Studia Scientiarum Mathematicarum Hungarica

VOLUME 23
NUMBERS 1—2
1988

1828

AKADÉMIAI KIADÓ, BUDAPEST

STUDIA SCIENTIARUM MATHEMATICARUM HUNGARICA

A QUARTERLY OF THE HUNGARIAN ACADEMY OF SCIENCES

Studia Scientiarum Mathematicarum Hungarica publishes original papers on mathematics mainly in English, but also in German, French and Russian.

Studia Scientiarum Mathematicarum Hungarica is published in yearly volumes of four issues (mostly double numbers published semiannually) by

AKADÉMIAI KIADÓ
Publishing House of the Hungarian Academy of Sciences
H-1054 Budapest, Alkotmány u. 21.

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Studia Scientiarum Mathematicarum Hungarica

Volume 23

Akadémiai Kiadó, Budapest
1988

STUDIA SCIENTIARUM MATHEMATICARUM HUNGARICA

Volume 22

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 1—6

BEMERKUNGEN ZUM PROBLEM DER DÜNNSTEN ÜBERDECKUNGEN DER HYPERBOLISCHEN EBENE DURCH KONGRUENTE HYPERZYKELBEREICHE

I. VERMES

Wir wollen in dieser Arbeit das Problem der dünnsten Überdeckungen der hyperbolischen Ebene untersuchen, wobei die Bereiche in den Überdeckungen kongruente Hyperzykelbereiche sind, für die gewisse zusätzliche Geräumigkeitsbedingung gefordert ist.

Man versteht unter einem *Hyperzykel* (oder einer *Abstandslinie* bzw. *Äquidistante*) die Gesamtheit derjenigen Punkte der Ebene, die von einer Geraden gleichen Abstand l haben, und alle auf derselben Seite von ihr gelegen sind. Die beiden kongruenten Äquidistanten auf verschiedenen Seiten von der Geraden (d. h. der *Grundlinie*) begrenzen einen Teil der Ebene, der als *Hyperzykelbereich* vom Abstand l heißt.

Unsere Untersuchungen verknüpfen sich zu den Arbeiten [2], [4] bzw. [3], [5] (s. noch [6] 224—238). In diesen Arbeiten beschäftigte L. Fejes Tóth sich mit den Kreisausfüllungen der hyperbolischen Ebene und mit der dichtesten Horozyklenlagerung bzw. mit den Kreisüberdeckungen der hyperbolischen Ebene und mit der dünnsten Horozyklenüberdeckung. Die Ergebnisse seiner Untersuchungen zeigen, daß die dichteste Ausfüllungen bzw. die dünnste Überdeckungen in den Fällen der regelmäßigen Bereichsysteme zustande kommen.

Auf die Anregung dieser Ergebnisse wurden die oberen Dichteschranken der Ausfüllungen der hyperbolischen Ebene durch kongruente Hyperzykelbereiche in [10] und die dichtesten Konfigurationen in [9] gegeben. In derartigen Untersuchungen sind die Abschätzungen der Dichten in jedem Fall auf je einer vieleckigen Zelle der geeigneten ausgewählten Zellenzerlegungen der hyperbolischen Ebene durch die obere bzw. untere Schranke der Dichten gegeben. K. Böröczky hat nämlich in [1] gezeigt, daß kein Dichtenbegriff bezüglich der ganzen hyperbolischen Ebene existiert werden kann, der die trivialen und natürlichen Bedingungen — mindestens im klassischen Sinne — erfüllen könnte.

Es ist im allgemeinen notwending einige Voraussetzungen für die Bereiche in den Untersuchungen der Überdeckungsprobleme stellen, daß die übertriebenen Häufungen der Bereiche ausschließbar seien.

In dieser Arbeit untersuchen wir diejenigen Überdeckungen der hyperbolischen Ebene durch die kongruente Hyperzykelbereiche vom Abstand l, zu den je eine Hyperzyklenausfüllung durch kongruente Exemplaren der Bereiche vom Abstand $l-d$ existiert, falls man den zu den Hyperzykelbereichen gehörigen Abstand l mit demselben Abstand d verkleinert ($l>d>0$). Also ist die Existenz von einem solchen d als

1980 *Mathematics Subject Classification*. Primary 52A45; Secondary 52A40.
Key words and phrases. Packing, covering, tiling, inequalities and extremum problems, convex sets.

zusätzliche Geräumigkeitsbedingung angenommen. Diese Geräumigkeitsbedingung ist wirklich eine unentbehrliche Voraussetzung für die Überdeckungen, denn kann solche Überdeckung gegeben werden, in der die Grundlinien der Hyperzykelbereiche paarweise je ein gemeinsames Lot haben, da aber existiert mindestens ein Punkt in der Ebene, der im Inneren unendlich vieler Bereiche liegt [13].

Fortan werden wir über geräumige Hyperzyklenüberdeckung sprechen.

Betrachten wir die Zerlegung der Ebene in die Dirichletschen Zellen, die zu einem solchen System der Hyperzykelbereiche gehört. Die Dirichletschen Zellen der Hyperzykelbereiche geben eine eindeutige Zerlegung der Ebene ebenso, wie im Falle der Kreissysteme, denn der Begriff der Potenzlinien zweier Abstandslinien kann auf ähnliche Weise erklärt werden. (Die Potenzlinie zweier kongruenter Abstandslinien ist die Symmetrieachse ihrer Grundlinien.) Die oben erwähnten Zellenzerlegungen übereinkommen mit der Dirichletschen Zellen der entsprechenden Hyperzyklenausfüllungen vom Abstand $l-d$.

Es wird gezeigt, daß die Eckpunkte der betrachteten Dirichletschen Zellen die eigentlichen Punkte der hyperbolischen Ebene sind, und die Anzahl der in einem Eckpunkt sich treffenden Seiten eine endliche — bezüglich der ganzen hyperbolischen Ebene — beschränkte natürliche Zahl ist.

In der Tat, falls ein Eckpunkt in den Zellenzerlegung ein uneigentlicher Punkt wäre, so könnte man keine Überdeckung aus einer Hyperzyklenausfüllung vom Abstand $l-d$ durch die Steigerung der Abstände der Hyperzykelbereiche erzeugen. Andererseits hätte die Anzahl der in einem Eckpunkt sich treffenden Seiten keine obere Schranke, so hätten die Größen der Winkel der Dirichletschen Zellen keine untere Schranke, folglich wären die Radien der die entsprechenden Grundlinien berührenden Kreise um die Eckpunkte der Zellen (s. Fig.) auch unbeschränkt. In diesem Falle könnte man auch keine Überdeckung aus einer Hyperzyklenausfüllung vom Abstand $l-d$ durch die Steigerung der Abstände der Hyperzykelbereiche erzeugen.

Es ist leicht zu sehen, daß die obere Schranke (N) der Anzahl der in einem Eckpunkt sich treffenden Seiten und der Abstand d voneinander abhängen. Man kann solche Überdeckung konstruieren, in deren die Eckpunkte der Dirichletschen Zellen n-tes Grades für jede Werte von n vorkommen können, wobei $3 \leqq n \leqq N$ ist.

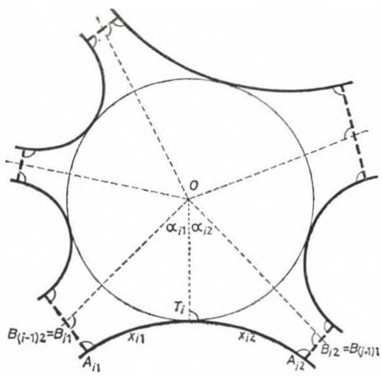

Fig. 1

Betrachten wir jetzt die zu den Dirichletschen Zellen gehörigen dualen Zerlegungen der hyperbolischen Ebene. In einer solchen dualen Zerlegung gehört je ein rechtwinkliges $2n$-Eck ($n \geqq 3$ ist und n hat eine obere Schranke, wie es früher gezeigt wurde) zu jedem Eckpunkt der Dirichletschen Zellen, dessen Seiten einerseits die gemeinsamen Lote der Grundlinien der Abstandslinien sind, deren Potenzlinien in diesem Eckpunkt sich treffen; andererseits gehören die Strecken zu den Seiten dieses Vieleckes, die zwischen den Fußpunkten der gemeinsamen Lote auf den Grundlinien liegen. Die Seiten eines solchen $2n$-Eckes liegen abwechselnd auf den Grundlinien bzw. auf ihren gemeinsamen Loten.

Wir wollen das Minimum der Überdeckungsdichten auf diesen $2n$-Ecken bestimmen. Wir zeigen, daß das Minimum sich auf einem geeigneten regelmäßigen $2n$-Eck verwirklicht.

Nehmen wir ein solches $2n$-Eck aus der dualen Zerlegung (s. Fig.), wobei der Punkt O ein Eckpunkt in der Zerlegung in die Dirichletschen Zellen, und deshalb der Mittelpunkt eines Kreises ist. Dieser Kreis berührt die Grundlinien der zum O gehörigen Abstandslinien. Es seien die Eckpunkte dieses $2n$-Eckes: $A_{11}, A_{12}, A_{21}, A_{22}, A_{31}, A_{32}, \ldots, A_{i1}, A_{i2}, \ldots, A_{n1}, A_{n2}$, das Radius des Kreises r und der Fußpunkt des aus dem Punkt O auf die Strecke A_{i1}, A_{i2} gefällten Lotes T_i. Die aus dem Punkt O ausgehenden Potenzlinien halbieren und senkrecht schneiden die Seiten $A_{(i-1)2}A_{i1}$ bzw. $A_{i2}A_{(i+1)1}$ in den Punkten $B_{i1}=B_{(i-1)2}$ bzw. $B_{i2}=B_{(i+1)1}$ $(i=1, 2, 3, \ldots, n$ und natürlich mit der Bezeichung $A_{n2}=A_{02})$. Es seien noch die Strecken $A_{i1}T_i$ und T_iA_{i2} mit x_{i1} bzw. x_{i2}, und die Winkel $B_{i1}OT_i$ und T_iOB_{i2} mit α_{i1} bzw. α_{i2} bezeichnet.

Auf Grund der trigonometrischen Beziehungen der Lambertschen Vierecke[1] ergibt sich die folgende Beziehung für das Viereck $A_{i1}T_1OB_{i1}$:

$$\operatorname{ctg} \alpha_{i1} = \operatorname{sh} r \operatorname{th} x_{i1}.$$

Daraus folgen für das Viereck $A_{1i}T_iOB_{i1}$ und auf ähnliche Weise für das Viereck $A_{i2}T_iOB_{i2}$ die Gleichungen:

$$x_{i1} = \operatorname{arth} \frac{\operatorname{ctg} \alpha_{i1}}{\operatorname{sh} r} \quad \text{bzw.} \quad x_{i2} = \operatorname{arth} \frac{\operatorname{ctg} \alpha_{i2}}{\operatorname{sh} r}.$$

Betrachten wir die Überdeckungsdichte auf dem Vieleck $A_{11}, A_{12}, A_{21}, A_{22}, A_{31}, A_{32}, \ldots, A_{i1}, A_{i2}, \ldots, A_{n1}, A_{n2}$:

$$D(l) = \frac{\operatorname{sh} l \sum\limits_{\substack{i=1 \\ j=1,2}}^{n} x_{ij}}{(2n-2)\pi - 2n\dfrac{\pi}{2}} = \frac{\operatorname{sh} l \sum\limits_{\substack{i=1 \\ j=1,2}}^{n} \operatorname{arth} \dfrac{\operatorname{ctg} \alpha_{ij}}{\operatorname{sh} r}}{(n-2)\pi},$$

wobei $r > \Delta(\alpha_{ij})$ und $\alpha_{ij} < \dfrac{\pi}{2}$ $\left(\text{also } \dfrac{\operatorname{ctg} \alpha_{ij}}{\operatorname{sh} r} < 1\right)$ bestehen[2].

[1] S. z. B. [8] § 10 und § 13 (S. 37—40 bzw. S. 71—78).

[2] $\Delta(\alpha_{ij})$ bezeichnet das zum Winkel α_{ij} gehörige Parallellot.

Jetzt beweisen wir, daß die Funktion $x \mapsto y = \operatorname{arth} \frac{\operatorname{ctg} x}{\operatorname{sh} r}$ im Intervall $0 < x < \frac{\pi}{2}$ $\left(\text{im Falle } \frac{\operatorname{ctg} x}{\operatorname{sh} r} < 1\right)$ konvex ist.

Die erste und zweite Ableitungen dieser Funktion sind

$$y' = -\frac{\operatorname{sh} r}{\operatorname{sh}^2 r \sin^2 x - \cos^2 x} \quad \text{und}$$

$$y'' = \frac{\operatorname{sh} r \sin 2x\,(\operatorname{sh}^2 r + 1)}{(\operatorname{sh}^2 r \sin^2 x - \cos^2 x)^2},$$

folglich besteht $y'' > \left(0 < x < \frac{\pi}{2}\right)$, deshalb ist die Funktion $x \mapsto y = \operatorname{arth} \frac{\operatorname{ctg} x}{\operatorname{sh} r}$ (von unten) konvex.

Wenden wir die Jensensche Ungleichung[3] für die Funktion $D(l)$ an:

$$D(l) = \frac{\operatorname{sh} l \sum\limits_{\substack{i=1\\ j=1,2}}^{n} \operatorname{arth} \frac{\operatorname{ctg} \alpha_{ij}}{\operatorname{sh} r}}{(n-2)\pi} = \frac{\operatorname{sh} l}{\frac{(n-2)\pi}{2n}} \, \frac{\sum\limits_{\substack{i=1\\ j=1,2}}^{n} \operatorname{arth} \frac{\operatorname{ctg} \alpha_{ij}}{\operatorname{sh} r}}{2n} \geqq$$

$$\geqq \frac{\operatorname{sh} l \cdot \operatorname{arth} \frac{\operatorname{ctg} \frac{\pi}{n}}{\operatorname{sh} r}}{\frac{(n-2)\pi}{2n}} = D(l, n, r).$$

Die Gleichheit besteht nur im Falle, wenn jeder Winkel $\alpha_{ij} = \frac{\pi}{n}$ ist, was die Regelmäßigkeit des Vieleckes bedeutet.

In [11] und [12] wurden die reguläre Überdeckungen der hyperbolischen Ebene durch kongruente Hyperzykelbereiche untersuch, die aus den regulären Ausfüllungen der Ebene durch kongruente Hyperzykelbereiche (s. [9]) abgeleitet werden können, wenn man den Abstand der Hyperzykelbereichen nur minimalermaßen zur Überdeckung steigert. Die Überdeckungsdichten wurden auch in diesen Untersuchungen bezüglich der den Direchletschen Zerlegungen entsprechenden dualen Zerlegungen betrachtet, und die Dichtefunktion ist

$$D(l, n) = \frac{\operatorname{sh} l \cdot \operatorname{arth} \frac{\operatorname{ctg} \frac{\pi}{n}}{\operatorname{sh} l}}{\frac{(n-2)\pi}{2n}}, \quad \text{wobei} \quad \Delta\left(\frac{\pi}{n}\right) < l < +\infty$$

und $n \geqq 3$ sind.

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[3] S. [7] S. 31—34.

In den zitierten Arbeiten wurde — unter anderem — die Ungleichung $D(l, n+1) > D(l, n)$ bewiesen, wo natürlich $l > \Delta\left(\frac{\pi}{n+1}\right)$ ist; daraus folgt, daß die dünnste reguläre Überdeckung der hyperbolischen Ebene durch kongruente Hyperzykelbereiche vom Abstand l sich im Falle $n=3$ verwirklicht. Ferner gilt $\lim_{l=+\infty} D(l, 3) = \frac{\sqrt{12}}{\pi}$, was die dünnste Horozyklenüberdeckungsdichte der hyperbolischen Ebene ist.

Es ist leicht zu sehen, daß die folgende Ungleichung für die regulären Überdeckungen zwischen den Funktionen $D(l, n)$ und $D(l, n, r)$ im Falle $l \geqq r$ gilt:

$$D(l, n, r) \geqq D(l, n)$$

und die Gleichheit nur im Falle $l=r$ besteht.

Daraus bereits ergibt es sich, daß die dünnste Hyperzyklenüberdeckung sich für gegebenen Abstand l der Hyperzykelbereichen so verwirklicht wird, falls $n=3$ und die Überdeckung regulär ist, ferner die entsprechenden Hyperzyklen sich in den Eckpunkten der Dirichletschen Zellen schneiden.

ANMERKUNG. Man soll bemerken, daß Bui Van Dung — nach dem Eingang des Manuskriptes dieser Arbeit — sich in seiner Dissertation mit der ϱ-Systeme der Hyperzykellagerungen — auf Grund der Idee von J. Molnár — beschäftigte. (Vgl. J. Molnár, On the ϱ-system of unit circles, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **20** (1977), 196—203 und Bui Van Dung, ϱ-система областей гиперциклов (im Druck).) Bui Van Dung hat sich noch mit den losen Überdeckungen durch kongruente Hyperzykel- bzw. Hypersphere Bereiche beschäftigt. (Vgl. Bui Van Dung, О рыхлости покрытия областями гиперциклов и гиперсфер в гиперболических плоскости и пространстве (im Druck).)

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(Eingegangen am 19. Dezember 1980)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 7—14

ON THE GENERALIZED HANKEL TRANSFORMATION

I. FENYŐ

There are several definitions of the Hankel transformation of certain classes of generalized functions. One [1] was given by us and is based on a relation between the Fourier and Hankel transformation. A quite different way was followed in order to give applicable definition of the Hankel transformation of generalized functions by A. H. Zemanian [2], [3], [4]. The present paper has the aim to give a new definition of the Hankel transformation of a certain class of distributions using an idea of Gelfand and Schilow [5, p. 153—155] defining the Fourier transformation of distributions. We define the Hankel transformation of distributions having their support on the positive axis but only for entire order in contrary to the definition of Zemanian [4], who elaborated a definition for arbitrary order. But it seems that the definition given below is much simpler than that of Zemanian and do not need to introduce very special testing function spaces.

1. Let $D_+(a)$ be the Schwartz testing function space whose functions have the support on $[0, a]$ $(a>0)$. For a given nonnegative entire n and $a\,(a>0)$ we denote by $H_n(a)$ the space of functions ψ fulfilling the following conditions:

a) ψ is an entire function;

b) $|s^k\psi(s)| \leqq C_k \exp(a\,|\mathrm{Im}\, s|)$ for great $|s|$, $(k = 0, 1, 2, \dots)$;

c$_n$) $\psi(-s) = (-1)^n \psi(s)$;

d$_n$) $\int\limits_0^\infty s^k \psi(s)ds = 0, \qquad k = n, n+2, n+4, \dots;$

e$_n$) $|\psi(s)| = O(|s|^n), \qquad (s \to 0).$

Under the Hankel transform (of order n) of a function $f\colon R_+ \to R$ we understand the following expression (if it exist for every $s \in R$)

$$\mathscr{H}_n(f)(s) = \int\limits_0^\infty t J_n(st) f(t)\, dt \qquad (n = 0, 1, 2, \dots) \tag{1}$$

where $J_n(t)$ denotes the Bessel function of the first kind of order n.

1980 *Mathematics Subject Classification.* Primary 46F12; Secondary 42B10, 44A15.
Key words and phrases. Generalized functions, Hankel transformations.

THEOREM 1. *The Hankel transformation $\mathcal{H}_n$ is an algebraic isomorphism between $D_+(a)$ and $H_n(a)$.*

PROOF. What we have to prove is that $\psi \in H_n(a)$ iff there exist a unique function $\varphi \in D_+(a)$ for which $\psi = \mathcal{H}_n(\varphi)$.

Let us consider first a function $\varphi \in D_+(a)$. Its Hankel transform (of order n) is

$$\mathcal{H}_n(\varphi)(s) = \psi(s) = \int_0^a t\, J_n(st)\, \varphi(t)\, dt \tag{2}$$

and therefore, J_n being an entire function, a) holds. From (2) also the property c_n) follows immediately.

In order to show b) let us consider the Bessel differential operator $\mathcal{B}_n$ defined for a function $x \in C^2$ as follows

$$\mathcal{B}_n(x)(t) = \frac{d^2 x}{dt^2} + \frac{1}{t}\frac{dx}{dt} - \frac{n^2}{t^2} x. \tag{3}$$

If we substitute now for x our testing function φ then we see

$$\mathcal{B}_n(\varphi) \in D_+(a)$$

as φ vanishes together with all his derivatives at the point $t=0$. By a well-known theorem (see e.g. [7, p. 61 (32)])

$$\mathcal{H}_n(\mathcal{B}_n(\varphi)) = -s^2 \mathcal{H}_n(\varphi)(s). \tag{4}$$

But all testing function of $D_+(a)$ fulfils conditions (α) and (β) in the theorem of Griffith [8] and therefore by the statement (ii) of the quoted theorem

$$|s^{5/2} \mathcal{H}_n(s)| = |s^{1/2} \mathcal{H}_n(\mathcal{B}_n(\varphi)(s))| \leqq C' e^{a|\mathrm{Im}\, s|} \tag{5}$$

for great values of $|s|$. From this immediately follows b) for $k=0, 1, 2$.

Using the relation (4) to the second iterated $\mathcal{B}_n^2$ of $\mathcal{B}_n$ we get

$$\mathcal{H}_n(\mathcal{B}_n^2(\varphi))(s) = s^4 \mathcal{H}_n(\varphi)(s). \tag{6}$$

As $\mathcal{B}_n^2(\varphi) \in D_+(a)$, we get

$$|s^{9/2} \mathcal{H}_n(\varphi)(s)| \leqq C'' e^{a|\mathrm{Im}\, s|}$$

and from this follows b) for $k=3, 4$. By induction we see the validity of b) for every $k=0, 1, 2, 3, \ldots$.

Let us now prove the statement d_n). By b) the Hankel transform (of order n) of $\psi \in H_n(a)$ exists and by the well-known inversion theorem of Hankel transformation (see e.g. [7]) (2) implies

$$\varphi(t) = \int_0^\infty s\, J_n(ts)\, \psi(s)\, ds. \tag{7}$$

All the derivatives of φ vanish at $t=0$. By the power series expansion of the Bessel function of order n (n is positive integer) this condition is automatically fullfilled for the derivatives of order $0, 1, 2, \ldots, n-1, n+1, n+3, \ldots$.

Let us now consider the integral

$$(8') \qquad \int_0^\infty s^2 J_n'(ts)\,\psi(s)\,ds.$$

For $n>0$ we have by a well-known formula (see e.g. [9] p. 360 (C)):

$$J_n'(t) = \frac{1}{2}\{J_{n-1}(t) - J_{n+1}(t)\}$$

and therefore (8′) becomes

$$(8'') \qquad \frac{1}{2}\int_0^\infty s^2 [J_{n-1}(ts) - J_{n+1}(ts)]\,\psi(s)\,ds$$

and by (8′) this exists uniformly with respect to t.

For $n=0$ we use the relation $J_0'(t) = -J_1(t)$ and this guarantees the uniform convergence of (8′) also in the case $n=0$. As (8′) equals (8″), therefore

$$(8''') \qquad \int_0^\infty s^3 J_n''(ts)\,\psi(s)\,ds = \frac{1}{2}\int_0^\infty s^3 [J_{n-1}'(ts) - J_{n+1}'(ts)]\,\psi(s)\,ds.$$

We use again the relation above which shows the uniform convergence with respect to t of the left-hand side of (8″). By induction we see the uniform convergence of

$$(8^k) \qquad \int_0^\infty s^{k+1} J_n^{(k)}(ts)\,\psi(s)\,ds \qquad (k = 0, 1, 2, \ldots).$$

Therefore from (7) it follows

$$(9) \qquad \varphi^{(k)}(t) = \int_0^\infty s^{k+1} J_n^{(k)}(ts)\,\psi(s)\,ds \qquad (k = 0, 1, 2, \ldots)$$

and so, as $J_n^{(k)}(0) \neq 0$ for $k = n, n+2, n+4, \ldots$, the assertion follows immediately.

The property e$_n$) is just the statement (v) in the theorem of Griffith.

Let us now show that also the converse is true. Let $\psi: C \to C$ be a function satisfying the properties a)—e). Then, obviously, the Hankel transform of ψ exists. But also all the integrals of the form (8^k) exist. Let us define $\varphi: R_+ \to R$ by (7), then $\varphi \in C^\infty(R_+)$ and the derivatives of φ are given by (9). If the conditions a)—e) are fulfilled all the other conditions (i)—(v) in [8] are fulfilled, and by this reason, by the theorem of Griffith, $\varphi(t) = 0$ if $t > a$. By the representation (9) and by d$_n$) $\varphi^{(k)}(0) = 0$ $(k = 0, 1, 2, \ldots)$ therefore φ belongs to $D_+(a)$.

The one-to-one correspondence between φ and ψ follows from the properties of the Hankel transformation.

The theorem has been proved.

2. Let us introduce in $D_+(a)$ the usual topology by the system of norms

$$(10) \qquad \|\varphi\|_p = \sup_{\substack{t \in (0, a) \\ k = 0, 1, \ldots, p}} |\varphi^{(k)}(t)|$$

and in $H_n(a)$ the topology generated by the following system of norms

$$\|\psi\|_p = \sup_{\substack{s\in C \\ k=0,1,\dots,p}} |s^k \psi(s)|\, e^{-a|\mathrm{Im}\, s|} \quad (\psi\in H_n(a), p = 0, 1, 2, \dots). \tag{11}$$

We see at once by (9) that the mapping $\mathscr{H}_n$ is not only an algebraic isomorphism between $H_n(a)$ and $D_+(a)$. We denote the inductive limit (for fixed n) of the spaces $H_n(a)$ by H_n and that of $D_+(a)$ for $0<a\to\infty$ by D_+. The notation of the dual spaces should be H_n' and D_+', respectively.

If we denote by Z the ultradistribution testing function space we see immediately by a) and b) that

$$H_n \subset Z \qquad (n = 0, 1, 2, \dots) \tag{12}$$

and this inclusion is proper. The topology induced by Z in H_n is just that what we defined above.

It is important to remark that the topology in H_n is independent from n.

3. Let $u\in D_+'$ an arbitrary distribution. We will now give a definition of $\mathscr{H}_n(u)$, the Hankel transformation of u of order n ($n=0, 1, 2, \dots$). For this purpose we need the following statement.

LEMMA. *Let $\psi\in H_n$. Then $s^{2p+1}\psi(s)\in H_{n+1}$, $s^{2p}\psi(s)\in H_n$ ($p=0, 1, 2, \dots$).*

PROOF. The conditions a) and b) are fulfilled for every n, further for $s^{2p+1}\psi$ obviously c_{n+1}), d_{n+1}) and e_{n+1}); for $s^{2p}\psi$ the properties c_n), d_n) and e_n) are valid.

Definition of the Hankel transformation of a distribution. Let $u\in D_+'$, we define $\mathscr{H}_n(u)$ as an element of H_{n+1}' by the following prescription:

$$\langle \mathscr{H}_n(u), s\psi\rangle = \langle u, t\mathscr{H}_n(\psi)\rangle \tag{14}$$

for every testing function $\psi\in H_n$. Here $\langle\, .\, ,\, .\,\rangle$ denotes the scalar product of a functional with the corresponding testing function. The definition (14) makes sense as $\mathscr{H}_n(\psi)=\varphi\in D_+$ and by the lemma $s\psi\in H_{n+1}$. We see at once that by (14) $\mathscr{H}_n(u)$ is a special ultradistribution, more precisely, it is the restriction of an ultradistribution to a subspace of Z. It is easy to see that the definition (14) goes into the classical definition of the Hankel transformation if u is a regular distribution generated by a function f for which $t^{1/2}f(t)\in L^2(R_+)$. In this case the right-hand side of (14) will be

$$\int_0^\infty f(t)t\varphi(t)\,dt \tag{15}$$

and this is, by the Parseval theorem concerning the Hankel transformation [6, Theorem 1], equal to

$$\int_0^\infty \mathscr{H}_n(f)(s)\, s\mathscr{H}_n(\varphi)(s)\, ds. \tag{16}$$

If φ runs over the functional space D_+ then the set of values of the integrals (15) uniquely defines f (a.e) [5] and therefore also (16) defines $\mathscr{H}_n(f)$ which is the classical Hankel transform of f.

4. We will now give a definition of the Hankel transformation $\mathscr{H}_n$ of a linear continuous functional v defined on H_{n+1}. Also this can be done by using the Parseval theorem. $\mathscr{H}_n(v)$ should be a distribution in D'_+ defined by the prescription

(17) $$\langle t\mathscr{H}_n(v), \mathscr{H}_n(\psi)\rangle = \langle v, s\psi\rangle \qquad (\psi \in H_n).$$

This definition makes sense by the fact $\mathscr{H}_n(\psi)\in D_+$ $(\psi\in H_n)$. Also this is a generalization of the classical Hankel transform by the Parseval theorem.

The following statement is valid:

THEOREM 2. *Let* $u\in D'_+$, *then*

(18) $$\mathscr{H}_n(\mathscr{H}_n(u)) = u$$

and for $v\in H'_{n+1}$

(19) $$\mathscr{H}_n(\mathscr{H}_n(v)) = v.$$

PROOF. Using the notation $\mathscr{H}_n(\psi)=\varphi$ from which $\psi=\mathscr{H}_n(\varphi)$, we get immediately from (17) and (14)

$$\langle \mathscr{H}_n(\mathscr{H}_n(u)), t\varphi\rangle = \langle \mathscr{H}_n(\mathscr{H}_n(u)), t\,\mathscr{H}_n(\psi)\rangle = \langle \mathscr{H}_n(u), s\psi\rangle = \langle u, t\varphi\rangle$$

for all $\varphi\in D_+$. This proves the statement.

The relation (19) can be proved in a similar way starting from (14).

5. We can now show that the Hankel transformation given by (14) possesses the formal properties of the Hankel transformation of usual functions. These properties provide the possibility of applications of the generalized Hankel transformation.

THEOREM 3. *Let* $u\in D'_+$ *and* $\mathscr{B}_n$ *the Bessel differential operator defined in* (3). *Then*

(20) $$\mathscr{H}_n(\mathscr{B}_n(u)) = -s^2\mathscr{H}_n(u).$$

PROOF. The adjoint operator of the linear differential operator $\mathscr{B}_n$ denoted by $\mathscr{B}^*_n$ is as follows:

(21) $$\mathscr{B}^*_n(f) = f'' - \left(\frac{f}{t}\right)' - \frac{n^2}{t^2}f \qquad (n = 0, 1, 2, \ldots)$$

$(f\in C^2)$. Then

(22) $$\langle \mathscr{B}_n(u), \varphi\rangle = \langle u, \mathscr{B}^*_n(\varphi)\rangle \qquad \varphi\in D_+$$

holds. A simple calculation shows

$$\frac{1}{t}\mathscr{B}^*_n(t\varphi) = \mathscr{B}_n(\varphi),$$

and therefore by a well-known theorem [7, p. 61 (32) and p. 62 (35)]

(23) $$\mathscr{H}_n\left(\frac{1}{t}\mathscr{B}^*_n(t\varphi)\right)(s) = \mathscr{H}_n\left(\mathscr{B}_n(\varphi)\right)(s) = -s^2\psi(s)$$

where $\psi = \mathscr{H}_n(\varphi)$. By the inversion theorem of Hankel transformation

$$-t\,\mathscr{H}_n(s^2\psi)(t) = \mathscr{B}_n^*(t\varphi)(t). \tag{24}$$

Using the definition (14) and the relations (22), (24) we get

$$\langle s\,\mathscr{H}_n(\mathscr{B}_n(u)), \psi\rangle = \langle \mathscr{B}_n(u), t\varphi\rangle = \langle u, \mathscr{B}_n^*(t\varphi)\rangle = -\langle u, t\,\mathscr{H}_n(s^2\psi)\rangle =$$
$$= -\langle s\mathscr{H}_n(u), s^2\psi\rangle = -\langle s^3[\mathscr{H}_n(u)], \psi\rangle$$

for every $\psi \in H_{n+1}$, this proves the statement (20).

4. An other well-known formula for Hankel transformation of functions can also be generalized for Hankel transformation of distributions.

THEOREM 4. *Let* $u \in D'_+$ *and* $\mathscr{D}u$ *its (distributional) derivative,* $n \geqq 1$. *Then*

$$\mathscr{H}_n(\mathscr{D}u) = -\frac{1+n}{2n} s\,\mathscr{H}_{n-1}(u) - \frac{1-n}{2n}\mathscr{H}_{n+1}(u).$$

PROOF. By (14) we have

$$\begin{aligned}&\langle s\,\mathscr{H}_n(\mathscr{D}u), \psi\rangle = \langle \mathscr{D}u, t\mathscr{H}_n(\psi)\rangle = \\ &= -\Big\langle u, \frac{d}{dt}[t\,\mathscr{H}_n(\psi)]\Big\rangle = -\langle u, \mathscr{H}_n(\psi)\rangle - \Big\langle u, t\frac{d}{dt}\mathscr{H}_n(\psi)\Big\rangle.\end{aligned} \tag{25}$$

But

$$\begin{aligned}\mathscr{H}_n(\psi) + t\frac{d}{dt}\mathscr{H}_n(\psi) &= \int_0^\infty [sJ_n(ts) + ts^2 J_n'(ts)]\,\psi(s)\,ds = \\ &= \int_0^\infty s\frac{d}{ds}(sJ_n(ts))\,\psi(t)\,dt.\end{aligned} \tag{26}$$

Applying the well-known relations (5) and (7) in [7, p. 512] we get

$$\mathscr{H}_n(\psi) + t\frac{d}{dt}\mathscr{H}_n(\psi) = \int_0^\infty [s(1-n)\,J_n(ts) + ts^2 J_{n-1}(ts)]\psi(s)\,ds =$$

$$= \int_0^\infty \left[\frac{1-n}{2n} ts^2 \frac{2n}{ts} J_n(ts) + ts^2 J_{n-1}(ts)\right]\psi(s)\,ds =$$

$$= \int_0^\infty \left[\frac{1-n}{2n} ts^2 J_{n-1}(ts) + \frac{1-n}{2n} ts^2 J_{n+1}(ts) + ts^2 J_{n-1}(ts)\right]\psi(s)\,ds =$$

$$= \frac{1+n}{2n} t\,\mathscr{H}_{n-1}(s\psi) + \frac{1-n}{2n} t\,\mathscr{H}_{n+1}(s\psi).$$

Substituting this expression into (25), we obtain

$$\langle s\,\mathscr{H}_n(\mathscr{D}u),\psi\rangle=-\frac{1+n}{2n}\langle u,t\mathscr{H}_{n-1}(s\psi)\rangle-\frac{1-n}{2n}\langle u,t\mathscr{H}_{n+1}(s\psi)\rangle=$$

$$=-\frac{1+n}{2n}\langle s\mathscr{H}_{n-1}(u),s\psi\rangle-\frac{1-n}{2n}\langle s\mathscr{H}_{n+1}(u),s\psi\rangle$$

for every $\psi\in H_{n+1}$. This completes the proof.

7. In order to show how to calculate with our results let us consider for example the classical Bessel differential equation

$$\frac{d^2u}{dt^2}+\frac{1}{t}\frac{du}{dt}+\left(1-\frac{n^2}{t^2}\right)u=0 \tag{27}$$

which equals to

$$\mathscr{B}_n u=-u. \tag{28}$$

We intend to look for the general distributional solution of (27) resp. (28) in D'_+. Using the generalized Hankel transformation to (28) and after this the result (20), we get

$$\mathscr{H}_n(\mathscr{B}_n(u))=-s^2\,\mathscr{H}_n(u)=-\mathscr{H}_n(u) \tag{29}$$

for every s, from which $\mathscr{H}_n(u)=0$. An other solution of (29) is δ_1 (i.e. $\langle\delta_1,\psi\rangle=$ $=\psi(1)$, $\psi\in H_{n+1}$). But

$$\langle\mathscr{H}_n(J_n),s\psi\rangle=\langle J_n,t\varphi\rangle=\int_0^\infty tJ_n(t)\varphi(t)dt=\mathscr{H}_n(\varphi)(1)=\psi(1)=\langle\delta_1,\psi\rangle,$$

therefore

$$\mathscr{H}_n(J_n)=\delta_1. \tag{30}$$

THEOREM 5. *The homogeneous differential equation* (27) *has no other solutions in the domain D'_+ than the classical ones.*

The function $J_n(t)$ has in the classical sense no Hankel transform but his generalized Hankel transform exists. Therefore the results (30) seems to be interesting.

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(Received July 1, 1982)

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ON THE DEGENERATION OF THE FOCAL LOCUS OF A SUBMANIFOLD IN EUCLIDEAN SPACE

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The concept of focal points of a submanifold in euclidean space has been applied in several cases, e.g., by J. Milnor [3] in his lecture on Morse theory and J. Szenthe [6] studied focal points of a principal orbit of submanifold in euclidean space. Recently, H. Singh [7] considered the locus of focal points of a submanifold in euclidean space. Singh [7] gave a sufficient condition under which the focal locus of an m-dimensional submanifold is the union of m hypersurfaces, the so-called focal hypersurfaces.

In this paper we shall consider the locus of all focal points of an m-dimensional submanifold in R^n, where $1 \leqq m \leqq n-1$ and obtain sufficient conditions under which it is the union of m submanifolds, the so-called sheets.

Milnor [3] has applied the fact that along any normal line at a point of a submanifold of dimension m embedded in a euclidean space, there are at most m focal points. We shall apply this results, too, and follow the notations used by Milnor [3] in this paper.

The author is thankful to Prof. J. Szenthe for providing useful discussion during the preparation of this paper.

1. Introduction

Let R^n be a euclidean space of dimension n and let f be an immersion of an m-dimensional differentiable connected manifold M into R^n. Let the normal bundle $T(M)^{\perp}$ of M be defined by

$$T(M)^1 = \{(p, w' | p \in M, w' \text{ normal to } M \text{ at } p\}.$$

Obviously, $T(M)^{\perp}$ is an n-dimensional bundle space differentiably embedded in the tangent bundle TR^n of R^n.

Consider the end-point map

$$\Phi\colon T(M)^{\perp} \to R^n,$$

defined by

$$\Phi(p, w') = p + w', \qquad (p, w') \in T(M)^{\perp}.$$

A point $s \in R^n$ is called a focal point of M if $s = p + w'$ is a critical value of the end-point map.

1980 *Mathematics Subject Classification*. Primary 53A05; Secondary 53B25, 53C40.
Key words and phrases. Submanifolds in euclidean space, focal locus of a submanifold.

Let $k_1, \ldots, k_m$ be the principal curvatures of M at p in the normal direction w. Then the reciprocals $k_1^{-1}, \ldots, k_m^{-1}$ of these principal curvatures are called principal radii of curvatures.

Consider the normal line L which contains all focal points $p+Rw$, where w is a fixed unit vector orthogonal to M at p and let R be a radius of curvature at the point on M in the direction w. Then the set of these focal points $(p, Rw) \in T(M)^{\perp}$ of M will be called focal locus of M in the normal bundle and the set of points $p+Rw \in R^n$ is called the focal locus of M in R^n.

The following well-known lemma ([3], pp. 34) is used in this paper.

LEMMA. *The focal points of M along L are precisely the points $p+k_i^{-1}w$, where $1 \leqq i \leqq m$, $k_i \neq 0$ and $p \in M$. Thus there are at most m focal points of M along L.*

Since there are at most m focal points of M along L, therefore the focal locus of the submanifold M can have at most m connected components.

Our aim in this paper is to find conditions under which these components are submanifolds of given dimension at least $n-m-1$ and at most $n-1$ in R^n.

In order to find the conditions we require the Weingarten map and Rodrigues formula in case of submanifold, which have already been defined and derived respectively by Singh [7]. *From now we suppose that the normal connection $\nabla^{\perp}$ of the normal bundle $T(M)^{\perp}$ is flat, i.e. $\nabla_X^{\perp} w = 0$.* So, according to [7], if X is a principal vector for the direction of the normal vector field w, then the Rodrigues formula yields that

$$\nabla'_X \cdot w = -kX,$$

where k is a principal curvature and ∇' is the covariant derivation with respect to the canonical connection on R^n.

2. Conditions for the focal locus to be a submanifold

Let $\varphi^1, \ldots, \varphi^{n-m-1}$ be a spherical coordinate system on a unit sphere in R^{n-m} with base $(e_1, \ldots, e_{n-m})$. Then any unit vector is given by

$$\sum_{i=1}^{n-m} \xi^i(\varphi^1, \ldots, \varphi^{n-m-1}) e_i,$$

where $\xi^i(\varphi^1, \ldots, \varphi^{n-m-1})$ are components of the unit vector in the base $(e_1, \ldots, e_{n-m})$.

If $(u^1, \ldots, u^m)$ is a local coordinate system in a neighbourhood U of $p \in M$. Let $(w_1, \ldots, w_{n-m})$ be defined in the neighbourhood U of p such that it is an orthonormal base for the normal space $T_p(M)^{\perp}$ of M at every point of the neighbourhood. Then a local coordinate system $(u^1, \ldots, u^m; \varphi^1, \ldots, \varphi^{n-m-1})$ can be defined in a neighbourhood V of a point q of the manifold of unit normals to M.

Consider an isomorphism of the normal space $T_p(M)^{\perp}$ to the space R^{n-m} such that $w_i \to e_i$. Then the unit vector $w(u^1, \ldots, u^m; \varphi_1, \ldots, \varphi^{n-m-1})$ at point p of M will be given by

$$w = \sum_{i=1}^{n-m} \xi^i w_i,$$

where $\xi^i(\varphi^1, \ldots, \varphi^{n-m-1})$ are components of its image vector in R^{n-m}.

Let R_j be the jth radius of curvature at the point r on M in the direction of w. Then the corresponding focal point q' on focal locus is given by

$$r'(u^1, \dots, u^m;\ \varphi^1, \dots, \varphi^{n-m-1}) = r(u^1, \dots, u^m) + R_j(u^1, \dots, u^m;\ \varphi^1, \dots, \varphi^{n-m-1})$$
$$w(u^1, \dots, u^m;\ \varphi^1, \dots, \varphi^{n-m-1}).$$

The partial derivative of this expression with respect to u^1 and φ^a, respectively, give

$$\partial_l r' = \partial_l r + (\partial_l R_j) w + R_j \partial_l w; \qquad \partial_l = \partial/\partial u^l, \quad l = 1, \dots, m,$$

and

$$\partial'_a r' = (\partial'_a R_j) w + R_j(\partial'_a w); \qquad \partial'_a = \partial/\partial\varphi^a, \quad a = 1, \dots, n-m-1,$$

since $\partial'_a r = 0$.

If $\partial_l r$ is in principal direction then by Rodrigues formula [7]

$$\partial_l w = -k_l \partial_l r,$$

where k_l are m principal curvatures of the submanifold M at p, i.e., $k_l = \dfrac{1}{R_l}$,

$$\partial_l r' = (1 - k_l R_j)\partial_l r + (\partial_l R_j) w.$$

Our next aim is to find general conditions for the jth sheet to include a submanifold of given dimension.

For the jth sheet we have

$$\partial_j r' = \partial_j \left(\frac{1}{k_j}\right) w,$$

and

$$\partial_l r' = \left(1 - \frac{k_l}{k_j}\right) \partial_l r + \partial_l \left(\frac{1}{k_j}\right) w, \qquad l = 1, \dots, j-1, j+1, \dots, m.$$

We shall now consider three cases.

Case I. The sheet includes a submanifold of dimension at least $n-m-1$.

Case II. The sheet includes a submanifold of dimension k $(n-m-1 < k < n-1)$.

Case III. The sheet is a submanifold of dimension $n-1$.

The Case III has already been considered in [7].

Case I. Consider the wedge product

$$\partial'_1 r' \wedge \dots \wedge \partial'_{n-m-1} r' =$$
$$= \{(\partial'_1 R_j) w + R_j(\partial'_1 w)\} \wedge \dots \wedge \{(\partial'_{n-m-1} R_j) w + R_j(\partial'_{n-m-1} w)\}.$$

This reduces to

$$\partial'_1 r' \wedge \dots \wedge \partial'_{n-m-1} r' = \sum_{a=1}^{n-m-1} \{(\partial'_a R_j) R_j^{n-m-2} \partial'_1 w \wedge \dots$$
$$\dots \wedge \partial'_{a-1} w \wedge w \wedge \partial'_{a+1} w \wedge \dots \wedge \partial'_{n-m-1} w\} + R_j^{n-m-1} \partial'_1 w \wedge \dots \wedge \partial'_{n-m-1} w,$$

since $w \wedge w = 0$.

The vectors $\partial_1' w, \ldots, \partial_{n-m-1}' w$ are linearly independent, since $(\varphi^1, \ldots, \varphi^{n-m-1})$ is a coordinate system on the unit sphere of dimension $n-m-1$; hence their wedge product is not zero. Moreover, the vectors $w, \partial_1' w, \ldots, \partial_{n-m-1}' w$ are linearly independent, and therefore the multivectors

$$\partial_1' w \wedge \ldots \wedge \partial_{a-1}' w \wedge w \wedge \partial_{a+1}' w \wedge \ldots \wedge \partial_{n-m-1}' w,$$

and

$$\partial_1' w \wedge \ldots \wedge \partial_{n-m-1}' w$$

are linearly independent, too. Thus the jth sheet $r' = r + R_j w$ is an $(n-m-1)$-dimensional submanifold if $\partial_l r' = 0$ for $l = 1, \ldots, m$, which is true if and only if $(1 - k_l R_j)\partial_l r + (\partial_l R_j) w = 0$. This holds if and only if the following conditions hold:

(1) $\qquad \partial_l R_j = 0, \quad \text{i.e.,} \quad R_j(u^1, \ldots, u^m; \varphi^1, \ldots, \varphi^{n-m-1})$

does not depend on $(u^1, \ldots, u^m)$,

(2) $\qquad R_l \equiv R_j \quad \text{for} \quad l = 1, \ldots, m,$

i.e., $R_1 \ldots, R_m$ coincide with jth sheet. Thus we have

THEOREM 2.1. *A sufficient condition for the jth sheet to be a submanifold of dimension* $n-m-1$ *is that*

(i) R_j *=constant with respect to u's.*
(ii) $\partial_a' R_j$, $a = 1, \ldots, n-m-1$ *are not all zero.*
(iii) R_j *coincides with* $R_1, \ldots, R_{j-1}, R_{j+1}, \ldots, R_m$.

From the above Theorem 2.1 the following questions arise.

QUESTION 2.1. Is there a submanifold of R_n where the jth sheet coincides with the remaining sheets and the jth sheet is a flat submanifold of dimension $n-m-1$.

The answer is yes. Proof is given below.

Let S^m be a unit sphere immersed in R^n with centre at the origin. Consider a decomposition of R^n into two disjoint subspaces R^{m+1} and R^{n-m-1} such that $R^n = R^{m+1} \times R^{n-m-1}$ and R^{n-m-1} is an orthogonal complement of R^{m-1}. Obviously, S^m is fixed in R^{m+1}. If $(x^1, \ldots, x^n)$ is a local coordinate system in the neighbourhood of a point in R^n, then S^m, R^{m+1} and R^{n-m-1} will be defined, respectively, by

$$S^m = \{(x^1, \ldots, x^{m+1}, 0, \ldots, 0) | (x^1)^2 + \ldots + (x^{m-1})^2 = 1\},$$

$$R^{m+1} = \{(x^1, \ldots, x^{m+1}, 0, \ldots, 0)\},$$

and

$$R^{n-m-1} = \{(0, \ldots, 0, x^{m-2}, \ldots, x^n)\}.$$

Let $(u^1, \ldots, u^m)$ be a local coordinate system of a point p on S^m with position vector r and let $R_1, \ldots, R_m$ be the principal radii of curvature of S^m at p, then it can easily be seen that R_j does not depend on $(u^1, \ldots, u^m)$ and R_j coincides with $R_1, \ldots, R_m$ as studied earlier. The remaining is to show that the jth sheet of focal locus is a flat submanifold of dimension $n-m-1$. To prove this it is sufficient to show that $r + R_j w$ is a point of R^{n-m-1}.

Let $N_pS^m = R^{n-m-1} \oplus \{\bar{w}\}$ be the normal space of S^m at the point p, where $\bar{w} \in N_pS^m \cap R^{m+1}$ is the unit normal vector to S^m at p in R^{m+1} passing through the centre of the sphere. Let $\hat{w} \in R^{n-m-1}$, a normal vector to S^m at p in R^{n-m-1}. Then for $w \in N_pS^m$ we have $w = \lambda\bar{w} + \hat{w}$, where λ is a scalar, so that $\lambda = \langle w, \bar{w}\rangle$, as $\bar{w}$ is orthogonal to $\hat{w}$.

Since R_j is the radius of curvature corresponding to the jth sheet, therefore $\dfrac{1}{R_j} = k_j$ is an eigenvalue of the second fundamental form $\langle w, \alpha(X, Y)\rangle$ in the direction of w, where

$$\alpha(X, Y) = \sum_{j,i=1}^{m} \partial_j\partial_i r\xi^i\eta^j, \qquad \partial_j\partial_i r = \frac{\partial^2 r}{\partial u^j \partial u^i}.$$

Using the values of w and $\alpha(X, Y)$ we have

$$\Big\langle w, \sum_{j,i=1}^{m} \partial_j\partial_i r\xi^i\eta^j\Big\rangle = \Big\langle \lambda\bar{w}+\hat{w}, \sum_{j,i}^{m} \partial_j\partial_i r\xi^i\eta^j\Big\rangle =$$

$$= \lambda\Big\langle \bar{w}, \sum_{j,i=1}^{m} \partial_j\partial_i r\xi^i\eta^j\Big\rangle$$

since $\hat{w} \in R^{n-m-1}$ and $\sum_{j,i} \partial_j\partial_i r\xi^i\eta^j R^{m+1}$. Thus the "second fundamental form of S^m at p in the direction w" is equal to λ times the "second fundamental form of S^m at p in the direction $\bar{w}$". This means that

$$R_j(w) = \frac{1}{\lambda} R_j(\bar{w}) = \frac{1}{\langle w, \bar{w}\rangle} R_j(\bar{w}),$$

i.e.,

$$\text{(A)} \qquad R_j(w)\cos(w, \bar{w}) = R_j(\bar{w}).$$

Since j is arbitrary, therefore this holds for all sheets. This concludes that R_j is the hypotenuse of the right angle triangle defined by pOw. Consequently, the focal point is the intersection of the ray defined by w with R^{n-m-1}. Since R^{n-m-1} is a flat submanifold of dimension $n-m-1$, therefore the sheet which lies on R^{n-m-1} is a flat submanifold of dimension $n-m-1$, too.

From Relation (A) the following remarks may be deduced.

Remark 1. If w coincide with $\bar{w}$, i.e., $\cos(w, \bar{w}) = 1$, then $R_j(w) = R_j(\bar{w})$ this means that the focal locus is a point, which is the centre of the sphere. This happens only if sphere is of dimension $n-1$.

Remark 2. If w is orthogonal to $\bar{w}$, i.e., $\cos(w, \bar{w}) = 0$, then $R_j(\bar{w}) = 0$ or $R_j(w) = \infty$. This implies that there is no focal point of S^m.

Question 2.2. Are conditions (i) and (iii) independent,

or

Does (iii) $\Rightarrow$ (i) in general?

ANSWER. Conditions (i) and (iii) are not independent, i.e., (iii) always implies (i).

For the answer we shall use the following Theorem ([5], pp. 110—111).

THEOREM. *For $n \geqq 2$, let $M^n \subset R^m$ be a connected immersed submanifold of R^m with all points umbilic. Then either M lies in some n-dimensional plane or else M lies in some n-dimensional sphere in some $(n+1)$-dimensional plane.*

Condition (iii) means that the principal radii of curvature are equal and this implies that the point p is umbilic. Since p is an arbitrary point on M, M is an umbilic submanifold of R^n. Then by the above theorem M either lies in some n-dimensional plane and then the principal curvatures will be zero, which means that the focal locus is empty; or M lies in some n-dimensional sphere, then R_j does not depend on $(u^1, \ldots, u^m)$.

Thus condition (iii) implies condition (i) in both cases.

Case II. Consider the wedge product for the jth sheet

$$\partial_1 r' \wedge \ldots \wedge \partial_j r' \wedge \ldots \wedge \partial_q r' \wedge \partial_1' r' \wedge \ldots \wedge \partial_{n-m-1}' r' =$$

$$= \left\{\left(1-\frac{k_1}{k_j}\right)\partial_1 r + \partial_1\left(\frac{1}{k_j}\right) w\right\} \wedge \ldots \wedge \partial_j\left(\frac{1}{k_j}\right) w \wedge \ldots \wedge \left\{\left(1-\frac{k_q}{k_j}\right)\partial_q r + \partial_q\left(\frac{1}{k_j}\right) w\right\} \wedge$$

$$\wedge \{R_j(\partial_1' w) + (\partial_1' R_j) w\} \wedge \ldots \wedge \{R_j(\partial_{n-m-1}' w) + (\partial_{n-m-1}' R_j) w\};$$

$$q \leqq m$$

$$= \partial_j\left(\frac{1}{k_j}\right) R_j^{n-m-1}\left\{\left(1-\frac{k_1}{k_j}\right)\ldots\left(1-\frac{k_q}{k_j}\right)\right\}$$

$$\partial_1 r \wedge \ldots \wedge w \wedge \ldots \wedge \partial_q r \wedge \partial_1' w \wedge \ldots \wedge \partial_{n-m-1}' w.$$

Since $\partial_1 r, \ldots, \partial_{j-1} r, w, \partial_{j+1} r, \ldots, \partial_q r, \partial_1' w, \ldots, \partial_{n-m-1}' w$ are $q+n-m-1$ linearly independent vectors, therefore their wedge product is not zero. Since $R_j \neq 0$,

$$\partial_1' r' \wedge \ldots \wedge \partial_q r' \wedge \partial_1' r' \wedge \ldots \wedge \partial_{n-m-1}' r' \neq 0$$

if

$$\partial_j\left(\frac{1}{k_j}\right)\left\{\left(1-\frac{k_1}{k_j}\right)\ldots\left(1-\frac{k_q}{k_j}\right)\right\} \neq 0.$$

Thus we have

THEOREM 2.2. *A sufficient condition for the jth sheet to include a submanifold of dimension $k = q+n-m-1$ is that*

(i) $\partial_j\left(\frac{1}{k_j}\right) \neq 0,$

(ii) $k_q \neq k_j, \quad q = 1, \ldots, m,$

i.e. k_j is different from $k_1, \ldots, k_q$.

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(Received December 22, 1982)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 23—35

EINE METHODE ZUR BESTIMMUNG DER DÜNNSTEN GITTERFÖRMIGEN k-FACHEN KREISÜBERDECKUNGEN

ÁGOTA H. TEMESVÁRI

1.1. Eine Menge von kongruenten abgeschlossenen Kreisen bildet eine k-fache Überdeckung in der Ebene, wenn jeder Punkt der Ebene zu mindestens k der Kreise gehört. Die k-fache Kreisüberdeckung ist gitterförmig, wenn die Kreismittelpunkte die Gitterpunkte eines Gitters Γ sind.

Wie im allgemeinen bei Überdeckungen, so ist es auch bei den gitterförmigen k-fachen Kreisüberdeckungen eine Grundaufgabe, das Minimum der Dichte und die dem Minimum entsprechende Kreisüberdeckung oder Kreisüberdeckungen zu bestimmen. (Die Definition der Dichte siehe z. B. in [2].) Die Kreisüberdeckung mit der minimalen Dichte wird die dünnste genannt. Kershner hat die dünnste einfache Kreisüberdeckung ohne die Voraussetzung der Gitterförmigkeit in [4] bestimmt. Die dünnsten k-fachen Kreisüberdeckungen sind für die folgenden Werte von k bekannt: für $k=2, 3, 4$ Blundon [1], für $k=5, 6$ Subak [6] und für $k=7$ Haas [3]. Die Ergebnisse von Subak und Haas wurden nicht publiziert. (Die Dissertation von Haas hat ungefähr 200 Seiten.) Eine andere Lösung für $k=5$ kann man in [7] finden.

Von Linhart [5] stammt ein Verfahren, mit dessen Hilfe wir das Minimum der Dichte der gitterförmigen k-fachen Kreisüberdeckung mit einer beliebigen Genauigkeit berechnen können. Das geht zur Zeit ungefähr auf 3—4 Stellen. (Linhart hat auf 2 Stellen gerechnet.) Auf Grund dieses Verfahrens gibt er gute Vermutungen für das Minimum der Dichte im Fall $k\leqq 20$ an.

In dieser Arbeit geben wir eine Methode an, mit deren Hilfe man die dünnste gitterförmige k-fache Kreisüberdeckung für einen beliebigen Wert von k bestimmen kann. Dazu müssen wir die Minima von endlich vielen Funktionen von einer Veränderlichen bestimmen. (Diese Funktionen können eventuell eine implizite Darstellung haben.)

Als Fortsetzung dieser Arbeit bestimmen wir die dünnsten gitterförmigen 6-, 7- und 8-fachen Kreisüberdeckungen mit Hilfe dieser Methode. Es scheint, daß wir eine einfachere Lösung im Fall $k=7$ angeben können als in [3]. Für $k=8$ sehen wir die Richtigkeit der Vermutung in [5] ein.

1.2. Zuerst führen wir einige Bezeichnungen ein. Es sei O ein beliebiger Punkt der Ebene. Mit X bezeichnen wir den Ortsvektor $\overrightarrow{OX}$ und auch seinen Endpunkt. Es sei $|X|$ die Länge von X.

Mit Γ bezeichnen wir das durch die linear unabhängigen Vektoren A und B bestimmte Gitter, das aus den Vektoren $mA+nB$ (m und n sind ganze Zahlen) be-

1980 *Mathematics Subject Classification.* Primary 52A45; Secondary 51M05.
Key words and phrases. Lattice covering, k-fold covering, thinnest covering.

steht. Das Gitter Γ ist in normaler Darstellung, wenn die folgenden Ungleichungen für seine Basisvektoren A und B gelten:

$$|A| \leqq |B| \leqq |B-A|, \quad \sphericalangle(AOB) \leqq \frac{\pi}{2}. \tag{1}$$

Es ist offenbar, daß man eine Basis zu einem beliebigen Gitter Γ angeben kann, die die Ungleichungen (1) befriedigt. Im weiteren nehmen wir immer an, daß Γ mit den Basisvektoren A und B in normaler Darstellung ist.

Es seien $|A|=a$, $|B|=b$, $\frac{a}{b}=x$, $\sphericalangle(AOB)=\alpha$. Mit $T(\Gamma)$ bezeichnen wir den Inhalt des Grundparallelogramms von Γ. Mit diesen Bezeichnungen bekommen wir die mit (1) äquivalenten Ungleichungen

$$0 < x \leqq 1, \quad 0 \leqq \cos\alpha \leqq \frac{x}{2}. \tag{2}$$

Auf solche Weise können wir das geordnetes Zahlenpaar $(x, \cos\alpha)$ zu einem beliebigen Gitter Γ mit Hilfe der normalen Darstellung zuordnen. Es ist offenbar, daß wir dasgleiche Zahlenpaar nur ähnlichen Gittern zuordnen. Wir betrachten das rechtwinklige Koordinatensystem $x, y=\cos\alpha$. Der Punkt mit den Koordinaten $(x, \cos\alpha)$ entspricht dem Gitter Γ. Weil Γ in normaler Darstellung ist, liegt der Punkt $(x, \cos\alpha)$ im rechtwinkligen Dreieck OPQ, wo $O(0,0)$, $P(1,0)$, $Q\left(1, \frac{1}{2}\right)$ (Abb. 1) sind. Und umgekehrt, wenn wir einen beliebigen von O verschiedenen Punkt des Dreiecks OPQ betrachten, dann entspricht ein Gitter Γ diesem Punkt, das abgesehen von einer Ähnlichkeit eindeutig ist.

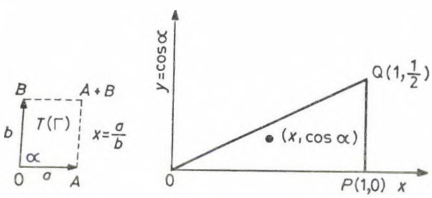

Abb. 1

Mit $k[XYZ]$ bzw. $\widehat{k}[XYZ]$ bezeichnen wir den durch die Punkte X, Y, Z bestimmten Kreis bzw. die entsprechende Kreislinie. Der Radius von $k[XYZ]$ sei $r[XYZ]$. Mit $k(X, r)$ bzw. $\widehat{k}(X, r)$ bezeichnen wir den Kreis vom Radius r und mit dem Mittelpunkt X bzw. den Rand von $k(X, r)$. Ist $r=1$, dann brauchen wir kurz die Bezeichnungen $k(X)$ und $\widehat{k}(X)$.

Ist T der Inhalt des Dreiecks XYZ und sind x, y, z die Seitenlängen des Dreiecks, dann ist der Umkreisradius

$$r[XYZ] = \frac{xyz}{4T}. \tag{3}$$

Es seien $\Delta_1, \Delta_2, \dots, \Delta_n, \dots$ solche Dreiecke, deren Ecken Gitterpunkte des Gitters Γ sind. Diese Dreiecke werden Gitterdreiecke genannt. Mit k_i bezeichnen wir den Umkreis des Gitterdreiecks Δ_i. Es sei r_i der Radius von k_i. Im folgenden wenden wir die Bezeichnung

$$Q_i := \frac{r_i^2}{T(\Gamma)} \tag{4}$$

oft an.

Wir betrachten einen Kreis K vom Radius R und ein Gitter Γ. Die Kreise $K+X$, $X \in \Gamma$ bilden eine gitterförmige Anordung von K. Mit $L(\Gamma, R)$ bezeichnen wir diese gitterförmige Kreisanordnung. Ist $R=1$, dann sei $L(\Gamma, R)=L(\Gamma)$.

Die Dichte der gitterförmigen Kreisanordnung $L(\Gamma, R)$ ist

$$\frac{R^2\pi}{T(\Gamma)}. \tag{5}$$

(Die Definition der Dichte siehe z. B. in [2].) Mit $D_{k,\Gamma}$ bezeichnen wir die Dichte der dünnsten gitterförmigen k-fachen Kreisüberdeckung mit dem Gitter Γ. Es ist also $D_{k,\Gamma} = \min \frac{R^2\pi}{T(\Gamma)}$, wo wir das Minimum für alle gitterförmige k-fache Kreisüberdeckungen betrachten.

Es sei XY die durch die Punkte X und Y bestimmte Gerade. Mit $XY|Z$ bezeichnen wir die Halbebene, die den Punkt Z enthält und die durch die Gerade XY begrenzt ist.

Am Ende der Beweise benutzen wir das Zeichen $\square$.

2. Das Wesen der Methode zeigen die folgenden Sätze. Auf Grund des folgenden Satzes können wir eine Zerlegung des Dreiecks OPQ (außer O) angeben, die angenehme Eigenschaften hat.

SATZ 1. *Es sei $k>0$ eine ganze Zahl. Wir haben die Punkte des Dreiecks OPQ (s. in 1.2) den Gittern Γ in normaler Darstellung zugeordnet. Es gibt eine Zerlegung des Dreiecks OPQ (außer O) in endlich vielen Bereichen $H_1, H_2, \dots, H_s$, die die folgenden Eigenschaften besitzt.*

1.1. Beliebige zweie von den Bereichen haben keinen gemeinsamen inneren Punkt.

1.2. Ein beliebiger, von O verschiedener Punkt des Dreiecks OPQ gehört zu mindestens einem dieser Bereiche.

1.3. Ein beliebiges Bereich H_i $(1 \leqq i \leqq s)$ besteht aus endlich vielen, einzeln zusammenhängenden Teilbereichen.

1.4. Zu einem beliebigen Bereich H_i $(1 \leqq i \leqq s)$ kann man ein nicht stumpfwinkliges Gitterdreieck Δ_i angeben, dessen Umkreis k_i in seinem Inneren höchstens $k-1$ Gitterpunkte, die abgeschlossene Kreisscheibe k_i aber mindestens $k+2$ Gitterpunkte enthält.

1.5. Der zum Bereich H_i zugeordnete Kreis hat die Eigenschaft, daß $L(\Gamma, r_i)$ eine k-fache Überdeckung ist, aber $L(\Gamma, R)$ für $R<r_i$ keine k-fache Überdeckung bildet.

1.6. Es gilt $Q_i = Q_j$ für die gemeinsamen Punkte von H_i und H_j. (Die Definition von Q_i siehe in (4).)

Der Beweis des Satzes gründet sich auf den folgenden Hilfssätzen, er ist ihre einfache Folgerung, deshalb legen wir ihn nicht ausführlich dar.

HILFSSATZ 1. *Ist die Kreisanordung $L(\Gamma, R)$ eine k-fache Überdeckung und enthält der Kreis k_i, der durch irgendein nicht stumpfwinkliges Gitterdreieck Δ_i bestimmt ist, in seinem Inneren höchstens $k-1$ Gitterpunkte von Γ, dann gilt $R \geqq r_i$.*

BEWEIS. Im entgegengesetzten Fall $(R<r_i)$ wäre der Mittelpunkt des Kreises k_i höchstens $(k-1)$-fach überdeckt, d. h., $L(\Gamma, R)$ könnte keine k-fache Kreisüberdeckung sein. □

HILFSSATZ 2. *Ist $L(\Gamma, R)$ eine k-fache Überdeckung und liegen höchstens $k-1$ Gitterpunkte im Inneren des Kreises k_i, dann ist die Dichte der Kreisüberdeckung mindestens $\frac{r_i^2}{T(\Gamma)}\pi$.*

BEWEIS. Aus dem Hilfssatz 1 folgt $R \geqq r_i$. Die Dichte von $L(\Gamma, R)$ ist $\frac{R^2}{T(\Gamma)}\pi$ auf Grund von (5) (s. in 2.1). Daraus ergibt sich $\frac{R^2}{T(\Gamma)}\pi \geqq \frac{r_i^2}{T(\Gamma)}\pi$. □

Die Gitterdreiecke Δ_i und Δ_j sind im Fall eines gegebenen Gitters Γ äquivalent, wenn eine Verschiebung oder eine Spiegelung an einem Gitterpunkt oder die Aufeinanderfolge dieser Abbildungen das eine Gitterdreieck in das andere überführt.

HILFSSATZ 3. *Es sei ein Gitter Γ gegeben. Dann gibt es höchstens*

$$3\binom{k+1}{2}\left[3\binom{k+1}{2}-1\right],$$

paarweise nicht äquivalente Gitterdreiecke, deren Umkreise in ihren Inneren höchstens $k-1$ Gitterpunkte enthalten.

BEWEIS. Wir betrachten ein Gitterdreieck Δ_i, bei dem der Kreis k_i in seinem Inneren höchstens $k-1$ Gitterpunkte enthält. Wir können offenbar annehmen, daß eine Ecke des Gitterdreiecks Δ_i der Anfangspunkt O der Basisvektoren von Γ ist. Mit X bzw. Y bezeichnen wir die anderen zwei Ecken von Δ_i.

Wir sehen ein, daß X und Y nur in einem beschränkten Teil der Ebene um O liegen können. Wir zeigen nämlich, wenn z. B. X außerhalb dieses Ebenenteiles ist, dann enthält $k[OXY]$ mindestens k Gitterpunkte in seinem Inneren für ein beliebiges Gitterdreieck OXY.

Wir betrachten die Ecke X, die ein Gitterpunkt ist, deshalb kann man X in der Form $X=mA+nB$ darstellen, wo m und n ganze Zahlen sind. Wegen der Symmetrie an O genügt es, die Fälle $m, n \geqq 0$ und $m \leqq 0$, $n \geqq 0$ zu untersuchen.

Zuerst seien $m, n \geqq 0$. Ist $X=kA$ bzw. $X=kB$, dann enthält der Umkreis des Gitterdreiecks Δ_i die Seite OX (Abb. 2) und damit auch die Gitterpunkte A, $2A, \dots, (k-1)A$ bzw. $B, 2B, \dots, (k-1)B$, d. h. mindestens $k-1$ Gitterpunkte in seinem Inneren. Folglich muß $m \leqq k$ bzw. $n \leqq k$ im Fall $n=0$ bzw. $m=0$ gelten. Es seien $m, n>0$. Die Winkel $\sphericalangle(X(nB)O) = (X(mA)O)$ sind wegen der normalen Darstellung von Γ nicht spitzwinklig. Deshalb enthält der Umkreis k_i des Dreiecks $\Delta_i=OXY$ das Dreieck $X(nB)O$ oder $X(mA)O$. Das bedeutet aber, daß k_i mindestens $m+n-1$ Gitterpunkte enthält. Es muß also $m+n-1 \leqq k-1$ gelten. Daraus ergibt sich $m+n \leqq k$, was bedeutet, daß X nur ein zum Dreieck $O(kA)(kB)$ gehöriger Gitterpunkt sein kann.

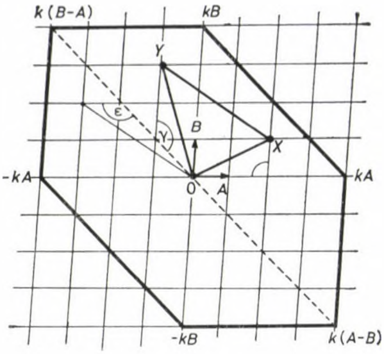

Abb. 2

Wir betrachten den Fall $m<0$, $n\geqq 0$. Ist $X=n(B-A)$, dann gilt $n\leqq k$, weil k_i die Gitterpunkte $B-A$, $2(B-A)$, ..., $(n-1)(B-A)$ in diesem Fall enthält. Weil Γ in normaler Darstellung ist, sind die Winkel $\gamma=\alpha+\sphericalangle(OAB)$ und $\varepsilon=\alpha+\sphericalangle(OBA)$ stumpfwinklig. Ist $|m|<n$, dann gilt $\sphericalangle\big(X(|m|(B-A))O\big)=\sphericalangle\big(X((n-|m|)B)O\big)=\gamma>\dfrac{\pi}{2}$. Deshalb enthält der Kreis k_i entweder das stumpfwinklige Dreieck $X\big(|m|(B-A)\big)O$ oder $O\big((n-|m|)B\big)X$, d. h. mindestens $n-1$ Gitterpunkte. Es muß also $n\leqq k$ gelten, was bedeutet, daß der Gitterpunkt X ein Punkt des Dreiecks $O(kB)\big(k(B-A)\big)$ ist. Wenn $n<|m|$ ist, dann gilt $\sphericalangle\big(X(n(B-A))O\big)=\sphericalangle\big(X((n-|m|)A)O\big)=\varepsilon>\dfrac{\pi}{2}$. Folglich liegen mindestens $|m|-1$ Gitterpunkte im Inneren des Kreises k_i. Weil k_i höchstens $k-1$ Gitterpunkte in seinem Inneren enthält, gilt $|m|\leqq k$. Deshalb kann X nur ein Gitterpunkt des Dreiecks $O(-kA)\big(k(B-A)\big)$ sein.

Zusammenfassend ergibt sich also, daß X, Y Gitterpunkte des durch kA, kB, $k(B-A)$, $-kA$, $-kB$, $k(A-B)$ bestimmten zentralsymmetrischen Sechsecks sein müssen. Die Zahl der von O verschiedenen Gitterpunkte, die im Sechseck oder auf seinem Rand liegen, ist $6\dbinom{k+1}{2}$. Die Anzahl der die Bedingungen befriedigenden Gitterdreiecke ist also höchstens $3\dbinom{k+1}{2}\left[3\dbinom{k+1}{2}-1\right]$ wegen der Zentralsymmetrie und weil O, X und Y verschiedene Punkte sind. □

HILFSSATZ 4. *Wir nehmen an, daß der Kreis k_i bei einem gegebenen Gitter Γ höchstens $k-1$ Gitterpunkte in seinem Inneren enthält. Die Kreisanordnung $L(\Gamma, r_i)$ kann eine k-fache Überdeckung nur dann sein, wenn der abgeschlossene Kreis k_i mindestens $k+2$ Gitterpunkte, eingerechnet auch die Ecken von Δ_i, enthält.*

Der BEWEIS geht auf indirekte Weise. Gehören höchstens $k+1$ Gitterpunkte zum Kreis k_i, dann können höchstens $k-2$ Gitterpunkte in seinem Inneren sein. Deshalb ist der Mittelpunkt des Kreises k_i ein innerer Punkt von höchstens $k-2$ Deckkreisen. Wir betrachten das Dreieck $\Delta_i=OXY$ und die Kreise $\hat{k}(O, r_i)$, $\hat{k}(X, r_i)$. Einer der Schnittpunkte dieser Kreislinien ist eben der Mittelpunkt von k_i. Es gibt aber einen Punkt in der Nähe des Mittelpunktes von k_i im durch $k(O, r_i)$

und $k(X, r_i)$ nicht überdeckten Teil, der höchstens $(k-1)$-fach überdeckt ist. (Die um höchstens $k-2$ innere Punkte geschlagenen Kreise und $k(Y, r_i)$ überdecken diesen Punkt.) □

HILFSSATZ 5. *Es sei $\bar{k}_i$ der Kreis vom größten Radius oder einer von diesen Kreisen bei einem gegebenen Gitter Γ in normaler Darstellung, der höchstens $k-1$ Gitterpunkte in seinem Inneren und mindestens drei Gitterpunkte auf seinem Rand enthält. Dann*

1. kann man drei Gitterpunkte auf dem Rand von $\bar{k}_i$ derart auswählen, daß das durch diese Punkte bestimmte Dreieck nicht stumpfwinklig sei und

2. ist $L(\Gamma, \bar{r}_i)$ eine k-fache Überdeckung, wenn wir mit $\bar{r}_i$ den Radius von $\bar{k}_i$ bezeichnet haben.

BEWEIS. 1. Wir nehmen indirekt an, daß beliebige drei von den Gitterpunkten, die auf dem Rand von $\bar{k}_i$ liegen, ein stumpfwinkliges Dreieck bestimmen. Das bedeutet, daß die Gitterpunkte auf dem Rand von $\bar{k}_i$ im Inneren eines Halbkreisbogens liegen. Es seien C und D die Endpunkte dieses Bogens, weiterhin sei $\bar{K}_i$ der Mittelpunkt des Kreises $\bar{k}_i$ (Abb. 3). Dann kann man offenbar einen Gitterkreis $k[CDU]$ vom größeren Radius als $\bar{r}_i$ angeben, wo $U \in CD|\bar{K}_i$ gilt und $k[CDU]$ höchstens $k-1$ Gitterpunkte in seinem Inneren enthält. Das widerspricht aber der Eigenschaft von $\bar{k}_i$, daß er den maximalen Radius hatte.

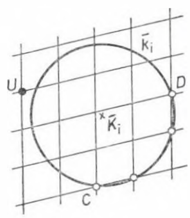

Abb. 3

2. Auch in diesem Fall geht der Beweis auf indirekte Weise. Wir nehmen an, daß es einen Punkt P in der Ebene gibt, der durch die Kreise von $L(\Gamma, \bar{r}_i)$ höchstens $(k-1)$-fach überdeckt ist. Dann enthält der Kreis $k(P, \bar{r}_i)$ höchstens $k-1$ Gitterpunkte. Wir vergrößern $k(P, \bar{r}_i)$ vom Punkt P ausgehend bis zu der Lage, in der schon ein Gitterpunkt auf seinem Rand liegt. Es sei G_1 dieser Gitterpunkt. Gibt es keinen weiteren Gitterpunkt auf seinem Rand, dann können wir diesen Kreis derart vergrößern, daß G_1 ein Randpunkt des Kreises bleibt und die Anzahl der inneren Gitterpunkte höchstens $k-1$ ist. Am Ende kommt noch ein Gitterpunkt z. B. G_2 auf den Rand dieses Kreises. Liegen keine weitere Gitterpunkte auf dem Rand des Kreises, dann können wir seinen Radius mit Beibehaltung von G_1 und G_2 weiter vergrößern. Endlich kommt noch ein Gitterpunkt G_3 auf den Rand dieses Kreises. Der Kreis $k[G_1 G_2 G_3]$ enthält aber höchstens $k-1$ Gitterpunkte in seinem Inneren und hat einen größeren Radius als $\bar{r}_i$. Das ist aber ein Widerspruch. □

Im folgenden reihen wir die Gitter in normaler Darstellung in Klassen ein. Die Gitter Γ_1 und Γ_2 in normaler Darstellung gehören zu derselben Klasse, wenn es

Gitterdreiecke $\Delta_{i,1}=OX_1Y_1$ bzw. $\Delta_{i,2}=OX_2Y_2$ in Γ_1 bzw. Γ_2 gibt, für die

$$X_1 = mA_1+nB_1 \qquad Y_1 = pA_1+qB_1$$

$$X_2 = mA_2+nB_2 \qquad Y_2 = pA_2+qB_2$$

gelten und die Umkreise $k_{i,1}$ und $k_{i,2}$ der Gitterdreiecke $\Delta_{i,1}$ und $\Delta_{i,2}$ den größten Radius unter den durch Gitterdreiecke bestimmten Kreisen haben, die höchstens $k-1$ Gitterpunkte in ihren Inneren enthalten.

Jedem Gitter können wir einen Punkt des rechtwinkligen Dreiecks OPQ im rechtwinkligen Koordinatensystem x, $y=\cos\alpha$ auf die in 1.2 beschriebene Weise zuordnen. Wir betrachten die Menge der Punkte im Dreieck OPQ, die zu dergleichen Klasse gehören. Mit H_i bezeichnen wir eine solche Menge.

Hilfssatz 6. *Betrachten wir alle Gitter in normaler Darstellung, dann gibt es eine Zerlegung des Dreiecks OPQ in endlich viele Mengen $H_1, H_2, \dots, H_s$, wo eine beliebige Menge H_i auch aus endlich vielen, einzeln zusammenhängenden und paarweise disjunkten Mengen $H_{i,1}, H_{i,2}, \dots, H_{i,r}$ bestehen kann.*

Beweis. Gilt $(x, \cos\alpha)\in H_i$, dann betrachten wir das dem Punkt $(x, \cos\alpha)$ entsprechende Gitter Γ. Im Gitter Γ hat der Umkreis $\bar{k}_i$ eines bestimmten Gitterdreiecks $\bar{\Delta}_i=OXY$ (Abb. 4) den maximalen Radius unter den Kreisen, die Umkreise von Gitterdreiecken sind und die höchstens $k-1$ Gitterpunkte in ihren Inneren enthalten. Aus der Definition der Gitter vom gleichen Typ und aus dem Hilfssatz 3 folgt, daß die Anzahl der Dreiecke $\bar{\Delta}_i$ vom verschiedenen Typ, d. h. die Anzahl der Bereiche H_i endlich ist.

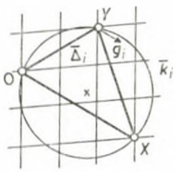

Abb. 4

Es seien $X=mA+nB$ und $Y=pA+qB$. Mit $\bar{r}_i$ haben wir den Radius des Kreises $\bar{k}_i$ bezeichnet. Wir betrachten den Quotient $\dfrac{\bar{r}_i^2}{T(\Gamma)}$. Auf Grund der Formel (3) kann man $\bar{r}_i^2$ mit den Seiten und mit dem Inhalt des Gitterdreiecks $\bar{\Delta}_i$ ausdrücken. Der Inhalt von $\bar{\Delta}_i$ ist $cT(\Gamma)$, wo c eine positive rationale Zahl ist. Nach (3) ist

$$\bar{Q}_i := \frac{\bar{r}_i^2}{T(\Gamma)} = \frac{|X|^2|Y|^2|Y-X|^2}{4^2c^2T^3(\Gamma)}. \tag{6}$$

Es ist $|Y-X|^2=s^2A^2+t^2B^2+2stAB=s^2a^2+t^2b^2+2stab\cos\alpha$, wo $s=m-p$ und $t=n-q$ sind. Wir können auch $|X|^2$ und $|Y|^2$ in ähnlicher Weise aufschreiben. Es gilt auch $T(\Gamma)=ab\sin\alpha$. Wir teilen den Zähler und den Nenner von (6) durch b^6.

Endlich bekommen wir mit den Bezeichnungen von 1.2, daß

$$\bar{Q}_i(x, \alpha) = \tag{7}$$
$$= \frac{(m^2x^2+n^2+2mnx\cos\alpha)(p^2x^2+q^2+2pqx\cos\alpha)(s^2x^2+t^2+2stx\cos\alpha)}{4^2c^2x^3\sin^3\alpha}$$

ist.

Es ist klar, daß (7) im Fall der zu dergleichen Klasse gehörigen Gitter nur von x und α hängt. Aus (7) ergibt sich

$$\bar{Q}_i(x, \alpha)x^3\sin^3\alpha = A_ix^3\cos^3\alpha + x^2\cos^2\alpha(B_ix^2+C_i)+ \tag{8}$$
$$+x\cos\alpha(D_ix^4+E_ix^2+F_i)+G_ix^6+J_ix^4+K_ix^2+L_i.$$

Die Koeffizienten auf der rechten Seite, die wir mit dem Index i bezeichnet hatten, sind reelle Zahlen und für die Gitter, bei denen $(x, \cos\alpha)\in H_i$ gilt, stimmen diese Koeffizienten nacheinander überein.

Ist $k>2$, dann kann H_i nicht das ganze Dreieck OPQ sein. Deshalb gibt es einen Bereich $H_j(i\neq j)$, so daß H_i und H_j gemeinsame Punkte haben. Es sei $P_{ij}\in H_i\cap \cap H_j$. Dann gibt es ein nicht stumpfwinkliges Gitterdreieck $\bar{\Delta}_j$ in dem Punkt P_{ij} entsprechenden Gitter, dessen Umkreis $\bar{k}_j$ höchstens $k-1$ Gitterpunkte in seinem Inneren enthält und der größte oder einer von den größten unter den diese Eigenschaft besitzenden Kreisen ist. Es gilt also $\bar{r}_i=\bar{r}_j$ in dem Punkt P_{ij} entsprechenden Gitter, folglich gilt auch

$$\bar{Q}_i = \bar{Q}_j. \tag{9}$$

Wir zeigen, daß die Kurven, die die Gleichheit (9) befriedigen, das Dreieck OPQ in endlich viele Teile zerlegen. Dazu ist es genug einzusehen, daß diese Kurven endlich viele gemeinsame Punkte haben oder übereinstimmen.

Es ist offenbar, daß wir auch $\bar{Q}_j$ in der Form (7) aufschreiben können. Die entsprechenden Koeffizienten sind andere (oder nicht alle gleich). Mit $A_j, B_j, \ldots, L_j$ bezeichnen wir diese Koeffizienten. $\bar{Q}_i=\bar{Q}_j$ ist offenbar zur Gleichung

$$(\bar{Q}_i(x, \alpha)-\bar{Q}_j(x, \alpha))x^3\sin^3\alpha = 0 \tag{10}$$

äquivalent. Es sei

$$\bar{Q}_{ij}(x, \alpha) := (\bar{Q}_i(x, \alpha)-\bar{Q}_j(x, \alpha))x^3\sin^3\alpha. \tag{11}$$

Auf Grund von (9) kann man (11) folgenderweise aufschreiben:

$$\bar{Q}_{ij}(x, \alpha) = A_{ij}x^3\cos^3\alpha+x^2\cos^2\alpha(B_{ij}x^2+C_{ij})+ \tag{12}$$
$$+x\cos\alpha(D_{ij}x^4+E_{ij}x^2+F_{ij})+G_{ij}x^6+J_{ij}x^4+K_{ij}x^2+L_{ij},$$

wo $A_{ij}, B_{ij}, \ldots, L_{ij}$ reelle Zahlen und nicht alle die Null sind.

Im folgenden verwenden wir die Bezeichnung $y=\cos\alpha$. Aus (12) kann man sehen, daß die Gleichung $\bar{Q}_{ij}(x, \alpha)=0$ in der Veränderlichen y höchstens dritten Grades ist.

Zuerst nehmen wir an, daß $A_{ij}\neq 0$ ist. Weil $x>0$ ist, kann man die Gleichung $\bar{Q}_{ij}(x,\alpha)=0$ mit der Substitution

$$z = y+\frac{x^2(B_{ij}x^2+C_{ij})}{3A_{ij}x^3}$$

folgenderweise aufschreiben:

(13) $$z^3+3\bar{p}z+2\bar{q} = 0,$$

wo

(14) $$2\bar{q} = \frac{2x^6(B_{ij}x^2+C_{ij})^3}{27A_{ij}^3x^9}-\frac{x^2(B_{ij}x^2+C_{ij})x(D_{ij}x^4+E_{ij}x^2+F_{ij})}{3A_{ij}^2x^6}+ +\frac{G_{ij}x^6+J_{ij}x^4+K_{ij}x^2+L_{ij}}{A_{ij}x^3}$$

und

(15) $$3\bar{p} = \frac{3A_{ij}x^3x(D_{ij}x^4+E_{ij}x^2+F_{ij})-x^4(B_{ij}x^2+C_{ij})^2}{3A_{ij}^2x^6}$$

sind. Aus (14) und (15) kann man ablesen, daß $\bar{q}$ und $\bar{p}$ gebrochene Funktionen von x sind. Die Lösungen der Gleichung (13) sind $z_1=u+v$, $z_2=\varepsilon_1 u+\varepsilon_2 v$, $z_3=\varepsilon_2 u+\varepsilon_1 v$, wo

(16) $$u = \sqrt[3]{-\bar{q}+\sqrt{\bar{q}^2+\bar{p}^3}}\,,\quad v = \sqrt[3]{-\bar{q}-\sqrt{\bar{q}^2+\bar{p}^3}}$$

und ε_1, ε_2 dritte Einheitswurzeln sind.

Wir sehen ein, daß sich die Kurven z_1, z_2, z_3 höchstens in endlich vielen Punkten schneiden oder gleich sind. Das bedeutet, daß die Kurven $\bar{Q}_{ij}(x,\alpha)=0$ das rechtwinklige Dreieck OPQ in endlich viele Teile zerlegen. Gilt nämlich z. B. $z_2=z_3$, dann ist $u(\varepsilon_1-\varepsilon_2)=v(\varepsilon_1-\varepsilon_2)$, woraus $u=v$ folgt. Auf Grund von (16) ist

(17) $$\bar{q}^2+\bar{p}^3 = 0$$

in diesem Fall. Wir substituieren $\bar{q}$ und $\bar{p}$ aus (14) bzw. (15) in (17). Mit äquivalenten Umformungen bekommen wir eine Gleichung 12-ten Grades für x. Es ist leicht zu zeigen, daß auch die Gleichheit $z_1=z_2$ nur dann auftreten kann, wenn (17) gilt.

Ist $A_{ij}=0$ in (12), dann bekommen wir eine Gleichung höchstens 2-ten Grades in der Variablen y. Wir können auch diesen Fall ähnlich dem Fall $A_{ij}\neq 0$ untersuchen.

Weil es nur endlich viele verschiedene Quotienten $\bar{Q}_i$ gibt, ist auch die Anzahl der Gleichungen vom Typ (9) endlich. Wir betrachten die Kurven, die die Gleichungen $\bar{Q}_i=\bar{Q}_j$ bzw. $\bar{Q}_r=\bar{Q}_s$ befriedigen und sehen ein, daß auch diese Kurven endlich viele gemeinsame Punkte haben oder übereinstimmen. In diesem Fall gelten also die Gleichungen

(19) $$\begin{aligned}\bar{Q}_{ij}(x,\alpha) &= A_{ij}x^3y^3+x^2y^2(B_{ij}x^2+C_{ij})+xy(D_{ij}x^4+E_{ij}x^2+F_{ij})+\\ &\quad +G_{ij}x^6+J_{ij}x^4+K_{ij}x^2+L_{ij} = 0\\ \bar{Q}_{rs}(x,\alpha) &= A_{rs}x^3y^3+x^2y^2(B_{rs}x^2+C_{rs})+xy(D_{rs}x^4+E_{rs}x^2+F_{rs})+\\ &\quad +G_{rs}x^6+J_{rs}x^4+K_{rs}x^2+L_{rs} = 0\end{aligned}$$

und wir suchen die Lösungen dieses Gleichungssystems mit zwei Unbekannten. Es ist bekannt, daß dieses Gleichungssystem im allgemeinen Fall endlich viele reelle und komplexe Lösungen hat. Es kann nur noch der Fall vorkommen, daß die Kurven $\bar{Q}_{ij}(x, \alpha)=0$, $\bar{Q}_{rs}(x, \alpha)=0$ einen gemeinsamen Bogen höchstens 2-ten Grades und außerdem endlich viele Schnittpunkte haben.

Wir haben gezeigt, daß sich die endlich vielen Kurven, die die gesamten möglichen endlich vielen Gleichungen $\bar{Q}_{ij}(x, \alpha)=0$ befriedigen, in endlich vielen Punkten schneiden oder zusammenfallen. Das bedeutet aber, daß diese Kurven das Dreieck OPQ in endlich viele Teile zerlegen. Daraus folgt schon, daß jeder Bereich H_i höchstens aus endlich vielen Bereichen $H_{i1}, H_{i2}, \dots, H_{ir}$ bestehen kann, die einzeln zusammenhängend und paarweise disjunkt sind. □

Bemerkung 1. Auf Grund des Satzes 1 können wir eine Zerlegung des rechtwinkligen Dreiecks OPQ machen. Aus dem Satz 1 und aus dem Hilfssatz 6 folgt, daß jeder Bereich H_i (vielleicht $H_{i1}, H_{i2}, \dots, H_{ir}$) durch endlich viele Kurven begrenzt ist. (Diese Kurven befriedigen die Gleichungen vom Typ $\bar{Q}_{ij}(x, \alpha)=0$.)

Wir betrachten die Gitter Γ, die dem Bereich H_i entsprechen. Nach dem Hilfssatz 5 ist $L(\Gamma, \bar{r}_i)$ eine k-fache Überdeckung und auf Grund des Hilfssatzes 2 ist die Dichte mindestens $\bar{Q}_i(x, \alpha)\pi$. Deshalb müssen wir das absolute Minimum der Funktion $\bar{Q}_i$, $(x, \cos\alpha)\in H_i$ mit zwei Veränderlichen finden. Das ist eine unangenehme Aufgabe. Auf die Erleichterung dieser Aufgabe bezieht sich der folgende Satz.

