and $\mathscr{T}$ replaced by co-$\mathscr{Q}$ and $\mathscr{P}$ in 4.10 (1) and asserting bi-continuity in the conclusion.)

c) Henceforth only one of dual statements like a) and b) in the lemma will be formulated.

4.12 NOTATIONS. Let $\boldsymbol{d}$ be a pseudo-direction and $\boldsymbol{D}$ a pseudo-directional structure on a set X. Then

$$\operatorname{dom}^* \boldsymbol{d} = \{G \in \operatorname{dom} \boldsymbol{d}\colon G = \bigcup\{F \in \operatorname{ran} \boldsymbol{d}\colon F \subset G\}\},$$

$$\operatorname{ran}^* \boldsymbol{d} = \{F \in \operatorname{ran} \boldsymbol{d}\colon F = \bigcap\{G \in \operatorname{dom} \boldsymbol{d}\colon F \subset G\}\},$$

$$\mathscr{P}^*_{\boldsymbol{D}} \triangleq \bigcup_{\boldsymbol{d}\in \boldsymbol{D}} \operatorname{dom}^* \boldsymbol{d},\ \mathscr{Q}^*_{\boldsymbol{D}} \triangleq \bigcup_{\boldsymbol{d}\in \boldsymbol{D}} \text{co-ran}^* \boldsymbol{d},\ \mathsf{X}^*_{\boldsymbol{D}} = (X; \mathscr{P}^*_{\boldsymbol{D}}, \mathscr{Q}^*_{\boldsymbol{D}}).$$

REMARKS. a) In other words, dom $\boldsymbol{d}$ consists of those $G \in \operatorname{dom} \boldsymbol{d}$ for which each $x \in G$ is separated by $\boldsymbol{d}$ from $X \setminus G$.

b) Assume $\boldsymbol{D}$ is a pseudo-directional structure of a bispace X and $\mathscr{P} = \mathscr{P}^*_{\boldsymbol{D}}$. Then we have $\mathscr{P} = \mathscr{P}_{\boldsymbol{D}}$ since $\mathscr{P}^*_{\boldsymbol{D}} \subset \mathscr{P}_{\boldsymbol{D}} \subset \mathscr{P}$. So if $\boldsymbol{D}$ is a pseudo-directional structure of $\mathsf{X}^*_{\boldsymbol{D}}$ then it is compatible. (In general, $\boldsymbol{D}$ need not be a pseudo-directional structure of $\mathsf{X}^*_{\boldsymbol{D}}$, as it may induce a bitopology strictly finer than $(\mathscr{P}^*_{\boldsymbol{D}}, \mathscr{Q}^*_{\boldsymbol{D}})$.) ∎

LEMMA. *For a bispace* X, *the following conditions are equivalent:*
(i) $\mathscr{P}$ *is* S_π *with respect to* $\mathscr{Q}$;
(ii) *there is an orderly directional structure* $\boldsymbol{D}$ *of* X *such that* $\mathscr{P} = \mathscr{P}_{\boldsymbol{D}}$;
(iii) *there is a normal pseudo-directional structure* $\boldsymbol{D}$ *of* X *such that* $\mathscr{P} = \mathscr{P}^*_{\boldsymbol{D}}$.

PROOF. (i)⇒(ii): For each $x \in U \in \mathscr{P}$, take an orderly direction $\boldsymbol{d}(x, U)$ of X separating $\{x\}$ from $X \setminus U$ (Lemma 4.10) and put

$$\boldsymbol{D} = \{\boldsymbol{d}(x, U)\colon x \in U \in \mathscr{P}\}.$$

For each $x \in U \in \mathscr{P}$, there is a $G \in \operatorname{dom} \boldsymbol{d}(x, U)$ with $x \in G \subset U$ (by the separating property of $\boldsymbol{d}(x, U)$), therefore $\mathscr{P} \subset \mathscr{P}_{\boldsymbol{D}}$. Conversely, $\mathscr{P}_{\boldsymbol{D}} \subset \mathscr{P}$ since $\boldsymbol{D}$ is a directional structure of X.

(ii)⇒(iii): If a pseudo-direction $\boldsymbol{d}$ is orderly then $\operatorname{dom}^* \boldsymbol{d} = \operatorname{dom} \boldsymbol{d}$.