Satz 2. *Wir betrachten die dem Bereich H_i entsprechenden gitterförmigen k-fachen Kreisüberdeckungen $L(\Gamma, \bar{r}_i)$. Dann ist die Dichte von $L(\Gamma, \bar{r}_i)$ entweder auf dem Rand des Bereiches H_i oder in einem einzigen (wohlbestimmten) inneren Punkt von H_i minimal.*

Vor dem Beweis des Satzes definieren wir eine Gittertransformation. Wir betrachten einen beliebigen Bereich H_i. Das nicht stumpfwinklige Gitterdreieck $\bar{\Delta}_i$ wurde diesem Bereich (Hilfssatz 5) zugeordnet. $\bar{\Delta}_i$ kann höchstens bei den Gittern rechtwinklig sein, die den Randpunkten von H_i entsprechen. Der Umkreis $\bar{k}_i$ von $\bar{\Delta}_i$ hatte den größten Radius unter den Kreisen, die Umkreise von Gitterdreiecken sind und die höchstens $k-1$ Gitterpunkte in ihren Inneren enthalten. Die Ecken von $\bar{\Delta}_i$ wurden mit O, X und Y (Abb. 4) bezeichnet, wo O der Anfangspunkt der Basisvektoren des Gitters Γ ($(x, \cos\alpha)\in H_i$) ist. Wenn das Dreieck OXY nicht regulär ist, dann hat es verschiedene Seiten. Wir nehmen an, daß z. B. $|Y|<|Y-X|$ gilt. Die Gittertransformation g_i wird für die Gitter definiert, bei denen $(x, \cos\alpha)\in H_i$ gilt. Wir halten die Gitterpunkte auf der Gittergerade OX fest und wir bewegen den Gitterpunkt Y auf einer zu OX parallelen Gerade derart, daß $|Y|$ wächst. Das Gitter Γ ändert sich entsprechend der Lageänderung von Y. Wir wenden die Transformation g_i höchstens bis zu der Lage an, wo der dem Gitter entsprechende Punkt ein Randpunkt von H_i ist oder $|Y|=|Y-X|$ gilt. Mit g_i^{-1} bezeichnen wir die inverse Transformation von g_i. Auch g_i^{-1} wird bis zu einer der obigen zwei Lagen angewandt.

Beweis des Satzes 2. Wir betrachten die Gitter, für die $(x, \cos)\alpha\in H_i$ gilt. Wir haben das nicht stumpfwinklige Gitterdreieck $\bar{\Delta}_i=OXY$ diesen Gittern zugeordnet. Das Dreieck $\bar{\Delta}_i$ kann nur höchstens im Fall der Gitter rechtwinklig sein, die den bestimmten Randpunkten von H_i entsprechen. Wir nehmen ein Gitter, für das der entsprechende Punkt $(x, \cos\alpha)$ ein innerer Punkt von H_i ist.

Aus der Definition der Transformationen g_i bzw. g_i^{-1} folgt, daß der Inhalt T des spitzwinkligen Gitterdreiecks $\bar{\Delta}_i$ konstant ist. Weil $\bar{\Delta}_i$ ein Gitterdreieck ist, gilt $T=cT(\Gamma)$, wo $c>0$ eine rationale Zahl ist.

Ist $|Y|<|Y-X|$, dann verwenden wir die Transformation g_i. Der Umkreisradius des Dreiecks OXY nimmt ab, weil Y sich in Richtung der Mittelsenkrechte von OX auf der zu OX parallelen Gerade bewegt. (Die Abnahme bedeutet eine streng monotone Abnahme auch in diesem Fall.) So nimmt $\bar{r}_i$ ab und auf Grund der obigen Überlegung nimmt auch $\bar{Q}_i$ ab. Ebenso ergibt sich die Abnahme von $\bar{Q}_i$ im Fall $|Y|>|Y-X|$, wenn wir die Transformation g_i^{-1} anwenden.

Während der Anwendung dieser Transformationen können zwei Fälle auftreten.

1. Wir erreichen ein Gitter, bei dem der entsprechende Punkt $(x, \cos\alpha)$ auf dem Rand von H_i liegt.

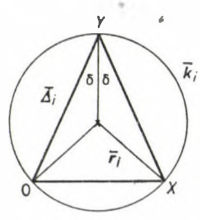

Abb. 5

2. Es gilt $|Y|=|Y-X|$ für die Seiten des Gitterdreiecks $\bar{\Delta}_i$. Wir untersuchen den letzteren Fall. Es sei $\delta := \frac{\sphericalangle(OYX)}{2}$ (Abb. 5). Dann gilt

$$\text{(20)} \qquad \frac{\bar{r}_i^2}{T(\Gamma)} = \frac{\bar{r}_i^2 c}{T} = \frac{\bar{r}_i^2 c}{\frac{1}{2}\bar{r}_i^2 \sin 2\delta \sin^2 \frac{\pi-2\delta}{2}} = \frac{c}{\sin\delta\cos^3\delta},$$

wo $\delta\in\left(0, \frac{\pi}{4}\right]$ ist. Wir betrachten die Funktion $f(\delta)=\sin\delta\cos^3\delta$, $\delta\in\left(0, \frac{\pi}{2}\right]$. Aus ihrer ersten Ableitung $f'(\delta)=\cos^2\delta(1-4\sin^2\delta)$ kann man ablesen, daß $f(\delta)$ im Fall $\delta\leqq\frac{\pi}{6}$ wächst und $f(\delta)$ für $\delta\geqq\frac{\pi}{6}$ abnimmt. Ist $\sphericalangle(OYX)<\frac{\pi}{3}$, dann halten wir die Seite OX fest und bewegen Y auf der Mittelsenkrechte von OX derart, daß $|Y|$ abnimmt. Die entstehenden Dreiecke werden gleichschenklig und δ wächst inzwischen, d. h. (20) nimmt auf Grund des vorhergehenden ab. Folglich bekommen wir entweder ein Gitter während der Anwendung der vorigen Transformation, bei dem der diesem Gitter entsprechende Punkt $(x, \cos\alpha)$ auf dem Rand von H_i liegt, oder das Dreieck $\bar{\Delta}_i=OXY$ regulär wird. Ist $\sphericalangle(OYX)>\frac{\pi}{3}$, dann bewegen wir Y wieder auf der Mittelsenkrechte von OX, aber in die umgekehrte Richtung, d. h. $|Y|$ wächst. δ nimmt ab und deshalb nimmt auch (20) ab. Auch hier erreichen wir einen der obigen zwei Fälle. Ist das Dreieck $\bar{\Delta}_i=OXY$ regulär, dann entspricht ein wohlbestimmter Punkt $(|Y|=|Y-X|=|X|)$ diesem Gitter im rechtwinkligen Koordinatensystem $x, y=\cos\alpha$. □

Bemerkung 2. $\bar{Q}_i$ kann also in einem inneren Punkt von H_i nur dann minimal sein, wenn das Dreieck Δ_i regulär ist. In diesem Punkt rechnen wir den Wert von $\bar{Q}_i$ aus und $\bar{Q}_i\pi$ gibt die Dichte der dünnsten Kreisüberdeckungen bei den Gittern an, die den Punkten von H_i entsprechen.

Die andere Möglichkeit ist, daß die stetige Funktion $\bar{Q}_i$, $(x, \cos\alpha)\in H_i$ mit zwei Veränderlichen in einem Randpunkt von H_i ihr Minimum annimmt. Auf dem Rand von H_i ist aber $\bar{Q}_i$ eine Funktion mit einer Veränderlichen. H_i ist nämlich durch $\cos\alpha=\frac{x}{2}$ oder $x=1$ oder $\cos\alpha=0$ oder durch die Kurven vom Typ $\bar{Q}_i=\bar{Q}_j$ begrenzt. In diesen Fällen hängt $\bar{Q}_i$ von einer Veränderlichen ab, die wir in einem bestimmten Intervall untersuchen müssen. In den ersten drei Fällen geht das einfach. Im letzteren Fall ($\bar{Q}_i=\bar{Q}_j$) können aber technische Schwierigkeiten vorkommen. (Z. B. es ist nicht zweckmäßig die Funktion mit einer Veränderlichen, die wir aus der Gleichung $\bar{Q}_i=\bar{Q}_j$ bekommen, in expliziter Form aufzuschreiben.)

Bemerkung 3. Aus den Sätzen 1 und 2 folgt, daß wir die Bestimmung der dünnsten gitterförmigen k-fachen Kreisüberdeckungen auf die Bestimmung der Minima von endlich vielen Funktionen in einer Veränderlichen zurückgeführt haben, und wir müssen das absolute Minimum unter diesen Minima auswählen. Der Hilfssatz 5 garantiert, daß die diesem absoluten Minimum entsprechende gitterförmige Kreislagerung (oder Kreislagerungen) eine k-fache Überdeckung bildet.

Als Fortsetzung dieses Artikels erörtern wir die dünnsten gitterförmigen 6-, 7- und 8-fachen Kreisüberdeckungen. In diesen konkreten Fällen verfolgen wir nicht den im vorhergehenden geschriebenen Gedankengang Schritt für Schritt. Die Ursache ist, daß die technische Ausführung mit einer kleinen Änderung einfacher ist. Diese Änderung ist sehr kurz die folgende. Wir müssen nicht entscheiden, ob der dem Bereich H_i zugeordnete Kreis k_i der größte unter den Kreisen ist, die durch Gitterdreiecke bestimmt sind und die mindestens $k-1$ Gitterpunkte in ihren Inneren enthalten. Es gilt nämlich $\frac{R^2\pi}{T(\Gamma)}\geqq\frac{r_i^2\pi}{T(\Gamma)}$ für jeden Gitterkreis k_i mit den obigen zwei Eigenschaften (s. den Hilfssatz 2). Dann schätzen wir den Quotient Q_i von unten. Das Wesen der Methode bleibt aber auch bei den Untersuchungen in den konkreten Fällen erhalten.

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(Eingegangen am 11. Februar 1983)

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ON TWO PROBLEMS OF KATONA CONCERNING SUMS OF VECTORS

HA LE ANH

In [1] Katona proved certain inequalities concerning the length of the sum of 3 random vectors. He used combinatorial and geometric tools. He proved two lemmas concerning sums of real numbers hoping that they are true for higher dimensional vectors, as well. We prove that the lemmas are true for two-dimensional spaces, but not for three- (or higher-) dimensional ones.

This means that some of the results of [1] can be extended for two-dimensional vectors, but (unfortunately) the method does not work for higher dimensions.

Let us first consider the question raised in Lemma 2.2 of [1]:

THEOREM 1. *Suppose $a_1, \ldots, a_5 \in \mathbf{R}^n$ with $|a_i| \geqq 1$ $(1 \leqq i \leqq 5)$. If $n=1$ or 2 then*

$$|a_i+a_j+a_k| \geqq 1 \tag{1}$$

holds for at least one of the triples $1 \leqq i<j<k \leqq 5$ different from (1, 2, 3), (1, 2, 4) *and* (3, 4, 5). *This is not true if $n \geqq 3$.*

The other theorem answers the question raised in Lemma 2.4 of [1]:

THEOREM 2. *Suppose $a_1, a_2, a_3, b_1, b_2 \in \mathbf{R}^n$ with $|a_i|, |b_j| \geqq 1$ $(1 \leqq i \leqq 3,\ 1 \leqq j \leqq 2)$. If $n=1$ or 2 then all the inequalities*

$$|a_1+a_2+a_3| < 1 \tag{2}$$

$$|a_i+b_1+b_2| < 1 \quad (1 \leqq i \leqq 3) \tag{3}$$

$$|a_i+a_j+b_k| \geqq 1 \quad (1 \leqq i < j \leqq 3,\ 1 \leqq k \leqq 2) \tag{4}$$

cannot hold simultaneously. This is not true if $n \geqq 3$.

The positive parts of both theorems are particular cases of the following lemma.

LEMMA. *Suppose $c_1, c_2, c_3, d \in \mathbf{R}^2$ with $|c_i| \geqq 1$ $(1 \leqq i \leqq 3)$. Then at least one of the inequalities*

$$|c_1+c_2+c_3| \geqq 1, \tag{5}$$

$$|d+c_1| \geqq 1 \tag{6}$$

$$|d+c_2| \geqq 1 \tag{7}$$

$$|d+c_3| \geqq 1 \tag{8}$$

holds.

1980 *Mathematics Subject Classification.* Primary 51M25.
Key words and phrases. Random vectors, length of vectors.

PROOF. As rotation does not change the length, it can be supposed that the second component of d is zero: $d=(\delta, 0)$ and its first component $\delta \geqq 0$. Introduce the notations $c_i=(\alpha_i, \beta_i)$ $(1 \leqq i \leqq 3)$.

Assume that (6) is not true. If $\alpha_1 \geqq 0$, then the angle between c_1 and d is $\leqq 90°$ therefore the length of their sum is $\geqq |c_i| \geqq 1$. By this contradiction we may suppose

$$\alpha_1 < 0. \tag{9}$$

On the other hand, the contrary of (6) implies that the absolute value of the second component of $d+c_1$ is <1. However, this is nothing else but β_1. Hence we have

$$|\beta_1| < 1. \tag{10}$$

Assume now that (7) is not true. By the same reasoning as above,

$$\alpha_2 < 0 \tag{11}$$

and

$$|\beta_2| < 1 \tag{12}$$

can be supposed.

Finally, if (8) is not true then

$$\alpha_3 < 0 \tag{13}$$

and

$$|\beta_3| < 1 \tag{14}$$

should hold.

Consider now the left-hand side of (5):

$$|c_1+c_2+c_3| = (\alpha_1+\alpha_2+\alpha_3)^2+(\beta_1+\beta_2+\beta_3)^2 =$$

$$= \alpha_1^2+\beta_1^2+\alpha_2^2+\beta_2^2+\alpha_3^2+\beta_3^2+2\alpha_1\alpha_2+2\alpha_1\alpha_3+2\alpha_2\alpha_3+2\beta_1\beta_2+2\beta_1\beta_3+2\beta_2\beta_3.$$

Using $|a_1|, |a_2|, |a_3| \geqq 1$, (9), (11) and (13) we obtain that this sum is greater than

$$3+2\beta_1\beta_2+2\beta_1\beta_3+2\beta_2\beta_3. \tag{15}$$

If all βs have the same sign then the latter sum and consequently $|a_1+a_2+a_3|$ are >1. If exactly two of the βs are negative, multiply all second components by -1, so $-1<\beta_3<0$, $0 \leqq \beta_1, \beta_2 < 1$ can be assumed by the symmetry of βs in (15). Using these assumptions, the following inequalities are obtained for (15):

$$3+2\beta_1\beta_2+\beta_3(2\beta_1+2\beta_2) \geqq 3+2\beta_1\beta_2-(2\beta_1+2\beta_2) =$$

$$= 3-\beta_2(2-2\beta_1)-2\beta_1 > 3-(2-2\beta_1)-2\beta_1 = 1.$$

This proves that (5) must hold if none of (6)—(8) holds. The proof is complete.

PROOF of Theorem 1. Apply the lemma with $c_1=a_2$, $c_2=a_3$, $c_3=a_4$ and $d=a_1+a_5$. Thus $|a_i+a_j+a_k| \geqq 1$ holds with one of the choices $(i, j, k)=(2, 3, 4)$, (1, 2, 5), (1, 3, 5), (1, 4, 5). This is actually a stronger statement than the first part of Theorem 1.

The second part is proved with the following construction:

$$a_1 = (1, 0, 0), \quad a_2 = (1, 0, 0),$$

$$a_3 = \left(-\frac{1}{2}, \sqrt{\frac{3}{4}-4\varepsilon}, -2\sqrt{\varepsilon}\right), \quad a_4 = \left(-\frac{1}{2}, -\sqrt{\frac{3}{4}-4\varepsilon}, -2\sqrt{\varepsilon}\right),$$

$$a_5 = (-1-\varepsilon, 0, \sqrt{\varepsilon}),$$

where $0<\varepsilon<\frac{1}{16}$. It is easy to see that $|a_1+a_2+a_3|>1$, $|a_1+a_2+a_4|>1$, $|a_3+a_4+a_5|>$ >1 but all other triple-sums have a length <1. The proof is complete.

PROOF of Theorem 2. To prove the first part of the theorem, take $c_1=a_1$, $c_2=a_2$, $c_3=a_3$, $d=b_1+b_2$. By the lemma, (2) or (3) should be violated. This is a stronger statement than the first part of the theorem.

The second part is proved with a construction of the 3-dimensional vectors satisfying $|a_1|, |a_2|, |a_3|, |b_1|, |b_2|\geqq 1$, (2), (3) and (4):

$$a_1 = \left(\varepsilon, \sqrt{\frac{1-\varepsilon^2}{2}}, -\sqrt{\frac{1-\varepsilon^2}{2}}\right), \quad a_2 = \left(\varepsilon, -\sqrt{\frac{1-\varepsilon^2}{2}}, -\sqrt{\frac{1-\varepsilon^2}{2}}\right),$$

$$a_3 = (\varepsilon, 0, \sqrt{1-\varepsilon^2})$$

$$b_1 = (1, 0, 0), \quad b_2 = (-1-\varepsilon, 0, 0).$$

It is easy to see that all these conditions are satisfied for $\varepsilon<\frac{4\sqrt{2}-5}{7}$. The proof is complete.

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(Received February 18, 1983; in revised form March 1, 1986)

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ON A CONJECTURE OF G. O. H. KATONA

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Abstract

Let H be a finite ordered set (say, different real numbers $|H|=n$), however, their ordering is unknown for us. In this paper we solve the following problems:

If $A=\{i_1, i_2, \dots, i_k\}$ $(1\leqq i_1<i_2\dots<i_k\leqq n)$ and x is an arbitrary element of H then the minimal number of comparisons needed to decide whether the index of element x is in A or not (say in decreasing order) is $n-1$.

If x, y are arbitrary elements of H then the minimal number of comparisons needed to decide whether the indexes of elements x, y are p, q or q, p in case $1\leqq p<q\leqq\left\lceil\frac{n}{2}\right\rceil$ is $n+q-3$.

1. Introduction

Let $H=\{z_1, z_2, \dots, z_n\}$ be a finite ordered set (say, different real numbers). However, their ordering is unknown for us. There are many situations where we want to obtain certain information concerning H using pairwise comparisons of the elements.

The simplest question of this type: Which is the largest (smallest) element in H. It is easy to prove that any strategy finding the largest element needs at least $n-1$ comparisons. Ira Pohl ([4]) proved that at least $n+\left\lceil\frac{n}{2}\right\rceil-2$ ($\lceil X\rceil$ denotes the smallest integer $\geqq x$) comparisons are needed if we want to determine the largest and smallest elements simultaneously (see also [3], [5], [7], [9]). In a recent paper [7] it is proved that we need $n+\left\lceil\frac{n-1}{2}\right\rceil-2$ $(n\geqq 3)$ comparisons if we want to decide only whether x_1 and x_2 are the largest and smallest elements. If we want to decide whether x_1 is the largest and x_2 is the smallest element in H we need at least $n+\left\lceil\frac{n}{2}\right\rceil-3$ $(n\geqq 3)$ and whether x_1, x_2 are neighbouring elements in H we need at least $2(n-l)$ $(n\geqq 3)$ comparisons ([8]).

To find the two largest elements $n+\lceil\log_2 n\rceil-2$ comparisons are needed ([2]) (for similar results see also [1], [3], [5]). Moreover it is proved [6] that it is impossible to find a pair of consecutive elements with a smaller number of comparisons.

1980 *Mathematics Subject Classification.* Primary 05A05; Secondary 05A15.
Key words and phrases. Finite ordered set, searching, optimal strategy.

We can easily prove that if x is an arbitrary element of H and we want to decide whether x is the i-st in H (say in decreasing order, $1 \leqq i \leqq n$) we need at least $n-1$ comparisons. G. O. H. Katona [10] raised and proved the following: Let $c_1, c_2, \ldots, c_n$ be the elements of H in decreasing order and let $A=\{r_1, r_2, \ldots, r_k\}$ $(1 \leqq r_1 < r_2 < \ldots < r_k \leqq n)$. The question is whether the index of x is in A or not. Katona proved that we need at least $n-1$ comparisons ([10]). In this paper we give a new proof.

G. O. H. Katona raised the following problem: if x, y are arbitrary elements of H and we want to decide whether x is p-st, y is q-st or x is q-st y is p-st in H how many comparisons are needed. He conjectures that $n+q-3$ comparisons are needed at least for this in case of $1 \leqq p < q \leqq \left[\frac{n}{2}\right]$. In this paper we prove that the conjecture of G. O. H. Katona is true.

2. Notations, definitions

For the present purposes it will be better to use the notations $x=z_1$, i.e. $H=\{x, z_2, z_3, \ldots, z_n\}$ and $A=\{r_1, r_2, \ldots, r_k\}$ $(1 \leqq r_1 < r_2 < \ldots < r_k \leqq n)$. The question is the following: whether the index of x is in A or not.

The first pair to be compared is denoted by $S_0=(c, d)$. If the result of the comparison is $c>d$ then the value of the variable ε_1 is 1. In the opposite case $c>d$, $\varepsilon_1=0$. The choice of the next pair $S_1(\varepsilon_1)$ depends on ε_1. Suppose $S_1(\varepsilon_1)=(e(\varepsilon_1), f(\varepsilon_1))$.

Define ε_2 to be 1 if $e(\varepsilon_1)>f(\varepsilon_1)$ and to be 0 otherwise. Continuing this procedure in the same way

$$S_{i-1}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{i-1}) \tag{1}$$

is defined for some 0, 1 sequences $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{i-1}$ with the restriction that if $S_{i-1}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{i-1})$ is defined then $S_{i-2}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{i-2})$ is defined, too. The value of ε_i is 1 or 0 according to whether the first or the second member is larger. A set of questions given in this way will be called a strategy suitable for deciding the question "whether the index of x is in A or not" iff for all sequences $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l$ when

(2) $S_{l-1}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{l-1})$ is determined, but

(3) $S_l(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l)$ is not

(4) then answers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l$ (together with the questions $S_0, S_1(\varepsilon_1), \ldots, S_{l-1}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{l-1})$) give a unique reply to the problem: the index of x is in A or not.

We use the notation $\mathcal{S}(A)$ for such a strategy. We say that the strategy $\mathcal{S}(A)$ is finished for the sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l$ if the conditions (2)—(4) are satisfied. The maximum length of the sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l$ finishing the strategy is called its length. It will be denoted by $L(\mathcal{S}(A))$.

Denote by $T_i(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i)$ the inequality set up from the pair $S_{i-1}(\varepsilon_1, \varepsilon_2, \ldots \varepsilon_{i-1})$ on the basis of the answer ε_i. Now we can express condition (4) in a modified way: The inequalities

$$T_1(\varepsilon_1), T_2(\varepsilon_1, \varepsilon_2), \ldots, T_l(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l) \tag{5}$$

uniquely decide whether the index of x is in A or not. Situation $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i)$ of $\mathscr{S}(A)$ is the situation after answering the question $S_{i-1}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{i-1})$. That is, we have the inequalities $T_1(\varepsilon_1), \ldots, T_i(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i)$ and denote it by $\mathscr{E}_i$.

Similarly we can define the strategy suitable for deciding if x, y are arbitrary elements of H, then the indexes in H of the elements x, y (say in decreasing order) are $p, q \left(1 \leqq p < q \leqq \left[\frac{n}{2}\right]\right)$ or not. Denote this strategy by $\mathscr{S}(p, q)$.

3. The results

We shall prove the following theorems:

THEOREM 1.

$$\min_{\mathscr{S}(A)} L(\mathscr{S}(A)) = n-1 \qquad (1 \leqq |A| < n). \tag{6}$$

THEOREM 2.

$$\min_{\mathscr{S}(p,q)} L(\mathscr{S}(p, q)) = n+q-3 \tag{7}$$

if

$$1 \leqq p < q \leqq \left[\frac{n}{2}\right].$$

PROOF of Theorem 1. It is easy to find a strategy $\mathscr{S}(A)$ satisfying (6). We compare all the z's with x. If the results of the comparisons are as follows : the element x is larger in $n-r_s$ cases $(1 \leqq s \leqq k)$ then x is the r_s-st otherwise not. The number of comparisons is $n-1$. This proves that $\min_{\mathscr{S}(A)} L(\mathscr{S}(A)) \leqq n-1$. It remains to prove

$$L(\mathscr{S}(A)) \geqq n-1 \tag{8}$$

for any strategy $\mathscr{S}(A)$. This will be done in the following way:

An algorithm will be given which determines a branch of the strategy, that is, a sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l$ finishing it. This branch will have a length of $\geqq n-1$. The algorithm determines the ε's recursively. Partitions of $H-\{x\}$ will be used. The partitions will also be defined recursively for any situation $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i)$ along the indicated branch. The branch and the partitions will be determined simultaneously. A partition has 3 classes:

$$H-\{x\} = A^i \cup K^i \cup N^i.$$

At the beginning $A^0 = H-\{x\}$, $K^0 = N^0 = \emptyset$. We now introduce the concept of graph-realization. Correspond the elements of the set H to the vertices of a graph $\vec{G}$. Let a comparison be an edge of $\vec{G}$ between the corresponding vertices. Let the answer be the orientation of this edge in the following way: if we compare two elements, say c and d in some state of $\vec{G}$ and the result of the comparison is $c > d$ then we direct the edge from c to d, conversely, when $c < d$ we direct the edge from d to c. In the state $(\varepsilon_1, \ldots, \varepsilon_i)$ let $\vec{G}^i$ denote the graph derived in this way. By the above correspondence we uniquely associate an oriented graph to all states of $\mathcal{S}(A)$. It follows from the correspondence that an arbitrary state $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i)$ of $\mathcal{S}(A)$ the relation $e>f$ is realized if and only if an oriented path leads in $\vec{G}^i$ from e to f. Denote G^i the graph obtained from $\vec{G}^i$ by cancelling the direction of edges.

In the situation $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i)$ the elements of $N^i\,(K^i)$ are those elements, which are greater (smaller) than x on the basis of $\mathcal{E}_i$ and let $A^i = H - (\{x\} \cup N^i \cup K^i)$. We distinguish two cases:

CASE I. $r_s \in A$, $r_s = s$ for all r_s. Because of the condition $r_1 < r_2 < \ldots < r_k$ this is equivalent with the fact that $r_k = k$.

CASE II. There exists such r_v $(r_v \in A)$ for which $r_v \neq v$. Let r_t be the first among these that is $r_t \neq t$ but $r_s = s$, if $s < t$. Suppose that $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i$ and N^i, K^i, A^i are defined. Then the next description determines ε_{i+1} and N^{i+1}, K^{i+1}, A^{i+1}. Let $S_i(\varepsilon_1, \ldots, \varepsilon_i) = (g, h)$. In the following we do not mention the cases which follow from the cases to be discussed by changing the roles of the elements g and h. We do not write the corresponding sets A^{i+1}, N^{i+1}, K^{i+1} neither, as they follow from $\mathcal{E}_{i+1}$.

$$\text{(1)} \qquad \left.\begin{array}{l} g, h \in A^i \\ g, h \in N^i \\ g, h \in K^i \end{array}\right| \begin{array}{l} \varepsilon_{i+1} = \text{arbitrary, except if} \\ \text{it is determined by the} \\ \text{extension of } \mathcal{E}_i. \end{array}$$

$$\text{(2)} \qquad g \in N^i, \quad h \in A^i \cup K^i \quad \varepsilon_{i+1} = 1$$

$$\text{(3)} \qquad g \in A^i, \quad h \in K^i \quad \varepsilon_{i+1} = 1$$

$$\text{(4)} \qquad g = x, \quad h \in N^i(K^i) \quad \varepsilon_{i+1} = O(1)$$

$$\text{(5)} \qquad g = x, \quad h \in A^i.$$

In Case I. $\varepsilon_{i+1} = 0$, if $|N^i| + |A_1^i| \leqq k-2$, where the elements of A_1^i are the elements from A^i being greater than h on the basis of ε_i and $\varepsilon_{i+1} = 1$ otherwise.

In Case II. $\varepsilon_{i+1} = 1$, if $|K^i| + |A_2^i| \leqq n - r_t - 1$, where the elements of A_2^i are the elements from A^i being smaller than h on the basis of $\mathcal{E}_i$ and $\varepsilon_{i+1} = 0$ otherwise.

In this way we have defined a branch $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l$ of the strategy $\mathcal{S}(A)$. It will be denoted by $P(A)$. The length $|P(A)|$ of $P(A)$ is l. We shall prove $l = |P(A)| \geqq$ $\geqq n-1$.

We discuss the cases I and II separately.

CASE I. *We prove that* $|N^l| \leqq k-1$. Suppose that $|N^l| > k-1$ in the contrary to our assumption.

Let $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i)$ be the situation for which $|N^i| \leqq k-1$ and $|N^{i+1}| > k-1$ hold. Obviously, there exists such situation. Let $S_i(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i) = (a, b)$. It follows from the definition of $P(A)$ that $a = x$ or $b = x$. Suppose that $a = x$ and from assumption $|N^i| \leqq k-1$ it follows that $b \in A^i$ and $\varepsilon_{i+1} = 0$. But we use point (5), and then $\varepsilon_{i+1} = 1$. From this contradiction it follows that $|N^l| \leqq k-1$.

Now we prove that $|K^l| \leqq n-k$.

Suppose that $|K^l| > n-k$ in contrast with our assumption. Let $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_j)$ be the situation for which $|K^j| \leqq n-k$, $|K^{j-1}| > n-k$ hold. Obviously, there exists such situation. Let $S_j(\varepsilon_1, \ldots, \varepsilon_j)=(c, d)$. From the definition of $P(A)$ it follows that $c=x$ or $d=x$ holds.

Suppose that $c=x$ and — as it is easy to see — $d\in A^j$, that is we use (5). Because $\varepsilon_{j+1}=1$ for this reason $|N^j|+|A_1^j|>k$ and — according to our assumption — $|K^j|+|A_2^j|\geqq n-k$, that is

$$|N^j|+|A_1^j| \geqq k-1$$

$$|K^j|+|A_2^j| \geqq n-k$$

and from these inequalities it follows that

$$|N^j|+|K^j|+|A_1^j|+|A_2^j| \geqq n-1.$$

But this is impossible, because

$$N^j \cup A_1^j \cup K^j \cup A_2^j \subseteqq H-\{x, b\}.$$

The contradiction proves our statement.

We shall prove that $|K^l|=n-k$.

Suppose that $|K^l|\neq n-k$. As $|K^l|\leqq n-k$ holds therefore $|K^l|<n-k$ and $|K^l|+|N^l|<n-1$ that is $A^l\neq\emptyset$. Let $a\in A^l$. From the definition of $P(A)$ it follows that neither $x>a$ nor $x<a$ follows from $\mathscr{E}_l$. Consequently if elements of A^l are smaller (larger) than element x then the index of x is (is not) in A. It follows from this that the $\mathscr{S}(A)$ is unfinished. The contradiction proves our assertion. Consequently $|K^l|=n-k$, $|N^l|\leqq k-1$.

Let — say — $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i)$ be that situation when $|K^i|\neq n-k, |K^{i+1}|=n$ hold. Suppose that

$$S_i(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i) = (a, b).$$

We can easily see that $a=x$ or $b=x$ holds and — say — $a=x$ then $b\in A^i$ and we use (5). The $\varepsilon_{i+1}=1$ because $|K^i|\neq n-k$, $|K^{i+1}|=n-k$ and in this way

$$|N^i|+|A_1^i| \geqq k-1$$

$$|K^i|+|A_2^i| = n-k-1$$

that is

$$|N^i|+|A_1^i|+|K^i|+|A_2^i| \geqq n-2.$$

On the other hand

$$|N^i|+|A_1^i|+|K^i|+|A_2^i| \leqq n-2$$

holds, therefore

$$|N^i|+|A_1^i|+|K^i|+|A_2^i| = n-2.$$

We prove that the graph G^{i+1} is connected.

From the definition of $P(A)$ it follows that there are directed paths in $\vec{G}^i$ from elements of N^i to x and there are directed paths in $\vec{G}^i$ from element x to elements of K^i and simultaneously there are directed paths in $\vec{G}^i$ from elements of A_1^i to b and there are directed paths from b to elements of A_2^i and $x>b\in\mathscr{E}_{i+1}$. That is G^{i+1} is connected and has at least $n-1$ edges. It follows from this that in $\mathscr{E}_{i+1}$ there are at least $n-1$ inequalities and so there are at least $n-1$ inequalities in $\mathscr{E}_l$, too. With this we have proved, that in Case I

$$L(\mathcal{S}(A)) \geqq n-1$$

holds where $\mathcal{S}(A)$ is arbitrary.

CASE II. *We proved that* $|K^l| \leqq n-r_t$. Suppose that $|K^l| > n-r_t$ in the contrary to our assumption.

Let $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i)$ be the situation for which $|K^i| \leqq n-r_t$, $|K^{i+1}| > n-r_t$ hold. Let

$$S_i(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i) = (a, b).$$

It follows from the definition of $P(A)$ that $a=x$ or $b=x$ holds. Suppose that $a=x$ and — as it is easy to see — $b \in A^i$. But we use (5) and $\varepsilon_{i+1}=0$, $|K^{i+1}| \leqq n-r_t$. From this contradiction it follows that $|K^l| \leqq n-r_t$.

We prove that $|N^l| = r_t - 1$. Suppose that $|N^l| > r_t - 1$.

Let $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i)$ be the situation for which $|N^i| \leqq r_t - 1$, $|N^{i+1}| > r_t - 1$ hold and let

$$S_i(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i) = (a, b).$$

We can easily see that $a=x$ or $b=x$ and if $a=x$ then $b \in A^i$. We use (5) and $\varepsilon_{i+1}=0$ (since $|N^{i+1}| > r_t - 1$) so

$$|K^i| + |A_2^i| > n - r_t - 1$$

that is

$$|K^i| + |A_2^i| \geqq n - r_t.$$

According to our assumption $|N^{i+1}| > r_t - 1$ that is $|N^i| + |A_1^i| + 1 > r_t - 1$ and $|N^i| + |A_1^i| + 1 \geqq r_t$. From the inequalities $|N^i| + |A_1^i| + 1 > r_t$, $|K^i| + |A_2^i| \geqq n - r_t$ it follows that

$$|N^i| + |A_1^i| + |K^i| + |A_2^i| + 1 \geqq n$$

and this is a contradiction, that is $|N^l| > r_t - 1$ is impossible.

We prove that $|N^l| < r_t$ is impossible, too.

Suppose that $|N^l| < r_t - 1$. Because $|K^l| \leqq n - r_t$ for this reason $|N^l| + |K^l| < n-1$ and so $A^l \neq \emptyset$. If $n_1(n_2)$ elements of $A^l(n_1+n_2=|A^l|)$ are smaller (larger) as x and $|N^l| + n_2 = r_t - 2$ then the index of $x \notin A$ and $|N^l| + n_2 = r_t - 1$ then the index of $x \in A$. It follows from this that the $\mathcal{S}(A)$ is unfinished. The contradiction proves our assertion and so $|N^l| = r_t - 1$.

Let $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i)$ be the situation when $|N^i| \neq r_t - 1$, $|N^{i+1}| = r_t - 1$. Suppose that

$$S_i(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i) = (a, b).$$

It is easy to see that $a=x$ or $b=x$ and if $a=x$ then $b \in A^i$ and we use (5). Because $|N^i| \neq r_t - 1$, $|N^{i+1}| = r_t - 1$ for this reason $\varepsilon_{i+1} = 0$ and $|K^i| + |A_2^i| > n - r_t - 1$ that is $|K^i| + |A_2^i| \geqq n - r_t$. Since

$$|N^i| + |A_1^i| + 1 = r_t - 1$$

for this reason

$$|K^i|+|A_2^i|+|N^i|+|A_1^i|+1 \geqq n-1.$$

On the other hand

$$N^i \cup A_2^i \cup K^i \cup A_2^i \subseteq H-\{x, b\}$$

therefore

$$|K^i|+|A_2^i|+|N^i|+|A_1^i| = n-2.$$

We can easily see that the graph G^l is connected and from this

$$L(\mathcal{S}(A)) \geqq n-1$$

follows.

The proof of Theorem 1 is completed.

Proof of Theorem 2. It is easy to find a strategy which satisfies (7) (strategy of Katona). We compare x and y. Suppose that $x>y$. After this we compare the elements $H-\{x, y\}$ with y. Suppose that $y<a_1, y<a_2, \dots, y<a_i$ and $y>b_1$, $y>b_2, \dots, y>b_j$ $(i+j=n-2)$. If $j=n-q$ then the y is q-st element in H, otherwise not. If $j=n-q$ then we compare element x with elements $a_1, \dots, a_i$. If x is smaller in $p-1$ cases, the x is p-st and y q-st element in H otherwise not. The number of comparisons is $n+q-3$.

This proves

$$\min_{\mathcal{S}(p,q)} L(\mathcal{S}(p, q)) \leqq n+q-3.$$

It remains to prove

(9) $$L(\mathcal{S}(p, q)) \geqq n+q-3$$

for any strategy. This will be done in the following way.

An algorithm will be given which determines a branch of the strategy that is a sequence $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_l$ finishing it. This branch will have a length $\geqq n+q-3$. The algorithm determines the ε's recursively.

Partitions of $H-\{x, y\}$ will be used. The partitions will also be defined recursively for any situation $(\varepsilon_1, \dots, \varepsilon_i)$ along the indicated branch. The branch and the partitions will be defined simultaneously. We suppose that $(p, q)\neq(1, 2)$ because if $(p, q)=(1, 2)$ then — we can easy see — (9) hold. A partition has 7 classes: N_{11}^i, N_{12}^i, N_2^i, K_{21}^i, K_{22}^i, K_1^i, A^i. The heuristic meaning of the classes is:

$N_{11}^i \cup K_{22}^i$: the set of elements which will be greater than both x and y;

$K_{21}^i \cup N_{12}^i$: the set of elements which will be greater than the elements y and smaller than element x;

$K_1^i \cup N_2^i$: the set of elements which will be smaller than both x and y.

At the beginning $A^0=H-\{x, y\}$, $N_{11}^0, N_{12}^0, N_2^0, K_{21}^0, K_{22}^0, K_1^0=\emptyset$.

Let $S_0=(a, b)$. If x, y is not in S_0 and — say — $a>b$ then $a=a^*$, $b\in K_1^1$ and in case $q=p+1$ $a^*\in N_{11}^1$ and in case $q\neq p+1$ $a^*\in N_{12}^1$. Suppose that some element b first occurs in the $S_i(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_i)$.

If b is larger and $|N_{11}^i|<p-1$ then $b\in N_{11}^{i+1}$, if $|N_{11}^i|=p-1$ and $|N_{12}^i|<q-p-1$ then $b\in N_{12}^{i+1}$, if $|N_{11}^i|=p-1$, $|N_{12}^i|=q-p-1$ then $b\in N_2^{i+1}$.

If b is smaller and $|K_1^i|<n-q$ then $b\in K_1^{i+1}$.

If there is not a^* and $|K_1^i|=n-q$, $|K_{21}^{i+1}|<q-p-1$ then $b\in K_{21}^{i+1}$ and if there is not a^* and $|K_{21}^{i+1}|=q-p-1$ then $b\in K_{22}^{i+1}$.

If there is an a^* and $|K_1^i|=n-q$ then if $a^*\in N_1^i$, then $b\in K_{22}^{i+1}$ (in this case $q=p+1$).

If $|K_1^i|=n-q-1$, $a^*\in N_{12}^i$ then $a^*\in K_{21}^{i+1}$, $b\in K_1^{i+1}$.

If $|K_1^i|=n-q$, $a^*\in K_{21}^i$ and if then $b\in K_{21}^{i+1}$ and if $|K_{21}^i|=q-p-1$ then $b\in K_{22}^{i+1}$

Let

$$N^{i+1} = N_{11}^{i+1}\cup N_{12}^{i+1}\cup N_2^{i+1}$$

$$K^{i+1} = K_1^{i+1}\cup K_{21}^{i+1}\cup K_{22}^{i+1}$$

From the definition of the sets it follows:

If — say — $b\neq a^*$ and $b\in N_{11}^i(N_{12}^i, \dots)$ then $b\in N_{11}^r$ $(N_{12}^r, \dots)(r>i)$.

If $N_2^i\neq\emptyset$ then $K_{21}^i\cup K_{22}^i=\emptyset$.

If $K_{22}^i\neq\emptyset$ then $N_{12}^i\cup N_2^i=\emptyset$.

Suppose that $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_i$ and $N_{11}^i, \dots$ are defined. We define ε_{i+1} and by this $N_{11}^{i+1}, \dots$ sets.

Let $S_i(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_i)=(g, h)$.

(1)	$g, h\in A^i$	$\varepsilon_{i+1} = 1$
(2)	$g\in N^i(K^i)$, $h\in A^i$	$\varepsilon_{i+1} = 1(0)$
(3)	$g\in N_{11}^i$, $h\in N_{12}^i\cup N_2^i\cup K^i$	$\varepsilon_{i+1} = 1$
(4)	$g\in N_{12}^i$, $h\in N_2^i\cup K^i$	$\varepsilon_{i+1} = 1$
(5)	$g\in N_2^i$, $h\in K^i$	$\varepsilon_{i+1} = 1$
(6)	$g\in K_{22}^i$, $h\in K_{21}^i\cup K_1^i$	$\varepsilon_{i+1} = 1$
(7)	$g\in K_{21}^i$, $h\in K^i$	$\varepsilon_{i+1} = 1$
(8)	$g, h\in N_{11}^i, N_{12}^i, N_2^i,$ $K_{22}^i, K_{21}^i, K_1^i$ $(g, h \neq a^*)$	$\varepsilon_{i+1} =$ arbitrary, except if it is determined by the extension of $\mathcal{E}_i$
(9)	$g = x$, $h = y$	$\varepsilon_{i+1} = 1$
(10)	$g = x$, $h\in N_{11}^i\cup K_{22}^i$ $(h\in N_{12}^i\cup K_{21}^i\cup K_1^i\cup N_2^i)$	$\varepsilon_{i+1} = 0(1)$
(11)	$g = x$, $h\in A^i$	$\varepsilon_{i+1} = 1$ if $\lvert K_1^i\cup K_{21}^i\rvert<n-p-1$ and $\varepsilon_{i+1} = 0$ otherwise
(12)	$g = y$, $h\in K_1^i\cup N_2^i$ $(h\in K_{21}^i\cup K_{22}^i\cup N_{12}^i\cup N_{11}^i)$	$\varepsilon_{i+1} = 1(0)$
(13)	$g = a^*(a^*\in N_{11}^i)$, $h\notin N_{11}^i$ $(h\in N_{11}^i)$	$\varepsilon_{i+1} = 1(0)$

(14) $g = a^*(a^* \in N_{12}^i \cup K_{21}^i)$ $\qquad \varepsilon_{i+1} = 0(1)$

$h \in N_{12}^i \cup \{x\} \cup N_{11}^i$

$(h \notin N_{12}^i \cup \{x\} \cup N_{11}^i \cup A^i)$

(15) $g = y, \quad h \in A^i$ $\qquad \varepsilon_{i+1} = 0$ if $|N_{11}^i \cup N_{12}^i| < q-2$ and $\varepsilon_{i+1} = 1$ otherwise.

In this way we have defined a branch $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_l$ of the strategy. It will be denoted by $P(p, q)$. The length $|P(p, q)|$ of $P(p, q)$ is l. We shall prove $l \leqq n+q-3$.

It is easy to see that for the nonempty sets there are the following possible cases:

(I) $N_{11}^i, K_{21}^i, K_1^i, K_{22}^i$

(II) $N_{11}^i, K_{21}^i, K_1^i$

(III) $N_{11}^i, N_{12}^i, K_{21}^i, K_1^i$

(IV) $N_{11}^i, N_{12}^i, K_1^i$

(V) $N_{11}^i, N_{12}^i, N_2^i, K_1^i$.

We illustrate these cases in Fig 1.

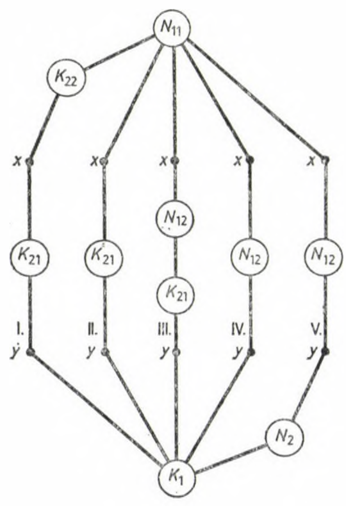

Fig. 1

The following lemma will be stated with reference to Fig. 1.

LEMMA 1. *An arbitrary element $a(\notin A^i)$ cannot act $\mathcal{E}_i$ as smaller (greater) together with an element from a subset being on a lower (upper) level than the subset which we have taken the element a from.*

PROOF. Suppose that — say in the case (III) — $a<b\in\mathscr{E}_i$ and — say — $a\in N_{12}^i$. Suppose that

$$S_j(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_j) = (a, b) \qquad (j < i).$$

From $a\in N_{12}^i$ it follows that $a\in A^j\cup N_{12}^j$ and $a\in N_{12}^{j+1}$. If $a\in A^j$ then — because of $a<b$ — $a\notin N_{12}^{j+1}$ and $a\notin N_{12}^i$. This is a contradiction. From this it follows $a\notin A^j$. Thus $a\in N_{12}^j$. Because $\varepsilon_{j+1}=0$ therefore we use (3), (8), (10) or (14) and from this $b\in N_{11}^j\cup\{x\}\cup N_{12}^j$, $b\in N_{11}^{j+1}\cup\{x\}\cup N_{12}^{j+1}$ and $b\in N_{11}^i\cup\{x\}\cup N_{12}^i$ follow.

Suppose that $a>b\in\mathscr{E}_i$ and $a\in N_{12}^i$. If

$$S_j(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_j) = (a, b) \qquad (j < i)$$

then — obviously — $a\in A^j\cup N_{12}^j$.

If $a\in A^j$ then we use (1), (2) or (11). In Case 1 and 2 $b\in A^j\cup K^j$, $b\in K^{j+1}$, $b\in K^i$. In case (11) $b=x$. Because $a>x$ therefore $|K_j^1\cup K_{21}^j|\geqq n-p-1$ and $a\in N_{11}^{j+1}$ follows. This is a contradiction (we supposed $a\in N_{12}^i$). In case $a\in A^j$ the statement is true.

If $a\in N_{12}^j$, then — because $\varepsilon_{j+1}=1$ — we use (2), (4), (12) or (14) and $b\in A^j\cup \cup N_2^j\cup K_j\cup N_{12}^j\cup\{y\}$. But $N_2^j=K_{22}^j=\emptyset$ thus $b\in A^j\cup K_{21}^j\cup K_1^j\cup N_{12}^j\cup\{y\}$, and $b\in K_{21}^{j+1}\cup K_1^{j+1}\cup N_{12}^{j+1}\cup\{y\}$, $b\in K_{21}^i\cup K_1^i\cup N_{12}^i\cup\{y\}$. This is the proof of the lemma if $a\in N_{12}^i$.

We can prove the statement of lemma if a is taken from some other set of (III) or branches (I), (II), (I), and (V). The lemma is proved.

LEMMA 2. *If the strategy $\mathscr{S}(p, q)$ is finished for the sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l$ then the element x is p-st and element y is q-st in H.*

PROOF. From Lemma 1 and definition $P(p, q)$ it follows that if $a>x$ $(a>y)$ follows from $\mathscr{E}_l$ then a is on a level above $x(y)$. Similarly if $a<x(a<y)$ follows from $\mathscr{E}_l$ then a is on a level under $x(y)$.

From the definition of $P(p, q)$ it follows that

$$|K_{22}^l\cup N_{11}^l| \leqq p-1$$

$$|K_1^l\cup N_2^l| \leqq n-q$$

$$|N_{12}^l\cup K_{21}^l| \leqq q-p-1$$

hold. The statement of the lemma follows from these inequalities. The lemma is proved.

LEMMA 3. *If the strategy $\mathscr{S}(p, q)$ is finished for the sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l$ then $A^l=\emptyset$.*

PROOF. From Lemma 3 it follows that element x is pst and element y is qst in H. Thus we can decide for all elements of $H-\{x, y\}$ whether they are smaller or larger than $x(y)$. From this it follows that all elements of $H-\{x, y\}$ occur in $\mathscr{E}_l$ and thus from definition of A^l it follows that $A^l=\emptyset$. The lemma is proved.

LEMMA 4. *In Cases* (IV), (V) *if $a\in N_{11}^l\cup N_{12}^l$ then there exists $b\in\{y\}\cup K_1^l$ such that $b<a\in\mathscr{E}_l$. In Cases* (I), (II), (III) *if $a\in K_1^l$ then there exists $b\in\{x\}\cup N_{11}^l\cup N_{12}^l$ such that $a<b\in\mathscr{E}_l$.*

PROOF. We prove the first half of the statement. In Cases (IV), (V) $K_{22}^l=K_{21}^l=\emptyset$ hold. Let the element a be an arbitrary element of $N_{11}^l\cup N_{12}^l$. Consider that pair in which element a occurs first time. Let $S_i(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_i)=(a, b)$. Because $a\in N_{11}^l\cup N_{12}^l$ and $a\in A^i$ this is why $\varepsilon_{i+1}=1$ — that is $a>b$ — and we use (1), (2), or (15) and $b\in K_1^{i+1}\cup\{y\}$, $b\in K_1^l\cup\{y\}$ hold. This proves the first half of the lemma.

We prove the other half of the lemma. In Cases (I), (II), (III) $N_2^l=\emptyset$ and $K_{21}^l\neq\emptyset$ hold. Let a be an arbitrary element of K_1^l and suppose that $a\notin K_1^i$ but $a\in K_1^{i+1}$. From this it follows that $a\in A^i$ and $S_i(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_i)=(a, b)$, $\varepsilon_{i+1}=0$ that is we use (1), (2) or (11) and $b\in\{x\}\cup N_{11}^{i+1}\cup N_{12}^{i+1}$, $b\in\{x\}\cup N_{11}^l\cup N_{12}^l$ $(N_2^l=\emptyset)$. This completes the proof of the lemma.

PROOF OF THEOREM 2. We distinguish two cases.

Cases (IV) or (V) hold. In the cases $K_{21}^l\cup K_{22}^l=\emptyset$. From Lemma 4 it follows that if $a\in N_{11}^l\cup N_{12}^l$ then there exists an inequality $a>b$ in $\mathscr{E}_l$ and $b\in K_1^l\cup\{y\}$. Since $|N_{11}^l\cup N_{12}^l|=q-2$ therefore the number of $a>b$ inequalities in $\mathscr{E}_l$ in which $a\in N_{11}^l\cup N_{12}^l$, $b\in K_1^l\cup\{y\}$ is at least $q-2$. We can easily see that the subgraph induced $K_1^l\cup\{y\}(N_{11}^l\cup N_{12}^l\cup\{x\})$ is connected. Consequently there are at least $n-q(q-2)$ edges among the vertices in $K_1^l\cup\{y\}(N_{11}^l\cup N_{12}^l\cup\{x\})$. That is the number of inequalities $a<b$ in $\mathscr{E}_l$ where $a, b\in K_1^l\cup\{y\}(a, b\in N_{11}^l\cup N_{12}^l\cup\{x\})$ is at least $n-q(q-2)$. Summing up our results:

$$l \geqq q-2+n-q+q-2 = n+q-4.$$

If there is

$$x > y\in\mathscr{L}_l \quad \text{or} \quad x > c\in\mathscr{L}_l \quad \text{and} \quad c\in K_2^l$$

then

$$l \geqq n+q-3$$

that is (9) hold.

Suppose that $x>y\notin\mathscr{E}_l$ and there is no $x>c$ in $\mathscr{E}_l$ where $c\in K_1^l$. Consider S_0.

Let $S_0=(a, b)$. From our condition it follows that a, $b\in A^0$. If — say — $a>b$ then $a=a^*$ and $b\in K_1^1$, $b\in K_1^l$ (we have supposed that $(p, q)=(1, 2)$, and y does not occur as x). We can easily verify that $a^*>y\in\mathscr{E}_l$. Indeed from definition of $P(p, q)$ it follows that if we compared elements a^*, e and $e\in N_{11}^l\cup N_{12}^l$ then $a^*<e$. Since $a^*>y$ follows from $\mathscr{E}_l$ therefore $a^*>y\in\mathscr{E}_l$. Thus the element a^* occurs with the y and b where $b\in K_1^l$ in $\mathscr{E}_l$. From this it follows that elements of $N_{11}^l\cup N_{12}^l$ occur with elements of $K_1^l\cup\{y\}$ at least in $q-1$ inequalities in $\mathscr{E}_l$. Since the subgraph induced $K_1^l\cup\{y\}(N_{11}^l\cup N_{12}^l\cup\{x\})$ is connected the number of inequalities in $\mathscr{E}_l$ is at least

$$q-1+n-q+q-2 = n+q-3.$$

That is (9) in case (IV), (V) holds.

Consider the Cases (I), (II), (III).

In these cases on the basis of Lemma 4 all elements of K_1^l occur with elements of $\{x\}\cup N_{11}^l\cup N_{12}^l$ in $\mathscr{E}_l$ and $K_{21}^l\neq\emptyset$, $N_2^l=\emptyset$ hold. From this it follows that the number of inequalities in $\mathscr{E}_l$ in which elements of K_1^l occur with elements of $\{x\}\cup N_{11}^l\cup N_{12}^l$ is $n-q$. We can easily see that the subgraph induced $\{x\}\cup N_{11}^l\cup N_{12}^l\cup K_{22}^l\cup K_{21}^l$ $(\{y\}\cup K_1^l)$ is connected. From this it follows that the number of inequalities $a>b$ in $\mathscr{E}_l$ where $a, b\in\{x\}\cup N_{11}^l\cup K_{22}^l\cup K_{21}^l\cup N_{12}^l$ $(\{y\}\cup K_1^l)$ is at least $q-2$ $(n-q)$. Thus the number of inequalities in $\mathscr{E}_l$ is at least

$$n-q+q-2+n-q = 2n-q-2.$$

On the other hand $q \leqq \left[\frac{n}{2}\right]$ and from this it follows that

$$l \geqq 2n-q-2 \geqq 2n-\left[\frac{n}{2}\right]-2 = n+\left\lfloor\frac{n}{2}\right\rfloor-2 \geqq$$

$$\geqq n+\left[\frac{n}{2}\right]-3 \geqq n+q-3.$$

From this it follows that (9) holds in cases (I), (II), (III), too. The proof of Theorem 2 is now complete.

REMARK. If the elements p, q are arbitrary then the problem is open. Proved ([9]) in case $p=1$, $q=n$ that

$$L(\mathscr{S}(p,q)) \geqq n+\left[\frac{n-1}{2}\right]-2.$$

ACKNOWLEDGEMENT. The author would like to thank professor G. O. H. Katona for his valuable remarks.

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(Received April 25, 1983)

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STRICTLY π-REGULAR NEAR-RINGS

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Abstract

In this paper we have obtained a generalized form of a regular near-ring named as strictly π-regular near-ring. It has been proved that every semisimple near-ring in the sense of Blackett is strictly π-regular. Some fruitful results on regular near-rings have been generalized and chain conditions, structure theorems and radical properties have been discussed.

Introduction

In this paper we have dealt with π-regular near-rings in which every non-zero R-subgroup contains at least one non-zero element which is associated with non-zero natural idempotents. Such near-rings are named as strictly π-regular near-rings. Clearly, every regular near-ring is strictly π-regular but not conversely. In Section 1, conditions have been obtained under which a strictly π-regular near-ring becomes regular. The result of Ligh ([11], Th. 4.4) has been generalized. In Section 2, we have dealt with chain conditions and structure theorems. Section 3 deals with the radical properties of this near-ring. Almost every radical of a strictly π-regular near ring behaves in a similar manner as in the case of a regular near-ring.

Preliminaries

Throughout R will denote a zero-symmetric left near-ring.

A near-ring R is called regular if for each $a\in R$, there exists $x\in R$ such that $a=axa$. R is called π-regular (semi π-regular) if for each $a\in R$, there exists $x\in R$ and a positive integer n such that $a^n=a^nxa^n$ $(a^n=a^nxa)$ [7]. If R is a semi π-regular near ring with no non-zero nilpotent elements, then R is regular [7].

We denote by $\operatorname{Ann}(a)=\{x\in R|ax=0\}$, the right annihilator of an element $a\in R$. It is easy to see that $\operatorname{Ann}(a)$ is a right ideal of R.

A near-ring R is called right duo if every right ideal of R is also a left ideal ([15], p. 278).

A near-ring R is called an S-near ring if $a\in aR$ for each $a\in R$. Every regular near-ring is an S-near ring.

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[1] Most of the part of this paper are included in author's doctoral dissertation at Sukhadia University, Udaipur. The author thanks Dr. S. C. Choudhary for his guidance.

AMS (MOS) subject classification (1970). Primary 16A56; Secondary 16A30.

Key words and phrases. Strictly π-regular, π-regular, semi π-regular, radical subgroup, quasi radical, the radical JCR, G-radical.

A near-ring R is called strongly semisimple (semisimple in the sense of Blackett [4]), if it contains no non-zero nilpotent R-subgroups and satisfies d.c.c. on R-subgroups of R. A strongly semisimple near-ring satisfies a.c.c. on R-subgroups of R [14]. A near-ring R is called simple if it contains no proper two-sided ideals.

Let R contain an identity 1. An element $a \in R$ is said to be right quasi-regular (r.q.r) if a is in the right ideal generated by the set $\{r - ar | r \in R\}$ [16]. The element a is called quasi-regular (q.r.) if there exists $x \in R$ such that $(1-a)x = 1$ [3]. Clearly a q.r. element is r.q.r. If a is a non-zero idempotent of R, then a cannot be r.q.r and hence cannot be q.r. [16].

The radical subgroup $A(R)$ of a near-ring R with identity is the intersection of all maximal R-subgroups of R. $A(R)$ is a quasi-regular R-subgroup that contains every q.r. right ideals of R [3].

Corresponding to the Jacobson radical in rings, the radical $J_0(R)$, $J_{1/2}(R)$ $(= D(R))$, $J_1(R)$ and $J_2(R)$ have been obtained in near-rings by Betsch (see [15], p. 136). If R contains an identity, then $J_1(R) = J_2(R)$ and it is denoted by $J(R)$. In general we have, $J_0(R) \subseteq D(R) \subseteq J_1(R) \subseteq J_2(R)$. The upper nil radical $\mathfrak{N}(R)$ is the sum of all the nil ideals of R while $G(R)$, the G-radical is the intersection of all modular maximal ideals of R ([15], p. 160, 164). Also $D(R) \subseteq G(R)$.

§ 1. Definitions, properties and characterizations

Definition 1.1. Let R be a π-regular near-ring and let a be a non-zero element in R such that $a^n = a^n x a^n$ for some $x \in R$ and some n. If the natural idempotents $a^n x$ and $x a^n$ for the element a are non-zero, then a is called a strictly π-regular element.

Clearly, a non-zero idempotent in a π-regular near-ring is a strictly π-regular element. We are now in a position to define a special type of π-regular near-ring which is a generalization of a regular near-ring.

Definition 1.2. A π-regular near-ring R in which every non-zero R-subgroup contains at least one strictly π-regular element is called a strictly π-regular near-ring.

Clearly a strictly π-regular near-ring is π-regular but not conversely (see [7], Ex. 1.3). Also a regular near-ring is strictly π-regular but not conversely as can be seen from the following examples:

Example 1.3 (Clay [6]). Let $R = \{0, 1, 2, 3, 4\}$ with addition modulo 5 and multiplication defined as follows:

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	4	3	2	1
3	0	1	2	3	4
4	0	0	0	0	0

Then R is a π-regular near-ring. The only non-zero R-subgroup of R is R itself in which 3 is a strictly π-regular element. Hence R is a strictly π-regular near-ring. But R is not a regular near-ring.

EXAMPLE 1.4 ([10]). Let $(R, +)$ be a group which contains an element y such that $y+y\neq 0$. Define multiplication on R as follows. For all $a\in R$, define $0.a=0$, $y.a=0$ and $x.a=a$ for all $x\in R$, $x\neq y$, $x\neq 0$. Then $(R, +, \cdot)$ is a near-ring. The only R-subgroups or right ideals of R are (0) and R. So R is the only non-zero R-subgroup of itself which clearly contains idempotent elements other than y. These idempotent elements are clearly strictly π-regular elements and thus R is a strictly π-regular near-ring. But R is not a regular near-ring.

LEMMA 1.5 ([10], Th. 4.3). *Let R be a strongly semisimple near-ring and let M be a non-zero R-subgroup of R. Then M contains an idempotent e such that $eR=M$.*

THEOREM 1.6. *Every strongly semisimple near-ring is strictly π-regular.*

PROOF. Let R be a strongly semisimple near-ring. Then R satisfies d.c.c. and hence a.c.c. on R-subgroups [14]. Thus R is π-regular by ([7], cor. 1.15& Th. 1.16). By Lemma 1.5, every non-zero R-subgroup of R contains an idempotent which is clearly a strictly π-regular element. Hence R is strictly π-regular.

THEOREM 1.7. *If R is a near-ring with no non-zero nilpotent elements, then the following are equivalent:*
(i) *R is regular;*
(ii) *R is strictly π-regular;*
(iii) *R is π-regular;*
(iv) *R is semi π-regular.*

PROOF. Clearly (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv) while (iv)$\Rightarrow$(i) follows from ([7]), Th. 1.8).

This theorem with the result of Heatherly ([9], cor. 3.8) immediately gives us the following

COROLLARY 1.8. *If R is a strictly π-regular near-ring with a right distributive element and without proper divisors of zero, then R is a near-field.*

We now consider the distributive case.

LEMMA 1.9 (Ligh [13]). *If R is a distributive near-ring, then R', the commutator subgroup of $(R, +)$, is an ideal of R and $RR'=R'R=(0)$.*

THEOREM 1.10. *A distributive strictly π-regular near-ring is a ring.*

PROOF. Let R' be the commutator subgroup of $(R, +)$ and let $R'\neq(0)$. Then R', being an ideal of R, contains a strictly π-regular element and hence contains a non-zero idempotent. This is contrary to $R'R'=(0)$ and so $R'=(0)$. This implies that $(R, +)$ is commutative and thus R becomes a ring.

We now generalize Ligh's result ([11], Th. 4.4) for strictly π-regular near-rings.

LEMMA 1.11. *If R' is a homomorphic image of a π-regular near-ring R whose idempotents are central, then the idempotents of R' are central.*

PROOF. Let $f\colon R\to R'$ be the homomorphism, then R' is π-regular. Let e' be any non-zero idempotent in R' and x' an arbitrary element in R'. Then there exists $a\in R$ and $y\in R$ such that $af=e'$ and $yf=x'$. Now $a\in R$ gives $a^n=a^nza^n$ for some $z\in R$

and some n. Here $a^n z$ is a non-zero idempotent in R (for $a^n z=0 \Rightarrow a^n=0 \Rightarrow a^n f=0f=0 \Rightarrow e'=0 \in R'$, a contradiction). Since idempotents in R are central, we have $a^n = a^{2n}z$. Again $a^n f=a^{2n}f=e'$ and $(a^n z)f=(a^n f)(zf)=(a^{2n}f)(zf)=(a^{2n}z)f=a^n f=e'$. Now $e'x'=((a^n z)f)(yf)=((a^n z)y)f=(y(a^n z))f=(yf)(a^n z)f=x'e'$. Thus idempotents in R' are central in R'.

PROPOSITION 1.12. *Let R be a subdirectly irreducible strictly π-regular near-ring whose idempotents are central. Then the following are true:*

(a) *R has an identity element;*

(b) *the idempotents of R are either the zero or the identity;*

(c) *R is a near-field.*

PROOF. (a) and (b) .Let e be a non-zero idempotent of R. Then $\operatorname{Ann}(e)$, the right annihilator of e, will be an ideal of R since e is central. If $\operatorname{Ann}(e)=(0)$, then e is an identity for $e(er-r)=0$ where r is an arbitrary element of R, and so $er-r \in \operatorname{Ann}(e)=(0)$, i.e. $er=r=re$. Now consider all those idempotents $e \neq 0$ of R for which $\operatorname{Ann}(e) \neq (0)$. Then $A=\bigcap \operatorname{Ann}(e) \neq (0)$ as R is subdirectly irreducible. Since A is a non-zero ideal of R and R is strictly π-regular, there exists a non-zero element x in A, which is a strictly π-regular element. Therefore $x^n=x^n z x^n$ for some $z \in R$ and some n, where $x^n z$ and zx^n are non-zero idempotents. Now $x \in A$ implies that $ex=0$ for all non-zero idempotents e for which $\operatorname{Ann}(e) \neq (0)$. Let $e'=x^n z$. If $\operatorname{Ann}(e')=(0)$, then e' is an identity as shown above. Then $e=ee'=e(x^n z)=0$ as $ex=0$. This is a contradiction. Hence $\operatorname{Ann}(e') \neq (0)$. Therefore $e'x=0$ as shown above and so $x^n=e'x^n=0$, again a contradiction (for $x^n=0 \Rightarrow e'=x^n z=0$ which is not true). We thus conclude that $\operatorname{Ann}(e)=(0)$ for each non-zero idempotent e of R. Therefore the idempotents of R are either the zero or the identity. Now as R is strictly π-regular, there exists a non-zero idempotent in R which is an identity. Thus R has an identity.

(c) Let x be any non-zero element of R. Then xR is a non-zero R-subgroup of R as R has an identity. Since R is strictly π-regular, there exists a strictly π-regular element and hence a non-zero idempotent in xR. This shows by (a), that xR contains the identity of R and therefore $xR=R$. By Ligh ([12], Th. 2.3), R is a near-field.

THEOREM 1.13. *A strictly π-regular near-ring R is isomorphic to a subdirect sum of near-fields iff every idempotent in R is central in R.*

PROOF. Necessity is quite clear. To prove the sufficiency, let R be a strictly π-regular near-ring whose idempotents are central. Now R is isomorphic to a subdirect sum of subdirectly irreducible near-rings R_i ([15], p. 26). Each R_i, being a homomorphic image of R, has all idempotents central in R_i by Lemma 1.11. Thus by Proposition 1.12, each R_i is a near-field. Hence the theorem.

§ 2. Chain conditions and structure theorems

LEMMA 2.1. *A strictly π-regular near-ring contains no non-zero nilpotent R-subgroups.*

PROOF. Let A be a non-zero nilpotent R-subgroup of a strictly π-regular near-ring R. Then $A^n=(0)$ for some n. Also there exists a strictly π-regular element say a

in A. Then $a^m = a^m x a^m$ for some $x \in R$ and some m where $a^m x$ and xa^m are non-zero idempotents. Now $a^m x \in A$ and so $a^m x = (a^m x)^n \in A^n = (0)$, a contradiction. Thus R contains no non-zero nilpotent R-subgroups.

Thus we immediately have the following

THEOREM 2.2. *A strictly π-regular near-ring R that satisfies d.c.c. on R-subgroups is strongly semisimple.*

COROLLARY 2.3. *A strictly π-regular near-ring R that satisfies d.c.c. on R-subgroups, satisfies a.c.c. on R-subgroups.*

COROLLARY 2.4. *If R is a strictly π-regular S-near-ring that satisfies d.c.c. on R-subgroups, then R is regular.*

PROOF. By Theorem 2.2, R is strongly semisimple and so $J_2(R) = (0)$ (see Lemma 3.5). The result now follows from ([10], Th. 4.5), which states that a near-ring R that satisfies d.c.c. on R-subgroups of R is regular iff $J_2(R) = (0)$ and $a \in R$, for all $a \in R$.

It is to be noted that in Corollary 2.4, the condition for R to be an S-near-ring is essential, otherwise R may not be regular. This can be seen from Ex. 1.3 and Ex. 1.4.

The following Lemma is due to Blackett [4].

LEMMA 2.5 ([4]). *Let R be a strongly semisimple near-ring. Then R is a finite direct sum of ideals $R_i (1 \leqq i \leqq k)$ where each R_i is a simple near-ring with d.c.c. on R_i-subgroups.*

We thus have the following

THEOREM 2.6. *Let R be a strictly π-regular near-ring with d.c.c. on R-subgroups of R. Then R is the direct sum of ideals $R = R_1 \oplus R_2 \oplus \ldots \oplus R_n$, where each $R_i (1 \leqq i \leqq n)$ is a simple and strictly π-regular near-ring with d.c.c. on R_i-subgroups.*

PROPOSITION 2.7. *Let R be a simple and right duo near-ring which is such that $aR \neq (0)$ for each $a\,(\neq 0) \in R$. Then R is without proper divisors of zero.*

PROOF. Let $a (\neq 0) \in R$ and $ab = 0$, $b \in R$. Then $b \in \operatorname{Ann}(a)$, where $\operatorname{Ann}(a)$ is an ideal of R as R is right duo. Since R is simple, either $\operatorname{Ann}(a) = (0)$ or $\operatorname{Ann}(a) = R$. But $\operatorname{Ann}(a) = R$ is ruled out for then $aR = (0)$ which is not true by the given condition. Hence $\operatorname{Ann}(a) = (0)$ and so $b = 0$. Thus R has no proper divisors of zero.

COROLLARY 2.8. *If R is simple, right duo and S-near-ring. Then R is without proper divisors of zero.*

LEMMA 2.9. (Heatherly [8]). *If R is a simple near-ring with no non-zero nilpotent elements and satisfies d.c.c. on R-subgroups, then* (a) *every non-zero idempotent of R is a left identity and R has at least one such idempotent* (b) *R is regular* (c) *if R has a non-zero right distributive element, then R is a near-field and* (d) *if R has a non-zero right distributive element and is d.g., then R is a division ring.*

THEOREM 2.10. *Let R be a strictly π-regular near-ring such that* (i) *R is right duo* (ii) *R satisfies d.c.c. on R-subgroups and* (iii) *$a'R' \neq (0)$ for each $a'(\neq 0) \in R'$ where R' is a homomorphic image of R. Then the following are true:*

(a) *R is the direct sum of regular near-rings with left identities.*
(b) *If any simple homomorphic image of R has a nonzero right distributive element, then R is a finite direct sum of near-fields.*
(c) *If R is d.g., and if any simple homomorphic image of R has a non-zero right distributive element, then R is a finite direct sum of division rings.*

PROOF. By Theorem 2.6, $R=R_1\oplus R_2\oplus\ldots\ldots\oplus R_n$, where each $R_i (1\leqq i\leqq n)$ is a simple near-ring with d.c.c. on R_i-subgroups. Also each R_i, being a homomorphic image of R, is right duo. Now consider one such direct summand say R_1. Then by condition (iii) of the theorem, $aR_1\neq(0)$ for each $a(\neq 0)\in R_1$. Thus by Proposition 2.7, R_1 is without proper divisors of zero and hence with no non-zero nilpotent elements. Thus each R_i satisfies all the four properties of the Lemma 2.9. Hence the theorem.

NOTE. It is to be noted that the condition (iii) of Theorem 2.10 always hold if R is an S-near-ring.

THEOREM 2.11. *If R is a strictly π-regular near-ring with d.c.c. on R-subgroups and every idempotent of R is central, then R is a finite direct sum of near-fields.*

PROOF. By Theorem 2.6, $R=R_1\oplus R_2\oplus\ldots\ldots\oplus R_n$, where each R_i is a simple and strictly π-regular near-ring with d.c.c. on R_i-subgroups. Also each R_i has central idempotents by Lemma 1.11. Consider one such direct summand, say R_1. By Theorem 1.13, R_1 is isomorphic to a subdirect sum of near-fields and since R_1 is simple, R_1 becomes a near-field. Hence the theorem.

§ 3. Radicals

We now discuss some radical properties of a strictly π-regular near-ring.

THEOREM 3.1. *If R is a strictly π-regular near-ring, then $D(R)$ is zero.*

PROOF. Suppose that $D(R)\neq(0)$. Then since R is strictly π-regular, there exists $a(\neq 0)\in D(R)$ where a is a strictly π-regular element. Then $a^n=a^nxa^n$ for some $x\in R$ and some n, where a^nx is a non-zero idempontent that belongs to $D(R)$. Therefore a^nx is a r.q.r. element which is a contradiction to the fact that no non-zero idempotent can be r.q.r. [16]. Hence $D(R)=(0)$.

COROLLARY 3.2. *If R is a strictly π-regular near-ring then*

$$\mathfrak{N}(R) = J_0(R) = D(R) = (0).$$

COROLLARY 3.3. *If R is a strictly π-regular and right duo near-ring, then the G-radical $G(R)=(0)$.*

PROOF. Since R is right duo, $D(R)=G(R)$.

THEOREM 3.4. *Let R be a strictly π-regular near-ring with identity. Then $A(R)$, the radical subgroup of R is zero.*

PROOF. Let $A(R)\neq(0)$. Then since R is strictly π-regular there exists $a(\neq 0)\in$ $\in A(R)$ which is a strictly π-regular element. Then $a^n=a^nxa^n$ for some $x\in R$ and some n, where a^nx and xa^n are non-zero idempotents. Now $a^nx\in A(R)$ and so a^nx is a q.r. element and hence a r.q.r. element which is a contradiction by [16]. Hence $A(R)=(0)$.

The following Lemma is due to Betsch ([1], Th. 4).

LEMMA 3.5. *Let R be a near-ring with d.c.c. on R-subgroups of R. Then R has no non-zero nilpotent R-subgroups iff $J_2(R)=(0)$.*

THEOREM 3.6. *Let R be a near-ring that satisfies d.c.c. on R-subgroups of R. Then R is strictly π-regular iff $J_2(R)=(0)$.*

PROOF. Let R be a strictly π-regular near-ring. Then R contains no non-zero nilpotent R-subgroups by Lemma 2.1 and since R satisfies the d.c.c. on R-subgroups of R, we have $J_2(R)=(0)$ by Lemma 3.5. Conversely if $J_2(R)=(0)$, then R contains no non-zero nilpotent R-subgroups by Lemma 3.5 and since R satisfies the d.c.c. on R-subgroups, R becomes strongly semisimple. Thus R is strictly π-regular by Theorem 1.6.

COROLLARY 3.7. *Let R be a strictly π-regular near-ring with d.c.c. on R-subgroups. Then R is a finite direct sum of right ideals which are R-modules of type 2.*

PROOF. By Theorem 3.6, $J_2(R)=(0)$ and the result thus follows by Betsch ([1,] Th. 3.4).

ACKNOWLEDGEMENT. The author is highly grateful to the referee for his many helpful suggestions.

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(Received September 16, 1983)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 61—71

SUR UNE FAMILLE DE NOMBRES HAUTEMENT COMPOSES SUPERIEURS

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1. Introduction

Soit $d(n)=\sum_{d|n} 1$, la fonction nombre de diviseurs. On sait ([HAR], p. 262) que l'ordre maximum de $\log d(n)/\log 2$ est $\log n/\log\log n$ et nous avons donné dans [ROB 1], [NIC 2], des majorations explicites de $\log d(n)$.

Dans cette étude la famille des nombres $\tilde{N}$ est intervenue de façon naturelle:

$$N \text{ est } \tilde{N} \Leftrightarrow \exists\alpha > 1 \text{ tel que } g_\alpha(n) = \frac{\log d(n)}{\log 2} - \alpha\frac{\log n}{\log\log n}$$

soit maximum en N.

Ces nombres constituent une famille infinie qui, excepté pour les trois premiers, sont hautement composés supérieurs, c'est-à-dire maximisent $\dfrac{\log d(n)}{\log 2} - \varepsilon \log n$ pour une certaine valuer de $\varepsilon > 0$. De façon précise, si N est $\tilde{N}$ pour le paramètre α alors N est hautement composé supérieur pour le paramètre ε tel que :

$$\varepsilon = \alpha\frac{\log\log N - 1}{(\log\log N)^2}. \tag{1}$$

Les nombres hautement composés supérieurs définis par Ramanujan ([RAM]), ont été étudiés par Alaoglu—Erdős ([ALA]), Erdős ([ERD]), Nicolas ([NIC 1]).

Monsieur le Professeur Erdős m'a encouragé à étudier la structure des nombres $\tilde{N}$ et nous présentons ici les principaux résultats obtenus.

Soit $(C_k)_{k\in\mathbf{N}}$ la suite des nombres hautement composés supérieurs, et $f(n) = \dfrac{\log d(n)}{\log 2} \Big/ \dfrac{\log n}{\log\log n}$.

Proposition 1. — $f(C_k) > 1$ *pour* $k \geqq 4$; *la fonction f est non seulement décroissante sur les nombres* $\tilde{N}$ *mais aussi sur les nombres* C_k *pour* $k \geqq 15$.

Proposition 2. — *Soit* $Q(X)$ *le nombre de nombres* $\tilde{N}$ *inférieurs à* X *alors* $Q(X) \geqq (\log X)^{1-\tau}$ *pour* $\tau > 7/12$.

Proposition 3. — *Il existe une infinité de nombres* C_k *qui ne sont pas des nombres* $\tilde{N}$.

1980 *Mathematics Subject Classification.* Primary 10H25.

Key words and phrases. Nombre de diviseurs, comportement asymptotique, nombres hautement composés supérieurs.

La démonstration de la proposition 1 suit celle faite pour l'étude de la fonction $\omega(n)$ ([ROB 2], Th. 10 et Th. 14). Signalons d'ailleurs que les propositions 2 et 3 s'appliquent aussi à $\omega(n)$ en faisant les changements adéquats $\left(C_k = \prod_{i=1}^{k} p_i,\ p_i\right.$ désignant le $i^{\text{ème}}$ nombre premier, $\left. f(n) = \frac{\omega(n) \log\log n}{\log n}\right)$.

La proposition 2 se démontre par l'intermédiaire de 3 lemmes d'analyse et en utilisant le résultat d'Huxley ([HUX])

$$\theta(x+x^\tau) = \theta(x) + x^\tau + o(x^\tau/\log x) \quad \text{pour} \quad \tau > 7/12 \tag{2}$$

où θ est la fonction de Tchebytchef, $\theta(x) = \sum_{p \leq x} \log p$.

Si l'on peut prouver que la formule (2) est vraie pour tout $\tau > 0$ alors les démonstrations de ce papier montreront que :

$$Q(X) > (\log X)^{1-\eta} \quad \text{pour tout} \quad \eta > 0$$

nombre à comparer à $\log X/\log\log X$ nombre de nombres hautement composés supérieurs, inférieurs à X.

Pour la dernière proposition on utilise :

$$\underline{\lim}\, \frac{p_{n+1} - p_n}{\log n} < 1 \tag{3}$$

propriété due à P. Erdős, ainsi qu'un lemme technique de concavité.

2. Démonstration de la première proposition

Lemme 1. — *Soit N un nombre hautement composé supérieur associé à ε, c'est-à-dire tel que $\frac{\log d(n)}{\log 2} - \varepsilon \log n$ soit maximum en N, alors:*

$$\frac{\log d(N)}{\log 2} - \varepsilon \log N \geq \frac{\varepsilon \log N}{\log\log N - 1},$$

pour N assez grand.

Preuve. Il suffit de faire la démonstration pour le plus grand nombre hautement composé supérieur associé à ε. Celui-ci est défini comme suit :

$$\text{Si } x = e^{1/\varepsilon} \quad \text{et} \quad v_i = \log(1+1/i)/\log 2$$

alors

$$\log N = \theta(x) + \sum_{i \geq 2} \theta(x^{v_i})$$

(cette somme étant en réalité finie). On a alors :

$$\log d(N)/\log 2 = \pi(x) + \sum_{i \geq 2} v_i\, \pi(x^{v_i})$$

où $\pi(x) = \sum_{p \leqq x} 1$. Comme

$$\pi(x^{\nu_i}) = \frac{\theta(x^{\nu_i})}{\nu_i \log x} + \int_2^{x^{\nu_i}} \frac{\theta(u)\, du}{u \log^2 u} \qquad \forall i$$

il vient

$$\frac{\log d(N)}{\log 2} - \varepsilon \log N = \sum_{i \geqq 1} \nu_i \int_2^{x^{\nu_i}} \frac{\theta(u)\, du}{u \log^2 u}$$

$$= \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + o\left(\frac{x}{\log^3 x}\right)$$

et

$$\frac{\varepsilon \log N}{\log \log N - 1} = \frac{1}{\log x} \frac{x}{\log x}\left(1 + o\left(\frac{1}{\log x}\right)\right).$$

Par suite :

$$\frac{\log d(N)}{\log 2} - \varepsilon \log N - \frac{\varepsilon \log N}{\log \log N - 1} = \frac{x}{\log^3 x} + o\left(\frac{x}{\log^3 x}\right)$$

expression positive pour x assez grand.

Lemme 2. — *La fonction* $f(n) = \dfrac{\log d(n)}{\log 2} \dfrac{\log \log n}{\log n}$ *est décroissante sur les* C_k *pour* k *assez grand.*

Preuve. Soit ε et b définis par les égalités :

$$\varepsilon b = \frac{\log d(C_{k+1})}{\log 2} - \varepsilon \log C_{k+1} = \frac{\log d(C_k)}{\log 2} - \varepsilon \log C_k$$

alors ε est le paramètre commun à C_k et C_{k+1}.

L'inégalité $f(C_{k+1}) < f(C_k)$ s'écrit:

$$(b + \log C_{k+1}) \frac{\log \log C_{k+1}}{\log C_{k+1}} < (b + \log C_k) \frac{\log \log C_k}{\log C_k}$$

et encore:

$$\log \log C_{k+1} + b \frac{\log \log C_{k+1}}{\log C_{k+1}} < \log \log C_k + b \frac{\log \log C_k}{\log C_k}.$$

La fonction $t \to t + bte^{-t}$ est, pour $b > e^2$, décroissante dans l'intervalle $[2, \alpha]$, α étant définie par $1 + be^{-\alpha}(1 - \alpha) = 0$.

Comme b tend vers l'infini avec k, il reste à prouver que $\log \log C_{k+1} < \alpha$ c'est-à-dire, compte tenu que $t \to 1 + be^{-t}(1 - t)$ est croissante pour $t \geqq 2$ que

$$1 + \frac{b}{\log C_{k+1}} (1 - \log \log C_{k+1}) < 0$$

et cette inégalité est celle démontrée au lemme 1.

Pour terminer la démonstration de la proposition 1, nous allons montrer que dans le lemme 1 on peut remplacer «pour N assez grand», par $N \geqq C_{15}$.

Posons

$$A = \frac{\log d(N)}{\log 2} - \varepsilon \log N; \quad B = \frac{\varepsilon \log N}{\log\log N - 1}.$$

Nous allons montrer que:

$$A > \frac{x}{\log^2 x} + a\frac{x}{\log^3 x} \quad \text{et} \quad B < \frac{x}{\log^2 x} + b\frac{x}{\log^3 x} \quad \text{avec} \quad b < a$$

d'abord pour $x \geqq 22\,016$ $(\Leftrightarrow x^{v_2} \geqq 347)$, par les formules de Rosser et Schoenfeld, puis à l'ordinateur pour $x \leqq 22\,016$.

Étude de A. Pour $y > 347$ nous avons $\theta(y) > y - \dfrac{dy}{\log y}$ avec $d = 5/9$ et l'on sait aussi que $\theta(y) > 3/4y$ ([SCH], p. 359).

On a donc pour $x > 22\,016$:

$$A > \int_2^x \frac{\theta(u)\,du}{u\log^2 u} + v_2 \int_2^{x^{v_2}} \frac{\theta(u)\,du}{u\log^2 u}$$

$$= (1+v_2)\int_2^{x^{v_2}} \frac{\theta(u)\,du}{u\log^2 u} \int_{x^{v_2}}^x \frac{\theta(u)\,du}{u\log^2 u}$$

$$> \frac{3}{4}(1+v_2)\int_2^{x^{v_2}} \frac{u\,du}{u\log^2 u} + \int_{x^{v_2}}^x \frac{du}{\log^2 u} - d\int_{x^{v_2}}^x \frac{du}{\log^3 u}$$

soit

$$A > \int_2^x \frac{du}{\log^2 u} - d\int_{347}^x \frac{du}{\log^3 u} = \frac{x}{\log^2 x} + (2-d)\int_2^x \frac{du}{\log^3 u} - \frac{2}{\log^2 2}$$

$$A \geqq \frac{x}{\log^2 x} + (2-d)\frac{x}{\log^3 x}, \quad 2-d = 1{,}444\ldots.$$

Étude de B. Nous utilisons les majorations de ([SCH], p. 357)

$$\theta(x) < 1{,}001\,093x$$

$$\theta(x) < x + 0{,}020\,14x/\log x$$

$$\pi(x) < 1{,}251\,2x/\log x.$$

$$\log N \leqq \theta(x) + \theta(x^{v_2}) + \frac{1}{\varepsilon(\log 2)^2}\pi(x^{v_3})$$

$$\log N \leqq x + 0{,}020\,2\frac{x}{\log x} + 1{,}001\,1x^{v_2} + \frac{1{,}252\,x^{v_3}}{v_3(\log 2)^2}$$

$$\log N \leqq x\left(1 + \frac{0{,}273}{\log x}\right) \text{ pour } x \geqq 22\,016.$$

Comme la fonction $t \to t/(\log t - 1)$ est croissante pour $t > e^2$ il vient

$$B \leqq \frac{1}{\log x} \frac{x(1+0{,}273/\log x)}{\log x - 1}$$

$$B \leqq \frac{x}{\log^2 x}\left(1+\frac{1{,}44}{\log x}\right) \text{ pour } x \geqq 22\,016.$$

3. Sur la proposition 2

LEMME 3. — *Soit $f(x)$ une fonction tendant vers l'infini avec x telle que $f(x) = o(x/\log x)$ et supposons que :*

$$\theta(x+f(x)) = \theta(x)+f(x)+o(f(x)/\log x) \quad \textit{lorsque} \quad x \to \infty.$$

Soit

$$\alpha(x) = \frac{1}{\log x}\left(\log\theta(x)+1+\frac{1}{\log\theta(x)-1}\right).$$

Alors

$$\alpha(x+f(x)) = \alpha(x)-(1+o(1))f(x)/(x\log^2 x).$$

DÉMONSTRATION. L'hypothèse donne:

$$\begin{aligned}\log\theta(x+f(x)) &= \log\theta(x)+f(x)/\theta(x)+o(f(x)/(x\log x))\\ &= \log\theta(x)+f(x)/x+o(f(x)/(x\log x)).\end{aligned}$$

D'où

$$\frac{1}{\log\theta(x+f(x))-1} = \frac{1}{\log\theta(x)-1}+O\left(\frac{f(x)}{x\log^2 x}\right).$$

On a aussi

$$\frac{1}{\log(x+f(x))} = \frac{1}{\log x}\left(1-\frac{f(x)}{x\log x}+O\left(\frac{f^2(x)}{x^2\log x}\right)\right)$$

et

$$\alpha(x+f(x)) = \frac{1}{\log x}\left(1-\frac{f(x)}{x\log x}+O\left(\frac{f^2(x)}{x^2\log x}\right)\right)$$
$$\times\left(\log\theta(x)+1+\frac{1}{\log\theta(x)-1}+\frac{f(x)}{x}+o\left(\frac{f(x)}{x\log x}\right)\right)$$

$$\alpha(x+f(x)) = \alpha(x)+\frac{1}{\log x}\left(\frac{f(x)}{x}-\frac{f(x)}{x\log x}\log\theta(x)-\frac{f(x)}{x\log x}+o\left(\frac{f(x)}{x\log x}\right)\right)$$

$$\alpha(x+f(x)) = \alpha(x)-\frac{f(x)}{x\log^2 x}+o\left(\frac{f(x)}{x\log^2 x}\right).$$

LEMME 4. — *Sous les mêmes hypothèses qu'au lemme 3, soit une fonction $A(x)$ telle que :*

(i) $A(x)=O(x/\log^2 x)$
(ii) $A(x+f(x))=A(x)+O(f(x)/\log^2 x)$

et soit

$$g(x) = \log(1+A(x)/\theta(x))/\log x$$

$$h(x) = g(x)(\log\theta(x)-1)^{-2}(1+g(x)\log x/(\log\theta(x)-1))^{-1}$$

$$\beta(x) = \alpha(x)+g(x)-h(x).$$

alors

$$\beta(x+f(x)) < \beta(x) \quad \textit{pour } x \textit{ assez grand.}$$

DÉMONSTRATION. On a

$$g(x+f(x)) = \frac{1}{\log x}\left(1+O\left(\frac{f(x)}{x\log x}\right)\right)\log\left(1+\frac{A(x)+O(f(x)/\log^2 x)}{\theta(x)+O(f(x))}\right)$$

$$g(x+f(x)) = \frac{1}{\log x}\log\left(1+\frac{A(x)}{\theta(x)}+O\left(\frac{f(x)}{x\log^2 x}\right)\right)\left(1+O\left(\frac{f(x)}{x\log x}\right)\right) =$$

$$= \frac{1}{\log x}\left(\log\left(1+\frac{A(x)}{\theta(x)}\right)+O\left(\frac{f(x)}{x\log^2 x}\right)\right)\left(1+O\left(\frac{f(x)}{x\log x}\right)\right)$$

d'où

$$g(x+f(x)) = g(x)+O\left(\frac{f(x)}{x\log^3 x}\right).$$

De même :

$$h(x+f(x)) = \left(g(x)+O\left(\frac{f(x)}{x\log^3 x}\right)\right)\times\left(\frac{1}{\log\theta(x)-1}+O\left(\frac{f(x)}{x\log^2 x}\right)\right)^2$$

$$\times\left(1+\frac{(g(x)+O(f(x)/x\log^3 x))(\log x+O(f(x)/x))}{\log\theta(x)-1+O(f(x)/x)}\right)^{-1}$$

$$h(x+f(x)) = \left\{g(x)\frac{1}{(\log\theta(x)-1)^2}+O\frac{f(x)}{x\log^6 x}\right)\right\}$$

$$\times\left\{1+\frac{g(x)\log x}{\log\theta(x)-1}+O\left(\frac{f(x)}{x\log^4 x}\right)\right\}^{-1}$$

$$h(x+f(x)) = h(x)+O\left(\frac{f(x)}{x\log^6 x}\right).$$

La propriété sur $\beta(x)$ se déduit alors du lemme 3.

LEMME 5. — *Soient* $0\leqq\tau<1$ *et* $\lambda=1/(1-\tau)$. *Soient* f_1 *et* g *deux fonctions définies sur* $\mathbf{R}^+$ *à valeurs dans* $\mathbf{R}$, *g croissante,* f_1 *positive et* C^2, *satisfaisant lorsque* $x\to\infty$ *aux conditions:*

(i) $g(x) = x+x^\tau f_1(x)+o(x^\tau f_1(x))$
(ii) $f_1'(x) = o(f_1(x)/x)$
(iii) $f_1''(x) = o(f_1(x)/x^2)$
(iv) $\forall a>0,\quad f_1(axf_1^\lambda(x))\sim f_1(x).$

Soit x_0 quelconque et posons $x_{n+1}=g(x_n)$, alors, lorsque $n\to\infty$, on a :

$$x_n \sim \left(nf_1(n^\lambda)/\lambda\right)^\lambda.$$

DÉMONSTRATION. Posons $h(x)=f_1(x^\lambda)$

$$k(x)=\left(\frac{x\,h(x)}{\lambda}\right)^\lambda$$

alors

$$h'(x)=\lambda x^{\lambda-1}f_1'(x^\lambda)=o\left(h(x)/x\right)$$

$$h''(x)=\lambda(\lambda-1)x^{\lambda-2}f_1'(x^\lambda)+\lambda^2x^{2\lambda-2}f_1''(x^\lambda)=o\left(h(x)/x^2\right)$$

d'où

$$k'(x)=\left(\frac{x\,h(x)}{\lambda}\right)^{\lambda-1}\left(h(x)+xh'(x)\right)=$$

$$=\left(\frac{x\,h(x)}{\lambda}\right)^{\lambda-1}h(x)+o\left(x^{\lambda-1}h^\lambda(x)\right)$$

$$k''(x)=\left(\frac{x\,h(x)}{\lambda}\right)^{\lambda-2}\frac{\lambda-1}{\lambda}\left(h(x)+xh'(x)\right)^2+\left(\frac{x\,h(x)}{\lambda}\right)^{\lambda-1}\left(2h'(x)+x\,h''(x)\right)$$

$$=O\left(x^{\lambda-2}h^\lambda(x)\right).$$

On peut écrire

$$k(n+1)=k(n)+k'(n)+O\left(k''(n)\right)$$

$$k(n+1)=k(n)+\left(\frac{n\,h(n)}{\lambda}\right)^{\lambda-1}h(n)+o\left(n^{\lambda-1}h(n)^\lambda\right).$$

Soit $c>0$ alors

$$g\left(ck(n)\right)=ck(n)+c^\tau\left(\frac{nh(n)}{\lambda}\right)^{\lambda-1}f_1\left(ck(n)\right)+o\left(n^{\lambda-1}h(n)^{\lambda-1}\right)f_1\left(ck(n)\right).$$

L'hypothèse (iv) donne:

$$f_1\left(ck(n)\right)=h(n)+o\left(h(n)\right)$$

donc

$$g\left(ck(n)\right)=ck(n)+c^\tau\left(\frac{n\,h(n)}{\lambda}\right)^{\lambda-1}h(n)+o\left(n^{\lambda-1}h(n)^\lambda\right).$$

Par suite

$$g\left(k(n)\right)-k(n+1)=o\left(n^{\lambda-1}h(n)^\lambda\right)\tag{4}$$

$$\left.\begin{array}{lll}(5)\quad g\left(ak(n)\right)<ak(n+1) & \text{si} & a>1\\ (6)\quad g\left(ak(n)\right)>ak(n+1) & \text{si} & a<1\end{array}\right\}\quad\text{pour } n\geqq n_0.$$

Supposons avoir choisi n_0 de telle sorte que sur $[n_0, \infty[$, $k(x)$ soit fonction croissante. Pour $a>1$, $\exists p$ tel que $x_{n_0}<ak(n_0+p)$ par suite d'après (5)

$$x_{n_0+1} = g(x_{n_0}) < g\big(ak(n_0+p)\big)$$

$$x_{n_0+1} < ak(n_0+1+p)$$

et par suite

$$\forall n \geqq n_0 \qquad x_n < ak(n+p).$$

La suite x_n étant croissante, on a pour $a<1$, $\exists q$ tel que $x_q>ak(n_0)$ et d'après (6) on conclut à:

$$\exists p' \quad \forall n \geqq q \quad x_n > ak(n+p').$$

D'où $x_n \sim k(n)$.

La proposition 2 est alors conséquence du théorème suivant en prenant $f_1=1$.

THÉORÈME. — *Supposons qu'il existe $f(x)$ telle que lorsque $x\to\infty$*

(i) $f(x) = o(x^{\log(3/2)/\log 2})$

(ii) $\theta\big(x+f(x)\big) = \theta(x)+f(x)+o\big(f(x)/\log x\big)$.

Soit C_k et $C_{k'}$ deux nombres $\tilde{N}$ consécutifs alors:

$$p(C_{k'}) \leqq p(C_k)+f\big(p(C_k)\big)$$

$p(n)$ désignant le plus grand facteur premier de n.

Si de plus $f(x)=x^{\tau}f_1(x)$ et si $f_1(x)$ vérifie les hypothèses du lemme 5 alors le nombre $Q(X)$ de nombres $\tilde{N}\leqq X$ vérifie

$$Q(X)\big(f_1(Q(X))\big)^{\lambda} \geqq \big(\lambda+o(1)\big)(\log X)^{1/\lambda}$$

avec $\lambda=1/(1-\tau)$.

PREUVE. Soit α le paramètre commun aux 2 nombres $\tilde{N}$ consécutifs, C_k et $C_{k'}$. Il existe alors x et y tels que:

$$\log C_k = \theta(x)+A(x)$$

$$\log C_{k'} = \theta(y)+A(y)$$

avec $A(z)=\sum\limits_{i\geqq 2}\theta(z^{\nu_i})$. De plus C_k est hautement composé supérieur pour $1/\log x$ et $C_{k'}$ pour $1/\log y$. D'après l'égalité (1), on a

$$\alpha = \beta(x) = \beta(y)$$

avec

$$\beta(x) = \frac{1}{\log x}\left(\log\log C_k+1+\frac{1}{\log\log C_k-1}\right),$$

soit avec les notations des lemmes 3 et 4

$$\beta(x) = \alpha(x)+g(x)-h(x)$$

et de même

$$\beta(y) = \alpha(y)+g(y)-h(y).$$

Les hypothèses du lemme 4 sont vérifiées: en effet

$$A(x) = O(x^{\nu_2}) = O(x/\log^2 x)$$

$$\theta\big((x+f(x))^{\nu_i}\big) = \theta(x^{\nu_i})+o\big(f(x)\,x^{\nu_i-1}\log x\big)$$

$$= \theta(x^{\nu_i})+o\big(f(x)/\log^3 x\big)$$

donc

$$A\big(x+f(x)\big) = A(x)+o\big(f(x)/\log^2 x\big).$$

Par suite $\beta\big(x+f(x)\big)<\beta(x)$ et donc $y<x+f(x)$.

Pour étudier $Q(X)$, désignons par N le $Q(X)^{\text{ième}}$ nombre $\tilde{N}$ alors:

$$\log N = \big(1+o(1)\big)\theta\big(p(N)\big)$$

donc

$$p(N) = \big(1+o(1)\big)\log N = \big(1+o(1)\big)\log X.$$

D'après la première partie de ce théorème et le lemme 5, $Q(X)$ vérifie donc:

$$\left(\frac{Q(X)\big(f(Q(X))\big)^{\lambda}}{\lambda}\right)^{\lambda} \geqq \big(1+o(1)\big)\log X$$

ce qui conduit au résultat final.

4. La troisième proposition

Commençons par démontrer un lemme.

LEMME 6. — *Soit* $\varphi\colon [x_0, +\infty[\to \mathbf{R}$, C^4, *concave et vérifiant* $\varphi^{(4)}<0$. *Soit* $z>y\geqq 0$ *vérifiant* $(z-y)\varphi'(x)+y^2\varphi''(x)<0$ *alors* $\varphi(x+z)+\varphi(x-y)<2\varphi(x)$ *pour* x *tel que* $x-y\geqq x_0$.

DÉMONSTRATION. Par la concavité nous pouvons écrire:

$$\varphi(x+z)-\varphi(x+y) < (z-y)\varphi'(x+y) < (z-y)\varphi'(x).$$

Comme $\varphi^{(4)}<0$ on a:

$$\varphi(x+y) < \varphi(x)+y\varphi'(x)+(y^2/2)\varphi''(x)+(y^3/6)\varphi'''(x)$$

et

$$\varphi(x-y) < \varphi(x)-y\varphi'(x)+(y^2/2)\varphi''(x)-(y^3/6)\varphi'''(x)$$

d'où par addition des trois inégalités

$$\varphi(x+z)+\varphi(x-y) < 2\varphi(x)+(z-y)\varphi'(x)+y^2\varphi''(x)$$

d'où le résultat.

PROPOSITION. — *Il existe une infinité de nombres hautement composés supérieurs qui ne sont pas* $\tilde{N}$.

DÉMONSTRATION. Soit N hautement composé supérieur et soit p son plus grand facteur premier; soit P le suivant de p. N est $\tilde{N}$ s'il existe α tel que g_α soit maximum en N ; en particulier

$$g_\alpha(N) \geqq g_\alpha(NP)$$

$$g_\alpha(N) \geqq g_\alpha(N/p)$$

ce que l'on peut écrire:

$$\frac{\log N}{\log\log N} - \frac{\log(N/p)}{\log\log(N/p)} < \frac{\log 2}{\alpha} < \frac{\log NP}{\log\log NP} - \frac{\log N}{\log\log N}.$$

Cette double inégalité ne peut être vérifiée que si

$$2\varphi(\log N) \leqq \varphi(\log N + \log P) + \varphi(\log N - \log p) \tag{7}$$

si φ désigne la fonction $x \to x/\log x$.

Or une telle inégalité n'est pas toujours possible. En effet, φ vérifie les hypothèses du lemme 6; par suite si

$$\log P - \log p < (\log p)^2 \frac{\log\log N - 2}{\log\log N - 1} \frac{1}{\log N \log\log N} \tag{8}$$

l'inégalité (7) n'est pas vérifiée.

Or l'inégalité (8) peut s'écrire $\log P - \log p \lesssim \dfrac{\log p}{p}$.

L'inégalité (3), $\varliminf \dfrac{p_{n+1} - p_n}{\log n} < 1$, montre qu'il existe une infinité de p tels que:

$$P - p \leqq c \log p \quad \text{avec} \quad c < 1.$$

Soit $\log P - \log p \leqq \dfrac{(P-p)}{p} \leqq \dfrac{c \log p}{p}$.

L'inégalité (8) est donc satisfaite pour une infinité de p; aussi les nombres N, hautement composés supérieurs, se terminant par de tels p, ne peuvent-ils être des nombres $\tilde{N}$.

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(Reçu le 10 novembre 1983)

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INTEGRAL MANIFOLDS AND DECOMPOSITION OF NONLINEAR DIFFERENTIAL SYSTEMS

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The purpose of this paper is to study the problem of nonlinear differential systems decomposition by the method of integral manifolds [1—3].

1. Throughout this paper we shall let E^n denote the real n-dimensional Euclidean space and $|\cdot|$ the Euclidean norm on this space. For any $r>0$ we shall let $B^n(r)$ denote the open ball $\{x\in E^n \,|\, |x|<r\}$.

Consider the differential system

$$\begin{aligned} \dot{x}_1 &= Ax_1+f_1(t, x_1, x_2) \\ \dot{x}_2 &= Bx_2+f_2(t, x_1, x_2) \end{aligned} \tag{1.1}$$

where x_i and f_i vary in E^{n_i}, A and B are real matrices, $t\in R$, $i=1, 2$.

Henceforth we shall assume that (1.1) conforms with the following hypotheses.

(i) The eigenvalues λ_i $(i=1, \ldots, n_1)$ of A satisfy the inequality $|\mathrm{Re}\,\lambda_i|<\alpha$ and the eigenvalues λ_j $(j=1, \ldots, n_2)$ of B satisfy the inequality $\mathrm{Re}\,\lambda_j<-\beta$, where $0\leqq\alpha<\beta$.

(ii) The functions f_i are continuous, bounded and uniformly Lipschitzian in x_1, x_2 on $R\times E^{n_1}\times B^{n_2}(r)$

$$|f_i(t, x_1, x_2)| \leqq M \tag{1.2}$$

$$|f_i(t, x_1, x_2)-f_i(t, \bar{x}_1, \bar{x}_2)| \leqq \lambda(|x_1-\bar{x}_1|+|x_2-\bar{x}_2|), \quad i = 1, 2 \tag{1.3}$$

where M and λ are sufficiently small and, also, $f_i(t, 0, 0)=0$.

THEOREM 1.1. *Let the hypotheses* (i), (ii) *hold, then the system* (1.1) *has the integral manifold represented in the form* $x_2=h(t, x_1)$ *where* h *is a function defined and continuous on* $R\times E^{n_1}$. *Moreover,* h *satisfies the inequalities*

$$|h(t, x_1)| \leqq DM \tag{1.4}$$

$$|h(t, x_1)-h(t, \bar{x}_1)| \leqq \lambda\Delta|x_1-\bar{x}_1| \tag{1.5}$$

with positive constants D, Δ *and, also,* $h(t, 0)=0$ [3—6].

1980 *Mathematics Subject Classification.* Primary 34C45; Secondary 34C20.
Key words and phrases. Integral manifold, stability, decomposition.

The flow on this manifold is described by the n_1-dimensional system

$$\dot{u} = Au + f_1(t, u, h(t, u)). \tag{1.6}$$

It is well-known (see [3, 6]) that for any solution $x_1 = x_1(t)$, $x_2 = x_2(t)$, $x_i(t_0) = x_i^0$, of (1.1) with sufficiently small $|x_2^0 - h(t_0, x_1^0)|$ there is a solution $u = u(t)$, $u(t_0) = u_0$ of (1.6) such that

$$x_1(t) = u(t) + \varphi_1(t), \quad x_2 = h(t, u(t)) + \varphi_2(t) \tag{1.7}$$

where $\varphi_i(t) = O(e^{-\gamma(t-t_0)})$ as $t - t_0 \to \infty$, $\gamma > 0$. Moreover, if the zero solution of (1.6) is stable (asymptotically stable, unstable), then the zero solution of (1.1) is stable (asymptotically stable, unstable). Later we shall obtain exact expressions for φ_i.

Let us suppose, additionally, that

(iii) The functions f_i, $i = 1, 2$ have first and second order continuous and bounded derivatives with respect to t, x_1, x_2 for $t \in R$, $x_1 \in E^{n_1}$, $x_2 \in B^{n_2}(r)$.

Then the function h has first and second order continuous and bounded derivatives with respect to t, x_1 for $t \in R$, $x_1 \in E^{n_1}$ [3, 4].

2. Let us introduce new variables u, y and z by the formulae $z = x_2 - h(t, x_1)$, $y = x_1 - u$, where u satisfies (1.6) and consider the following auxiliary differential system

$$\begin{aligned} \dot{u} &= Au + f_1(t, u, h(t, u)) \\ \dot{y} &= Ay + Y(t, u, y, z) \\ \dot{z} &= Bz + Z(t, u, y, z) \end{aligned} \tag{2.1}$$

where

$$Y = f_1(t, u+y, z+h(t, u+y)) - f_1(t, u, h(t, u))$$

$$Z = f_2(t, u+y, z+h(t, u+y)) - f_2(t, u+y, h(t, u+y)) -$$
$$- \frac{\partial h}{\partial x}(t, u+y)[f_1(t, u+y, z+h(t, u+y)) - f_1(t, u+y, h(t, u+y))].$$

We shall show that this last system has an integral manifold $y = H(t, u, z)$ such that the function H satisfies the inequalities

$$|H(t, u, z)| \leqq \lambda a |z| \tag{2.2}$$

$$|H(t, u, z) - H(t, u, \bar{z})| \leqq \lambda c |z - \bar{z}| \tag{2.3}$$

$$|H(t, u, z) - H(t, \bar{u}, z)| \leqq b\,|z|\,|u - \bar{u}| \tag{2.4}$$

$$t \in R, \quad u, \bar{u} \in E^{n_1}, \quad z, \bar{z} \in B^{n_2}(\varrho)$$

with $a > 0$, $b > 0$, $c > 0$, $\varrho > 0$.

The flow on this manifold is governed by the $(n_1 + n_2)$-dimensional system

$$\dot{u} = Au + f_1(t, u, h(t, u)) \tag{2.5}$$

$$\dot{v} = Bv + Z(t, u, H(t, u, v), v). \tag{2.6}$$

Moreover, every solution of (1.1) with sufficiently small $|x_2^0-h(t_0, x_1^0)|$ can be represented as the superposition of corresponding solutions of (2.5) and (2.6) of form

$$\begin{aligned} x_1 &= u+H(t, u, v) \\ x_2 &= v+h(t, x_1) = v+h(t, u+H(t, u, v)). \end{aligned} \tag{2.7}$$

Our proofs of this facts are modelled on Pliss [3] and Kelley [4].

From (i) there follows the existence of a constant $K \geqq 1$, such that

$$|e^{At}| \leqq Ke^{\alpha|t|}, \qquad -\infty < t < \infty \tag{2.8}$$

$$|e^{Bt}| \leqq Ke^{-\beta t}, \qquad 0 \leqq t < \infty. \tag{2.9}$$

Next, by means of our assumptions with respect to f_1, f_2 and Theorem 1.1 it is easy to see that there exists a constant $N>0$ such that Y and Z satisfy the inequalities

$$|Y(t, u, y, z)| \leqq \lambda N(|y|+|z|) \tag{2.10}$$

$$|Z(t, u, y, z)| \leqq \lambda N|z| \tag{2.11}$$

$$|Y(t, u, y, z)-Y(t, u, \bar{y}, \bar{z})| \leqq \lambda N(|y-\bar{y}|+|z-\bar{z}|) \tag{2.12}$$

$$|Z(t, u, y, z)-Z(t, u, y, z)| \leqq \lambda N(|y-\bar{y}|+|z-\bar{z}|) \tag{2.13}$$

$$\begin{aligned} &|Y(t, u, y, z)-Y(t, \bar{u}, \bar{y}, \bar{z})| \leqq \\ &\leqq N(\mu+|y|+|z|)[|z-\bar{z}|+(1+|y|)|y-\bar{y}|+(|y|+|z|)|u-\bar{u}|] \end{aligned} \tag{2.14}$$

$$|Z(t, u, y, z)-Z(t, \bar{u}, \bar{y}, \bar{z})| \leqq N[\mu|z-\bar{z}|+|z|(|u-\bar{u}|+|y-\bar{y}|)] \tag{2.15}$$

where $t \in R$; $u, \bar{u} \in E^{n_1}$; $y, \bar{y} \in E^{n_1}$; $z, \bar{z} \in B^{n_2}(\varrho_0)$; $\mu=\mu(\lambda, \varrho_0)>0$, $\mu(\lambda, \varrho_0) \to 0$ as $\lambda \to 0$, $\varrho_0 \to 0$.

Let S be the set of functions $H\colon R^+ \times E^{n_1} \times B^{n_2}(\varrho) \to E^{n_1}$ such that H is continuous and satisfies (2.2)—(2.4). Let d be the metric on S defined by

$$d(H, \bar{H}) = \sup_{H, \bar{H} \in S} \left\{ \frac{1}{|z|} |H(t, u, z)-\bar{H}(t, u, z)|,\, t \in R,\, u \in E^{n_1},\, z \in B^{n_2}(\varrho) \right\}$$

and note that, with respect to d, S is a complete metric space.

For each $H \in S$ we consider the differential system

$$\dot{u} = Au+f_1(t, u, h(t, u)) \tag{2.16}$$

$$\dot{z} = Bz+Z(t, u, H(t, u, z), z) \tag{2.17}$$

the solutions of which we denote by $u=\Phi(t, t_0, u_0)$, $z=\Psi(t, t_0, u_0, z_0|H)$ with the understanding that $\Phi(t_0, t_0, u_0)=u_0$, $\Psi(t_0, t_0, u_0, z_0|H)=z_0$.

The functions $f_1(t, u, h(t, u))$, $Z(t, u, H(t, u, z), z)$ are uniformly bounded on their domains, hence, any solution of (2.16), (2.17) is defined for all t.

Let $\varphi(t)=\Phi(t, \tau, u)$, $\psi(t)=\Psi(t, \tau, u, z|H)$ then by the variation of constants formula,

$$\psi(t) = e^{B(t-\tau)}z+\int_\tau^t e^{B(t-s)}Z(s, \varphi(s), H(s, \varphi(s), \psi(s)), \psi(s))\, ds.$$

By (2.9) and (2.11) there holds for all $-\infty<\tau\leqq t<\infty$,

$$|\psi(t)| \leqq Ke^{-\beta(t-\tau)}|z|+\int_\tau^t Ke^{-\beta(t-s)}\lambda N|\psi(s)|\,ds.$$

Therefore, by Gronwall's Lemma, we obtain

$$|\psi(t)| \leqq Ke^{-\beta_1(t-\tau)}|z|,\quad \beta_1=\beta-K\lambda N,\quad -\infty<\tau\leqq t<\infty. \tag{2.18}$$

It is clear that $\beta_1\geqq\gamma>\alpha$ for sufficiently small λ. Now define an operator T on S by setting

$$T(H)(\tau,u,z)=-\int_\tau^\infty e^{-A(t-\tau)}Y\big(t,\varphi(t),H(t,\varphi(t),\psi(t)),\psi(t)\big)\,dt \tag{2.19}$$

$$\varphi(t)=\Phi(t,\tau,u),\quad \psi(t)=\Psi\big(t,\tau,u,z|H\big).$$

The improper integral here converges by virtue (2.8)—(2.10), (2.2) and (2.18).

It is a straightforward exercise to show that $T(H)$ as defined in (2.19) is continuous on $R\times E^{n_1}\times B^{n_2}(\varrho)$. Also, using (2.2), (2.10), (2.8) and (2.18), it is easy to see that

$$\begin{aligned}|T(H)(\tau,u,z)| &\leqq K\int_\tau^\infty e^{-\alpha(\tau-t)}\lambda N(1+\lambda a)|\psi(t)|\,dt\leqq\\ &\leqq\frac{K^2\lambda N}{\beta_1-\alpha}(1+\lambda a)|z|\leqq\lambda a|z|\end{aligned} \tag{2.20}$$

if $\nu=K^2N/(\beta_1-\alpha)<1/\lambda$ and $a\geqq\nu/(1-\lambda\nu)$.

Thus $T(H)$ satisfies the boundedness condition required by (2.2). To prove that $T(H)$ satisfies the Lipschitz conditions required by (2.3), (2.4) and that T is a contraction mapping, we reason as follows.

Let $u\in E^{n_1}$, $z,\bar z\in B^{n_2}(\varrho)$, $\psi_1(t)=\Psi(t,\tau,u,\bar z|H)$. Then by (2.3), (2.8) and (2.12)

$$|T(H)(\tau,u,z)-T(H)(\tau,u,\bar z)|\leqq K\int_\tau^\infty e^{-\alpha(\tau-t)}\lambda N(1+\lambda c)\,|\psi(t)-\psi_1(t)|\,dt. \tag{2.21}$$

Now, using (2.13), (2.8) and (2.3) we find that

$$|\psi(t)-\psi_1(t)|\leqq Ke^{-\beta(t-\tau)}|z-\bar z|+\int_\tau^t Ke^{-\beta(t-s)}\lambda N(1+\lambda c)\,|\psi(s)-\psi_1(s)|\,ds.$$

Therefore, by Gronwall's Lemma, we obtain

$$|\psi(t)-\psi_1(t)|\leqq Ke^{-\beta_2(t-\tau)}|z-\bar z|,\qquad -\infty<\tau\leqq t<\infty$$

$$\beta_2=\beta-K\lambda N(1+\lambda c).$$

It is clear that for sufficiently small λ we can choose c such that $\beta_2>\gamma>\alpha$ and $\dfrac{K^2N}{\beta_2-\alpha}(1+\lambda c)\leqq c$. From this last inequality it follows that $T(H)$ satisfies the Lipschitz condition required by (2.3).

In exactly the same way by the simple inequality

$$|\Phi(t, \tau, u)-\Phi(t, \tau, \bar{u})| \leqq K e^{\alpha_1(t-\tau)}|u-\bar{u}|, \qquad \alpha_1=\alpha+K\lambda(1+\lambda\Delta), \qquad -\infty<\tau \leqq t<\infty$$

and (2.14), (2.15) it is easy to show that $T(H)$ for some constant $b>0$ and sufficiently small μ, λ, ϱ satisfies the condition required by (2.4).

Now, let $H, \bar{H} \in S$, $\psi_2(t)=\Psi(t, \tau, u, z|\bar{H})$. Then by (2.12), (2.18) and (2.2)

$$|T(H)(\tau, u, z)-T(\bar{H})(\tau, u, z)| \leqq \tag{2.22}$$

$$\leqq \int_\tau^\infty K e^{-\alpha(\tau-t)} \lambda N\left[(1+\lambda c)|\psi(t)-\psi_2(t)|+\left|H(t, \varphi(t), \psi(t))-\bar{H}(t, \varphi(t), \psi(t))\right|\right] dt \leqq$$

$$\leqq \int_\tau^\infty K e^{-\alpha(\tau-t)} \lambda N[(1+\lambda c)|\psi(t)-\psi_2(t)|+K e^{-\gamma(t-\tau)}|z|\, d(H, \bar{H})]\, dt.$$

Using (2.13), (2.8) and (2.3) we find that

$$|\psi(t)-\psi_2(t)| \leqq \int_\tau^t K e^{-\beta(t-s)} \lambda N[(1+\lambda c)|\psi(s)-\psi_2(s)|+K e^{-\gamma(t-s)}|z|\, d(H, \bar{H})]\, ds.$$

Multiplying both sides of this inequality by $e^{\beta(t-\tau)}$ and then applying the reasoning of the type used to deduce Gronwall's Lemma [7] we obtain

$$|\psi(t)-\psi_2(t)| \leqq \frac{K^2 \lambda N}{\beta_2-\gamma} e^{-\gamma(t-\tau)}|z|\, d(H, \bar{H}).$$

Substitution of this into (2.22) yields

$$|T(H)(\tau, u, z)-T(\bar{H})(\tau, u, z)| \leqq \frac{K^2 \lambda N}{\gamma-\alpha}\left[(1+\lambda c) \frac{K \lambda N}{\beta_2-\gamma}+1\right] d(H, \bar{H})\, |z|.$$

From this last inequality it follows easily that T is a contraction mapping if λ is sufficiently small.

Thus, T is a contraction mapping of S into itself and so, by the Banach Contraction Mapping Principle, T must have a unique fixed point $H \in S$.

The function H is the solution of the equation

$$H(\tau, u, z)=-\int_\tau^\infty e^{A(\tau-t)} Y(t, \varphi(t), H(t, \varphi(t), \psi(t)), \psi(t))\, dt$$

$$\varphi(t)=\Phi(t, \tau, u), \qquad \psi(t)=\Psi(t, \tau, u, z|H)$$

and, therefore, the equation $y=H(t, u, z)$ represents an integral manifold for (2.1).

The flow on this manifold is governed by the equations

$$\dot{u}=Au+F(t, u) \tag{2.23}$$

$$\dot{v}=Bv+G(t, u, v) \tag{2.24}$$

where

$$F=f(t, u, h(t, u)), \; G=Z(t, u, H(t, u, v), v).$$

3. Our next aim is to obtain the representation (2.7). Let $x_1=x_1(t)$, $x_2=x_2(t)$ be a solution of (1.1) with $x_i(t_0)=x_i^0$, $i=1, 2$, where $|x_2^0-h(t_0, x_1^0)|\leqq\varrho$. Then there exists a solution $u=u(t)$, $v=v(t)$ of (2.23), (2.24) with $u(t_0)=u_0$, $v(t_0)=v_0$ such that $x_1(t)=u(t)+H\big(t, u(t), v(t)\big)$, $x_2(t)=v(t)+h\big(t, x_1(t)\big)$. It is sufficient to show that this last representation holds for $t=t_0$. Let $t=t_0$, then

$$x_1^0 = u_0+H(t_0, u_0, v_0)$$

$$x_2^0 = v_0+h(t_0, x_1^0)$$

and, therefore $v_0=x_2^0-h(t_0, x_1^0)$. For u_0 we obtain the equation

(3.1) $$x_1^0 = u_0+H\big(t_0, u_0, x_2^0-h(t_0, x_1^0)\big).$$

Consider the auxiliary equation

(3.2) $$u = P(u)$$

where $u\in E^{n_1}$, $P(u)=x_1^0-H\big(t_0, u, x_2^0-h(t_0, x_1^0)\big)$.

Let $q=\varrho b<1$, then by (2.4) $|P(u)-P(\bar{u})|\leqq q|u-\bar{u}|$.

Thus P is a contraction mapping on E^{n_1} and so, by the Banach Contraction Mapping Principle, P must have a unique fixed point u_0, which is the required solution of (3.1).

Now we have the exact expressions for φ_1 and φ_2 in (1.7)

(3.3) $$\varphi_1 = H\big(t, u(t), v(t)\big)$$

(3.4) $$\varphi_2 = v(t)+h\big(t, u(t)+H(t, u(t), v(t))\big)-h\big(t, u(t)\big).$$

This and (1.5), (2.2), (2.18) allow us to write

$$|\varphi_1(t)| \leqq \lambda a K e^{-\gamma(t-t_0)}\,|x_2^0-h(t_0, x_1^0)|, \quad t\geqq t_0$$

$$|\varphi_2(t)| \leqq (1+\lambda\Delta\lambda a)\,|v(t)| \leqq (1+\lambda^2\Delta a)Ke^{-\gamma(t-t_0)}\,|x_2^0-h(t_0, x_1^0)|.$$

Therefore

$$|x_1(t)| \leqq |u(t)|+\lambda a K\,|x_2^0-h(t_0, x_1^0)|\,e^{-\gamma(t-t_0)}, \qquad t\geqq t_0$$

$$|x_2(t)| \leqq \lambda\Delta\,|u(t)|+(1+\lambda^2\Delta a)K\,|x_2^0-h(t_0, x_1^0)|\,e^{-\gamma(t-t_0)}.$$

From these last inequalities it follows easily that the stability problem for (1.1) is equivalent to the stability problem for (2.23) (see Section 1).

Now we can summarize our results in the following

THEOREM 3.1. *Let* (1.1) *satisfy the hypotheses* (i) *through* (iii). *Then there exist numbers* λ_0, ϱ_1 *such that* $0\leqq\lambda\leqq\lambda_0$, $0<\varrho\leqq\varrho_1$, $t_0\in R$ *imply that the following assertions are true*:

(1) *There exists for* (2.1) *an integral manifold represented by an equation of the form* $y=H(t, u, z)$ *where* H *is a function defined and continuous on* $R^+\times E^{n_1}\times B^{n_2}(\varrho)$ *and, moreover,* H *satisfies* (2.2)—(2.4).

(2) *Every solution* $x_1=x_1(t)$, $x_2=x_2(t)$ *of* (1.1) *with* $x_i(t_0)=x_i^0$, $i=1, 2$, $|x_2^0-h(t_0, x_1^0)|<\varrho$ *can be represented in the form* (2.7) *where* $u=u(t)$, $u(t_0)=u_0$ *is a solution of* (2.23), u_0 *is a solution of* (3.1); $v=v(t)$ *is a solution of* (2.24) *with* $v(t_0)=x_2^0-h(t_0, x_1^0)$.

(3) *If the zero solution of* (2.23) *is stable (asymptotically stable, unstable) then the zero solution of* (1.1) *is stable (asymptotically stable, unstable).*

Note that our proof of this theorem is based on the Banach Contraction Mapping Principle, and therefore it is easy to extend it on the system (1.1), where x_1, x_2 are elements of Banach spaces. Moreover, in this proof we do not use the boundedness of B. So the theorem of the type Theorem 3.1 can be proved for systems like (1.1) in Banach space with the unbounded operator B, if $A=0$ and B is the generator of a strongly continuous linear semigroup $S_B(t)$ such that $|S_B(t)|\leqq Ke^{-\beta t}$, $t\geqq 0$ (see, also, [6, 8, 9]).

Now let us suppose that (2.23) can be represented as

$$\begin{aligned}\dot{u}_1 &= A_1 u_1 + F_1(t, u_1, u_2)\\ \dot{u}_2 &= A_2 u_2 + F_2(t, u_1, u_2),\end{aligned} \tag{3.5}$$

where

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u, \qquad \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = A, \qquad \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = F,$$

and the eigenvalues of A_1 and A_2 have zero and positive real parts, respectively. It is clear that (3.5) conforms with the hypotheses (i)—(iii) with respect to the new time $\tau=-t$. Therefore, this system can be reduced to the system of the form

$$\begin{aligned}\dot{w}_1 &= A_1 w_1 + F_3(t, w_1)\\ \dot{w}_2 &= A_2 w_2 + F_4(t, w_1, w_2),\end{aligned}$$

which is analogous to (2.23), (2.24).

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(Received August 2, 1984)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 81—83

ON THE DECOMPOSITION OF INFINITE SERIES INTO MONOTONE DECREASING PARTS

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Abstract

We prove that any non-negative series in l_2 can be broken into decreasing finite parts.

It is easy to prove if $a_n \geqq 0$ $(n=1, 2, \dots)$ and $\sum_{n=1}^{\infty} a_n < \infty$, then there are indices $0=j_0<j_1<\dots$ such that the sequence $\sum_{k=j_{i-1}+1}^{j_i} a_k$ is monotone decreasing. We can ensure strict monotonicity if $\{a_i\}$ has infinitely many positive terms.

The aim of our paper is to prove the conjecture of the authors of [1] that similar property holds for the sequences in l_2.

THEOREM. *Suppose $a_i \geqq 0$ for $i=1, 2, \dots$ and $\sum_{i=1}^{\infty} a_i^2 < \infty$. Then there exist indices $0=j_0<j_1<\dots$ such that the sequence of the sums $X_i = \sum_{k=j_{i-1}+1}^{j_i} a_k$ is monotone decreasing and*

$$X_1 \leqq 2\sqrt{\sum_{i=1}^{\infty} a_i^2}. \tag{1}$$

If there are infinitely many positive elements in the series we can ensure strict monotonicity and strict inequality in (1).

First we prove a finite version of the theorem in the next

LEMMA. *Let $a_i \geqq 0$ for $i=1, 2, \dots, n$ and $\sum_{i=1}^{n} a_i > 0$. Then there exist indices $0=j_0<j_1<\dots<j_k=n$ such that the sequence $\{X_i\}_1^k$ is strictly monotone decreasing and $X_1 < 2\sqrt{\sum_{i=1}^{n} a_i^2}$. $(X_i = \sum_{m=j_{i-1}+1}^{j_i} a_m.)$*

PROOF of the lemma. For each sequence of indices $0=j_0<j_1<\dots<j_k=n$ we define two weights:

$$W(\{j_i\}_0^k) = \sum_{i=1}^{k} X_i^2 + 2\sum_{i=1}^{k-1} \sum_{m=j_i+1}^{n} a_m^2$$

$$W'(\{j_i\}_0^k) = \sum_{i=1}^{k} (n-j_i).$$

(The weight W' is introduced only to prove the strict monotonicity.)

1980 *Mathematics Subject Classification.* Primary 40A05.
Key words and phrases. l_2-series, monotonicity.

Let us choose a strictly monotone sequence $\{j_i\}_0^k$ which minimizes W and such that among the sequences with the same minimal value W it minimizes W' as well. It is easy to see that $\{j_i\}_0^k$ has this property among the monotone but not necessarily strictly monotone sequences, too, because replacing the multiple indices with simple ones the weights W and W' will not increase. We prove that this choice of sequence $\{j_i\}_0^k$ fulfils the lemma.

(a) $X_i > X_{i+1}$ for $i = 1, \dots, k-1$.

For a fixed i we define

$$\{J_t'\}_0^k = (j_0, j_1, \dots, j_{i-1}, j_i+1, j_{i+1}, \dots, j_k).$$

Since $\{J_t\}_0^k$ is a strictly increasing sequence, $\{J_t'\}_0^k$ is still monotone increasing. Here $X_m' = X_m$ if $m \neq i$, $m \neq i+1$; $X_i' = X_i + a_{j_i+1}$; $X_{i+1}' = X_{i+1} - a_{j_i+1}$. It is easy to see, that $W'(\{J_t\}_0^k) - W'(\{J_t'\}_0^k) = 1$. Because of the minimum property of $\{J_t\}_0^k$ it follows that

$$0 < W(\{j_i'\}_0^k) - W(\{j_i\}_0^k) = X_i'^2 + X_{i+1}'^2 - X_i^2 - X_{i+1}^2 - 2a_{j_i+1}^2 = 2a_{j_i+1}(X_i - X_{i+1})$$

and hence by $a_{j_i+1} \geqq 0$ we have $X_i > X_{i+1}$.

(b) We have to show that $X_1 < 2\sqrt{\sum_{i=1}^{n} a_i^2}$.

From $\sum_{i=1}^{n} a_i > 0$ we have $X_1 > 0$. There exists a unique $0 < j \leqq j_1$ such that $\sum_{i=1}^{j-1} a_i < \frac{X_1}{2} \leqq \sum_{i=1}^{j} a_i$. Let $d = \sum_{i=1}^{j} a_i - \frac{X_1}{2}$. Clearly, $0 \leqq d < a_j$. Let us consider the sequence $\{j_i'\}_0^{k+1} = (j_0, j, j_1, \dots, j_k)$. This sequence is monote. Here $X_m' = X_{m-1}$ for $m = 3, \dots, k+1$ and $X_1' = \frac{X_1}{2} + d$; $X_2' = \frac{X_1}{2} - d$. Using the minimum property of $\{j_i\}_0^k$ we have

$$0 \leqq W(\{j_i'\}_0^{k+1}) - W(\{j_i\}_0^k) = -\frac{X_1^2}{2} + 2d^2 + 2\sum_{i=j+1}^{n} a_i^2 <$$

$$< -\frac{X_1^2}{2} + 2a_j^2 + 2\sum_{i=j+1}^{n} a_i^2 \leqq -\frac{X_1^2}{2} + 2\sum_{i=1}^{n} a_i^2,$$

hence $X_1 < 2\sqrt{\sum_{i=1}^{n} a_i^2}$.

For proving the infinite case we shall use König's lemma.

Proof of the theorem. We can suppose that $a_{n_0} > 0$ for some $n_0 > 0$. We construct a directed tree. The vertices will be the sequences $0 = j_0 < j_1 \dots < j_k$ with the property, that for the sequence $X_i = \sum_{m=j_{i-1}+1}^{j_i} a_m$ $(i = 1, \dots, k)$ the inequalities

$$2\sqrt{\sum_{i=1}^{\infty} a_i^2} > X_1 > X_2 > \dots > X_k$$

hold. We set an edge from $\{j_i\}_0^k$ to $\{j_i'\}_0^{k'}$ iff $k'=k+1$ and $j_i'=j_i$ for $i=1, \dots, k$. Our lemma says that for every fixed $n>n_0$ there is a vertex $\{j_i\}_0^k$ in the digraph where $j_k=n$, thus we have an infinite graph. Obviously, we have a directed path to any vertex from the vertex (0), since for any vertex $\{j_i\}_0^k$ of our graph, also $\{j_i\}_0^m$ are vertices for $m \leqq k$, and they form a directed path. We separate two cases.

The first case is that there is a vertex $\{j_i\}_0^k$ with infinite outdegree. Then $X_k > \sum_{i=j_k+1}^{n} a_i$ for infinitely many $n>j_k$. Now the statement follows from the case when $\sum_{i=1}^{\infty} a_i < \infty$.

In the opposite case we have a digraph such that there are directed paths from (0) to infinitely many vertices, but the outdegree of each vertex is finite. The König lemma says that there is an infinite directed path from (0) in this digraph. Denote the m-th vertex of this path by $\{j_i^m\}_{i=0}^{k_m}$ and let $j_i=j_i^i$. It is obvious, that $k_m=m$ and $j_i^m=j_i$ for $m \geqq i$ and therefore j_i satisfies the statement of the theorem.

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(Received December 12, 1984)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 85—87

A REMARK CONCERNING STRONG UNIQUENESS OF APPROXIMATION

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Introduction

Let us recall the definition of strongly unique approximation.

Let $(X, \|.\|)$ be a real normed linear space, $K \subset X$ a non-void subset.

We say that for some $x_0 \in X$ the approximation by K is strongly unique if the following assumptions are satisfied:

(i) the set

$$\{k \in K;\ \|x_0 - k\| = \inf_{h \in K} \|x_0 - h\|\}$$

contains exactly one element (say, k_0);

(ii) there exists a constant $c > 0$ (depending on x_0) such that for all $k \in K$

$$\|x_0 - k\| \geqq \|x_0 - k_0\| + c\,\|k_0 - k\|.$$

We call a set $K \subset X$ strongly Chebyshev if conditions (i) and (ii) are satisfied for every $x_0 \in X$.

This property has been investigated by several authors, see for example [3], [4].

In [5] it is proved that there is no proper nontrivial strongly Chebyshev subspace in any smooth normed linear space.

On the contrary, in [3] it is shown that there are points for which the approximation by K is strongly unique even in Hilbert space, if K is not a subspace.

But — as we shall show in this paper — this cannot be true for all points of the space.

Namely, we shall prove that there is no non-singleton non-trivial strongly Chebyshev set in any separable Banach space with Fréchet differentiable norm.

The result

THEOREM. *Let $(X, \|.\|)$ be a separable Banach space with Fréchet differentiable norm, and $K \subset X$ a proper subset which is strongly Chebyshev. Then K is a singleton.*

PROOF. We proceed by contradiction.

Let us denote the nearest element to $x \in X$ by $P_K(x)$. First, for the sake of completeness, we show that the strong Chebyshev property of K implies the continuity of P_K in the norm topology.

1980 *Mathematics Subject Classification.* Primary 41A65.

Key words and phrases. Chebyshev subspace, strongly Chebyshev subspace.

Let $\|x-x_0\|<\varepsilon$. Clearly,

$$\begin{aligned}\|x_0-P_K(x_0)\|+2\|x-x_0\| &\geqq \|x-P_K(x_0)\|+\|x-x_0\| \geqq \\ &\geqq \|x-P_K(x)\|+\|x-x_0\| \geqq \|x_0-P_K(x)\| \geqq \\ &\geqq \|x_0-P_K(x_0)\|+c_0\|P_K(x)-P_K(x_0)\|.\end{aligned}$$

This — using the fact that $c_0>0$ — implies the continuity of P_K at x_0.

Secondly, using the indirect hypothesis we prove that $P_K(X)$ is uncountable.

Assuming the contrary, we have

$$P_K(X)=\{k_1, k_2, \dots\}.$$

Introducing the sets $X_i=\{x\in X;\ P_K(x)=k_i\}$, these sets X_i are closed, pairwise disjoint, and we have

$$\bigcup_{i=1}^{\infty} X_i = X.$$

This — using an elementary, but ingenious lemma of Asplund [1] — implies that all but one X_i is void. So, $P_K(X)$ is uncountable.

Elementary reasoning shows that there exist $c^*>0$, $H\subset X\setminus K$ such that H is uncountable,

$$\|x-k\| \geqq \|x-P_K(x)\|+c^*\|P_K(x)-k\| \tag{1}$$

for all $x\in H$, $k\in K$,

$$P_K(x_1)\neq P_K(x_2)$$

for any $x_1\neq x_2$, $x_1, x_2\in H$, and H has a condensation point $x^*\in H$ (because P_K is continuous).

Let $f_{x^*}\in(X, \|\,.\,\|)^*$, $\|f_{x^*}\|=1$, be such that

$$f_{x^*}(P_K(x^*)-x^*)=\|P_K(x^*)-x^*\|.$$

Choosing $\varepsilon>0$ small enough, the differentiability of the norm implies that

$$\begin{aligned}f_{x^*}(k-x^*)+\frac{c^*}{2}\|k-P_K(x^*)\| &\geqq \|k-x^*\| \geqq \\ &\geqq f_{x^*}(k-x^*)-\frac{c^*}{2}\|k-P_K(x^*)\|\end{aligned} \tag{2}$$

for $\|k-P_K(x^*)\|<\varepsilon$. (This latter follows from the fact that f_{x^*} is the supporting functional of the sphere at $P_K(x^*)-x^*$.)

Let $x_1\in H$ be such that $\|x_1-x^*\|<\varepsilon$, $\|P_K(x_1)-P_K(x^*)\|<\varepsilon$. Then, using (1) and (2),

$$\begin{aligned}f_{x^*}(P_K(x_1)-x^*)+\frac{c^*}{2}\|P_K(x_1)-P_K(x^*)\| &\geqq \|x^*-P_K(x_1)\| \geqq \\ \geqq \|x^*-P_K(x^*)\|+c^*\|P_K(x^*)-P_K(x_1)\| &= f_{x^*}(P_K(x^*)-x^*)+c^*\|P_K(x^*)-P_K(x_1)\|.\end{aligned}$$

This inequality yields

$$f_{x^*}(P_K(x_1)) \geqq f_{x^*}(P_K(x^*)) + \frac{c^*}{2}\|P_K(x_1) - P_K(x^*)\|. \tag{3}$$

With the same reasoning, we also have

$$f_{x_1}(P_K(x^*)) \geqq f_{x_1}(P_K(x_1)) + \frac{c^*}{2}\|P_K(x_1) - P_K(x^*)\|. \tag{4}$$

On the other hand, we have

$$(f_{x^*} - f_{x_1})(P_K(x_1) - P_K(x^*))| \leqq \|f_{x^*} - f_{x_1}\|\,\|P_K(x_1) - P_K(x^*)\|.$$

In case ε is sufficiently small, the continuity of the derivative f_x implies [2, p. 30]

$$\|f_{x^*} - f_{x_1}\| < \frac{c^*}{2}. \tag{5}$$

Clearly, formulas (3), (4), (5) contradict each other.

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(Received February 20, 1985)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 89—91

A CHARACTERIZATION OF THE NORMAL DISTRIBUTION

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In a recent paper (Glänzel and al. (1984)) a characterization theorem for non-negative random variables based on a simple connection of two truncated moments was given. In this letter that result will be extended for arbitrary real valued random variables (Proposition 1) and applied to characterize the normal distribution (Proposition 2).

PROPOSITION 1. *Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a given probability space and let $H=[a, b]$ be an interval for some $a<b$ ($a=-\infty$ and $b=+\infty$ might as well allowed). Let $X\colon \Omega\to H$ be a continuous random variable with the distribution function F and let g and h be two real functions defined on H such that*

$$\mathsf{E}\{g(X)|X \geqq x\} = \mathsf{E}\{h(X)|X \geqq x\}\lambda_h^g(x), \qquad x\in H$$

is defined with some real function λ_h^g. Assume that $g, h\in C^1(H)$, $\lambda_h^g\in C^2(H)$ and F is a twice continuously differentiable and strictly monotone function on the set H. Finally assume that the equation $h\lambda_h^g=g$ has no solution on int *H. Then F is uniquely determined by the functions g, h and λ_h^g, particularly*

$$F(x) = \int_a^x C\left|\frac{\lambda'(u)}{\lambda(u)h(u)-g(u)}\right| \exp(-s(u))\,du,$$

where the function s is a solution of the differential equation $s'=\dfrac{\lambda' h}{\lambda h-g}$ and C is a constant, chosen to make $\int_H dF=1$.

PROOF. The proof is completely analogous to that of Theorem 2.1 in Glänzel and al. (1984).

PROPOSITION 2. *Let $X\colon \Omega\to\mathbf{R}$ be a continuous random variable and let*

$$g(x) = x^2-mx-\sigma^2, \qquad x\in\mathbf{R}$$

and

$$h(x) = x-m, \qquad x\in\mathbf{R}$$

1980 *Mathematics Subject Classification* (1985 Revision). Primary 62E10.
Key words and phrases. Characterization, normal distribution, truncated moments.

be two real functions with the parameters $m \in \mathbf{R}$ and $\sigma \in \mathbf{R}^+$. The distribution of the random variable X is normal if and only if the function λ_h^g defined in Proposition 1 has the form

$$\lambda_h^g(x) = x, \qquad x \in \mathbf{R}.$$

PROOF. The functions F, g, h and λ_h^g obviously satisfy the conditions of Proposition 1.

Assume that the distribution of the random variable is normal with expectation m and standard deviation σ. Consider the transformed random variable $Y = \dfrac{X-m}{\sigma}$ and use the notation $y = \dfrac{x-m}{\sigma}$. Note that $\mathsf{E}\{h(X)|X \geqq x\} \neq 0$ for all real x. From the definition we obtain that

$$\lambda_h^g(x) = \frac{\mathsf{E}\{g(X)|X \geqq x\}}{\mathsf{E}\{h(X)|X \geqq x\}} = \frac{\mathsf{E}\{\sigma Y^2 + mY - \sigma | Y \geqq y\}}{\mathsf{E}\{Y|Y \geqq y\}} =$$

$$= \frac{\int\limits_y^\infty (\sigma t^2 + mt - \sigma) \exp\left(-\frac{t^2}{2}\right) dt}{\int\limits_y^\infty t \exp\left(-\frac{t^2}{2}\right) dt} = \frac{\sigma y \exp\left(-\frac{y^2}{2}\right) + m \exp\left(-\frac{y^2}{2}\right)}{\exp\left(-\frac{y^2}{2}\right)} =$$

$$= \sigma y + m = x \quad \text{for all} \quad x \in \mathbf{R}.$$

Assume that λ_h^g has the form $\lambda_h^g(x) = x$. Then the equation

$$\frac{\lambda'(x) h(x)}{\lambda(x) h(x) - g(x)} = \frac{x-m}{\sigma^2}, \qquad x \in \mathbf{R}$$

follows from Proposition 1. Thus,

$$f(x) = \frac{C}{\sigma^2} e^{-\frac{(x-m)^2}{2\sigma^2}}, \qquad x \in \mathbf{R}$$

with $C = \dfrac{\delta}{\sqrt{2\pi}}$, i.e., the distribution is normal.

REMARK. Let us use the notations

$$e_X^{(1)}(x) = \mathsf{E}\{X|X \geqq x\}, \qquad x \in \mathbf{R}$$

and

$$e_X^{(2)}(x) = \mathsf{E}\{X^2|X \geqq x\}, \qquad x \in \mathbf{R}.$$

We define the functions $d_X(x) = e_X^{(1)}(x) - x$ and $D_X(x) = e_X^{(2)}(x) - x e_X^{(1)}(x)$ for all real x. The statement of Proposition 2 can be reformulated as follows. The random variable $X\colon \Omega \to \mathbf{R}$ has a normal distribution if and only if

$$D_X = m d_X + \sigma^2$$

holds for some real m and $\sigma \in \mathbf{R}^+$.

Substituting the functions D_X and d_X by the corresponding sample statistics, normality tests for random variables with unknown parameters can readily be constructed.

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(Received March 27, 1985)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 93—103

ON HOMOMORPHISMS OF *PL*-SEMIGROUPS

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McAlister [2, 3] proved that every inverse semigroup is an idempotent separating homomorphic image of an E-unitary inverse semigroup, and his well-known P-theorem describes the structure of E-unitary inverse semigroups by means of groups, partially ordered sets and semilattices. The former result was generalized for orthodox semigroups, independently by Takizawa [10] and the author [7]. On the other hand, Takizawa [9] introduced the notion of a PL-semigroup and generalized McAlister's P-theorem for E-unitary $\mathcal{R}$-unipotent semigroups. However, while in the P-theorem the structural data (i.e. the group, the partially ordered set on which the group acts and the semilattice) are essentially uniquely determined, in Takizawa's structure theorem this is not the case.

The problem whether there exists a "simplest" PL-triple determining a given E-unitary $\mathcal{R}$-unipotent semigroup led to the investigation of the homomorphisms of PL-semigroups, in particular, to the solution of the isomorphism problem of PL-semigroups (Section 2). The question of the existence of a "simplest" PL-triple is answered, in general, in the negative (Section 3).

1. Preliminaries

For a regular semigroup S, denote by E_S the set of idempotents in S. For every s in S, let $V_S(s)$ be the set of inverses of s in S. We shall denote by σ_S, or simply by σ, the least group congruence on S.

A semigroup S is called *$\mathcal{R}$-unipotent* if each $\mathcal{R}$-class of S contains a unique idempotent. Clearly, $\mathcal{R}$-unipotent semigroups are regular, and, what is more, they are orthodox by a result of Venkatesan [11]. An $\mathcal{R}$-unipotent semigroup S is termed *E-unitary* if E_S is a unitary subset in S.

For the sake of completeness we define PL-semigroups and state Takizawa's structure theorem [9]. However, it was observed in [8] that one of the conditions in the definition of a PL-triple [9] is "essentially" implied by the others. So we omit this condition from the definition. Note that triples obtained in this way were termed pre-PL-triples in [8].

By a *partial idempotent semigroup* $X=(X;\ \circ)$ we mean a partial groupoid such that (i) for any $x\in X$, $x\circ x$ $(=x^2)$ is defined and equals x, (ii) if $x\circ y$ and $y\circ z$

1980 *Mathematics Subject Classification* (1985 Revision). Primary 20M10.
Key words and phrases. E-unitary $\mathcal{R}$-unipotent semigroups, McAlister triple, P-semigroup, PL-triple, PL-semigroup.

are defined for $x, y, z \in X$ then $(x \circ y) \circ z$ and $x \circ (y \circ z)$ are also defined and are equal.

Let X be a partial idempotent semigroup. In the same way as in the case of bands, one can define Green's equivalences $\mathcal{R}$ and $\mathcal{L}$ on X and partial orders $\leqq_{\mathcal{R}}$ and $\leqq_{\mathcal{L}}$ on the set of $\mathcal{R}$-classes and $\mathcal{L}$-classes, respectively. For $x \in X$ and $Y \subseteq X$, the $\mathcal{L}$-class of x will be denoted by $x\mathcal{L}$ and the set of $\mathcal{L}$-classes $\{y\mathcal{L} : y \in Y\}$ by $Y\mathcal{L}$.

A partial idempotent semigroup X is called *$\mathcal{R}$-unipotent* if the $\mathcal{R}$-relation is trivial on X. In this case $\leqq_{\mathcal{R}}$ is considered to be a partial order on X.

Consider a triple $(G, \mathcal{X}, \mathcal{Y})$ consisting of a group G, an $\mathcal{R}$-unipotent partial idempotent semigroup $(\mathcal{X}; \circ)$ and a subband $\mathcal{Y}$ of $\mathcal{X}$ such that

(I) $\mathcal{Y}$ is an order ideal of $\mathcal{X}$ under $\leqq_{\mathcal{R}}$,

(II) G acts on $\mathcal{X}$, on the left, by automorphism in the sense that if $a \circ b$ $(a, b \in \mathcal{X})$ is defined then $ga \circ gb$ is defined for any $g \in G$ and $ga \circ gb = g(a \circ b)$,

(III) $G\mathcal{Y} = \mathcal{X}$,

(IV) for all $g \in G$, there exists $a \in \mathcal{Y}$ with $(ga)\mathcal{L} \in \mathcal{Y}\mathcal{L}$.

Such a triple will be called a *PL-triple*.

For a PL-triple $(G, \mathcal{X}, \mathcal{Y})$, define a multiplication on

$$PL(G, \mathcal{X}, \mathcal{Y}) = \{(a, g) \in \mathcal{Y} \times G : (g^{-1}a)\mathcal{L} \in \mathcal{Y}\mathcal{L}\}$$

by

$$(a, g)(b, h) = (a \circ gb, gh).$$

RESULT 1.1 ([9] and [8]). *Let $(G, \mathcal{X}, \mathcal{Y})$ be a PL-triple. Then $PL(G, \mathcal{X}, \mathcal{Y})$ is an E-unitary $\mathcal{R}$-unipotent semigroup. Moreover,*

(i) *the band of idempotents in $PL(G, \mathcal{X}, \mathcal{Y})$ is $\{(a, 1) : a \in \mathcal{Y}\}$ which is isomorphic to $\mathcal{Y}$,*

and, for every $(a, g), (b, h) \in PL(G, \mathcal{X}, \mathcal{Y})$, we have

(ii) *$(a, g)\mathcal{R}(b, h)$ if and only if $a = b$,*

(iii) *$(a, g)\mathcal{L}(b, h)$ if and only if $g^{-1}a\mathcal{L}h^{-1}b$,*

(iv) *$(a, g)\sigma(b, h)$ if and only if $g = h$.*

The E-unitary $\mathcal{R}$-unipotent semigroup $PL(G, \mathcal{X}, \mathcal{Y})$ is termed a *PL-semigroup*.

RESULT 1.2 ([9]). *Every E-unitary $\mathcal{R}$-unipotent semigroup is isomorphic to a PL-semigroup.*

The proof of this result is based on the following construction:

Let S be an E-unitary $\mathcal{R}$-unipotent semigroup. Put $G_S = S/\sigma_S$. Define a partial binary operation $\circ$ on $\mathcal{X}_S = E_S \times G_S$ such that $(a, g) \circ (b, h)$ is defined if and only if there exists $s \in S$ and $s' \in V_S(s)$ with $ss' = a$, $s\sigma_S = g^{-1}h$ and in this case let $(a, g) \circ (b, h) = (sbs', g)$. Let G_S act on $\mathcal{X}_S$ so that, for any $k \in G_S$ and $(a, g) \in \mathcal{X}_S$, we have $k(a, g) = (a, kg)$. Then $(G_S, \mathcal{X}_S, \mathcal{Y}_S)$ with $\mathcal{Y}_S = \{(e, 1) : e \in E_S\}$ is a PL-triple which we shall call the *Takizawa triple* corresponding to S, and $PL(G_S, \mathcal{X}_S, \mathcal{Y}_S)$ is isomorphic to S.

As we have mentioned, in the original definition of a PL-triple Takizawa [9] required that $(G, \mathcal{X}, \mathcal{Y})$ satisfied one more condition, namely:

(0) $\mathcal{Y}\mathcal{L}$ is an order ideal of $\mathcal{X}\mathcal{L}$ under $\leqq_{\mathcal{L}}$.

It was shown in [8] that if in a *PL*-triple $(G, \mathcal{X}, \mathcal{Y})$ the "unessential" products in $\mathcal{X}$ are omitted then the *PL*-triple obtained satisfies (0). More precisely, the following holds:

Let $(G, \mathcal{X}, \mathcal{Y})$ be a *PL*-triple. Define a partial groupoid $\mathcal{X}_m=(\mathcal{X};\ \circ_m)$ by restricting the operation on $\mathcal{X}$ in such a way that $x \circ_m y$ is defined if and only if

(1) there exists $g \in G$ such that $(gx)\mathcal{L} \in \mathcal{Y}\mathcal{L}$ and $gy \in \mathcal{Y}$ in $(\mathcal{X};\ \circ)$,

and if this is the case then let $x \circ_m y = x \circ y$. One can verify that if (1) holds for x and y then $x \circ y$ is defined in $\mathcal{X}$. Let G act on $\mathcal{X}_m$ in the same way as on $\mathcal{X}$.

A *PL*-triple $(G, \mathcal{X}, \mathcal{Y})$ is called *reduced* if, for all $x, y \in \mathcal{X}$, the product $x \circ y$ in $\mathcal{X}$ is defined if and only if (1) holds in $\mathcal{X}$. For example, the Takizawa triples are reduced.

Result 1.3 ([8]). *For any PL-triple* $(G, \mathcal{X}, \mathcal{Y})$, *the triple* $(G, \mathcal{X}_m, \mathcal{Y})$ *is a reduced PL-triple satisfying condition* (0), *and* $PL(G, \mathcal{X}, \mathcal{Y}) = PL(G, \mathcal{X}_m, \mathcal{Y})$.

The following assertion, essentially due to Takizawa [9] and modified for *PL*-triples in our sense in [8], shows a connection between *PL*-triples determining an inverse *PL*-semigroup and McAlister triples. For the definition of a McAlister triple the reader is referred to [4] or [5].

Result 1.4 ([9] and [8]). *Let* $(G, \mathcal{X}, \mathcal{Y})$ *be a PL-triple such that* $\mathcal{Y}$ *is a semilattice. Put* $\mathcal{X}' = \mathcal{X}_m/\mathcal{L}_m$ *and* $\mathcal{Y}' = \mathcal{Y}\mathcal{L}_m$ *where* $\mathcal{L}_m$ *is used to denote Green's* $\mathcal{L}$*-relation on* $\mathcal{X}_m$. *For any* $g \in G$ *and* $x\mathcal{L}_m \in \mathcal{X}'$ *we define* $g(x\mathcal{L}_m) = (gx)\mathcal{L}_m$. *Then* $(G, (\mathcal{X}';\ \leqq_{\mathcal{L}_m}), \mathcal{Y}')$ *is a McAlister triple and* $PL(G, \mathcal{X}, \mathcal{Y})$ *is isomorphic to* $P(G, \mathcal{X}', \mathcal{Y}')$.

We shall need to think of a McAlister triple as a *PL*-triple. In the following proposition we show that McAlister triples are just the "commutative" *PL*-triples.

We term a partial idempotent semigroup $X=(X,\ \circ)$ *commutative* if, for every $x, y \in X$, $x \circ y$ is defined if and only if $y \circ x$ is defined, and in this case $x \circ y = y \circ x$. A *PL*-triple $(G, \mathcal{X}, \mathcal{Y})$ is called *commutative* if $\mathcal{X}$ is commutative.

Proposition 1.5. (i) *Let* $(G, (\mathcal{X};\ \leqq), \mathcal{Y})$ *be a McAlister triple. Then* $(G, (\mathcal{X};\ \wedge_{\leqq}), \mathcal{Y})$ *with* $\wedge_{\leqq}$ *the partial operation of forming the greatest lower bound in* $(\mathcal{X};\ \leqq)$ *is a commutative PL-triple.*

(ii) *Let* $(G, (\mathcal{X};\ \circ), \mathcal{Y})$ *be a commutative PL-triple. Define a relation* $\leqq_0$ *on* $\mathcal{X}$ *by the rule that* $x \leqq_0 y$ *if and only if* $x \circ y = x$. *Then* $\leqq_0$ *is a partial order and* $(G, (\mathcal{X};\ \leqq_0), \mathcal{Y})$ *is a McAlister triple.*

(iii) *For every McAlister triple* $(G, (\mathcal{X};\ \leqq), \mathcal{Y})$, *the relation* $\leqq_{\wedge_{\leqq}}$ *is just* $\leqq$.

(iv) *For every commutative PL-triple* $(G, (\mathcal{X};\ \circ), \mathcal{Y})$, *the partial operation* $\circ$ *is a restriction of* $\wedge_{\leqq_0}$, *and, consequently,* $\circ_m = (\wedge_{\leqq_0})_m$.

Proof. (i) is proved in [9]. (ii) immediately follows if we observe that in a commutative partial idempotent semigroup $\mathcal{X}$, both $\mathcal{R}$ and $\mathcal{L}$ are trivial. Hence $\leqq_{\mathcal{R}}$ and $\leqq_{\mathcal{L}}$ are partial orders on $\mathcal{X}$ and $\leqq_{\mathcal{R}} = \leqq_{\mathcal{L}} = \leqq_0$. (iii) and (iv) can be checked by straightforward calculations.

For the undefined notions and notations the reader is referred to [1].

2. Homomorphisms of PL-semigroups

In this section we describe the homomorphisms of PL-semigroups by means of homomorph-like mappings of PL-triples.

Let S and $\bar{S}$ be two E-unitary $\mathscr{R}$-unipotent semigroups and $\Psi\colon S\to\bar{S}$ a homomorphism. Obviously, by restricting Ψ onto E_S, Ψ determines a homomorphism of E_S into $E_{\bar{S}}$ which is injective [surjective] provided Ψ is injective [surjective]. Since $\Psi\sigma_{\bar{S}}$ is a homomorphism of S into a group, there exists a unique group-homomorphism $\psi\colon S/\sigma_S\to\bar{S}/\sigma_{\bar{S}}$ making the diagram below commutative:

$$\begin{tikzcd} S \arrow[r, "\Psi"] \arrow[d, "\sigma_S^\natural"'] & \bar{S} \arrow[d, "\sigma_{\bar{S}}^\natural"] \\ S/\sigma_S \arrow[r, dashed, "\psi"] & \bar{S}/\sigma_{\bar{S}} \end{tikzcd}$$

Clearly, the surjectivity of Ψ is inherited by ψ. The fact that the same holds for injectivity is shown as follows. By a result due to Saitô [6], we have $\sigma_S=\{(s,t)\in S\times S\colon s't\in E_S$ for some/for every $s'\in V_S(s)\}$ and, similarly, for $\sigma_{\bar{S}}$. Hence if $s\Psi\sigma_{\bar{S}}t\Psi$ and $s'\in V_S(s)$ then $s'\Psi\in V_{\bar{S}}(s\Psi)$ and $(s't)\Psi=s'\Psi\cdot t\Psi\in E_{\bar{S}}$. If Ψ is injective then we obtain $s't\in E_S$ which implies that $s\sigma_S t$. Thus ψ is also injective.

Applying this argument for PL-semigroups, we can state the following.

Proposition 2.1. *Let $\Psi\colon PL(G,\mathscr{X},\mathscr{Y})\to PL((\bar{G},\bar{\mathscr{X}},\bar{\mathscr{Y}})$ be a homomorphism of PL-semigroups. Then there exist uniquely determined homomorphisms $\varphi\colon G\to\bar{G}$ and $\Theta\colon\mathscr{Y}\to\bar{\mathscr{Y}}$ such that $(a,g)\Psi=(a\Theta,g\varphi)$ for any $(a,g)\in PL(G,\mathscr{X},\mathscr{Y})$. Moreover, φ and Θ are injective [surjective] provided Ψ is injective [surjective].*

Proof. Denote the ith $(i=1,2)$ projection of $S=PL(G,\mathscr{X},\mathscr{Y})$ and $\bar{S}=PL(\bar{G},\bar{\mathscr{X}},\bar{\mathscr{Y}})$ by π_i and $\bar{\pi}_i$, respectively. Suppose the mappings $\varphi\colon G\to\bar{G}$, $\Theta\colon\mathscr{Y}\to\bar{\mathscr{Y}}$ have the property that $(a,g)\Psi=(a\Theta,g\varphi)$ for any $(a,g)\in S$. Since, by Result 1.1 (i), $\pi_1|E_S\colon E_S\to\mathscr{Y}$ and $\bar{\pi}_1|E_{\bar{S}}\colon E_{\bar{S}}\to\bar{\mathscr{Y}}$ are isomorphisms, we have

$$\Theta=(\pi_1|E_S)^{-1}(\Psi|E_S)(\bar{\pi}_1|E_{\bar{S}}). \tag{2}$$

On the other hand, by Result 1.1 (iv), the kernels of π_2 and $\bar{\pi}_2$ are just the least group congruences σ_S and $\sigma_{\bar{S}}$, respectively. Therefore

$$\varphi=\iota^{-1}\psi\bar{\iota} \tag{3}$$

where ψ is the homomorphism $S/\sigma_S\to\bar{S}/\sigma_{\bar{S}}$ induced by Ψ and $\iota\colon S/\sigma_S\to G$, $\bar{\iota}\colon\bar{S}/\sigma_{\bar{S}}\to\bar{G}$ are the isomorphisms corresponding to π_2 and $\bar{\pi}_2$, respectively. Conversely, it is clear, that the mappings Θ and φ defined by (2) and (3), respectively, are homomorphisms. Let $(a,g)\in S$ and $(a,g)\Psi=(\bar{a},\bar{g})$. By definition, we have $\bar{g}=g\varphi$. Moreover, Result 1.1 (ii) implies $(a,g)\mathscr{R}(a,1)$ and $(a,g)\Psi\mathscr{R}(\bar{a},\bar{1})$. However, the former relation ensures $(a,g)\Psi\mathscr{R}(a,1)\Psi=(a\Theta,\bar{1})$ by (2) whence we obtain $(\bar{a},\bar{1})\mathscr{R}(a\Theta,\bar{1})$. As $\bar{S}$ is $\mathscr{R}$-unipotent we conclude $\bar{a}=a\Theta$. Thus the first assertion is proved. The second assertion immediately follows from the argument preceding the proposition.

In the case of *P*-semigroups and McAlister triples, each pair of homomorphisms $\varphi: G \to \bar{G}$ and $\Theta: \mathscr{Y} \to \bar{\mathscr{Y}}$ obtained from a homomorphism $\Psi: P(G, \mathscr{X}, \mathscr{Y}) \to P(\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$ can be (uniquely) extended to a homomorphism $\eta: \mathscr{X} \to \bar{\mathscr{X}}$ by defining $(ga)\eta = g\varphi \cdot a\Theta$ for every $g \in G$ and $a \in \mathscr{Y}$.

Unfortunately, this is far not the case with *PL*-semigroups and *PL*-triples. We shall see below "non-equivalent " *PL*-triples which determine isomorphic *PL*-semigroups and both of which have no non-equivalent "isomorphic image". First of all, define the notion of a "homorphism" of *PL*-triples.

Let $(G, (\mathscr{X}; \circ), \mathscr{Y})$ and $(\bar{G}, (\bar{\mathscr{X}}; \bar{\circ}), \bar{\mathscr{Y}})$ be two *PL*-triples. Let $\varphi: G \to \bar{G}$ be a group-homomorphism and $\eta: \mathscr{X} \to \bar{\mathscr{X}}$ a homomorphism of partial groupoids in the sense that $x\eta \bar{\circ} y\eta$ is defined whenever $x \circ y$ is defined and $(x \circ y)\eta = x\eta \bar{\circ} y\eta$. The pair (φ, η) will be called a *PL-homomorphism* of $(G, \mathscr{X}, \mathscr{Y})$ into $(\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$ — in notation: $(\varphi, \eta): (G, \mathscr{X}, \mathscr{Y}) \to (\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$ — provided $\mathscr{Y}\eta \subseteq \bar{\mathscr{Y}}$ and $(gx)\eta = g\varphi \cdot x\eta$ for any $g \in G$ and $x \in \mathscr{X}$. By a *PL-monomorphism* we shall mean a *PL*-homomorphism $(\varphi, \eta): (G, \mathscr{X}, \mathscr{Y}) \to (\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$ with the property that (m) both φ and $\eta|\mathscr{Y}: \mathscr{Y} \to \bar{\mathscr{Y}}$, the restriction of η onto $\mathscr{Y}$, are injective. A *PL*-homomorphism $(\varphi, \eta): (G, \mathscr{X}, \mathscr{Y}) \to (\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$ will be termed a *PL-epimorphism* if (e) for every $\bar{g} \in \bar{G}$ and $\bar{a} \in \bar{\mathscr{Y}}$ with $(\bar{g}\bar{a})\mathscr{L} \in \bar{\mathscr{Y}}\mathscr{L}$, there exist $g \in G$ and $a \in \mathscr{Y}$ such that $g\varphi = \bar{g}$, $a\eta = \bar{a}$ and $(ga)\mathscr{L} \in \mathscr{Y}\mathscr{L}$. If a *PL*-homomorphism $(\varphi, \eta): (G, \mathscr{X}, \mathscr{Y}) \to (\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$ has the properties that (i1) both φ and $\eta|\mathscr{Y}$ are bijective, and (i2) $x\eta \mathscr{L} y\eta$ implies $x \mathscr{L} y$ for every $x, y \in \mathscr{X}$ then it is called a *PL-isomorphism.* A *PL-equivalence* is defined to be a *PL*-homomorphism $(\varphi, \eta):(G, \mathscr{X}, \mathscr{Y}) \to (\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$ where both φ and η are bijective and $x \circ y$ is defined in $\mathscr{X}$ for every x, y in $\mathscr{X}$ whenever $x\eta \bar{\circ} y\eta$ is defined in $\bar{\mathscr{X}}$.

It is easily seen that a *PL*-equivalence is necessarily a *PL*-isomorphism. Moreover, if $(\varphi, \eta):(G, \mathscr{X}, \mathscr{Y}) \to (\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$ and $(\bar{\varphi}, \bar{\eta}): (\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}}) \to (\bar{\bar{G}}, \bar{\bar{\mathscr{X}}}, \bar{\bar{\mathscr{Y}}})$ are *PL*-equivalences then both $(\varphi\bar{\varphi}, \eta\bar{\eta})$ and $(\varphi^{-1}, \eta^{-1})$ are *PL*-equivalences. So the existence of a *PL*-equivalence between *PL*-triples determines an equivalence relation on the class of all *PL*-triples. We say that two *PL*-triples are *PL-equivalent* if there exists a *PL*-equivalence of one of them into the other one.

PROPOSITION 2.2. *A PL-homomorphism is a PL-isomorphism if and only if it is both a PL-monomorphism and a PL-epimorphism.*

PROOF. Let $(\varphi, \eta):(G, \mathscr{X}, \mathscr{Y}) \to (\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$ be a *PL*-homomorphism. Suppose it to be a *PL*-isomorphism. Clearly, (i1) implies (m). If $\bar{g} \in \bar{G}, \bar{a}, \bar{b} \in \bar{\mathscr{Y}}$ with $\bar{g}\bar{a} \mathscr{L} \bar{b}$ then, by (i1), there exist $g \in G$, $a, b \in \mathscr{Y}$ with $g\varphi = \bar{g}$, $a\eta = \bar{a}$ and $b\eta = \bar{b}$. Here $(ga)\eta = g\varphi \cdot a\eta = \bar{g}\bar{a} \mathscr{L} \bar{b} = b\eta$ which implies by (i2) that $ga \mathscr{L} b$ in $\mathscr{X}$. Thus (e) also follows.

Conversely, suppose that (φ, η) satisfies (m) and (e). First of all, observe that (e) implies both φ and $\eta|\mathscr{Y}$ to be surjective. On the one hand, apply (e) for any $\bar{g} \in \bar{G}$ and an $\bar{a} \in \bar{\mathscr{Y}}$ with $(\bar{g}\bar{a})\mathscr{L} \in \bar{\mathscr{Y}}\mathscr{L}$ which exists by (IV), and, on the other hand, apply (e) for $\bar{1} \in \bar{G}$ and any $\bar{a} \in \bar{\mathscr{Y}}$. Hence, by (m), we deduce that (i1) holds. Now assume that $x, y \in \mathscr{X}$ and $x\eta \mathscr{L} y\eta$ in $\bar{\mathscr{X}}$. By (III), we have $x = ga$ and $y = hb$ for some $g, h \in G$ and $a, b \in \mathscr{Y}$. Thus $g\varphi \cdot a\eta = (ga)\eta \mathscr{L} (hb)\eta = h\varphi \cdot b\eta$ whence

$(h^{-1}g)\varphi\cdot a\eta\mathcal{L}b\eta$ follows where $b\eta\in\bar{\mathcal{Y}}$. Applying (e) we obtain $k\in G$, $c, d\in\mathcal{Y}$ such that $k\varphi=(h^{-1}g)\varphi$, $c\eta=a\eta$ and $kc\mathcal{L}d$ in $\mathcal{X}$. Property (m) implies $k=h^{-1}g$ and $c=a$. Thus $(h^{-1}g)a\mathcal{L}d$. Since η is a homomorphism we infer that $(h^{-1}g)\varphi\cdot a\eta= =((h^{-1}g)a)\eta\mathcal{L}d\eta$, that is, $b\eta\mathcal{L}d\eta$. Making use of the fact that $\eta|\mathcal{Y}$ is an injective homomorphism by (m), we deduce that $b\mathcal{L}d$ in $\mathcal{Y}$. Therefore $(h^{-1}g)a\mathcal{L}b$ and hence $x=ga\mathcal{L}hb=y$. Thus (i2) also holds which completes the proof.

REMARK. It turned out in the previous proof that both φ and $\eta|\mathcal{Y}$ are surjective provided (φ,η) is a *PL*-epimorphism. This clearly implies by (III) that if (φ,η) is a *PL*-epimorphism or, in particular, a *PL*-isomorphism then η is surjective.

Given a *PL*-homomorphism $(\varphi,\eta):(G,\mathcal{X},\mathcal{Y})\to(\bar{G},\bar{\mathcal{X}},\bar{\mathcal{Y}})$ we can define a mapping $\Phi_{\varphi,\eta}: PL(G,\mathcal{X},\mathcal{Y})\to PL(\bar{G},\bar{\mathcal{X}},\bar{\mathcal{Y}})$ of *PL*-semigroups in a natural way: for each $(a,g)\in PL(G,\mathcal{X},\mathcal{Y})$, let $(a,g)\Phi_{\varphi,\eta}=(a\eta, g\varphi)$. Here $(a\eta,g\varphi)$ is indeed in $PL(\bar{G},\bar{\mathcal{X}},\bar{\mathcal{Y}})$ as $(a,g)\in PL(G,\mathcal{X},\mathcal{Y})$ implies that there exists $b\in\mathcal{Y}$ with $g^{-1}a\mathcal{L}b$ whence it follows that $(g\varphi)^{-1}a\eta=(g^{-1}a)\eta\mathcal{L}b\eta\in\bar{\mathcal{Y}}$.

PROPOSITION 2.3. (i) *For every PL-homomorphism* $(\varphi,\eta)\colon(G,\mathcal{X},\mathcal{Y})\to(\bar{G},\bar{\mathcal{X}},\bar{\mathcal{Y}})$, *the mapping* $\Phi_{\varphi,\eta}$ *is a homomorphism of* $PL(G,\mathcal{X},\mathcal{Y})$ *into* $PL(\bar{G},\bar{\mathcal{X}},\bar{\mathcal{Y}})$.
Moreover,

(ii) $\Phi_{\varphi,\eta}$ *is injective if and only if* (φ,η) *is a PL-monomorphism;*
(iii) $\Phi_{\varphi,\eta}$ *is surjective if and only if* (φ,η) *is a PL- epimorphism;*
(iv) $\Phi_{\varphi,\eta}$ *is an isomorphism if and only if* (φ,η) *is a PL-isomorphism.*

PROOF. (i) and (iii) are immediate. The "if" part of (ii) is also clear and its converse part follows from Proposition 2.1. Statement (iv) is implied by (ii), (iii) and Proposition 2.2.

We shall say that $\Phi_{\varphi,\eta}$ is the homomorphism induced by the *PL*-homomorphism (φ,η).

As we have already mentioned, not all homomorphism of *PL*-semigroups are induced by *PL*-homomorphisms. The following simple example shows how much more the case is complicated.

EXAMPLE. Let $G=\{1,g\}$ be a cyclic group of order 2 and $Z=\{e,f\}$ a left zero semigroup. Clearly, the left group $S=Z\times G$ is an E-unitary $\mathcal{R}$-unipotent semigroup. Consider the *PL*-triples $(G,Z,Z)_1$ and $(G,Z,Z)_2$ where the only difference between them is that in $(G,Z,Z)_1$ the action of G on Z is defined trivially, that is, $g\cdot_1 z=z$ for every $z\in Z$, and in $(G,Z,Z)_2$ the action is defined by $g\cdot_2 e=f$, $g\cdot_2 f=e$. Obviously, we have $S=PL(G,Z,Z)_1=PL(G,Z,Z)_2$.

Now we verify that $(G,Z,Z)_1$ and $(G,Z,Z)_2$ are not *PL*-equivalent. Suppose $(\varphi,\eta)\colon(G,Z,Z)_1\to(G,Z,Z)_2$ to be a *PL*-equivalence. Then φ is the identity automorphism of G and η is either the identity automorphism ι of Z or the automorphism ζ interchanging the elements of Z. Since $(g\cdot_1 e)\iota=e\iota=e\neq f=g\cdot_2 e=g\varphi\cdot_2 e\iota$ and $(g\cdot_1 e)\zeta= =e\zeta=f\neq e=g\cdot_2 f=g\varphi\cdot_2 e\zeta$, neither (φ,ι) nor (φ,ζ) is a *PL*-equivalence which contradicts our assumption.

Finally, we notice that each *PL*-isomorphism of $(G,Z,Z)_i$ $(i=1,2)$ into a *PL*-triple is trivially a *PL*-equivalence.

Now we turn to describing homomorphisms of *PL*-semigroups by means of *PL*-homomorphisms of *PL*-triples. Let $(G, \mathscr{X}, \mathscr{Y})$ and $(\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$ be *PL*-triples and $\Psi: PL(G, \mathscr{X}, \mathscr{Y}) \to PL(\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$ a homomorphism. If Ψ is not induced by a *PL*-homomorphism then it is natural to try to find a *PL*-triple $(\tilde{G}, \tilde{\mathscr{X}}, \tilde{\mathscr{Y}})$, a *PL*-isomorphism (φ, η) and a *PL*-homomorphism $(\bar{\varphi}, \bar{\eta})$ of $(\tilde{G}, \tilde{\mathscr{X}}, \tilde{\mathscr{Y}})$ into $(G, \mathscr{X}, \mathscr{Y})$ and $(\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$, respectively, such that we have $\Psi = \Phi_{\varphi,\eta}^{-1}\Phi_{\bar{\varphi},\bar{\eta}}$.

$$\begin{tikzcd}[column sep=small] & (\tilde{G}, \tilde{\mathscr{X}}, \tilde{\mathscr{Y}}) \arrow[dl, dashed, "(\varphi,\eta)"'] \arrow[dr, dashed, "(\bar{\varphi},\bar{\eta})"] & \\ (G, \mathscr{X}, \mathscr{Y}) & & (\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}}) \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & PL(\tilde{G}, \tilde{\mathscr{X}}, \tilde{\mathscr{Y}}) \arrow[dl, dashed, "\Phi_{\varphi,\eta}"'] \arrow[dr, dashed, "\Phi_{\bar{\varphi},\bar{\eta}}"] & \\ PL(G, \mathscr{X}, \mathscr{Y}) \arrow[rr, "\Psi"'] & & PL(\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}}) \end{tikzcd}$$

If this is the case then $(a, g)\Psi = \bigl(a(\eta|\tilde{\mathscr{Y}})^{-1}(\bar{\eta}|\tilde{\mathscr{Y}}), g\varphi^{-1}\bar{\varphi}\bigr)$ for every $(a, g) \in PL(G, \mathscr{X}, \mathscr{Y})$, that is, Ψ can be obtained from (φ, η) and $(\bar{\varphi}, \bar{\eta})$ in a fairly simple way. The main result of this section states that this idea can be carried over for any homomorphisms of *PL*-semigroups.

THEOREM 2.4. *Let $(G, \mathscr{X}, \mathscr{Y})$ and $(\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$ be two PL-triples. Denote by S and $\bar{S}$ the respective PL-semigroups $PL(G, \mathscr{X}, \mathscr{Y})$ and $PL(\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$. Then, for every homomorphism $\Psi: S \to \bar{S}$, there exist a PL-triple $(\tilde{G}, \tilde{\mathscr{X}}, \tilde{\mathscr{Y}})$, a PL-isomorphism $(\varphi, \eta): (\tilde{G}, \tilde{\mathscr{X}}, \tilde{\mathscr{Y}}) \to (G, \mathscr{X}, \mathscr{Y})$ and a PL-homomorphism $(\bar{\varphi}, \bar{\eta}): (\tilde{G}, \tilde{\mathscr{X}}, \tilde{\mathscr{Y}}) \to (\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$ such that $\Psi = \Phi_{\varphi,\eta}^{-1}\Phi_{\bar{\varphi},\bar{\eta}}$. Moreover, Ψ is injective [surjective, an isomorphism] if and only if $(\bar{\varphi}, \bar{\eta})$ is a PL-monomorphism [PL-epimorphism, PL-isomorphism].*

REMARK. The *PL*-triple $(\tilde{G}, \tilde{\mathscr{X}}, \tilde{\mathscr{Y}})$ can be chosen to be the Takizawa triple corresponding to S.

PROOF. Let $\tilde{G} = G$. Define $\tilde{\mathscr{X}} = (\mathscr{Y} \times G; \tilde{\circ})$ to be the partial groupoid with the following partial operation: for any $a, b \in \mathscr{Y}$ and $g, h \in G$, the product $(a, g)\tilde{\circ}(b, h)$ is defined if and only if $((h^{-1}g)a)\mathscr{L} \in \mathscr{Y}\mathscr{L}$ in $\mathscr{X}$, and if this holds then $(a, g)\tilde{\circ}(b, h) = \bigl(a \circ (g^{-1}h)b, g\bigr)$. Moreover, let $\tilde{\mathscr{Y}} = \{(a, 1): a \in \mathscr{Y}\}$ and define the action of $\tilde{G}$ on $\tilde{\mathscr{X}}$ by $k(a, g) = (a, kg)$ for every $k \in \tilde{G}$ and $(a, g) \in \tilde{\mathscr{X}}$. Observe that $((h^{-1}g)a)\mathscr{L} \in \mathscr{Y}\mathscr{L}$ if and only if $(a, g^{-1}h) \in S$ where the latter is equivalent to saying that there exist $s \in S$ and $s' \in V_S(s)$ with $ss' = (a, 1)$ and $s\sigma_S = g^{-1}h$. Furthermore, in this case $(a, g^{-1}h)(b, 1)s' = \bigl(a \circ (g^{-1}h)b, 1\bigr)$ in S for any $s' \in V_S\bigl((a, g^{-1}h)\bigr)$. Therefore $(\tilde{G}, \tilde{\mathscr{X}}, \tilde{\mathscr{Y}})$ is a *PL*-triple and (ν, ε) with ν the natural isomorphism of G_S onto $G = \tilde{G}$ and $\varepsilon: \mathscr{X}_S \to \tilde{\mathscr{X}}$ defined by $((a, 1), g)\varepsilon = (a, g)$ is a *PL*-equivalence of the Takizawa triple $(G_S, \mathscr{X}_S, \mathscr{Y}_S)$ of S onto $(\tilde{G}, \tilde{\mathscr{X}}, \tilde{\mathscr{Y}})$. We have chosen to work with the latter *PL*-triple because the calculations are simpler in this way. Define $\eta: \tilde{\mathscr{X}} \to \mathscr{X}$ by $(a, g)\eta = ga$ for every $g \in G$, $a \in \mathscr{Y}$. Making use of the fact that, for any x, y in $\mathscr{X}$, the product $x \circ y$ exists provided (1) holds, we immediately obtain that (ι, η) with ι the identity automorphism of $\tilde{G} = G$ is a *PL*-homomorphism of $(\tilde{G}, \tilde{\mathscr{X}}, \tilde{\mathscr{Y}})$ into $(G, \mathscr{X}, \mathscr{Y})$. Since $\Phi_{\nu,\varepsilon}$ is an isomorphism by Proposition 2.3 (iv) and $\Phi_{\nu,\varepsilon}\Phi_{\iota,\eta} = \Phi_{\nu,\varepsilon\eta}$ is the isomorphism of $PL(G_S, \mathscr{X}_S, \mathscr{Y}_S)$ onto S used by Takizawa, we infer $\Phi_{\iota,\eta}$ to be an isomorphism. Hence, by Proposition 2.3 (iv), (ι, η) is a PL-isomorphism.

Now let us define $\bar\varphi\colon G \to \bar G$ and $\Theta\colon \mathscr{Y} \to \bar{\mathscr{Y}}$ to be the homomorphisms induced by $\Psi\colon S \to \bar S$ (see Proposition 2.1) and $\bar\eta\colon \tilde{\mathscr{X}} \to \bar{\mathscr{X}}$ to be the mapping assigning $g\bar\varphi \cdot a\Theta$ to (a, g) for every $(a, g) \in \tilde{\mathscr{X}}$. We intend to verify that $(\bar\varphi, \bar\eta)$ is a PL-homomorphism. It is immediate by definition that $\tilde{\mathscr{Y}}\bar\eta \subseteq \bar{\mathscr{Y}}$ and $(h(a, g))\bar\eta = h\bar\varphi(a, g)\bar\eta$ holds for every $h \in \tilde G$ and $(a, g) \in \tilde{\mathscr{X}}$. Suppose $(a, g), (b, h) \in \tilde{\mathscr{X}}$ such that $(a, g)\tilde\circ(b, h)$ is defined. Then there exists $c \in \mathscr{Y}$ with $(h^{-1}g)a\mathscr{L}c$ whence it follows that $(a, g^{-1}h) \in S$ and $(a, g^{-1}h)\mathscr{L}(c, 1)$ in S by Result 1.1 (iii). This implies $(a\Theta, (g^{-1}h)\bar\varphi) = (a, g^{-1}h)\Psi\mathscr{L}(c, 1)\Psi = (c\Theta, \bar 1)$ in $\bar S$. Consequently, by Result 1.1 (iii), we have $(h^{-1}g)\bar\varphi a\Theta \mathscr{L} c\Theta$ in $\bar{\mathscr{X}}$. Then, again applying the fact that if (1) holds for some elements x, y in a PL-triple then their product is defined, we deduce that $((h^{-1}g)\bar\varphi a\Theta)\bar\circ b\Theta$ is defined in $\bar{\mathscr{X}}$ and therefore $(g\bar\varphi a\Theta)\bar\circ(h\bar\varphi b\Theta)$ is also defined. Moreover, we have

$$((a, g)\tilde\circ(b, h))\bar\eta = g\bar\varphi\big(a\circ(g^{-1}h)b\big)\Theta \tag{4}$$

and

$$(a, g)\bar\eta\bar\circ(b, h)\bar\eta = g\bar\varphi(a\Theta\bar\circ(g^{-1}h)\bar\varphi b\Theta). \tag{5}$$

In order to prove that the right-hand sides of (4) and (5) are equal, consider the first component of the image of the product $(a, g^{-1}h)(b, 1)$ in S under Ψ. We have

$$\begin{aligned}\big((a\circ(g^{-1}h)b)\Theta, (g^{-1}h)\bar\varphi\big) &= \big(a\circ(g^{-1}h)b, g^{-1}h\big)\Psi = \big((a, g^{-1}h)(b, 1)\big)\Psi = \\ &= (a, g^{-1}h)\Psi \cdot (b, 1)\Psi = (a\Theta, (g^{-1}h)\bar\varphi)(b\Theta, \bar 1) = \\ &= \big(a\Theta\bar\circ((g^{-1}h)\bar\varphi \cdot b\Theta), (g^{-1}h)\bar\varphi\big).\end{aligned}$$

This completes the proof of the fact that $(\bar\varphi, \bar\eta)$ is a PL-homomorphism.