(iii)⇒(i): Take some $x \in U \in \mathscr{P} = \mathscr{P}_{\boldsymbol{D}}$, $U \neq X$. Then there are a positive integer n and $G_i \in \operatorname{dom}^* \boldsymbol{d}_i$ with $\boldsymbol{d}_i \in \boldsymbol{D}$ $(1 \leq i \leq n)$ such that $\bigcap_{i=1}^{n} G_i \subset U$. Now $\boldsymbol{d}_i$ separates $\{x\}$ from $X \setminus G_i$ $(1 \leq i \leq n)$, thus there exists a normal pseudo-direction of X separating $\{x\}$ from $X \setminus U$ (Lemma 4.11 b)), consequently $\mathscr{P}$ is S_π with respect to $\mathscr{Q}$ (Lemma 4.10).

REMARKS. c) In (iii), it is not superfluous to assume that $\boldsymbol{D}$ is normal. In fact, the existence of a pseudo-directional structure $\boldsymbol{D}$ of X with $\mathscr{P} \subset \mathscr{P}^*_{\boldsymbol{D}}$ (which means the same as $\mathscr{P} = \mathscr{P}^*_{\boldsymbol{D}}$) is equivalent to the following condition strictly weaker than (i): if $x \in U \in \mathscr{P}$ then there are $G_i \in \mathscr{P}$ $(i \in \omega)$ such that

$$x \in G_0,\ \operatorname{cl}_{\mathscr{Q}} G_i \subset G_{i+1} \subset U \quad (i \in \omega).$$

To see that this condition does not imply (i), take $(X; \mathscr{T}, \mathscr{T})$ where $(X, \mathscr{T})$ is the space defined below.

Let $X = \mathbf{R}^2$,

$$A = \{n + 1/k\colon n \in \omega, 0 \neq k \in \omega\},$$

$$p_{nkl} = n + 1/(k+1) + 1/l, \quad q_{nkl} = n + 1/k - 1/l.$$

For $z=(x, y)\in X$, put

$$\mathcal{N}(z) = \begin{cases} \{]x-\varepsilon, x+\varepsilon[\times\{y\}: \varepsilon > 0\} & \text{if } x\in A, y \neq 0, \\ \{\{z\} \cup [(\{p_{nkl}, q_{nkl}\}\times \mathbf{R})\setminus S]: |S| < \omega\} & \\ \quad \text{if } n\in\omega, 0 \neq k\in\omega, 2k(k+1) < l\in\omega, x = p_{nkl}, y = 0, & \\ \{\{z\} \cup (]n, \to[\times \mathbf{R}): n\in\omega\} & \text{if } x = y = 0, \\ \{\{z\}\} & \text{otherwise.} \end{cases}$$

Let $\mathcal{T}$ be the topology for which $\mathcal{N}(z)$ is a neighbourhood base at z. Observe that, with $z_0=(0, 0)$, the sets

$$H_t = \{z_0\}\cup(]t, \to[\times \mathbf{R}) \quad (t\in A)$$

are open and

$$z_0\in H_{n+1/k}, \ \mathrm{cl}\, H_{n+1/k} \subset H_{n+1/(k+1)} \subset H_n \quad (n\in\omega, \ 0 \neq k\in\omega);$$

furthermore, the elements of $\mathcal{N}(z)$ are open-closed for $z\neq(0, 0)$, excepting some rational εs in the first line of the definition. But $\mathcal{T}$ is not S_π, since if f is a continuous real function then $f(z)=f((0, 0))$ for $z\in(A\times \mathbf{R})\setminus C$ where C is countable; this can be shown applying the standard argument used in similar spaces, see e.g. [104] Example 94.

d) $\mathcal{P}_D^*$ cannot be replaced by $\mathcal{P}_D$ in (iii), since any bispace has a compatible normal directional structure, namely the one given in the proof of Proposition 4.5.

4.13 E. Deák and Hamburger [35] have proved that several internal characterizations (i.e. characterizations in terms of objects in the space) of complete regularity in topological spaces are internally equivalent (i.e. their equivalence can be proved without using external tools such as the set of the real numbers). On the model of their method, we intend to show that some internal characterizations of the bitopological complete regularity are also internally equivalent. For this purpose, let us replace Definition 1.7 ab) and part of Definition 1.7 ea) by the following:

Definition. a) In a bispace X, $\mathcal{P}$ is *S_π with respect to* $\mathcal{Q}$ if for any $x\in U\in\mathcal{P}$, there is a normal pseudo-direction of X separating $\{x\}$ from $X\setminus U$.

b) The bispace X is S_π if $\mathcal{P}$ is S_π with respect to $\mathcal{Q}$ and vice versa.

4.14 Check that the proof of Lemma 4.12 is internal if (i) is understood in the sense of Definition 4.13 a). [The other lemmas used in the proof have been proved internally, including (ii)⇒(iii) in Lemma 4.10.] So we have:

Theorem. *For a bispace X, the following conditions are internally equivalent:*

(i) *X is S_π in the sense of Definition 4.13;*

(ii) *X has a compatible orderly directional structure;*

(iii) *X has a normal pseudo-directional structure $\mathbf{D}$ with* $\mathsf{X}=\mathsf{X}_D^*$.

Proof. (i)⇒(ii): Take orderly directional structures $\mathbf{D}$ and $\mathbf{E}$ of X such that $\mathcal{P}=\mathcal{P}_D$ and $\mathcal{Q}=\mathcal{Q}_E$ (Lemma 4.12). Now $\mathsf{X}=\mathsf{X}_{D\cup E}$.

(ii)⇒(iii): Evident.

(iii)⇒(ii): Lemma 4.12.

REMARKS. a) It does not change (iii) if we assume that $\boldsymbol{D}$ is compatible, too (cf. Remark 4.12 b)).

b) The normality is needed in (iii) (cf. Remark 4.12 c)).

c) The condition $\mathsf{X}=\mathsf{X}_D^*$ cannot be dropped from (iii) even if $\boldsymbol{D}$ is supposed to be compatible (cf. Remark 4.12 d)).

d) It is necessary to assume in (iii) that $\boldsymbol{D}$ is a pseudo-directional structure *of the bispace* X, and not just *on the set X*. Indeed, if X is an arbitrary set, $\mathcal{Q}$ is indiscrete and $\mathcal{P} \trianglelefteq \{G\}$ where $\emptyset \neq G \neq X$ then there is a normal pseudo-directional structure $\boldsymbol{D}$ on X with $\mathsf{X}=\mathsf{X}_D^*$ iff G is infinite. (Consequently, the existence of such a $\boldsymbol{D}$ does not depend only on $\varrho\mathsf{X}$).

COROLLARY. *$\mathcal{P}$ is S_π with respect to $\mathcal{Q}$ iff there is a topology $\mathcal{Q}_1 \subset \mathcal{Q}$ such that $(\mathcal{P}, \mathcal{Q}_1)$ is S_π.*

PROOF. First assume that $\mathcal{P}$ is S_π with respect to $\mathcal{Q}$. Then there is an orderly directional structure $\boldsymbol{D}$ of X such that $\mathcal{P}=\mathcal{P}_D$ (Lemma 4.12). We have $\mathcal{Q}_D \subset \mathcal{Q}$. Moreover, $(\mathcal{P}, \mathcal{Q}_D)$ is S_π by the above theorem.

b) Assume now that $(\mathcal{P}, \mathcal{Q}_1)$ is S_π and $\mathcal{Q}_1 \subset \mathcal{Q}$. Then there is a compatible orderly directional structure $\boldsymbol{D}$ of $(X; \mathcal{P}, \mathcal{Q}_1)$. Now $\boldsymbol{D}$ is an orderly directional structure of X and $\mathcal{P}=\mathcal{P}_D$, therefore $\mathcal{P}$ is S_π with respect to $\mathcal{Q}$ (Lemma 4.12).

PROBLEM. Is a statement analogous to this corollary true for S_2, S_3, or S_4? (Below we give a counterexample for S_1.)

EXAMPLE. If

$$X=\mathbf{J} \quad \mathcal{P} \trianglelefteq \{]0,1]\},\ \mathcal{Q}_1 \subset \mathcal{Q} \trianglelefteq \{[0, 1/n[: 0 \neq n \in \omega\}$$

then $\mathcal{P}$ is S_1 with respect to $\mathcal{Q}$, but $(X; \mathcal{P}, \mathcal{Q}_1)$ is not S_1. Indeed, if $\mathcal{Q}_1$ is indiscrete then $\mathcal{P}$ is not S_1 with respect to $\mathcal{Q}_1$; on the other hand, if $\mathcal{Q}_1$ is not indiscrete then $\mathcal{Q}_1$ is not S_1 with respect to $\mathcal{P}$.

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