The equality of mappings $\Psi = \Phi_{\iota,\eta}^{-1}\Phi_{\bar\varphi,\bar\eta}$ is obvious by the definitions. The last assertion of the theorem follows from Proposition 2.3.

An immediate consequence of Theorem 2.4 is the following solution of the isomorphism problem of PL-semigroups.

COROLLARY 2.5. *The PL-semigroups $PL(G, \mathscr{X}, \mathscr{Y})$ and $PL(\bar G, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$ are isomorphic to each other if and only if there exist a PL-triple $(\tilde G, \tilde{\mathscr{X}}, \tilde{\mathscr{Y}})$ and two PL-isomorphisms of $(\tilde G, \tilde{\mathscr{X}}, \tilde{\mathscr{Y}})$ into the PL-triples $(G, \mathscr{X}, \mathscr{Y})$ and $(\bar G, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$, respectively.*

3. On the class of PL-triples determining a given E-unitary $\mathscr{R}$-unipotent semigroup

In this section we apply the notions of a PL-homomorphism, PL-isomorphism and PL-equivalence to show that, in the class $\mathscr{PL}_S$ of all reduced PL-triples determining a given E-unitary $\mathscr{R}$-unipotent semigroup S, the Takizawa triple corresponding to S is the "free" PL-triple in the sense that each PL-triple in $\mathscr{PL}_S$ is its PL-isomorphic image. Moreover, if S is an inverse semigroup then the McAlister triple corresponding to S is the "simplest" element of $\mathscr{PL}_S$ in the sense that it is a PL-isomorphic image of all PL-triples in $\mathscr{PL}_S$.

Let S be an E-unitary $\mathscr{R}$-unipotent semigroup. Denote by $\mathscr{PL}_S$ the class of all reduced PL-triples which determine a PL-semigroup isomorphic to S. Let us define a quasi-order $\precsim$ on $\mathscr{PL}_S$ in such a way that $(G, \mathscr{X}, \mathscr{Y}) \precsim (\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$ if and only if there exists a PL-isomorphism of $(\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$ into $(G, \mathscr{X}, \mathscr{Y})$. The equivalence relation corresponding to $\precsim$ will be denoted by $\equiv$. Clearly, we have $(G, \mathscr{X}, \mathscr{Y}) \equiv (\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$ provided $(G, \mathscr{X}, \mathscr{Y})$ and $(\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$ are PL-equivalent. However, the following example shows that there can exist PL-triples in $\mathscr{PL}_S$ which are not PL-equivalent but $\equiv$-related, even if S is an inverse semigroup.

Example. Denote by I the meet-semilattice of integers and by $\mathbf{Z}$ the additive group of integers. Let S be the direct product $I \times \mathbf{Z}$ which is clearly an E-unitary inverse semigroup. One can easily check that the reduced PL-triple $(G_S, \mathscr{X}_S, \mathscr{Y}_S)$ is PL-equivalent to the PL-triple $(\mathbf{Z}, \mathscr{X}, \mathscr{Y})$ defined as follows. Let $\mathscr{X} = (I \times \mathbf{Z}; \circ)$ with the following (full) multiplication: $(i, s) \circ (j, t) = (i \wedge j, s)$ for every (i, s), $(j, t) \in \mathscr{X}$. Define the action of $\mathbf{Z}$ on $\mathscr{X}$ by $t(i, s) = (i, t+s)$ for every $t \in \mathbf{Z}$ and $(i, s) \in \mathscr{X}$. Let $\mathscr{Y} = \{(i, 0) : i \in I\}$. Note that $(i, s) \mathscr{L} (j, t)$ in $\mathscr{X}$ if and only if $i = j$.

Define an equivalence relation ϱ on $\mathscr{X}$ such that, for any (i, s), $(j, t) \in \mathscr{X}$, we have $(i, s) \varrho (j, t)$ if and only if either $i = j$ and $s = t$ or $i = j < 0$. One can immediately verify that ϱ is compatible both with the multiplication and the action. Thus we can define a multiplication and an action of $\mathbf{Z}$ on $\mathscr{X}^\varrho = \mathscr{X}/\varrho$ in the natural way. Observe that ϱ separates the elements of $\mathscr{Y}$. Therefore we easily see that $(\mathbf{Z}, \mathscr{X}^\varrho, \mathscr{Y}^\varrho)$ with $\mathscr{Y}^\varrho = \{a\varrho : a \in \mathscr{Y}\}$ is a PL-triple belonging to $\mathscr{PL}_S$.

Similarly, we can define an equivalence relation τ on $\mathscr{X}$ in such a way that, for any (i, s) $(j, t) \in \mathscr{X}$, we have $(i, s) \tau (j, t)$ if and only if either $i = j = 0$ and $s \equiv t$ (mod 2) or $(i, s) \varrho (j, t)$. Obviously, τ is also compatible both with the multiplication and the action, and thus $(\mathbf{Z}, \mathscr{X}^\tau, \mathscr{Y}^\tau)$ also belongs to $\mathscr{PL}_S$. Observe that $|\{t((i, s)\varrho) : t \in \mathbf{Z}\}|$ is equal to 1 or is infinite according to whether $i < 0$ or not, while $|\{t((0, s)\tau) : t \in \mathbf{Z}\}| = 2$. Therefore $(\mathbf{Z}, \mathscr{X}^\varrho, \mathscr{Y}^\varrho)$ and $(\mathbf{Z}, \mathscr{X}^\tau, \mathscr{Y}^\tau)$ are not PL-equivalent. However, define $\eta_1 : \mathscr{X}^\varrho \to \mathscr{X}^\tau$ by $(i, s)\varrho \mapsto (i, s)\tau$ and $\eta_2 : \mathscr{X}^\tau \to \mathscr{X}^\varrho$ by $(i, s)\tau \mapsto (i-1, s)\varrho$ for every $(i, s) \in \mathscr{X}$. Denote the identity automorphism of $\mathbf{Z}$ by ι. It is not difficult to check that both (ι, η_1) and (ι, η_2) are PL-isomorphisms, that is, $(\mathbf{Z}, \mathscr{X}^\varrho, \mathscr{Y}^\varrho) \equiv (\mathbf{Z}, \mathscr{X}^\tau, \mathscr{Y}^\tau)$.

Theorem 3.1. *Let S be an E-unitary $\mathscr{R}$-unipotent semigroup. For any PL-triple $(G, \mathscr{X}, \mathscr{Y})$ in $\mathscr{PL}_S$, we have $(G, \mathscr{X}, \mathscr{Y}) \precsim (G_S, \mathscr{X}_S, \mathscr{Y}_S)$ and, moreover, $(G, \mathscr{X}, \mathscr{Y}) \equiv (G_S, \mathscr{X}_S, \mathscr{Y}_S)$ if and only if they are PL-equivalent.*

Proof. In the first half of the proof of Theorem 2.4 we have seen that $(G, \mathscr{X}, \mathscr{Y}) \precsim (\tilde{G}, \tilde{\mathscr{X}}, \tilde{\mathscr{Y}})$ and the latter is PL-equivalent to the Takizawa triple $(G_S, \mathscr{X}_S, \mathscr{Y}_S)$. This proves the first assertion. The second statement easily follows from the fact that in $(G_S, \mathscr{X}_S, \mathscr{Y}_S)$ we have $ga = hb$ for some $g, h \in G_S$ and $a, b \in \mathscr{Y}_S$ if and only if $g = h$ and $a = b$. For this property implies by (III) that in each PL-isomorphism $(\varphi, \eta) : (G, \mathscr{X}, \mathscr{Y}) \to (G_S, \mathscr{X}_S, \mathscr{Y}_S)$, the mapping η is injective.

In the sequel let S be an E-unitary inverse semigroup. By McAlister's well-known result, if $(G, (\mathscr{X}; \leqq), \mathscr{Y})$ and $(\bar{G}, (\bar{\mathscr{X}}; \leqq), \bar{\mathscr{Y}})$ are McAlister triples which define P-semigroups isomorphic to S then there exists a group-isomorphism $\varphi: G \to \bar{G}$ and a bijective isotone mapping $\eta: \mathscr{X} \to \bar{\mathscr{X}}$ such that η maps $\mathscr{Y}$ onto $\bar{\mathscr{Y}}$ and $(gx)\eta = g\varphi \cdot x\eta$ for every $g \in G$ and $x \in \mathscr{X}$. Obviously, (φ, η) is a PL-equivalence between the commutative PL-triples $(G, (\mathscr{X}; \wedge_{\leqq}), \mathscr{Y})$ and $(\bar{G}, (\bar{\mathscr{X}}; \wedge_{\leqq}), \bar{\mathscr{Y}})$, and, in particular, between the reduced commutative PL-triples $(G, (\mathscr{X}; (\wedge_{\leqq})_m), \mathscr{Y})$ and $(\bar{G}, (\bar{\mathscr{X}}; (\wedge_{\leqq})_m), \bar{\mathscr{Y}})$. Thus we infer that, up to PL-equivalence, there exists a unique commutative PL-triple in $\mathscr{PL}_S$.

Let $(G, \mathscr{X}, \mathscr{Y}) \in \mathscr{PL}_S$. Denote $(\mathscr{X}/\mathscr{L}; (\wedge_{\leqq_{\mathscr{L}}})_m)$ by $\mathscr{X}'$ and $\{a\mathscr{L}: a \in \mathscr{Y}\}$ by $\mathscr{Y}'$. Results 1.3, 1.4 and Proposition 1.5 ensure that $(G, \mathscr{X}', \mathscr{Y}') \in \mathscr{PL}_S$. The proof of Result 1.4 is based on the fact which can be formulated in our terminology such that (ι, η) with ι the identity automorphism of G and η the natural mapping of $\mathscr{X}$ onto $\mathscr{X}/\mathscr{L}$ is a PL-isomorphism of $(G, \mathscr{X}, \mathscr{Y})$ into $(G, \mathscr{X}', \mathscr{Y}')$. Since there exists an, up to PL-equivalence, unique commutative PL-triple in $\mathscr{PL}_S$ — denote it by M_S —, we have shown that $M_S \preceq (G, \mathscr{X}, \mathscr{Y})$ for any $(G, \mathscr{X}, \mathscr{Y})$ in $\mathscr{PL}_S$.

Finally, we note that every PL-homomorphism $(\bar{\varphi}, \bar{\eta}):(G, \mathscr{X}, \mathscr{Y}) \to (\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$ into a commutative PL-triple can be (uniquely) factorized through $(\iota, \eta):(G, \mathscr{X}, \mathscr{Y}) \to \to (G, \mathscr{X}', \mathscr{Y}')$.

Let $(\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$ be any commutative PL-triple and $(\bar{\varphi}, \bar{\eta})$ a PL-homomorphism of $(G, \mathscr{X}, \mathscr{Y})$ into $(\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$. Define $\eta': \mathscr{X}' \to \bar{\mathscr{X}}$ such that $(x\mathscr{L})\eta' = x\bar{\eta}$ for any $x \in \mathscr{X}$. If $x \mathscr{L} y$ in $\mathscr{X}$ then, by commutativity, we have $x\bar{\eta} = (x \circ y)\bar{\eta} = x\bar{\eta} \circ y\bar{\eta} = y\bar{\eta} \circ x\bar{\eta} = (y \circ x)\bar{\eta} = y\bar{\eta}$. Hence η' is, indeed, a mapping. Making use of the facts that $(G, \mathscr{X}', \mathscr{Y}')$ is reduced and both (ι, η) and $(\bar{\varphi}, \bar{\eta})$ are PL-homomorphisms, it is easily seen that $(\bar{\varphi}, \eta'):(G, \mathscr{X}', \mathscr{Y}') \to (\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$ is a PL-homomorphism. Thus $(\bar{\varphi}, \bar{\eta}) = (\iota, \eta)(\bar{\varphi}, \eta')$. Unicity of $(\bar{\varphi}, \eta')$ is clear.

We can sum up what we have proved as follows:

THEOREM 3.2. *Let S be an E-unitary inverse semigroup and $(G, \mathscr{X}, \mathscr{Y}) \in \mathscr{PL}_S$. Define $\mathscr{X}' = (\mathscr{X}/\mathscr{L}; (\wedge_{\leqq_{\mathscr{L}}})_m)$ and $\mathscr{Y}' = \{a\mathscr{L}: a \in \mathscr{Y}\}$. Let G act on $\mathscr{X}'$ in the way that $g(x\mathscr{L}) = (gx)\mathscr{L}$ for any $g \in G$ and $x \in \mathscr{X}$. Then $(G, \mathscr{X}', \mathscr{Y}')$ is the, up to equivalence, unique reduced commutative PL-triple in $\mathscr{PL}_S$. Moreover, $(\iota, \eta):(G, \mathscr{X}, \mathscr{Y}) \to (G, \mathscr{X}', \mathscr{Y}')$ with ι the identity automorphism of G and η the natural mapping of $\mathscr{X}$ onto $\mathscr{X}'$ is a PL-isomorphism which has the property that, for any commutative PL-triple $(\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$ and any PL-homomorphism $(\bar{\varphi}, \bar{\eta}):(G, \mathscr{X}, \mathscr{Y}) \to (\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$, there exists a unique PL-homomorphism $(\varphi', \eta'):(G, \mathscr{X}', \mathscr{Y}') \to (\bar{G}, \bar{\mathscr{X}}, \bar{\mathscr{Y}})$ such that $(\bar{\varphi}, \bar{\eta}) = (\iota, \eta)(\varphi', \eta')$.*

$$
\begin{tikzcd}
(G,\mathscr{X},\mathscr{Y}) \arrow[r, "(\bar{\varphi},\bar{\eta})"] \arrow[d, "(\iota,\eta)"'] & (\bar{G},\bar{\mathscr{X}},\bar{\mathscr{Y}}) \\
(G,\mathscr{X}',\mathscr{Y}') \arrow[ur, dashed, "(\varphi',\eta')"'] &
\end{tikzcd}
$$

COROLLARY 3.3. *Let S be an E-unitary inverse semigroup. For every PL-triple $(G, \mathscr{X}, \mathscr{Y})$ in $\mathscr{PL}_S$, we have $M_S \preceq (G, \mathscr{X}, \mathscr{Y})$ where M_S is used to denote the up to PL-equivalence unique commutative PL-triple in $\mathscr{PL}_S$.*

This corollary says that M_S which is, essentially, the McAlister triple corresponding to S is the "simplest" PL-triple in $\mathcal{PL}_S$, that is, the least element of $\mathcal{PL}_S$ with respect to the quasi-order $\preceq$, provided S is an E-unitary inverse semigroup. If S is an E-unitary $\mathcal{R}$-unipotent semigroup then, in general, there does not exist a least element in $\mathcal{PL}_S$. The example in Section 2 shows that the PL-triples $(G, Z, Z)_1$ and $(G, Z, Z)_2$ are uncomparable minimal elements in $\mathcal{PL}_{Z\times G}$ with respect to the quasi-order $\preceq$.

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(Received June 17, 1985)

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STABLE TRANSVERSALS AND STOCHASTIC FUNCTIONS IN POLYOMINOES

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Abstract

We show that not every polyomino has a stochastic function (a labelling of its cells by non-negative real numbers so that the labels in every maximal rectangle sum to 1). We also show that determining whether a polyomino has a stochastic function can be done in polynomial time, but that determining whether it has a stable transversal (a stochastic function with integer labels) is NP-complete. This settles some open questions posed by Berge, Chen, Chvatal, and Seow.

1. Introduction

A *polyomino* is a finite set of cells in the infinite planar square grid. Polyominoes have an ancient tradition as a game or puzzle [4], but recently they have attained new importance in digital image processing and in circuit design. An image or a circuit layout can be thought of as a polyomino for some purposes, and combinatorial properties of polyominoes, such as the minimum number of rectangles whose union equals a given polyomino, influence the efficiency with which an image or circuit can be represented or processed in some way.

Berge et al. surveyed many combinatorial results about polyominoes, and posed many more open questions [2]. This paper answers two of these questions, as well as a third related question that Berge et al. did not explicitly pose.

Any polyomino can be thought of as a hypergraph in a natural way. The cells of the polyomino correspond to the vertices of a hypergraph; its maximal rectangles correspond to the edges. (A maximal rectangle of a polyomino P is simply any rectangle contained in P that is not strictly contained in some larger rectangle within P.) Using the language of hypergraphs, define a *transversal* of a polyomino P to be a set of cells of P that has at least one cell in common with each maximal rectangle of P. The set is a *stable transversal* if it contains exactly one cell in common with each maxima rectangle.

Equivalently, a stable transversal is a function X of the cells of P that maps each cell c to $\{0, 1\}$ in such a way that

$$\sum_{c \in R} X(c) = 1$$

Research partially supported by National Science Foundation Grant MCS 8203238.

1980 *Mathematics Subject Classification.* Primary 05B50; Secondary 05C65, 68C25.
Key words and phrases. Polyomino, stochastic function, stable transversal, NP-complete, circuit design.

for each maximal rectangle R in P. Allowing X to take on non-integer values yields a *stochastic function*, namely a function X mapping the cells of P to $[0, 1]$ such that

$$\sum_{c \in R} X(c) = 1 \tag{1}$$

for each maximal rectangle R in P. Clearly a stable transversal is a special case of a stochastic function.

Berge et al. give an example of a polyomino that has no stable transversal, although it does have a stochastic function. They pose the following open questions:

Q1 Is there a polyomino with no stochastic function?

Q2 How difficult is it to determine whether a polyomino has a stable transversal? If the answer to the first question is "yes", a third question follows naturally:

Q3 How difficult is it to determine whether a polyomino has a stochastic function?

We provide the answers "yes", "NP-complete", and "polynomial", respectively, to the three questions.

2. Notation and a useful lemma

Let P be a polyomino equipped with a stochastic function X. Number the n cells of P with the integers $\{1, 2, \ldots, n\}$. A *top cell* of P is a cell whose upper neighbor is not in P. Similarly, a *bottom* (respectively *left, right*) *cell* is a cell in P whose lower (respectively left, right) neighbor is not in P. A rectangle is determined by the locations of two diagonally opposite corners, so let $\langle a, b\rangle$ denote the rectangle with corners numbered a and b. (In most cases, we will only refer to $\langle a, b\rangle$ when it is a maximal rectangle of P with a as its upper left corner.) Let x_a denote the value of X at cell a, and let $x_{\langle a, b\rangle}$ denote

$$\sum_{c \in \langle a, b\rangle} X(c).$$

In Figure 1, the top cells are 1, 2, and 5; the right cells are 2, 5, and 7; and the maximal rectangles are $\langle 1, 4\rangle$, $\langle 2, 6\rangle$, $\langle 3, 5\rangle$, and $\langle 4, 7\rangle$. Thus we must have $x_{\langle 2,6\rangle} = x_2 + x_4 + x_6 = 1$.

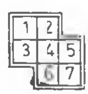


Fig. 1. A simple polyomino

The following lemma formalizes and slightly generalizes an arguments used by Berge et al. to exhibit a polyomino with no stable transversal.

LEMMA 1 (Berge et al. [2]). *Let P be a polyomino with stochastic function X. Suppose R_1 (with corners a, b, c, and d, reading clockwise from upper left) and R_2 (with corners e, f, g, and h) are rectangles in P such that*

(i) $c \in R_2$ *and* $e \in R_1$,

(ii) $x_{\langle a,c\rangle} = x_{\langle e,g\rangle} = 1$,

(iii) $x_{\langle b,h\rangle} = x_{\langle d,f\rangle} = 1$.

Then $x_q = 0$ *for all cells* q *in* $(R_1 \cup R_2) - (\langle b, h\rangle \cup \langle d, f\rangle)$. (See Figure 2.)

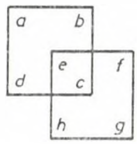


Fig. 2. Overlapping rectangles

PROOF. By (i), $\langle b, h\rangle$ and $\langle d, f\rangle$ are actually rectangles within P, so we are allowed to refer to them in (iii). We have

$$x_{\langle a,c\rangle} + x_{\langle e,g\rangle} = 2, \quad \text{by (ii)},$$

and

$$x_{\langle b,h\rangle} + x_{\langle d,f\rangle} = 2 \quad \text{by (iii)}.$$

The first equation counts cells in $R_1 \cup R_2$, counting cells in $R_1 \cap R_2$ twice; the second counts cells in $\langle b, h\rangle \cup \langle d, f\rangle$, also counting cells in $R_1 \cap R_2$ twice. Subtract the second from the first and recall that $x_q \geqq 0$ for all cells q to complete the proof.

We will almost always apply Lemma 1 in situations where $\langle a, c\rangle$, $\langle e, g\rangle$, $\langle b, h\rangle$, and $\langle d, f\rangle$ are maximal rectangles, although in one case $\langle b, h\rangle$ and $\langle d, f\rangle$ will be contained in larger maximal rectangles whose cells are known to have value 0 outside the region shown in Figure 2.

If s is a real number, let $\bar{s}$ denote $1-s$.

3. Wires, signals, and gates

The constructions in the main theorems are best described by analogy with digital circuits. This section describes the building blocks of circuit design via polyominoes. Each piece of circuit is described as if it were part of a large polyomino P with stochastic function X.

A *wire* is a series of overlapping 2×2 rectangles, as in Figure 3(a). By Lemma 1 applied to the maximal rectangles $\langle 1, 4\rangle$ and $\langle 4, 7\rangle$, we find that $x_1 = x_7 = 0$. Applying Lemma 1 to each pair of overlapping 2×2 rectangles, we find successively that $x_4 = x_{10} = 0$ and $x_7 = x_{13} = 0$. It is not necessary that cell 1 be a top or left cell, or that cell 13 be a bottom or right cell; as long as there are at least three 2×2 rectangles, we can extend the wire as far as we like, knowing that the center cells must have value 0.

A wire propagates a *signal* s along one side as follows. If $x_3 = s$, then because $x_{\langle 1,4\rangle} = 1$ and $x_1 = x_4 = 0$, we must have $x_2 = \bar{s}$. This forces $x_6 = s$, because $x_{\langle 2,6\rangle} = 1$ and $x_4 = 0$. Similarly this forces $x_5 = \bar{s}$ and $x_9 = s$, forcing $x_8 = \bar{s}$ and $x_{12} = s$, finally forcing $x_{11} = \bar{s}$. These values satisfy equation (1) for all the rectangles in the wire. Thus the wire propagates signal s down the lower left side, or, equivalently, $\bar{s}$ up the

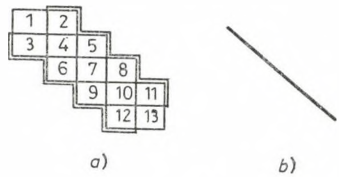


Fig. 3. Wire

upper right; the signal is determined by facing in the direction of propagation and reading the value on the right of the wire. We symbolize a wire as in Figure 3(b).

A *tab* is a group of three 2×2 rectangles attached to a wire as in Figure 4(a). Its purpose is to restrict the signal on the side of the wire nearest the tab to be at least 1/2; this restriction is symbolized as in Figure 4(b). It does this as follows. In the wire, Lemma 1 shows that

$$x_1 = x_4 = x_7 = x_{10} = x_{18} = x_{24} = x_{30} = x_{35} = 0;$$

note that x_{13} is not included because it does not belong to a 2×2 maximal rectangle. In the tab, Lemma 1 shows that

$$x_{32} = x_{27} = x_{21} = x_{16} = 0.$$

Suppose $x_3 = s$ and $x_{33} = t$. Then as before we find that

$$x_6 = x_9 = s; \qquad x_2 = x_5 = x_8 = x_{11} = \bar{s};$$

$$x_{28} = x_{22} = t; \quad x_{26} = x_{20} = x_{15} = \bar{t}.$$

But then

$$0 = x_{\langle 10,17\rangle} - x_{\langle 12,18\rangle} = (x_{10}+x_{11}) - (x_{14}+x_{18}) = \bar{s} - x_{14},$$

so

$$x_{14} = x_{19} = x_{25} = x_{31} = \bar{s}; \quad x_{23} = x_{29} = x_{34} = s.$$

Thus the tab does not stop signal s from propagating. However, by looking at part of the rectangles $\langle 8, 22\rangle$ and $\langle 15, 19\rangle$, we see that

$$x_8 + x_{22} = \bar{s} + t \leqq 1$$

and

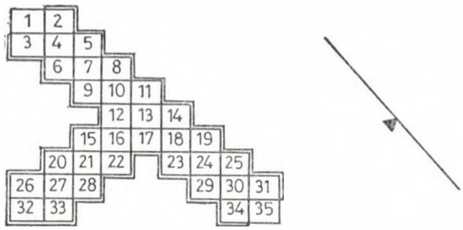
$$x_{19} + x_{15} = \bar{s} + \bar{t} \leqq 1.$$

Adding, we find that

$$2\bar{s} + 1 \leqq 2,$$

which implies that $s \geqq 1/2$, as claimed. The remaining cells (12, 13, and 17) can be assigned values in many ways, the easiest being $x_{12} = x_{17} = 0$, $x_{13} = s$. This forces $t = s$, but the value of t really does not matter.

The next building block consists of a central 3×3 rectangle with three 2×2 rectangles attached to each corner, as shown in Figure 5(a). These attachments can be extended as wires in adjacent pairs to form a *turn* (Figure 5(b)), in opposite pairs to form an *inverter* (Figure 5(c)), or in threes (Figure 5(d)). If signal s enters on the upper



a) b)

Fig. 4. Tab

left wire, then signal $\bar{s}$ leaves on any other attached wire. To see this, use Lemma 1 to find that

$$x_1 = x_6 = x_{12} = x_{18} = x_4 = x_9 = x_{15} =$$

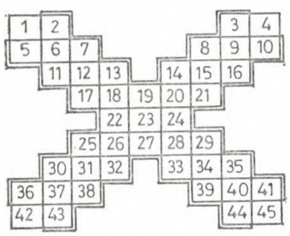
$$= x_{20} = x_{42} = x_{37} = x_{31} = x_{26} = x_{45} = x_{40} = x_{34} = x_{28} = 0.$$

Then if $x_5 = s$, the signal propagates toward the center, yielding $x_{17} = s$, and $x_{13} = \bar{s}$. Now $x_{17} + x_{21} \leqq 1$, so $x_{21} \leqq 1 - x_{17} = \bar{s}$, which forces $x_{14} = 1 - x_{21} \geqq s$. Similarly, $x_{33} \leqq 1 - x_{14} \leqq \bar{s}$, forcing $x_{29} \geqq s$, forcing $x_{25} \leqq \bar{s}$, and finally forcing $x_{32} \geqq s$. But since $x_{32} \leqq 1 - x_{13} = s$, we must have $x_{32} = s$, which forces all the other inequalities, too. In short,

$$x_{17} = x_{14} = x_{29} = x_{32} = s, \quad x_{13} = x_{21} = x_{33} = x_{25} = \bar{s}.$$

These signals then propagate down their respective wires as claimed. The remaining values must be $x_{19} = x_{22} = x_{24} = x_{27} = 0$ and $x_{23} = 1$.

A *crossover* (Figure 6(a)) allows two wires to cross each other without interference, except that the signals are inverted as they cross. It consists of two elongated



a)

b) c) d)

Fig. 5. Turn/Inverter

overlapping inverters, symbolized as in Figure 6(b). As in the previous paragraph, a signal entering the "vertical" inverter, say $x_3=s$, starts a chain reaction forcing $x_3=x_2=x_{15}=x_{16}=s$ and $x_1=x_4=x_{17}=x_{14}=\bar{s}$. A signal entering the "horizontal" inverter, say $x_7=t$, forces $x_7=x_6=x_{11}=x_{12}=t$ and $x_5=x_8=x_{13}=x_{10}=\bar{t}$. The two signals never need to interact, since we can set $x_9=1$ to take care of the large central rectangles.

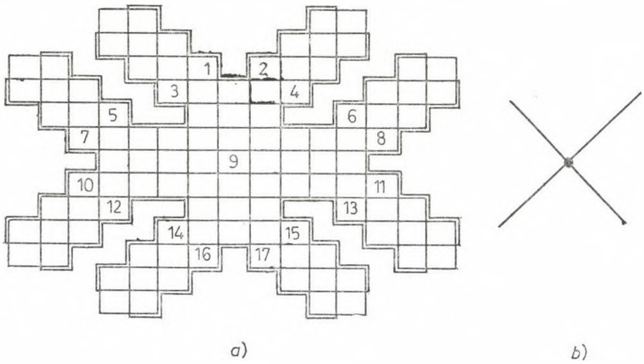


Fig. 6. Crossover

A NAND *gate* (Figure 7(a), symbolized in Figure 7(b)) takes two input signals s and t, and produces an output signal r. Besides the usual restriction that these signals must lie in [0, 1], a NAND gate enforces the restriction

$$r+s+t \leqq 2, \tag{2}$$

but allows any values that obey this restriction. In other words, if $s=t=1$ then r must be 0; otherwise r can be nonzero. In Section 5 we will see why the name NAND is appropriate. To verify its behavior, suppose $x_1=s$, $x_{19}=t$, and $x_{14}=r$. By Lemma 1 applied to the input and output wires,

$$x_2 = x_3 = x_{11} = x_{15} = x_{17} = x_{18} = 0,$$

hence

$$x_4 = \bar{s}, \quad x_{16} = \bar{t}, \quad \text{and} \quad x_{12} = x_8 = \bar{r}.$$

Applying Lemma 1 to $\langle 6, 11\rangle$ and $\langle 11, 15\rangle$ yields

$$x_6 = x_7 = x_{15} = 0.$$

We must have $x_{10}\leqq 1-x_1=\bar{s}$, and $x_9\leqq 1-x_{19}=\bar{t}$. But since

$$1 = x_{\langle 6,11\rangle} = x_8+x_9+x_{10} \leqq \bar{r}+\bar{t}+\bar{s},$$

equation (2) must hold. If r decreases from $2-(s+t)$, then $\bar{r}$ increases, and x_9 or x_{10} must decrease. Increasing x_{13} or x_5 compensates for this, allowing smaller values for r.

Finally, a *supertab* (Figure 8(a), symbolized in Figure 8 (b)) forces entering signals to be 1 and departing signals to be 0. Wires may be attached at any of the corners.

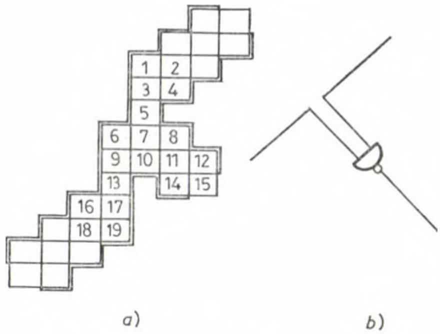


Fig. 7. NAND gate

To verify its behavior, apply Lemma 1 to $\langle 2, 36\rangle$ and $\langle 10, 44\rangle$ to find that

$$x_2 = x_3 = x_{43} = x_{44} = 0.$$

By symmetry,

$$x_{13} = x_{19} = x_{27} = x_{33} = 0.$$

Now Lemma 1 applies to $\langle 1, 10\rangle$ and $\langle 8,38\rangle$, even though $\langle 7, 12\rangle$ is not maximal, because $x_{13}=0$. Thus

$$x_1 = x_{17} = x_{18} = x_{24} = x_{25} = x_{31} = x_{32} = x_{37} = x_{38} = 0.$$

By symmetry, every cell has value 0 except for $x_4, x_5, x_7, x_{20}, x_{23}, x_{26}, x_{39}, x_{41}$, and x_{42}. This implies that $1 = x_{\langle 7,13\rangle} = x_7$, and by symmetry that

$$x_5 = x_7 = x_{41} = x_{39} = 1,$$

which in turn forces

$$x_4 = x_{20} = x_{26} = x_{42} = 0, \quad \text{and} \quad x_{23} = 1.$$

Since wires run at a 45° angle from the coordinate axes of the planar grid, it is convenient to rotate the schematic diagrams of circuits by 45°. The remaining figures are drawn with this convention.

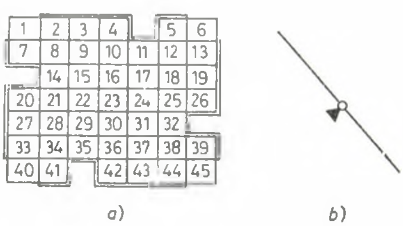


Fig. 8. Supertab

4. A polyomino with no stochastic function

Berge et al. give an example of a polyomino with no stable transversal. Using the components of the previous section, we can view their example as a wire with tabs on opposite sides (Figure 9). Let s be the signal propagating to the right, the first tab forces $s \leqq 1/2$ and the second forces $s \geqq 1/2$, so we must have $s=1/2$. A stable transversal has only integer values, so this polyomino cannot have one.

Fig. 9. A polyomino with no stable transversal

Similar reasoning shows that the polyomino shown schematically in Figure 10 cannot have a stochastic function. If the wire begins by propagating signal s to the right, the first supertab inverts s to $\bar{s}$ and forces $s=1$, then the second inverts $\bar{s}$ to s and forces $\bar{s}=1$. But it is impossible to have both $s=1$ and $\bar{s}=1$.

Fig. 10. A polyomino with no stochastic function

5. NP-completeness and polynomial algorithms

A polyomino may or may not have a stable transversal or a stochastic function. Theorems 1 and 2 below show that determining whether it has a stable transversal is NP-complete, whereas determining if it has a stochastic function can be done in polynomial time. For definitions and standard results about NP-completeness, consult the excellent book by Garey and Johnson [3].

THEOREM 1. *Determining whether a polyomino has a stochastic function can be done in polynomial time.*

PROOF. Let P be a polyomino with n cells. Since a rectangle is determined by two opposite corners, there are at most n^2 rectangles in P. Determining whether P has a stochastic function is equivalent to determining whether the system of linear inequalities

$$\sum_{c \in R} x_c = 1, \quad \text{for all maximal rectangles } R \text{ in } P;$$

$$0 \leqq x_c \leqq 1, \quad \text{for all cells } c \text{ in } P$$

has a feasible solution $(x_1, \ldots, x_n)$. This system has at most $2n^2+2n$ inequalities, n variables, and integer coefficients. A feasible solution can be found, if it exists, in polynomial time using (for example) Khachiyan's ellipsoid algorithm for linear programming [1].

THEOREM 2. *Determining whether a polyomino has a stable transversal is NP-complete.*

PROOF. Clearly, the problem is in NP, since given a polyomino with n cells we can guess the n values of a stable transversal X and verify that equation (1) holds for each maximal rectangle. There are at most n^2 such rectangles (as noted in the proof of Theorem 1), each of size at most n, so the verification can easily be done in time $O(n^3)$.

We prove the problem is NP-hard by giving a reduction from 3SAT [3]. Let F be a Boolean formula in 3CNF with n variables and m clauses. We construct a polyomino P with $O(m^2)$ cells that has a stable transversal if and only if F is satisfiable.

For each variable v in F, form a "variable component" consisting of a short wire leading into a sequences of inverters, as in Figure 11. Use as many inverters as there are occurrences of v in F, leaving enough room between them to allow the descending wires to contain turns. Call the signal entering from the left v; then signal v leaves from the even numbered inverters, $\bar{v}$ from the odd. In a stable transversal, signal v must be either 0 or 1, which we interpret as *false* and *true* to obtain a truth assignment to the variable v.

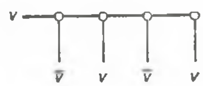


Fig. 11. Variable component

For each clause $(a \vee b \vee c)$ in F, form a "clause component" consisting of two NAND gates, input wires labelled $\bar{a}$, $\bar{b}$, and $\bar{c}$, and an output wire with a tab attached, all connected with turns and inverters as shown in Figure 12. The tab forces the output signal to be at least 1/2, so it must be 1, representing *true*. A NAND gate with *true* output forces at least one of its inputs to be *false*, by equation (2). Thus the component forces at least one of the three input signals $\bar{a}$, $\bar{b}$, and $\bar{c}$ to be *false*, which means at least one of a, b, and c must be *true*.

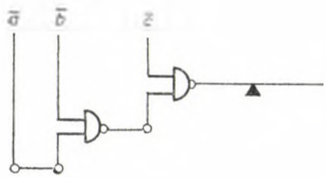


Fig. 12. Clause component

Finally, connect the $3m$ wires descending from the variable components to the $3m$ wires ascending from the clause components, using turns, crossovers, and inverters as necessary to ensure that the signal leaving a variable component arrives at a clause component with the correct value. Figure 13 shows a possible connection for the formula $(x \vee \bar{y} \vee z)(\bar{x} \vee \bar{z} \vee w)$. By the observations here and in Section 4, the resulting polyomino has a stable transversal if and only if F is satisfiable.

The strategy used in Figure 13 to connect the wires partitions the space between the variables and the clauses into $3m$ layers, each used to route one wire to the right until it is above its destination. There are $3m$ vertical strips near variables and another

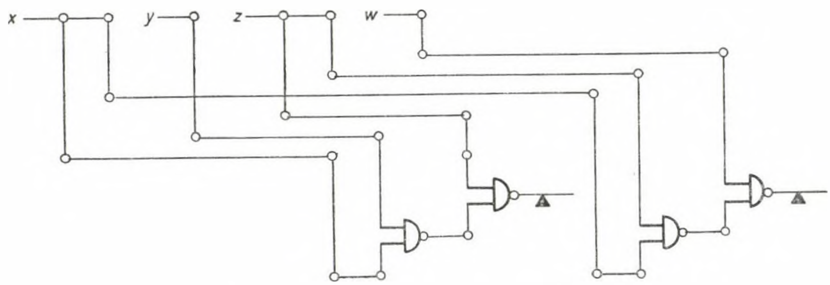


Fig. 13. Polyomino constructed for $(x\vee\bar{y}\vee z)(\bar{x}\vee\bar{z}\vee w)$

$3m$ near clauses. Each layer and strip has constant thickness (enough to hold a cross-over or turn), so the whole construction lies within a rectangle of area $O(m^2)$. Thus the number of cells in the resulting polyomino is $O(m^2)$, and the construction can be done easily in polynomial time.

6. Acknowledgement

I am grateful to Li Qiao and Richard Brualdi for bringing these problems to my attention.

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(Received February 5, 1985)

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REAL CONFORMAL SPIN STRUCTURES ON MANIFOLDS

PIERRE ANGLÈS

Abstract

This paper is divided into three parts. In the first one, we study conformal orthogonal flat geometry, in a purely algebraic way, avoiding systematically any matricial formalism. We introduce and give geometrical characterizations of groups called *conformal spinoriality groups.* In the second part, *real conformal spin structures* on riemannian or pseudo-riemannian manifolds V are defined, in a purely differential geometrical scope, without any algebraic topology machinery. We obtain necessary and sufficient conditions for the existence of a real conformal spin structure on V, in which the conformal spinoriality groups play an essential part. The third section is assigned to the investigation of the *connections between real spin structures and real conformal spin ones.*

Foreword

The notion of spin structure on a manifold V has been introduced by A. Haefliger who specified an idea from Ehresmann (Sur l'extension du groupe structural d'un espace fibré, *C. R. A. S. Paris* **243** (1956), p. 558—560). J. Milnor (Spin structure on manifolds, *Enseignement mathématique,* Genève, 2° série **9** (1963), p. 198—203) and A. Lichnerowicz [10], [11] have taken an interest in those structures. In a self-contained way, A. Crumeyrolle [4], [5],]6], [7], has developed the study of vector bundles associated with spin structures, in any dimension and signature. He introduced the general defini tions of spin structures on a real paracompact n-dimensional smooth pseudo-riemannian (in particular riemannian) manifold and drew up necessary and sufficient conditions for their existences in a purely geometrical way. More precisely, he defined the notion of spinoriality groups such that the existence of a spin structure on V be submitted to the reduction of the structure group $O(p, q)$ of "the bundle of orthonormal frames of V", to a spinoriality group, once being got into the complexified. One of the main guiding principles is that the study of fields over curved spaces is nothing but the consideration of spin-orthogonal, or symplectic, fibrations. According to the same guidance, there appears the problem of the investigation of conformal spin-structures, in which the part previously assigned to the group $O(p, q)$ will be now given to the conformal one: $C_n(p, q)$.

1980 *Mathematics Subject Classification.* Primary 53A30; Secondary 15A66, 22E70.

Key words and phrases. Real conformal group $C_n(p, q)$ isomorphic to $PO(p+1, q+1)$ (Möbius group), real conformal restricted group $(C_n(p, q))_r$, real conformal spinoriality groups, V pseudo-Riemannian (or Riemannian) manifold, bundle ξ of orthornormal frames of V, Greub-extension of ξ.

I. CONFORMAL SPINORIALITY GROUPS

1. A summary of previous results

The following works of reference [1], [3], [4, 5, 7], [12], [17] contain the facts concerning Clifford algebras and spinors. The present paper is selfcontained. Let $E_n(p, q)$, $(p+q=n, n>2)$, be $\mathbf{R}^n$ with a quadratic form Q of arbitrary signature (p, q). $\mathrm{Cl}(E_n)$ denotes the Clifford algebra of E_n with the quadratic form Q; α is the principal automorphism of $\mathrm{Cl}(E_n)$, β the principal anti-automorphism of $\mathrm{Cl}(E_n)$. B is the fundamental bilinear form associated with Q chosen so that for all $x\in\mathbf{R}^n$, $B(x, x)=Q(x)$. It is well-known that the group $\mathrm{Pin}\, Q=\mathrm{Pin}\,(p, q)$ constitutes the 2-fold covering of the orthogonal group $O(p, q)$. If $g\in\mathrm{Pin}\,(p, q)$, we define $\psi_1(g)x=gxg^{-1}$, $x\in\mathbf{R}^n$, $\psi_1(g)\in O(p, q)$ and $\psi(g)=\alpha(g)xg^{-1}$, $\psi(g)\in O(p, q)$. We introduce an orthonormal basis of $E_n(p, q)$ such that $Q(e_i)=e_i^2=\varepsilon_i(\varepsilon_i=1, 1\leq i\leq p, \varepsilon_i=-1, p+1\leq i\leq n)$. In $\mathbf{R}^2$ with a quadratic form Q_2 of signature $(1, 1)$, we consider an orthonormal basis $\{e_0, e_{n+1}\}$ such that $Q(e_0)=(e_0)^2=1$, $Q(e_{n+1})=(e_{n+1})^2=-1$. Then $\{e_1, \dots, e_n, e_0, e_{n+1}\}$ is an orthonormal basis of $\mathbf{R}^{n+2}=E_{n+2}(p+1, q+1)=E_n(p,q)\oplus E_2(1, 1)$; e_0 and e_{n+1} are chosen once and for all. $C_n(p, q)$ stands for the conformal Lie group [1], [2], [7] of $\mathbf{R}^n$ isomorphic to $PO(p+1, q+1)=\dfrac{O(p+1, q+1)}{\mathbf{Z}_2}$, [1] which we agree to call the Möbius group of $E_n(p, q)$ [2]. More precisely, we have constructed in [1] an injective map u from $E_n(p, q)$ into the isotropic cone C_{n+2} of $E_{n+2}(p+1, q+1)$ defined for all $x\in E_n(p, q)$ by

$$\text{(B)} \qquad u(x) = \frac{1}{2}(x^2-1)e_0+x+\frac{1}{2}(x^2+1)e_{n+1}.$$

The "projection" φ called "twistor projection" or "conformal spinor projection" from $\mathrm{Pin}\,(p+1, q+1)$ onto $C_n(p, q)$ is such that for almost all $x\in E_n(p, q)$ and for all $g\in\mathrm{Pin}\,(p+1, q+1)$

$$\text{(A)} \qquad \alpha(g)u(x)g^{-1} = \psi(g)u(x) = \sigma_g(x)u(\varphi(g)x),$$

with $\sigma_g(x)\in\mathbf{R}$ [1]. We set $e_N=e_0e_{n+1}e_1\dots e_n$: the kernel of φ is: $\mathscr{A}=\{1, -1, e_N, -e_N\}$ isomorphic to $\mathbf{Z}_2\times\mathbf{Z}_2$ if $(e_N)^2=1$, or to $\mathbf{Z}_4$ if $(e_N)^2=-1$ [1].

$\varphi\,(\mathrm{Spin}\,(p+1, q+1))$ is called *the real conformal restricted group.*

If we set $x_0=\dfrac{e_0+e_{n+1}}{2}$ and $y_0=\dfrac{e_0-e_{n+1}}{2}$, $\{x_0, y_0\}$ is then a special "real Witt-Basis" of $\mathbf{C}^2$ associated with $\{e_0, e_{n+1}\}$. From (B): $u(x)=x^2x_0+x-y$, we deduce:

$$u(x)y_0 = x^2x_0y_0+xy_0 \quad \text{and} \quad y_0u(x) = x^2y_0x_0+y_0x,$$

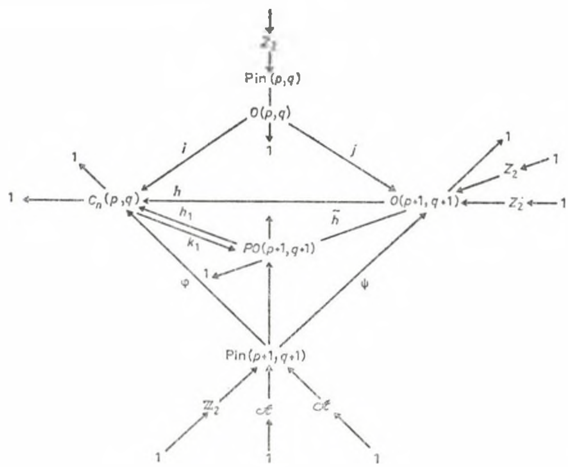
whence we obtain:

$$u(x)y_0+y_0u(x) = 2B(u(x), y_0) = x^2.$$

Thus, (A) is equivalent to

$$(\mathrm{A}_1) \qquad x = u(x)-2B(u(x), y_0)x_0+y_0.$$

It is, now, possible to construct an explicit homomorphism h from the orthogonal Lie group $O(p+1, q+1)$ onto $C_n(p, q)$, in order to obtain a commutative diagramm [2, p. 46] i, respectively, j denotes the identity map from $O(p, q)$ into $C_n(p, q)$, respectively, $O(p+1, q+1)$.



First, we construct h. For any ω belonging to $O(p+1, q+1)$, there exists g(modulo ± 1) belonging to $\mathrm{Pin}\,(p+1, q+1)$ such that $\omega=\psi(g)$. As, according to [1] for any g, g' in $\mathrm{Pin}\,(p+1, q+1)$ such that $\varphi(g)=f\in C_n(p, q)$ and $\varphi(g')=$ $=f'\in C_n(p, q)$, $\varphi(g'g)=f'\circ f$ and for almost all $x\in E_n(p, q)$, $\sigma_{g'g}(x)=$ $=\sigma_{g'}(f(x))\sigma_g(x)$, we obtain that $\sigma_g(x)\neq 0$ when $f(x)$ is defined and that (A) is equivalent to

(A$_2$) $$u(f(x)) = \lambda_g(x)\psi(g)u(x) \quad \text{for} \quad f=\varphi(g)$$

where $\lambda_g(x)=(\sigma_g(x))^{-1}$. So with any $\omega\in O(p+1, q+1)$ we can associate $f=$ $=\varphi(g)\in C_n(p, q)$ such that:

$$f(x) = \lambda_g(x)\{\omega\cdot u(x)-2B(\omega\cdot u(x), y_0)x_0\}+y_0 \quad \text{with} \quad 2\lambda_g(x)B(\omega\cdot u(x), x_0) = -1.$$

One obtains a map h from $O(p+1, q+1)$ into $C_n(p, q)$.

We agree to denote $\lambda_g=\lambda_{-g}=\lambda_\omega$ where $\omega=\psi(g)=\psi(-g)\in O(p+1, q+1)$ and we can easily verify that it is possible to write

(C) $$f(x) = \lambda_\omega(x)\{\omega\cdot u(x)-2B(\omega\cdot u(x), y_0)x_0\}+y_0$$
$$\lambda_\omega(x) = \frac{-1}{2B(\omega\cdot u(x), x_0)}$$

when $f(x)$ is defined.

One can verify that $\omega\to h(\omega)=f=\varphi(g)$ *is a homomorphism from* $O(p+1, q+1)$ *onto* $C_n(p, q)$ *such that* $i=h\circ j$, $\varphi=h\circ j$, $\varphi=h\circ\psi$. *Thus we obtain an isomorphism* h_1 *of Lie groups from* $PO(p+1, q+1)$ *onto* $C_n(p, q)$ *by using quotient groups such* *that* $h = h_1 \circ \bar{h}$ *where* $\bar{h}$ *is the homomorphism associated with the classical exact sequence of groups:*

$$1 \to Z_2 \to O(p+1, q+1) \xrightarrow{\bar{h}} PO(p+1, q+1) \to 1.$$

Let k_1 be the inverse of h_1. In the same way if C'_n stands for the complex conformal group and $O'(n+2)$ for the complex orthogonal group, $\mathrm{Pin}'(n+2)$ is an 8-fold covering of C'_n, with kernel $\mathscr{A}' = \{1, -1, e_N, -e_N, i, -i, ie_N, -ie_N\}$. φ', respectively ψ', denotes the "complex conformal spinor-projection", respectively the "twisted spinor-projection", from $\mathrm{Pin}'(n+2)$ onto C'_n, respectively $O'(n+2)$. Thus, we obtain an isomorphism of Lie groups h'_1 from $PO'(n+2)$ onto C'_n. Its inverse is denoted by k'_1.

Let us, last, recall [2, p. 41] the following remark: if $n = 2r$, then $e_N f_{r+1} = (-i)^{r-p} f_{r+1}$, where $f_{r+1} = y_1 \dots y_r y_0$ is an $(r+1)$-isotropic vector and $f_{r+1} e_N = (-1)^{r+1} (-i)^{r-p} f_{r+1}$ according to a result given in [3, p. 91].

2. Definitions of real conformal spinoriality groups (n is even, $n = 2r$, $r > 1$)

Let $f_{r+1} = y_1 \dots y_r y_0 = f_r y_0$ be an $(r+1)$-isotropic vector.

a) Let H_C be the set of elements $\gamma \in \mathrm{Spin}(p+1, q+1)$ such that $\gamma^f{}_{r+1} = \varepsilon_1 f_{r+1}$ where $\varepsilon_1 \in \mathscr{A} = \{1, -1, e_N, -e_N\}$. We agree to call, by definition, the subgroup $S_C = \varphi(H_C)$ of $(C_n(p, q))_r$, the *real conformal group associated with* $f_{r+1} = y_1 \dots y_r y_0$.

Following the result given above (according which $e_N f_{r+1} = (-i)^{r-p} f_{r+1}$) such a definition is equivalent to this one:

$S_C = \varphi(H_C)$, where H_C is the set of elements $\gamma \in \mathrm{Spin}(p+1, q+1)$ such that if $r-p \equiv 0$ or 2 (modulo 4) $\gamma^f{}_{r+1} = \pm f_{r+1}$ and if $r-p \equiv 1$ or 3 (modulo 4) $\gamma^f{}_{r+1} = \varepsilon^f{}_{r+1}$ with $\varepsilon = \pm 1$ or $\pm i$

b) Let $(H_C)_e$ be the set of elements $\gamma \in \mathrm{Spin}(p+1, q+1)$ such that $\gamma^f{}_{r+1} = \mu^f{}_{r+1}$, where $\mu \in \mathbf{C}^*$. *We agree to call by definition the enlarged real conformal spinoriality group associated with* f_{r+1} *the subgroup* $(S_C)_e = \varphi((H_C)_e)$ of $(C_n(p, q))_r$.

We can observe that e_0 and e_{n+1} being chosen once and for all, these definitions are associated with the choice of an r-isotropic vector $f_r = y_1 \dots y_r$ of $E_n(p, q)$.

c) Remark. Let us observe that these subgroups, at first sight "bigger" than those defined in [5, 6, 7] are subgroups of $(C_n(p, q))_r$ which cannot be reduced to subgroups of $SO(p, q)$ defined in [4] as real spinoriality-groups. More precisely, one can easily verify, for example, that any real conformal spinoriality group contains the following elements:

α) the special conformal transformation $x \to f(x) = x(1+ax)^{-1}$ where

$$f = \varphi\left(1 + \frac{1}{2}(e_{n+1} - e_0)\, a\right) \quad \text{with} \quad a = e_1 + \dots + e_n,$$

β) the translation $x \to x + y$ where $y = e_1 + \dots + e_p - e_{n-p+1} \dots - e_n$.

3. Description of the enlarged real conformal spinoriality groups

The abbreviation s.t.i.m. stands for maximal totally isotropic subspace as in [3].

PROPOSITION 1. *Any enlarged real conformal group of spinoriality* $(S_C)_e$ *is the stabilizer of the s.t.i.m. associated with the r-isotropic vector* $y_1 \dots y_r$, *for the action of* $(C_n(p,q))_r$. *If pq is even,* $(S_C)_e$ *is connected, if pq is odd,* $(S_C)_e$ *has two connected components. For* $0<p<r$, $\dim (S_C)_e=(r+1)^2+\frac{p(p+1)}{2}$. $(\sigma)_e$, *the enlarged real group of spinoriality associated with* f_r [7] *is a normal subgroup of* $(S_C)_e$.

PROOF. The demonstration can be led in two steps. Let us write $f_1=h(\omega)$ for $\omega\in O(p+1,q+1)$.

a) *First, we suppose that* $u(f_1(y_i))=f_1(y_i)-y_0$ *is well determined for all* i, $1\leqq i\leqq r$, *idest, equivalently,* $f_1(0)$ *and* $f_1(y_i)$ *well defined for all* i, $1\leqq i\leqq r$.

According to [7], $\gamma^f{}_{r+1}=\pm\mu f_{r+1}$, $\mu\in \mathbf{C}^*$ is equivalent to $\gamma^f{}_{r+1}\gamma^{-1}=N(\gamma)\mu^2 f_{r+1}$. Thus, $(H_c)_e$ is the set of element $g\in \mathrm{Spin}\,(p+1,\ q+1)$ such that $gf_{r+1}g^{-1}=\sigma f_{r+1}$ where $\sigma=N(g)\mu^2=\pm\mu^2$.

One can easily notice that $\alpha(g)f_{r+1}g^{-1}=\sigma f_{r+1}$ is equivalent to

$$\text{(I)}\qquad \alpha(g)y_1g^{-1}\alpha(g)y_2g^{-1}\dots\alpha(g)y_rg^{-1}\alpha(g)y_0g^{-1}=\sigma f_{r+1}.$$

We set $\psi(g)=\omega$, $\omega=SO(p+1,q+1)$, so that $\alpha(g)y_ig^{-1}=\omega(y_i)$, $1\leqq i\leqq r$ and $\alpha(g)y_0g^{-1}=\omega(y_0)$. So, g belongs to $(H_c)_e$ iff $\psi(g)=\omega$ belongs to σ_e *the real enlarged spinoriality group associated with* $f_{r+1}=y_1\dots y_ry_0$.

According to the diagram given above, we obtain $(S_C)_e=h(\sigma_e)$.

By an easy computation, taking account of the formulas (C), we obtain that (I) is equivalent to

$$\sigma_\omega(y_1)\sigma_\omega(y_2)\dots\sigma_\omega(y_r)(-\sigma_\omega(0))u(f_1(y_1))\dots u(f_1(y_r))u(f_1(0))=$$

$$=\sigma y_1\dots y_ry_0=-\sigma u(y_1)\dots u(y_r)u(0)$$

as $u(y_1)\dots u(y_r)u(0=-y_1\dots y_ry_0$. So we have the following relation equivalent to (I):

$$\text{(II)}\quad \sigma_\omega(y_1)\dots\sigma_\omega(y_r)\sigma_\omega(0)\,u(f_1(y_1)\dots u(f_1(y_r)u(f_1(0))))=\sigma u(y_1)\dots u(y_r)u(0),$$

which means, [3, 15], that the vectors $u(f_1(y_i)),\dots,u(f_1(y_0)),u(f_1(0))$ belong to the $(r+1)$ - s.t.i.m. associated with f_{r+1}.

As one can notice that u operates on the set of isotropic subspaces as the translation of vector $u(0)=-y_0$, if $u(z)$ belongs to the $(r+1)$-s.t.i.m., so z belongs to $F'=\{y_1,\dots,y_r\}$.

According to our assumption, $\sigma_\omega(y_i)\neq 0$ for all i, $1\leqq i\leqq r$ and $\sigma_\omega(0)\neq 0$. As, taking account of [7], ω belongs to σ_e which stabilizes the $(r+1)$- s.t.i.m: $\{y_1,\dots,y_r,y_0\}$ for the action of $SO(p+1,q+1)$, the restriction of ω to $F'=\{y_1,\dots,y_r\}$ stabilizes F'. So we find that for all i, $1\leqq i\leqq r$, $\sigma_\omega(y_i)=\sigma_\omega(0)\neq 0$.

(I) is equivalent to

$$\text{(III)}\qquad \omega(y_1)\dots\omega(y_r)(-\sigma_\omega(0))f_1(0)+\omega(y_1)\dots\omega(y_r)\sigma_\omega(0)y_0=\sigma y_1\dots y_ry_0,$$

as $\omega(y_0)=-\sigma_\omega(0)(f_1(0)-y_0)$. As $\omega(y_1), \ldots, \omega(y_r)$ are independent in F', according to the definition of ω, and as $f_1(0)$ belongs to F',

(III bis) $$\omega(y_1)\ldots\omega(y_r)f_1(0)=0$$

necessarily [15, p. 14 Th. 4.3; 3 chapter II]. Thus (III) means that

$$\omega(y_1)\ldots\omega(y_r)y_0=\frac{\sigma}{\sigma_\omega(0)}y_1\ldots y_r y_0$$

whence we deduce [3, chapter II] that

(IV) $$\omega(y_1)\ldots\omega(y_r)=\frac{\sigma}{\sigma_\omega(0)}y_1\ldots y_r.$$

An easy computation gives the following result:

(V) $$\omega(y_1)\ldots\omega(y_r)=(\sigma_\omega(0))^r\prod_{i=1}^{r}(f_1(y_i)-f_1(0))$$

as $\omega(y_i)=\sigma_\omega(0)(f_1(y_i)-f_1(0))$ for all i, $1\leqq i\leqq r$.

If $f_1(0)=0$, we obtain that

$$f_1(y_1)\ldots f_1(y_r)=\frac{\sigma}{(\sigma_\omega(0))^{r+1}}y_1\ldots y_r,$$

where $\sigma=\pm\mu^2$ belongs to $\mathbf{C}^*$. So μ_1 such that $f_1(y_1)\ldots f_1(y_r)=\mu_1 y_1\ldots y_r$ is any element of $\mathbf{C}^*$.

If not, observing that

$$\prod_{i=1}^{r}(f_1(y_i)-f_1(0))f(0)=\frac{1}{(\sigma_\omega(0))^r}\omega(y_1)\ldots\omega(y_r)f_1(0)=0,$$

according to (III bis) and equals $f_1(y_1)\ldots f_1(y_r)f_1(0)$ we find that the vectors $f_1(y_1), \ldots, f_1(y_r)$, $f_1(0)$ are dependent in $\{y_1, \ldots y_r, y_0\}$ [15, Th. 4.2, p. 15]. As [see IV and V],

$$\prod_{i=1}^{r}(f_1(y_i)-f_1(0))=\frac{\sigma}{(\sigma_\omega(0))^{r+1}}y_1\ldots y_r=\mu_1 y_1\ldots y_r$$

where $\mu_1\in\mathbf{C}^*$, taking account of the dependence of the vectors $f_1(y_1), \ldots f_1(y_r)$, $f_1(0)$, we obtain that $f_1(y_1)\ldots f_1(y_r)=\mu_2 y_1\ldots y_r$ where $\mu_2\in\mathbf{C}^*$.

b) Let us prove, now, that for any f_1 belonging to $(S_c)_e$, it is permissible to suppose that $f_1(0)$ is well-defined and to find $z_1, \ldots, z_r$ linearly independent, belonging to F' such that $f_1(z_i)$ be well-defined if some of the elements $f_1(y_1), \ldots f_1(y_r)$ are not defined. Let us recall that, classically, for $x=x^1e_1+\ldots+x^pe_p+x^{p+1}e_{p+1}+\ldots+x^ne_n$ the x^i, $1\leqq i\leqq p$ are called "spatial coordinates of x $E_n(p,q)$" and those for $p+1\leqq i\leqq n$ the "temporal coordinates of x". It is well-known (see for example [16, p. 341] that $\dfrac{O(p+1,q+1)}{\underline{|(O(p+1,q+1))|}}\approx\mathbf{Z}_2\times\mathbf{Z}_2$ where $\underline{|G|}$, denotes the connected component of the Lie group G, and that $\underline{|O(p+1,q+1)|}=SO_+(p+1,q+1)$.

(i) Let us assume that p and q are even ($n=p+q$ is even).

We know [1, 2] that $\overline{|\mathrm{Pin}\,(p+1, q+1)|} = G_0^+(p+1, q+1)$, $\varphi^{-1}(\overline{|C_n(p, q)|}) = \mathrm{Spin}\,(p+1, q+1)$, $C_n(p, q)$ has two connected components, and, at last, that $\mathrm{Pin}\,(p+1, q+1)$ has four connected components. We introduce the following elements $\pm e_p$, $\pm e_Q$, where $e_p = e_0 e_1 \ldots e_p$ and $e_Q = e_{p+1} \ldots e_n e_{n+1}$. One verify that ± 1 belong to $G_0^+(p+1, q+1)$, $\pm e_p$ belong to $G_0(p+1, q+1) - G_0^+(p+1, q+1)$, $\pm e_Q$ belong to $\mathscr{C}_C G_0(p+1, q+1)$ where C is $\mathrm{Pin}\,(p+1, q+1) - \mathrm{Spin}\,(p+1, q+1)$. So, we find, again, that $O(p+1, q+1)$ has 4 connected components and that any element $\omega \in O(p+1, q+1)$ can be written $\omega = \omega^* \omega_0$, $\omega_0 \in \overline{|O(p+1, q+1)|}$ and $\omega^* = \mathrm{Id}_{E_{n+2}} = \psi(\pm 1)$ or $\omega^* = \psi(\pm e_p) = (\mathrm{Sym})_e$ (space-symmetry), or, $\omega^* = \psi(\pm e_Q) = (\mathrm{Sym})_t$ (time-symmetry), or $\omega^* = -\mathrm{Id}_{E_{n+2}} = \psi(\pm e_N) = (\mathrm{Sym})_{et}$ (space-time symmetry).

We observe that $h(\overline{|O(p+1, q+1)|}) \subset \overline{|C_n(p, q)|}$ and that any element $f \in C_n(p, q)$ can be written $f = f^* \circ f_0$ where $f^* = h(\omega^*)$ and $f_0 = h(\omega_0) \in \overline{|C_n(p, q)|}$ and $f^* = \varphi(\pm 1) = \varphi(\pm e_N) = (\mathrm{Id})_{E_n}$ or $f^* = \varphi(\pm e_p) = \varphi(\pm e_Q) = \mathrm{Inv}\,(0, 1) \circ (\mathrm{Sym})_e = (\mathrm{Sym})_t \circ \mathrm{Inv}\,(0, -1)$ in the space $E_n(p, q)$, where $\mathrm{Inv}\,(0, 1)$, respectively $\mathrm{Inv}\,(0, -1)$, denotes the inversion of pole O and power 1, respectively of pole O and power -1. According to [1, 2] $f_0 = \Omega \circ \mathscr{T} \circ \mathscr{H} \circ S$ where Ω belongs to $SO_+(p, q)$, T is a translation of vector $b \in E_n(p, q)$, $\mathscr{H}$ is a dilatation and S is the special conformal transformation $x \to x(1+ax)^{-1}$ where $a \in E_n(p, q)$. We remark that $(S_c)_e \subset \overline{|C_n(p, q)|}$ and that we are led to study the case where f_0 belongs to $\overline{|C_n(p, q)|}$, $f_0 = \Omega \circ \mathscr{T} \circ \mathscr{H} \circ S$, with $\sigma_{g_0}(x) = \underbrace{\sigma_\Omega(\mathscr{T} \circ \mathscr{H} \circ S(x))}_{=1} \underbrace{\sigma_{\mathscr{T}}(\mathscr{H} \circ S(x))}_{=1} \underbrace{\sigma_{\mathscr{H}} S(x)}_{=\lambda^{-1}} \underbrace{\sigma_S(x)}_{=N(1+ax)}$. As $\sigma_{g_0}(x) = \sigma_{\omega_0}(x) = 0$ is the equation of singular points of f_0, [1], we find that for any isotropic vector x, singular point of f_0 we have $B(a, x) = -\frac{1}{2}$. As $f_0 = h(\omega_0)$ stabilizes F' [see 4], *we observe that* $f_0(0)$ is *well-defined,* according to the fact that $f_0(0) = \Omega \circ \mathscr{T} \circ \mathscr{H} \circ S(0)$ with $S(0) = 0$, $\mathscr{H}(0) = 0$, and that $\Omega \circ \mathscr{T}(0)$ *belongs to* F'. Thus, we can assume that $f_1(0)$ is well-defined according to the writing of f_0.

If y_i, $1 \leqq i \leqq r$, is a singular point for f_0 we can find an isotropic vector $a_1 = \sum_{i=1}^{r} \alpha^i y_i$ belonging to F' such that $z_i = y_i + a_1$, $1 \leqq i \leqq r$, satisfy the conditions $B(a, z_i) \neq -\frac{1}{2}$, the vectors z_i being linearly independent. We denote that setting $a_1 = y_1 + \ldots + y_r$ belonging to F', $B(a_1 z_i) \neq -\frac{1}{2}$ for any i, $1 \leqq i \leqq r$ and these vectors z_i translated from the y_i's are linearly independent.

(ii) Let us now assume that p and q are odd.

$C_n(p, q)$ has 4 connected components and $\varphi^{-1}(\overline{|C_n(p, q)|}) = G_0^+(p+1, q+1)$, [1]. We observe that ± 1, $\pm e_N$, $\pm e_p$, $\pm e_Q$ belong to $G_0^+(p+1, q+1)$ and that $G_0^+(p+1, q+1) = \psi^{-1}(\overline{|O(p+1, q+1)|})$, [4]. $\mathrm{Pin}\,(p+1, q+1)$ has four connected components as previously. One can easily see that any element ω of $O(p+1, q+1)$

can be written $\omega=\omega^*\omega_0$, where ω_0 belongs to $\boxed{O(p+1, q+1)}$ and

$$\omega^* = \mathrm{Id}_{E_{n+2}} = \psi^{(\pm 1)} = \psi(\pm e_N) = \psi(\pm e_p) = \psi(\pm e_Q)$$

or

$$\omega^* = (\mathrm{Sym})_0 \circ (\mathrm{Sym})_{n+1} = \psi(e_0 e_{n+1})$$

or

$$\omega^* = (\mathrm{Sym})_0 \circ (\mathrm{Sym})_{n+1} \circ (\mathrm{Sym})_1 = \psi(e_0 e_{n+1} e_1)$$

or

$$\omega^* = (\mathrm{Sym})_0 \circ (\mathrm{Sym})_1 \circ (\mathrm{Sym})_2 = \psi(e_0 e_1 e_2).$$

Thus, any element f belonging to $C_n(p, q)$ can be written $f=f^* \circ f_0$ when $f_0=h(\omega_0)$ belongs to $\boxed{C_n(p, q)}$ and where $f^*=\mathrm{Id}_{E_n}$ or $f^*=-\mathrm{Id}_{E_n}$ or $(-\mathrm{Id})_{E_n} \circ (\mathrm{Sym})_1$ or $\mathrm{Inv}\,(0, 1) \circ (\mathrm{Sym})_1 \circ (\mathrm{Sym}_2)$ with obvious notations.

Thus, any element f belonging to $(S_C)_s$ can be written $f=(\pm \mathrm{Id})_{E_n} \circ f_0$, with $f_0 \in \boxed{C_n(p, q)}$ as $e_0 e_{n+1} e_1$ and $e_0 e_1 e_2$ are odd. We are so led to the previous demonstration (i).

The results concerning the dimension come from those given in [7] for the spinoriality groups. As for the connected components the same method as in [1] leads to the determination of their number.

4. Description of the real conformal groups of spinoriality in a strict sense

As in [7] normalization conditions appear. We obtain the following statement:

PROPOSITION 2. *S_c is the subgroup of $(C_n(p, q)_r)$ of elements f_1 which stabilize the s.t.i.m. associated with the r-isotropic vector $f_r=y_1 \dots y_r$ and satisfy*

$$f_1(y_1) \dots f_1(y_r) = \pm y_1 \dots y_r.$$

In elliptic signature S_c has 2 connected components. $\dim S_c=r^2+2r$. *If Q is a neutral form $(p=r)$, S_c has 2 connected components if r is even and 4 connected components if r is odd.* $\mathrm{Dim}\, S_c=\dfrac{r(3r+5)}{2}$. *In signature (p, q), $p \leqq n-q$, p positive terms $r>2$, if pq is even S_c has 2 connected components and if pq is odd S_c has 4 connected components.* $\mathrm{Dim}\, S_c=(r+1)^2-2+\dfrac{p(p+1)}{2}$.

5. Remarkable factorization of elements of $(S_c)_e$ and S_c and topological remarks if $n=2r$

If pq is even any element $f_0 \in (S_c)_e$ can be written in the form $f_0=\Omega \circ \mathcal{T} \circ \mathcal{H} \circ S$ where $\Omega \in O(p, q)$ and stabilizes $F'=\{y_1, \dots, y_r\}$ and thus belongs to σ_e the spinoriality group associated with f_r, $\mathcal{T}$ translation, $\mathcal{H}$ dilatation, S conformal transformation: $x \to x(1+ax)^{-1}$.

If pq is odd f_0, belonging to $(S_c)_e$ can be written $f_0=(\pm \mathrm{Id}_{E_n}) \circ \Omega \circ \mathcal{T} \circ \mathcal{H} \circ S$ with Ω belonging to σ_e.

Thus, we obtain that $\dfrac{\overline{|(S_c)_e|}}{\overline{|\sigma_e|}}$ is homeomorphic to $\mathbf{R}^{2n+1}$, taking account of the fact that the group of the translations of E_n has n parameters, that $\mathbf{R}^+$ is homeomorphic to $\mathbf{R}$, and last, that $\overline{|C_n(p,q)|}$ is homeomorphic to $\overline{|O(p,q)|}\times\mathbf{R}^{2n+1}$ — [2, p. 35 or part III of this paper] — and that $C_n(p,q)$ has $\dfrac{(n+1)(n+2)}{2}$ parameters. We shall use this remark later.

One can easily extend such factorizations to the case of the real conformal spinoriality groups in a strict sense S_c.

II. REAL CONFORMAL SPIN STRUCTURE ON MANIFOLDS

1. Definitions

V is a real paracompact a-dimensional pseudo-riemannian (in particular, riemannian) manifold. Its fundamental tensor field is called, abusively, Q. We denote by $\xi(E, V, O(p,q), \pi)$, or simply ξ, the principal bundle of orthonormal frames of V.

a) Let $i\colon O(p,q)\to C_n(p,q)$ be the canonical injective homomorphism. The group $O(p,q)$ acts on $C_n(p,q)$ by: $(\omega, f)\in O(p,q)\times C_n(p,q)\to i(\omega)f\in C_n(p,q)$. Let $\xi_1(A_1, V, C_n(p,q), \omega_1)$ be the principal bundle with structure group $C_n(p,q)$ over the same base V, obtained by i-extension of ξ, [9]. $\xi_1(V)=i(\xi(V))=\xi_i(V)=$ $=\xi_1(A_1, V, C_n(p,q), \overline{\omega}_1)$, is a principal bundle with structure soup $C_n(p,q)$ in the following way: let us choose a covering $(\mathscr{U}_{\alpha'})_{\alpha'\in A}$ of V with a system of local cross-sections $\sigma_{\alpha'}$ and transition functions $g_{\alpha'\beta'}$. Let us define maps $g'_{\alpha'\beta'}=i\circ g_{\alpha'\beta'}$. Then, for all $x\in U_{\alpha'}\cap U_{\beta'}\cap U_{\gamma'}$ the $g'_{\alpha'\beta'}$ satisfy the relation: $g'_{\alpha'\beta'}(x)g'_{\beta'\gamma'}(x)=g'_{\alpha'\gamma'}(x)$ and consequently, there is a principal bundle ξ_i with a system of local sections such that the $g'_{\alpha'\beta'}$ are the corresponding transition functions, according to the general result of [9].

b) Let us recall that $C_n(p,q)$ is isomorphic to $PO(p+1, q+1)$.
Using, with previous notations, the classical sequence of groups:

$$1\to Z_2\to O(p+1, q+1)\xrightarrow{\tilde{h}} PO(p+1, q+1)\to 1,$$

let us define $\tilde{\lambda}=\tilde{h}\circ j=k_1\circ i$ and let $P\xi_1(V)=\tilde{\lambda}(\xi(V))=\xi_{\tilde{\lambda}}(V)$ be the $\tilde{\lambda}$-extension of the principal bundle $\xi(V)$. $P\xi_1(V)=\xi_{\tilde{\lambda}}(V)=P\xi_1(E_1', V, PO(p+1, q+1), \pi_1)$ is a principal bundle with structure group $PO(p+1, q+1)$ over the same base V. Thus, e_0 and e_{n+1} being chosen once and for all, the two bundles ξ_1 and $P\xi_1$ are isomorphic. Subsequently, as the action of $PO(p+1, q+1)$ on the set of projective frames of $P(E_{n+2})$ is simply transitive, it is suitable to retain $P\xi_1$ the principal bundle, $\tilde{\lambda}$-extension of ξ, with structure group $PO(p+1, q+1)$.

c) Let us introduce $\theta(V)$ the trivial bundle with typical fibre $\mathbf{R}^2$ with a quadratic form Q_2 of signature $(1, 1)$, and let us write: $\theta(V)=\xi_0\oplus\xi_{n+1}$, as a Whitney sum of two bundles with typical fibre $\mathbf{R}$ and the required condition of orthogonality for Q_2.

We define, then, $T_1(V)=T(V)\oplus\theta(V)=\bigcup_{x\in V} T_1(x)(V)$, where $T(V)$ is the tangent bundle of V and $T_1(x)(V)=T(x)\oplus(\zeta_0)_x\oplus(\zeta_{n+1})_x$, with obvious notations.

We denote by Clif (V, Q) or simply Clif (V), the Clifford bundle of V and we introduce another bundle $\mathrm{Clif}_1(V)$ in the following way. In any point $x\in V$, let us consider $\otimes T_1(x)$ and the Clifford algebra $(Cl_{n+2})_x$ obtained as a quotient algebra of $\otimes T_1(x)$ by the ideal generated by $X_1(x)\otimes X_1(x)-Q_{n+2}(X_1(x))$, where $X_1(X)\in T_1(X)$ and Q_{n+2} is the quadratic form of signature $(p+1, q+1)$ defined on $\mathbf{R}^{n+2}$. The collection of the Clifford algebras $(Cl_{n+2})_x$ is naturally a vector bundle of typical fibre $Cl_{n+2}(p+1, q+1)$, which we denote by $\mathrm{Clif}_1(V)$ and which is an "amplified Clifford bundle" in the same way as $T_1(V)$ is an "amplified tangent bundle". It is possible to define the action of the group $C_n(p, q)$ on such a bundle by means of the representation K_1 so settled. For any ω belonging to $Cl_{n+2}(p+1, q+1)$, for any $\varphi(g)\in C_n(p, q)$ we set: $K_1\varphi(g)\omega=\alpha(g)\omega g^{-1}$, which defines a representation of $C_n(p, q)$ into $Cl_{n+2}(p+1, q+1)$. Thus, $PO(p+1, q+1)$ isomorphic to $C_n(p, q)$ operates on $\mathrm{Clif}_1(V)$. $\mathrm{Clif}'_1(V)$ denotes its complexified and in the same way as previously, we can define the action of $PO'(n+2)$ isomorphic to C'_n on this bundle.

2. Flat conformal spin structures in even dimension

1) Let $\mathbf{R}^{2r}$ be endowed with a quadratic form of signature (p, q): we suppose that $p\leqq n-p\ (n=2r)$. As in [2, p. 40], we introduce a "real" "special Witt-decomposition" of $\mathbf{C}^{n+2}=E'_{n+2}=(E_{n+2})_{\mathbf{C}}$, naturally associated with the previous basis of E_{n+2}: $\{e_1, \dots, e_n, e_0, e_{n+1}\}$: $(W_1)_{n+2}=\{x_i, y_j\}$ with:

$$\begin{cases} x_1=\dfrac{e_1+e_n}{2}, \dots, x_p=\dfrac{e_p+e_{n-p+1}}{2},\ x_{p+1}=\dfrac{ie_{p+1}+e_{n-p}}{2}, \dots, x_r=\dfrac{ie_r+e_{n-r+1}}{2}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x_0=\dfrac{e_0+e_{n+1}}{2} \\ y_1=\dfrac{e_1-e_n}{2}, \dots, y_p=\dfrac{e_p-e_{n-p+1}}{2},\ y_{p+1}=\dfrac{ie_{p+1}-e_{n-p}}{2}, \dots, y_r=\dfrac{ie_r-e_{n-r+1}}{2}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad y_0=\dfrac{e_0-e_{n+1}}{2}. \end{cases}$$

So that for all i and j, $B(x_i, y_j)=\dfrac{\delta_{ij}}{2}$ and $x_iy_j+y_jx_i=\delta_{ij}=2B(x_i, y_j)$ $0\leqq i\leqq r$, $0\leqq j\leqq r$.

We know [6, 7] that, for each Witt-decomposition of E'_{n+2}, $E'_{n+2}=F\oplus F'$, we can find a basis of isotropic vectors $\{\eta_1, \dots, \eta_r, \eta_0\}$ in F' and a basis of isotropic vectors $\{\zeta_1, \dots, \zeta_r, \zeta_0\}$ in F, such that $\{\zeta_i, \eta_j\}$ is a "real" Witt-basis of E'_{n+2}. With the same notations as in I) 1, we consider $\eta=k_1\circ\varphi$ from $\mathrm{Pin}\,(p+1)$ onto $PO(p+1, q+1)$ via the exact sequence:

$$1\to\mathscr{A}\to\mathrm{Pin}\,(p+1, q+1)\xrightarrow{\eta} PO(p+1, q+1)\to 1$$

and $\eta'=k'_1\circ\varphi'$, so that we have the corresponding exact sequence:

$$1\to\mathscr{A}'\to\mathrm{Pin}'\,(n+2)\xrightarrow{\eta'} PO'(n+2)\to 1.$$

Let Cl'_{n+2} be the complexified algebra of Cl_{n+2}, and let ϱ be the classical spin-representation, [3], of Cl'_{n+2} corresponding to the left action of Cl'_{n+2} on the minimal ideal $Cl'_{n+2}f_{r+1}$, (where $f_{r+1}=y_1y_2\ldots y_ry_0$ is an isotropic $(r+1)$-vector), called [cf. 2] *"the space of conformal spinors" associated* with $E_n(p,q)$.

2) In [2] we consider the projective space $P(E'_{n+2})$ and *projective Witt-frames of* $P(E'_{n+2})$, associated with Witt-basis of F'_{n+2} and in particular *projective orthogonal Witt-frames* of $P(E'_{n+2})$.

Let $(\tilde{\Omega}_{n+2})_1$ and $(\tilde{W}_{n+2})_1$ be two projective orthogonal Witt-frames of $P(E'_{n+2})$ so that $(\tilde{W}_{n+2})_1=\tau_1^{-1}(\tilde{\Omega}_{n+2})_1$ where $\tau_1^{-1}\in PO(p+1,q+1)$. As in [4], we identify the complexified of τ_1 with τ_1. Thus, we determine the action of $\mathrm{Pin}\,(p+1,q+1)$ on $PO(p+1,q+1)$. Let g be one of the four elements of $\mathrm{Pin}\,(p+1,q+1)$ such that $\eta(g)=\tau_1\in PO(p+1,q+1)$. We observe that $\tau_1=\eta(g)=\eta(-g)=\eta(e_Ng)=\eta(-e_Ng)$; [2].

If $(\tilde{W}_{n+2})$ is a projective Witt-frame of $P(E'_{n+2})$ associated with an orthogonal projective frame of $P(E'_{n+E})$ and with a "real" orthonormal basis $(\mathscr{B}'_1)_{n+2}$ of E'_{n+2} and with a "real" orthonormal basis $(\mathscr{B}'_1)$ of E'_n, (e_0, e_{n+1} being chosen once and for all), *we define* [1, 2], "over" the *orthonormal "real" basis* $(\mathscr{B}'_{1n})$ *of* E'_n *the four spinor-frames called conformal spinor-frames or* E_n:

$$\{\varepsilon_1(x_{i_0}x_{i_1}\ldots x_{i_h}f_{r+1})\} \quad \text{where} \quad \varepsilon_1=\begin{cases}\pm 1 & \text{or}\\ \pm e_N\end{cases} \quad i_0<i_1<\ldots<i_h$$

such that if $\eta(g)=\tau_1\in PO(p+1,q+1)$ and if $\delta\in\{g,-g,ge_N,-ge_N\}$ we have: $x_{i_0}x_{i_1}\ldots x_{i_h}f_{r+1}=\delta^{-1}\xi_{i_0}\xi_{i_1}\ldots\xi_{i_h}\delta f_{r+1}=\varrho(\delta^{-1})\xi_{i_0}\xi_{i_1}\ldots\xi_{i_h}\delta f_{r+1}$. This is equivalent to: $\varrho(\delta)[x_{i_0}x_{i_1}\ldots x_{i_h}f_{r+1}]=\delta x_{i_0}x_{i_1}\ldots x_{i_h}f_{r+1}=\xi_{i_0}\xi_{i_1}\ldots\xi_{i_h}\delta f_{r+1}$. Thus, $(\tilde{\mathscr{R}}_{n+2})_1=\eta(\delta)(\tilde{\mathscr{R}}'_{n+2})_1$ is equivalent to $S_{n+2}=\varrho(\delta)S'_{n+2}$ where $(\tilde{\mathscr{R}}_{n+2})_1$ and $(\tilde{\mathscr{R}}'_{n+2})_1$ — [respectively S_{n+2} and S'_{n+2}] are projective orthogonal frames in the projective space $P(E'_{n+2})$, respectively "conformal spinor-frames" with $S'_{n+2}=x_{i_0}\ldots x_{i_h}f_{r+1}$ and $S_{n+2}=\xi_{i_0}\ldots\xi_{i_h}\delta f_{r+1}$.

DEFINITION 1. A conformal spinor of E_n, associated with a complex representation ϱ of $\mathrm{Pin}\,(p+1,q+1)$ in a space of spinors for the Clifford algebra Cl'_{n+2}, is by definition an equivalence class $((\tilde{\mathscr{R}}_{n+2})_1,g,\chi_{n+2})$, where $(\tilde{\mathscr{R}}_{n+2})_1$ is a projective orthogonal frame of $P(E'_{n+1})$, $g\in\mathrm{Pin}\,(p+1,q+1)$, $\chi_{n+2}\in\mathbf{C}^{2^{r+1}}$ and where $((\tilde{\mathscr{R}}'_{n+2})_1,g',\chi'_{n+2})$ is equivalent to $((\tilde{\mathscr{R}}_{n+2})_1,g,\chi_{n+2})$ if and only if we have: $(\tilde{\mathscr{R}}'_{n+2})_1=\sigma(\tilde{\mathscr{R}}_{n+2})_1$, $\sigma=\eta(\gamma)\in PO(p+1,q+1)$ with $\gamma=g'g^{-1}$ and $\chi'_{n+2}={}^t\varrho(\gamma))^{-1}\chi_{n+2}$, where ${}^t\varrho^{-1}$ is the dual representation of ϱ and where $(\varrho(\gamma))^{-1}$ is identified with an endomorphism of $\mathbf{C}^{2^{r+1}}$.

We can also write: $(\tilde{\mathscr{R}}'_{n+2})_1=(\tilde{\mathscr{R}}_{n+2})\sigma$ instead of $(\tilde{\mathscr{R}}'_{n+2})_1=\sigma(\tilde{\mathscr{R}}_{n+2})_1$, which defines a right action and, in the same way, we can use the associated projective orthogonal Witt-frames of $P(E'_{n+2})$: $(\tilde{\Omega}_{n+2})_1$, $(\tilde{\Omega}'_{n+2})_1$.

DEFINITION 2. We agree to call by definition an equivalence class $((\tilde{\mathscr{R}}_{n+2})_1,g)$ *where g is in* $\mathrm{Pin}\,(p+1,q+1)$ *and* $(\tilde{\Omega}_{n+2})_1$ *is a projective orthogonal frame of* $P(E'_{n+2})$ *a conformal spinor frame of* E_n associated with the "real" orthonormal basis $(\mathscr{B}_1)_n$ of E'_n. $((\tilde{\mathscr{R}}_{n+2})_1,g)$ is equivalent to $(\tilde{\mathscr{R}}'_{n+2})_1,g')$ if and only if: $(\tilde{\mathscr{R}}'_{n+2})_1=(\tilde{\mathscr{R}}_{n+2})_1\sigma$ and $\sigma=\eta(\gamma)$ with $g,g'\in\mathrm{Pin}\,(p+1,q+1)$, and $\gamma=g'g^{-1}$.

We remark that $((\tilde{\mathscr{R}}_{n+2})_1, g) \sim ((\tilde{\mathscr{R}}_{n+2})_1, -g) \sim ((\tilde{\mathscr{R}}_{n+2})_1, e_N g) \sim ((\tilde{\mathscr{R}}_{n+2})_1, -e_N g)$. *If we suppose* $g, g' \in \mathrm{Pin}\,(n+2)$ *with* $\gamma = g'g^{-1} \in \mathrm{Pin}\,(p+1, q+1)$, *we can consider the action of* $\mathrm{Pin}\,(p+1, q+1)$ *on every spinor frame* of $Cl'_{n+2} f_{r+1}$.

Definition 3. With obvious notations, $(\tilde{\Omega}_{n+2})_1$ and $(\tilde{\Omega}'_{n+2})_1$ being projective orthogonal Witt-frames of $P(E'_{n+2})$, $((\tilde{\Omega}_{n+2})_1, g)$ *and* $(\tilde{\Omega}'_{n+2})_1, g')$ *define the same flat conformal spin structure if and only if* $(\tilde{\Omega}'_{n+2})_1 = \sigma(\tilde{\Omega}_{n+2})_1$, $\eta'(\gamma) = \sigma$, $\gamma = g'g^{-1}$, $g, g' \in \mathrm{Pin}'\,(n+2)$, $\gamma \in \mathrm{Pin}\,(p+1, q+1)$.

(Thus $((\tilde{\Omega}_{n+2})_1, g) \sim ((\tilde{\Omega}_{n+2})_1, -g) \sim ((\tilde{\Omega}_{n+2})_1, e_N g) \sim ((\tilde{\Omega}_{n+2})_1, -e_N g)$). In the same way as in [2] we define complex conformal spin flat structures, using the mapping η' from $\mathrm{Pin}'\,(n+2)$ onto $PO'(n+2)$ with kernel $\mathscr{A}'$.

3. Manifolds of even dimension admitting a real conformal spin structure in a strict sense

Let V be a real paracompact n-dimensional smooth pseudo-Riemannian (in particular Riemannian) manifold. In this paragraph and the next three we assume that n is even, $n = 2r$. As in 1, ξ stands for the bundle of orthonormal frames of V; $P\xi_1(E'_1, V, PO(p+1, q+1), \pi_1)$ is the principal bundle obtained as the $\tilde{\lambda}$-extension of ξ. We agree to give the following definitions which generalize those given in [4], [10, 11] for the orthogonal case to the conformal orthogonal one.

Definition 4. V admits a real conformal spin structure in a strict sense if there exists a principal fibre bundle $S_1(E_1, V, \mathrm{Pin}\,(p+1, q+1), q_1)$ and a morphism of principal bundles $\tilde{\eta}\colon S_1 \to P\xi_1$, such that S_1 be a 4-fold covering of $P\xi_1$ with the following commutative diagram where the horizontal mappings correspond to right translations:

$$\begin{tikzcd}
E_1 \times \mathrm{Pin}\,(p+1, q+1) \arrow[r] \arrow[d, "\tilde{\eta} \times \eta"'] & E_1 \arrow[d, "\tilde{\eta}"'] \arrow[dr, "q_1"] & \\
E'_1 \times PO(p+1, q+1) \arrow[r] & E'_1 \arrow[r, "\pi_1"'] & V
\end{tikzcd}$$

S_1 is called the bundle of conformal spinor frames of V.

Definition 5. According to this definition, we introduce the *bundle of conformal spinors* $\sigma_1 = \left(\dfrac{S_1(V) \times \mathbf{C}^{2^{r+1}}}{\mathrm{Pin}\,(p+1, q+1)}, V, \mathrm{Pin}\,(p+1, q+1), \mathbf{C}^{2^{r+1}}\right)$, complex vector bundle of dimension 2^{r+1} with typical fibre $\mathbf{C}^{2^{r+1}}$ associated with *the bundle* $S_1(V)$ of "*conformal spinor frames*". We write: $\sigma_1 = (\sigma'_1, V, \mathrm{Pin}\,(p+1, q+1), s_1)$.

Remarks. It is always possible to define the two fibrations $P\xi_1$ and S_1 by means of the same trivialising neighbourhoods $(U_{\alpha'})_{\alpha' \in A}$ and local cross-section $z_{\alpha'}$, $\tilde{\mathscr{R}}_{\alpha'}$, with transition functions, $\gamma_{\alpha'\beta'}$, respectively $\eta(\gamma_{\alpha'\beta'})$.

$$z_{\beta'}(x) = z_{\alpha'}(x)\gamma_{\alpha'\beta'}(x), \qquad \gamma_{\alpha'\beta'}(x)\in \mathrm{Pin}\,(p+1,\, q+1)$$

$$\tilde{\eta}(z_{\beta'}(x)) = \tilde{\mathscr{R}}_{\beta'}(x) = \tilde{\eta}(z_{\alpha'}(x))\eta(\gamma_{\alpha'\beta'}(x)) =$$

$$= \tilde{\mathscr{R}}_{\alpha'}(x)\eta(\gamma_{\alpha'\beta'}(x)), \qquad \eta(\gamma_{\alpha'\beta'})\in PO(p+1,\, q+1)$$

$\tilde{\mathscr{R}}_{\alpha'}(x)$ and $\tilde{\mathscr{R}}_{\beta'}(x)$ are "projective orthogonal frames" of $P(E'_{n+2})$.

Let us consider the Clifford algebra Cl_{n+2} of $E_{n+2}(p+1, q+1)$ and the complexified algebra Cl'_{n+2} isomorphic to $Cl_{n+2}(Q')$ where Q' is the complexification of Q [2].

The sequence

$$\{x_{i_0}x_{i_1}\dots x_{i_h}y_{j_0}y_{j_1}\dots y_{i_h}\}\begin{cases}0\leqq h\leqq r\\ 0\leqq i_0<i_1<i_2\dots\leqq r\\ 0\leqq j_0<j_1<j_2\dots\leqq r\end{cases}$$

is a basis of Cl'_{n+2} where $\{x_i, y_j\}$ is the special Witt-basis of $\mathbf{C}^{n+2}$.

The choice of the above basis establishes a linear isomorphism μ between Cl'_{n+2} and $\mathbf{C}^{2^{n+2}}$.

We can observe that the spinorial bundle σ_1 associated with the bundle S_1 is a principal bundle with typical fibre $\mathbf{C}^{2^{r+1}}$ and structure group $\mathrm{Pin}\,(p+1, q+1)$, which operates effectively in $\mathbf{C}^{2^{r+1}}$ ($\mathbf{C}^{2^{r+1}}$ is an irreductible Cl'_{n+2}-representation-space).

It is permissible to choose any irreducible representation of Cl'_{n+2} in $C^{2^{r+1}}$ and convenient to choose the representation corresponding to the left action of Cl'_{n+2} in the minimal ideal of conformal spinors, $Cl'_{n+2}f_{r+1}=Cl'_{n+2}y_1y_2\dots y_r y_0$ of which the

$$\{\varepsilon_1 x_{i_0}x_{i_1}\dots x_{i_h}f_{r+1}\} \quad (i_0<i_1<\dots<i_h), \quad \text{where } \varepsilon_1=\begin{cases}\pm 1 \text{ or (cf. II, 1 and 2)}\\ \pm e_N\end{cases}$$

constitute *"four conformal spinor frames"*.

By restriction of μ to $Cl'_{n+2}f_{r+1}$ we obtain a linear identification of $Cl'_{n+2}f_{r+1}$ *with* $\mathbf{C}^{2^{r+1}}$.

Over an open set of V, endowed with the cross-section $z\colon x\to z(x)$ of S_1 a conformal spinor field χ will be defined by a smooth application χ from E_1 into $\mathbf{C}^{2^{r+1}}\colon z\to \to\chi(z)$ such that ([10]) if $\chi(z)=\mu(u)$ $u\in Cl'_{n+2}f_{r+1}, (u=vf_{r+1})$, then

$$\chi(z\gamma^{-1}) = \gamma\chi(z) = \mu(\gamma u), \qquad (\forall\gamma), \quad (\gamma\in\mathrm{Pin}\,(p+1, q+1)).$$

We denote χ_x the restriction of χ to $\mathscr{S}_{1x}=s_1^{-1}(x)$ and observe that

$$\text{(I)} \qquad (\chi_x(z))^{i_0i_1\dots i_h}x_{i_0}x_{i_1}\dots x_{i_h}f_{r+1}=(\chi_x(z))^{i_0i_1\dots i_h}(\gamma x_{i_0}x_{i_1}\dots x_{i_h})f_{r+1}.$$

4. Necessary conditions for the existence of a real conformal spin structure in a strict sense on manifolds of even dimension

Let $x \to z_x$ be a local cross-section over $\mathscr{U}$, a trivializing open set in the bundle S_1. As in [7] we set: $z_x = v(x, g(x)) = v^x(g(x))$, $g(x) \in \mathrm{Pin}\,(p+1, q+1)$: according to the construction of associated bundles $[z_x, x_{i_0} \ldots x_{i_h} f_{r+1}]$, identified to $[z_x \gamma^{-1}, \gamma^x_{(i)} f_{r+1}]$, is a cross-section over $\mathscr{U}$ in the bundle σ_1 which we denote by $[z_x, x_{(i)} f_{r+1}]$ or $M^x(x_{(i)} f_{r+1})$. Let also $\tilde{\mathscr{R}}_x = \tilde{\eta}(z_x)$.

Let $(\mathscr{U}_{\alpha'})_{\alpha' \in A}$ be a trivializing atlas for the bundle $P\xi_1$. We can always suppose that there exists over $\mathscr{U}_{\alpha'}$ a cross-section $z_{\alpha'}$ in S_1; we take again $\tilde{\mathscr{R}}_{\alpha'}(x) = \tilde{\eta}(z_{\alpha'}(x))$. If $\tilde{W}_{\alpha'}(x)$ is the projective "real" Witt-frame associated with the projective orthogonal frame $\tilde{\mathscr{R}}_{\alpha'}(x)$, we write, *abusively*, $\tilde{\eta}(z_{\alpha'}(x)) = \tilde{W}_{\alpha'}(x)$.

In the projective space $P(\mathbf{C}^{n+2})$ the projective "real" frame

$$\underbrace{\{\pi(x_0), \ldots, \pi(x_r), \pi(y_0), \ldots, \pi(y_r), \pi(x_0 + \ldots + x_r + y_0 + \ldots + y_r)\}}_{(2r+3)\ \text{elements}}$$

correspond to the "real" Witt-basis $\{x_i, y_j\}$ $0 \leq i \leq r$, $0 \leq j \leq r$ of $\mathbf{C}^{n+2}$. (π is the canonical map: $E_{n+2} \to P(E_{n+2})$.) We agree to denote such a projective "real" frame by $\widetilde{\{x_i, y_j\}}$. As the action of $PO(p+1, q+1)$ on the set of projective orthogonal frames is simply transitive, we can write: $\tilde{W}_{\alpha'}(x) = \tilde{\eta}(z_{\alpha'}(x) = \tilde{\Theta}^x_{\alpha'}(\widetilde{\{x_i, y_j\}})$, where the $\tilde{\Theta}^x_{\alpha'}$ admit the transition functions $\eta(\gamma_{\alpha'\beta'})$ in $PO(p+1, q+1)$.

If there exists over V a real conformal spin structure in a strict sense, this structure induces in the "amplified" tangent space $T_1(x)$ at x a flat real conformal spin structure (in a purely algebraic way (see II.2)) defined by an equivalence class of $(\tilde{\Omega}_x, g_x)$, $g_x \in \mathrm{Pin}'(n+2)$, $\tilde{\Omega}_x$, "projective Witt frame", depending differentiably on x. Let us recall that

$$(\tilde{\Omega}_x, g_x) \sim (\tilde{\Omega}_x, -g_x) \sim (\tilde{\Omega}_x, e_N g_x) \sim (\tilde{\Omega}_x, -e_N g_x) \qquad \text{(see II.2).}$$

We note that $PO'(n+2)$ operates transitively in the vector space of "real" or complex projective Witt frames, and that in the above class there will always be "real" projective Witt frames.

With the previous notations, at $x \in U_{\alpha'} \cap U_{\beta'}$, we must obtain two "equivalent frames", which necessarily determine the same flat real conformal spin structure in the "amplified" tangent space at $x: T_1(x)$,

$$(\tilde{\Omega}^x_{\alpha'} = \tilde{\Theta}^x_{\alpha'}\{\alpha(\lambda_{\alpha'}(x))\widetilde{\{x_i, y_j\}}\lambda^{-1}_{\alpha'}(x)\},\ g_{\alpha'}(x))$$

and

$$(\tilde{\Omega}^x_{\beta'} = \tilde{\Theta}^x_{\beta'}\{\alpha(\lambda_{\beta'}(x))\widetilde{\{x_i, y_j\}}\lambda^{-1}_{\beta'}(x)\},\ g_{\beta'}(x))$$

$\lambda_{\alpha'}(x)$, $\lambda_{\beta'}(x)$, $g_{\alpha'}(x)$, $g_{\beta'}(x) \in \mathrm{Pin}'(n+2)$, $\lambda_{\alpha'}$, $\lambda_{\beta'}$ defined respectively over $U_{\alpha'}$ and $U_{\beta'}$ and $g_{\alpha'}$, $g_{\beta'}$ over a neighbourhood of x included in $U_{\alpha'} \cap U_{\beta'}$, with $(g_{\beta'} g^{-1}_{\alpha'})_x \in$ $\in \mathrm{Pin}\,(p+1, q+1)$ and $\eta(g_{\beta'}(x)) = \eta(g_{\alpha'\beta'}(x) g_{\alpha'}(x))$, denoting $\eta(g_{\alpha'\beta'})$ the transition functions of $\tilde{\Omega}^x_{\alpha'}$, $\tilde{\Omega}^x_{\beta'}$, $g_{\alpha'\beta'}$ with values in $\mathrm{Pin}\,(p+1, q+1)$, and $\eta(\alpha(\lambda_{\alpha'}) g_{\alpha'\beta'} \lambda^{-1}_{\beta'}) =$ $= \eta(\lambda_{\alpha'\beta'})$.

We also set $\tilde{\Theta}^x_{\alpha'}(\alpha(\lambda_{\alpha'}(x))\widetilde{\{x_i, y_j\}}\lambda^{-1}_{\alpha'}(x) = \tilde{\mu}^x_{\alpha'}(\widetilde{\{x_i, y_j\}})$.

With the notations of II.3, if $\chi(\tilde{\Omega}^x_{\alpha'}, g_{\beta'}(x)) = \mu(f_{r+1})$, then

$$\chi(\tilde{\Omega}^x_{\beta'}, g_{\beta'}(x)) = \mu(\varepsilon_1 g^{-1}_{\alpha'\beta'}(x) f_{r+1}) \quad \text{where} \quad \varepsilon_1 = \begin{cases} \pm 1 & \text{or} \\ \pm e_N. \end{cases}$$

As the spinor thus defined at x is a well determined element in $(Cl'_{n+2} f_{r+1})_x$, $\chi(\tilde{\Omega}^x_{\alpha'}, g_{\alpha'}(x)) = \chi(\tilde{\Omega}^x_{\beta'}, g_{\beta'}(x))$ whence we deduce:

(I) $$f_{r+1} = \varepsilon_2 \varepsilon_1 f_{r+1} g^{-1}(x),$$

where $\varepsilon_2 = \pm 1$ if r is even and $\varepsilon_2 = 1$ if r is odd.

As matter of fact, let us recall, first, that e_N anticommutes with every element of E_{n+2}, [3], and that for all $g \in \mathrm{Pin}\,(p+1, q+1)$, $e_N g = \pm g e_N$, and that $\alpha(g) = \pm g$ for all $g \in \mathrm{Pin}\,(p+1; q+1)$. Moreover, $\tilde{\mu}^x_{\alpha'}$ and $\tilde{\mu}^x_{\beta'}$ satisfy the relation

$$\tilde{\mu}^x_{\beta'}(u) = \tilde{\mu}^x_{\alpha'}(\alpha(g_{\alpha'\beta'})(x)) u g^{-1}_{\alpha'\beta'}(x))$$

for all $u \in Cl'_{n+2}$. Consequently,

$$\tilde{\mu}^x_{\alpha'}(f_{r+1}) = \tilde{\mu}^x_{\beta'}(\varepsilon_1 g^{-1}_{\alpha'\beta'}(x) f_{r+1}) = \tilde{\mu}^x_{\alpha'}(\alpha(g_{\alpha'\beta'})(x) \varepsilon_1 g^{-1}_{\alpha'\beta'}(x) f_{r+1} g^{-1}_{\alpha'\beta'}(x)) =$$
$$= \tilde{\mu}^x_{\alpha'}(\varepsilon_1 \alpha(g_{\alpha'\beta'}(x) g^{-1}_{\alpha'\beta'}(x) f_{r+1} g^{-1}_{\alpha'\beta'}(x)) = \tilde{\mu}^x_{\alpha'}(\varepsilon_1 fr_{+1} g^{-1}_{\alpha'\beta'}(x)).$$

Therefore, we obtain (I), denoting that via the projective space, there appears a quotient and, then, the factor $\varepsilon_2 = \pm 1$ corresponding to the ambiguity of sign for homogeneous elements of the Clifford algebra. Using the principal antiautomorphism β of the Clifford algebra and observing that for all g belonging to $\mathrm{Pin}\,(p+1, q+1)$, $\beta(g) = g^{-1} N(g)$, [3], and that

$$\beta(e_N) = (-1)^{\frac{(n+1)(n+2)}{2}} e_N = (-1)^{r+1} e_N, \qquad (\text{for } n = 2r),$$

further

$$\beta(f_{r+1}) = (-1)^{\frac{r(r+1)}{2}} f_{r+1},$$

and

$$\beta(g^{-1}_{\alpha'\beta'}(x)) = \frac{g_{\alpha'\beta'}(x)}{N(g_{\alpha'\beta'}(x))},$$

we obtain

$$\varepsilon_2 f_{r+1} = \frac{g_{\alpha'\beta'}(x)}{N(g_{\alpha'\beta'}(x))} f_{r+1} \beta(\varepsilon_1).$$

As $f_{r+1} e_N = (-1)^{r+1} (-i)^{r-p} f_{r+1}$ [see I.1] we get, if $\varepsilon_1 = e_N$,

$$\varepsilon_2 f_{r+1} N(g_{\alpha'\beta'}(x)) = g_{\alpha'\beta'}(x) (-i)^{r-p} f_{r+1},$$

or equivalently,

$$g_{\alpha'\beta'}(x) f_{r+1} = \varepsilon_2 (i)^{r-p} N(g_{\alpha'\beta'}(x)) f_{r+1}$$

and then, in any case,

(II) $$(g_{\alpha'\beta'}(x) f_{r+1}) = \varepsilon_2 \varepsilon N(g_{\alpha'\beta'}(x)) f_{r+1}$$

where

$$\varepsilon = \pm 1 \quad \text{if} \quad r-p \equiv \begin{cases} 0 \\ 2 \end{cases} \quad \text{or} \quad (\text{modulo } 4)$$

and

$$\varepsilon = \begin{cases} \pm 1 \\ \pm i \end{cases} \quad \text{or if} \quad r-p \equiv \begin{cases} 1 \\ 3 \end{cases} \quad \text{or} \quad (\text{modulo } 4)$$

cf. [2].

Thus, $g_{\alpha'\beta'}(x)$ belongs to a subgroup H_C of Spin $(p+1, q+1)$ which is mapped by η onto a subgroup of $PSO\,(p+1, q+1)$-special projective orthogonal group isomorphic to a subgroup S_C called in [2] "the conformal spinoriality group S_C" in a strict sense, [see part I], associated with the r-isotropic vector $f_r = y_1 \dots y_r$. (We observe that $\varphi(H_C) = S_C \subset (C_n(p, q))_r$, the restricted conformal group, [1, 2] where φ is the "projection" from Pin $(p+1, q+1)$ onto $C_n(p, q)$, (cf. I)).

We note that $\alpha(g_{\alpha'\beta'}(x)) = g_{\alpha'\beta'}(x)$ as $H_C \subset \text{Spin}\,(p+1, q+1)$. According to [7, p. 158],

$$g_{\alpha'\beta'}(x) f_{r+1} = \varepsilon_2 \varepsilon N(g_{\alpha'\beta'}(x)) f_{r+1}$$

implies

$$g_{\alpha'\beta'}(x) f_{r+1} g^{-1}_{\alpha'\beta'}(x) = N(g_{\alpha'\beta'}(x)) \varepsilon^2 f_{r+1}, \tag{III}$$

as $(N(g_{\alpha'\beta'}(x)))^2 = 1$ and $\varepsilon_2^2 = 1$, where

$$\varepsilon^2 = (i^{r-p})^2 = (-1)^{r-p} = (e_N)^2 = (-1)^{r+q},$$

for $n = p+q = 2r$. Therefore we have, applying $\tilde{\mu}^x_{\beta'}$ to f_{r+1}, with the following notations:

$$\tilde{\mu}^x_{\beta'}(f_{r+1}) = \tilde{f}_{\beta'}(x) \quad \text{and} \quad \tilde{\mu}^x_{\alpha'}(g_{\alpha'\beta'}(x)) = \tilde{g}_{\alpha'\beta'}(x),$$

taking account of the fact that g is in Pin $(p+1, q+1)$ iff $g = v_1 \dots v_k$, $v_1, \dots, v_k \in E_{n+2}$ with $Q(v_i) = \pm 1$, $1 \leq i \leq k$ [3, 4]—[17]

$$\tilde{f}_{\beta'}(x) = \varepsilon_2 \tilde{g}_{\alpha'\beta'}(x) \tilde{f}_{\alpha'}(x) \tilde{g}^{-1}_{\alpha'\beta'}(x). \tag{IV}$$

Then applying $\tilde{\mu}^x_{\alpha'}$ to the previous relation (III), and observing that $N(\tilde{g}_{\alpha'\beta'}(x)) = N(g_{\alpha'\beta'}(x))$, we obtain

$$\tilde{f}_{\beta'}(x) = \varepsilon_2 (e_N)^2 N(\tilde{g}_{\alpha'\beta'}(x)) \tilde{f}_{\alpha'}(x). \tag{V}$$

We observe that $\eta(g_{\alpha'\beta'}(x))$ are transition functions for cross-sections in the complexified bundle $(P\xi_1)_C$ of $P\xi_1$. The cocycle $\eta(\gamma_{\alpha'\beta'})$ which defines $P\xi_1$ and the cocycle $\eta(g_{\alpha'\beta'})$ are cohomologous in $PO'(n+2)$. Thus, we have obtained:

Proposition 1. *If there exists on V a real conformal spin structure in a strict sense,*

1° *there exists over V an isotropic $(r+1)$-vector pseudo-field modulo a factor ε_2, $\varepsilon_2 = \pm 1$ if r is even, $\varepsilon_2 = 1$ if r is odd pseudo-cross section in the bundle $Clif_1'(V)$.*

2° *The group of the principal bundle $P\xi_1$ is reducible in $PO'(n+2)$ to a subgroup isomorphic to S_C — the conformal spinoriality group in a strict sense associated with the r-isotropic vector $f_r = y_1 \dots y_r$ — which is a subgroup of $(C_n(p, q))_r$, the restricted conformal group.*

3° *The complexified bundle* $(P\xi_1)_C$ *admits local cross-sections over trivializing open sets with transition functions* $\eta(g_{\alpha'\beta'})$, $g_{\alpha'\beta'}(x) \in \mathrm{Spin}\,(p+1, q+1)$ *such that if the mappings* $\tilde{f}_{\alpha'}: x \in U_{\alpha'} \cap U_{\beta'} \to \tilde{f}_{\alpha'}(x)$ *locally define the previous* $(r+1)$*-isotropic pseudo-field, then*

$$\tilde{f}_{\beta'}(x) = (e_N)^2 N(g_{\alpha'\beta'}(x))\tilde{f}_{\alpha'}(x), \qquad \textit{modulo} \quad \varepsilon_2,$$

and

$$\tilde{f}_{\beta'}(x) = \tilde{g}_{\alpha'\beta'}(x)\tilde{f}_{\alpha'}(x)\tilde{g}^{-1}_{\alpha'\beta'}(x) \qquad (\textit{modulo} \ \ \varepsilon_2),$$

where $\varepsilon_2 = \pm 1$ *if r is even and* $\varepsilon_2 = 1$ *if r is odd.*

5. Sufficient conditions for the existence of real conformal spin structures in a strict sense on manifolds of even dimension

Let us consider the bundle $P\xi_1$.

Proposition 2. *Let* $(U_{\alpha'}, \tilde{\mu}_{\alpha'})_{\alpha' \in A}$ *be a trivializing atlas for the complexified bundle* $(P_1\xi)_C$ *on V, with transition functions* $\eta(g_{\alpha'\beta'}(x)) \in PO(p+1, q+1)$.

If there exists over V an isotropic $(r+1)$*-vector pseudo-field, modulo a factor* $\varepsilon_2 = \pm 1$ *if r is even and* $\varepsilon_2 = 1$ *if r is odd, locally determined by means of* $x \in U_{\alpha'} \to \tilde{f}_{\alpha'}(x)$ *such that if* $x \in U_{\alpha'} \cap U_{\beta'} \neq \emptyset$ *we have* $\tilde{f}_{\beta}(x) = \tilde{g}_{\alpha'\beta'}(x)\tilde{f}_{\alpha'}(x)\tilde{g}^{-1}_{\alpha'\beta'}(x)$, *modulo* ε_2,

$$\tilde{\mu}^x_{\alpha'}(g_{\alpha'\beta'}(x)) = \tilde{g}_{\alpha'\beta'}(x), \qquad \tilde{f}_{\beta'}(x) = (e_N)^2 N(\tilde{g}_{\alpha'\beta'}(x))\tilde{f}_{\alpha'}(x), \quad \textit{modulo} \ \varepsilon_2,$$

then the manifold V admits a real conformal spin structure in a strict sense.

All the following algebraic calculations are made modulo ε_2, which we omit for simplicity. Shortly, we put $\tilde{f}_{\beta'}(x) = f'$, $\tilde{f}_{\alpha'}(x) = f$, $\tilde{g}_{\alpha'\beta'}(x) = \delta$. Then

$$\left.\begin{cases} f' = \delta f \delta^{-1} \\ f' = (e_N)^2 N(\delta) f \end{cases}\right\} \Rightarrow \delta f \delta^{-1} = (e_N)^2 N(\delta) f \Rightarrow \delta f = (e_N)^2 N(\delta) f,$$

whence we deduce since the intersection of any right minimal ideal with any left minimal ideal is of dimension 1 [3, p. 71]: $\delta f = \tilde{\varepsilon}(x) f$, $\tilde{\varepsilon}(x) \in \mathbf{C}^*$. Then, $(e_N)^2 N(\delta) f \delta \delta^{-1} = \tilde{\varepsilon}(x) f \delta^{-1}$; therefore we obtain

$$f\delta^{-1} = \frac{(e_N)^2 N(\delta) f}{\tilde{\varepsilon}(x)}.$$

Applying the principal anti-automorphism β to $f\delta^{-1}$ we get:

$$\beta(\delta^{-1})\beta(f) = \frac{(e_N)^2 N(\delta)}{\tilde{\varepsilon}(x)} \beta(f),$$

or equivalently,

$$\frac{\delta}{N(\delta)} f = \frac{(e_N) N(\delta)}{\tilde{\varepsilon}(x)} f,$$

as

$$\beta(\delta^{-1}) = \frac{\delta}{N(\delta)}, \qquad [3], [17]$$

thus

$$\delta f = \tilde{\varepsilon}(x) f = (e_N)^2 \frac{N^2(\delta)}{\tilde{\varepsilon}(x)} f,$$

which gives $(\tilde{\varepsilon}(x))^2 = (e_N)^2$, as $(N(\delta))^2 = 1$, with

$$(e_N)^2 = (-1)^{r-p} = \begin{cases} 1 & \text{if } \; r-p = 0 \quad (\text{modulo } 2) \\ -1 & \text{if } \; r-p = 1 \quad (\text{modulo } 2). \end{cases}$$

Then we obtain $\tilde{\varepsilon}(x) = \pm 1$ if $r-p$ is even and $\tilde{\varepsilon}(x) = \pm i$ if $r-p$ is odd. So, we write $\tilde{\varepsilon}(x) = \tilde{\varepsilon}$ and then $g_{\alpha'\beta'}(x) f_{r+1} = \varepsilon_2 \tilde{\varepsilon} f_{r+1}$ where $\tilde{\varepsilon} = \begin{cases} \pm 1 \text{ or} \\ \pm i \end{cases}$ and $g_{\alpha'\beta'}(x) \in H_C$ (with the notations of II.4).

If at $x \in U_{\alpha'} \cap U_{\beta'}$, $\tilde{\mu}^x_{\alpha'}(y'_1 y'_2 \ldots y'_r y'_0) = \tilde{f}_{\alpha'}(x)$, we can complete the set of vectors $\{y'_1, y'_2, \ldots y'_r, y'_0\}$ with $\{x'_1, x'_2, \ldots, x'_r, x'_0\}$ so that $\tilde{\mu}^x_{\alpha'}\{\widetilde{x'_i, y'_j}\}$ and $\tilde{\mu}^x_{\beta'}\{\widetilde{x'_i, y'_j}\}$ constitute Witt-projective frames in the complexified bundle $(P\xi_1)_C$, with transition functions $\eta(g_{\alpha'\beta'})$. (This is a consequence of the extension of the Witt-Theorem to projective orthogonal classical group and to projective orthogonal frames.) Therefore we shall omit the prime and suppose that $\tilde{\mu}^x_{\alpha'}(y_1 y_2 \ldots y_r y_0) = \tilde{f}_{\alpha'}(x)$.

Let us consider over $U_{\alpha'}$ the local cross-section in $Clif'_1(V)$:

$$x \to (x_{(i)} f_{r+1})^x_{\alpha'} = \tilde{\mu}^x_{\alpha'}(x_{(i)} f_{r+1}).$$

As for any $\alpha' \in A$, if $x \in U_{\alpha'} \cap U_{\beta'}$, $\tilde{g}_{\alpha'\beta'}(x) \tilde{f}_{\alpha'}(x) = \varepsilon_2 \tilde{\varepsilon} \tilde{f}_{\alpha'}(x)$ where $\tilde{\varepsilon} = \pm 1$ if $r-p$ is even and $\tilde{\varepsilon} = \pm i$ if $r-p$ is odd; using β the principal antiautomorphisms of the Clifford algebra we obtain, modulo ε_2,

$$\beta\big(\tilde{f}_{\alpha'}(x) \beta(\tilde{g}_{\alpha'\beta'}(x))\big) = \tilde{\varepsilon}_\beta\big(\tilde{f}_{\alpha'}(x)\big),$$

or, equivalently, $\tilde{f}_{\alpha'}(x) g^{-1}_{\alpha'\beta'}(x) N\big(\tilde{g}_{\alpha'\beta'}(x)\big) = \tilde{\varepsilon} f_{\alpha'}(x)$ modulo ε_2 — as $\beta(g) = g^{-1} N(g)$ — and then

$$\tilde{f}_{\beta'}(x) g^{-1}_{\alpha'\beta'}(x) = (e_N)^2 N\big(\tilde{g}_{\alpha'\beta'}(x)\big) \tilde{f}_{\alpha'}(x) g^{-1}_{\alpha'\beta'}(x) =$$

$$= (e_N)^2 \frac{N\big(\tilde{g}_{\alpha'\beta'}(x)\big)}{N\big(g_{\alpha'\beta'}(x)\big)} \tilde{\varepsilon} \tilde{f}_{\alpha'}(x) \qquad (\text{modulo } \varepsilon_2),$$

and, therefore,

$$\tilde{f}_{\beta'}(x) g^{-1}_{\alpha'\beta'}(x) = (e_N)^2 \tilde{\varepsilon} \tilde{f}_{\alpha'}(x) \qquad (\text{modulo } \varepsilon_2),$$

where $(e_N)^2 \tilde{\varepsilon} = (-1)^{r-p} \tilde{\varepsilon}$. We shall write

$$\tilde{f}_{\beta'}(x) \tilde{g}^{-1}_{\alpha'\beta'}(x) = \varepsilon' \tilde{f}_{\alpha'}(x) \qquad (\text{modulo } \varepsilon_2),$$

where $\varepsilon' = \tilde{\varepsilon}$ if $r-p$ is even and $\varepsilon' = -\tilde{\varepsilon}$ if $r-p$ is odd.

Then

$$(x_{(i)} f_{r+1})^x_{\beta'} = \varepsilon' \tilde{g}_{\alpha'\beta'}(x) (x_{(i)} f_{r+1})^x_{\alpha'} \qquad (\text{modulo } \varepsilon_2),$$

where ε' is determined in any case $[(x_{(i)} f_{r+1})^x_{\beta'}$ is known, $(x_{(i)} f_{r+1})^x_{\alpha'}$ is known and one can find an element of the kernel which gives such a relation].

We can associate to each x in V a 2^{r+1} dimensional subspace, in $T_1(x)$ the amplified tangent space at x, differentiable in x, such that $\tilde{\mu}^x_{\alpha'}(x_{(i)}f_{r+1})=(x_{(i)}f_{r+1})^x_{\alpha'}$ and the transition functions of $\tilde{\mu}^x_{\alpha'}$ are $\eta(g_{\alpha'\beta'})$. Therefore we have constructed a spinorial bundle over V, with typical fibre $\mathbf{C}^{2r+1}$.

With the frame $\{x_{(i)}f_{r+1}\}^x_{\alpha'}$, we associate the frame $\tilde{\mu}^x_{\alpha'}\{\widetilde{x_i,x_j}\}$. Then with $\{g_{\alpha'\beta'}(x)x_{(i)}f_{r+1}\}^x_{\alpha'}$ is associated $\tilde{\mu}^x_{\beta'}\{\widetilde{x_i,y_j}\}$.

We can determine $\lambda_{\alpha'}(x)\in \mathrm{Pin}'(n+2)$ such that with the frame $\{\lambda_{\alpha'}x_{(i)}f_{r+1}\}$ is associated the frame $\tilde{\mu}^x_{\alpha'}\{\alpha(\lambda_{\alpha'})\{\widetilde{x_i,y_j}\}\lambda^{-1}_{\alpha'}\}$, where $\tilde{\mu}^x_{\alpha'}\{\alpha(\lambda_{\alpha'})\{\widetilde{x_i,y_j}\}\lambda^{-1}_{\alpha'}\}$ is "a real" projective Witt-frame in $(P\xi_1)_{\mathbf{C}}$. We have got a real conformal spin structure in a strict sense.

REMARK. We can observe that the $g_{\alpha'\beta'}(x)$ are defined modulo $\varepsilon_{1\alpha'\beta'}(x)=\begin{cases}\pm 1 & \text{or}\\ \pm e_N.\end{cases}$ According to previous results (see I, 1) any real conformal structure will be obtained from the one associated with the choice of $\varepsilon_{1\alpha'\beta'}$ such that $\varepsilon_{1\alpha'\beta'}$ determine a cocyle with values in $\mathbf{Z}_2\times\mathbf{Z}_2$ if $(e_N)^2=1$, respectively in $\mathbf{Z}_4$ if $(e_N)^2=-1$.

Therefore the set of conformal spin structures is of the same cardinality as $H^1(V,\mathbf{Z}_2\times\mathbf{Z}_2)$ if $r-p$ is even, respectively as $H^1(V,\mathbf{Z}_4)$ if $r-p$ is odd.

PROPOSITION 3. *Let us assume that the structure group of the bundle $P\xi_1$ reduces in $PO'(n+2)$ to a subgroup isomorphic to a conformal spinoriality group $S_{\mathbf{C}}$ in a strict sense; then the manifold V admits a real conformal spin structure in a strict sense.*

If we have transition functions $\eta(g_{\alpha'\beta'})$, $g_{\alpha'\beta'}(x)\in H_{\mathbf{C}}$, according to $g_{\alpha'\beta'}(x)f_{r+1}=\varepsilon f_{r+1}$, with $\varepsilon=\pm 1$ if $r-p\equiv\begin{cases}0 & \text{or}\\ 2\end{cases}$ (modulo 4) and $\varepsilon=\begin{cases}-1 & \text{or}\\ \pm i\end{cases}$ if $r-p\equiv\begin{cases}1 & \text{or}\\ 3\end{cases}$ (modulo 4); on account of previous remarks, we get

$$\tilde{g}_{\alpha'\beta'}(x)\tilde{f}_{\alpha'}(x)=\varepsilon_2\varepsilon\tilde{f}_{\alpha'}(x)$$

(where $\varepsilon_2=\pm 1$ if r is even and $\varepsilon_2=1$ if r is odd).

Using β the principal antiautomorphism of the Clifford algebra, as $\beta(g)=$ $=g^{-1}N(g)$ for all $g\in \mathrm{Pin}\,(p+1,q+1)$, we get successively,

$$\tilde{f}_{\alpha'}(x)\tilde{g}^{-1}_{\alpha'\beta'}(x)N(\tilde{g}_{\alpha'\beta'}(x))=\varepsilon\tilde{f}_{\alpha'}(x)\qquad(\text{modulo }\varepsilon_2),$$

$$\tilde{f}_{\alpha'}(x)\tilde{g}^{-1}_{\alpha'\beta'}(x)=\frac{\varepsilon\tilde{f}_{\alpha'}(x)}{N(\tilde{g}_{\alpha'\beta'}(x))}\qquad(\text{modulo }\varepsilon_2).$$

As

$$\tilde{f}_{\beta'}(x)=\tilde{g}_{\alpha'\beta}(x)\tilde{f}_{\alpha'}(x)\tilde{g}^{-1}_{\alpha'\beta'}(x)\qquad(\text{modulo }\varepsilon_2),$$

we find

$$\tilde{f}_{\beta'}(x)=\tilde{g}_{\alpha'\beta'}(x)\frac{\varepsilon\tilde{f}_{\alpha'}(x)}{N(\tilde{g}_{\alpha'\beta'}(x))}=\frac{\varepsilon}{N(\tilde{g}_{\alpha'\beta'}(x))}\tilde{g}_{\alpha'\beta'}(x)\tilde{f}_{\alpha'}(x)=$$

$$=\frac{\varepsilon^2}{N(\tilde{g}_{\alpha'\beta'}(x))}\tilde{f}_{\alpha'}(x),\quad(\text{modulo }\varepsilon_2).$$

And then,

$$\tilde{f}_{\beta'}(x)\tilde{g}^{-1}_{\alpha'\beta'}(x) = \frac{\varepsilon^2}{N(\tilde{g}_{\alpha'\beta'}(x))}\tilde{f}_{\alpha'}(x)\tilde{g}^{-1}_{\alpha'\beta'}(x) =$$

$$= \frac{\varepsilon^3}{(N(\tilde{g}_{\alpha'\beta'}(x)))^2}\tilde{f}_{\alpha'}(x) = \varepsilon^3\tilde{f}_{\alpha'}(x), \quad (\text{modulo } \varepsilon_2)$$

(as $(N(\tilde{g}_{\alpha'\beta'}(x))^2 = 1)$, where

$$\varepsilon^3 = \varepsilon \quad \text{if} \quad r-p \equiv \begin{cases} 0 & \text{or} \\ 2 \end{cases} \quad (\text{modulo } 4) \text{ and if} \quad r-p \equiv \begin{cases} 1 & \text{or} \\ 3 \end{cases} \quad (\text{modulo } 4),$$

$$\varepsilon^2 = \{\pm 1 \quad \text{and} \quad \varepsilon^3 = \pm\varepsilon.$$

Starting with this result it is possible to take up again the proof of Proposition 2.

REMARK. We observe that the auxiliary bundle $\Theta(V)$ previously introduced does not occur in such a statement which is, so, self-contained as the conformal spinoriality group is only defined by elements of $E_n(p, q)$ and of its complexified E'_n (see [2] or part I).

6. Manifolds of even dimension with a real conformal spin structure in a broad sense

Let $(C_n(p, q))_r$ be the restricted conformal group (see I, 1). Let $f_{r+1} = y_1 y_2 \ldots \ldots y_r y_0$ be an isotropic $(r+1)$-vector. The enlarged conformal group of spinoriality $(S_C)_e$ associated with the isotropic $(r+1)$-vector f_{r+1} is the subgroup $\varphi((H_C)_e)$ of $(C_n(p, q))_r$ where $(H_C)_e$ is the subgroup to the elements of $\text{Spin}(p+1, q+1)$ such that $\gamma f_{r+1} = \mu f_{r+1}$, $\mu \in \mathbf{C}^*$ [see part I].

In I we proved that $(S_C)_e$ is the "stabilizer" for the action of $(C_n(p, q))_r$ of the s.t.i.m. associated with the isotropic r-vector $y_1 y_2 \ldots y_r$. (We recall that the abbreviation s.t.i.m. stands for maximal totally isotropic subspace.)

DEFINITION. V admits a real conformal spin structure in a broad sense if and only if the structural group $PO(p+1, q+1)$ of the principal bundle $P\xi_1$ — the $\tilde{\lambda}$-extension of the principal bundle ξ of orthonormal frames of V — is reducible to a subgroup of $PO'(n+2)$ isomorphic to $(S_C)_e$, the enlarged conformal group of spinoriality associated with the isotropic r-vector $y_1 \ldots y_r$.

According to Proposition 3 such a definition is a generalization of definitions given in II.3.

PROPOSITION 4. *V admits a real conformal spin structure in a broad sense if and only if there exists over V an $(r+1)$-s.t.i.m.-field idest a sub-bundle of $T_1^{\mathbf{C}}(V)$ such that, with the same notation as in Proposition 2, we have:*

$$\tilde{f}_{\beta'}(x) = \tilde{g}_{\alpha'\beta'}(x)\tilde{f}_{\alpha'}(x)\tilde{g}^{-1}_{\alpha'\beta'}(x),$$

modulo $\varepsilon_2 = \pm 1$ *if r is even*, $\varepsilon_2 = 1$ *if r is odd*, $g_{\alpha'\beta'}(x) \in \mathrm{Pin}(p+1, q+1)$, $\tilde{f}_{\beta'}(x) = = \mu_{\alpha'\beta'}(x)\tilde{f}_{\alpha'}(x)$, $\mu_{\alpha'\beta'}(x) \in \mathbf{C}^*$.

As in the proof of Proposition 2, we obtain $g_{\alpha'\beta'}(x) f_{r+1} = \lambda_{\alpha'\beta'}(x) f_{r+1}$, $\lambda_{\alpha'\beta'}(x) \in \mathbf{C}^*$. Then, taking again the method given in the proof of Proposition 2 above, we get the result.

Conversely, if it is possible to reduce the structure group $PO(p+1, q+1)$ to a subgroup isomorphic to $(S_C)_e$ in $PO'(n+2)$, the same method as in the proof of Proposition 3 leads to the existence of an $(r+1)$-s.t.i.m.-field, locally defined by means of the maps $\tilde{f}_{\alpha'}$.

7. Manifolds of odd dimension admitting a conformal spin special structure

Let us assume that V is an orientable manifold of dimension $2r+1$. We extend the definitions given in II.3 replacing respectively $\mathrm{Pin}(p+1, q+1)$, $C_n(p, q)$ and $PO(p+1, q+1)$ by $\mathrm{Spin}(p+1, q+1)$, $(C_n(p, q))_r$, and $PSO(p+1, q+1)$.

Cl^+_{n+2} is central, simple, [3]. $Cl^+_{n+2}(Q')(n=2r+1$, Q' the complexified of $Q)$, is isomorphic to $Cl_{n+2}(Q')(n=2r)$, [3]. As in [4, 7] and in [2, p. 76], we introduce the associated Witt basis $\{x_i, y_j, z_n\}$ and projective Witt frame and the representation of Cl'^+_{n+2} in the space $x_{i_0}x_{i_1}\dots x_{i_h} f_{r+1}, f_{r+1} = y_1 y_2 \dots y_r y_0$. The bundles S_1 and σ_1 are defined in the same way. In the study of necessary and sufficient existence conditions, only few details are modified: one arrives at identical statements, the $g_{\alpha'\beta'}(x)$ belonging to $\mathrm{Spin}(p+1, q+1)$. (Let us now recall that e_N belongs to the center of Cl_{n+2} and that $e_N f_{r+1} = f_{r+1} e_N = (-i)^{r-p} f_{r+1}$ [2, p. 77].)

III. CONNECTIONS BETWEEN SPIN STRUCTURES AND CONFORMAL SPIN STRUCTURES

Let us assume that $n = p+q = 2r$.

We only study here the case of real conformal spin structures in a strict sense. $\boxed{G}$ stands for the identity component of the Lie group G.

1. Topological remarks

It is known [16, p. 335—341] that $O(p)\times O(q)$ is a maximal compact subgroup of $O(p, q)$ and that every compact subgroup of $O(p, q)$ is conjugate to a subgroup of $O(p)\times O(q)$. More precisely, $O(p, q)$ *is homeomorphic to* $O(p)\times O(q)\times \mathbf{R}^{pq}$.

Thus, as the Poincaré group $P(p, q)$ is the semi-direct product of the Lorentz group $O(p, q)$ and the group of translations of E_n which has n parameters, we got that $P(p, q)$ *is homeomorphic to* $O(p)\times O(q)\times \mathbf{R}^{pq+n}$, therefore, as the conformal affine group $CO(p, q)$ is the semi-direct product of $P(p, q)$ and of the group dilatations, observing that $\mathbf{R}^+$ is homeomorphic to $\mathbf{R}$, we obtain that $CO(p, q)$ *is homeomorphic to* $O(p)\times O(q)\times \mathbf{R}^{pq+n+1}$.

Moreover, $\boxed{O(p, q)} = SO_+(p, q)$ the identity component of $O(p, q)$ contains the identity component $SO(p)\times SO(q)$ of the maximal compact subgroup $O(p)\times$

$\times O(q)$. Let us recall, [1], that every element in the identity component of the conformal group $C_n(p,q)$ is a composition of one "orthochronous" rotation belonging to $SO_+(p,q)$ one translation, one dilatation and one special conformal transformation. Thus, via the results concerning the Lie algebra $LC_n(p,q)$ given in [1], we obtain that $C_n(p,q)$ is homeomorphic to $SO_+(p,q)\times \mathbf{R}^{pq+2n+1}$ and homeomorphic to $SO(p)\times \times SO(q)\times \mathbf{R}^{pq+2n+1}$.

Such a homeomorphism is in agreement with a general result of Cartan—Iwasawa—Mostow [14, p. 59], according which *a bundle whose structure group is a connected Lie group is equivalent in its group to a bundle whose group is a compact subgroup.*

2. Connections between spin structures and conformal spin structures

a) In the same way as in II.1, we introduce the "Greub extension" ξ_j of $\xi - j$-extension of ξ, and ξ_i, i-extension of ξ and, then, $P\xi_1 = \xi_\lambda$ $\tilde{\lambda}$-extension of ξ.

Clif_2 is the auxiliary bundle the typical fibre of which is $Cl_2(1,1)$. Clif (V, Q) is the Clifford bundle of (V, Q). According to the classical isomorphism — (see for example [7]) — which we denote by λ from $Cl_n(p,q)\otimes Cl_2(1,1)$ onto $Cl_{n+2}(p+1,q+1)$, we still abusively denote by λ the isomorphism from $\mathrm{Clif}(V)\otimes \mathrm{Clif}_2$ onto $\mathrm{Clif}_1(V)$ and from $\mathrm{Clif}'_1(V)\otimes \mathrm{Clif}'_2$ onto $\mathrm{Clif}'_1(V)$.

As $\Theta(V)$ is a trivial bundle, let us recall that then there exists a Pin (1, 1)-spin structure on $\Theta(V)$. ψ denotes the "twisted projection" from Pin Q onto $O(Q)$. We shall use the following two statements given in [7].

There exists a Pin (p,q)-spin structure in a strict sense on V iff:

(i) There exists on V, modulo a factor ± 1, an isotropic r-vector field, pseudo-cross-section in the bundle Clif(V); the complexified pseudo-riemannian bundle ξ_C admits local cross-sections, over a trivialization open set $(U_{\alpha'})_{\alpha'\in A}$ with transition functions $\psi(g_{\alpha'\beta'})$, $g_{\alpha'\beta'}\in \mathrm{Spin}\,(p,q)$ such that if $x\in U_{\alpha'}\cap U_{\beta'}\neq\emptyset \to f_{\alpha'}(x)$ locally define the previous r-vector field, then $f_{\beta'}(x) = N(\dot{g}_{\alpha'\beta'}(x)) f_{\alpha'}(x)$; $f_{\beta'}(x) = g_{\alpha'\beta'}(x) f_{\alpha'}(x) g^{-1}_{\alpha'\beta'}(x)$, where

$$f_{\alpha'}(x) = \mu^x_{\alpha'}(f_r), \qquad \dot{g}_{\alpha'\beta'}(x) = \mu^x_{\alpha'}(g_{\alpha'\beta'}(x)),$$

$f_r = y_1 \ldots y_2$; $\mu^x_{\alpha'}$ isomorphism well defined [see 7] from Cl'_n onto $Cl'_n(x)$.

(ii) The structure group of the bundle ξ is reducible in $O'(n)$ to a real spinoriality group $\sigma(p,q)$ in a strict sense.

b) Let us denote by a $C_n(p,q)$ spin structure, respectively by a Pin (p,q)-spin structure, a real conformal spin structure in a strict sense, respectively a real Pin (p,q) spin structure in a strict sense, on V.

In the same way, we agree to denote by a Pin $(p+1, q+1)$ spin structure a real Pin $(p+1, q+1)$-spin structure over the bundle ξ_j of orthonormal frames of the amplified tangent bundle $T_1(V)$. We want to prove the following statement:

Proposition. (1) *If there exists a* Pin (p,q) *spin structure on* V, *then there exists a* Pin $(p+1, q+1)$*-spin structure on* ξ_j.

(2) *If there exists a* Pin $(p+1, q+1)$*-spin structure on* ξ_j*, then there exists a* $C_n(p, q)$*-spin structure on* V.

(3) *If there exists a* $C_n(p, q)$*-spin structure on* V*, if* r *and* p *are odd, then there exists* a Pin $(p+1, q+1)$*-spin structure on* ξ_j.

PROOF. (1) Let us assume that there exists a Pin (p, q)-spin structure on V. Let $f_r = y_1 \dots y_2$ be an isotropic r-vector. By assumption, there exists a pseudo-cross section in the bundle $\mathrm{Clif}'(V)$; so we can naturally form a pseudo-cross section in the bundle $\mathrm{Clif}'(V) \otimes \mathrm{Clif}'_2$, locally determined by

$$x \to \mu^x_{\alpha'}(f_r) \otimes \mu^{2x}_{\alpha'}(y_0) = f_{\alpha'}(x) \otimes f^2_{\alpha'}(x) = \tilde{f}_{\alpha'}(x),$$

where $x \to f^2_{\alpha'}(x) = \mu^2_{\alpha'}(y_0)$ locally determines a cross-section in the bundle Clif'_2, with obvious notations. By using λ, isomorphism form $\mathrm{Clif}'(V) \otimes \mathrm{Clif}'_2$ onto $\mathrm{Clif}'_1(V)$, we obtain a pseudo-cross-section in the bundle $\mathrm{Clif}'_1(V)$ locally determined by means of $x \in U_{\alpha'} \to f'_{\alpha'}(x) = \lambda(\tilde{f}_{\alpha'}(x))$ which satisfy the required conditions for the existence of a Pin (p, q)-spin structure on ξ_j (see [2]).

Moreover, we observe that the reduction of $O(p, q)$ to $\sigma(p, q)$ in $O'(n)$, and that of $O(1, 1)$ to $\sigma(1, 1)$ in $O'(2)$ imply the reduction of $O(p+1, q+1)$ to $\sigma(p+1, q+1)$, associated with $y_1, \dots y_r y_0$, in $O'(n+2)$.

(2) Let us assume that there is a Pin $(p+1, q+1)$-spin structure on V. We observe that $\eta = \bar{h} \circ \psi$ is a projection from Pin $(p+1, q+1)$ onto $PO(p+1, q+1)$ with kernel $\mathscr{A} = \{1, -1, e_N, -e_N\}$.

There exists a principal bundle S_1 twofold covering of ξ_j and a morphism of principal bundle $\psi' \colon S_1 \to \xi_j$. So we can set $\tilde{\eta} = h \circ \psi'$ which is a morphism of principal bundles from S_1 onto $P\xi_1$ and S_1 is a fourfold covering of $P\xi_1$. Thus, we have got the existence of a $C_n(p, q)$-spin structure on V. We can also observe, according to [2], that the reduction of $O(p+1, q+1)$ to $\sigma(p+1, q+1)$ in $O'(n+2)$ implies the reduction in $PO'(n+2)$ of $PO(p+1, q+1)$ to $h(\sigma(p+1, q+1))$ which is isomorphic by h_1 to $S_C(p, q)$ the real conformal spinoriality group in a strict sense associated with $f_r = y_1 \dots y_r$.

(3) At last, let us assume that there exists a $C_n(p, q)$-spin structure on V and that r and p are odd. If r is odd, then $\varepsilon_2 = 1$. According to the previous paragraph 5 of the second part of this paper, there exists an isotropic $(r+1)$-pseudo-vector field (so defined modulo $\varepsilon_2 = 1$), locally determined by means of $x \in U_{\alpha'} \to \tilde{f}_{\alpha'}(x)$ such that for any $x \in U_{\alpha'} \cap U_{\beta'} \neq \emptyset$ we have

$$\tilde{f}_{\alpha'}(x) = \tilde{g}_{\alpha'\beta'}(x) \tilde{f}_{\alpha'}(x) \tilde{g}^{-1}_{\alpha'\beta'}(x)$$

and

$$\tilde{f}_{\beta'}(x) = (e_N)^2 N(\tilde{g}_{\alpha'\beta'}(x)) \tilde{f}_{\alpha'}(x) \quad \text{modulo } \varepsilon_2 = 1.$$

Thus, as r is odd, as $(e_N)^2 = (-1)^{r-p}$ if p is odd then $(e_N)^2 = 1$. So we get the existence of an isotropic pseudo-vector field which satisfy the required sufficient condition [7] for the existence of a Pin $(p+1, q+1)$ spin structure on ξ_j.

REMARK. Let us recall (I.5 of this paper) that $\dfrac{\overline{|S_C(p, q)|}}{\overline{|\sigma(p, q)|}}$ is homeomorphic to $\mathbf{R}^{2n+1}$ and so is a solid spare according to [14, p. 54]. Following the corollary 12-6

[14, p. 56] any bundle with structure group $\overline{|S_C(p,q)|}$ is reducible in $\overline{|S_C(p,q)|}$ to a bundle with structure group $\overline{|\sigma(p, q)|}$. If there exists a $C_n(p, q)$-spin structure on V, according to the paragraph 5, Part II, $\overline{|C_n(p,q)|}$ is reductible to $\overline{|S_C(p,q)|}$ in C'_n. Moreover, the previous reduction of $\overline{|S_C(p,q)|}$ to $\overline{|\sigma(p, q)|}$ is made in $\overline{|S_C(p,q)|}$ and not in $O'(n)$ as $S_C(p, q)$ is obviously "extended out" of $O'(n)$ so that it is not permissible to use the sufficient condition given in [7] for the existence of a Pin (p, q)-spin structure on V.

Another paper [18] deals with real conformal symplectic geometry and conformal symplectic spin structures on a smooth real $2r$-dimensional manifold.

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(Received February 20, 1984)

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LEAVES OF PRECODES ASSIGNED TO FINITE MOORE AUTOMATA

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To investigate the structure of finite Moore automata, the concepts of code, precode and complexity are introduced by Ádám [1] and investigated in [1—6].

The notions of code and precode give a constructive description of all (representatives of isomorphism classes of) initially connected Moore automata. The problem is to give a constructive description of all reduced initially connected Moore automata. An initially connected Moore automaton is reduced if and only if it has finite complexity. We say that a code is of finite complexity if the corresponding Moore automaton is of finite complexity. Hence the above problem is equivalent to:

Basic Problem [1]. Give a constructive description of all codes C with finite complexity.

A precode corresponds to a step of construction of a code (hence a step of construction of a Moore automaton). If we can proceed to construct a code of finite complexity (i.e., a reduced Moore automaton), then the precode is said to be of finite complexity. Hence it is a problem to obtain a condition so that a given precode is of finite complexity. Also, it is a problem to determine when we loose the possibility to continue the procedure to construct a reduced Moore automaton, in other words, when the complexity changes from finite to infinite. We shall study about these problems.

1

In this section, we provide fundamental definitions and some results due to Ádám [1, 2] with a little modification.

For a finite nonempty set Z, the cardinality of Z is denoted by $|Z|$. Z^* is the free monoid generated by Z. $\mathfrak{N}$ is the set of all positive integers and $\mathfrak{N}_0$ is the set of all nonnegative integers. For $t, k \in \mathfrak{N}_0$, we denote $[t:k]=\{a \in \mathfrak{N}_0 | t \leqq a \leqq k\}$.

In this paper, a *partial automaton* means a 5-tuple $B=([1:v], X, Y, \delta, \lambda)$ such that:

(1) v is a positive integer. $[1:v]$ is called a state set of B.

(2) X and Y are finite nonempty sets, called an input set and an output set of B, respectively.

1980 *Mathematics Subject Classification.* Primary 68D30; Secondary 68D40.
Key words and phrases. Finite Moore automata, codes, precodes, complexity.

(3) δ is a partial mapping of $[1:v]\times X$ into $[1:v]$ called a state transition function (δ is extended as usual to a partial mapping of $[1:v]\times X^*$ into $[1:v]$).

(3) λ is a mapping of A *onto* Y called an output function.

(4) For any $a\in[1:v]$ there exists $p\in X^*$ such that $\delta(1,p)=a$.

If δ is defined for any element of $[1:v]\times X$, then B is said to be an *(initially connected finite) Moore automaton*.

Let $A=([1:v], X, Y, \delta, \lambda)$ be a Moore automaton. If $\lambda(\delta(a,p))\neq\lambda(\delta(b,p))$ holds for $a,b\in[1:v]$ and $p\in X^*$, then we say that p distinguishes between a and b. $\omega(a,b)$ is the minimal length of p which distinguishes between a and b. If there is no word which distinguishes between a and b, then we write $\omega(a,b)=\infty$. The *complexity* $\Omega(A)$ of A is defined by $\Omega(A)=\min\{\omega(a,b)|a,b\in[1:v],\ a\neq b\}$. If $v=1$ then $\Omega(A)=0$.

LEMMA 1.1. *Let* $A=([1:v], X, Y, \delta, \lambda)$ *be a Moore automaton. For* $a,b\in[1:v]$ *and* $p\in X^*$, *if* $\omega(\delta(a,p),\delta(b,p))<\infty$ *then* $\omega(a,b)<\infty$. □

The notions of codes and precodes are introduced in [1] as tools to describe Moore automata constructively.

Let $n\in\mathfrak{N}$. A 6-tuple $D=(r,s,\beta,\gamma,\varphi,\mu)$ is said to be an n-input *precode* if the following eight postulates are fulfilled:

(A) r, s are nonnegative integers.

(B) β and φ are mappings of $[2:r+s+1]$ into $[1:r+1]$.
γ is a mapping of $[2:r+s+1]$ into $[1:n]$.
μ is a mapping of $[1:r+1]$ into $\mathfrak{N}$.

(C) $\beta(a)<a$ for any $a\in[2:r+1]$.

(D) For $a,b\in[2:r+1]$, if $a<b$ then $(\beta(a),\gamma(a))<(\beta(b),\gamma(b))$ in the lexicographic order.

(E) For $a\in[r+2:r+s+1]$, $(\beta(a),\gamma(a))$ is the lexicographically smallest element in $([1:r+1]\times[1:n])-\{(\beta(b),\gamma(b))|b\in[2:a-1]\}$.

(F) $\varphi(a)=a$ for any $a\in[2:r+1]$.

(G) $\varphi(a)=1$ or $(\beta(\varphi(a)),\gamma(\varphi(a)))<(\beta(a),\gamma(a))$ for any $a\in[r+2:r+s+1]$.

(H) $\mu(a)\in\{1\}\cup\{\mu(b)+1|b\in[1:a-1]\}$ for any $a\in[1:r+1]$.

We denote $\mu(D)=\max\{\mu(a)|a\in[1:r+1]\}$.

It follows from Postulates (D) and (E) that $a\neq b$ implies $(\beta(a),\gamma(a))\neq$ $\neq(\beta(b),\gamma(b))$. Hence we have $r+s\leqq n(r+1)$, i.e., $s\leqq nr+n-r$. If $s=nr+n-r$, then the precode is said to be a *code*.

Let $D=(r,s,\beta,\gamma,\varphi,\mu)$ and $D'=(r',s',\beta',\gamma',\varphi',\mu')$ be n-input precodes. If $r+s\leqq r'+s'$ and $\beta',\gamma',\varphi',\mu'$ are extensions of β,γ,φ,μ then we denote $D\leqq D'$. (In such a situation, in what follows, we shall write $\beta',\gamma',\varphi',\mu'$ simply by β,γ,φ,μ, respectively. We denote $D<D'$ if $D\leqq D'$ and $r+s<r'+s'$. If $D<D'$ and $r'+s'=$ $=r+s+1$ then we denote $D\prec D'$.

In what follows, we sometimes define a precode $D'=(r,s',\beta,\gamma,\varphi,\mu)$ for a given precode $D=(r,s,\beta,\gamma,\varphi,\mu)$ and $s'\in[s:nr+n-r]$. In such a case, $(\beta(a),\gamma(a))$ for any $a\in[r+s+2:r+s'+1]$ is determined uniquely by Postulate (E). Hence we need only determine values $\varphi(a)$ for $a\in[r+s+2:r+s'+1]$ so as to satisfy Postulate (G). (The other postulates are obviously fulfilled.) If we define $\varphi(a)=1$ for any $a\in[r+s+2:r+s'+1]$ then Postulate (G) is satisfied. Hence we can always find at least one such D'. Especially, for any precode D, there exists a code C such that $D \leqq C$.

Let $D=(r, s, \beta, \gamma, \varphi, \mu)$ be an n-input precode with $\mu(D)=m$. Let $X=\{x_1, x_2, \ldots, x_n\}$ and $Y=\{y_1, y_2, \ldots, y_m\}$. Define a partial mapping δ_D of $[1:r+1]\times X$ into $[1:r+1]$ by $\delta_D(\beta(a), x_{\gamma(a)})=\varphi(a)$ for any $a\in[2:r+s+1]$. Define a mapping λ_D of $[1:r+1]$ onto Y by $\lambda_D(a)=y_{\mu(a)}$ for any $a\in[1:r+1]$. Then it is not difficult to verify that $\Psi(D)=([1:r+1], X, Y, \delta_D, \lambda_D)$ is a partial automaton. $\Psi(D)$ is an automaton if and only if D is a code.

The *complexity* $\Omega(D)$ of a precode D is defined by

$$\Omega(D) = \min\{\Omega(\Psi(C))|C \text{ is a code with } D \leqq C\}.$$

Hence $\Omega(C)=\Omega(\Psi(C))$ for a code C. It is evident that $D\leqq D'$ implies $\Omega(D)\leqq \Omega(D')$.

2

In the present section, we introduce the notion of essential leaves of a precode. It is shown that the finiteness of complexity is closely related to the behaviour on essential leaves.

Let $D=(r, s, \beta, \gamma, \varphi, \mu)$ be a precode. $a\in[1:r+1]$ is said to be a *leaf* of D if $a\notin\{\beta(c)|c\in[2:r+1]\}$. The set of all leaves of D is denoted by $\mathfrak{L}_D$. $r+1$ is always a leaf of D, and thus $\mathfrak{L}_D$ is nonempty.

LEMMA 2.1. *Let $D=(r, s, \beta, \gamma, \varphi, \mu)$ be an n-input precode and let $1\leqq a<b\leqq r+1$. Assume that b is not a leaf. Furthermore, assume that, for any $j'\in[1:n]$, there exists $c\in[2:r+s+1]$ such that $(\beta(c), \gamma(c))=(a, j')$. Then there exists $j\in[1:n]$ such that $\delta_D(a, x_j)<\delta_D(b, x_j)$ and $\delta_D(b, x_j)>b$.*

PROOF. There exists $c\in[2:r+1]$ and $j\in[1:n]$ such that $(\beta(c), \gamma(c), \varphi(c))=(b, j, c)$. By Postulate (C), we have $\delta_D(b, x_j)=c>\beta(c)=b$. By the assumption, there exists $d\in[2:r+s+1]$ such that $(\beta(d), \gamma(d))=(a, j)$. Assume $d\in[2:r+1]$. Since $(\beta(d), \gamma(d))=(a, j)<(b, j)=(\beta(c), \gamma(c))$, we have $d<c$ by Postulate (D). Hence $\delta_D(a, x_j)=d<c=\delta_D(b, x_j)$. Assume $d\in[r+2:r+s+1]$. If $\varphi(d)=1$ then $\delta_D(a, x_j)=1<c$. Otherwise, by Postulate (G),

$$(\beta(\varphi(d)), \gamma(\varphi(d)))<(\beta(d), \gamma(d))=(a, j)<(b, j)=(\beta(c), \gamma(c)).$$

Hence we have $\delta_D(a, x_j)=\varphi(d)<c=\delta_D(b, x_j)$ by Postulate (D). □

LEMMA 2.2. *Let $C=(r, s, \beta, \gamma, \varphi, \mu)$ be an n-input code and let $1\leqq a<b\leqq r+1$. Assume that b is not a leaf. Then there exists $p\in X^*$ such that $\delta_C(b, p)\in\mathfrak{L}_D$, $\delta_C(a, p)<\delta_C(b, p)$ and $\delta_C(b, p)>b$.*

PROOF. By consecutive use of Lemma 2.1. □

A leaf e of a precode $D=(r, s, \beta, \gamma, \varphi, \mu)$ is said to be an *essential leaf* if $\lambda(e)=\lambda(a)$ for some $a\in[1:e-1]$. $\mathfrak{E}_D$ is the set of all essential leaves of D. Let $\mathfrak{E}_D=\{e_1, \ldots, e_k\}$ with $e_1<e_2<\ldots<e_k$. Then e_t is said to be the *t-th essential leaf of* $D(t\in[1:k])$. The k-th essential leaf e_k is also called the *last essential leaf*.

The following lemma is an easy consequence of the definition:

LEMMA 2.3. *Let* $D=(r, s, \beta, \gamma, \varphi, \mu)$ *and* $D'=(r, s', \beta, \gamma, \varphi, \mu)$ *be precodes with* $D \leqq D'$. *Then* $\mathfrak{L}_D=\mathfrak{L}_{D'}$, $\mathfrak{E}_D=\mathfrak{E}_{D'}$ *and* $\mu(D)=\mu(D')$. □

PROPOSITION 2.4. *Let* $C=(r, s, \beta, \gamma, \varphi, \mu)$ *be a code. Then* $\Omega(C)<\infty$ *if and only if the following condition is satisfied:*

$(\mathscr{C})$ $\qquad \omega(e, a)<\infty$ *for any* $e \in \mathfrak{E}_C$ *and* $a \in [1: e-1]$.

PROOF. Assume that $(\mathscr{C})$ holds. Let $b, c \in [1: r+1]$ and $b<c$. Then, by Lemma 2.2, there exists $p \in X^*$ such that $\delta_C(c, p) \in \mathfrak{L}_C$ and $\delta_C(b, p)<\delta_C(c, p)$. If $\delta_C(c, p)$ is not essential then $\lambda_C(\delta_C(b, p)) \neq \lambda_C(\delta_C(c, p))$. Hence $\omega(b, c)<\infty$. If $\delta_C(c, p)$ is essential then $\omega(\delta_C(b, p), \delta_C(c, p))<\infty$ by the assumption. Hence, by Lemma 1.1, $\omega(b, c)<\infty$. Thus we have $\Omega(C)<\infty$. The converse is obvious. □

PROPOSITION 2.5. *If a precode* $D=(r, s, \beta, \gamma, \varphi, \mu)$ *has no essential leaf then* $\Omega(D)<\infty$.

PROOF. Let $C=(r, s', \beta, \gamma, \varphi, \mu)$ be a code with $D \leqq C$. Since $\mathfrak{E}_C=\mathfrak{E}_D=\emptyset$ the condition $(\mathscr{C})$ is automatically satisfied. Hence we have $\Omega(D) \leqq \Omega(C)<\infty$. □

Let $D=(r, s, \beta, \gamma, \varphi, \mu)$ be a precode. Concerning the t-th essential leaf e_t of D, consider the following conditions $(\mathscr{C}_t)$ and $(\mathscr{D}_t)$ for D:

$(\mathscr{C}_t)$ For any $a \in [1: e_t-1]$ there exists $p \in X^*$ such that one of the following holds:

(1) $\lambda_D(\delta_D(a, p)) \neq \lambda_D(\delta_D(e_t, p))$.

(2) $\delta_D(a, p) \neq \delta_D(e_t, p)$ and $\delta_D(a, p)>e_t$.

(3) $\delta_D(a, p) \neq \delta_D(e_t, p)$ and $\delta_D(e_t, p)>e_t$.

$(\mathscr{D}_t)$ For any $a, b \in [1: e_t]$ with $a \neq b$, there exists $p \in X^*$ such that one of the following holds:

(4) $\lambda_D(\delta_D(a, p)) \neq \lambda_D(\delta_D(b, p))$.

(5) $\delta_D(a, p) \neq \delta_D(b, p)$ and $\delta_D(a, p)>e_t$.

(6) $\delta_D(a, p) \neq \delta_D(b, p)$ and $\delta_D(b, p)>e_t$.

For the sake of convenience, we assume that $(\mathscr{C}_0)$ and $(\mathscr{D}_0)$ are the empty condition, i.e., every precode satisfies $(\mathscr{C}_0)$ and $(\mathscr{D}_0)$.

LEMMA 2.6. *Let* $D=(r, s, \beta, \gamma, \varphi, \mu)$ *be an n-input precode and let* e_t *be the t-th essential leaf of D. If D satisfies the condition* $(\mathscr{C}_t)$ *then* $s \geqq 1$ *and*

$$(\beta(r+s+1), \gamma(r+s+1)) \geqq (e_t, 1).$$

PROOF. In the condition $(\mathscr{C}_t)$, it is evident that $p=1$ satisfies neither of (1), (2) and (3). Hence there exists $j \in [1: n]$ such that $\delta_D(e_t, x_j)$ is defined. Thus there exists $c \in [2: r+s+1]$ such that $(\beta(c), \gamma(c))=(e_t, j)$. Since e_t is an essential leaf, we have $c \in [r+2: r+s+1]$. Hence $s \geqq 1$. By Postulate (E), we have

$$(e_t, 1) \leqq (e_t, j)=(\beta(c), \gamma(c)) \leqq (\beta(r+s+1), \gamma(r+s+1)). \quad □$$

Let $D=(r, s, \beta, \gamma, \varphi, \mu)$ and $D'=(r, s', \beta, \gamma, \varphi, \mu)$ be precodes with $D \leqq D'$. If e_t is the t-th essential leaf of D then it is also the t-th essential leaf of D'. If $\delta_D(a, x_j)$ is defined then $\delta_{D'}(a, x_j)$ is also defined and $\delta_{D'}(a, x_j)=\delta_D(a, x_j)$. Hence we have:

LEMMA 2.7. *Let* $D=(r, s, \beta, \gamma, \varphi, \mu)$ *and* $D'=(r, s', \beta, \gamma, \varphi, \mu)$ *be precodes such that* $D \leqq D'$. *Then we have the following:*

(7) *If D satisfies* $(\mathscr{C}_t)$ *then D' satisfies* $(\mathscr{C}_t)$.

(8) *If D satisfies* $(\mathscr{D}_t)$ *then D' satisfies* $(\mathscr{D}_t)$. □

LEMMA 2.8. *Let* $D=(r, s, \beta, \gamma, \varphi, \mu)$ *be an n-input precode and let e be a leaf of D. Then the following conditions are equivalent:*

(9) $s \geqq 1$ *and* $(\beta(r+s+1), \gamma(r+s+1)) \geqq (e, n)$.

(10) *For any* $a \in [1:e]$ *and* $j \in [1:n]$, *there exists* $c \in [2:r+s+1]$ *such that* $(\beta(c), \gamma(c))=(a, j)$.

(11) $\delta_D(a, x_j)$ *is defined for any* $a \in [1:e]$ *and* $j \in [1:n]$.

(12) *For any* $j \in [1:n]$, *there exists* $c \in [2:r+s+1]$ *such that* $(\beta(c), \gamma(c))=(e, j)$.

(13) $\delta_D(e, x_j)$ *is defined for any* $j \in [1:n]$.

PROOF. (9) implies (10) by Postulate (E). (10) implies (12) obviously. Assume that (12) holds. Then there exists $c \in [2:r+s+1]$ such that $(\beta(c), \gamma(c))=(e, n)$. Since e is a leaf, we have $c \in [r+2:r+s+1]$. By Postulate (E), $(\beta(r+s+1), \gamma(r+s+1)) \geqq$ $\geqq (\beta(c), \gamma(c))=(e, n)$. Thus we have (9). The equivalences (10)⇔(11) and (12)⇔(13) are direct from definition. □

Let $D=(r, s, \beta, \gamma, \varphi, \mu)$ be a precode. Let $\mathfrak{E}_D=\{e_1, \ldots, e_k\}$ with $e_1<\ldots<e_k$. Determine a nonnegative integer $t(D)$ as follows.

If $s=0$ then $t(D)=0$.

If $s \geqq 1$ then

$$t(D)=\max\big(\{0\} \cup \{t \in [1:k] \mid (\beta(r+s+1), \gamma(r+s+1)) \geqq (e_t, n)\}\big).$$

In other words, $t(D)$ is the maximal integer t such that $\delta_D(e_t, j)$ is defined for any $j \in [1:n]$. (If there is no such essential leaf then $t(D)=0$.) It is evident that $D \leqq D'$ implies $t(D) \leqq t(D')$.

LEMMA 2.9. *Let* $D=(r, s, \beta, \gamma, \varphi, \mu)$ *and* $D'=(r, s', \beta, \gamma, \varphi, \mu)$ *be n-input precodes with* $D \leqq D'$. *Then, for any* $t \in [0: t(D)]$, *we have the following:*

(14) *D' satisfies* $(\mathscr{C}_t)$ *if and only if D satisfies* $(\mathscr{C}_t)$.

(15) *D' satisfies* $(\mathscr{D}_t)$ *if and only if D satisfies* $(\mathscr{D}_t)$.

PROOF. It is evident that $\lambda_{D'}(a)=\lambda_D(a)$ and $\delta_{D'}(a, x_j)=\delta_D(a, x_j)$ for any $a \in [1:e_t]$ and $j \in [1:n]$ where e_t is the t-th essential leaf. The conclusions are immediate from these facts. □

LEMMA 2.10. *Let* $D=(r, s, \beta, \gamma, \varphi, \mu)$ *be an n-input precode and let* e_t *be the t-th essential leaf of D. The following conditions are equivalent:*

(16) *D satisfies* $(\mathscr{C}_i)$ *for any* $i\in[1:t]$.

(17) *D satisfies* $(\mathscr{D}_t)$.

PROOF. (17) implies (16) obviously. Assume that D satisfies $(\mathscr{C}_i)$ for any $i\in[1, t]$. For each $a, b\in[1: e_t]$ with $a<b$, we have to show the existence of $p\in X^*$ which satisfies (4), (5) or (6). We shall show this by induction on b. When $b=e_t$, we have the conclusion directly from $(\mathscr{C}_t)$. Let $a, b\in[1: e_t-1]$ and $a<b$. Assume that one of (4), (5) and (6) holds for any $a', b'\in[1: e_t]$ with $b'>b$ and $a'<b'$.

Case 1; $b\notin\mathfrak{L}_D$. By Lemma 2.6, we have $s\geqq 1$ and $(\beta(r+s+1), \gamma(r+s+1))\geqq$ $\geqq(e_t, 1)$. It follows that $\delta_D(c, x_j)$ is defined for any $c\in[1: e_t-1]$ and $j\in[1: n]$. Hence, by Lemma 2.1, there exists $j\in[1: n]$ such that $\delta_D(a, x_j)<\delta_D(b, x_j)$ and $\delta_D(b, x_j)>b$. If $\delta_D(b, x_j)>e_t$ then we have the conclusion. If not, by the inductive hypothesis, there exists $q\in X^*$ such that (4), (5) or (6) holds where $\delta_D(a, x_j)$, $\delta_D(b, x_j)$, q play the roles of a, b, p, respectively. Putting $p=x_jq$, it can easily be seen that (4), (5) or (6) is fulfilled.

Case 2; $b\in\mathfrak{L}_D-\mathfrak{E}_D$. Obvious from the fact that $\lambda_D(a)\neq\lambda_D(b)$.

Case 3; $b\in\mathfrak{E}_D$. Assume that b is the i-th essential leaf of D $(i\in[1: t-1])$. By $(\mathscr{C}_i)$, there exists $p\in X^*$ such that one of the following holds:

(1′) $\lambda_D(\delta_D(a, p))\neq\lambda_D(\delta_D(b, p))$.

(2′) $\delta_D(a, p)\neq\delta_D(b, p)>b$.

(3′) $\delta_D(b, p)\neq\delta_D(a, p)>b$.

(1′) implies (4) obviously. In cases (2′) and (3′), the proof is carried out by the same way as the last part of Case 1. □

PROPOSITION 2.11. *Let* $C=(r, s, \beta, \gamma, \varphi, \mu)$ *be a code. Assume that C has at least one essential leaf. Let* e_k *be the last essential leaf of C. Then the following three conditions are equivalent:*

(18) $\Omega(C)<\infty$.

(19) *C satisfies* $(\mathscr{C}_i)$ *for any* $i\in[1, k]$.

(20) *C satisfies* $(\mathscr{D}_k)$.

PROOF. The equivalence of (19) and (20) is shown in Lemma 2.10.

(18)⇒(20). Assume that $(\mathscr{D}_k)$ does not hold. Then it can easily be seen that three exist $a, b\in[1: e_k]$ such that $a\neq b$ and $\lambda_C(\delta_C(a, p))=\lambda_C(\delta_C(b, p))$ for any $p\in X^*$. This means that $\omega(a, b)=\infty$. Hence $\Omega(C)=\infty$.

(20)⇒(18). First assume $e_k=r+1$. Let $a, b\in[1: r+1]$ and $a<b$. There exists $p\in X^*$ which satisfies one of (4), (5) and (6). However, neither (5) nor (6) takes place, and (4) is equivalent to $\omega(a, b)<\infty$. Thus we have $\Omega(C)<\infty$.

Next assume $e_k\neq r+1$. Let $a, b\in[1: r+1]$ and $a<b$. We show $\omega(a, b)<\infty$.

Case 1; $b > e_k$. If b is a leaf then it is not essential. Hence $\lambda_C(a) \neq \lambda_C(b)$ and thus $\omega(a, b) = 0$. If b is not a leaf then, by Lemma 2.2, there exists $p \in X^*$ such that $\delta_C(b, p) \in \mathfrak{L}_C$, $\delta_C(b, p) > b$ and $\delta_C(a, p) < \delta_C(b, p)$. Since $\delta_C(b, p) > e_k$, $\delta_C(b, p)$ is not essential. Hence $\lambda_C(\delta_C(a, p)) \neq \lambda_C(\delta_C(b, p))$, i.e., $\omega(a, b) < \infty$.

Case 2; $b \leqq e_k$. There exists $p \in X^*$ which satisfies (4), (5) or (6). In case (4), we have $\omega(a, b) < \infty$. In cases (5) and (6), we have $\omega(\delta_C(a, p), \delta_C(b, p)) < \infty$ by the consideration in Case 1. Hence we have $\omega(a, b) < \infty$. □

3

In the previous section, we have characterized finiteness of complexity of codes by using conditions $(\mathscr{C}_t)$ and $(\mathscr{D}_t)$. In this section, we make similar considerations on precodes.

LEMMA 3.1. *Let $D = (r, s, \beta, \gamma, \varphi, \mu)$ be an n-input precode. Let $j \in [1 : n]$ and let e_t be the t-th essential leaf. Assume $e_t \neq r+1$. Moreover, assume that $(\beta(c), \gamma(c)) < (e_t, j)$ holds for any $c \in [r+2 : r+s+1]$. Then there exists a precode $D = (r, s', \beta, \gamma, \varphi, \mu)$ with $D < D'$ and $(\beta(r+s'+1), \gamma(r+s'+1)) = (e_t, j)$ which satisfies $(\mathscr{C}_t)$.*

PROOF. The integer s' is uniquely determined by Postulate (E). For such s', take a precode $D'' = (r, s'-1, \beta, \gamma, \varphi, \mu)$ with $D \leqq D''$. Then $(\beta(c), \gamma(c)) < (e_t, j)$ for any $c \in [r+2 : r+s']$. Let

$$S = \{a \in [1:\ e_t+1] \mid \text{ there exists } f(a) \in \mathfrak{N}_0 \text{ such that } \delta_{D''}(a, x_j^{f(a)}) = e_t+1\}.$$

Since $\delta_{D''}(e_t+1, x_j^0) = e_t+1$, S is nonempty. If $\delta_{D''}(a, x_j^{f(a)}) = \delta_{D''}(a, x_j^{g(a)}) = e_t+1$ with $f(a) \neq g(a)$, then there exists $a' \in [e_t+1 : r+1]$ such that $\delta_{D''}(a', x_j) \leqq a'$. Thus there exists $c \in [2 : r+s']$ such that $(\beta(c), \gamma(c)) = (a', j)$ and $\varphi(c) \leqq a'$. Since $a' > e_t$, we have $c \in [2 : r+1]$. By Postulates (C) and (F), $c > \beta(c) = a' \geqq \varphi(c) = c$ which is a contradiction. Consequently, $f(a)$ is uniquely determined for each $a \in S$. Let b be an element of S such that $f(b)$ takes the maximal value.

Define a precode $D' = (r, s', \beta, \gamma, \varphi, \mu)$ by

$$(\beta(r+s'+1), \gamma(r+s'+1), \varphi(r+s'+1)) = (e_t, j, b).$$

By Postulate (C), $\beta(b) < b \leqq e_t+1$. Since e_t is a leaf, $\beta(b) \neq e_t$. Hence we have

$$(\beta(b), \gamma(b)) < (e_t, j) = (\beta(r+s'+1), \gamma(r+s'+1)).$$

Hence Postulate (G) is satisfied, and thus D' is actually a precode. Now we show that D' satisfies the condition $(\mathscr{C}_t)$. We have

$$\delta_{D'}(e_t, x_j^{f(b)+1}) = \delta_{D'}(b, x_j^{f(b)}) = e_t+1.$$

Let $a \in [1 : e_t-1]$. If $\delta_{D'}(a, x_j^{f(b)+1}) = e_t+1$ then it contradicts the maximality of $f(b)$. Consequently, we have

$$\delta_{D'}(e_t, x_j^{f(b)+1}) = e_t+1 \neq \delta_{D'}(a, x_j^{f(b)+1}).$$

$(\mathscr{C}_t)$ follows easily from this fact. □

LEMMA 3.2. *Let* $D=(r, s, \beta, \gamma, \varphi, \mu)$ *be an n-input precode with* $\mu(D)\geqq 2$. *Let* $j\in[1:n]$. *Assume that* $r+1$ *is the k-th (and the last) essential leaf, and* $(\beta(c), \gamma(c))<$ $<(r+1, j)$ *holds for any* $c\in[r+2:r+s+1]$. *Then there exists a precode* $D'=$ $=(r, s', \beta, \gamma, \varphi, \mu)$ *with* $D<D'$ *and* $(\beta(r+s'+1), \gamma(r+s'+1))=(r+1, j)$ *which satisfies* $(\mathscr{C}_k)$.

PROOF. Take a precode $D''=(r, s'-1, \beta, \gamma, \varphi, \mu)$ with $D\leqq D''$. Then $\delta_{D''}(a, x_j)$ is defined for any $a\in[1:r]$ and $\delta_{D''}(r+1, x_j)$ is not defined. Let $\mu(r+1)=z$ and fix $u\in[1, \mu(D)]$ such that $u\neq z$. Each $a\in[1:r]$ satisfies exactly one of the following:

(1) For any $f\in\mathfrak{N}_0$, $\lambda_{D''}(\delta_{D''}(a, x_j^f))=y_z$.

(2) There exists $f(a)\in\mathfrak{N}_0$ such that $\lambda_{D''}(\delta_{D''}(a, x_j^i))=y_z$ for any $i\in[0: f(a)-1]$ and $\lambda_{D''}(\delta_{D''}(a, x_j^{f(a)}))=y_u$.

(3) There exists $g(a)\in\mathfrak{N}_0$ such that $\lambda_{D''}(\delta_{D''}(a, x_j^i))=y_z$ for any $i\in[0: g(a)-1]$ and $\lambda_{D''}(\delta_{D''}(a, x_j^{g(a)}))\neq y_z, y_u$.

(4) There exists $h(a)\in\mathfrak{N}_0$ such that $\lambda_{D''}(\delta_{D''}(a, x_j^i))=y_z$ for any $i\in[0: h(a)-1]$ and $\delta_{D''}(a, x_j^{h(a)})=r+1$.

There exists $a\in[1:r]$ such that $\mu(a)=u$. a satisfies (2) with $f(a)=0$. Hence there exists at least one element which satisfies (2). Take $b\in[1:r]$ which satisfies (2), and which has the maximal value $f(b)$. Define a precode D' by $(\beta(r+s'+1), \gamma(r+s'+1),$ $\varphi(r+s'+1))=(r+1, j, b)$. We show that D' satisfies the condition $(\mathscr{C}_k)$. We have

$$\lambda_{D'}(\delta_{D'}(r+1, x_j^i))=y_z$$

for any $i\in[1: f(b)]$, and

$$\lambda_{D'}(\delta_{D'}(r+1, x_j^{f(b)+1}))=y_u\neq y_z.$$

Let $a\in[1:r]$. If a satisfies (2) then, since $f(a)\leqq f(b)$,

$$\lambda_{D'}(\delta_{D'}(a, x_j^{f(a)}))=y_u\neq y_z=\lambda_{D'}(\delta_{D'}(r+1, x_j^{f(a)})).$$

If a satisfies (3) and $g(a)\leqq f(b)$ then

$$\lambda_{D'}(\delta_{D'}(a, x_j^{g(a)}))\neq y_z=\lambda_{D'}(\delta_{D'}(r+1, x_j^{g(a)})).$$

If a satisfies (3) and $g(a)\geqq f(b)+1$ then

$$\lambda_{D'}(\delta_{D'}(a, x_j^{f(b)+1}))\neq y_u=\lambda_{D'}(\delta_{D'}(r+1, x_j^{f(b)+1})).$$

If a satisfies (4) and $h(a)\geqq f(b)+1$, or if a satisfies (1) then

$$\lambda_{D'}(\delta_{D'}(a, x_j^{f(b)+1}))=y_z\neq y_u=\lambda_{D'}(\delta_{D'}(r+1, x_j^{f(b)+1})).$$

If a satisfies (4) and $h(a)\leqq f(b)$ then

$$\lambda_{D'}(\delta_{D'}(a, x_j^{f(b)+1}))=\lambda_{D'}(\delta_{D'}(r+1, x_j^{f(b)+1-h(a)}))=y_z\neq$$
$$\neq y_u=\lambda_{D'}(\delta_{D'}(r+1, x_j^{f(b)+1})). \quad \square$$

PROPOSITION 3.3. *Let* $D=(r, s, \beta, \gamma, \varphi, \mu)$ *be an n-input precode with* $\mu(D)\geqq 2$. *Then the following conditions are equivalent:*

(5) $\Omega(D)<\infty$.

(6) *D satisfies* $(\mathscr{C}_i)$ *for any* $i\in[1: t(D)]$.

(7) *D satisfies* $(\mathscr{D}_{t(D)})$.

PROOF. Equivalence of (6) and (7) is shown in Lemma 2.10.

(5)$\Rightarrow$(6). There exists a code $C=(r, s', \beta, \gamma, \varphi, \mu)$ such that $D\leqq C$ and $\Omega(C)<\infty$. By Proposition 2.11, C satisfies $(\mathscr{C}_i)$ for any $i\in[1:k]$ where $k=|\mathfrak{E}_D|$. By Lemma 2.9, D satisfies $(\mathscr{C}_i)$ for any $i\in[1: t(D)]$.

(6)$\Rightarrow$(5). If $t(D)=k$, take a code $C=(r, s', \beta, \gamma, \varphi, \mu)$ such that $D\leqq C$. Then, by Lemma 2.9, C satisfies $(\mathscr{C}_i)$ for any $i\in[1: k]$. Hence, by Proposition 2.11, $\Omega(D)\leqq\Omega(C)<\infty$. Assume $t(D)<k$. Put $D=D_{t(D)}$. By Lemmas 3.1 and 3.2, for each $i\in[t(D)+1: k]$, we can construct a precode $D_i=(r, s_i, \beta, \gamma, \varphi, \mu)$ such that $D_{i-1}<D_i$, $(\beta(r+s_i+1), \gamma(r+s_i+1))=(e_i, n)$ and D_i satisfies $(\mathscr{C}_i)$. Take a code $C=(r, s', \beta, \gamma, \varphi, \mu)$ with $D_k\leqq C$. Then, by Lemma 2.7, C satisfies $(\mathscr{C}_i)$ for any $i\in[1: k]$. By Proposition 2.11, $\Omega(C)<\infty$. Since $D<C$, we have $\Omega(D)<\infty$. □

Let $D=(r, s, \beta, \gamma, \varphi, \mu)$ be an n-input precode. We can decide whether D satisfies $(\mathscr{C}_t)$ (or $(\mathscr{D}_t)$) or not by the values $\delta_D(a, x_j)$ for $a\in[1: e_t]$, $j\in[1: n]$ and $\lambda_D(a)$ for $a\in[1: e_t]$. Hence, when $\mu(D)\geqq 2$, the above proposition shows that finiteness of a precode D is decided only by the following values:

$$\big(\beta(c), \gamma(c), \varphi(c)\big) \quad \text{for} \quad \beta(c)\in[1: e_{t(D)}],$$

$$\mu(c) \quad \text{for} \quad c\in[1: e_{t(D)}].$$

We have an alternate proof of the following result in [2].

PROPOSITION 3.4. *Every precode* $D=(r, 0, \beta, \gamma, \varphi, \mu)$ *is of finite complexity.*

PROOF. Let $D'=(r+1, 0, \beta, \gamma, \varphi, \mu)$ be a precode with $D\prec D'$ defined by

$$\big(\beta(r+2, \gamma(r+2), \varphi(r+2), \mu(r+2)\big) = \big(r+1, 1, r+2, \mu(D)+1\big).$$

Then $\mu(D')\geqq 2$ and $t(D)=0$. Since any precode satisfies $(\mathscr{D}_0)$, it follows from Proposition 3.3 that $\Omega(D)\leqq\Omega(D')<\infty$. □

When $\mu(D)=1$, we have the following result [6].

PROPOSITION 3.5. *Let* $D=(r, s, \beta, \gamma, \varphi, \mu)$ *be a precode with* $\mu(D)=1$. *Then* $\Omega(D)<\infty$ *if and only if* $r=0$ *or* $s=0$.

PROOF. $s=0$ implies $\Omega(D)<\infty$ by Proposition 3.4. The other parts can easily be seen. □

The assumption $\mu(D)\geqq 2$ in Proposition 3.3 is indispensable. Let $D=$ $=(r, 1, \beta, \gamma, \varphi, \mu)$ be an n-input precode with $r\geqq 1$, $n\geqq 2$ and $\mu(D)=1$. Then clearly $t(D)=0$ and D satisfies $(\mathscr{D}_0)$. However, $\Omega(D)=\infty$.

By using our results, we can give a simple constructive proof of Proposition 3.4. Namely, for a given precode $D=(r, 0, \beta, \gamma, \varphi, \mu)$, we can give the following procedure to construct a code $C=(r+1, s, \beta, \gamma, \varphi, \mu)$ with finite complexity such that $D<C$. It seems that our procedure is much more simple than the construction in [2].

PROPOSITION 3.6. *Let* $D=(r, 0, \beta, \gamma, \varphi, \mu)$ *be an n-input precode. Define a code* $C=(r+1, s, \beta, \gamma, \varphi, \mu)$ *with* $D<C$ *as follows:*

(8) *Put* $\mathfrak{L}=[1:r]-\{\beta(c)|c\in[1:r+1]\}$. *(Notice that if* $n=1$ *then* $\mathfrak{L}=\emptyset$*).*

(9) *For each* $e\in\mathfrak{L}$, *determine* j_e *as follows:*
If $\gamma(e+1)=1$ *then* $j_e=2$. *Otherwise* $j_e=1$.

(10) $(\beta(r+2), \gamma(r+2), \varphi(r+2), \mu(r+2))=(r+1, 1, r+2, \mu(D)+1)$.

(11) *If* $(\beta(c), \gamma(c))=(e, j_e)$ *for* $e\in\mathfrak{L}$ *then* $\varphi(c)=e+1$.

(12) *Otherwise* $\varphi(c)=1$.

Then C is of finite complexity.

PROOF. We have $\mathfrak{L}=\mathfrak{L}_D-\{r+1\}$. By (10), $r+1$ is not a leaf of C. Hence $\mathfrak{L}_C=\mathfrak{L}\cup\{r+2\}$. By (10), $r+2$ is not an essential leaf. Thus $\mathfrak{E}_C\subseteq\mathfrak{L}$. Let e_t be the t-th essential leaf of C. Assume $\gamma(e_t+1)=1$. Then it can easily be seen that $(\gamma(c), \varphi(c))=(2, e_t+1)$ if and only if $\beta(c)=e_t$. Hence $\delta_C(e_t, x_2)=e_t+1$ and $\delta_C(a, x_2)\neq e_t+1$ for any $a\in[1:e_t-1]$. Thus we have $(\mathscr{C}_t)$. When $\gamma(e_t+1)\neq 1$, by considering similarly with respect to x_1, we have also $(\mathscr{C}_t)$. Hence $\Omega(C)<\infty$ by Proposition 2.11. □

4

Now consider the problem when it may take place that $D\prec D'$, $\Omega(D)<\infty$ and $\Omega(D')=\infty$. If $D\prec D'$ with $D=(r, 0, \beta, \gamma, \varphi, \mu)$ and $D'=(r+1, 0, \beta, \gamma, \varphi, \mu)$ then $\Omega(D)$, $\Omega(D')<\infty$ by Proposition 3.4. Hence if we have the above situation, then D and D' are of the following type: $D=(r, s, \beta, \gamma, \varphi, \mu)$, $D'=(r, s+1, \beta, \gamma, \varphi, \mu)$. Moreover, if $\mathfrak{E}_D=\emptyset$ then $\Omega(D)$, $\Omega(D')<\infty$ by Lemma 2.3 and Proposition 2.5. When $\mu(D)=1$, we have the following:

PROPOSITION 4.1. *Let* $D=(r, s, \beta, \gamma, \varphi, \mu)$ *and* $D'=(r, s+1, \beta, \gamma, \varphi, \mu)$ *be precodes such that* $D\prec D'$ *and* $\mu(D)=1$. *Then the following conditions are equivalent:*

(1) $\Omega(D)<\infty$ *and* $\Omega(D')=\infty$.

(2) $r\leqq 1$ *and* $s=0$.

PROOF. By Proposition 3.5. □

When $\mu(D)\geqq 2$ and $\mathfrak{E}_D\neq\emptyset$, we have the following:

PROPOSITION 4.2. *Let* $D=(r, s, \beta, \gamma, \varphi, \mu)$ *and* $D'=(r, s+1, \beta, \gamma, \varphi, \mu)$ *be n-input precodes such that* $D\prec D'$. *Assume* $\mu(D)\geqq 2$ *and* $\mathfrak{E}_D\neq\emptyset$. *Let* $\mathfrak{E}_D=\{e_1, \dots, e_k\}$ *and* $e_1<\dots<e_k$. *Then the following conditions are equivalent:*

(3) $\Omega(D)<\infty$ *and* $\Omega(D')=\infty$.

(4) $(\beta(r+s+2), \gamma(r+s+2))=(e_t, n)$ *for some* $t\in[1:k]$, F *satisfies* $(\mathscr{D}_{t-1})$ *and* D' *does not satisfy* $(\mathscr{D}_t)$.

PROOF. (3)$\Rightarrow$(4). Put $t=t(D)+1$. By Proposition 3.3, D satisfies $(\mathscr{D}_{t-1})$. By Lemma 2.7, D' satisfies $(\mathscr{D}_{t-1})$. If $t(D')=t-1$ then $\Omega(D')<\infty$ by Proposition 3.3. Hence $t(D')\neq t-1$. Since $D\prec D'$, we have $t(D')=t$ and $(\beta(r+s+2), \gamma(r+s+2))=(e_t, n)$. By Proposition 3.3, D' does not satisfy $(\mathscr{D}_t)$.

(4)$\Rightarrow$(3). Since $t(D)=t-1$ and $t(D')=t$, the conclusion is immediate from Proposition 3.3. □

PROPOSITION 4.3. *Let* $D_1=(r, s_1, \beta, \gamma, \varphi, \mu)$ *be an n-input precode with finite complexity. Assume that* $\mu(D_1)\geqq 2$ *and* $\mathfrak{E}_{D_1}\neq\emptyset$. *Let* e_t *be the t-th essential leaf of* D_1. *Assume that* $s_1=0$ *or* $(\beta(r+s_1+1), \gamma(r+s_1+1))<(e_t, 1)$. *Then there exist precodes* $D=(r, s-1, \beta, \gamma, \varphi, \mu)$ *and* $D'=(r, s, \beta, \gamma, \varphi, \mu)$ *such that* $(\beta(r+s+1), \gamma(r+s+1))=(e_t, n)$, $D_1\leqq D\prec D'$, $\Omega(D)<\infty$ *and* $\Omega(D')=\infty$.

PROOF. The integer s is uniquely determined by Postulate (E), and we have $s_1\leqq s-n$. There exists a code $C=(r, nr+n-r, \beta, \gamma, \varphi, \mu)$ such that $D_1<C$ and $\Omega(C)<\infty$. Let $D_2=(r, s-n, \beta, \gamma, \varphi, \mu)$ be a precode such that $D_2<C$. Then $D_1\leqq D_2$ and $\Omega(D_2)<\infty$. D_2 satisfies the following:

For any $a\in[1:e_t-1]$ and $j\in[1:n]$, there exists $c\in[2:r+s-n+1]$ such that $(\beta(c), \gamma(c))=(a, j)$.

For any $j\in[1:n]$, there exists no $c\in[2:r+s-n+1]$ such that $(\beta(c), \gamma(c))=(e_t, j)$.

Since e_t is an essential leaf of D_1, it is an essential leaf of D_2. Hence there exists $a\in[1:e_t-1]$ such that $\mu(a)=\mu(e_t)$. For each $j\in[1:n]$, determine $c_j\in[2:r+s-n+1]$ by $(\beta(c_j), \gamma(c_j))=(a, j)$. Determine precodes $D=(r, s-1, \beta, \gamma, \varphi, \mu)$ and $D'=(r, s, \beta, \gamma, \varphi, \mu)$ with $D_2\leqq D\prec D'$ by

$$(\beta(r+s-n+j), \gamma(r+s-n+j), \varphi(r+s-n+j)) = (e_t, j, \varphi(c_j))$$

for any $j\in[1:n]$. Then D and D' are well-defined as precodes, and we have $t(D)=t-1$, $t(D')=t$. It can easily be seen that $\lambda_{D'}(e_t)=\lambda_{D'}(a)$ and $\delta_{D'}(e_t, x_j)=\delta_{D'}(a, x_j)$ for any $j\in[1:n]$. Hence D' does not satisfy $(\mathscr{D}_t)$, and thus $\Omega(D')=\infty$ by Proposition 3.3. Since $t(D_2)=t-1$ and $\Omega(D_2)<\infty$, D_2 satisfies $(\mathscr{D}_{t-1})$ by Proposition 3.3. Hence D satisfies $(\mathscr{D}_{t-1})$ by Lemma 2.7. Thus $\Omega(D)<\infty$ by Proposition 3.3. □

By our results, when we construct a code of finite complexity or of infinite complexity, only a few values of the function φ should be carefully chosen, and the other values of φ can be taken arbitrarily.

Let $D=(r, 0, \beta, \gamma, \varphi, \mu)$ be given such that $\mu(D)\geqq 2$ and $\mathfrak{E}_D\neq\emptyset$. We can construct a code $C=(r, s, \beta, \gamma, \varphi, \mu)$ with $\Omega(C)<\infty$ and $D<C$ by the following way:

I. For each $e\in\mathfrak{E}_D$, take $j_e\in[1:n]$ arbitrarily.

II. For each $c\in[r+2:r+s+1]$ with $(\beta(c), \gamma(c))=(e, j_e)$, determine $\varphi(c)$ suitably (e.g., according to Lemmas 3.1 and 3.2).

III. Other values $\varphi(c)$ are taken arbitrarily so as to satisfy Postulate (G).

It can easily be seen that every code $C=(r, s, \beta, \gamma, \varphi, \mu)$ with $\Omega(C)<\infty$ and $D<C$, is obtained by the above way. (Moreover, it is enough to take $j_e=n$ for every

$e \in \mathfrak{E}_D$ in I.) However, in general, it seems difficult to state the procedure in the step II. Lemmas 3.1 and 3.2 are simple example to obtain some of such codes C.

We can construct a code $C=(r, s, \beta, \gamma, \varphi, \mu)$ with $\Omega(C)=\infty$ and $D<C$ by the following way:

I. Take $e \in \mathfrak{E}_D$ arbitrarily.

II. For each $c \in [r+2: r+s+1]$ with $\beta(c)=e$, determine $\varphi(c)$ suitably (e.g., according to Proposition 4.3).

III. Other values $\varphi(c)$ are taken arbitrarily so as to satisfy Postulate (G).

It can easily be seen that every code $C=(r, s, \beta, \gamma, \varphi, \mu)$ with $\Omega(C)=\infty$ and $D<C$, is obtained by the above way. (Moreover, it is enough to take e to be the last essential leaf of D in I.) However, in general, it seems difficult to state the procedure in the step II. Proposition 4.3. is a simple example to obtain some of such codes C.

PROPOSITION 4.4. *Let $r, s \in \mathfrak{N}_0$ and $n, m \in \mathfrak{N}$ such that $2 \leqq m \leqq r+1$ and $s \leqq$ $\leqq nr+n-r$. Then the following conditions are equivalent:*

(5) *There exists an n-input precode $D=(r, s, \beta, \gamma, \varphi, \mu)$ such that $\mu(D)=m$ and $\Omega(D)=\infty$.*

(6) *There exists an n-input precode $D=(r, s, \beta, \gamma, \varphi, \mu)$ such that $\mu(D)=m$ and $t(D) \geqq 1$.*

(7) *$m \leqq r$, $s \geqq n$ and $s \geqq 2n-r$.*

PROOF. (5)$\Rightarrow$(6). By Proposition 3.3, D does not satisfy $(\mathscr{D}_{t(D)})$. Since every precode satisfies $(\mathscr{D}_0)$, we have $t(D) \geqq 1$.

(6)$\Rightarrow$(7). If $m=r+1$ then $\mathfrak{E}_D=\emptyset$ and thus $t(D)=0$. Hence we have $m \leqq r$. For any $j \in [1: n]$ there exists $c \in [r+2: r+s+1]$ such that $(\beta(c), \gamma(c))=(e_{t(D)}, j)$. Hence we have $s \geqq n$. For any $j \in [1: n]$ there exist $c, c' \in [2: r+s+1]$ such that $(\beta(c), \gamma(c))=(1, j)$ and $(\beta(c'), \gamma(c'))=(e_{t(D)}, j)$. Since $e_{t(D)} \neq 1$, we have $r+s \geqq 2n$.

(7)$\Rightarrow$(5). See the proof of [6, Theorem 2]. □

By Propositions 3.5 and 4.4, we have the following which is one of the main result of [6].

PROPOSITION 4.5. *Let $r, s \in \mathfrak{N}_0$ and $n, m \in \mathfrak{N}$ such that $m \leqq r+1$ and $s \leqq nr+n-r$. Then the following conditions are equivalent:*

(8) *Every n-input precode $D=(r, s, \beta, \gamma, \varphi, \mu)$ with $\mu(D)=m$ is of finite complexity.*

(9) *One of the following holds:* (a) $r=0$. (b) $s=0$. (c) $m=r+1$. (d) $m \geqq 2$ *and* $s<n$. (e) $m \geqq 2$ *and* $s<2n-r$. □

Let $D=(r, s, \beta, \gamma, \varphi, \mu)$ be an n-input precode with $\mu(D) \geqq 2$. Let e_t be the t-th essential leaf of D. If $s=0$ or $(\beta(r+s+1), \gamma(r+s+1))<(e_t, 1)$ then $(\mathscr{D}_t)$ is not satisfied by Lemma 2.6.

If $(e_t, 1) \leqq (\beta(r+s+1), \gamma(r+s+1))<(e_t, n)$ and $(\mathscr{D}_t)$ is satisfied then every precode $D'=(r, s', \beta, \gamma, \varphi, \mu)$ with $D<D'$ and $(\beta(r+s'+1), \gamma(r+s'+1))=(e_t, n)$, satisfies $(\mathscr{D}_t)$ and thus of finite complexity.

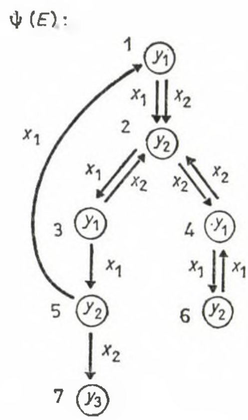
We may expect that the following holds:

If $(e_t, 1) \leqq (\beta(r+s+1), \gamma(r+s+1)) < (e_t, n)$ and $(\mathscr{D}_t)$ is not satisfied then there exists a precode $D'=(r, s', \beta, \gamma, \varphi, \mu)$ with $D<D'$ and $(\beta(r+s'+1), \gamma(r+s'+1))= =(e_t, n)$ which does not satisfy $(\mathscr{D}_t)$, i.e., D' is of infinite complexity.

However, this statement is not true. We shall show this by an example.

Example 4.6. Let $E=(6, 5, \beta, \gamma, \varphi, \mu)$ be a 2-input precode given in the following table:

	β	γ	φ	μ
1	—	—	—	1
2	1	1	2	2
3	2	1	3	1
4	2	2	4	1
5	3	1	5	2
6	4	1	6	2
7	5	2	7	3
8	1	2	2	—
9	3	2	2	—
10	4	2	2	—
11	5	1	1	—
12	6	1	4	—



6 and 7 are leaves and 6 is the only essential leaf. E does not satisfy $(\mathscr{D}_1)$ because

$$\lambda_E(\delta_E(6, (x_1^2)^* x_1)) = \lambda_E(\delta_E(2, (x_1^2)^* x_1)) = \lambda_E(\delta_E(5, (x_1^2)^* x_1)) = y_1,$$

$$\lambda_E(\delta_E(6, (x_1^2)^*)) = \lambda_E(\delta_E(2, (x_1^2)^*)) = \lambda_E(\delta_E(5, (x_1^2)^*)) = y_2,$$

$$\delta_E(6, (x_1^2)^* x_1 x_2) = \delta_E(2, (x_1^2)^* x_1 x_2) = \delta_E(5, (x_1^2)^* x_1 x_2) = 2,$$

$$\delta_E(6, (x_1^2)^* x_2) \quad \text{is not defined.}$$

Let $E'=(6, 6, \beta, \gamma, \varphi, \mu)$ be a precode with $E \prec E'$. If $\varphi(13) \neq 7$ then

$$\lambda_{E'}(\delta_{E'}(6, x_2)) \neq y_3 = \lambda_{E'}(\delta_{E'}(5, x_2)),$$

and

$$\lambda_{E'}(\delta_{E'}(6, x_1^2 x_2)) \neq y_3 = \lambda_{E'}(\delta_{E'}(2, x_1^2 x_2)).$$

If $\varphi(13)=7$ then

$$\lambda_{E'}(\delta_{E'}(6, x_1^2 x_2)) = y_3 \neq y_1 = \lambda_{E'}(\delta_{E'}(5, x_1^2 x_2)),$$

and

$$\lambda_{E'}(\delta_{E'}(6, x_2)) = y_3 \neq y_1 = \lambda_{E'}(\delta_{E'}(2, x_2)).$$

Hence E' satisfies $(\mathscr{D}_1)$, and thus $\Omega(E') < \infty$. □

To end the paper, we consider the following condition on a precode D:

$$(\mathscr{E}) \qquad \Omega(C) < \infty \quad \text{for any code } C \text{ with } \quad D \leqq C.$$

PROPOSITION 4.7. *Let* $D=(r, s, \beta, \gamma, \varphi, \mu)$ *be a precode. We have the following:*

(10) *If* $s=0$ *then D does not satisfy* $(\mathscr{E})$.

(11) *Let* $\mu(D)=1$. *Then D satisfies* $(\mathscr{E})$ *if and only if* $r=0$ *and* $s\geqq 1$.

(12) *If* $\mu(D)\geqq 2$, $s\geqq 1$ *and* $\mathfrak{E}_D\neq\emptyset$ *then D satisfies* $(\mathscr{E})$.

Assume $\mu(D)\geqq 2$, $s\geqq 1$ *and* $\mathfrak{E}_D\neq\emptyset$. *Let* e_k *be the last essential leaf of D.*

(13) *If D satisfies* $(\mathscr{D}_k)$ *then it satisfies* $(\mathscr{E})$.

(14) *If* $(\beta(r+s+1), \gamma(r+s+1))<(e_k, 1)$ *then D satisfies neither* $(\mathscr{D}_k)$ *nor* $(\mathscr{E})$.

(15) *When* $(\beta(r+s+1), \gamma(r+s+1))\geqq(e_k, n)$, *$D$ satisfies* $(\mathscr{E})$ *if and only if D satisfies* $(\mathscr{D}_k)$, *i.e.,* $\Omega(D)<\infty$.

PROOF. (10) There exists a code $C=(r+1, s', \beta, \gamma, \varphi, \mu)$ with $D<C$ which satisfies the following:

$$(\beta(r+2), \gamma(r+2), \varphi(r+2), \mu(r+2)) = (r+1,\ 1,\ r+2,\ \mu(r+1)).$$

If $\beta(c)=r+1$ then $\varphi(c)=r+2$.
If $\beta(c)=r+2$ then $\varphi(c)=r+2$.
We have $\lambda_C(r+1)=\lambda_C(r+2)$ and $\delta_C(r+1, x_j)=\delta_C(r+2, x_j)=r+2$ for any $j\in[1:n]$. Hence we have $\omega(r+1, r+2)=\infty$ and thus $\Omega(C)=\infty$.

(11) Assume that $r=0$ and $s\geqq 1$. Let $C=(r', s', \beta, \gamma, \varphi, \mu)$ be a code such that $D<C$. Then clearly $r'=r=0$. Hence $\Omega(C)=0$. Assume $r\geqq 1$. Let $C=$ $=(r, s', \beta, \gamma, \varphi, \mu)$ be a code such that $D\leqq C$. Then $\mu(C)=1$. Hence $\Omega(C)=\infty$. If $r=s=0$ then the conclusion follows from (10).

(12) Let $C=(r', s', \beta, \gamma, \varphi, \mu)$ be a code such that $D\leqq C$. Then $r=r'$ and $\mathfrak{E}_C=\mathfrak{E}_D$. Hence $\Omega(C)<\infty$ by Proposition 2.5.

(13) Let $C=(r', s', \beta, \gamma, \varphi, \mu)$ be a code such that $D\leqq C$. Then $r=r'$ and C satisfies $(\mathscr{D}_k)$ by Lemma 2.9. Hence $\Omega(C)<\infty$ by Proposition 2.11.

(14) By Lemma 2.6, D does not satisfy $(\mathscr{D}_k)$. By Proposition 4.3, there exists a precode D' such that $\Omega(D')=\infty$.

(15) Assume that D does not satisfy $(\mathscr{D}_k)$. Since $t(D)=k$, we have $\Omega(D)=\infty$ by Proposition 3.3. Hence D does not satisfy $(\mathscr{E})$. The converse follows from (13). □

The converse of the assertion (13) in the above proposition does not hold, i.e. when $(e_k, 1)\leqq(\beta(r+s+1), \gamma(r+s+1))<(e_k, n)$, $(\mathscr{E})$ does not necessarily imply $(\mathscr{D}_k)$. For the precode E in Example 4.6 does not satisfy $(\mathscr{D}_1)$ but satisfies $(\mathscr{E})$.

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(Received July 8, 1985)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 157—160

A NOTE ON THE RADICAL THEORY OF INVOLUTION ALGEBRAS

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In [4] we have dealt with a fundamental problem of the general radical theory of involution algebras over a commutative ring K with 1. This problem concerns the Anderson—Divinsky—Suliński (briefly A—D—S) property of a radical class $\mathbf{R}$: does $I^* \lhd A^*$ imply $\mathbf{R}(I^*) \lhd A^*$? In [4] necessary and sufficient conditions have been established for a radical class $\mathbf{R}$ to have the A—D—S property. J. Wichmann (private communication) asked about the A—D—S property of radical classes of involution algebras over a commutative ring K^* with 1 and with involution *. As in a commutative ring the identical mapping $x \xrightarrow{\text{id}} x$ is an involution, the case considered in [3] and [4] corresponds to that of involution algebras over K^{id}.

It is the purpose of this note to answer Wichmann's question in the case when K^* is a field with non-identical involution *. We shall show that in this case a radical class is either hypernilpotent or hypoidempotent, and that every radical class of involution algebras over a field K^* with involution $* \neq \text{id}$, has the A—D—S property.

Let K^* be any field with involution *. A K^*-algebra A^* is an *involution K^*-algebra* if in A there is defined a unary operation * such that $x^{**}=x$, $(x+y)^*=x^*+y^*$, $(xy)^*=y^*x^*$ and $(kx)^*=k^*x^*$ for all $x, y \in A$ and $k \in K^*$. Without the fear of ambiguity we shall denote by * both the involutions defined in K and A, respectively.

An involution K^*-subalgebra I^* of A^* is called an *ideal* of A^* if it is a ring-ideal of A^*. By a *homomorphism* φ we mean an algebra-homomorphism such that $(\varphi(kx))^* = k^*\varphi(x^*)$.

Let us recall that a subclass $\mathbf{R}$ of involution K^*-algebras is called a *radical class* (in the sense of Kurosh and Amitsur) if

(i) $\mathbf{R}$ is *homomorphically closed:* if $A^* \in \mathbf{R}$ then $\varphi(A^*) \in \mathbf{R}$ for every homomorphism φ,

(ii) $\mathbf{R}$ is *inductive:* if an involution K^*-algebra A^* contains an ascending chain of ideals I_α^* such that

$$\cup I_\alpha^* = A^* \quad \text{and} \quad I_\alpha^* \in \mathbf{R} \quad \text{for each } \alpha, \text{ then} \quad A^* \in \mathbf{R},$$

(iii) $\mathbf{R}$ is *closed under extensions:* if $I^* \lhd A^*$, $I^* \in \mathbf{R}$ and $(A/I)^* \in \mathbf{R}$, then $A^* \in \mathbf{R}$.

Research partially supported by Hungarian National Foundation for Scientific Research Grant no. 1813.

1980 *Mathematics Subject Classification.* Primary 16A21; Secondary 46K99.
Key words and phrases. A—D—S property, involution algebras, radical, semisimple classes.

It is clear that if $\mathbf{R}$ is a radical class, then for any involution K^*-algebra A^*:

$$\mathbf{R}(A^*) = \sum (I^* | I^* \lhd A^*,\ I^* \in \mathbf{R}) \in \mathbf{R}.$$

This ideal is called the *radical* of A^*.

As in [2] a radical class $\mathbf{R}$ will be called *hypernilpotent (hypoidempotent)* if $\mathbf{R}$ contains every nilpotent involution K^*-algebra (if $\mathbf{R}$ consists only of idempotent involution K^*-algebras). Moreover, we say that the radical class $\mathbf{R}$ has the A—D—S *property* if $\mathbf{R}(I^*) \lhd A^*$ for any involution K^*-algebra A^* and ideal $I^* \lhd A^*$.

Radical classes of involution rings were studied in [5] and radical classes of involution K-algebras over a field K (with identical involution) were investigated in [3] and [4]. In this note *we shall work with involution algebras over a field K^* such that the involution * of K^* is not the identity.*

One can easily check the validity of the Andrunakievich Lemma for involution K^*-algebras.

LEMMA 1. *If $K^* \lhd I^* \lhd A^*$ and $\overline{K^*}$ denotes the ideal of A^* generated by K^*, then $(\overline{K^*})^3 \subseteq K^*$ holds.*

PROPOSITION 1. *If $\mathbf{R}$ is a hypernilpotent or hypoidempotent radical class of involution K^*-algebras, then $\mathbf{R}$ has the A—D—S property.*

PROOF. By Lemma 1 the standard proof carries over (e.g. [2]).

For any involution K^*-algebra A^*, let A_0^* denote the additive group of A^* with the given involution. Thus A_0^* can be considered as an involution K^*-algebra with zero-multiplication. We shall use the notation:

$$T(A^*) = \{x + x^* | x \in A^*\}$$

for the set of trace-elements of A^*. Let us notice that $T(A^*) = T(A_0^*)$.

LEMMA 2. $T(A^*) \neq \{0\}$ *for any non-zero involution K^*-algebra A^*.*

PROOF. It will suffice to show that $T(A_0^*) \neq \{0\}$. Since the involution * of K^* is not the identity, there is a non-zero element $a \in K^*$ such that $a - a^* \neq 0$. Suppose indirectly that $T(A_0^*) = \{0\}$. In this case $x = -x^*$ for every $x \in A_0^*$. We get

$$\begin{aligned} -(a-a^*)x &= ((a-a^*)x)^* \\ &= -(a-a^*)x^* = (a-a^*)x. \end{aligned}$$

This implies that $2(a-a^*)x = 0$, hence $2x = 0$, that is, $x = -x$ for all $x \in A_0^*$. Since $x = -x^*$, we have $x = x^*$ for all $x \in A_0^*$. Thus the involution * of A_0^* is the identity. Let $x_0 \in A_0^*$ be any non-zero element and consider the one-dimensional involution subalgebra $H^{\mathrm{id}} = K^* x_0$ of $A_0^* = A_0^{\mathrm{id}}$. By the involution rules, the involution of H^{id} is not the identity, which is a contradiction. Therefore $T(A_0^*) \neq \{0\}$, that is, $T(A^*) \neq \{0\}$.

PROPOSITION 2. *If A^* is an involution K^*-algebra with zero-multiplication, then A^* is isomorphic to a direct sum of copies of K_0^*.*

PROOF. A is a vector space over the field K. It suffices to show that in A a basis $\{\mathcal{U}_\alpha|\alpha\in\Lambda\}$ can be chosen such that each $\mathcal{U}_\alpha$ is invariant under the involution *, that is, $\mathcal{U}_\alpha^*=\mathcal{U}_\alpha$. Let $\{\mathcal{U}_\alpha|\alpha\in\Lambda\}$ be a maximal linearly independent system of elements of A which are invariant under the involution *. Then $\{\mathcal{U}_\alpha|\alpha\in\Lambda\}$ spans a subspace V of A. Assume that $V\neq A$. By the choice of the elements $\mathcal{U}_\alpha$, the subspace V is closed under the involution *, hence $(A/V)^*=A^*/V^*\neq\{0\}$ is an involution K^*-algebra. Applying Lemma 2 we get $T(A^*/V^*)\neq\{0\}$, therefore there exists an element $x\in A^*$ such that $x+x^*\notin V^*$. One can check easily that the system $\{\mathcal{U}_\alpha|\alpha\in\Lambda\}\cup\{x+x\}^*$ is linearly independent, contrary the maximality of $\{\mathcal{U}_\alpha|\alpha\in\Lambda\}$. Hence the case $V\neq A$ is not possible, and so $V^*=A^*$.

THEOREM 1. *If* **R** *is a radical class of involution K^*-algebras, then* **R** *is either hypernilpotent or hypoidempotent.*

PROOF. In view of Proposition 2 it is straightforward to see that if **R** contains an involution K^*-algebra ($\neq 0$) with zero-multiplication, then **R** contains also K_0^*. Therefore, by the inductive property of **R**, every direct sum of copies of K_0^* belongs to **R**. Hence, by Proposition 2, in this case **R** contains every involution K^*-algebra with zero-multiplication. Otherwise, if **R** does not contain any involution K^*-algebra with zero-multiplication, then **R** is clearly hypoidempotent.

COROLLARY. *In the variety of involution K^*-algebras every radical class has the A—D—S property.*

PROOF. Straightforward by Proposition 1 and Theorem 1.

In [2] Ánh and Wiegandt proved the same result as Theorem 1 for (not necessarily associative) algebras over an arbitrary field. In the case of K^{id} every hypernilpotent or hypoidempontent radical has the A—D—S property, but in [3] a radical class is constructed which does not have the A—D—S property if char $K\neq 2$.

Now we turn to the investigation of semisimple classes. Let **R** be any radical class of involution K^*-algebras. The class

$$\mathcal{S}\mathbf{R}=\{A^*|\mathbf{R}(A^*)=0\}$$

is called the *semisimple class* of the radical **R**. In the cases of K^{id} semisimple classes were investigated in [4]. If the involution * of K^* is not the identity, then by the Corollary we can show the following for involution K^*-algebras.

THEOREM 2. *A class* **S** *of involution K^*-algebras is the semisimple class of a radical class if and only if*

(i) **S** *is coinductive: if an involution K^*-algebra A^* contains a descending chain of ideals I_α^* such that $\cap I_\alpha^*=0$ and $A^*/I_\alpha^*\in\mathbf{S}$ for each α, then $A^*\in\mathbf{S}$,*

(ii) **S** *is regular: if $A^*\in\mathbf{S}$, then every non-zero ideal of A^* has a non-zero homomorphic image in* **S**,

(iii) **S** *is closed under extensions: if I^*, $A^*/I^*\in\mathbf{S}$ then also $A^*\in\mathbf{S}$.*

The proof is analogous to that of [3] Theorem 3 using the fact that every semisimple class is hereditary (this follows from the Corollary).

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(Received July 29, 1985)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 161—168

ON THE NUMBER OF PRIME FACTORS OF INTEGERS OF THE FORM a_i+b_j

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1. Throughout this paper we use the following notations:

For any real number x let $[x]$ denote the greatest integer less than or equal to x, and let $\|x\|$ denote the distance from x to the nearest integer:

$$\|x\| = \min(x-[x], 1+[x]-x).$$

We write $e^{2\pi i x}=e(x)$. The cardinality of the set X is denoted by $|X|$. $\mu(n)$ is the Möbius function. $\omega(n)$ denotes the number of the prime factors of n counting multiplicity, and $\lambda(n)$ denotes the Liouville λ function:

$$\lambda(n) = (-1)^{\omega(n)}.$$

$\left(\frac{n}{p}\right)$ is the Legendre symbol.

The purpose of this paper is to show that if $a_1<a_2<\dots$, $b_1<b_2<\dots$ are "dense" sequences of positive integers then both equations $\lambda(a_x+b_y)=+1$ and $\lambda(a_u+b_v)=-1$ must be solvable. (See [1], [2], [3] and [9] for other somewhat related results. In fact, in all these papers arithmetic properties of sums of sequences of integers are studied.)

We will prove the following results:

THEOREM 1. *For any real number $\gamma>0$, there exists a real number N_0 such that if N is a positive integer with $N>N_0$, $\mathcal{A}\subset\{-N,-N+1,\dots,N\}$, $\mathcal{B}\subset\{-N,-N+1,\dots,N\}$ and*

(1) $$|\mathcal{A}||\mathcal{B}| > N^2(\log N)^{-\gamma},$$

then both equations

(2) $$\lambda(a_x+b_y) = +1, \qquad a_x\in\mathcal{A}, \quad b_y\in\mathcal{B}$$

and

(3) $$\lambda(a_u+b_v) = -1, \qquad a_u\in\mathcal{A}, \quad b_v\in\mathcal{B}$$

are solvable.

Research partially supported by Hungarian National Foundation for Scientific Research Grant no. 1811.

Setting here $\mathcal{B}=\mathcal{A}$ and $\mathcal{B}=-\mathcal{A}$ (where $-\mathcal{A}$ consists of the negatives of the elements of $\mathcal{A}$), we obtain:

COROLLARY 1. *For any real number $\gamma>0$, there exists a real number N_1 such that if N is a positive integer with $N>N_1$, $\mathcal{A}\subset\{1, 2, \dots, N\}$ and*

$$|\mathcal{A}| > N(\log N)^{-\gamma}, \tag{4}$$

then each of the equations

$$\lambda(a_x+a_y)=+1,$$

$$\lambda(a_u+a_v)=-1,$$

$$\lambda(a_q-a_r)=+1,$$

$$\lambda(a_s-a_t)=-1$$

is solvable.

The lower bounds in (1) and (4) seem to be far from the best possible. In fact, it can be shown that if the generalized Riemann hypothesis is true, then (1) can be replaced by

$$|\mathcal{A}|\,|\mathcal{B}| > c(\varepsilon)N^{5/3+\varepsilon}. \tag{1'}$$

On the other hand, (1) cannot be replaced by

$$|\mathcal{A}|\,|\mathcal{B}| > (1-\varepsilon)N.$$

(To see this, set $\mathcal{A}=\{0\}$, and let $\mathcal{B}$ denote the set of the integers n with $|n|\leqq N$, $\lambda(n)=+1$.) Perhaps, it is enough to assume that

$$|\mathcal{A}|\,|\mathcal{B}|\,N^{-1}\to+\infty.$$

Unfortunately, I have not been able to prove this.

Replacing (1) by (1′) in Theorem 1, we may derive Corollary 1 with

$$|\mathcal{A}| > N^{5/6+\varepsilon} \tag{4'}$$

in place of (4) (under the assumption that the generalized Riemann hypothesis is true). One may guess that in Corollary 1, (4) or (4′) can be replaced by $|\mathcal{A}|\to+\infty$. This is not so, as the following theorem shows:

THEOREM. *If $N\geqq 4$, then there exist sequences*

$$\mathcal{A}\subset\{1, 2, \dots, N\},\quad \mathcal{B}\subset\{1, 2, \dots, N\} \tag{5}$$

such that

$$|\mathcal{A}| = |\mathcal{B}| = \left[\frac{\log N}{\log 4}\right], \tag{6}$$

$$\lambda(a-a')=+1 \quad \textit{for all} \quad a\in\mathcal{A},\ a'\in\mathcal{A},\ a\neq a' \tag{7}$$

and

$$\lambda(b-b')=-1 \quad \textit{for all} \quad b\in\mathcal{B},\ b'\in\mathcal{B},\ b\neq b'. \tag{8}$$

The lower bound in Theorem 2 seems to be nearer to the truth than the upper bound in Corollary 1. In fact, perhaps, the right-hand side of (4) in Corollary 1 can be replaced by $(\log N)^c$ but again, it seems to be hopeless to prove this.

2. The proof of Theorem 1 will be based on a result of Hajela and Smith:

LEMMA 1. (i) *For any real number* $\varrho>0$, *for* $x>x_0(\varrho)$ *we have*

$$\Big|\sum_{n\le x}\mu(n)e(n\alpha)\Big|<x(\log x)^{-\varrho}\quad \textit{for all}\quad 0\le\alpha\le 1. \tag{9}$$

(ii) *Under the generalized Riemann hypothesis, for* $\varepsilon>0$, $x>x_1(\varepsilon)$ *we have*

$$\Big|\sum_{n\le x}\mu(n)e(n\alpha)\Big|<x^{5/6+\varepsilon}\quad \textit{for all}\quad 0\le\alpha\le 1. \tag{10}$$

PROOF. (i) is the first part of Theorem 4.1 in [5], while (ii) is Corollary 4.2 in [5]. (In fact, for α's belonging to the "minor arcs", (9) can be proved by using Vaughan's identity, while for α's belonging to the "major arcs", the estimate of $\Big|\sum_{n\le x}\mu(n)e(n\alpha)\Big|$ can be reduced to the estimate of sums of the form $\sum_{n\le x}\mu(n)\chi(n)$ where χ is a character belonging to a "small" modulus, and these last sums can be estimated by using standard contour integral technics.)

We shall need the following consequence of Lemma 1:

LEMMA 2. (i) *For any real number* $\varrho>0$, *for* $x>x_2(\varrho)$ *we have*

$$\Big|\sum_{n\le x}\lambda(n)e(n\alpha)\Big|<x(\log x)^{-\varrho}\quad \textit{for all}\quad 0\le\alpha\le 1. \tag{11}$$

(ii) *Under the generalized Riemann hypothesis, for* $\varepsilon>0$, $x>x_3(\varepsilon)$ *we have*

$$\Big|\sum_{n\le x}\lambda(n)e(n\alpha)\Big|<x^{5/6+\varepsilon}\quad \textit{for all}\quad 0\le\alpha\le 1. \tag{12}$$

PROOF. Let us define the multiplicative function $f(n)$ in the following way: let $f(n)=1$ if $n=k^2$ where k is a positive integer and let $f(n)=0$ if n is not a square. Then it is well-known (and it can be seen easily) that we have

$$f(n)=\sum_{d|n}\lambda(d).$$

Hence, by the Möbius inversion formula,

$$\lambda(n)=\sum_{d|n}f(d)\mu(n/d)=\sum_{d^2|n}\mu(n/d^2)$$

so that

$$\Big|\sum_{n\le x}\lambda(n)e(n\alpha)\Big|=\Big|\sum_{n\le x}\Big(\sum_{d^2|n}\mu(n/d^2)\Big)e(n\alpha)\Big|=$$

$$=\Big|\sum_{d^2\le x}\sum_{k\le x/d^2}\mu(k)e(kd^2\alpha)\Big|\le\sum_{d^2\le x}\Big|\sum_{k\le x/d^2}\mu(k)e(kd^2\alpha)\Big|. \tag{13}$$

Thus by using (i) in Lemma 1 (with 2ϱ in place of ϱ) we obtain that if x is large enough (in terms of ϱ) then

$$\Big|\sum_{n\le x}\lambda(n)e(n\alpha)\le$$

$$\le\sum_{d\le x^{1/4}}\Big|\sum_{k\le x/d^2}\mu(k)e(kd^2\alpha)\Big|+\sum_{x^{1/4}<d\le x^{1/2}}\Big|\sum_{k\le x/d^2}\mu(k)e(kd^2\alpha)<$$

$$<\sum_{d\le x^{1/4}}\frac{x}{d^2(\log(x/d^2))^{2\varrho}}+\sum_{x^{1/4}<d\le x^{1/2}}\sum_{k\le x/d^2}1\le$$

$$\le\sum_{d\le x^{1/4}}\frac{x}{d^2(\log x^{1/2})^{2\varrho}}+\sum_{x^{1/4}<d\le x^{1/2}}\frac{x}{d^2}=$$

$$=2^{2\varrho}\frac{x}{(\log x)^{2\varrho}}\sum_{d\le x^{1/4}}\frac{1}{d^2}+x\sum_{x^{1/4}<d}\frac{1}{d^2}<\frac{x}{(\log x)^{\varrho}}$$

which completes the proof of (11).

By using (13), (12) can be derived from (10) in the same way.

3. In this section, we complete the proof of Theorem 1. Let R_+ and R_- denote the number of solutions of (2) and (3), respectively, so that

$$R_++R_-+\sum_{\substack{a+b=0\\ a\in\mathcal{A},\,b\in\mathcal{B}}}1=\sum_{a\in\mathcal{A},\,b\in\mathcal{B}}1=|\mathcal{A}||\mathcal{B}|, \tag{14}$$

and put

$$F(\alpha)=\sum_{a\in\mathcal{A}}e(a\alpha),\quad G(\alpha)=\sum_{b\in\mathcal{B}}e(b\alpha).$$

Then we have

$$R_+=\sum_{\substack{-2N\le n\le 2N\\ \lambda(n)=+1}}\ \sum_{\substack{a+b=n\\ a\in\mathcal{A},\,b\in\mathcal{B}}}1=$$

$$=\sum_{\substack{-2N\le n\le 2N\\ \lambda(n)=+1}}\ \sum_{a\in\mathcal{A},\,b\in\mathcal{B}}\int_0^1 e((a+b-n)\alpha)\,d\alpha=$$

$$=\int_0^1\Big(\sum_{\substack{-2N\le n\le 2N\\ \lambda(n)=+1}}e(-n\alpha)\Big)\Big(\sum_{a\in\mathcal{A}}e(a\alpha)\Big)\Big(\sum_{b\in\mathcal{B}}e(b\alpha)\Big)\,d\alpha=$$

$$=\int_0^1\Big(\sum_{0<|n|\le 2N}\frac{1}{2}(1+\lambda(n))e(-n\alpha)\Big)F(\alpha)G(\alpha)\,d\alpha= \tag{15}$$

$$=\frac{1}{2}\int_0^1\Big(\sum_{-2N\le n\le 2N}e(-n\alpha)\Big)F(\alpha)G(\alpha)\,d\alpha+$$

$$+\frac{1}{2}\int_0^1\Big(-1+2\,Re\sum_{n=1}^{2N}\lambda(n)e(n\alpha)\Big)F(\alpha)G(\alpha)\,d\alpha.$$

Clearly, we have

$$\int_0^1 \Big(\sum_{-2N\leq n\leq 2N} e(-n\alpha)\Big) F(\alpha)G(\alpha)\,d\alpha =$$

$$(16) \qquad = \int_0^1 \sum_{-2N\leq n\leq 2N} \sum_{a\in\mathscr{A}} \sum_{b\in\mathscr{B}} e\big((a+b-n)\alpha\big)\,d\alpha =$$

$$= \sum_{\substack{a\in\mathscr{A},\, b\in\mathscr{B}\\ -2N\leq a+b\leq 2N}} 1 = \sum_{a\in\mathscr{A},\, b\in\mathscr{B}} 1 = |\mathscr{A}|\,|\mathscr{B}|.$$

Furthermore, by using Lemma 2 (with 2γ in place of ϱ), Cauchy's inequality and Parseval's formula, we obtain that if N is large enough in terms of γ, then

$$\left|\frac{1}{2}\int_0^1 \Big(-1+2Re\sum_{n=1}^{2N}\lambda(n)e(n\alpha)\Big)F(\alpha)G(\alpha)\,d\alpha\right| \leq$$

$$\leq \frac{1}{2}\int_0^1 \Big(1+2\Big|\sum_{n=1}^{2N}\lambda(n)e(n\alpha)\Big|\Big)|F(\alpha)|\,|G(\alpha)|\,d\alpha <$$

$$(17) \qquad < 2\int_0^1 2N(\log 2N)^{-2\gamma}|F(\alpha)|\,|G(\alpha)|\,d\alpha < \frac{N}{(\log N)^{\gamma}}\int_0^1 |F(\alpha)G(\alpha)|\,d\alpha \leq$$

$$\leq \frac{N}{(\log N)^{\gamma}}\Big(\int_0^1 |F(\alpha)|^2\,d\alpha \int_0^1 |G(\alpha)|^2\,d\alpha\Big)^{1/2} = \frac{N}{(\log N)^{\gamma}}(|\mathscr{A}|\,|\mathscr{B}|)^{1/2}.$$

In view of (1), (15), (16) and (17) yield for large N that

$$(18) \qquad \left|R_+ - \frac{1}{2}|\mathscr{A}|\,|\mathscr{B}|\right| < \frac{N}{(\log N)^{\gamma}}(|\mathscr{A}|\,|\mathscr{B}|)^{1/2} < \frac{1}{(\log N)^{\gamma/2}}|\mathscr{A}|\,|\mathscr{B}|$$

so that (2) can be solved, in fact we have

$$(19) \qquad R_+ \sim \frac{1}{2}|\mathscr{A}|\,|\mathscr{B}|.$$

Finally, in view of (1) we have

$$\sum_{\substack{a+b=0\\ a\in\mathscr{A},\, b\in\mathscr{B}}} 1 \leq \min(|\mathscr{A}|, |\mathscr{B}|) \leq (|\mathscr{A}|\,|\mathscr{B}|)^{1/2} =$$

$$(20) \qquad = \frac{1}{(|\mathscr{A}|\,|\mathscr{B}|)^{1/2}}|\mathscr{A}|\,|\mathscr{B}| < \frac{(\log N)^{\gamma/2}}{N^{1/2}}|\mathscr{A}|\,|\mathscr{B}| \ (= o(|\mathscr{A}|\,|\mathscr{B}|)).$$

It follows from (14), (19) and (20) that also

$$R_- \sim \frac{1}{2}|\mathscr{A}|\,|\mathscr{B}|$$

holds, which completes the proof of Theorem 1.

By using (ii) in Lemma 2 in place of (i), it can be shown in the same way that (1) in Theorem 1 can be replaced by (1′).

4. The proof of Theorem 2 will be based on the following Ramsey-type theorem of Erdős and Szekeres [4]:

LEMMA 3. *Let K, M be positive integers and G_K a graph of K vertices. If*

$$K \geqq \binom{2M-2}{M-1}, \tag{21}$$

then either G_K or the complement of G_K contains a complete subgraph of M vertices.

We use this lemma with $K=[N/2]$, $M=\left[\frac{\log N}{\log 4}\right]$. Then by $N \geqq 4$ we have

$$\binom{2M-2}{M-1} \leqq 2^{2M-2} = 2^{2\left[\frac{\log N}{\log 4}\right]-2} \leqq 2^{2\frac{\log N}{\log 2}-2} = \frac{N}{4} \tag{22}$$

and

$$K = \left[\frac{N}{2}\right] > \frac{N}{2} - 1 \geqq \frac{N}{2} - \frac{N}{4} = \frac{N}{4}. \tag{23}$$

(21) follows from (22) and (23) so that, in fact, Lemma 3 can be applied.

Let us define the graph G_K in the following way: Denoting the vertices of G_K by $P_1, P_2, \dots, P_K$, we connect the vertices P_i, P_j if and only if $\lambda(i-j)=+1$. Then by Lemma 3, either G_K or the complement of it contains a complete subgraph of M vertices. Assume first that G_K contains a complete subgraph of M vertices, and denote the vertices of it by $P_{i_1}, P_{i_2}, \dots, P_{i_M}$. Then clearly, $\mathcal{A}=\{i_1, i_2, \dots, i_M\}$, $\mathcal{B}=\{2i_1, 2i_2, \dots, 2i_M\}$ satisfy (5), (6), (7) and (8). Assume now that the complement of G_K contains a complete subgraph of M vertices, and denote the vertices of it by P_{j_1}, $P_{j_2}, \dots, P_{j_m}$. Then again, $\mathcal{A}=\{2j_1, 2j_2, \dots, 2j_M\}$, $\mathcal{B}=\{j_1, j_2, \dots, j_M\}$ satisfy (5), (6), (7) and (8), which completes the proof of Theorem 2.

5. Note that Erdős and Sárközy proved in [3] that for "dense" sequences $\mathcal{A}$,

$$\mu(a_x+a_y)=0$$

is solvable. On the other hand, the equations

$$\mu(a_u+a_v)=+1, \quad \mu(a_q+a_r)=-1,$$

$$\mu(a_s-a_t)=+1, \quad \mu(a_z-a_w)=-1$$

need not be solvable, as the sequence $\mathcal{A}=\{4, 8, \dots, 4k, \dots\}$ shows.

Furthermore, Erdős and Sárközy proved in [2] that if p is a prime number greater than 2, $\mathcal{A}\subset\{1, 2, \dots, p-1\}$ and

$$|\mathcal{A}| > 6p^{7/8}(\log p)^{1/2},$$

then each of the equations

$$\left(\frac{a_x+a_y}{p}\right)=+1,\quad \left(\frac{a_u+a_v}{p}\right)=-1,\quad \left(\frac{a_q-a_r}{p}\right)=+1,\quad \left(\frac{a_s-a_t}{p}\right)=-1$$

is solvable. This result and Theorem 1 suggest the following conjecture:

CONJECTURE 1. If $f(n)$ is a multiplicative arithmetic function which assumes only the values -1 and $+1$,

$$\sum_{f(p)=-1}\frac{1}{p} \tag{24}$$

is divergent, and $\mathscr{A}$ is an infinite sequence of positive integers with

$$\limsup_{N\to+\infty}\frac{1}{N}\sum_{\substack{a\le N\\ a\in\mathscr{A}}}1>0,$$

then each of the equations

$$f(a_x+a_y)=+1,\qquad f(a_u+a_v)=-1,\qquad f(a_q-a_r)=+1,\qquad f(a_s-a_t)=-1$$

is solvable.

In fact, this would follow from

CONJECTURE 2. If $f(n)$ is a multiplicative arithmetic function which assumes only the values -1 and $+1$ and for which the series (24) is divergent, then we have

$$\lim_{N\to+\infty}\Big(\max_{0\le\alpha\le1}\Big|\sum_{n=1}^{N}f(n)e(n\alpha)\Big|\Big)=0.$$

For α's belonging to the "minor arcs",

$$\Big|\sum_{n=1}^{N}f(n)e(n\alpha)\Big| \tag{25}$$

can be estimated by using a quantitative version of Daboussi's theorem proved by Montgomery and Vaughan in [8]. On the other hand, for α's belonging to the "major arcs", a sharper form of Halász's mean value theorems [6], [7] (a mean value theorem with a good error term) would be needed for the estimate of (25).

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(Received August 13, 1985)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 169—174

A REMARK ON NORMED ALMOST LINEAR SPACES

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Abstract

G. Godini introduced a concept of normed almost linear spaces. In this paper we prove that the analytic dual of such a space with some additional properties is not trivial.

Introduction

In [1], the concept of a normed almost linear space (NALS) is introduced. For the sake of completeness, we shall repeat here the necessary definitions.

An almost linear space is a set X together with two mappings

$$s\colon X\times X\to X \quad \text{and} \quad m\colon \mathbf{R}\times X\to X$$

with the properties L_1-L_8 below. We shall denote $s(x,y)$ by $x+y$, and $m(\lambda, x)$ by λx.

We write $(-1)x=-x$, and $x+(-y)=x-y$.

Let $x, y, z\in X$, $\lambda, \mu\in \mathbf{R}$.

L_1 $(x+y)+z = x+(y+z)$,

L_2 $x+y = y+x$,

L_3 there exists $0\in X$ such that $x+0 = x$ for all $x\in X$,

L_4 $1x = x$,

L_5 $0x = 0$,

L_6 $\lambda(x+y) = \lambda x+\lambda y$,

L_7 $\lambda(\mu x) = (\lambda\mu)x$,

L_8 $(\lambda+\mu)x = \lambda x+\mu x$ for $\lambda \geqq 0$, $\mu \geqq 0$.

For an almost linear space X we introduce the following sets:

$$V_X = \{x\in X;\ x-x = 0\}$$

$$W_X = \{x\in X;\ x = -x\}.$$

1980 *Mathematics Subject Classification*. Primary 46B99.
Key words and phrases. Almost linear spaces.

A norm on X is a functional $\|\,.\,\|:X\to\mathbf{R}$ with properties N_1-N_4 below. Let $x, y, z\in X$ and $\lambda\in\mathbf{R}$.

N_1 $\|x-z\| \leqq \|x-y\|+\|y-z\|$,

N_2 $\|\lambda x\| = |\lambda|\,\|x\|$,

N_3 $\|x\| = 0$ iff $x = 0$.

Here we remark that N_1 implies

$$\|x+y\| \leqq \|x\|+\|y\|$$

using the substitutions $-z=y$ and $y=0$.

So, V_X is a normed linear space. Then, denoting the weak convergence in V_X by $\rightharpoonup$, we can take a v_α net in V_X such that $v_\alpha\rightharpoonup v$, and $v\in V_X$.

N_4 With the above notations, we have

$$\|x-v\| \leqq \liminf \|x-v_\alpha\| \quad \text{for all } x\in X.$$

We call an almost linear space X a NALS if we have a norm on X.

Now, we introduce the concept of an almost linear functional on an almost linear space.

Let X be an almost linear space. A functional $f\colon X\to\mathbf{R}$ is called almost linear if satisfies the following properties: $(x, y\in X, \lambda\geqq 0)$

$$f(x+y)=f(x)+f(y),$$

$$f(\lambda x) = \lambda f(x),$$

$$-f(-x)\leqq f(x).$$

We define the norm of an almost linear functional by

$$\|f\| = \sup\,\{|f(x)|,\ x\in X,\ \|x\| \leqq 1\}.$$

We say that an almost linear functional is bounded if $\|f\|<+\infty$.

In [1], the following problem is mentioned: Does there exist any nontrivial NALS X such that the only bounded almost linear functional on X is the null-functional?

In this paper we show that on a special class of NALS, we always have nontrivial bounded almost linear functionals. We give also an example for such a NALS.

We say, that in a NALS X the system (X_α) is independent if any two different finite linear combinations of the vectors (X_α) are different vectors.

We say that the NALS X is finite dimensional if we have a finite maximal independent system in X.

The result

THEOREM. *Let X be a NALS with the following properties.*

(i) $X=W_X$;

(ii) *X is finite dimensional;*

(iii) $x+z=y+z$ *implies* $x=y$ *for all* $x, y, z\in X$.

Then we have a nontrivial almost linear functional on X, which is bounded.

PROOF. We shall need some lemmas.

LEMMA 1. *For all $x, y \in X$,*

$$\|x+y\| \geqq \|x\|.$$

PROOF. Using property N_1 for $x=z$, we have

$$\|x-x\| \leqq \|x-y\|+\|y-x\|.$$

Using $X=W_X$, we get

$$\|2x\| \leqq \|x+y\|+\|y+x\|,$$

thus the desired conclusion.

Now, let $x_1, \ldots, x_n$ be a maximal independent system in X. Introducing the set

$$K = \{\lambda_1 x_1+\ldots+\lambda_n x_n;\ \lambda_1, \ldots, \lambda_n \geqq 0\} \subset X,$$

we can easily construct an n-dimensional normed linear space Y with basis $y_1, \ldots, y_n$ such that the mapping $T: K \to Y$

$$T(\lambda_1 x_1+\ldots+\lambda_n x_n) = \lambda_1 y_1+\ldots+\lambda_n y_n$$

is additive, positively homogeneous and norm-preserving onto a cone $C \subset Y$.

Fixing an element $y \in C$ such that

$$y = \mu_1 y_1+\ldots+\mu_n y_n$$

with all $\mu_i > 0$, the Banach—Hahn theorem guarantees the existence of a linear functional f on Y such that $\|f\|=1$, and $f(y)=\|y\|$.

Clearly, the functional $f^* = f \circ T$ is an additive and positive homogeneous functional on K such that

$$|f^*(k)| \leqq \|k\| \quad \text{for all} \quad k \in K,$$

and for the element $x^* = \mu_1 x_1+\ldots+\mu_n x_n$,

$$f^*(x^*) = \|x^*\|.$$

In the remaining part of the proof we shall extend this functional to X, proving that the extension $\bar{f}$ is almost linear and $\|\bar{f}\|=1$.

LEMMA 2. *Let $y \in X$ be arbitrary. Then there exists $z \in K$ such that*

$$y+z \in K.$$

PROOF. Let us assume indirectly that for all $z \in K$,

$$y+z \notin K.$$

We show that $y, x_1, \ldots, x_n$ is an independent system in X.

In the opposite case we have

$$\lambda_0 y+\lambda_1 x_1+\ldots+\lambda_n x_n = \lambda_0' y+\lambda_1' x_1+\ldots+\lambda_n' x_n$$

for some non-negative $(n+1)$-tuples $(\lambda_0, \lambda_1, \ldots, \lambda_n) \neq (\lambda_0', \ldots, \lambda_n')$.

Now, we distinguish between two cases.

Case 1. $\lambda_0=\lambda_0'$. Then by (iii)

$$\lambda_1 x_1+\ldots+\lambda_n x_n = \lambda_1' x_1+\ldots+\lambda_n' x_n$$

which contradicts the independence of the system $x_1, \ldots, x_n$.

Case 2. $\lambda_0\neq\lambda_0'$. We can assume without loss of generality that $\lambda_0<\lambda_0'$. Using (iii), we have

$$\lambda_1 x_1+\ldots+\lambda_n x_n = (\lambda_0'-\lambda_0)y+\lambda_1' x_1+\ldots+\lambda_n' x_n.$$

This implies that for

$$z = \frac{\lambda_1'}{\lambda_0'-\lambda_0}x_1+\frac{\lambda_2'}{\lambda_0'-\lambda_0}x_2+\ldots+\frac{\lambda_n'}{\lambda_0'-\lambda_0}x_n$$

$y+z\in K$, which is absurd.

The following lemma is an easy consequence of Lemma 2.

LEMMA 3. *For all $y\in X$ there exists $\lambda>0$ and $z\in K$ such that*

$$y+z = \lambda x^*.$$

PROOF. Using Lemma 2, we have a z such that

$$y+z\in K,$$

so

$$y+z = \lambda_1 x_1+\ldots+\lambda_n x_n.$$

Since

$$x^* = \mu_1 x_1+\ldots+\mu_n x_n$$

with all $\mu_i>0$, the proof is an elementary computation, which is left to the reader.

LEMMA 4. *For all $y\in X$, there exists $\varepsilon>0$ such that*

$$x^*+\varepsilon y\in K.$$

PROOF. Let us assume the contrary. Without loss of generality we have $y\notin K$. We prove that $y, x_1, \ldots, x_n$ is an independent system.

In the case when this latter is not fulfilled, there exist different $(n+1)$-tuples

$$(\lambda_0, \lambda_1, \ldots, \lambda_n) \quad\text{and}\quad (\lambda_0', \lambda_1', \ldots, \lambda_n')$$

such that

$$\lambda_0 y+\lambda_1 x_1+\ldots+\lambda_n x_n = \lambda_0' y+\lambda_1' x_1+\ldots+\lambda_n' x_n.$$

Similarly, we can exclude the case $\lambda_0=\lambda_0'$ as in the proof of Lemma 3. Assuming now $\lambda_0<\lambda_0'$, we have

$$\lambda_1 x_1+\ldots+\lambda_n x_n = (\lambda_0'-\lambda_0)y+\lambda_1' x_1+\ldots+\lambda_n' x_n.$$

Multiplying both sides with a constant $c>0$ such that

$$\max_{1\leq i\leq n} c\lambda_i' \leq \min_i \mu_i,$$

an easy computation shows that

$$(\lambda_0' - \lambda_0)cy + x^* \in K,$$

which is again absurd.

Now, we shall extend f^* to X. The extension is denoted by $\bar{f}$.

Let be $y \in X$ arbitrary. Using Lemma 2, we have $z \in K$ such that

$$y + z = u \in K.$$

Now, we define

$$\bar{f}(y) = f^*(u) - f^*(z).$$

First, we show that this definition is correct. Let us assume

$$y + z_1 = u_1$$

$$y + z_2 = u_2$$

for some

$$z_1, z_2, u_1, u_2 \in K.$$

We can write $y + z_1 + z_2 = u_1 + z_2$, and, on the other hand

$$y + z_2 + z_1 = u_2 + z_1.$$

These together imply

$$u_1 + z_2 = u_2 + z_1.$$

By the additivity of f^*

$$f^*(u_1) + f^*(z_2) = f^*(u_2) + f^*(z_1)$$

and finally

$$f^*(u_1) - f^*(z_1) = f^*(u_2) - f^*(z_2).$$

We have proved that the definition is correct.

The positive homogeneity of $\bar{f}$ is an easy consequence of f^*, the proof is left to the reader.

We have the same situation with additivity.

Now we show that $\bar{f}$ is non-negative on X.

Let $y \in X$ be arbitrary. Using Lemma 3, there exists $z \in K$, $\lambda > 0$ such that

$$y + z = \lambda x^*.$$

Applying Lemma 1,

$$\|\lambda x^*\| = \lambda \|x^*\| = \|y + z\| \geqq \|z\|.$$

Also we have

$$\bar{f}(y) + f^*(z) = f^*(\lambda x^*) = \lambda \|x^*\|,$$

$$f^*(z) \leqq \|z\| \leqq \lambda \|x^*\|.$$

The two latter relations imply $\bar{f}(y) \geqq 0$.

Summing up, $\bar{f}$ is an almost linear functional on X.

Finally, we show that

$$\bar{f}(y) \leqq \|y\|$$

for all $y \in X$.

Lemma 4 implies that we have $\varepsilon > 0$ such that

$$x^* + \varepsilon y = u \in K.$$

Clearly we have

$$f^*(u) = \bar{f}(u) \leqq \|u\|.$$

Using Lemma 1 and the triangle inequality,

$$\|x^*\| \leqq \|u\| \leqq \|x^*\| + \varepsilon \|y\|$$

so

(1) $$\bar{f}(u) \leqq \|x^*\| + \varepsilon \|y\|.$$

Applying

$$\bar{f}(x^*) = \|x^*\|, \qquad x^* + \varepsilon y = u$$

we have

$$\|x^*\| + \varepsilon \bar{f}(y) = \bar{f}(u).$$

Using (1),

$$\bar{f}(y) \leqq \|y\|.$$

The Theorem is proved.

Finally, we give an example for a NALS with properties (i), (ii) and (iii).

Let K be a convex cone in an arbitrary finite dimensional normed linear space. Introducing $(-1)x = x$, this is a NALS in the case when K is sufficiently "narrow". (The latter assumption is necessary for N_1). All the necessary proofs are elementary.

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(Received August 14, 1985)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 175—179

ON THE CHARACTERIZATION OF k-DIMENSIONAL ADDITIVE FUNCTIONS I

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In [2] and [3] we examined the "maximum functions" g of additive functions $f\colon \mathbf{N}\to\mathbf{R}$, where $0\leqq r_1<r_2<\ldots<r_m$ are fixed integers and

(1) $$g(n)\in G_n := \{f(n+r)\colon r\in\{r_1, r_2, \ldots, r_m\},\ |f(n+r_i)| \text{ maximal}\}.$$

The results of these papers can be generalized for the k-dimensional space.

In this paper let $f\colon \mathbf{N}\to\mathbf{R}^k$ denote an additive function and $|f|$ its Euclidean norm.

We can prove the following theorem similar to the proof of Theorem 2 in [3]:

THEOREM 1. *For any monotonically increasing function* $h\colon \mathbf{N}\to\mathbf{R}$ *there exists a sequence* $A=\{a_1<a_2<\ldots\}\subset\mathbf{N}$ *such that* $a_n>h(n)$ *and if*

(2) $$\lim_{a_n\in A,\, n\to\infty} g(a_n) = \underline{c},$$

then $g(m)=\underline{c}$ *for all* $m\in\mathbf{N}$. *The same holds for all sets* $A=\{a_1, a_2, \ldots\}$ *having upper density one.*

THEOREM 2. *If* $g=\underline{c}$, *then, with the exception of at most* $[m/2]$ *primes* p, $f(p^\alpha)=\underline{0}$ *for all* α.

REMARK 1. The definition in (1) permits the construction of different maximum functions g_i with different $\underline{c}_i$. We have only $g_i=\underline{c}_i$ and $|f(n)|\leqq|\underline{c}_i|=c$ for all $n\in\mathbf{N}$. If t_i are coprime numbers with $|f(t_i)|=c$, then the angle of $f(t_j), f(t_s)$ must be $\geqq\frac{2\pi}{3}$ for all $j\neq s$. This permits only one g in the case $k=1$ and at most three g_i's in the cases $k\geqq2$. (Let for example $f(2^\alpha)=\varepsilon$, $f(3^\alpha)=\varepsilon^2$, $f(5^\alpha)=1$ with $\varepsilon=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ and $r_j=j$ for $m\geqq30$. Then we can choose $g_j=\varepsilon^j$ $(j=1, 2, 3)$.)

It is possible to prescribe the maximum on the multiples of a number q. In this case the cardinality of the g_i's depends only from the choice of the pairwise coprime s_i's for which $|f(s_iq)|=c$. (For example in the case $r_j=j$ with $j\leqq m=2^t$ and by the definition $f(p^\alpha)=\underline{0}$ for $p\neq2$ and $f(2^\alpha)=c\varepsilon^\alpha$ for $\alpha\leqq t$ and $f(2^\beta)=c\varepsilon^t$ if $\beta>t$, where ε is a primitive t-th unit root and $c>0$, we have $g_i=c\varepsilon^i$ $(i=1, \ldots, t)$.) This

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1980 *Mathematics Subject Classification*. Primary 11A25

Key words and phrases. Characterization of additive functions, the function $c\log n$.

construction does not permit another q_1 with the properties $(s_i, q_1)=1$ and $|f(q_1)|=c$, except the case of the third unit roots.

REMARK 2. The proof of Theorem 1 yields the following result as well.

THEOREM 1′. *For any* $h\colon \mathbf{N}\to\mathbf{R}$ *there exists a sequence* $A=\{a_1<a_2<\dots\}\subset\mathbf{N}$ *such that* $a_n>h(n)$ *and if*

$$\lim_{a_n\in A,\, n\to\infty} |g(a_n)| = c, \tag{3}$$

then $|g(n)|=c$ *for all* $n\in\mathbf{N}$.

The weaker condition (3) does not yield $g=\underset{\sim}{c}$, for example in the case $f(2^\alpha)=$ $=(-1)^\alpha$ and $f(p^\alpha)=0$ for the other primes p, whenever $r_1, r_2, \dots, r_m$ contains a complete residue system mod 2.

COROLLARY 1. *From Theorem 1′ follows that if* $|g|$ *is monotonically decreasing, then* $|g|=c$. *So* $|g|$ *cannot be strictly monotonically decreasing. Similarly, if* $|g|$ *is monotonically increasing and bounded, then* $|g|=c$.

If $|g|\to\infty$ monotonically, then we have a general result only in the case that $|g|$ is strictly monotonic. For the dimensions $k\geqq 2$ we need a generalized form ([4]) of a well-known theorem of Erdős ([1]):

(4) *If the function* $f\colon \mathbf{N}\to\mathbf{R}^k$ *is additive and its Euclidean norm is monotonic from a point on, then* $f(n)=\underset{\sim}{c}\log n$ *with a constant* $\underset{\sim}{c}\in\mathbf{R}^k$.

Applying this theorem, we shall prove

THEOREM 3. *If*

$$|g|\to\infty \text{ is strictly increasing from a point on, then} \tag{5}$$

$$f(n) = \underset{\sim}{c}\log n \tag{6}$$

with a constant $\underset{\sim}{c}\in\mathbf{R}^k$.

If f *is completely additive, then we can find a set* A *with the rarity* $a_n>h(n)$, *such that* (5) *on* A *implies* (6) *for all* $n\in\mathbf{N}$.

REMARK 3. The strict monotonicity is a necessary condition. Consider

$$f(n) = \begin{cases} \log(n/2) & \text{for } 2|n \\ \log n & \text{for } 2\nmid n, \end{cases}$$

where $m=2$, $r_1=0$, $r_2=1$.

The definition (1) differs from

$$g^*(n) = \max\{|f(n+r_i)|\colon 1\leqq i\leqq m,\ 0\leqq r_1<\dots<r_m\},$$

which was examined in [5] for real-valued functions. The case $k=1$ gives a stronger result than Theorem 3:

THEOREM 4. *Let* $k=1$. *Then* (5) *on an arbitrary set having upper density one resp. on a suitable rare set with the rarity* $a_n>h(n)$ *implies* (6).

Proofs

PROOF OF THEOREM 2. Without restricting generality, assume that

$$\underline{c} = (c, 0, \ldots, 0), \quad c > 0.$$

Let f_1 be the first coordinate of f. By (1)

$$|f_1(n)| \leqq |f(n)| \leqq c \quad \text{for} \quad n \geqq 1+r_1. \tag{7}$$

(7) yields that $|f(p^\alpha)| > \delta > 0$ cannot hold for an infinity of powers of different primes. (Otherwise selecting an infinite set of $f(p^\alpha)$ pointing to approximately the same direction, we would have $|f(\prod p^\alpha)| \to \infty$, contradicting (7).) This yields

$$|f_1(n)| \leqq |f(n)| \leqq c \quad \text{for all} \quad n \in \mathbf{N}. \tag{8}$$

(In the opposite case there is an $m \in \mathbf{N}$, for which $|f(m)| > c$ with $m < 1+r_1$ and then $|f(mp^\alpha)| > c$ with a suitable prime power contradicts (7).)

Let us consider the smallest $n_0 \in \mathbf{N}$ for which $f_1(n_0) = c$. By (8) for all $(t, n_0) = 1$ yields

$$f_1(t) < 0 \quad \text{or} \quad f(t) = \underline{0}. \tag{9}$$

The minimal choice of n_0 implies that for all $p_i^{\alpha_i} \| n_0$

$$f_1(p_i^{\alpha_i}) > f_1(p_i^\beta) \quad \text{for all} \quad \beta < \alpha_i,$$

resp. (9) gives

$$f_1(p_i^{\alpha_i}) \geqq f_1(p_i^\beta) \quad \text{for all} \quad \beta > \alpha_i.$$

So if $c \neq 0$, then

(10) $|f|$ has its maximum only on the multiples of n_0.

We divide the primes p into 3 classes:

(a) $p | n_0$;

(b) $p \nmid n_0$, $f_1(p^\alpha) < 0$ for some α;

(c) $f(p^\alpha) = \underline{0}$ for all α.

(9) just means that every prime p belongs to one of these classes.

The class (a) contains at most $\frac{\log n_0}{\log 2}$ primes.

We shall prove that (b) is finite, too. First we show that $r_1, r_2, \ldots, r_m$ contains a complete residue system mod n_0 by the exception $c = 0$. Otherwise there is an i $(1 \leqq i \leqq n_0)$ such that $r_j \not\equiv i \mod n_0$ $(j = 1, \ldots, m)$. If now x is any solution of the congruence

$$x \equiv -i \mod n_0,$$

then no $x + r_j$ is a multiple of n_0, which contradicts (10).

As a byproduct, we obtain $n_0 \leqq m$. Let m_i be the number of the r_j's, $r_j \equiv i \mod n_0$. Clearly $\sum_{i=1}^{n_0} m_i = m$, hence $m_i \leqq [m/n_0]$ for some i.

We show now that the number of primes in (b) is smaller than m_i for this i, too. Otherwise, let $q_1, \ldots, q_d$ $(d=m_i)$ be primes from (b) such that $f_1(q_j^{\beta_j})<0$. Let $R_1, \ldots, R_d$ be the r_j's that are $\equiv i \pmod{n_0}$.

Let x be the solution of the system of congruences

$$x \equiv -i \bmod n_0$$

$$x+R_j \equiv q_j^{\beta_j} \bmod q_j^{\beta_j+1}.$$

Then $f_1(x+r_h)<c$ for all h; namely if $r_h \not\equiv i \bmod n_0$, then $n_0 \nmid x+r_h$, if $r_h \equiv i \bmod n_0$, then $r_h=R_j$ for some j, so $n_0 | x+R_j=q_j^{\beta_j} y_j$ and therefore

$$f_1(x+R_j) = f_1(q_j^{\beta_j})+f_1(y_j) < f_1(y_j) \leqq c,$$

which contradicts $g \equiv \underline{c}$.

We know now that the number of primes in (a) and (b) together is at most

$$\frac{\log n_0}{\log 2}+\left[\frac{m}{n_0}\right]-1 \leqq \left[\frac{m}{2}\right] \quad \text{for} \quad 2 \leqq n_0 \leqq m.$$

If $n_0=1$ or $r_1, \ldots, r_m$ does not contain a complete residue system mod n_0, then $f=\underline{0}$.

Proof of Theorem 3. We know

$$|f(n-r_m+r_i)| \leqq |g(n-2r_m+r_i)| \quad \text{for all} \quad n > r_m,$$

so, by (5), we have

$$|g(n-r_m)| = \max\{|f(n-r_m+r_1), \ldots, |f(n)|\} = |f(n)|, \tag{11}$$

i.e. $|f|$ is strictly monotonic for $n>r_m$. Thereafter (4) yields (6).

If f is completely additive, then let

$$A_n := \{n^{s_n}-2r_m+r_i,\ (n+1)^{s_n}-2r_m+r_i\colon\ 1 \leqq i \leqq m, \text{ with a suitable large } s_n\},$$

and $A=\bigcup_{n=1}^{\infty} A_n$.

The rarity $a_n>h(n)$ can be guaranteed with the suitable choice of the s_n's. Similarly to the above proof we obtain

$$|f(n^{s_n})| \leqq |f(n+1)^{s_n})|,$$

i.e. $|f(n)| \leqq |f(n+1)|$ for all $n \in \mathbf{N}$, which, by (4), gives (6).

Proof of Theorem 4. Let (t_n) be a sequence satisfying $(t_n, n(n+1))=1$. Let

$$B_n := \{t_n-2r_m+r_i,\ nt_n-2r_m+r_i,\ (n+1)t_n-2r_m+r_i\colon\ 1 \leqq i \leqq m\}$$

and $B=\bigcup_{n=1}^{\infty} B_n$.

B can be arbitrarily thin by choosing t_n sufficiently large, or, given a sequence of density one we can achieve that B be contained in our sequence by a suitable choice of t_n.

Similarly to the proof of Theorem 3 we have

$$|f(t_n)| \leqq |f(nt_n)| \leqq |f(n+1)t_n|, \tag{12}$$

which for real f gives $|f(n)| \leqq |f(n+1)|$. This yields $f(n)=c \log n$ by (4).

REMARK 4. We can prove Theorem 4 without application of (4), too. Using (12), it can be shown that f is positive, resp. f is negative everywhere, i.e. f is monotonic. Then the original theorem of Erdős [1] implies the validity of the theorem. (If f is additive and monotonic, then $f(n)=c \log n$.)

I am indebted to Imre Ruzsa for his valuable remarks.

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(Received August 14, 1985)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 181—187

ON THE CHARACTERIZATION OF k-DIMENSIONAL ADDITIVE FUNCTIONS II

K. KOVÁCS

In this paper let $f\colon \mathbf{N}\to\mathbf{R}^k$ denote an additive arithmetical function and $|f|$ its Euclidean norm.

In [2] we proved a generalized form of a well-known theorem of Erdős ([1]):

(1) *If $|f|$ is monotonic from a point on, then $f(n)=\underline{c}\log n$ with a constant $\underline{c}\in\mathbf{R}^k$.*

In [3] we examined the "maximum function" of additive functions using (1). We can similarly define the "minimum function" v by

$$v(n)\in V_n=\{f(n+r_i)\colon\ i\in\{1,\dots,m\},\ |f(n+r_i)| \text{ is minimal}\}$$

with some fixed $0\leq r_1<\dots<r_m$.

If V_n contains more than one element, we can take any of them. $|v(n)|$ is unique anyway.

The proofs demand other methods as for the "maximum function" and the results are in general much weaker than in [3].

THEOREM 1. (a) *If*

$$|v| \textit{ is strictly increasing,} \tag{2}$$

then

$$f(n)=\underline{c}\log n \tag{3}$$

with a constant $\underline{c}\in\mathbf{R}^k$.

(b)

$$|v| \textit{ cannot be strictly decreasing.} \tag{4}$$

(c) *Let $k=1$ (f real). If* (2) *holds on an arbitrary set A having upper density one, then* (3) *holds. There are also arbitrarily rare sets A, in the sense that we can have $A=\{a_1<a_2<\dots\}\subset\mathbf{N}$ and $a_n>h(n)$ with any prescribed $h(n)$, for which* (2) *on A implies* (3).

For the property (4), similarly an arbitrary set of density 1 suffices and there are arbitrarily rare suitable sets.

1980 *Mathematics Subject Classification.* Primary 11A25

Key words and phrases. Characterization of additive functions, the function $c\log n$.

REMARK 1. The "strict monotonicity" is a necessary condition. If $|v|$ is only "increasing", then

$$f(p^\alpha) = \begin{cases} c \log p^\alpha & \text{if } p \neq p_0 \\ c \log p^\alpha + \delta & \text{with } \delta > 0, \text{ if } p = p_0 \end{cases}$$

gives for $m=2$, $r_1=0$ and $r_2=1$

$$v(n) = \begin{cases} c \log n & \text{if } p_0 \nmid n \\ c \log (n+1) & \text{if } p_0 \mid n, \end{cases}$$

which is increasing.

We managed to prove the following theorems only for the special case that $m=2$, $r_1=0$ and $r_2=1$.

THEOREM 2. (a) *If* $m=2$, $r_1=0$, $r_2=1$ *and*

$$\lim_{a_n \in A, n \to \infty} v(a_n) = \underline{c} \tag{5}$$

(i) *on a suitable set with the rarity* $a_n > h(n)$, *resp.*,
(ii) *on an arbitrary set having upper density one and* $\underline{c} = \underline{0}$, *then*

$$v = \underline{0} \tag{6}$$

and this implies

(7) $f = \underline{0}$ *except at most at the multiples of one prime.*

(b) *If* $k=1$, *then* $\lim_{n \to \infty} |v(a_n)| = c$ *implies* (6) *and* (7) *on a suitable set with the rarity* $a_n > h(n)$.

(c) *If* $k=2$ *and f is completely additive, then* $\lim_{n \to \infty} |v(n)| = c$ *yields* (6) *and* (7).

Proofs

PROOF OF THEOREM 1. (a) If $|v|$ is strictly increasing, then $|v(n)| = |f(n+r_1)|$, namely if there exists an n_0, such that $v(n_0) = f(n_0+r_i)$ with $i>1$, then

$$|v(n_0+r_i-r_1)| \leqq |f(n_0+r_i)| = |v(n_0)|$$

contradicts (2). So $|v(n)| = |f(n+r_1)|$ is strictly increasing, which by (1) yields (3).

(b) We prove that $|v|$ cannot be strictly decreasing. If there exists an n_0 such that $v(n_0) = f(n_0+r_i)$ with $i<m$, then

$$|v(n_0+r_i-r_m)| \leqq |f(n_0+r_i)| = |v(n_0)|$$

contradicts the strict monotonicity of $|v|$. So $|v(n)| = |f(n+r_m)|$ is strictly decreas ing, which by (1) yields (3). This gives (4).

(c) Choose a sequence (t_n) such that $(t_n, n(n+1))=1$ and let

$$A_n = \{t_n+r_i-2r_1,\ nt_n+r_i-2r_1,\ (n+1)t_n+r_i-2r_1,\ 1 \leqq i \leqq m\}$$

a set of $3m$ elements and let $A=\bigcup_{n=1}^{\infty} A_n$.

For $a=t_n, nt_n, (n+1)t_n$ we obtain, as in (a), that

$$f(a) = v(a-r_1) \quad \text{in the increasing case, resp.}$$
$$f(a) = v(a-r_m) \quad \text{in the decreasing case.}$$

Hence

$$|f(t_n)| \leqq |f(nt_n)| \leqq |f((n+1)t_n)| \quad (|v| \text{ increasing}), \text{ resp.}$$
$$|f(t_n)| \geqq |f(nt_n)| \geqq |f((n+1)t_n)| \quad (|v| \text{ decreasing}).$$

For a real f this implies that $|f|$ itself is increasing, resp. decreasing. Thus by (1) we are ready.

Proof of Theorem 2. We prove the theorem in three steps on $\mathbf{N}$ and meanwhile we show how the proof can be applied to the desired sets.

I. *If* $\lim_{n\to\infty} v(n)=\underline{c}$, *then* $\underline{c}=0$.

Proof. Let us assume that $c\neq 0$ $(c=|\underline{c}|)$. This implies

(8) $$|f(n)| \geqq c-\varepsilon_0 \quad \text{for all} \quad n \geqq n_0(\varepsilon_0).$$

Lemma. *Let* $Q=\{m{:}m \text{ even},\ v(m)=f(m)\}$ *and*

$$T=\{t{:}\ t \text{ even},\ t\notin Q\}.$$

If T is an infinite set then $\lim_{t\in T, t\to\infty} f(t^2)=\underline{c}$.

Proof. If there are infinitely elements t in T then $f(t)\neq v(t)$ and so

$$\lim_{t\in T, t\to\infty} f(t\pm 1) = \underline{c}.$$

Using that $(t-1, t+1)=1$, we have

$$\lim_{t\in T, t\to\infty} v(t^2-1) = \lim_{t\in T, t\to\infty} f(t^2) = \underline{c}, \qquad \text{q.e.d.}$$

Assume first that there is an infinity of primes p_i such that

(9) $$\lim_{p_i\to\infty} f(2p_i) = \underline{c}.$$

Then $f(p_i)\to\underline{c}_1=\underline{c}-f(2)$, and $\underline{c}_1\neq\underline{0}$ by (8). Consequently, $f(\prod_{i=1}^{s} p_i)\to\infty$, and hence $\prod_{i=1}^{s} p_i\in T$ for large s.

Let us choose the p_i's such that $f(p_i^2)$ point to approximately the same direction. So by (8) $f(4\prod_{i=1}^{s} p_i^2)\to\infty$, which contradicts the Lemma with the choice $(t)=$ $=(2\prod_{i=1}^{s} p_i)$.

If (9) does not hold then $f(2p)$ avoides a neighbourhood of $\underline{c}$, thus $f(2p)\in T$ for large p and by the Lemma $f(4p^2)\to\underline{c}$. From here we proceed taking "$4p^2$" into the place of "$2p$" in (9).

For rare sets: The Lemma is valid for all even t, for which $t-1$, t and t^2-1 are in the rare set, too. Taking pairwise coprime odd numbers t_i into the place of p_i in (9), we obtain the contradiction in the same way as soon as $2t_i-1, 2t_i, 4t_i^2-1$ further t_i, t_i^2, t_i^4 (to ensure (8) for these numbers) and $2\prod_{i=1}^{s} t_i-1$, $2\prod_{i=1}^{s} t_i$, $4\prod_{i=1}^{s} t_i^2-1$, $4\prod_{i=1}^{s} t_i^2$, $16\prod_{i=1}^{s} t_i^4-1$ $(s=1, \dots)$ are in the rare set. The suitable choice of the t_i's guarantees the condition $a_n>h(n)$.

II. *If*

$$\lim_{n\to\infty} v(n)=\underline{0}, \tag{10}$$

then $v=\underline{0}$.

Proof. If $v\neq\underline{0}$, then there exists an $s\in\mathbf{N}$ such that $f(s)\neq\underline{0}$ and $f(s+1)\neq\underline{0}$. Consider the neighbouring numbers

$$a_{sz}=s[s(s+1)^2z+1] \quad\text{and}\quad a_{sz}+1=(s+1)[s^2(s+1)z+1].$$

It is enough to find some infinite sequences (m) such that

$$f(s^2(s+1)m)\nrightarrow\underline{0} \quad\text{and}\quad f(s(s+1)^2m)\nrightarrow\underline{0}. \tag{11}$$

This namely implies for infinitely many $m\in\mathbf{N}$

$$f(s^2(s+1)m+1)\to\underline{0} \quad\text{and}\quad f(s(s+1)^2m+1)\to\underline{0},$$

which gives

$$f(a_{sm})\to f(s) \quad\text{and}\quad f(a_{sm}+1)\to f(s+1)$$

contradicting (10).

If f is not bounded then on the set of numbers coprime to $s(s+1)$, there is an infinite sequence (u) such that $f(u)\to\infty$. So we have (11) with $(m):=(u)$.

If f is bounded on the n's coprime to $s(s+1)$ then, for any sequence (q_i), where the q_i's are coprime to $s(s+1)$ and each other, $f(q_i)\to\underline{0}$, We want to find a number b such that

$$(b, s(s+1))=1,\quad f(s^2(s+1)b)\neq\underline{0},\quad f(s(s+1)^2b)\neq\underline{0}.$$

Having found this b, we can choose $(m):=(bq_i)$.

To find this b, select any two numbers d_1, d_2 coprime to each other and to $s(s+1)$ such that $f(d_1)\neq\underline{0}$, $f(d_2)\neq\underline{0}$ (if you can). Then

$$b:=\begin{cases}1, & \text{if } f(s^2(s+1))\neq\underline{0} \text{ and } f(s(s+1)^2)\neq 0,\\ d_j, & \text{if } \{f(s^2(s+1)), f(s(s+1)^2)\}\neq\{\underline{0}, -f(d_j)\} \text{ for } j=1 \text{ or } 2,\\ d_1 d_2 & \text{otherwise}\end{cases} \tag{12}$$

is a suitable choice.

If there is only one d_1 (or none), consider the congruence

$$x \equiv s \bmod s^2(s+1)^2 \tag{13}$$

$$x \equiv 1 \bmod d_1 \tag{13'}$$

(the second is considered only if d_1 exists).

For any solution we have $v(x)=v(s)\neq \underline{0}$, otherwise $x'=(x-s)/s^2(s+1)^2$ would provide d_1, and this contradicts (10).

For an arbitrary set A having upper density one: If f is not bounded on the subset of A contains the natural numbers coprime to $s(s+1)$, then there exist infinitely many pairwise coprime Q's in the above subset such that $f(Q)\to\infty$. So there are Q_1, $Q_2\in\mathbf{N}$, too, such that

(14) $|f(Q_1)|, |f(Q_2)|$ and $|f(Q_1Q_2)|$ are different and greater than

$$2\max\{f(s^2(s+1)),\ f(s(s+1)^2)\}.$$

For infinitely many $q_i\in\mathbf{N}$ coprime to $s(s+1)$ and each other

$$s^2(s+1)dq_i, \quad s(s+1)^2dq_i, \quad s[s(s+1)^2dq_i+1]\in A \tag{15}$$

with all $d\in\{Q_1, Q_2, Q_1Q_2\}$. If there are infinitely many q_i's such that $f(q_i)\to\infty$ or $\underline{0}$, then with the choice $(m)=(Q_1q_i)$ — in the last case using (14) — we obtain (11).

Otherwise we find a sequence (q_i) with $f(q_i)\to \underline{c}_1$, $\underline{c}_1\neq\underline{0}$, ∞. Let us try

$$(m^{(1)})=(Q_1q_i), \quad (m^{(2)})=(Q_2q_i), \quad (m^{(3)})=(Q_1Q_2q_i). \tag{16}$$

By (15) the sequences $f(m^{(j)})$ converge to three different limits. Hence, for one of $j=1, 2, 3$, the limits of the sequences

$$f(s(s+1)^2m^{(j)}) \quad\text{and}\quad f(s^2(s+1)m^{(j)})$$

are both nonzero. From here we proceed as at (11).

If f is bounded on the above set, then it is easy to show that f is bounded on the set of all natural numbers coprime to $s(s+1)$ either. From here we proceed as in the proof for $\mathbf{N}$, using that for all b in (12) there exist infinitely many pairwise coprime q_i, for which $f(q_i)\to\underline{0}$ such that (15) is valid with $d:=b$, too. Thus (11) is satisfied with $(m)=(bq_i)$, resp. infinitely many solutions of (13) are in A.

For rare set: First we show an A with the rarity

$$a_{n+1}-a_n > h(n) \text{ such that } \lim_{a_n\in A,\, n\to\infty} v(a_n)=\underline{0} \text{ implies } v=\underline{0}.$$

We take a double sequence (p_{ni}) of primes and a triple sequence (q_{nzj}) of positive integers. We put

$$T_n^1=\{\text{all products of different } p_{ni}\text{'s}\},$$

$$T_n^2=\{\text{all numbers of the form } (yp_{ni}) \text{ and } (yy'p_{ni}), \text{ where } y, y'\in\{2, \dots, i\},\ (y, y')=1\},$$

$$T_n^3=\{n^2(n+1)^2zq_{nzj}+n \text{ for all } z \text{ such that } (z, n(n+1))=1\}.$$

Let $T_n=T_n^1\cup T_n^2\cup T_n^3$,

$$A_n=\{t;\ n^2(n+1)t,\ n(n+1)^2t,\ n[n(n+1)^2t+1] \text{ for all } t\in T_n\}$$

and

$$A = \bigcup_{n=2}^{\infty} A_n.$$

The rarity $a_{n+1} - a_n > h(n)$ can be guaranteed with the suitable choice of p_{ni} and q_{znj}.

If there exists an $s \in \mathbf{N}$ such that $f(s) \neq \underline{0}$ and $f(s+1) \neq \underline{0}$, then let us consider A_s. If there exist infinitely many p_i's such that $f(p_i) \to \infty$, then by $(m) := (p_{si})$ we have (11), using that

(17) $$s^2(s+1)m, \quad s(s+1)^2 m \quad \text{and} \quad s[s(s+1)^2 m + 1] \in A_s.$$

If there exist infinitely many primes p_{si} such that $f(p_{si}) \to \underline{c}_1 \neq \underline{0}$, then $f(\prod_{i=1}^{l} p_{si}) \to \infty$, so with the choice $(m) := (\prod p_{si}) \subset T_s^1$ we have by (17) the validity of (11) again.

If $f(p_{si}) \to \underline{0}$, then we prove like for $\mathbf{N}$, using that (11) is valid with $(m) := (bp_{si})$ for arbitrary d_1, d_2 from (12) resp. if there exists only d_1 such that $(d_1, s(s+1)) = 1$ and $f(d_1) \neq \underline{0}$, then we can use that infinitely many solutions of (13) are in T_s^3 with the choice $z := d_1$.

To prove that $\lim_{b_n \in B, n \to \infty} v(b_n) = \underline{c}$ implies $v = \underline{0}$, let us consider the union of the above rare set A and the rare set in I. This gives a rare set with $b_n > h(n)$.

III. *We prove that* $v = \underline{0}$ *implies that* $f = \underline{0}$ *except at most at the multiplies of one prime.*

Proof. Let q_0 denote the smallest prime power for which $f(q_0) \neq \underline{0}$. Let q_1 be the next prime power (if there is any) $q_1 > q_0$ with $(q_1, q_0) = 1$ and $f(q_1) \neq \underline{0}$. Choose $r_\delta \in [1, q_0 - 1] (\delta = \pm 1)$ so that

$$q_1 r_\delta \equiv \delta \bmod q_0.$$

For at least one of the solutions we have

$$q_1 r_\delta - \delta \not\equiv 0 \bmod q_0^2$$

because

$$(q_1 r_1 - 1) + (q_1 r_{-1} + 1) = q_0 q_1 \not\equiv 0 \bmod q_0^2.$$

So

$$q_1 r_\delta - \delta = q_0 x, \quad (x, q_0) = 1.$$

Also we have

$$1 \leqq x \leqq \frac{q_1(q_0 - 1) + 1}{q_0} < q_1$$

therefore $f(x) = \underline{0}$.

So, using $r_1 \leqq q_0 - 1$,

$$|v(q_0 x)| = \min \{|f(q_0 x)|, |f(q_1 r_1)|\} = \min \{|f(q_0)|, |f(q_1)|\} \neq 0$$

contradicts $v = \underline{0}$.

If $\delta = -1$, we have the contradiction similarly by $v(q_0 x - 1) \neq \underline{0}$.

b) The unique step which demands commentary how $|f(t\pm 1)|\to c$ implies $|f(t^2-1)|\to 2c$ in the LEMMA. Using that f is real-valued, we obtain

$$|f(t^2-1)|\to 2c \quad \text{or} \quad 0.$$

From there (8) gives $|f(t^2-1)|\to 2c$, too.

c) It is sufficient to prove that $\lim\limits_{n\to\infty}|v(n)|=c$ implies $c=\underline{0}$, since then II and III complete the proof.

Let us assume that $c\neq\underline{0}$. Take an $a\in\mathbf{N}$ with $f(a)\neq\underline{0}$. This gives $f(a^n)\to\infty$, consequently $|f(a^{tn}\pm 1)|\to c$ for both $t=1$ and 2. So $|f(a^{2tn}-1)|=|f(a^{tn}+1)+$ $+f(a^{tn}-1)|\to c$ yields that the angle of $f(a^{tn}-1)$, $f(a^{tn}+1)$ converges to $2\pi/3$ for both $t=1$ and 2. Hence the angle of $f(a^{2n}-1)$, $f(a^n\pm 1)$ converges to $\pi/3$. Therefore

$$f[(a^{2n}+1)(a^n+\delta)]\to\underline{0} \quad \text{with} \quad \delta=1 \quad \text{or} \quad -1,$$

which contradicts (8).

I am indebted to Imre Ruzsa for his valuable remarks.

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(Received August 14, 1985)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 189—194

THE JOINT DISTRIBUTION OF THE BUSY AND FREE PERIODS OF THE MASS SERVICE SYSTEM

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The joint distribution of the busy and free periods for the queueing system $GI/M/1$ was obtained in [3, 3]. In the present paper a $GI/M/2$ queueing system with waiting time is studied. We suppose that the interarrival time between two successive arriving customers is a random variable having the general distribution $G(x)$ and that the service intensity of each server is equal to λ. Suppose that the first customer joins the system at time 0. We denote by $v(t)$ the number of customers in the system at time t. The time interval ζ from the begining $(t=0)$ to the first time the system is free is called the busy period;

$$\zeta = \inf\{t\colon\ v(t+0) = 0,\ v(t-0) \geqq 1\},$$

and the free period ξ is the interval from the first time the system is free to the time of a new arrival:

$$\xi = \inf\{t\colon\ v(t+0) \geqq 1,\ v(t-0) = 0\}-\zeta.$$

The sum $\eta=\zeta+\xi$ of the busy period and the free period is called a cycle.

To find the joint distribution $Q(s, t)=P\{\xi<s, \zeta<t\}$ of the free and busy periods, we should obtain the joint distribution $F_1(x, y)=\mathsf{P}\{\eta<x, \zeta<y\}$ of the cycle and the busy period.

Since the busy period depends on the number of customers in the system and therefore the cycle, we define ξ_n and η_n to be the busy period and cycle if there are n customers in the system, such that the joint probability distribution is

$$F_n(x, y) = \mathsf{P}\{\eta_n < x,\ \zeta_n < y\}.$$

We write the system of equations satisfied by the function $F_n(x, y)$ for the particular cases $x<y$ and $x>y$. First we introduce the notations:

$$e_1(s) = 2\lambda\frac{(2\lambda s)^{n-2}}{(n-2)!}e^{-2\lambda s},\quad e(a) = e^{-\lambda a}.$$

I. If $x>y$, from the theorem of total probability it follows that:

$$F_1(x, y) = \int_0^y [1-e(t)]\,dG(t)+\int_0^y e(t)F_2(x-t, y-t)\,dG(t)+\int_y^x [1-e(y)]\,dG(t);$$

(1)

1980 *Mathematics Subject Classification.* Primary 60K25.
Key words and phrases. Queueing system, busy and free periods.

for $n \geqq 2$.

$$F_n(x, y) = \int_0^y \int_0^t e_1(s)[1-e(t-s)]\, ds\, dG(t) +$$

$$+\int_y^x \int_0^y e_1(s)[1-e(y-s)]\, ds\, dG(t)+$$

(2)

$$+\int_0^y \int_0^t e_1(s) e(t-s) F_2(s-t, y-t)\, ds\, dG(t)+$$

$$+\int_0^y e(2t) \sum_{k=0}^{n-2} \frac{(2\lambda t)^k}{k!} F_{n-k+1}(s-t, y-t)\, dG(t).$$

II. If $x \leqq y$, denote $F_n^*(x) = \mathsf{P}\{\eta_n < x\}$. Since, by definition, the busy period does not exceed the cycle, we have $F_n(x, y) = F_n^*(x)$. In this case, from the theorem of total probability, we obtain the following integral equations:

$$F_1^*(x) = \int_0^x [1-e(t)]\, dG(t) + \int_0^x e(t) F_2^*(x-t)\, dG(t); \tag{3}$$

for $n \geqq 2$

$$F_n^*(s) = \int_0^x \int_0^t e_1(s)[1-e(t-s)]\, ds\, dG(t)+$$

(4)

$$+\int_0^x \int_0^t e_1(s) e(t-s) F_2^*(x-t)\, ds\, dG(t)+$$

$$+\int_0^x e(2t) \sum_{k=0}^{n-2} \frac{(2\lambda t)^k}{k!} F_{n-k+1}^*(x-t)\, dG(t).$$

For $|z| < 1$, the probability generating functions $F^*(x, z)$ and $F(x, y, z)$ are given by

$$F^*(x, z) = \sum_{n=1}^{\infty} z^n F_n^*(x),$$

$$F(x, y, z) = \sum_{n=1}^{\infty} z^n F_n(x, y).$$

Multiplying the equations (1), (3) and (2), (4) by z and z^n, respectively, and summing over all values of n, from (1) and (2) we obtain:

$$f(x, y, z) = z \int_0^y [1-e(t)]\, dG(t) + z \int_0^y e(t) F_2(x-t, y-t)\, dG(t)+$$

$$+z \int_y^x [1-e(y)]\, dG(t) + 2\lambda z^2 \int_0^y \int_0^t e[2s(1-z)][1-e(t-s)]\, ds\, dG(t)+$$

$$+2\lambda z^2 \int_y^x \int_0^y e[2s(1-z)][1-e(y-s)]\,ds\,dG(t)+ \tag{5}$$

$$+2\lambda z^2 \int_0^y \int_0^t e[2s(1-z)]e(t-s)F_2(x-t,\ y-t)\,ds\,dG(t)+$$

$$+\int_0^y e[2t(1-z)]\left[\frac{1}{z}F(x-t,\ y,\ z)-F_1(x-t,\ y-t)-zF_2(x-t,\ y-t)\right]dG(t).$$

From (3) and (4) we obtain:

$$F^*(x,\ y) = z\int_0^x [1-e(t)]\,dG(t)+z\int_0^x e(t)F_2^*(x-t)\,dG(t)+$$

$$+2\lambda z^2 \int_0^x \int_0^t e[2s(1-z)][1-e(t-s)]\,ds\,dG(t)+ \tag{6}$$

$$+2\lambda z^2 \int_0^x \int_0^t e[2s(1-z)]e(t-s)F_2^*(x-t)\,ds\,dG(t)+$$

$$+\int_0^x e[2t(1-z)]\left[\frac{1}{z}F^*(x-t,\ z)-F_1^*(x-t)-2F_2^*(x-t)\right]dG(t).$$

Now let:

$$F_n^+(x,\ y) = \begin{cases} F_n(x,\ y), & \text{if} \quad x > y, \\ 0, & \text{if} \quad x \leqq y, \end{cases} \tag{7}$$

$$F_n^-(x,\ y) = \begin{cases} 0, & \text{if} \quad x > y, \\ F_n^*(x), & \text{if} \quad x \leqq y, \end{cases}$$

and consider the Laplace transforms

$$g(r) = \int_0^\infty e^{-rx}G(x)\,dx,$$

$$\tilde{g}(r) = \int_0^\infty e^{-rx}\,dG(r) = rg(r),$$

$$f^*(r,\ z) = \int_0^\infty d^{-xr}F^*(x,\ z)\,dx,$$

$$f_1(r,\ v) = \int_0^\infty \int_0^\infty e^{-rx-y}F_1(x,\ y)\,dx\,dy.$$

Related to the definition of $F^+(x, y, z)$ and $F^-(x, y, z)$, we introduce the following notations:

$$\tilde{g}_1 = \tilde{g}(r+v+2\lambda(1-z)), \qquad \tilde{g}_2 = \tilde{g}(r+v+\lambda), \qquad \tilde{g}_3 = \tilde{g}(r),$$
$$g_3 = g(r), \qquad g_4 = g(r+v), \qquad g_2 = g(r+v+\lambda),$$
$$\tilde{g}_5 = \tilde{g}(r+2\lambda(1-z)), \qquad \tilde{g}_6 = \tilde{g}(r+\lambda).$$

Applying the Laplace transforms of $F_1(x, y)$ and $F^+(r, v, z)$ we obtain

$$\begin{aligned} f^+(r, v, z) &= \int_0^\infty \int_0^\infty e^{rx-vy} F^+(x, y, z)\, dx\, dy = f^+(r, v, z)\frac{1}{z}\tilde{g}_1 - f_1^+(r, v)\tilde{g}_1 + \\ &+ f_2^+(r, v)\left[\frac{2\lambda z^2}{1-2z}\{\tilde{g}_2-\tilde{g}_1\} - z\tilde{g}_1 + z\tilde{g}_2\right] + \frac{z}{v}[g_3-g_4] - \frac{z}{r+v}g_2 - \\ &- \frac{2}{1-z}\left[\frac{\tilde{g}_1}{r(r+v)} + \frac{g_3}{v+2\lambda(1-z)} - \frac{\tilde{g}_1}{r(v+2\lambda(1-z))} - \frac{g_3}{r} + \frac{g_4}{v}\right] + \\ &+ \frac{2z^2}{1-2z}\left[\frac{g_3}{v+2\lambda(1-z)} - \frac{\tilde{g}_1}{r(v+2\lambda(1-z))} + \frac{\tilde{g}_1}{r(r+v)} - \right. \\ &\left. - \frac{g_3}{v+\lambda} + \frac{\tilde{g}_2}{r(v+\lambda)} - \frac{\tilde{g}_2}{r(r+v)}\right], \end{aligned} \tag{8}$$

$$f_1^+(r, v) = f_2^+(r, v)\tilde{g}_2 - \frac{\lambda}{(r+v)r}g_2 + \frac{g_3-g_4}{v}. \tag{9}$$

Using the Laplace transform of $F_1^*(x)$ and $F^*(x, z)$, we obtain

$$\begin{aligned} f^*(r, z) &= f^*(r, z)\frac{1}{z}\tilde{g}_5 - f_1^*(r)\tilde{g}_5 + f_2^*(r)[z\tilde{g}_6 - z\tilde{g}_5] + \\ &+ \frac{2z^2}{1-2z}[\tilde{g}_6-\tilde{g}_5] + \frac{z^2}{1-z}\left[g_3 - \frac{\tilde{g}_5}{r}\right] + \frac{2z^2}{1-2z}\,\frac{\tilde{g}_5-\tilde{g}_6}{r} + zg_3 - \frac{z}{r}\tilde{g}_6. \end{aligned} \tag{10}$$

$$f_1^*(r) = f_2^*(r)\tilde{g}_6 + g_3 - \frac{\tilde{g}_6}{r}. \tag{11}$$

By solving (10) with respect to $f^*(r, z)$, we obtain

$$\begin{aligned} f^*(r, z) &= \left\{\frac{1}{z}(z-\tilde{g}_5)\right\}^{-1}\left(\{f_2^*(r)\}\left\{z\tilde{g}_6 + \frac{2z^2}{1-2z}[\tilde{g}_6-\tilde{g}_5] - z\tilde{g}_5\right\} - \right. \\ &\left. - f_1^*(r)\tilde{g}_5 + \frac{z^2}{1-z}\left[g_3 - \frac{\tilde{g}_5}{r}\right] + \frac{2z^2}{1-2z}\cdot\frac{\tilde{g}_5-\tilde{g}_6}{r} + zg_3 - \frac{z}{r}\tilde{g}_6\right). \end{aligned} \tag{12}$$

The equation

$$z = \tilde{g}(r+2\lambda(1-z)) \tag{13}$$

is studied in [2]. From the existence of the unique solution for equation (13) and from the analysis of $f^*(r, z)$ we obtain the equations for $f_1^*(r)$ and $f_2^*(r)$. Putting the denominator of equation (12) equal to zero at the point $z=z(r)$, the solution of equation (13) follows.

Therefore, from (11), we obtain

$$f_1^*(r) = \left[\frac{1}{\tilde{g}_6}\left\{z\tilde{g}_6 + \frac{2z^2}{1-2z}(\tilde{g}_6-\tilde{g}_5) - zg_5\right\} - \tilde{g}_5\right]^{-1} \times$$

$$\times\left[\left(\frac{g_3}{\tilde{g}_6} - \frac{1}{r+\lambda}\right)\left\{z\tilde{g}_6 + \frac{2z^2}{1-2z}(\tilde{g}_6-\tilde{g}_5) - z\tilde{g}_5\right\} -\right.$$

$$\left. - \frac{z^2}{1-z}\,\frac{\tilde{g}_3-\tilde{g}_5}{r} - \frac{2z^2}{1-2z}\,\frac{\tilde{g}_5-\tilde{g}_6}{r} - zg_3 + \frac{z}{r}\tilde{g}_6\right].$$

Similarly, from (8) and (9), we obtain

$$f_1^+(r, v) = \left[\tilde{g}_1 - \frac{1}{\tilde{g}_2}\left\{\frac{2z^2}{1-2z}(\tilde{g}_2-\tilde{g}_1) - z\tilde{g}_1 + z\tilde{g}_2\right\}\right]^{-1} \times$$

$$\times\left[\frac{1}{\tilde{g}_2}\left\{\frac{\lambda}{(r+v)r}\,g_2 - \frac{g(r)-g(r+v)}{v}\right\}\left\{\frac{2z^2}{1-2z}(\tilde{g}_2-\tilde{g}_1) - z\tilde{g}_1 + z\tilde{g}_2\right\} + \frac{z}{v}(g_3-g_4) -\right.$$

$$- \frac{z}{r+v}\,g_2 + \frac{z^2}{1-z}\left[\frac{\tilde{g}_1}{r(v+2\lambda+(1-z))} - \frac{g_3}{v+2\lambda(1-2)} - \frac{\tilde{g}_1}{r(r+v)} + \frac{g_3}{v} - \frac{g_4}{v}\right] +$$

$$\left. + \frac{2z^2}{1-2z}\left[\frac{g_3}{v+2\lambda(1-z)} - \frac{\tilde{g}_1}{r(v+2\lambda(1-z))} + \frac{\tilde{g}_1}{r(r+v)} - \frac{g_3}{v+\lambda}\right]\right].$$

Also $z=z(r+v)$ is the solution of the equation

$$z = \tilde{g}(r+v+2\lambda(1-z)).$$

The joint distribution of the cycle and the busy period has the form

$$F_1(x, y) = F_1^+(r, v) + F_1^-(r, v).$$

The corresponding Laplace transform is

$$f_1(r, v) = f_1^+(r, v) + f_1^-(r, v),$$

where

$$f_1^-(r, v) = \frac{1}{v} f_1^*(r+v).$$

The Laplace—Stieltjes transform for the joint distribution of the cycle and the busy period is given by $\tilde{f}_1(r, v) = rvf_1(r, v)$. It remains to obtain $\tilde{f}_n(r, v)$ which is determined from (2) and (4). Thus, the Laplace—Stieltjes transform for the joint distribution of the busy period and the free period is given by:

$$\tilde{g}(s, t) = \tilde{f}_1(s, t-s).$$

In a similar way the joint distribution of the cycle and the busy period for the queueing system $GI/M/n$ with waiting time can be obtained.

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(Received August 28, 1985)

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THE RECOVERY QUESTION FOR LOCAL INCIDENCE RINGS

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As defined by Rota [9] and Belding [4], we may associate with any locally finite preordered set $(X, \leqq)$ and ring A the *incidence ring* $I(X, A)$ *of* X *over* A, consisting of all functions $f\colon X \times X \to A$ such that $f(x, y) = 0$ if $x \not\leqq y$. The local finiteness condition on X allows multiplication to be defined in $I(X, A)$ via convolution; $I(X, A)$ thus becomes an associative ring with multiplicative identity. Rota introduced such rings in order to examine the classical Möbius function in a more general setting; since the appearance of [9], many authors have investigated $I(X, A)$ from a purely ring-theoretic point of view, eschewing the combinatorial aspects which motivated Rota. For instance, Belding, Stanley [10], Nachev [8], Voss [11] and Anderson [1] have each examined various forms of what has been called the "recovery question" for incidence rings: namely, if $I(X, A)$ is isomorphic (or more generally, Morita equivalent) to $I(Y, A)$, under what circumstances may we conclude that X and Y are order isomorphic?

In this article we will redefine the notion of an incidence ring in such a way as to render the local finiteness condition on X unnecessary. The resulting "local incidence rings" (introduced in section 2) will no longer have multiplicative identities, but will contain *sets of local units* (see section 1). The problem of recovering a preordered set from the class of rings Morita equivalent to the local incidence ring of that set is examined in section 3; this yields a result quite similar in flavor to Freyd's observation concerning the recovery question for amenable categories [7, page 18]. We finish with section 4, in which we mention how the usual incidence ring $I(X, A)$, along with the classical Möbius function μ, still arise in the context of local incidence rings for all germane partially ordered sets.

1. Preliminaries

Throughout this article we will adhere to the standard notation and definitions used in the theory of ordered structures (see for example [11]). Recall that if X is a preordered set, then $[x, y]$ denotes the set $\{z \in X \mid x \leqq z \leqq y\}$, called the *interval between* x *and* y. If for each $x, y \in X$ the interval $[x, y]$ is finite we call X *locally finite*. The equivalence relation $\sim$ defined on X via $x \sim y$ iff $[x, x] = [y, y]$ induces the partially ordered set $\tilde{X} = X/\sim$; elements of $\tilde{X}$ will be denoted by $[x]$. Note $[x] \leqq [y]$ in $\tilde{X}$ iff $x \leqq y$ in X.

1980 *Mathematics Subject Classification*. Primary 16A89; Secondary 06A10.
Key words and phrases. Incidence ring, Möbius function, preordered set, category ring.

If A is any unital ring, we define the *additive category of* $(X, \leqq)$ *with coefficients in* A (denoted $A(X, \leqq)$) to be the category whose objects are the elements of X, and for $x, y \in X$

$$\mathrm{Mor}_{A(X,\leqq)}(x, y) = \begin{cases} A & \text{if } x \leqq y \\ 0 & \text{otherwise} \end{cases}$$

with multiplication as composition. One can easily check that the canonical map $A(X, \leqq) \to A(\bar{X}, \leqq)$ given by $x \to [x]$ is an equivalence of additive categories.

Let A be any ring, and let $F = \{f_i | i \in I\}$ be a set of orthogonal idempotents in A. Define a relation $\leqq$ on F by $f \leqq f'$ if there exists a sequence of idempotents $f = f_1, f_2, \ldots, f_n = f'$ in F with $f_i A f_{i+1} \neq 0$ for $1 \leqq i \leqq n-1$. A straightforward calculation yields that $\leqq$ is in fact a preorder on F.

Let A be an associative ring. A subset E of A is called a *set of local units* for A (abbreviated slu) in case E is a set of idempotents such that for each $x \in A$ there exists $e \in E$ with $x \in eAe$ (see for example [3]). Note that any ring with identity has slu $\{1\}$. By a *left A-module M* over a ring A with slu we mean a module in the usual sense, with the additional unitary condition that $AM = M$. One can easily verify that if e is an idempotent in A and M is a left A-module, then $\mathrm{Hom}_A(Ae, M)$ is isomorphic to eM as abelian groups via the map $f \to (e)f$.

2. The local incidence ring

In this section we redefine the usual notion of an incidence ring (see [9, §3]).

Definition 2.1. Let X be a preordered set, and let A be any associative ring with identity. Let $LI(X, A)$ denote the set of all functions $f\colon X \times X \to A$ such that $f(x, y) = 0$ if $x \not\leqq y$, and $f(x, y) \neq 0$ for at most finitely many $(x, y) \in X \times X$. For f, g in $LI(X, A)$ define addition and multiplication via:

$$(f+g)(x, y) = f(x, y) + g(x, y)$$

$$(f \cdot g)(x, y) = \sum_{x \leqq z \leqq y} f(x, z) g(z, y).$$

Note that the latter sum is well defined due to the finiteness condition imposed on f and g. With addition and multiplication so defined, $LI(X, A)$ is an associative ring, called the *local incidence ring of X over A*.

Note that if $A(X, \leqq)$ is the additive category described in section 1, then $LI(X, A)$ is precisely the category ring of $A(X, \leqq)$ as defined by Gabriel [6, p. 346]. Also note that if X is a finite set, then $LI(X, A)$ is precisely the usual incidence ring $I(X, A)$.

If $f \in LI(X, A)$ and $a \in A$, then $af\colon X \times X \to A$ defined via $(af)(u, v) = af(u, v)$ is an element of $LI(X, A)$. Further, if $x \leqq y$ in X, then the function $\delta_{xy}\colon X \times X \to A$ defined via $\delta_{xy}(x, y) = 1$, $\delta_{xy}(u, v) = 0$ for $(x, y) \neq (u, v)$ is also in $LI(X, A)$. For any $f \in LI(X, A)$ we can clearly write

$$f = \sum_{x, y \in X} f(x, y) \delta_{xy}, \tag{2.2}$$

with at most finitely many terms in this sum being nonzero. For $a \in A$ and $x \leqq y$ in X we denote $a\delta_{xy}$ by a_{xy}; also, we denote the idempotent δ_{xx} by e_x. For S any finite subset of X we define $e_S = \sum_{x \in S} e_x$; a straightforward calculation demonstrates that the collection $E = \{e_S | S \subseteq X,\ S \text{ finite}\}$ is a set of local units for $LI(X, A)$.

If S is any finite subset of X, then S inherits a preorder via restriction. One can easily show that the map

$$\Theta_S\colon\ e_S LI(X, A) e_S \to LI(S, A) \quad \text{via} \quad \Theta_S(e_S f e_S) = f|_{S \times S}$$

is an isomorphism of rings. Specifically, if $c = [x]$ has $\operatorname{card}(c) = n \in N$, then $e_c LI(X, A) e_c$ is isomorphic to $LI([x], A)$. But $[x]$ is trivial; that is, $y \leqq z$ and $z \leqq y$ for every $y, z \in [x]$. Thus we have $LI([x], A) \simeq M_n(A)$. In particular, for each $x \in X$, $e_x LI(X, A) e_x \simeq A$ as rings. Further, a short calculation demonstrates that $e_x LI(X, A) e_y \neq 0$ if and only if $x \leqq y$; thus if R denotes $LI(X, A)$, we have $x \leqq y$ if and only if $\operatorname{Hom}_R(Re_x, Re_y)$ is nonzero. This observation will play a key role in recovering the structure of $(X, \leqq)$ from RMod.

3. The recovery question and Morita equivalence

Two rings A and B are said to be *Morita equivalent* in case their full left-module categories AMod and BMod are equivalent. It is well-known (see for example [2, Corollary 22.6]) that if $n \in N$ then A and $M_n(A)$ are Morita equivalent for any unital ring A. Recall that if X is a finite preordered set with $\operatorname{card}(X) = n$ and $\operatorname{card}(\tilde{X}) = 1$, then $LI(X, A) \simeq M_n(A)$. With these facts in mind, Nachev [8, Theorem1] proved

THEOREM. *If the length of the structure of two-sided ideals of A is finite, and the rings A and $I(X, A)$ are Morita equivalent, then $\tilde{X}$ is a one-element set.*

Let $\{x\}$ denote a one-element set. Then we may reinterpret Nachev's result as follows: if A has "enough structure", and the rings $I(\{x\}, A)$ and $I(X, A)$ are Morita equivalent, then $\tilde{X} \cong \widetilde{\{x\}}\ (= \{x\})$ as partially ordered sets. That is, in this special setting we can recover $\tilde{X}$ from the class of incidence rings over A which are Morita equivalent to $I(X, A)$. Anderson proved a similar result in [1, Corollary 2.9]: namely, if X and Y are finite preordered sets and A is a field such that the rings $I(X, A)$ and $I(Y, A)$ are Morita equivalent, then $\tilde{X} = \tilde{Y}$. In the context of local incidence rings we will generalize these results to *all* preordered sets, while requiring much less structure on A than Anderson used.

As noted in section 2, $LI(X, A)$ is precisely the category ring of $A(X, \leqq)$. By a result of Gabriel [6, Proposition II. 1.2], the category of (unitary) modules over this ring is equivalent to $FUN(A(X, \leqq), Ab)$, the category of additive functors from $A(X, \leqq)$ to Abelian Groups. Since $A(X, \leqq)$ and $A(\tilde{X}, \leqq)$ are equivalent categories, we conclude that $FUN(A(X, \leqq), Ab)$ and $FUN(A(\tilde{X}, \leqq), Ab)$ are equivalent as well. Thus we have shown

PROPOSITION 3.1. *Let A be any unital ring, and let X be any preordered set. Then $LI(X, A)$ is Morita equivalent to $LI(\tilde{X}, A)$.* ∎

The above proposition can also be proved directly, using ring-theoretic techniques. Let R denote $LI(X, A)$, and let X' be any subset of X consisting of exactly one member from each $\sim$ equivalence class of X. Then $R'=\bigoplus_{x\in X'} Re_x$ is a compatible locally projective generator for R Mod (see [3]), and its "limit endomorphism ring" is isomorphic to $LI(\bar{X}, A)$. Now apply [3 ,Theorem 2.5].

The unital ring A is called *semiperfect* if there exists a set of orthogonal idempotents $G=\{g_1, \dots, g_n\}$ in A such that $1=g_1+\dots+g_n$ and each $g_i A g_i$ is local. A subset $\bar{A}=\{f_1, \dots, f_m\}$ of G is called *basic* in case the set $\{Af_1, \dots, Af_m\}$ is an irredundant collection of the isomorphism classes of finitely generated indecomposable projective left A-modules. The *associated reduced preordered set of A* is the set $\bar{A}$ with the preordering described in section 1; $(\bar{A}, \leqq)$ is independent of the choice of $\{f_1, \dots, f_m\}$. By [2, Theorem 7.9], A is indecomposable if and only if $\bar{A}$ is connected.

For $f_i\in\bar{A}$ and $x\in X$ let f_{ix} denote the idempotent $f_i e_x$ in $LI(X, A)$. If we define a preorder on the set $\overline{LI(X, A)}=\{f_{ix} \mid x\in X,\ 1\leqq i\leqq m\}$ as described in section 1, then $\overline{LI(X, A)}$ is order isomorphic to $X\times\bar{A}$ by [11, Lemma 4.2].

Lemma 3.2. *Let A be a semiperfect ring with basic set $\{f_1, \dots, f_m\}$. Let X be any partially ordered set, and let R denote $LI(X, A)$. Then the set $C=\{Rf_{ix}|x\in X, 1\leqq i\leqq m\}$ is a complete irredundant collection of the isomorphism classes of finitely generated indecomposable projective left R-modules.*

Proof. Since each f_{ix} is idempotent, each Rf_{ix} is finitely generated and projective. If $Rf_{ix}\simeq Rf_{jy}$, then both $\mathrm{Hom}_R(Re_x, Re_y)$ and $\mathrm{Hom}_R(Re_y, Re_x)$ are nonzero (since Rf_{ix} and Rf_{jy} are summands of Re_x and Re_y, respectively); thus $x\leqq y$ and $y\leqq x$, so that $x=y$ since X is partially ordered. This gives $Rf_{ix}\simeq Rf_{jx}$, which upon multiplication by e_x yields $e_x Rf_{ix}\simeq e_x Rf_{jx}$ as left $e_x Re_x\simeq A$-modules. Thus $Af_i\simeq Af_j$, so $i=j$ by the hypothesis that $\{f_1, \dots, f_m\}$ is basic. We conclude that the collection C is irredundant. Furthermore, $\mathrm{End}_R(Rf_{ix})\simeq f_i e_x Re_x f_i\simeq f_i A f_i$, which is local; thus each Rf_{ix} is indecomposable.

If P is any finitely generated projective indecomposable left R-module, then there exists a finite subset $S=\{x_1, \dots, x_n\}$ of X (possibly with repeats) and a left R-module P' with

$$\bigoplus_{i=1}^{n}\bigoplus_{j=1}^{m} Rf_{jx_i}\simeq P\oplus P'.$$

Now apply [2, Lemma 12.2 and Theorem 12.6] to conclude that $P\simeq Rf_{jx_i}$ for some $1\leqq j\leqq m$ and $x_i\in S$. This completes the proof of the lemma. ∎

Proposition 3.3. *Let X and Y be partially ordered sets, and let A be any indecomposable semiperfect ring. If $LI(X, A)$ is Morita equivalent to $LI(Y, A)$, then X is order isomorphic to Y.*

Proof. Let $E(X)=\{f_{ix}|x\in X,\ 1\leqq i\leqq m\}$ and $E(Y)=\{f_{iy}|y\in Y, 1\leqq i\leqq m\}$ denote the sets of idempotents described above in $R=LI(X, A)$ and $S=LI(Y, A)$, respectively. Let H: R Mod $\to S$ Mod be an equivalence.

By Lemma 3.2 the sets $\{Rf_{ix}|f_{ix}\in E(X)\}$ and $\{Sf_{iy}|f_{iy}\in E(Y)\}$ are complete irredundant collections of the isomorphism classes of finitely generated projective indecomposable left R-modules and S-modules, respectively. Since H is an equivalence it preserves categorical properties; thus the sets $\{H(Rf_{ix}) | f_{ix} \in E(X)\}$ and $\{Sf_{iy} | f_{iy} \in E(Y)\}$ are the same, up to isomorphism. Using [11, Lemma 4.2] we may construct a bijection $\Theta: X \times \bar{A} \to Y \times \bar{A}$ defined via $\Theta(x, f_i) = (y, f_j)$ iff $H(Rf_{ix}) \simeq \simeq Sf_{jy}$. Since H is an equivalence, the abelian groups $\mathrm{Hom}_R(Rf_{ix}, Rf_{i'x'})$ and $\mathrm{Hom}_S(H(Rf_{ix}), H(Rf_{i'x'}))$ are isomorphic; along with the definition of the preorder on $X \times \bar{A}$, this allows us to conclude that both Θ and Θ^{-1} are order-preserving. Hence $X \times \bar{A}$ is order isomorphic to $Y \times \bar{A}$, so that X is order isomorphic to Y by a result of Voss ([11, Theorem 3.4]; note the local finiteness condition on X and Y is not used in the proof). ∎

Proposition 3.1 together with Proposition 3.3 now easily yield the main result of this article.

THEOREM 3.4. *Let X and Y be preordered sets, and let A be any indecomposable semiperfect ring. Then $LI(X, A)$ is Morita equivalent to $LI(Y, A)$ if and only if $\tilde{X}$ is order isomorphic to $\tilde{Y}$. Thus $\tilde{X}$ can be recovered from the class of local incidence rings over A which are Morita equivalent to $LI(X, A)$.* ∎

In [7, page 18] Freyd observes that if C and C' are amenable categories, such that $FUN(C, Ab)$ is equivalent to $FUN(C', Ab)$, then C and C' are equivalent categories; the key idea involved here is that C can be realized as the finitely generated projective objects in $FUN(C, Ab)$. Theorem 3.4 along with [6, Proposition II. 1.2] gives an analogous result: if $A(X, \leqq)$ and $A(Y, \leqq)$ are the additive categories of the preordered sets X and Y with coefficients in an indecomposable semiperfect ring A, such that $FUN(A(X, \leqq), Ab)$ is equivalent to $FUN(A(Y, \leqq), Ab)$, then $A(X, \leqq)$ and $A(Y, \leqq)$ are equivalent categories. Similar to Freyd's result, we have realized $A(X, \leqq)$ as the *indecomposable* finitely generated projective objects in $FUN(A(X, \leqq), Ab)$.

4. Möbius functions and local incidence rings

As mentioned in the introduction, incidence rings were first introduced by Rota [9] in order to view the classical Möbius function μ in a more general setting. We show in this section that those incidence rings which are relevant to such considerations arise quite naturally in the context of local incidence rings.

Let R be a ring with local units, and let $\mathrm{End}(R_R)$ denote the ring of endomorphisms of the right regular module. Then the left multiplication map $\lambda: R \to \mathrm{End}(R_R)$ is an injective ring homomorphism. Thus we may associate with $R = LI(X, A)$ a canonical ring with identity into which R can be embedded. Note that λ is an isomorphism if and only if R is unital; hence if X is infinite, λ is a proper embedding.

For the remainder of this section R will denote $LI(X, A)$. If X is any preordered set and A is a ring, we call a function $h: X \times X \to A$ *lower finite* in case for each $x \in X$ the set $S_h(x) = \{y \in X | h(y, x) \neq 0\}$ is finite. Also, we call a function $h: X \times X \to A$ *order preserving* in case $h(x, y) = 0$ whenever $x \not\leqq y$.

LEMMA 4.1. *Let $h: X \times X \to A$ be lower finite and order preserving. Then for each $f \in LI(X, A)$ the product $h \cdot f$ is a well-defined map from $X \times X$ to A, and $h \cdot f \in LI(X, A)$.*

PROOF. Using (2.2) it suffices to prove this result for functions of the form δ_{xy}. Let $(u, v)\in X\times X$. Then

$$(h\cdot\delta_{xy})(u, v) = \sum_{u\leqq w\leqq v} h(u, w)\delta_{xy}(w, v) = \begin{cases} h(u, x) & \text{if } v = y \\ 0 & \text{otherwise,}\end{cases}$$

so that the product $h\cdot\delta_{xy}$ is well defined. Further, we see by this equation that $(h\cdot\delta_{xy})(u, v)\neq 0$ only if $v=y$ and $u\in S_h(x)$. Since this latter set is finite, $h\cdot\delta_{xy}$ is indeed in $LI(X, A)$. ∎

We conclude from this lemma that left multiplication by an order preserving, lower finite function is an element of $\operatorname{End}(R_R)$. For certain preordered sets, the converse is also true. Following [5], we call a preordered set X *lower finite* in case for each $x\in X$ the set $S(x)=\{y\in X\mid y\leqq x\}$ is finite. Note that any lower finite preordered set X is necessarily locally finite. Also, if X is lower finite and $h\colon X\times X\to A$ is order preserving, then h is lower finite.

PROPOSITION 4.2. *Let X be a lower finite preordered set, and let A be any ring. Then $I(X, A)\simeq\operatorname{End}(R_R)$ via $\lambda\colon h\to\lambda_h$, where λ_h denotes left multiplication by h.*

PROOF. The fact that λ_h is in $\operatorname{End}(R_R)$ for $h\in I(X, A)$ follows directly from the above remark and Lemma 4.1. An easy check shows that λ is an injective ring homomorphism. For surjectivity, let $\psi\in\operatorname{End}(R_R)$ and define $\psi(\delta_{xy})=f^{xy}\in R$ for each $x\leqq y$ in X. Then for u, v in X we have

$$\begin{aligned} f^{xy}(u, v) &= (\psi(\delta_{xy}))(u, v) = (\psi(e_x\cdot\delta_{xy}))(u, v) \\ &= (\psi(e_x)\cdot\delta_{xy})(u, v) = (f^{xx}\cdot\delta_{xy})(u, v) \\ &= \begin{cases} f^{xx}(u, x) \text{ if } v = y \\ 0 \text{ otherwise.}\end{cases}\end{aligned}$$

Now define $h\in I(X, A)$ via $h(u, v)=f^{vv}(u, v)$. Then for each $x\leqq y$ in X,

$$\begin{aligned} (h\cdot\delta_{xy})(u, v) &= \begin{cases} h(u, x) \text{ if } v = y \\ 0 \text{ otherwise}\end{cases} \\ &= \begin{cases} f^{xx}(u, x) \text{ if } v = y \\ 0 \text{ otherwise}\end{cases} \\ &= f^{xy}(u, v) \text{ (by above)} = (\psi(\delta_{xy}))(u, v).\end{aligned}$$

Thus $h\cdot\delta_{xy}=\psi(\delta_{xy})$ for each $x\leqq y$ in X. A straightforward calculation along with (2.2) now gives $\lambda_h=\psi$, which completes the proposition. ∎

We conclude by demonstrating that the lower finiteness restriction imposed on X in Proposition 4.2 is indeed a natural one. Recall the classical Möbius Inversion Formula (see for example [9, Proposition 2]): Let $f(x)$ be a real-valued function defined for x ranging in a locally finite partially ordered set P. Let an element p exist with the property that $f(x)=0$ unless $x\geqq p$. Suppose that $g(x)=\sum_{y\leqq x} f(y)$. Then $f(x)=$ $=\sum_{y\leqq x} g(y)\mu(y, x)$.

The "zero condition" placed on the function f is imposed solely to ensure that the set $\mathrm{Supp}(f)=\{x\in X \mid f(x)\neq 0\}$ is contained in a lower finite partially ordered set. Therefore, in any discussion involving the Möbius Inversion Formula, there is absolutely no loss of generality in assuming the underlying partially ordered set is lower finite. This observation along with Proposition 4.2 indicates that the standard incidence ring $I(X, A)$ can be recovered explicitly from the local incidence ring $LI(X, A)$ whenever $I(X, A)$ is of computational interest.

Acknowledgement. The author is grateful to Dr. F. W. Anderson, whose timely encouragement and thoughtful suggestions proved pivotal during the preparation of this paper.

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(Received September 27, 1985)

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ON SYMMETRIC FUNCTIONS OF k VARIABLES

LUDGER RÜSCHENDORF

Abstract

A famous theorem of E. Schmidt says that each symmetric square integrable function $f(x_1, x_2)$ has a representation of the from $\sum_{n=1}^{\infty} \lambda_n \varphi_n(x_1)\varphi_n(x_2)$, where the φ_n are orthonormal solutions of the eigenvalue equation with non zero eigenvalues. We deal with the question to what extent this result can be generalized to functions of $k \geqq 2$ variables.

Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space and let

(1) $$S_k := \{h\colon (X^k, \mathscr{A}^k) \to (\mathbf{R}^1, \mathbf{B}^1);\ h \in L^2(\mu^k),\ h \text{ symmetric}\}$$

where h symmetric means that $h(\pi x) = h(x)$ for all permutations $\pi \in \mathfrak{S}_k$ of the components of x. Define for $\varphi\colon X \to \mathbf{R}^1$, $x \in X^k$

(2) $$\bigotimes_{i=1}^{k} \varphi(x) := \prod_{i=1}^{k} \varphi(x_i), \quad x = (x_1, \ldots, x_k),$$

and

(3) $$(SO)_k := \{h \in S_k\colon \text{there exists an orthonormal sequence } (\varphi_n) \subset L^2(\mu) \text{ and}$$
$$(\lambda_n) \subset \mathbf{R}^1 \text{ such that } \Sigma \lambda_n^2 < \infty \text{ and } h = \Sigma \lambda_n \bigotimes_{i=1}^{k} \varphi_n,$$

where convergence holds in quadratic mean in $L^2(\mu^k)$.

For $k=2$ a famous theorem of E. Schmidt says that $S_2 = (SO)_2$. This theorem and some extensions by Mercer have found important applications in probability and statistics as well as in many other branches of mathematics; examples of applications concern, e.g., asymptotic distributions and expansions of symmetric statistics (U-statistics, mixture of χ^2-distributions), definition of multiple stochastic integrals, and representation of stochastic processes (by first representing the covariance kernels). We want to discuss to what extent this and related results can be generalized to $k \geqq 2$.

If $h \in S_k$, we can decide whether h lies in $(SO)_k$ by considering the following generalized eigenvalue equation:

(4) $$[h, \varphi]_{k-1} = \lambda \bigotimes_{i=1}^{k-1} \varphi, \quad \varphi \in L^2(\mu), \quad \lambda \in \mathbf{R}^1, \quad \lambda \neq 0, \text{ where}$$
$$[h, \varphi]_{k-1}(x) := \int h(x, y)\varphi(y)\,d\mu(y), \quad x \in X^{k-1}.$$

1980 *Mathematics Subject Classification.* Primary 62B40; Secondary 41A58.

Key words and phrases. Eigenvalue expansion of symmetric kernels, Mercer's theorem, tensor product.

LEMMA 1. *If* $h=\sum_{n=1}^{N}\lambda_n\bigotimes_{i=1}^{k}\varphi_n\in(SO)_k$, $N\leqq\infty$, $\lambda_n\neq0$, $1\leqq n\leqq N$, $k\geqq3$, *then* $\{\varphi_n;\,1\leqq n\leqq N\}$ *are the only normed solutions of the eigenvalue equation* (4) *with* $\lambda\neq0$ *for k odd while for k even* $\{\pm\varphi_n;\,1\leqq n\leqq N\}$ *are the only normed solutions.*

PROOF. Let $\varphi\in L^2(\mu)$ satisfy $[h,\varphi]_{k-1}=\lambda\bigotimes_{i=1}^{k-1}\varphi$, $\|\varphi\|=1$, $\lambda\neq0$. Then from (4) we obtain

$$\sum_{n=1}^{N}\lambda_n\bigotimes_{i=1}^{k-1}\varphi_n\langle\varphi,\varphi_n\rangle=\lambda\bigotimes_{i=1}^{k-1}\varphi \tag{5}$$

$\langle\,,\,\rangle$ denoting scalar product. Therefore,

$$\sum_{n=1}^{N}\lambda_n\langle\varphi,\varphi_n\rangle^{k-1}\varphi_n=\lambda\varphi, \tag{6}$$

and

$$\begin{aligned}\sum_{n=1}^{N}\lambda_n\langle\varphi,\varphi_n\rangle^{k-2}\varphi_n\otimes\varphi_n&=\lambda\varphi\otimes\varphi\\&=\lambda^{-1}\sum_{n,m=1}^{N}\lambda_n\lambda_m\langle\varphi,\varphi_n\rangle^{k-1}\langle\varphi,\varphi_m\rangle^{k-1}\varphi_n\otimes\varphi_m.\end{aligned}\tag{7}$$

Let $M:=\{n\in\mathbf{N};\,1\leqq n\leqq N,\,\langle\varphi,\varphi_n\rangle\neq0\}$ and let $n_0\in M$. Then (7) implies that $\lambda_{n_0}\langle\varphi,\varphi_{n_0}\rangle^{k-2}\varphi_{n_0}=\lambda^{-1}\sum_{n=1}^{N}\lambda_n\lambda_{n_0}\langle\varphi,\varphi_n\rangle^{k-1}\langle\varphi,\varphi_{n_0}\rangle^{k-1}\varphi_n$ and, therefore, by the orthogonality of (φ_n): $M=\{n_0\}$, i.e. $\lambda\varphi=\lambda_{n_0}\langle\varphi,\varphi_{n_0}\rangle^{k-1}\varphi_{n_0}$. Since $\|\varphi\|=1$, we obtain $\varphi=\pm\varphi_{n_0}$. For k even $-\varphi_{n_0}$ is a solution while for k odd it is not. □

Lemma 1 implies that for $k\geqq3$, a symmetric function h lies in $(SO)_k$ if and only if all solutions of the eigenvalue equation (4) are orthogonal (up to equivalence by the factor -1 for k even) and h is identical to the eigenvalue expansion. In the following Proposition we construct some typical examples in S_k which are not in $(SO)_k$ for $k\geqq3$; so a direct generalization of Schmidt's orthogonal expansion valid for $k=2$ is not possible.

PROPOSITION 2. *Let* $\varphi,\psi\in L^2(\mu)$ *satisfy*

(i) $\mu\{\varphi=0\}=\mu\{\psi=0\}=0$

(ii) φ,ψ *are linear independent, i.e.* $a\varphi+b\psi=0$ $[\mu]$ *implies* $a=b=0$

(iii) $\langle\varphi,\psi\rangle\neq0.$

Then for $k\geqq3$ *the function*

$$h:=\bigotimes_{i=1}^{k}\varphi+\bigotimes_{i=1}^{k}\psi\in S_k\ \textit{but}\ h\notin(SO)_k. \tag{8}$$

PROOF. Assume that h has a representation $h=\sum_{n=1}^{N}\lambda_n\bigotimes_{i=1}^{k}\varphi_n$ as in Lemma 1, $\lambda_n\neq0$, $1\leqq n\leqq N$. Then from the orthogonality of φ_n, we derive with $a_n:=\langle\varphi,\varphi_n\rangle$, $b_n:=\langle\psi,\varphi_n\rangle$

$$a_n^{k-2}\varphi\otimes\varphi+b_n^{k-2}\psi\otimes\psi=\lambda_n\varphi_n\otimes\varphi_n\,[\mu^2]. \tag{9}$$

Since $\|\varphi_n\|=1$ we get that either a_n or b_n is not zero and, therefore, we obtain from assumption 1 that $\varphi_n \neq 0\,[\mu]$. Now (9) implies

$$a_n^{k-2}\left(\frac{\varphi(x_1)}{\varphi_n(x_1)}-\frac{\varphi(x_2)}{\varphi_n(x_2)}\right)\varphi(y)+b_n^{k-2}\left(\frac{\psi(x_1)}{\varphi_n(x_1)}-\frac{\psi(x_2)}{\varphi_n(x_2)}\right)\psi(y)=0 \tag{10}$$

for almost all $x_1, x_2, y \in X$. (10) for x_1, x_2 fixed implies by the linear independence Assumption 2 that

$$a_n\left(\frac{\varphi(x_1)}{\varphi_n(x_1)}-\frac{\varphi(x_2)}{\varphi_n(x_2)}\right)=0,\ b_n\left(\frac{\psi(x_1)}{\varphi_n(x_1)}-\frac{\psi(x_2)}{\varphi_n(x_2)}\right)=0 \text{ for almost all } x_1, x_2 \in X.$$

If $a_n \neq 0$, then $\varphi(x)=\dfrac{\varphi(x_2)}{\varphi_n(x_2)}\varphi_n(x)\,[\mu]$, if $b_n \neq 0$, then $\psi(x)=\dfrac{\psi(x_2)}{\varphi_n(x_2)}\varphi_n(x)\,[\mu]$. So for all $n \leqq N$ we obtain either $\varphi_n=c_n\varphi$ or $\varphi_n=d_n\psi$. The orthogonality of (φ_n) and Assumption 3, therefore, imply $\varphi_n=c_n\varphi$ for all $n \leqq N$ or $\varphi_n=d_n\psi$ for all $n \leqq N$, i.e. $N=1$, implying in the first case

$$\bigotimes_{i=1}^{k}\varphi+\bigotimes_{i=1}^{k}\psi=\left(\sum_{n=1}^{N}c_n\lambda_n\right)\bigotimes_{i=1}^{k}\varphi,$$

a contradiction to linear independence. □

REMARK 1. In order to decide whether $h \in S_k$ lies in $(SO)_k$ one can construct (φ_n) as solutions of the following sequence of max-problems:

1. Let φ_1 be a solution of

$$|\langle [h,\varphi]_{k-1}, \bigotimes_{i=1}^{k}\varphi\rangle| = \max_{\|\varphi\|_{L^2(\mu)}=1} \tag{11}$$

then $\lambda_1 := \left|\langle [h,\varphi_1]_{k-1}, \bigotimes_{i=1}^{k-1}\varphi_1\rangle\right|$ satisfies for $h \in (SO)_k$: $\lambda_1=\|[h,\varphi_1]_{k-1}\|_{L^2(\mu^{k-1})}$ and φ_1 is an eigenfunction with eigenvalue μ_1, $|\mu_1|=\lambda_1$.

2. If orthonormal solutions $(\varphi_1,\lambda_1), \dots, (\varphi_{n-1},\lambda_{n-1})$ are found, let $F_n := \langle \varphi_1, \dots, \varphi_{n-1}\rangle^{\perp}$ be the orthogonal complement in $L^2(\mu)$ and solve the max-problem (11) under the additional restriction $\varphi \in F_n$ obtaining solutions μ_n, φ_n. If $h \in (SO)_k$, then

$$\lambda_n := |\langle [h,\varphi_n]_{k-1}, \bigotimes_{i=1}^{k-1}\varphi_n\rangle| = \sup\{\|[h,\varphi]_{k-1}\|_{L^2(\mu^{k-1})};\ \|\varphi\|=1,\ \varphi \in F_n\}. \tag{12}$$

The proof can be given as for the case $k=2$ (cf. Riesz—Sz.-Nagy [6], p. 217). □

The following theorem shows that each symmetric function $h \in S_k$ has an expansion by the 'typical' symmetric functions $\bigotimes_{i=1}^{k}\varphi$ if one weakens the orthogonality condition.

THEOREM 3. *Let $h \in S_k$; then there exist $\varphi_n \in L^2(\mu)$, $\lambda_n \in \mathbf{R}$ with the following properties:*

1 $\sum_{n=1}^{\infty} \lambda_n^2 < \infty, \quad \|\varphi_n\|_{L^2(\mu)} = 1,\ n \in \mathbf{N}$

2 $h = \sum_{n=1}^{\infty} \lambda_n \bigotimes_{i=1}^{k} \varphi_n$ *(convergence in* $L^2(\mu^k)$*)*

3 *(weak orthogonality) If* $m \neq r$, *then*

$$\sum_{n=m(2^k-1)}^{(m+1)(2^k-1)-1} \lambda_n \bigotimes_{i=1}^{k} \varphi_n \quad \textit{and} \quad \sum_{n=r(2^k-1)}^{(r+1)(2^k-1)-1} \lambda_n \bigotimes_{i=1}^{k} \varphi_n$$

are orthogonal.

PROOF. We use the identification of $L^2(\mu^k)$ with $\bigotimes_{i=1}^{k} L^2(\mu)$, the k-fold topological (Hilbert-) tensor product of $L^2(\mu)$. For the pertaining results on tensor products we refer to Neveu [4]. Let $(\psi_i)_{i \in I}$ be an orthonormal basis of $L^2(\mu)$. Then $\{\psi_{i_1} \otimes \ldots \otimes \psi_{i_k};\ i_j \in I,\ 1 \leq j \leq k\}$ is an orthonormal basis of $L^2(\mu^k)$ and, therefore, there exists a countable set $T \subset I^k$ such that

$$h = \sum_{t \in T} a_t\, g_t, \quad \text{where } g_t = \psi_{t_1} \otimes \ldots \otimes \psi_{t_k}. \tag{13}$$

Now define for $f\colon X^k \to \mathbf{R}^1$ the symmetrization $Sf\colon X^k \to \mathbf{R}^1$ by $Sf(x) := \sum_{\pi \in \mathfrak{S}_k} f(\pi x)$. If $f(x) = \bigotimes_{i=1}^{k} f_i(x)$, then for $R \subset \{1, \ldots, k\}$ define

$$S_R f\colon X^k \to \mathbf{R}^1 \text{ by } S_R f := \bigotimes_{i=1}^{k} \Big(\sum_{j \in R} f_j\Big) \text{ where } \Big(\sum_{j \in R} f_j\Big)(y) := \sum_{j \in R} f_j(y),\ y \in X. \tag{14}$$

$S_R f$ is a 'typical' symmetric function and we get the following identity:

$$f = \bigotimes_{i=1}^{k} f_i \text{ implies } Sf = \sum_{l=0}^{k-1} (-1)^l \sum_{\substack{R \subset \{1, \ldots, k\} \\ |R| = k-l}} S_R f. \tag{15}$$

For the proof of (15) let $A \in \mathbf{R}^{k \times k}$, $A = (a_{ij})$ be a $k \times k$ matrix and let per $(A) = \sum_{\pi \in \mathfrak{S}_k} \sum_{i=1}^{k} a_{i\pi(i)}$ denote the permanent of A, $\mathfrak{S}_k$ again denoting the permutations of $\{1, \ldots, k\}$. A consequence of the inclusion exclusion principle is the following representation of per (A) (cf. Jacobs [3], Satz 3.13, p. 31):

$$\operatorname{per}(A) = \sum_{l=0}^{k-1} (-1)^l \sum_{\substack{R \subset \{1, \ldots, k\} \\ |R| = k-l}} \prod_{i=1}^{k} \Big(\sum_{j \in R} a_{ij}\Big). \tag{16}$$

Defining $a_{ij} := f_i(x_j)$, $1 \leq i, j \leq k$, (15) follows from (14).

Now by symmetry of h we obtain

$$(k!)h = Sh = \sum_{\pi \in \mathfrak{S}_k} h \circ \pi = \sum_{t \in T} a_t\, Sg_t,$$

since the series in (13) can be rearranged. Note that there exists an increasing sequence of finite subsets $T_n \subset T$ with $\|h - \sum_{t \in T_n} a_t g_t\|^2 = \sum_{t \in T_n^c} a_t^2 \|g_t\|^2 \to 0$ implying that

$$\sum_{t \in T_n^c} a_t^2 \|Sg_t\|^2 \leqq (k!)^2 \sum_{t \in T_n^c} a_t^2 \|g_t\|^2 \to 0.$$

One can even choose the summands Sg_t to be orthogonal by rearranging the g_t-sum according to terms with identical Sg_t (cf. Neveu [4], Lemma 6.13). Now (15) applied to $f = g_t$ implies that

$$h = \frac{1}{k!} \sum_{t \in T} a_t \sum_{i=0}^{k-1} (-1)^i \sum_{\substack{R \subset \{1, \ldots, k\} \\ |R| = k-i}} S_R g_t. \tag{17}$$

$S_R g_t$ is a 'typical' symmetric function and $\|S_R g_t\|^2_{L^2(\mu^k)} \leqq |R|^{2k} \leqq k^{2k}$. Renorming the $S_R g_t$ and choosing a suitable enumeration by natural numbers we get the desired representation of h, the weak orthogonality condition following from the orthogonality of (Sg_t) $t \in T$ and the fact that the number of summands $S_R g_t$ for the representation of Sg_t is $2^k - 1$.

REMARK 2. a) We remark that relation (15) can be derived also from a relation which is stated (without proof) on p. 52 of Dellacherie, Meyer [1] and which there is attributed to P. Cartier.

b) For $k=2$ the Mercer theorem says that the eigenvalue expansion of $h \in S_2 = (SO)_2$ is uniformly convergent (in the a'. s. sense) if X is a compact space with Borel σ-algebra $\mathscr{A}$ and $h(x,y)$ is continuous and positive definite. For $k \geqq 2$, we define $h \in S_k$ to be k-positive (definite) if

$$\langle h, \bigotimes_{i=1}^{k} \varphi \rangle \geqq 0, \quad \varphi \in L^2(\mu). \tag{18}$$

Then for $k \in 2\mathbf{N}+1$ the only k-positive function is $h=0$. Substituting in (17) $-\varphi$ and φ, we obtain $\langle h, \bigotimes_{i=1}^{k} \varphi \rangle = 0$ for all $\varphi \in L^2(\mu)$ and, therefore, using Theorem 3, $\langle h, h \rangle = \sum_{n=1}^{\infty} \lambda_n \langle h, \bigotimes_{i=1}^{k} \varphi_n \rangle = 0$. If $k \in 2\mathbf{N}$, then a Mercer type theorem for $h \in (SO)_k$ can immediately be inferred from the case $k=2$.

c) If for $A \in \mathscr{A}$, $A^k := A \times \ldots \times A$ denotes the k-fold product of A, define $\mathscr{A}_k := \sigma(A^k; A \in \mathscr{A})$ the σ-algebra generated by the 'typical'' symmetric sets and $\mathscr{B}_k$ the σ-algebra of all measurable symmetric sets in $(X^k, \mathscr{B}^k)$. A measure-free version of Theorem 3 on the level of sets is not true by results of Rao [5] and Grzegorek [2] stating that

(19) a) $\mathscr{A}_2 = \mathscr{B}_2$
b) For $k \geqq 3$, $|\mathscr{A}| \geqq 3$, $\mathscr{A}_k$ is a strict subset of $\mathscr{B}_k$.

Define $\overline{\mathscr{B}}_k := \{A_1 \times \ldots \times A_k; A_i \in \mathscr{A}, \; A_i \cap A_j = \emptyset \text{ for } i \neq j\}$ and let μ_i be nonatomic measures on $(X, \mathscr{A})$, $i = 1, 2$, then

$$\mu_1^k / \mathscr{A}_k = \mu_2^k / \mathscr{A}_k \quad \text{implies that} \quad \mu_1 = \mu_2. \tag{20}$$

For the proof of (20) let $A_1\times\ldots\times A_k\in\overline{\mathscr{B}_k}$; then by (15):

$$S1_{A_1\times\ldots\times A_k} = S\bigotimes_{i=1}^{k} 1_{A_i} = \sum_{l=0}^{k-1}(-1)^l \sum_{\substack{R\subset\{1,\ldots,k\}\\ |R|=k-l}} \bigotimes_{i=1}^{k}\Big(\sum_{j\in R} 1_{A_j}\Big).$$

Since $\sum_{j\in R} 1_{A_j}=1_{\bigcup_{j\in R} A_j}$, $\mu_1^k/\mathscr{A}_k=\mu_2^k/\mathscr{A}_k$ implies that

$$\mu_1^k(A_1\times\ldots\times A_k) = \frac{1}{k!}\int (S1_{A_1\times\ldots\times A_k})\,d\mu_1^k = \mu_2^k(A_1\times\ldots\times A_k).$$

Now the assumption that μ_i are nonatomic implies as it is well-known that $\overline{\mathscr{B}_k}$ is a determining system for μ_i^k, i.e. $\mu_1^k=\mu_2^k$ or, equivalently, $\mu_1=\mu_2$.

Relation (20) shows that 'diagonals' make the difference between $\mathscr{A}_k$ and $\mathscr{B}_k$. Relation (20) is true for all symmetric measures having $\overline{\mathscr{B}_k}$ as a determining class.

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(Received October 3, 1985)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 209—213

ON THE DISTRIBUTION OF A STOCHASTIC INTEGRAL ARISING IN THE THEORY OF FINANCE

THEODORE ARTIKIS

Introduction

Let $\{X(t), t\geqq 0\}$ be a stochastic process with stationary independent increments and $E(X(t))=\lambda t$, $V(X(t))=\sigma^2 t$ where $-\infty<\lambda<\infty$ and $0<\sigma^2<\infty$. We assume that $\{X(t), t\geqq 0\}$ is continuous in probability and that its sample paths are right continuous and have left limits. The stochastic integral

$$Z = \int_0^\infty e^{-t}\,dX(t) \tag{1}$$

exists and is finite almost surely. The distribution of Z is continuous and its characteristic function is

$$\gamma(u) = \exp\left\{\int_0^u \frac{\log v(y)}{y}\,dy\right\} \tag{2}$$

where $v(u)$ is the characteristic function of the increment $X(t+1)-X(t)$, [3]. The purpose of the present paper is to establish the distribution of Z, with $\{X(t), t\geqq 0\}$ a compound Poisson process, as convolution of certain transformed renewal distributions.

Let $F(x)$ be a distribution function on $(0, \infty)$ with finite mean μ and $\varphi(u)$ its characteristic function. Consider an equilibrium renewal process $\{N(t), t\geqq 0\}$ whose successive time intervals between renewals are distributed according to $F(x)$. The characteristic function of the time of the nth renewal is

$$\frac{\varphi(u)-1}{i\mu u}\,\varphi^{n-1}(u) \tag{3}$$

(see [2]). We shall use the distribution of the nth renewal in order to construct an infinitely divisible distribution related to the distribution of Z.

1980 *Mathematics Subject Classification.* Primary 60E
Key words and phrases. Stochastic integral, renewal distribution, characteristic functions.

The transformation

A characteristic function $a(u)$ is said to be infinitely divisible if for every positive integer n it is the nth power of some characteristic function. This means that there exists for every integer n a characteristic function $a_n(u)$ such that $a(u)=[a_n(u)]^n$ with $a_n(u)$ uniquely determined by $a(u)$, $a_n(u)=[a(u)]^{1/n}$ provided that one selects for the nth root the principal branch. The family o of infinitely divisible characteristic functions includes the important class L of self-decomposable characteristic functions [5]. The family o also includes the class U which is an interesting extension of the class L [1].

The following result on unimodal Lévy spectral functions has been established by Alf and O'Connor [1].

LEMMA. *Let $a(u)$ be an infinitely divisible characteristic function with Lévy spectral function M. Then M is unimodal if and only if there exists an infinitely divisible characteristic function $\beta(u)$ such that*

$$a(u) = \exp\left\{\frac{1}{u}\int_0^u \log\beta(y)\,dy\right\}. \tag{4}$$

Below we establish certain relationships of a transformed renewal distribution with the classes L and U.

THEOREM. *Let $F(x)$ be a distribution function on $(0,\infty)$ with finite mean μ and $\varphi(u)$ its characteristic function. Set*

$$\varphi_n(u) = \exp\left\{\int_0^u \frac{\varphi(y)-1}{y}\varphi^n(y)\,dy\right\}, \qquad n=0,1,2,\ldots$$

$$\vartheta_n(u) = \prod_{k=0}^{n-1}\varphi_k(u)$$

$$\psi_n(u) = \exp\left\{\frac{1}{u}\int_0^u \varphi^n(y)\,dy-1\right\}$$

$$\omega_n(u) = \exp\left\{\frac{1}{u}\int_0^u\int_0^y \frac{\varphi(x)-1}{x}\varphi^n(x)\,dx dy\right\}.$$

Then $\varphi_n(u)$ is an infinitely divisible characteristic function, $\vartheta_n(u)$ is of class L, $\psi_n(u)$ and $\omega_n(u)$ are of class U and satisfy

$$\psi_n(u)\varphi_n(u) = \psi_{n+1}(u)\omega_n(u), \qquad n=0,1,2,\ldots.$$

PROOF. Let $F^{(n)}(x)$ be the nth convolution of $F(x)$ with itself. Consider the function $\Lambda(x)$ defined by

$$\Lambda(x) = \int_0^\infty F^{(n)}(x-y)(1-F(y))\,dy. \tag{5}$$

The function $\Lambda(x)$ is well-defined since

$$\int_0^\infty y\,dF(y) < \infty$$

if and only if

$$\int_0^\infty (1-F(y))\,dy < \infty.$$

Furthermore consider the function $M(x)$ defined by

$$M(x) = -\int_x^\infty \frac{d\Lambda(y)}{y}. \tag{6}$$

The function $M(x)$ is non-decreasing on $(0, \infty)$ and satisfies

$$\int_0^1 x\,dM(x) < \infty.$$

Since

$$\frac{1}{u}\int_0^u \frac{\varphi(y)-1}{i\mu y}\varphi^n(y)\,dy = \int_0^\infty \frac{e^{iux}-1}{i\mu ux}\,d\Lambda(x)$$
$$= \int_0^\infty \frac{e^{iux}-1}{i\mu u}\,dM(x)$$

from [5] Theorem 11.2.2 it follows that

$$\begin{aligned} \varphi_n(u) &= \exp\left\{\int_0^u \frac{\varphi(y)-1}{y}\varphi^n(y)\,dy\right\} \\ &= \exp\left\{\int_0^\infty (e^{iux}-1)\,dM(x)\right\} \end{aligned} \tag{7}$$

is an infinitely divisible characteristic function. Set

$$\begin{aligned} \vartheta_n(u) &= \prod_{k=0}^{n-1} \varphi_k(u) \\ &= \exp\left\{\int_0^u \frac{\varphi^n(y)-1}{y}\,dy\right\}. \end{aligned} \tag{8}$$

Then $\vartheta_n(u)$ is of the form (7) with $M(x)$ defined by

$$M(x) = -\int_x^\infty \frac{1-F^{(n)}(y)}{y}\,dy. \tag{9}$$

Since $xM'(x)$ is non-increasing on $(0, \infty)$, from [5] Theorem 5.11.2 it follows that $\vartheta_n(u)$ is a self-decomposable characteristic function.

Consider the infinitely divisible characteristic functions $\varphi_n(u)$ and $\exp\{\varphi^n(u)-1\}$. Using the integral representation (4) for the members of the class U we get that

$$\psi_n(u) = \exp\left\{\frac{1}{u}\int_0^u \varphi^n(y)\,dy - 1\right\} \tag{10}$$

$$\omega_n(u) = \exp\left\{\frac{1}{u}\int_0^u \int_0^y \frac{\varphi(x)-1}{x}\varphi^n(x)\,dx\,dy\right\} \tag{11}$$

are members of the class U. Integrating by parts in the exponent of $\omega_n(u)$ we get that

$$\psi_n(u)\varphi_n(u) = \psi_{n+1}(u)\,\omega_n(u). \qquad \square$$

Using the expression (8) for the characteristic function $\vartheta_n(u)$ we have

$$\begin{aligned} \gamma(u) &= \prod_{n=1}^{\infty} \{\vartheta_n(u)\}^{p_n} \\ &= \prod_{n=1}^{\infty}\left\{\prod_{k=0}^{n-1} \varphi_k(u)\right\}^{p_n} \\ &= \exp\left\{\int_0^u \frac{P(\varphi(y))-1}{y}\,dy\right\} \end{aligned} \tag{12}$$

where $P(z)$ is the probability generating function of the discrete distribution $\{p_n\colon n=1, 2, \ldots\}$ having finite mean. Since the class L is closed under multiplication, raising to a positive power and passage to the limit we conclude that $\gamma(u)$ is a self-decomposable characteristic function. The characteristic function of the stochastic integral Z coincides with $\gamma(u)$ in (12) when $\{X(t), t\geqq 0\}$ is a compound Poisson process and the characteristic function of the increment $X(t+1)-X(t)$ is $v(u)=\exp\{[P(\varphi(u))-1]\}$.

Integrating by parts in the exponent of the characteristic function

$$\exp\left\{\frac{1}{u}\int_0^u\int_0^y \frac{P(\varphi(x))-1}{x}\,dx\,dy\right\},$$

which is a member of the class U, we get

$$\begin{aligned} &\exp\left\{\int_0^u \frac{P(\varphi(y))-1}{y}\,dy\right\} = \\ &= \exp\left\{\frac{1}{u}\int_0^u [P(\varphi(y))-1]\,dy\right\}\exp\left\{\frac{1}{u}\int_0^u\int_0^y \frac{P(\varphi(x))-1}{x}\,dx\,dy\right\}. \end{aligned} \tag{13}$$

We consider a stochastic process $\{Y(t), t\geqq 0\}$ with stationary and independent increments which is independent of the stochastic process $\{X(t), t\geqq 0\}$. We suppose that $\{Y(t), t\geqq 0\}$ is continuous in probability and that $\gamma(u)$ in (12) is the characteristic function of the increment $Y(t+1)-Y(t)$.

The stochastic integral $\int_0^1 t\,dY(t)$ exists in the sense of convergence in probability and its characteristic function is $\exp\left\{\frac{1}{u}\int_0^u \log\gamma(y)\,dy\right\}$, [4]. Hence the decomposition (13) is equivalent to the convolution model

$$\int_0^\infty e^{-t}\,dX(t) = \int_0^1 t\,dX(t)+\int_0^1 t\,dY(t). \tag{14}$$

Application

The theory of finance is concerned with the determination of the value of the firm as a going concern, the identification and analysis of factors with direct and indirect influence on this value, and with the valuation of investment opportunities. The economic value of the firm as a going concern is the present value of income that the firm will generate in the future. Assuming that the income of the firm is given by the stochastic process $\{X(t), t\geqq 0\}$ and since the corporate firm has an indefinite life, its economic value can be approximated by the stochastic integral

$$\int_0^\infty e^{-rt}\,dX(t),$$

where r is the rate of interest.

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(Received October 16, 1985)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 215—224

ERROR ESTIMATES OF A GENERAL LACUNARY TRIGONOMETRIC INTERPOLATION ON EQUIDISTANT NODES

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Dedicated to Professor A. Sharma on the occasion of his 65th birthday

Let q be a positive integer, $1 \leqq m_1 < \ldots < m_p$ be a given sequence of integers, $M_p = \{m_1, \ldots, m_p\}$, $C_{2\pi}$ the set of all 2π-periodic continuous functions, $\mathcal{T}_n$ the set of all trigonometric polynomials of degree at most n. The problem of M_q-interpolation can be stated in the following form: Let $x_{kn} = x_k = \dfrac{2\pi k}{n}$, $k = 0, 1, \ldots, n-1$ and $f(x) \in C_{2\pi}$. Construct polynomials

$$(1) \qquad R_n(f, x) \in \mathcal{P}_M, \quad M = \left[\frac{n(q+1)}{2}\right],$$

such that

$$(2) \qquad R_n(f, x_k) = f(x_k), \quad R_n^{(m_j)}(f, x_k) = 0, \quad k = 0, 1, \ldots, n-1; \quad j = 1, \ldots, q.$$

We mention that the more general case when prescribing certain values (i.e. not necessarily 0) for the derivatives does not yield essential novelties, neither from the point of view of existence and uniqueness, nor for error estimates. Therefore we restrict ourselves to the case (2). We also remark that interesting contributions on lacunary trigonometric interpolation were first obtained by O. Kis [3] and A. Sharma and A. K. Varma [5].

The most general existence and uniqueness theorems for the M_q-polynomials are given in [1]. Without explicitly mentioning these necessary and sufficient conditions for existence and uniqueness, *in what follows we shall assume that these conditions on the M_q and n are always satisfied.*

Our primary goal is to give error estimates for the R_n-polynomials in terms of the best polynomial approximation

$$E_n(f) = \inf_{T \in \mathcal{T}_n} \|f - T\|$$

of $f(x)$. Here and in the sequel $\|\,.\,\|$ will always mean supremum norm. $\tilde{f}(x)$ will denote the trigonometric conjugate of $f(x)$ (if it exists). First we give an estimate for trigonometric polynomials.

LEMMA 1. *Assume that M_q and n are such that the corresponding lacunary interpolation problem is uniquely solvable. Then there exists a δ, $0 < \delta < 1$, depending*

1980 *Mathematics Subject Classification* (1985 Revision). Primary 51A05; Secondary 42A15.
Key words and phrases. Lacunary interpolation, rate of convergence, saturation.

only on M_q such that for any $T\in\mathscr{T}_{[\delta n]}$ we have

$$\|T(x)-R_n[T,x]\| = O(n^{-m_1})\{\|T^{(m_1)}\|+\|\tilde{T}^{(m_1)}\|\}$$

where the constant involved in "O" depends only on δ.

It is interesting that the estimate is independent of $m_2, \ldots, m_p$ (see also Theorem 1 below).

PROOF of Lemma 1. Our basic references will be recent papers by S. Riemenschneider, A. Sharma and P. W. Smith [4] and A. Sharma and A. K. Varma [6]. According to [4] (1.4) – (1.5), if $T\in\mathscr{T}_M$ then

$$T(x)-R_n(T,x) = \sum_{j=1}^{q}\sum_{k=0}^{n-1} T^{(m_j)}(x_k)\,\varrho_{m_j}(x-x_k) \tag{3}$$

where $\varrho_{m_j}(x)\in\mathscr{T}_M$ are such that

$$\varrho_{m_j}^{(m_\nu)}(x_k) = \delta_{\nu j}\delta_{0k}, \qquad j, \nu = 1, \ldots, q; \quad k = 0, 1, \ldots, n-1.$$

These fundamental polynomials $\varrho_{m_j}(x)$ can be represented in the following form (see [4], p. 33 and (2.11), (2.12), (2.6)):

$$\varrho_{m_j}(x) = i^{-m_j}n^{-m_j-1}z^{-M}\sum_{\lambda=0}^{q}\sum_{\nu=0}^{n}\frac{\varepsilon_\nu H_{j,\lambda}\left(\frac{\nu}{n}\right)}{\Phi_q\left(\frac{N-\nu}{n}\right)}z^{\nu+\lambda n}, \qquad j=1,2,\ldots,q; \quad z=e^{ix}$$

where

$$\varepsilon_0=\frac{1}{2} \ \text{ or } \ 1, \quad \varepsilon_\nu = 1, \quad 1\leqq \nu\leqq n-1 \quad \text{and} \quad \varepsilon_n = 0 \ \text{ or } \ \frac{1}{2} \tag{5}$$

(the precise specification of ε_0 and ε_n is indifferent for us). $H_{j,\lambda}(x)$ and $\Phi_q(x)$ are algebraic polynomials of degree at most $\sum_{k=1}^{q} m_k - m_j$ and $\sum_{k=1}^{q} m_k$, respectively, with coefficients independent of n, and

$$N = \begin{cases} \left[\frac{n}{2}\right] & \text{if } q \text{ is even} \\ n & \text{if } q \text{ is odd.}\end{cases}$$

In particular, for $T(x)=e^{ilx}=z^l$, $l\leqq M$ we have from (3) and (4)

$$\begin{aligned} &z^l - R_n(z^l, x) = \\ &= z^{-M}\sum_{j=1}^{q} n^{-m_j-1}l^{m_j}\sum_{\nu=0}^{n}\frac{\varepsilon_\nu z^\nu}{\Phi_q\left(\frac{N-\nu}{n}\right)}\sum_{\lambda=0}^{q} H_{j,\lambda}\left(\frac{\nu}{n}\right)z^{\lambda n}\sum_{k=0}^{n-1} z_k^{l-\nu+M}, \quad l\leqq M \end{aligned} \tag{6}$$

where $z_k = e^{ix_k}$, $k=0, 1, \ldots, n-1$. Let

(7) $$\alpha = l + \left[\frac{n}{2}\right] \frac{1+(-1)^q}{2}$$

then for $l \leq \left[\frac{n}{2}\right]$

$$\sum_{k=0}^{n-1} z_k^{l-\nu+M} = \begin{cases} n & \text{if} \quad \nu = \alpha \\ 0 & \text{otherwise.} \end{cases}$$

Thus from (6)

(8) $$z^l - R_n(z^l, x) = \frac{z^{\alpha-M}}{\Phi_q\left(\frac{N-\alpha}{n}\right)} \sum_{j=1}^{q} \left(\frac{l}{n}\right)^{m_j} \sum_{\lambda=0}^{q} H_{j,\lambda}\left(\frac{\alpha}{n}\right) z^{\lambda n}, \quad l \leq \left[\frac{n}{2}\right].$$

Consider the functions

(9) $$\chi_{j,n}(\zeta, z) = \sum_{\lambda=0}^{q} \frac{H_{j,\lambda}\left(\zeta + \frac{1+(-1)^q}{2n}\left[\frac{n}{2}\right]\right) z^{\lambda n}}{\Phi_q\left(\frac{1-(-1)^q}{2} - \zeta\right)}, \quad j = 1, 2, \ldots, q$$

of the complex variable ζ. Here, according to Lemma 4 in [4]

$$\Phi_q\left(\frac{1-(-1)^q}{2}\right) \neq 0.$$

Since Φ_q is a polynomial, there exists a δ, $0 < \delta < \frac{1}{2}$, such that

$$\Phi_q\left(\frac{1-(-1)^q}{2} - \zeta\right) \neq 0 \quad \text{if} \quad |\zeta| \leq 2\delta.$$

Being $H_{j,\lambda}$ also polynomials we can expand $\chi_{j,n}(\zeta, z)$ into the Taylor series

(10) $$\chi_{j,n}(\zeta, z) = \sum_{k=0}^{\infty} a_{jkn}(z)\zeta^k, \qquad |\zeta| \leq 2\delta, \quad j = 1, \ldots, q,$$

where

(11) $$|a_{jkn}(z)| = O((2\delta)^{-k}), \qquad k = 0, 1, \ldots; \quad j = 1, 2, \ldots q; \quad z = e^{ix}$$

Thus (8), (9) and (10) yield

(12) $$\begin{aligned} z^l - R_n(z^l, x) &= z^{\alpha-M} \sum_{j=1}^{q} \left(\frac{l}{n}\right)^{m_j} \sum_{k=0}^{\infty} a_{jkn}(z) \left(\frac{l}{n}\right)^k = \\ &= z^{-n\left[\frac{q+1}{2}\right]} \sum_{j=1}^{q} \sum_{k=0}^{\infty} (in)^{-k-m_j} a_{jkn}(z) \frac{d^{k+m_j}}{dx^{k+m_j}} e^{ilx}, \quad l \leq \left[\frac{n}{2}\right], \end{aligned}$$

since by (7) and (1), $\alpha-M=l-n\left[\frac{q+1}{2}\right]$. Now if $T\in\mathscr{T}_{[\delta n]}$, then

$$w(x) = T(x)+i\tilde{T}(x) = \sum_{l=0}^{[\delta n]} c_l e^{ilx}$$

with certain complex coefficients c_l. Thus from (12) we obtain

$$w(x)-R_n(w, x) = z^{-n\left[\frac{q+1}{2}\right]} \sum_{j=1}^{q} \sum_{k=0}^{\infty} (in)^{-k-m_j} a_{jkn}(z)\, w^{(k+m_j)}(x), \tag{13}$$

and by the Bernstein—Szegö inequality

$$\|T(x)-R_n(T, x)\| \leqq \|w(x)-R_n(w, x)\| \leqq$$

$$\leqq \sum_{j=1}^{q} \sum_{k=0}^{\infty} n^{-k-m_j} |a_{jkn}(z)| \{\|T^{(k+m_j)}\| + \|\tilde{T}^{(k+m_j)}\|\} =$$

$$= \sum_{j=1}^{q} \sum_{k=0}^{\infty} n^{-k-m_j} (2\delta)^{-k} (\delta n)^{k+m_j-m_1} \{\|T^{(m_1)}\| + \|\tilde{T}^{(m_1)}\|\} =$$

$$= O(n^{-m_1}) \sum_{j=1}^{q} \delta^{m_j-m_1} \sum_{k=0}^{\infty} 2^{-k} \{\|T^{(m_1)}\| + \|\tilde{T}^{(m_1)}\|\} =$$

$$= O(n^{-m_1}) \{\|T^{(m_1)}\| + \|\tilde{T}^{(m_1)}\|\}.$$

This proves Lemma 1.

Our main result, which is a generalization of Theorem 1 in [7] is as follows.

THEOREM 1. *Assume that M_q and n are such that the corresponding lacunary interpolation problem is uniquely solvable. Then there exists a δ, $0<\delta<1$, depending only on M_q such that*

$$\|f(x)-R_n(f, x)\| = A_n E_{[\delta n]}(f) + O(n^{-m_1}) \sum_{k=0}^{n} (k+1)^{m_1-1} E_k(f) \quad (f\in C_{2\pi})$$

where

$$A_n = \begin{cases} 0 & \text{if} \quad \sum\limits_{m_j\in M_q \text{ even}} 1 - \sum\limits_{m_j\in M_q \text{ odd}} 1 = -1, \\ O(\log n) & \text{if} \quad \sum\limits_{m_j\in M_q \text{ even}} 1 - \sum\limits_{m_j\in M_q \text{ odd}} 1 = 0, \\ O(n) & \text{if} \quad \sum\limits_{m_j\in M_q \text{ even}} 1 - \sum\limits_{m_j\in M_q \text{ odd}} 1 = 1. \end{cases} \tag{14}$$

PROOF. Let δ be the number obtained from Lemma 1, and let $T\in\mathscr{T}_{[\delta n]}$ be the best approximating polynomial of $f(x)$. Then, according to Lemma 2 in [7]

$$\|T^{(m_1)}\| + \|\tilde{T}^{(m_1)}\| = O\Big(\sum_{k=0}^{n} (k+1)^{m_1-1} E_k(f)\Big).$$

Combining this with our Lemma 1, we obtain

$$\|f(x)-R_n(f,x)\| \leqq \|f(x)-T(x)\|+\|T(x)-R_n(T,x)\|+\|R_n(T-f,x)\|=$$

$$=(1+\|R_n\|)E_{[\delta n]}(f)+O(n^{-m_1})\sum_{k=0}^{n}(k+1)^{m_1-1}E_k(f).$$

Here $\|R_n\|$ is $O(1)$, $O(\log n)$ and $O(n)$, respectively, according to the cases listed in (14) (see [4] Lemmas 6, 7, 8 and [6] Lemma 7). We only have to remark that in case $\|R_n\|=O(1)$

$$E_{[\delta n]}(f)=O(n^{-m_1})\sum_{k=0}^{n}(k+1)^{m_1-1}E_k(f),$$

and thus we can take $A_n=0$. Q. E. D.

When the function is finitely many times differentiable, the differentiated series $R_n^{(r)}(f,x)$ will converge to the corresponding derivative. This is expressed by the following

THEOREM 2. *Under the conditions of Theorem 1, if* $f^{(p)}(x)\in C_{2\pi}$ *then*

$$\|f^{(r)}(x)-R_n^{(r)}(f,x)\|=O\left\{A_nE_{[\delta n]}(f^{(r)})+n^{-m_1}\sum_{k=0}^{n}(k+1)^{m_1-1}E_k(f^{(r)})\right\}$$

$$(r=0,1,\ldots,p).$$

PROOF. This is an easy consequence of Theorem 1, the relation $E_k(f)=O(k^{-r}E_k(f^{(r)}))$ and a result of J. Czipszer and G. Freud [2] which states that for $f^{(p)}\in C_{2\pi}$, $T\in\mathcal{T}_n$ one has

$$\|f^{(r)}(x)-T^{(r)}(x)\|=O(n^{(r)})\|f(x)-T(x)\|+4E_n(f^{(r)}),\qquad r=0,1,\ldots,p.\qquad \text{Q. E. D.}$$

Under certain assumption on the function, we can give a better estimate than the one obtained from Theorem 1. The next result can be considered as a generalization of Theorem 1 in [8] and Theorem 3 in G. Sunouchi [9]. See also the contributions made in [11] concerning $(0,m)$ trigonometric interpolation.

THEOREM 3. *If* $f(x)$ *and* $\tilde{f}(x)$ *are in* $C_{2\pi}$, *and*

$$\sum_{m_j\in M_q\text{ even}}1-\sum_{m_j\in M_q\text{ odd}}1=-1,$$

then

$$\|f(x)-R_n(f,x)\|=O\left\{w_{m_1}\left(f,\frac{1}{n}\right)+w_{m_1}\left(\tilde{f},\frac{1}{n}\right)\right\}$$

where w_{m_1} *denotes the* m_1^{th} *modulus of smoothness.*

PROOF. It is well-known that the de la Vallée-Poussin means $\tau_k(f,x)$ have the property

$$\|f(x)-\tau_k(f,x)\|\leqq 4E_k(f)=O\left(w_{m_1}\left(f,\frac{1}{k}\right)\right), \tag{15}$$

by the generalized Jackson theorem. Also

$$\|\tilde{f}(x)-\tilde{\tau}_k(f,x)\| = O\left(w_{m_1}\left(\tilde{f},\frac{1}{k}\right)\right).$$

Using the estimates

$$k^{-m_1}\|\tau_k^{(m_1)}(f,x)\| = O\left(w_{m_1}\left(\tau_k,\frac{1}{k}\right)\right) = O\left\{w_{m_1}\left(\tau_k-f,\frac{1}{k}\right)+w_{m_1}\left(f,\frac{1}{k}\right)\right\} =$$
$$= O\left(w_{m_1}\left(f,\frac{1}{k}\right)\right)$$

(cf. e.g. A. F. Timan [10], formula 4.8 (18)) as well as

$$k^{-m_1}\|\tilde{\tau}_k^{(m_1)}(f,x)\| = O\left(w_{m_1}\left(\tilde{f},\frac{1}{k}\right)\right)$$

we get from Lemma 1

$$\|\tau_{[\delta n]}(f,x)-R_n(\tau_{[\delta n]},x)\| = O\left(w_{m_1}\left(f,\frac{1}{n}\right)+w_{m_1}\left(\tilde{f},\frac{1}{n}\right)\right). \tag{16}$$

Hence if we write

$$f(x)-R_n(f,x) = f(x)-\tau_{[\delta n]}(f,x)+\tau_{[\delta n]}(f,x)-R_n(\tau_{[\delta n]},x)+R_n((\tau_{[\delta n]}-f,x)),$$

and use (15), (16) and the fact that by our assumptions $\|R_n\|=O(1)$ we get the statement.

Theorem 3 helps to solve the saturation problem of our operator under the stated assumptions. Again, the following result can be considered as a generalization of Theorems 1 and 2 in [7].

THEOREM 4. *Under the condition of Theorem 3 on M_q,*

(a) $\|f(x)-R_n(f,x)\| = O(n^{-m_1})$ *iff* $f^{(m_1-1)}(x)$ *and* $\tilde{f}^{(m_1-1)}(x)\in \mathrm{Lip}\,1$;

(b) $\|f(x)-R_n(f,x)\| = o(n^{-m_1})$ *iff* $f(x) = \text{const}$.

PROOF. (a) If $f^{(m_1-1)}(x)$ and $\tilde{f}^{(m_1-1)}(x)\in \mathrm{Lip}\,1$ then by Theorem 3 we have

$$\|f(x)-R_n(f,x)\| \leqq Kn^{-m_1} \qquad (K>0). \tag{17}$$

Conversely, assume that (17) holds. Let

$$u_0(x) = R_1(f,x), \quad u_k(x) = R_{2^k}(f,x)-R_{2^{k-1}}(f,x), \quad k = 1, 2, \dots. \tag{18}$$

Then by (17), with an absolute constant c we get

$$\|u_k\| \leqq cK2^{-(k-1)m_1}, \qquad k = 0, 1, \dots, \tag{19}$$

and hence by the Bernstein—Szegö inequality

$$\max\{\|u_k^{(j)}\|, \|\tilde{u}_k^{(j)}\|\} \leqq cK2^{k(j-m_1)+m_1}, \qquad j = 1, 2, \dots, m_1+1, \quad k = 1, 2, \dots.$$

Since by (18) $R_{2^s}(f, x) = \sum_{k=0}^{s} u_k(x)$, we obtain

$$(20)\quad \|R_{2^s}^{(m_1+1)}(f, x)\| \leqq \sum_{k=0}^{s} \|u_k^{(m_1+1)}\| \leqq cK2^{m_1} \sum_{k=0}^{s} 2^k < cK2^{m_1+s+1}, \quad s = 1, 2, \ldots$$

and similarly

$$(21)\qquad \|\tilde{R}_{2^s}^{(m_1+1)}(f, x)\| \leqq cK2^{m_1+s+1}, \qquad s = 1, 2, \ldots.$$

Now let $x \in [0, 2\pi)$ be arbitrary, and $0 < k \leqq 2^s$ such that $|x - x_{k, 2^s}| \leqq \pi 2^{-s}$. Then (2), (21) and the mean value theorem yield

$$|R_{2^s}^{(m_1)}(f, x)| = |R_{2^s}^{(m_1)}(f, x) - R_{2^s}^{(m_1)}(f, x_{k, 2^s})| \leqq$$

$$\leqq |x - x_k| \|R_{2^s}^{(m_1+1)}(f, x)\| \leqq cK\pi 2^{m_1+1},$$

i.e.

$$(22)\qquad \|R_{2^s}^{(m_1)}(f, x)\| \leqq cK\pi 2^{m_1+1}, \qquad s = 1, 2, \ldots.$$

Since by (17) and (18)

$$(23)\qquad f(x) = \sum_{k=0}^{\infty} u_k(x)$$

and by (19) the series $\sum_{k=1}^{\infty} u_k^{(m_1-1)}(x)$ uniformly converges, evidently

$$f^{(m_1-1)}(x) = \sum_{k=0}^{\infty} u_k^{(m_1-1)}(x).$$

With respect to (18), this means that

$$\lim_{s \to \infty} \|f^{(m_1-1)}(x) - R_{2^s}^{(m_1-1)}(f, x)\| = 0.$$

Thus if $h > 0$ is arbitrary then by (22) we get

$$(24)\qquad \begin{aligned} &|f^{(m_1-1)}(x+h) - f^{(m_1-1)}(x)| \leqq |f^{(m_1-1)}(x+h) - R_{2^s}^{(m_1-1)}(f, x+h)| + \\ &+ |R_{2^s}^{(m_1-1)}(f, x+h) - R_{2^s}^{(m_1-1)}(f, x)| + |R_{2^s}^{(m_1-1)}(f, x) - f^{(m_1-1)}(x)| \leqq cK2^{m_1+3}h, \end{aligned}$$

provided s is large enough. This proves that $f^{(m_1-1)}(x) \in \mathrm{Lip}\ 1$.

In order to prove the same for the conjugate function, we apply (13) for

$$(25)\qquad w(x) = R_m(f, x) + i\tilde{R}_m(f, x), \quad \text{where} \quad m = 2^{\left[\log \frac{2\delta n}{q+1}\right]}$$

(for large enough n). We obtain by (13), (25), (17), (11) and the Bernstein—Szegö inequality

$$n^{-m_1}|\mathrm{Re}\{z^{-n\frac{q+1}{2}}a_{10n}(z)w^{(m_1)}(x)i^{-m_1}\}| \leqq$$

$$\leqq (\|R_m(f,x)-R_n(R_m(f),x))\| + n^{-m_1}\sum_{k=1}^{\infty} n^{-k}|a_{1kn}(z)|\,\|w^{k+m_1}(x)\| +$$

$$+\sum_{j=2}^{q}\sum_{k=0}^{\infty} n^{-k-m_j}|a_{jkn}(z)|\,\|w^{(k+m_j)}(x)\| =$$

$$= \|R_m(f,x)-f(x)\| + \|f(x)-R_n(f,x)\| + \|R_n(f-R_m(f),x)\| +$$

$$+n^{-m_1}\sum_{k=1}^{\infty} n^{-k}(2\delta)^{-k}\left(\frac{2\delta n}{q+1}\right)^k \|R_m^{(m_1)}(f,x)\| +$$

$$+\sum_{j=2}^{q}\sum_{k=0}^{\infty} n^{-k-m_j}(2\delta)^{-k}\left(\frac{2\delta n}{q+1}\right)^{k+m_j-m_1}\|R_m^{(m_1)}(f,x)\| =$$

$$= O(n^{-m_1})+O(n^{-m_1})\sum_{k=1}^{\infty}(k+1)^{-q}+O(n^{-m_1})\sum_{j=2}^{q}(2\delta)^{m_j}\sum_{k=0}^{\infty}(q+1)^{-k},$$

i.e.

$$|\mathrm{Re}\{i^{-m_1}z^{-n\frac{(q+1)}{2}}a_{10n}(z)w^{(m_1)}(x)\}| = O(1). \tag{26}$$

Here by (9) and (10) we get for $z=z_k e^{\frac{ia}{n}}$, $k=0, 1, \dots, n-1$,

$$z^{-n\left(\frac{q+1}{2}\right)}a_{10n}(z) = e^{-ia\frac{q+1}{2}}(\psi_q(0))^{-1}\sum_{\lambda=0}^{q} H_{1\lambda}(0)e^{i\lambda a} =$$

$$= (\psi_q(0))^{-1}\{H_{1,\frac{q+1}{2}}(0)+2\sum_{\lambda=1}^{(q-1)/2}[H_{1,\lambda+\frac{q+1}{2}}(0)+H_{1,-\lambda+\frac{q+1}{2}}(0)]\cos\lambda a +$$

$$+H_{10}(0)\cos\frac{q+1}{2}a +$$

$$+i[2\sum_{\lambda=1}^{(q-1)/2}[H_{1,\lambda+\frac{q+1}{2}}(0)-H_{1,-\lambda+\frac{q+1}{2}}(0)]\sin\lambda a - H_{1,0}(0)\sin\frac{q+1}{2}a)],$$

where a is an arbitrary real number. Since here

$$H_{1,0}(0) = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 2^{m_2} & \dots & (q-1)^{m_2} \\ \cdot & \cdot & \cdot & \dots & \dots \\ 0 & 1 & 2^{m_q} & \dots & (q-1)^{m_q} \end{vmatrix} = \begin{vmatrix} 1 & 2^{m_2} & \dots & (q-1)^{m_q} \\ \cdot & \cdot & \dots & \dots \\ \cdot & \cdot & \dots & \dots \\ 1 & 2^{m_q} & \dots & (q-1)^{m_q} \end{vmatrix} \neq 0$$

(cf. Lemma 1 in [4]), the real and imaginary parts on the right-hand side of (27) as trigonometric polynomials in a are not identically constants. Hence there exists an a_0 such that

$$i^{-m_1} z^{-n\frac{q+1}{2}} a_{10n}(z) = \mu - i\nu, \quad \nu \neq 0, \quad z = z_k e^{i\frac{a_0}{n}}, \quad k = 0, 1, \ldots, n-1.$$

This means (see (25) and (26)) that

$$|\mu R_m^{(m_1)}(f, x) - \nu \tilde{R}_m^{(m_1)}(f, x)| = O(1) \quad \text{for} \quad x = x_k + \frac{a_0}{n}, \quad k = 0, 1, \ldots, n-1,$$

i.e. by (22)

$$\tilde{R}_m^{(m_1)}(f, y_k) = O(1) \quad \text{for} \quad y_k = x_k + \frac{a_0}{n}, \quad k = 0, 1, \ldots, n-1. \tag{28}$$

Now if $x \in [0, 2\pi)$ is arbitrary and y_k is such that $|x - y_k| \leqq \frac{\pi}{n}$, then by (28) and the mean value theorem

$$|\tilde{R}_m^{(m_1)}(f, x)| \leqq |\tilde{R}_m^{(m_1)}(f, x) - \tilde{R}_m^{(m_1)}(f, y_k)| + |\tilde{R}_m^{(m_1)}(f, y_k)| \leqq$$

$$\leqq |x - y_k| \, \|\tilde{R}_m^{(m_1+1)}(f, x)\| + O(1) = O\left(\frac{1}{n}\right) \frac{2\delta n}{q+1} + O(1) = O(1),$$

i.e.

$$\|\tilde{R}_m^{(m_1)}(f)\| = O(1).$$

By (19), $\|\tilde{u}_k\| = O(k 2^{-k m_1})$; hence the series $\sum_{k=0}^{\infty} \tilde{u}_k(x)$ uniformly converges. Thus by (23)

$$\tilde{f}(x) = \sum_{k=0}^{\infty} \tilde{u}_k(x)$$

exists. From this point the proof is the same as that for $f(x)$, and we readily conclude that $\tilde{f}^{(m-1)}(x) \in \text{Lip } 1$.

(b) Since R_n reproduces constants, the "if" part of the statement is obvious. Now if

$$\|f(x) - R_n(f, x)\| = o(n^{-m_1}),$$

then from the proof of part (a) follows that (24) holds with an arbitrary $K > 0$; hence $f^{(m_1-1)}(x) = \text{constant}$, i.e. $f(x) = \text{constant}$. Q. E. D.

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(Received October 17, 1985)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 225—229

ON COPS AND ROBBER GAME

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Let G be a finite connected graph and $c(G)$ the cop number of G, i.e. the minimum number of cops needed to catch the robber in the cops and robber game on G. It is shown that $c(G_1+G_2)\leqq c(G_1)+c(G_2)$, where G_1+G_2 is the cartesian product of G_1 and G_2. As a consequence we obtained that for each natural k there is a natural n such that $c(Q_n)=k$, where Q_n is n-cube.

1. Introduction

A finite connected graph G is given and two players, C and R, play the following game: C chooses some vertices and put m white pebbles (cops) on them (more than one cop may be placed on the same vertex). Then R chooses a vertex and put a black pebble (robber) on it. Then the players move alternately beginning with C. A move of C consists of choosing k cops, $0<k\leqq m$, and moving each of them along an edge of G to an adjacent vertex. A move of R consists of moving the robber along an edge of G to an adjacent vertex. Each move is seen by both players. C wins if he manages to occupy the same vertex as R after a finite number of moves, and R wins if he avoids this forever.

In some variants of this game each player is allowed to omit his move, i.e. all pebbles may sit still in their places. In that case it is convenient to regard all graphs as reflexive, i.e. equipped with loops at every vertex.

Since there is complete information in this game, either C or R must have a winning strategy: in the former case G will be called an (m cops)-win graph, otherwise G is robber(m)-win.

Let $c(G)$ denote the cop number of G, i.e. the minimum number of cops needed to catch the robber in a given graph G after a finite number of moves. In [1] and [2], the graphs for which $c(G)=1$ are characterized. In [1], it is proved that for each natural m there is a graph G for which $c(G)\geqq m$, and that for each planar graph G, $c(G)\leqq 3$.

1980 *Mathematics Subject Classification.* Primary 90D42; Secondary 90D26, 90D05.
Key words and phrases. Cops and robber game.

2. On cop number of cartesian product of graphs

Let $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ are two finite connected graphs. Denote by G_1+G_2 the sum of the graphs G_1 and G_2 (somewhere called cartesian product), i.e.:

$$G = G_1+G_2 = (V, E),$$

where

$$V = V_1 \times V_2$$

and

$$E = \{(x_1, y_1)(x_2, y_2) | (x_1 x_2 \in E_1 \text{ and } y_1 = y_2) \text{ or } (x_1 = x_2 \text{ and } y_1 y_2 \in E_2)\}.$$

Then we nave the following result, which can be naturally extended to the sum of more than two graphs:

THEOREM 1. *If G_1 and G_2 are two finite connected graphs, then*

$$c(G_1+G_2) \leq c(G_1)+c(G_2).$$

PROOF. Let $G_1=(V_1, E_1)$, $G_2=(V_2, E_2)$, where $V_1=\{x_1, x_2, \ldots, x_p\}$ and $V_2=\{y_1, y_2, \ldots, y_q\}$. Suppose that $c(G_1)=m$, $c(G_2)=n$, and that the player C has at his disposal $m+n$ cops. Then he can catch the robber in the graph $G=G_1+G_2$ in the following way.

Let the vertices $x_{i_1}, x_{i_2}, \ldots, x_{i_m}$ (not necessarily different) make a starting position in a winning strategy of the player C on the graph G_1 and the vertices $y_{j_1}, y_{j_2}, \ldots, y_{j_n}$ make a starting position in a winning strategy of the player C on the graph G_2. (Of course, if a graph is (m cops)-win, then any set of at most m vertices can serve as a starting position in a winning strategy of the player C, but the theorem also holds for digraphs. The theorem holds independently of the fact whether the graph is reflexive or not.) Denote by G_{x_k} the copy of G_2 induced by the vertices (x_k, y_1), (x_k, y_2), ..., (x_k, y_q) of the graph $G=G_1+G_2$. Similarly, G_{y_h} is the copy of G_1 induced by the vertices (x_1, y_h), (x_2, y_h), ..., (x_p, y_h) of the graph G.

First, the player C puts n cops in the vertices (x_{i_1}, y_{j_1}), (x_{i_1}, y_{j_2}), ..., (x_{i_1}, y_{j_n}) of the copy $G_{x_{i_1}}$ of the graph G_2 and begins to play applying his winning strategy for the graph G_2. This strategy enables him to cath at least the second coordinate of the robber. If the robber is not in $G_{x_{i_1}}$, then from this moment on one cop stays in this copy as the "shadow" of the robber, i.e. he always goes to the vertex (x_{i_1}, y_r) whenever the robber goes to some vertex (x_l, y_r).

Now, the player C repeats the procedure with the copies $G_{x_{i_2}}, G_{x_{i_3}}, \ldots, G_{x_{i_m}}$. After that, the player C achieves the position in which he has m cops placed in m vertices (x_{i_1}, y_r), (x_{i_2}, y_r),)..., (x_{i_m}, y_r) while the robber is in a vertex (x_l, y_r). Denote these m cops by $C_1, C_2, \ldots, C_m$, and the remaining n cops by $C_1', C_2', \ldots, C_n'$.

From now on, whenever the player R by his move changes the second coordinate of the vertex, so will do the cops $C_1, C_2, \ldots, C_m$, all staying in the same copy G_{y_t} of G_1 with the robber. If, being in a copy G_{y_r}, the robber by his move changes the first coordinate (staying in the same copy of G_1 and changing the copy of G_2), the player C moves the cops $C_1, C_2, \ldots, C_m$, according to the winning strategy for the graph G_1.

The robber must, from time to time, to change the copy of G_2 by his move, i.e. to make a move in the game on G_1. On the contrary, if the robber stays in a copy G_{x_l} of G_2 long enough, the cops $C_1', C_2', \ldots, C_n'$ will come into the copy G_{x_l} and either to catch the robber by applying a winning strategy for the graph G_2 or to "push" him into another copy of G_2, i.e. to force him to make a move in the game on G_1.

So, $m+n$ cops are sufficient for winning in the graph $G=G_1+G_2$. ∎

3. Cop number of n-cube

Let $c'(Q_n)$ denote the minimum number of cops needed to catch the robber in the irreflexive n-cube Q_n (the players are not allowed to omit the moves). Then we have the following consequence of Theorem 1.

COROLLARY 1.

$$c'(Q_n) \leqq \left\lceil \frac{n}{2} \right\rceil, \quad \textit{if} \quad n \not\equiv 2 \pmod 4,$$

$$c'(Q_n) \leqq \left\lceil \frac{n}{2} \right\rceil + 1, \quad \textit{if} \quad n \equiv 2 \pmod 4.$$

PROOF. It can be checked that $c'(Q_1)=1$, $c'(Q_2)=c'(Q_3)=c'(Q_4)=2$. Having in mind that $Q_n=Q_k+Q_{n-k}$, for $k=1, 2, \ldots, n-1$, we consider the sequence

$$a_1 = 1, \quad a_2 = a_3 = a_4 = 2; \; a_n = \min_{1 \leqq k \leqq n-1} (a_k + a_{n-k}), \quad \text{for} \quad n > 4.$$

We conclude that

$$a_n = \begin{cases} \left\lceil \frac{n}{2} \right\rceil, & \text{if} \quad n \not\equiv 2 \pmod 4 \\ \left\lceil \frac{n}{2} \right\rceil + 1, & \text{if} \quad n \equiv 2 \pmod 4, \end{cases}$$

hence follows the statement. ∎

On the other hand, for reflexive n-cube Q_n we have the following consequence of Theorem 1.

COROLLARY 2.

$$c(Q_n) \geqq \left\lceil \frac{n+1}{2} \right\rceil.$$

PROOF. It is sufficient to prove that $k<c(Q_{2k})$; then $k<c(Q_{2k+1})$ follows immediately.

Suppose that the player C put his k cops in some vertices of Q_{2k}. All cops together cover at most $k(2k+1)$ vertices. Since $k(2k+1)<2^{2k}$, for natural k, there is an uncovered vertex u (neither occupied by a cop nor adjacent to an occupied vertex) on which the player R can put the robber. Each vertex adjacent to the vertex u can be attacked only by the cops which are at the distance 2 from the vertex u. Each cop can cover at most two vertices adjacent to the vertex u. If there are not cops at all on the vertices adjacent to the vertex u, the player R will omit his move when in turn. At the moment when at least one cop come to some vertex adjacent to u, at least one adjacent vertex will not be covered by any cop. Now the robber can escape to that uncovered vertex. Playing in this manner, the robber avoids catching forever. ∎

Similarly, for irreflexive n-cube Q_n we obtain the following result.

COROLLARY 3.

$$\left\lceil \frac{n}{2} \right\rceil \leqq c'(Q_n). \quad ∎$$

Combining corollaries 1 and 3, we obtain the following theorem.

THEOREM 2.

$$c'(Q_n) = \left\lceil \frac{n}{2} \right\rceil, \quad \textit{if} \quad n \not\equiv 2 \pmod 4,$$

$$\left\lceil \frac{n}{2} \right\rceil \leqq c'(Q_n) \leqq \left\lceil \frac{n}{2} \right\rceil + 1, \quad \textit{if} \quad n \equiv 2 \pmod 4. \quad ∎$$

LEMMA 1.

$$c(Q_{n+1}) \leqq c'(Q_n) + 1.$$

PROOF. We consider a given cube Q_{n+1} as the cartesian product $Q_n + Q_1$, where the set of vertices of Q_1 is $\{1, 2\}$. Let $c'(Q_n) = a$ and suppose that the player C has at his disposal $a+1$ cops $C_0, C_1, \ldots, C_a$. The player C begins by applying a winning strategy in irreflexive cube Q_{n1}, i.e. in a copy of Q_n, playing with cops $C_1, C_2, \ldots, C_a$. His aim is to catch the first coordinate of the robber. Each time when the player R omits his move in this game, the player C will also omit his move, i.e. all cops C_1, $C_2, \ldots, C_a$ will sit still in their places, but the cop C_0 will move toward the robber. After several moves, the cop will either to catch the robber or to force him to make a move in the irreflexive game on Q_n. In such a way the player C will catch at least the first coordinate of the robber. If the robber is not catched completely, then from now on a cop will move as a "shadow" of the robber, while the other cops will go to the copy Q_{n2} of Q_n in Q_{n+1} and continue the game by applying a winning strategy in irreflexive game on Q_n. Now, the player R cannot omit his moves because in that case the "shadow" will catch the robber immediately . ∎

Combining Corollaries 1 and 2 and Lemma 1, we obtain the following statement.

THEOREM 3.

$$c(Q_n) = \left\lceil \frac{n+1}{2} \right\rceil, \quad \textit{if} \quad n \not\equiv 3 \pmod 4,$$

$$\left\lceil \frac{n+1}{2} \right\rceil \leqq c(Q_n) \leqq \left\lceil \frac{n+1}{2} \right\rceil + 1, \quad \textit{if} \quad n \equiv 3 \pmod 4. \quad ∎$$

From Theorem 2 (3) it follows that for each natural k there is a natural n such that $c'(Q_n) = k \left(c(Q_n) = k\right)$.

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(Received October 18, 1985)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 231—235

RADICAL CLASSES AND SEMISIMPLE CLASSES FOR HEMIRINGS

S. M. YUSUF and M. SHABIR

D. M. Olson and T. L. Jenkins [3] discussed Radical Theory for Hemirings. In this paper we have obtained some results on semisimple classes of hemirings. Using the structure of ideals in Euclidean hemirings [2] we have shown that a radical class P defined for hemirings either contains all Euclidean hemirings or contains only the Euclidean hemiring 0.

A *hemiring* is a semiring $(S, +, \cdot)$ with two additional properties:
(1) $(S, +)$ is a commutative semigroup with identity 0.
(2) $0 \cdot a = a \cdot 0 = 0$ for all $a \in S$.

We shall write hemiring S instead of hemiring $(S, +, \cdot)$.

A non-empty subset I of a hemiring S is called a *left semi-ideal if*
(i) $a, b \in I \Rightarrow a+b \in I$
(ii) $a \in I$ and $s \in S \Rightarrow sa \in I$.

If $I \neq S$, it is called a *proper* left semi-ideal of S.

Right semi-ideals and *two-sided semi-ideals* are defined in the usual manner.

The following relation $\sim$ yields a quotient hemiring modulo any semi-ideal I:
For all $a, b \in S$, $a \sim b \pmod I$ if $a+i_1 = b+i_2$ for some $i_1, i_2 \in I$. The quotient hemiring is denoted by S/I. Here I is not a congruence class but is contained in the congruence class $[i]$, where $i \in I$. However, when I is a k-ideal, we have $[i] = I$.

A semi-ideal I of a hemiring S is called a *k-ideal* of S if for every $s \in S$, whenever $s+i \in I$ for some $i \in I$, then $s \in I$. Not all semi-ideals are k-ideals Given a semi-ideal A of a hemiring S, there exist a minimal k-ideal of S containing A and is denoted by A^*. It can be verified that

$$A^* = \{x \in S \mid x+a \in A \text{ for some } a \in A\}.$$

Definition for homomorphism from a hemiring S into a hemiring T and for kernel of a homomorphism are analogous to those used for rings. However, we require additional condition that they preserve the additive identity.

The importance of k-ideals is that they are kernels of homomorphisms and that $S/A = S/A^*$ for any semi-ideal A of S.

1980 *Mathematics Subject Classification.* Primary 16A78; Secondary 16A21.
Key words and phrases. Hemirings, radical classes, semisimple classes.

As pointed out in [3], a homomorphism with zero kernel need not be injective. Such homomorphisms are called *semi-isomorphisms* and are symbolized by $\xrightarrow{\simeq}$, where the arrow is used to indicate the direction of the homorphism. Olson and Jenkins [3] have stated without proof the following results:

(1) *If* $\Phi: S \to T$ *is an epimorphism with kernel* K, *then* $S/K \xrightarrow{\simeq} T$.
(2) *If* I *and* J *are semi-ideals in* S, *then* $I/I \cap J^* \xrightarrow{\simeq} I+J/J$.
(3) *If* $\eta: S \to S/I$ *is the natural homomorphism, then* $\ker \eta = I^*$.
(4) *If* A *and* B *are semi-ideals of* S *with* $A \subseteqq B$, *then* $S/B \cong S/A/B/A$.

The result (4), as stated, is not necessarily true.

EXAMPLE. Let S be the hemiring defined by

$+$	0	e	f	g	h
0	0	0	0	0	0
e	0	e	0	e	h
f	0	0	f	f	0
g	0	e	f	g	h
h	0	h	0	h	h

and by $xy = g$ for all $x, y \in S$. $A = \{g, e\}$, $B = \{g, e, f\}$ are semi-ideals of S and $A \subset B$. Here

$$S/A = \{\{h\}, \{0, f\}, \{g, e\}\}$$

and

$$B/A = \{\{f\}, \{g, e\}\}$$

and $B/A \subsetneqq S/A$. Thus, $S/A/B/A$ is not meaningful.

The following form of result (4) is true:

(4) *If* A *and* B *are semi-ideals of* S *with* $A \subseteqq B$, *then* $S/B \cong S/A/B^*/A$.

This correction has to be applied to all the results where the authors in [3] have used the incorrect form of the result (4).

Radical and semisimple classes

A radical class is defined in [3] as follows:

ASSUMPTIONS. (1) All hemirings mentioned belong to a class μ of hemirings with the following two properties:

(a) If S is a hemiring in μ, then every homomorphic image of S is in μ.
(b) If S is a hemiring in μ and I is a semi-ideal of S, then I is a hemiring in μ.

(2) All homomorphisms will be considered to preserve only the two hemiring operations and the additive identity.

A non-empty subclass P of hemirings μ is called a *radical class* if

(R1) P is homomorphically closed.
(R2) If $A \notin P$, then A contains a proper k-ideal K such that A/K has no non-zero P-semi-ideals (semi-ideals which as hemirings are in the class P).

Using the results of [3], the radical class P can also be characterized as follows:

P is a radical class if and only if P has properties (R1), (R3) and (R4), where

(R3) Every hemiring S has a P-semi-ideal M which is a k-ideal of S and contains every other P-semi-ideal of S.
(R4) S/M has no non-zero P-semi-ideal.

The following characterization of radical classes, though not included in [3] can be proved as in the theory of rings.

P is a radical class if and only if P satisfies (R1), (R3) and (R5) where

(R5) *If I is a semi-ideal of S such that S/I and I are in P, then $S \in P$.*

As usual, the semisimple class SP of a radical class P is defined as the class of all hemirings having zero P-radical.

In the case of hemirings one can easily prove all the standard result concerning radical classes, upper radicals, lower radicals, semisimple classes and semisimple closures. Here we recall only one of them.

THEOREM 1. *Every semisimple class S is hereditary: if I is a semi-ideal of an S-hemiring A, then also I is an S-hemiring.*

LEMMA. *Let A be a hemiring and J and K be semi-ideals of A then A/K can be mapped onto A/J naturally iff $K \subseteqq J^*$.*

PROOF. Let $[x]_J, [x]_K$ be the congruence classes corresponding to semi-ideals J and K, respectively. Suppose A/K is mapped homomorphically onto A/J. Let $x \in K$ then $[x]_K = K^*$ is mapped onto the zero-element of A/J, i.e., $[x]_K \to J$ implies $x \in J^*$ and hence $K \subseteqq J^*$.

Conversely, if $K \subseteqq J^*$ then

$$S/K/J^*/K \cong S/J^* \cong S/J$$

and consequently S/K is mapped onto S/J.

Let $\{I_\alpha\}_{\alpha \in A}$ be a family of semi-ideals. Then the semi-ideal $\sum_{\alpha \in A} I_\alpha$ generated by $\bigcup_{\alpha \in A} I_\alpha$ has the following characteristic properties:

i) Every I_α is contained in the semi-ideal $\sum_{\alpha \in A} I_\alpha$.

ii) If every I is contained in a semi-ideal A then also

$$\sum_{\alpha \in A} I_\alpha \subseteqq A.$$

Now we define the union of factor hemirings dually to be semi-ideal $\sum_{\alpha \in A} I_\alpha$ as follows:

The union of factor hemirings A/J_α of a hemiring A is defined as the factor hemiring $A/\bigcap_{\alpha\in A} J_\alpha^*$. The hemiring $A/\bigcap_\alpha J_\alpha^*$ is obviously characterized by the properties:

i) $A/\bigcap_{\alpha\in A} J_\alpha^*$ can be mapped onto every A/J_α by natural homomorphism.

ii) If a factor hemiring A/K can be mapped onto every factor hemiring A/J_α by natural homomorphism then it can be mapped onto $A/(\bigcap_\alpha J_\alpha^*)$.

We shall call a factor hemiring an *$\mathcal{S}$-factor hemiring* if it is an S-hemiring.

THEOREM 2. *A class S of hemirings is a semi-simple class if and only if S satisfies*

S_1') *S is hereditary.*

S_2') *Every hemiring A has an $\mathcal{S}$-factor hemiring $(A)\mathcal{S}$ which can be mapped onto every $\mathcal{S}$-factor hemiring of A by a natural homomorphism.*

S_3') *The kernel of the mapping $A\to(A)\mathcal{S}$ has no non-zero S-factor hemiring.*

PROOF. The Lemma enables us to follow the line of the proof of the corresponding result in the theory of rings (cf. [4] Theorem 8.3).

Euclidean hemirings

Let S be a commutative hemiring with identity 1. The set $S_p=\{x\in S|$ there exists $y\in S$ such that $x=y+1\}\cup\{0\}$ is called the *principal part* of S. A commutative hemiring S with identity is called a *principal hemiring* if $S=S_p$. A *Euclidean hemiring* E is a principal hemiring with a function $\Phi\colon E\to Z^+$ satisfying the following properties:

(i) for $a\in E$, $\Phi(a)=0$ if and only if $a=0$,
(ii) for all $a, b\in E$, if $a+b\neq 0$, then $\Phi(a+b)\geqq\Phi(a)$,
(iii) for all $a, b\in E$, $\Phi(ab)=\Phi(a)\,(b)$,
(iv) for all $a, b\in E$, $b\neq 0$, there exists $q, r\in E$ such that $a=qb+r$, where $r=0$ or $\Phi(r)<\Phi(b)$.

Let E be a Euclidean hemiring, $a\in E$ and $T_a=\{x\in E|\Phi(x)\geqq\Phi(a)\}\cup\{0\}$. It has been proved in [2] that each T_a is a semi-ideal and for each $b\in E$, bT_a is also a semi-ideal.

THEOREM 3. *No non-zero proper semi-ideal in a Euclidean hemiring is a k-ideal.*

PROOF. Let A be a non-zero proper semi-ideal in a Euclidean hemiring E. Then by [2, Theorem 14], we have

$$A = L\cup dT_p$$

where $d\in E$ and dT_p is maximal in A, $L=\{t\in A|\Phi(t)<\Phi(dp)\}$ and $L\cap dT_p=0$.

Let $x\in E-A$. Then $\Phi(x+dp)\geqq\Phi(dp)$. Hence $x+dp\in dT_p$. As $dp\in dT_p\subseteq A$ and $x\notin A$, A is not a k-ideal.

We know that if P is a radical class, then for any hemiring S, $P(S)$ is a k-ideal of S. Using this result and [2, Theorem 14], we have

THEOREM 4. *A radical class P defined for hemirings either contains all Euclidean hemirings or contains only the Euclidean hemiring* 0.

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(Received January 20, 1985; in revised form October 23, 1985)

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PATH PARTITIONS AND CYCLE PARTITIONS OF EULERIAN GRAPHS OF MAXIMUM DEGREE 4

ODILE FAVARON and MEKKIA KOUIDER

Abstract

In this article we prove Gallai's and Hajós' conjectures about path and cycle partitions in the case of eulerian graphs with maximum degree 4; and Hajós' conjecture in the case of minimally 2-connected and minimally 2-edge-connected eulerian graphs.

1. Introduction

Let $G=(V, E)$ be an undirected graph of order $|V|=n$, of size $|E|=m$, of maximum degree Δ, without loops and, except in paragraph 3, without parallel edges. *Paths* and *cycles* are elementary: if they are not necessarily elementary, they will be called *trails*.

We recall some conventions: $N_U(x)$ is the set of the neighbours of x contained in a subset U of $V(G)$. $|P|$ denotes the length (number of edges) of a path or a cycle P. $\lfloor x \rfloor$ is the greatest integer less than or equal to x and $\lceil x \rceil$ the least integer greater than or equal to x. If A is a subgraph of G, the graph $G-E(A)$ is obtained from G by removing the edges of A and all the isolated vertices which may appear.

Let $p(G)$ be the minimum number of paths necessary to partition the edges of G; $pc(G)$ the minimum number of paths and cycles necessary to partition the edges of G. When all the degrees of G are even, $c(G)$ is the minimum number of cycles necessary to partition the edges of G. In this case G is said to be *even* and when, moreover, it is connected, G is said to be *eulerian*.

Two old conjectures are still unsettled.

C1 (Hajós). *If G is an even graph of order n, then* $c(G) \leqq \left\lfloor \frac{n}{2} \right\rfloor$.

C2 (Gallai). *If G is a connected graph of order n then* $p(G) \leqq \left\lfloor \frac{n+1}{2} \right\rfloor$.

Donald [2] and Lovász [3] proved the following

THEOREM. *For every graph G of order n, with u vertices of odd degree*

$$pc(G) \leqq \left\lfloor \frac{n}{2} \right\rfloor \quad \textit{and} \quad p(G) \leqq \frac{u}{2} + \left\lfloor \frac{3(n-u)}{4} \right\rfloor \leqq \left\lfloor \frac{3n}{4} \right\rfloor.$$

1980 *Mathematics Subject Classification* (1985 Revision). Primary 05C38.
Key words and phrases. Graph, eulerian path, cycle.

Clearly, every path-cycle partition contains at least $\frac{u}{2}$ paths. In an attempt to prove C1, Donald had conjectured that there exists a minimum path-cycle partition with exactly $\frac{u}{2}$ paths. Dom de Caen [1] gave a counterexample to this, but the following weaker conjecture implies C1.

C3 *Every graph of order n has a path-cycle partition with at most $\frac{n}{2}$ elements of which exactly $\frac{u}{2}$ are paths, where u is the number of odd vertices.*

Indeed, this conjecture is equivalent to the following:

C′1 *If G is an even graph of order n, then $c(G) \leqq \frac{n-1}{2}$.*

In order to show the equivalence it is sufficient to consider connected graphs. If C3 is true, let x be a vertex of the eulerian graph G and let $\{y_1, \ldots, y_{2p}\}$ be the neighbours of x. The graph $G-\{x\}$ has a path-cycle partition P with at most $\frac{n-1}{2}$ elements, and exactly p paths the endpoints of which are the y_i's. Then P induces a partition of G into at most $\frac{n-1}{2}$ cycles. Conversely one can see that C′1 implies C3 by joining an auxiliary vertex to all the odd vertices of G.

Conjecture C′1 seems stronger than C1. In fact N. Dean observed (private communication) that the assertion C′1 for n even is equivalent to C1 for every n.

Our purpose here is to study C2 and C′1 for some eulerian graphs. First let us note

PROPOSITION 1.1. *If G is an eulerian graph, minimally 2-connected or minimally 2-edge-connected, then $c(G) \leqq \frac{n-1}{2}$.*

PROOF. An eulerian minimally 2-connected graph non isomorphic to K_8 contains no triangle and satisfies $m \leqq 2n-4$ [4] thus $c(G) \leqq \frac{m}{4} \leqq \frac{n-2}{2}$. An eulerian minimally 2-edge-connected graph is either minimally 2-connected or separable; in this last case the proposition is verified by induction on n and applying the induction hypothesis to two eulerian subgraphs separated by a cutvertex.

We shall now prove C2 and C′1 for eulerian graphs of maximum degree at most 4.

2. Path partitions of eulerian graphs of maximum degree 4

An eulerian closed trail is called an *eulerian walk* when it is given by its ordered sequence of edges. A path P of an eulerian graph belongs to an eulerian walk if and only if $G-E(P)$ is connected. Furthermore a path P which belongs to an eulerian walk W is called *maximal* at its endpoint v if $v^+\in P$, where v^+ is the successor of v on W. Since G is even with maximum degree 4, we have $m=n_2+2n_4\leqq 2n$, where n_2 (resp. n_4) is the number of vertices of degree 2 (resp. of degree 4). We want to get an upper bound for $p(G)$ of about $\frac{m}{4}$. The idea of the proof is to take the edges, following an eulerian walk, in groups of at least four, and rearrange them in order to obtain paths.

LEMMA 2.1. *Let k and l be two integers with $l\in\{3, 4\}$. Let $T=u_0u_1\dots u_k u_{k+1}\dots$ $\dots u_{k+l}$ be a trail not of type G_1, G_2 or G_3 (Fig. 1) such that the degree in T of each vertex u_i is at most 4 and that $u_0u_1\dots u_k$ is a path P maximal at u_k, and assume that T is of the type G_1, G_2 or G_3 (Fig. 1), then the edges of T can be partitioned into 2 paths P' starting in u_0 and Q' ending in u_{k+l} with $|P'|=k$ and $|Q'|=l$.*

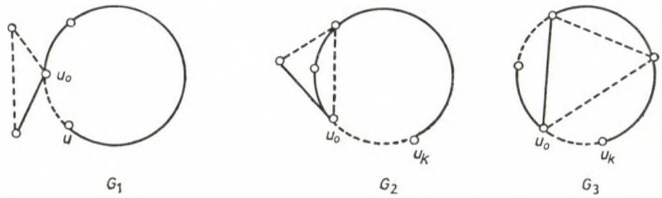


Fig. 1. G_i is the union of a triangle K_3 and a cycle $C_q (q\geqq 3)$ with i vertices in common

PROOF. If $Q=u_{k+1}\dots u_{k+l}$ is elementary, then $Q'=Q$, $P'=P$. Otherwise Q is isomorphic to C_3, C_4 or to the graph H (Fig. 2).

Let z be that point of $N_Q(u_k)-\{u_{k+l}\}$ contained in P, which is nearest to u_k on the path P. Then $z^+\neq u_k$. The different cases which may occur are given in Figure 3 if Q is isomorphic to C_4, and Figure 4 if Q is isomorphic to C_3 or to H. In these figures, P is represented by an arc of a circle, Q' by dotted lines and P' by full lines.

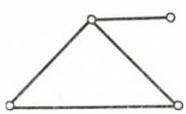


Fig. 2

1. $l=4$ and Q isomorphic to C_4 (Fig. 3).

One can take $P'=(P-zz^+)\cup zu_k$. Then $|P'|=|P|$ and $|Q'|=|Q|=4$.

2. $l=4$ and Q isomorphic to H (Fig. 4).

If the initial point u_k of Q coincides with b (Fig. 4.0), then u_{k+1} which is a point of P coincides with a. But a cannot be an inner point of P, otherwise its degree would be

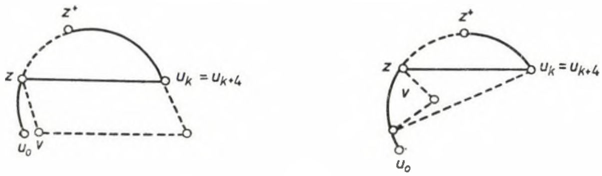
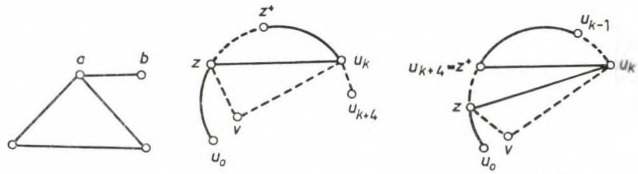


Fig. 3. The vertex v may belong to P



4.0 4.1 4.2

Fig. 4. The vertices v and u_{k+4} may belong to P

at least 5; neither be u_0, otherwise T would be of the type G_1, G_2 or G_3. Thus $u_k=a$ and $u_{k+4}=b$.

If $z^+\neq u_{k+4}$ (Fig. 4.1), one can still take $P'=(P-zz^+)\cup zu_k$. Then $|P'|=|P|$ and $|Q'|=|Q|=4$.

If $z^+=u_{k+4}$ (Fig 4.2), one can take $P'=(P-zz^+-u_ku_{k-1})\cup zu_k\cup u_kz^+$. Then $|P'|=|P|$ and $|Q'|=|Q|=4$.

3. $l=3$ and Q isomorphic to C_3.

This case can be reduced to the case 2 by deleting the edge u_ku_{k+4}. Then $|P'|=|P|$ and $|Q'|=|Q|=3$. □

We can now partition G.

THEOREM 2.2. *If G is an eulerian graph of size m and maximum degree at most 4, not isomorphic to C_3, C_4 or C_5 then* $p(G)\leqq\left\lfloor\frac{m+2}{4}\right\rfloor$.

PROOF. We see easily that if G is isomorphic to C_6 or to a graph of kind G_1, G_2 or G_3 (Fig. 1), then $m\geqq 6$ and $p=2\leqq\left\lfloor\frac{m+2}{4}\right\rfloor$.

In the other cases, $m\geqq 7$. By the lemma (take $k=2$, $l=4$), there exists a path P of length at least 4 which can be extended into an eulerian walk W of G. By the following rules, we shall, recursively, define a partition of the edges of W into paths P^i.

1. $L^0=P$.
2. We extend L^i until it becomes a path L^i_M maximal at its endpoint in W.
3. If possible, we take the trail Q^i formed by the 4 edges following L^i_M in W and not yet taken, and we apply the lemma to the trail $L^i_M\cup Q^i$; so we get two elementary

paths P^{i+1} and L^{i+1} with $P^{i+1} \cup L^{i+1} = L_M^i \cup Q^i$, $|P^{i+1}| = |L_M^i| \geqq 4$ and $|L^{i+1}| = |Q^i| = 4$.

4. We return to 2 (with incremented i).

This process must stop. Then either all the edges have been taken and $p(G) \leqq \frac{m}{4}$, or there remains a trail Q of at most 3 edges; in this last case, if Q is not elementary we apply again the lemma to get one more path and thus

$$p(G) \leqq \frac{m-|Q|}{4} + 1 \leqq \left\lfloor \frac{m+3}{4} \right\rfloor = \left\lceil \frac{m}{4} \right\rceil .$$

If $m \not\equiv 1 \pmod 4$ then $\left\lfloor \frac{m+3}{4} \right\rfloor = \left\lfloor \frac{m+2}{4} \right\rfloor$ and we are finished. If $m \equiv 1 \pmod 4$, there exists at least a vertex a of degree 2. Its neighbours a_1 and a_2 can be supposed adjacent otherwise $p(G) \leqq \frac{m+2}{4}$. When $d(a_1)$ or $d(a_2)$ is equal to 2 or when the even graph $G' = G - E(aa_1a_2a)$ is not connected, since $m \geqq 9$, it is easy to construct a path L^0 of length 5 which belongs to an eulerian walk of G. When $d(a_1) = d(a_2) = 4$ and G' is connected, one can verify that such a path still exists. So

$$p(G) \leqq \frac{(m-5)-|Q|}{4} + 2 \leqq \frac{m+2}{4} . \quad \square$$

Corollary 2.3. *If G is an eulerian graph of order n and maximum degree at most 4, then* $p(G) \leqq \left\lfloor \frac{n+1}{2} \right\rfloor$.

Proof. If $G = C_3$, C_4 or C_5, $p(G) = 2 \leqq \left\lfloor \frac{n+1}{2} \right\rfloor$. In the other cases, Theorem 2.2 implies $p(G) \leqq \left\lfloor \frac{n+1}{2} \right\rfloor$, because $m \leqq 2n$. $\square$

The bound $\left\lfloor \frac{m+2}{4} \right\rfloor$ of Theorem 2.2 is reached for example by any graph H_q in the infinite family of figure 5.

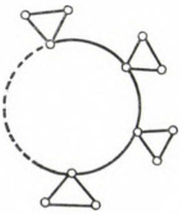


Fig. 5. The graph H_q is obtained from C_q by joining each of its vertices to a K_2.

3. Cycle partitions of even graphs with maximum degree 4.

Let G be an even graph with maximum degree $\Delta \leqq 4$ without loops, but possibly with parallel edges. A cycle partition may contain cycles of length 2. We denote k parallel edges between x and y by $k(xy)$.

We say that we "lift a vertex x" of degree 2 adjacent to y and z if we delete x and add an edge yz.

Our purpose is to prove

THEOREM 3.1. *If G is an even multigraph of order n, of size m, with $\Delta \leqq 4$, then $c(G) \leqq \frac{n+M-1}{2}$ where $M = m - m^*$ and m^* is the size of the simple graph induced by G.*

PROOF. We prove the theorem by induction on n assuming without loss of generality that G is connected.

If $n=2$ or 3, the result can be checked easily on each of the only six possible graphs (Fig. 6).

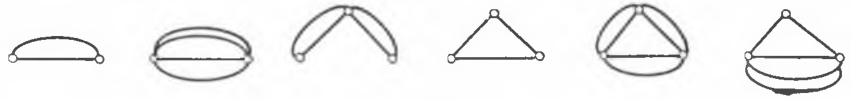


Fig. 6

Suppose the theorem true for every order less than n ($n \geqq 4$) and let G be an eulerian multigraph of order n.

Case 0. The graph has a cutvertex a.

There exist two eulerian subgraphs L_1 and L_2 such that $V(L_1) \cap V(L_2) = \{a\}$. Applying the induction hypothesis to L_1 and L_2, we get $c(G) \leqq c(L_1) + c(L_2) \leqq \frac{n+M-1}{2}$.

In each of the following cases, we shall apply the induction hypothesis to a graph G' associated to G, and construct a cycle partition P of G from a minimum cycle partition P' of G'.

Case 1. There exists a point x of degree 2.

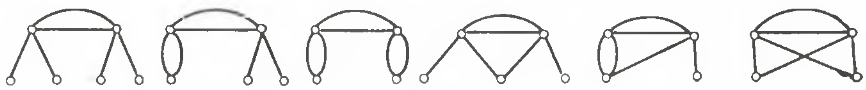
Let y and z be the neighbours of x. We take $G' = (G-x) \cup yz$ obtained from G by lifting x, and thus $c(G') \leqq \frac{(n-1)+(M+1)-1}{2} = \frac{n+M-1}{2}$. Then we see that $c(G) \leqq c(G') \leqq \frac{n+M-1}{2}$.

Henceforth we can assume that G is 2-connected and 4-regular.

Case 2. There exists a triple edge $3(xy)$.

Since $n \geqq 4$ and G is 2-connected, x and y have no common neighbour. Let $N(y) - \{x\} = \{z\}$ and $G' = (G-y) \cup xz$. Then $c(G') \leqq \frac{(n-1)+(M-2)-1}{2}$. We get P from P' by dividing the edge xz of G' by a vertex y and adding, as new element in P, the 2-cycle xyx. Thus $c(G) \leqq c(G')+1 \leqq \frac{n+M-2}{2}$.

Case 3. There exists a double edge $2(xy)$ (Fig. 7).



7.1 *7.2*

Fig. 7

We form G' by removing the 2-cycle xyx and identifying y with x. To get P from P', we choose two appropriate edges xa_i, $i \in \{1, 2\}$, of G', such that dividing each of them by a new vertex y_i and identifying y_1 and y_2 to form y, we obtain the starting configuration.

(a) If $N(x) \cap N(y) = \emptyset$ (Fig. 7.1), then $c(G') \leqq \frac{(n-1)+(M-1)-1}{2}$ and in every possible case it is easy to see that $c(G) \leqq c(G')+1$.

(b) If $N(x) \cap N(y) \neq \emptyset$ (Fig. 7.2), to get P we choose the two edges xa_i not contained in a same cycle of P'; then we can see that $c(G) \leqq c(G') \leqq \frac{(n-1)+(M+1)-1}{2}$.

Case 4. The graph is now simple, 2-connected and 4-regular.

(a) If there exists a vertex x with two adjacent neighbours a, b and two non adjacent neighbours c and d, then the graph $G'' = (G - cx - dx) \cup cd$ has a vertex of degree 2. By Case 1, we have $c(G'') \leqq \frac{n-1}{2}$. Given a partition P'' of G'' we can suppose that axb and cd are not in a same cycle otherwise we transform P'' by exchanging axb and ab. Returning to G, we get a partition with the same number of cycles.

Henceforth we assume that such a vertex x does not exist.

(b) If there exists x such that $N(x)$ is complete, then G is isomorphic to K_5 and $c(G) = \frac{n-1}{2}$.

If there exists x such that $N(x)$ is isomorphic to C_4, then, because of (a), one can verify that G is the graph of Fig. 8 and that $c(G) = 2 = \left\lfloor \frac{n-1}{2} \right\rfloor$.

(c) Now for each vertex x, either $N(x)$ is isomorphic to $2K_2$ or $N(x)$ is an independent set. If there exists a vertex x such that $N(x)$ is an independent set, then, since the case (a) is solved, $N(y)$ is also an independent set for every vertex y, and there exists a chordless cycle of length at least 4. If $N(x)$ is isomorphic to $2K_2$ for every x,

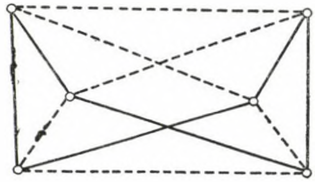


Fig. 8

then one can verify that G contains a cycle with the same properties. In either case, let C be such a cycle. We remove its edges and in the remaining graph we lift the vertices of C. The graph G' so obtained is simple since C is chordless and since no edge joins two vertices not in $V(C)$ but adjacent to the same vertex of C. By the induction hypothesis, $c(G') \leqq \frac{(n-4)-1}{2}$ and then, replacing the vertices of C, we see that $c(G) \leqq$ $\leqq c(G')+1 \leqq \frac{n-3}{2}$.

Now the proof is complete. □

COROLLARY 3.2. *If G is an even simple graph of order n, with $\Delta \leqq 4$, then $c(G) \leqq \leqq \frac{n-1}{2}$.*

PROOF. Clear from Theorem 3.1 with $M=0$. □

ACKNOWLEDGEMENTS. The authors are grateful to D. Sotteau for her helpful advice.

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(Received November 5, 1985)

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ANSWER TO A PROBLEM OF I. JOÓ

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If we have a continuous non-negative function $f\colon [0, \infty)\to \mathbf{R}$, it is easy to define a sequence $x_i \nearrow \infty$ such that

$$\int_{x_n}^{x_{n+1}} f \geqq \int_{x_{n+1}}^{x_{n+2}} f \qquad (n = 0, 1, \ldots).$$

The analogous discrete problem, found by I. Joó, is much more complicated. Namely, if we have a series

$$\sum_{n=1}^{\infty} a_n = \infty \quad \text{and} \quad a_n > 0 \quad \text{for} \quad n \geqq 1,$$

is there a sequence of natural numbers $N_0=0$, $N_i \nearrow +\infty$ such that

$$\sum_{j=N_i+1}^{N_{i+1}} a_j \geqq \sum_{j=N_{i+1}+1}^{N_{i+2}} a_j \qquad (i = 0, 1, \ldots)\,?$$

This question is partially answered in [1]. The authors showed that this problem is closely related to the following one:

Let for $c>0$ and $k\in\mathbf{N}$ $n_k(c)$ be the smallest number $m\in\mathbf{N}$ such that

$$kc \leqq \sum_{j=1}^{m} a_j.$$

What is the relation between the convergence of $\sum_{n=1}^{\infty} a_n^2$ and that of $\sum_{k=1}^{\infty} a_{n_k(c)}$ for the "typical" c's ($c>0$)? In [1] it is proved that if $\sum a_n^2=\infty$ then

$$Y := \{c > 0\colon\ \sum a_{n_k(c)} < \infty\}$$

is a set of first category. The question, can the set Y be of positive measure or not, is still unanswered.

To attack this last problem, I. Joó proposed in [2] the investigation of its continuous version: Suppose that $f\colon [0, \infty) \to \mathbf{R}$ is continuous, $f > 0$ and

$$\int_0^\infty f^2 = \infty.$$

Define the number $x_k(c)$ by the equality

$$\int_0^{x_k(c)} f = kc.$$

Then the set

$$Y := \{c > 0\colon \sum f(x_k(c)) < \infty\}$$

can be of positive measure?

The aim of the present paper is to give a positive answer to this question. We remark that Z. Magyar, independently of the author, has got the same result.

We shall prove the following

THEOREM. *There exists a continuous function* $f\colon [0, \infty) \to \mathbf{R}$ *such that* $0 < f(x) < 1$ $(x \in \mathbf{R})$,

$$\int_0^\infty f^2 = \infty \quad \textit{and} \quad \sum_{k=1}^\infty f(x_k(c)) < \infty \quad \textit{for a.e.} \quad c > 0.$$

The proof is based on the use of Beppo Levi's theorem, similarly to the proof of Theorem 3 in [1] and on a construction of J. A. Haight [3]. We shall argue in three steps.

1. Let $f(x) := \dfrac{1}{(x+1)^{3/4}}$; then $f \in L^2(0, \infty) \setminus L^1(0, \infty)$. Define further

$$F(t) := \int_0^t f = \frac{(t+1)^{1/4} - 1}{1/4};$$

then $F^{-1}(z) = \left(\frac{1}{4} z + 1\right)^4 - 1$.

Suppose that a system $\mathcal{J}$ of disjoint intervals is given in the segment $[0, n]$ and

$$|\mathcal{J}| \geqq cn$$

($|\mathcal{J}|$ denotes the sum of the lengths of the intervals of $\mathcal{J}$). Then $F^{-1}(\mathcal{J})$ is also a system of disjoint intervals and

$$|F^{-1}(\mathcal{J})| \geqq F^{-1}(|\mathcal{J}|) \geqq F^{-1}(cn)$$

because F^{-1} is monotone increasing and convex. Hence

(1)
$$\int_{F^{-1}(\mathcal{J})} f \geqq \int_{F^{-1}(n)-F^{-1}(cn)}^{F^{-1}(n)} f =$$
$$= F\big(F^{-1}(n)\big)-F\big(F^{-1}(n)-F^{-1}(cn)\big) \geqq$$
$$\geqq n\big(1-(1-c^4)^{1/4}\big)-4.$$

2. In this section we shall prove the following

LEMMA. *Let* $\varepsilon_k \searrow 0$, $\sum_{k=1}^{\infty} \varepsilon_k < \infty$. *We can define for each* $k \in \mathbf{N}$ *a number* $n_k \in \mathbf{N}$ *and a system* $\mathcal{J}_k$ *of disjoint intervals such that*

(a) *The intervals of* $\mathcal{J}_k$ *are left to that of* $\mathcal{J}_{k+1}$ *for all* k.

(b)
$$\mathcal{J}_k \subset [0, n_k], \quad |\mathcal{J}_k| \geqq \varepsilon_k n_k \quad \textit{and}$$
$$\lim_{k\to\infty} \big(1-(1-\varepsilon_k^4)^{1/4}\big) n_k = \infty.$$

(c)
$$\left|\left[\frac{1}{2^{k_0}}, 1\right] \cap \left(\bigcup_{k=k_0}^{\infty} \bigcup_{n=1}^{\infty} \frac{1}{n} \mathcal{J}_k\right)\right| \leqq c_0 \sum_{k=k_0}^{\infty} \varepsilon_k \qquad (k_0 = 1, 2, \ldots)$$

(where, of course, c_0 *is independent of* k_0*).*

PROOF. Introduce the notation ([3])

$$A(x_1, x_2, q, \varepsilon) := [x_1, x_2] \cap \bigcup_{t\in\mathbf{Z}} \left(e^{\frac{t}{q}}, e^{\frac{t+\varepsilon}{q}}\right).$$

We set

$$\mathcal{J}_k := A\left(\frac{n_k}{2}, n_k, q_k, 8\varepsilon_k\right)$$

where the parameters n_k, q_k are to be precised. If

(2)
$$n_{k+1} > 2n_k$$

then (a) is satisfied. The condition

(3)
$$q_k > 32$$

implies by [3], Lemma 1 the estimate $|\mathcal{J}_k| \geqq \varepsilon_k n_k$; further the condition

(4)
$$\frac{k}{\big(1-(1-\varepsilon_k)^4\big)^{1/4}} < n_k$$

yields that (b) is also satisfied. Fix a sequence (n_k) satisfying (2) and (4) and choose the numbers $q_k \in \mathbf{N}$ so that (c) be also assured.

Define

$$C_k^{(k_0)} := \left[\frac{1}{2^{k_0}}, 1\right] \cap \left(\bigcup_{n=1}^{\infty} \frac{1}{n} \mathcal{J}_k\right) \qquad (k \geqq k_0).$$

The arguments used in [3] show that for large q_k

$$|C_k^{(k_0)}| \leqq c_0 \varepsilon_k \qquad (k \geqq k_0).$$

Indeed, if $x \in C_k^{(k_0)}$ then $e^{\frac{t}{q_k}} < nx < e^{\frac{t+8\varepsilon_k}{q_k}}$. Here $nx < n_k$, hence $n < 2^{k_0} n_k < 2^k n_k$. By the Dirichlet theorem on simultaneous approximation there exists $q_k \in \mathbf{N}$,

$$q_k > (8\varepsilon_k)^{-2^k n_k} \tag{5}$$

such that

$$\left| \ln n - \frac{p_k^{(n)}}{q_k} \right| < q_k^{-1-\frac{1}{2^k n_k}} < \frac{8\varepsilon_k}{q_k} \qquad (n = 1, 2, \ldots, 2^k n_k)$$

is satisfied with some $p_k^{(n)} \in \mathbf{Z}$. Consequently,

$$e^{\frac{s-8\varepsilon_k}{q_k}} < x < e^{\frac{s+16\varepsilon_k}{q_k}}$$

$$e^{\frac{s}{q_k}} < e^{\frac{8\varepsilon_k}{q_k}} x < e^{\frac{s+24\varepsilon_k}{q_k}}$$

where s is integer. This inequality means that

$$e^{\frac{8\varepsilon_k}{q_k}} \cdot x \in A\left(\frac{1}{2^{k_0}} e^{\frac{8\varepsilon_k}{q_k}},\ e^{\frac{8\varepsilon_k}{q_k}},\ q_k,\ 24\varepsilon_k \right)$$

and hence

$$|C_k^{(k_0)}| \leqq e^{-\frac{8\varepsilon_k}{q_k}} \left| A\left(\frac{1}{2^{k_0}} e^{\frac{8\varepsilon_k}{q_k}},\ e^{\frac{8\varepsilon_k}{q_k}},\ q_k,\ 24\varepsilon_k \right) \right| < c_0 \varepsilon_k$$

(again by Lemma 1 of [3]) and we are ready.

3. Consider the function f given in 1. We shall modify it on each interval of $F^{-1}(\mathcal{J}_k)$ $(k=1, 2, \ldots)$ where the systems $\mathcal{J}_k$ are constructed in the above Lemma. If $I=(x, y)$ is an interval of $F^{-1}(\mathcal{J}_k)$, we define $\tilde{f}\colon [x, y] \to \mathbf{R}$ such that $\tilde{f}(x) = f(x)$, $\tilde{f}(y) = f(y)$, $\tilde{f}$ is continuous,

$$0 < \tilde{f} < 1, \quad \int_x^y \tilde{f} = \int_x^y f \quad \text{and} \quad \int_x^y \tilde{f}^2 < \frac{1}{2} \int_x^y \tilde{f}.$$

Such a modification is obviously possible. We shall see that the function $\tilde{f}\colon [0, \infty) \to \mathbf{R}$ satisfies the conditions of the Theorem. It is obvious from (1) and the point (b) of the Lemma that

$$\int_0^\infty \tilde{f}^2 = \infty.$$

Define

$$Y_{k_0} := \left[\frac{1}{2^{k_0}}, 1 \right] \setminus \left(\bigcup_{k=k_0}^\infty \bigcup_{n=1}^\infty \frac{1}{n} \mathcal{J}_k \right).$$

Then by the point (c) of the Lemma $\bigcup\limits_{k_0=1}^{\infty} Y_{k_0}$ covers the segment [0, 1] up to a set of measure 0. Hence it will be enough to prove that

(6) $$\sum_{k=1}^{\infty} \tilde{f}(\tilde{x}_k(c)) < \infty \quad \text{for a.e.} \quad c \in Y_{k_0}.$$

Returning to the original function f we have

$$\int\limits_{Y_{k_0}} f(x_k(c))\,dc = \int\limits_{Y_{k_0}} f(F^{-1}(kc))\,dc = \frac{1}{k} \int\limits_{F^{-1}(kY_{k_0})} f^2$$

hence

$$\sum_{k=1}^{\infty} \int\limits_{Y_{k_0}} f(x_k(c))\,dc = \sum_{k=1}^{\infty} \frac{1}{k} \int\limits_{F^{-1}(kY_{k_0})} f^2 = \int\limits_0^{\infty} f^2(t) \Big(\sum_{F(t)\in kY_{k_0}}^{k} \frac{1}{k} \Big) dt$$

and analogously

$$\sum_{k=1}^{\infty} \int\limits_{Y_{k_0}} \tilde{f}(\tilde{x}_k(c))\,dc = \int\limits_0^{\infty} \tilde{f}^2(t) \Big(\sum_{\tilde{F}(t)\in kY_{k_0}}^{k} \frac{1}{k} \Big) dt$$

where $\tilde{F}(t) = \int\limits_0^t \tilde{f}$. We see from the construction of $\tilde{f}$ that if $F(t) \notin \bigcup\limits_{k=1}^{\infty} \mathscr{I}_k$ then $\tilde{f}(t) = f(t)$ and $F(t) = \int\limits_0^t f = \int\limits_0^t \tilde{f} = \tilde{F}(t)$. Consequently, $F^{-1}(\mathscr{I}_k) = \tilde{F}^{-1}(\mathscr{I}_k)$. Secondly, the definition of Y_{k_0} implies that

$$kY_{k_0} \cap \mathscr{I}_n = \emptyset \qquad (n \geqq k_0).$$

For large $t\,(t \geqq t_{k_0})$ $F(t)$ and $\tilde{F}(t)$ do not meet $\mathscr{I}_1, \dots, \mathscr{I}_{k_0-1}$ hence $F(t) \in kY_{k_0}$ (or $\tilde{F}(t) \in kY_{k_0}$) implies $F(t) = \tilde{F}(t)$ and $f(t) = \tilde{f}(t)$. So we have

$$\int\limits_{t_{k_0}}^{\infty} \tilde{f}^2(t) \Big(\sum_{\tilde{F}(t)\in kY_{k_0}}^{k} \frac{1}{k} \Big) dt = \int\limits_{t_{k_0}}^{\infty} f^2(t) \Big(\sum_{F(t)\in kY_{k_0}}^{k} \frac{1}{k} \Big) dt \leqq$$

$$\leqq \int\limits_0^{\infty} f^2(t) \Big(\sum_{k(1/2^{k_0}) \leqq F(t) \leqq k} \frac{1}{k} \Big) dt < \infty$$

since

$$\sum_{F(t) \leqq k \leqq 2^{k_0} F(t)} \frac{1}{k} \leqq \text{const} \cdot k_0 < \infty.$$

Now

$$\sum_{k=1}^{\infty} \int\limits_{Y_{k_0}} \tilde{f}(\tilde{x}_k(c))\,dc = \int\limits_0^{\infty} \tilde{f}^2(t) \Big(\sum_{\tilde{F}(t)\in kY_{k_0}} \frac{1}{k} \Big) dt < \infty$$

and (6) follows from the theorem of Beppo Levi. The theorem is proved.

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(Received November 10, 1985)

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CONGRUENCES ON INVERSIVE HEMIRINGS

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Abstract

In the present paper we study congruences on inversive hemirings. In the first part we use Petrich's techniques to get a characterization. In the second part a special class of congruences is discussed, and a theorem is given about the quotient hemiring; in this part an interesting decomposition of hemirings is also introduced. We refer to [4] for theorems on congruences and to [1] for background on semigroups.

1

A *hemiring* is a nonempty set A with two operations $+$ and $\cdot$, where $(A, +)$ is a commutative semigroup with identity 0, $(A, \cdot)$ is a semigroup, $\cdot$ distributes over $+$ from both sides and $0\cdot a=a\cdot 0=0$ holds for all $a\in A$. A hemiring A is called an *additively inversive hemiring* if $(A, +)$ is an inverse semigroup (i.e. for every element a in A there exists exactly one element $(-a)$ such that $a+(-a)+a=a$ and $(-a)+ +a+(-a)=-a$. If, in addition, $(A, \cdot)$ is commutative, A is called a *commutative additively inversive hemiring*. Throughout this paper, when speaking of a *(commutative) hemiring,* we always mean for short a *(commutative) additively inversive hemiring;* we often write $-a$ instead of $(-a)$ to indicate the additively inverse element of a, $a-a$ instead of $a+(-a)$ to denote the additively idempotent element generated by a and $a-b$ instead of $a+(-b)$.

As far as additively idempotent or inverse elements of a hemiring $(A, +, \cdot)$ are concerned, we have the following trivial, but extremely useful lemma:

LEMMA 1.1. *For every additive idempotent $e\in A$, and for every $c\in A$, the product $e\cdot c$ is additively idempotent. If $a\in A$ and $-a$ is its additive inverse, then, for every $c\in A$, $c\cdot(-a)=-(c\cdot a)$ is the additive inverse of $c\cdot a$.*

Yusuf ([6]) has studied special subsets of a hemiring A: A subset I of A is said to be an *ideal* of A if I is an additive inversive subhemiring of A such that I contains all additively idempotent elements of A and such that $A\cdot I+I\cdot A\subset I$. The set E of all additive idempotents of a hemiring A is an ideal of A (by Lemma 1.1; see also [6]) and is called the *trivial ideal* of A. In the theory of hemirings, ideals play a very important role, when studying congruences on hemirings. A *congruence* on a hemiring $(A, +, \cdot)$ is a congruence ϱ on the additive semigroup $(A, +)$, such that for every $a, b, c\in A$, $a\varrho b$ implies $a\cdot c\varrho b\cdot c$ and $c\cdot a\varrho c\cdot b$.

As is well-known, the study of homomorphisms in groups is greatly simplified by the correspondence between homomorphisms and normal subgroups; the same holds for rings, where the corresponding role is played by two-sided ideals. After several

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1980 *Mathematics Subject Classification.* Primary 16A78; Secondary 20M25.
Key words and phrases. Hemirings, inversive hemirings, congruences.

attempts, an elegant analogue has been found for inverse semigroups, and "normal" subsemigroups were associated with congruences in a very natural way. Let us remember some definitions, which can be found in [4].

Let S be an inverse additive semigroup and let ϱ be a congruence on S. The restriction $\varrho|_E=\xi$ of ϱ to E (the subsemigroup of idempotents) is said to be the *trace* of ϱ; the subset N of the elements of S, which are ϱ-congruent to some idempotent, is said to be the *kernel* of ϱ. If we say that a congruence ξ on E is *normal* when $e\xi f$ implies that, $\forall a\in S$, $(-a)+e+a\xi(-a)+f+a$, then *every trace is normal;* and if we say an inverse subsemigroup N of S to be *normal* when it is *full* (i.e. it contains every idempotent of S) and *self-conjugate* (i.e. for every $x\in N$, and for every a in S, $(-a)+x+a$ belongs to N), then *every kernel of any congruence on S is a normal subsemigroup*. Besides, it is readily seen that the trace ξ and the kernel N of any congruence ϱ on S fulfil the following two properties:

(c) $(\forall a\in S)(\forall e\in E)$ [if $a+e\in N$ and $(-a)+a\xi e$, then $a\in N$]

(d) $(\forall a\in N)(-a)+a\xi a+(-a)$.

Following [4], we say that *a congruence pair* is a pair (ξ, N) where:

(a) ξ is a normal congruence on E;

(b) N is a normal subsemigroup of S;

(c) and *(d)* hold.

We may summarize the previous remarks by

LEMMA 1.2. *For every congruence ϱ on S, the pair (trace ϱ, kernel ϱ) is a congruence pair.*

Conversely:

THEOREM 1.3 ([4]). *For every congruence pair (ξ, N) there exists exactly one congruence ϱ on S, such that $\xi=$ trace ϱ and $N=$kernel ϱ namely:*

$$\varrho = \{(a\cdot b)\in S\times S\colon a-a\xi b-b \text{ and } a-b\in N\}.$$

Now, when S is commutative (a fortiori when S is the additive semigroup of a hemiring), conditions *(a)* and *(d)* are trivially satisfied. So we can give the following definition of a *congruence pair* on a hemiring:

DEFINITION 1.4. Let $(A, +, \cdot)$ be a hemiring. If ξ is a congruence on the additive semilattice of idempotents E of A, and N is an ideal of A, then (ξ, N) is said to be a *congruence pair* on A if the following condition holds:

(*) for every $a\in A$, if there exists $e\in E$: $e\xi a-a$ and $a+e\in N$, then $a\in N$.

The importance of this definition is emphasized by the following theorem, where the notions of trace and kernel of a congruence on a hemiring are the immediate extensions of the above ones for inverse semigroups:

THEOREM 1.5. *For every congruence ϱ on a hemiring A, (trace ϱ, Ker ϱ) is a congruence pair.*

Conversely, for every congruence pair (ξ, N) on A, there exists exactly one congruence ϱ such that trace $\varrho=\xi$ and ker $\varrho=N$, namely:

$$\varrho = \{(a, b)\in A\times A\colon a-a\xi b-b \text{ and } a-b\in N\}.$$

PROOF. A large part of the theorem follows from what we already observed about congruence pairs on commutative inverse semigroups and from Theorem 1.3. Here we have to prove that $\ker \varrho$ is an ideal of A. Let $a \in \ker \varrho$. Then there exists an additively idempotent element $e \in E$ such that $a \varrho e$. For every $c \in A$, by Lemma 1.1, $c \cdot e$ is an additive idempotent; since ϱ is a congruence on A, thus, $c \cdot a \varrho c \cdot e$, and $c \cdot a \in \ker \varrho$. A similar argument can be used for the right multiplication.

Conversely, let (ξ, N) be a congruence pair and let us prove that ϱ, as defined above, is a congruence on A and $\xi = \operatorname{trace} \varrho$, $N = \ker \varrho$. Again, by Theorem 1.3, the only thing we have to prove is that the additive congruence ϱ on $(A, +)$ is preserved by multiplication.

Firstly, we prove that every congruence ξ on the semilattice E is preserved by multiplication on A: let $e, f \in E$ and $c \xi A$. Then $c - c = g$ is an element of E and $g \cdot e \xi g \cdot f$ since ξ is a congruence on E. But $g \cdot e = (c-c) \cdot e = c \cdot e - c \cdot e = c \cdot e$, by Lemma 1.1; again, $g \cdot f = c \cdot f$ and therefore, if $e \xi f$, then $c \cdot e \xi c \cdot f$.

Now assume $a \varrho b$. It means that $a - a \xi b - b$ and $a - b \in N$. So, for every $c \in A$, $c \cdot a - c \cdot a = c \cdot (a-a) \xi c \cdot (b-b) = c \cdot b - c \cdot b$ and $c \cdot [a + (-b)] = c \cdot a + c \cdot (-b) = = c \cdot a + (-c \cdot b)$ belongs to N. Then $c \cdot a \varrho \cdot cb$. The same holds on the right side, completing the proof.

2

If $\xi = \varepsilon_E$, the identity relation on E, it is trivial to see that it is normal and property (*) holds. So we have the following (see also [3] for the corresponding theorem on semigroups):

COROLLARY 2.1. *For every ideal N of a hemiring A, the pair (ε_E, N) is a congruence pair, which corresponds to the following congruence:*

$$\varphi = \{(a, a) \in A \times A \colon a - a = b - b \text{ and } a - b \in N\}.$$

In this second part of the paper we want to get more information about this special class of congruences, whose trace equals the identity, by applying to hemirings some results about congruences on Clifford semigroups. Recall that:

DEFINITION 2.2. A semigroup is called a *Clifford semigroup* when it is inverse and its idempotent elements are central. A semigroup A is called a *strong semilattice of groups* if $A = \bigcup_{i \in Y} A_i$, where each A_i is a group, Y a semilattice and the following conditions hold:

(1) for each $i, j \in Y$, $i \geqq j$, there exists a homomorphism $\varphi_{i,j} \colon A_i \to A_j$, such that $\varphi_{i,i}$ is the identical automorphism for every A_i and $\varphi_{i,j} \cdot \varphi_{j,k} = \varphi_{i,k}$;

(2) for each $i, j \in Y$, if $a \in A_i$ and $b \in A_j$, then $a + b = (a\varphi_{i,i+j}) + (b\varphi_{j,i+j})$, (here $i+j$ denotes the greatest lower bound of i and j in the semilattice Y, and the addition of $(a\varphi_{i,i+j})$ and $(b\varphi_{j,i+j})$ takes place in the subgroup A_{i+j}).

PROPOSITION 2.3 ([1]. IV. 2.1). *A semigroup A is a Clifford semigroup if and only if A is a strong semilattice of groups.*

It is readily seen that the additive semigroup of a hemiring A is a Clifford semigroup, since it is commutative. The additive idempotents are the zero elements of its subgroups A_i. In the following we always call e the zero of A_i, f the zero of A_j, g the zero of A_h.

The following theorem points out a connection between the product on A and such a semilattice structure:

THEOREM 2.4.
(i) *For every* $i, j \in Y$, *there exists* $h \in Y$ *such that* $A_i \cdot A_j \subseteq A_h$; *we say* $h = i^*j$;
(ii) *For every* $i, j, h \in Y$, *if* $i+j$ *indicates the greatest lower bound of the two elements, then* $(i+j)^*h = (i^*h)+(j^*h)$;
(iii) *For every* $i, j, h \in Y$, *if* $i \leqq j$, *then* $i^*h \leqq j^*h$ *and* $h^*i \leqq h^*j$;
(iv) *For every* $i, j \in Y$, *then* $i^*i \leqq i^*j \leqq j^*j$ *and* $i^*i \leqq j^*i \leqq j^*j$.

PROOF. From Lemma 1.1 it follows that the product $e \cdot f (e \in A_i, f \in A_j)$ is an additive idempotent; hence it is the zero element of some subgroup, say A_h, of A and $e \cdot f = g$. If $a \in A_j$, then, again by Lemma 1.1, $g = e \cdot f = e \cdot (a-a) = e \cdot a - e \cdot a$ and $e \cdot a = g \in A_h$. If c is any element in A_i, then $e \cdot a = (c-c) \cdot a = c \cdot a - c \cdot a$, $c \cdot a \in A_h$ and part (i) is true.

Part (ii) is an easy consequence of distributivity on A and part (iv) is a corollary to (iii). Thus we only have to prove (iii). Let $e \in A_i$ and $f \in A_j$ and let c be an element of A_h. If $i \leqq j$, then $e+f=e$ and therefore $e \cdot c = (e \cdot c)+(f \cdot c)$ implies that $i^*h \leqq j^*h$.

Any ideal N of a hemiring A is certainly a normal subsemigroup of $(A, +)$; whence, by [2] 1.3, its intersections $N_i = N \cap A_i$ are normal subgroups of A_i, for every $i \in Y$. Besides, for every two elements $a, b \in A$, if $a-a=b-b$, then it must be true that $a, b \in A_i$ for some $i \in Y$. This implies

PROPOSITION 2.5. *Every ideal N of a hemiring A is the kernel of at least one congruence on A, namely* $N = \text{kernel}\ \varphi$, *where:* $\varphi = \{(a, b) \in A \times A$: *for some* $i \in Y$, $a, b \in A_i$ *and* $a-b \in N_i\}$.

But we can say more. Using again a result in [2], we can describe the quotient image by this congruence.

THEOREM 2.6. *Let N be an ideal of a hemiring A and let* φ *be the congruence associated to the congruence pair* (ε_E, N) *and let* $T = A/\varrho$. *Then* $T = \bigcup_{i \in Y} T_i$, *where* $T_i = A_i/N_i$ *and the homomorphisms* $\theta_{i,j}\colon T_i \to T_j$ *are induced naturally by the corresponding* $\varrho_{i,j}\colon A_i \to A_j$.

PROOF. By Lemma [2] 1.7 this is true if we consider the (Clifford) semigroups $(A, +)$ and $(T, +)$, defining the homomorphisms $\theta_{i,j}$ by: $t\theta_{i,j} = N_j + s\varrho_{i,j}$, where $t = N_i + s$. To complete the proof we only have to show that this decomposition of T is compatible with multiplication.

Let $t_1 = N_i + s_1$, $t_2 = N_j + s_2$; we have that:

$$t_1 \cdot t_2 = (N_i + s_1) \cdot (N_j + s_2) = N_i \cdot N_j + N_i \cdot s_2 + s_1 \cdot N_j + s_1 \cdot s_2.$$

Since N is an ideal of A, then $N_i \cdot N_j$, $N_i \cdot s_2$, $s_1 \cdot N_j$ are included in N. But $s_1 \in S_i$ and $s_2 \in S_j$; hence $t_1 \cdot t_2 = N_{i*j} + s_1 \cdot s_2$.

By using Theorem 2.4 it is just a matter of calculation to verify the distributivity of multiplication.

ACKNOWLEDGEMENT. The author wishes to thank Prof. R. Wiegandt, for drawing his attention to this problem.

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(Received November 11, 1985)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 257—263

SUBDIRECTLY IRREDUCIBLE LOCALLY BOOLEAN ALGEBRAS

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0. In [5] J. Płonka introduced the notion of a *locally Boolean algebra* as an algebra $\underset{\times}{A}=(A; \vee, \wedge, ')$ of the type $\langle 2, 2, 1\rangle$ where $(A; \vee, \wedge)$ is a distributive lattice and there exists a congruence R of $\underset{\times}{A}$ such that any congruence class $[a]_R$, $a \in A$ is a Boolean algebra with respect to the operations $\vee$, $\wedge$ and $'$ restricted to $[a]_R$.

Locally Boolean algebras have an interesting application in logic and were investigated from this point of view in [4]. In [8] a representation theorem for some algebras of this kind was given.

In this paper we describe all subdirectly irreducible locally Boolean algebras. We prove that such algebras can be constructed by means of disjunctive and codisjunctive distributive lattices. We use the notion of a disjunctive (and dually codisjunctive) lattice utilizing the notion of a disjunctive poset (see [1], [6] and [7]) for lattices.

Our terminology and nomenclature is basically that of G. Grätzer [2] and [3].

1. It was proved in [5] that the class $L(B)$ of all locally Boolean algebras is a variety. More precisely (see [5]), if $\underset{\times}{A}=(A; \vee, \wedge, ')$ is an algebra of the type $\langle 2, 2, 1\rangle$ then $\underset{\times}{A}$ belongs to $L(B)$ iff $\underset{\times}{A}$ satisfies the following identities:

(1) identities in $\vee$ and $\wedge$ which define distributive lattices;

(2) $(x')' = x$;

(3) $(x \vee x')' = x \wedge x'$;

(4) $(x \vee y) \wedge (x \vee y)' = (x \wedge x') \vee (y \wedge y')$;

(5) $(x \wedge y) \vee (x \wedge y)' = (x \vee x') \wedge (y \vee y')$.

A suitable congruence of an algebra $\underset{\times}{A}$ satisfying (1)—(5) is the relation R defined as follows:

(i) $$a \equiv b(R) \quad \text{iff} \quad a \wedge a' = b \wedge b' \quad \text{for all} \quad a, b \in A.$$

Moreover, the relation defined in (i) is the only one congruence of an algebra of type $\langle 2, 2, 1\rangle$, which decomposes $\underset{\times}{A}$ into its Boolean subalgebras and consequently $\underset{\times}{A}$ satisfies (1)—(5) (see [5]). Following J. Płonka we shall call R the *bounding congruence of* $\underset{\times}{A}$.

1980 *Mathematics Subject Classification.* Primary 06DXX; Secondary 06EXX.
Key words and phrases. Ordered sum of lattices, congruence, ideal, Boolean algebra, subdirectly irreducible algebra.

Let $\mathbf{A}=(A; \vee, \wedge, ')$ be a locally Boolean algebra and R be the bounding congruence of $\mathbf{A}$. The zero-element $a\wedge a'$ and the unit $a\vee a'$ of the Boolean algebra $([a]_R; \vee, \wedge, ')$, $a\in A$ we shall denote by 0_a and 1_a, respectively. The reduct $(A; \vee, \wedge)$ of $\mathbf{A}$ will be denoted by $\bar{\mathbf{A}}$. If $(L; \vee, \wedge)$ is a distributive lattice and $a, b\in L$, $a\leqq b$ then $\Theta(a, b)$ will denote the least congruence of $(L; \vee, \wedge)$ such that $a\equiv b(\Theta(a, b))$. By G. Grätzer—E. T. Schmidt theorem (see [2] p. 74) we have:

(ii) $$x \equiv y(\Theta(a, b)) \quad \text{iff} \quad a\wedge x = a\wedge y \quad \text{and} \quad b\vee x = b\vee y$$

and

(iii) $$\text{if} \quad x\equiv y(\Theta(a,b)) \quad \text{and} \quad (x\leqq y\leqq a\leqq b \ \text{ or } \ a\leqq b\leqq x\leqq y) \quad \text{then} \quad x=y$$

for all $x, y\in L$.

Finally we denote by $\operatorname{Con}\mathbf{A}$ and $\operatorname{Con}\bar{\mathbf{A}}$ the lattices of congruences of the algebra $\mathbf{A}$ and the lattice $\bar{\mathbf{A}}$, respectively. Of course we have $\operatorname{Con}\mathbf{A}\subseteq\operatorname{Con}\bar{\mathbf{A}}$.

2. In [1] the notion of a poset satisfying the disjunctivity condition was studied. By means of such posets J. R. Buchi characterized dense subsets of Boolean algebras (cf. in [6]). In [7] posets satisfying the disjunctivity condition and having the least element have been called disjunctive. We apply this notion for lattices in the following way.

A lattice $(L; \vee, \wedge)$ with the least element 0 is called to be *disjunctive* if for any two elements $a, b\in L$, $a<b$ there exists an element $c\in L$ such that

(C) $$0<c\leqq b \quad \text{and} \quad a\wedge c = 0.$$

Dually, a lattice $(L; \vee, \wedge)$ with the greatest element 1 we shall call *codisjunctive* if for any $a, b\in L$, $a<b$ there exists $c\in L$ such that

(C′) $$a\leqq c<1 \quad \text{and} \quad b\vee c=1.$$

We have

(iv) If Θ is a nontrivial congruence of a disjunctive lattice $(L; \vee, \wedge)$ then there exists $c\in L$ such that $0<c$ and $0\equiv c(\Theta)$.

(iv′) If Θ is a nontrivial congruence of a codisjunctive lattice $(L; \vee, \wedge)$ then there exists $c\in L$ such that $c<1$ and $c\equiv 1(\Theta)$.

In fact, if Θ is such a congruence then there exist $a, b\in L$ such that $a<b$ and $a\equiv b(\Theta)$. Hence by the definition of a disjunctive lattice we get (C) for some $c\in L$. Thus $0=a\wedge c\equiv b\wedge c(\Theta)$ and consequently $0\equiv c(\Theta)$. The proof of (C') is analogous.

Let us denote by DS and CDS the classes of all distributive disjunctive and all distributive codisjunctive lattices, respectively. Let $O_L(1_L)$ be the least (the greatest) element of a lattice $\mathbf{L}\in DS\,(\mathbf{L}\in CDS)$. For lattices $\mathbf{L}_1=(L_1; \vee, \wedge)\in CDS$ and $\mathbf{L}_2=(L_2; \vee, \wedge)\in DS$ such that $L_1\cap L_2=\emptyset$ we denote by $\mathbf{L}_1\oplus\mathbf{L}_2$ the ordered sum of $\mathbf{L}_1$ and $\mathbf{L}_2$.

We recall that $\mathbf{L}_1\oplus\mathbf{L}_2$ is the lattice $(L_1\cup L_2; \vee, \wedge)$ in which the lattice order $\leqq$ is defined as follows:

$$a\leqq b \quad \text{iff} \quad a\in L_1 \quad \text{and} \quad b\in L_2 \quad \text{or} \quad a\leqq_i b \quad \text{and} \quad a, b\in L_i,$$

$i=1, 2$, where $\leqq_i$ is the lattice order in L_i, $i=1, 2$.

Now we can define a unary operation $'$ on the set $L_1\cup L_2$ putting $a'=0_{L_2}$ if $a=1_{L_1}$, $a'=1_{L_1}$ if $a=0_{L_2}$ and $a'=a$ otherwise.

It is easy to see that the algebra $\widetilde{\underset{\times}{L}_1\oplus\underset{\times}{L}_2}=(L_1\cup L_2;\vee,\wedge,')$ is a locally Boolean algebra. The bounding congruence R of $\widetilde{\underset{\times}{L}_1\oplus\underset{\times}{L}_2}$ has exactly one 2-element congruence class $[0_{L_2}]_R=[1_{L_1}]_R=\{0_{L_2}, 1_{L_1}\}$ and other congruence classes are 1-element.

THEOREM 1. *If* $\underset{\times}{L}_1=(L_1;\vee,\wedge)\in CDS$, $\underset{\times}{L}_2=(L_2;\vee,\wedge)\in DS$ *and* $L_1\cap L_2=\emptyset$ *then the algebra* $\widetilde{\underset{\times}{L}_1\oplus\underset{\times}{L}_2}$ *is subdirectly irreducible.*

PROOF. It is easy to see that the bounding congruence R of $\widetilde{\underset{\times}{L}_1\oplus\underset{\times}{L}_2}$ is an atom in $\mathrm{Con}\,\widetilde{\underset{\times}{L}_1\oplus\underset{\times}{L}_2}$. Let Θ be a nontrivial congruence of $\widetilde{\underset{\times}{L}_1\oplus\underset{\times}{L}_2}$. Hence Θ is a nontrivial congruence of the lattice $\underset{\times}{L}_1\oplus\underset{\times}{L}_2$. Therefore there exist elements $a, b\in L_1\cup L_2$ such that $a<b$ and $a\equiv b(\Theta)$. If $a\in L_1$ and $b\in L_2$ then $1_{L_1}\equiv 0_{L_2}(\Theta)$ since a congruence class of a lattice must be convex. Thus $R\subseteq\Theta$.

If $a, b\in L_1$ then we consider a restriction Θ_1 of Θ to the sublattice L_1 of $\underset{\times}{L}_1\oplus\underset{\times}{L}_2$. Obviously, Θ_1 is a nontrivial congruence of the codisjunctive lattice $\underset{\times}{L}_1$ and by (iv') there exists $c\in L_1$ such that $c<1_{L_1}$ and $c\equiv 1_{L_1}(\Theta_1)$. Hence $c\equiv 1_{L_1}(\Theta)$ and $c=c'\equiv 0_{L_2}(\Theta)$ since Θ satisfies the substitution law for $'$. Thus $1_{L_1}\equiv c\equiv 0_{L_2}(\Theta)$ what proves that $R\subseteq 0$. The same result we get in the case when $a, b\in L_2$. This shows that R is the unique atom in the lattice $\mathrm{Con}\,\widetilde{\underset{\times}{L}_1\oplus\underset{\times}{L}_2}$ and consequently $\widetilde{\underset{\times}{L}_1\oplus\underset{\times}{L}_2}$ is subdirectly irreducible (cf. [3] p. 124).

3. From now on let us assume that $\bar{\underset{\times}{A}}=(A;\vee,\wedge,')$ is a fixed locally Boolean algebra with the bounding congruence R.

LEMMA 1. *If* $\Theta\in\mathrm{Con}\,\bar{\underset{\times}{A}}$ *and* $\Theta\subseteq R$ *then* $\Theta\in\mathrm{Con}\,\underset{\times}{A}$.

PROOF. Let $a\equiv b(\Theta)$ and Θ_1 be the restriction of Θ to the sublattice $[a]_R$ of $\bar{\underset{\times}{A}}$. Since Θ_1 is a congruence of the Boolean lattice $([a]_R;\vee,\wedge)$ so Θ_1 is a congruence of the Boolean algebra $([a]_R;\vee,\wedge,')$. Hence $a'\equiv b'(\Theta_1)$ because $a, b\in[a]_R$. Thus $a'\equiv b'(\Theta)$.

LEMMA 2. *If* $a, b\in A$, $a\leqq b$ *and* $a\equiv b(R)$ *then* $\Theta(a, b)\in\mathrm{Con}\,\underset{\times}{A}$.

PROOF. We shall prove that $\Theta(a, b)\subseteq R$ and apply Lemma 1. Let $c\equiv d(\Theta(a,b))$. By (ii) we have $a\wedge c=a\wedge d$ and $b\vee c=b\vee d$. Hence $[a]_R\wedge[c]_R=[a\wedge c]_R=[a\wedge d]_R=[a]_R\wedge[d]_R$ and $[a]_R\vee[c]_R=[b]_R\vee[c]_R=[b\vee c]_R=[b\vee d]_R=[b]_R\vee[d]_R=[a]_R\vee[d]_R$. Thus $[c]_R=[d]_R$ since the quotient lattice $\underset{\times}{A}/R$ is distributive.

Let us accept the following notations:

$$A_1=\{x\in A:\ |[x]_R|=1\},$$

$$A_2=\{x\in A:\ |[x]_R|=2\},$$

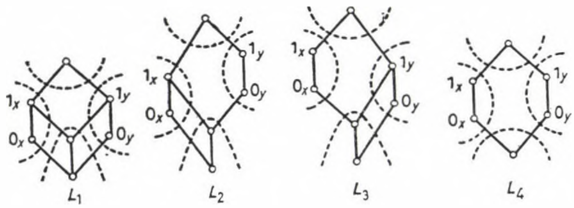
$$A_3=\{x\in A:\ |[x]_R|>2\}.$$

LEMMA 3. *If the algebra $\underset{\times}{A}$ is subdirectly irreducible then $A_3=\emptyset$.*

PROOF. If $a\in A_3$ then there exists $b\in[a]_R$ such that $0_a<b<1_a$. Hence by Lemma 2 and (ii) the relations $\Theta(0_a, b)$ and $\Theta(b, 1_a)$ are nontrivial congruences of $\underset{\times}{A}$ such that $\Theta(0_a, b)\cap\Theta(b, 1_a)$ is the trivial congruence of $\underset{\times}{A}$. Thus $\underset{\times}{A}$ is subdirectly reducible.

LEMMA 4. *If $A_3=\emptyset$ and $A_2\neq\emptyset$ then the following conditions (a) and (b) hold:*
(a) *A_2 is a sublattice of $\underset{\times}{\bar{A}}$,*
(b) *if $x\in A_1$ and $y\in A_2$ then $x\vee y\in A_2$ or $x\wedge y\in A_2$.*

PROOF. (a). Let $x, y\in A_2$. Then $x\vee y\in A_1\cup A_2$ and $x\wedge y\in A_1\cup A_2$ since $A_3=\emptyset$. But if $x\vee y\in A_1$ then the lattice $\underset{\times}{\bar{A}}$ is not distributive since in view of (4) and (5) it must contain one of the following sublattices L_1, L_2, L_3, L_4 which diagrams are presented in pictures (the classes of congruences of R are indicated by dash lines):



Thus $x\vee y\in A_2$. Similarly, $x\wedge y\in A_2$.

(b) If $x\in A_1$, $y\in A_2$, $x\vee y\in A_1$ and $x\wedge y\in A_1$ then $0_x=1_x$, $0_{x\vee y}=1_{x\vee y}$ and $0_{x\wedge y}=1_{x\wedge y}$. Hence by (5) and (4) we have $1_y=1_{y\wedge(x\vee y)}=1_y\wedge 1_{x\vee y}=1_y\wedge\wedge(0_x\vee 0_y)=(1_y\wedge 0_x)\vee(1_y\wedge 0_y)=(1_y\wedge 1_x)\vee 0_y=1_{x\wedge y}\vee 0_y=0_{x\wedge y}\vee 0_y=0_y$ — a contradiction.

LEMMA 5. *Let $A_3=\emptyset$, $|A_2|>2$ and $a\in A_2$. If all elements $x\in A_2$ satisfy the following identities*

$$1_a\vee 0_x=0_a\vee 1_x=1_a\vee 1_x=1_{a\vee x};\tag{6}$$

$$1_a\wedge 0_x=0_a\wedge 1_x=0_a\wedge 0_x=0_{a\wedge x},\tag{7}$$

then there exists a nontrivial congruence $\Theta(a)$ of $\underset{\times}{A}$ such that $0_a\not\equiv 1_a(\Theta(a))$.

PROOF. It follows from Lemma 4 that A_2 is a sublattice of $\underset{\times}{\bar{A}}$. Hence there exists $b\in A_2$ such that $[a]_R<[b]_R$ or $[b]_R<[a]_R$ since $|A_2|>2$. Let us put $\Theta(a)=\Theta(1_a, 1_b)$ if $[a]_R<[b]_R$ and $\Theta(a)=\Theta(1_b, 1_a)$ if $[b]_R<[a]_R$. To prove that $\Theta(a)$ is a required congruence of $\underset{\times}{A}$ we assume that $\Theta(a)=\Theta(1_a, 1_b)$ for $[a]_R<[b]_R$. The proof in the second case is similar.

By (ii), (6) and (7) we have $\Theta(1_a, 1_b)=\Theta(0_a, 0_b)$ since

$$0_a\equiv 0_b\,\Theta(1_a, 1_b)\quad\text{and}\quad 1_a\equiv 1_b\,\Theta(0_a, 0_b).$$

Let $c \equiv d(\Theta(1_a, 1_b))$. Hence by (ii) we have:

$$1_a \wedge c = 1_a \wedge d, \tag{8}$$

$$1_b \vee c = 1_b \vee d, \tag{9}$$

$$0_a \wedge c = 0_a \wedge d, \tag{10}$$

$$0_b \vee c = 0_b \vee d, \tag{11}$$

and consequently, $[a \wedge c]_R = [a \wedge d]_R$ and $[b \vee c]_R = [b \vee d]_R$.

If $c, d \in A_1$ then $c = c'$ and $d = d'$ so $c' \equiv d'(\Theta(1_a, 1_b))$.

If $c \in A_1$ and $d \in A_2$ then $c = 0_c = 1_c$ and the element d must satisfy (6) and (7). Thus $d = 1_d$ since $1_a \wedge d = 1_a \wedge c = 1_a \wedge 1_c = 1_{a \wedge c} = 1_{a \wedge d} \neq 0_{a \wedge d}$. Hence by (11), $0_b \vee \vee 1_d = 0_b \vee c = 0_b \vee 0_c = 0_{b \vee c} = 0_{b \vee d}$. Furthermore, $0_a = 0_{a \wedge b}$ since $[a]_R = [a]_R \wedge [b]_R = = [a \wedge b]_R$. Now we observe that the elements $b \vee d$, b, $a \wedge d \in A_2$ also satisfy (6) and (7) so we have $0_a = 0_{a \vee (a \wedge d)} = 0_a \vee 0_{a \wedge d} = 0_{a \wedge b} \vee 0_{a \wedge d} = 0_{(a \wedge b) \vee (a \wedge d)} = 0_{a \wedge (b \vee d)} = 1_a \wedge \wedge 0_{b \vee d} = 1_a \wedge (0_b \vee 1_d) = (1_a \wedge 0_b) \vee (1_a \wedge 1_d) = 0_{a \wedge b} \vee 1_{a \wedge d} = 0_a \vee 1_{a \wedge d} = 1_{a \vee (a \wedge d)} = 1_a$ — a contradiction. The argument in the case when $c \in A_2$ and $d \in A_1$ is similar.

If $c, d \in A_2$ then $c = 0_c$ and $d = 0_d$ or $c = 1_c$ and $d = 1_d$ because of (5), (8) and (7). We shall prove that

$$1_b \vee 1_c = 1_{b \vee c} = 1_{b \vee d} = 1_b \vee 1_d \tag{12}$$

and

$$1_b \vee 0_c = 1_{b \vee c} = 1_{b \vee d} = 1_b \vee 0_d. \tag{13}$$

In fact, the element $b \in A_2$ must satisfy (6) so $1_a \vee 1_b = 1_{a \vee b} = 1_b$. Moreover, $[a \vee b \vee c]_R = [a]_R \vee [b]_R \vee [c]_R = [b]_R \vee [c]_R = [b \vee c]_R$ so $1_{a \vee b \vee c} = 1_{b \vee c}$. If $1_b \vee 1_c = 0_{b \vee c}$ then $0_{b \vee c} = 1_b \vee 1_c = 1_a \vee 1_b \vee 1_c = 1_a \vee 0_{b \vee c} = 1_{a \vee b \vee c}$ since the element $b \vee c \in A_2$ satisfies (6). Hence $0_{b \vee c} = 1_{b \vee c}$ — a contradiction. Thus $1_b \vee 1_c = 1_{b \vee c}$. Similarly we prove that $1_b \vee 1_d = 1_{b \vee d}$. The equality $1_{b \vee c} = 1_{b \vee d}$ is obvious. The proof of (13) is analogous.

Now, if $c = 0_c$ and $d = 0_d$ then by (5) $1_a \wedge c' = 1_a \wedge 1_c = 1_{a \wedge c} = 1_{a \wedge d} = 1_a \wedge 1_d = = 1_a \wedge d'$ and by (12) $1_b \vee c' = 1_b \vee 1_c = 1_b \vee 1_d = 1_b \vee d'$. Thus $c' \equiv d'(\Theta(1_a, 1_b))$ because of (ii). The same result we get for $c = 1_c$ and $d = 1_d$ but in this case we apply (13).

We proved that $\Theta(1_a, 1_b)$ is a congruence of $\underset{\times}{A}$. Finally observe that $0_a \not\equiv \not\equiv 1_a(\Theta(1_a, 1_b))$ since otherwise we get $0_a = 1_a$ because of (iii). Q. E. D.

LEMMA 6. *If* $A_3 = \emptyset$ *and* $|A_2| > 2$ *then for any* $a \in A_2$ *there exists a nontrivial congruence* $\Theta(a)$ *of* $\underset{\times}{A}$ *such that* $0_a \not\equiv 1_a(\Theta(a))$.

PROOF. Let $a \in A_2$. Hence by (*a*) of Lemma 4 the set $A(a) = \{x \in A_2 \colon [a]_R < [x]_R$ or $[x]_R < [a]_R\}$ is not empty. We consider two cases.

Case 1. There exists $b \in A(a)$ such that $1_a < 0_b$ or $1_b < 0_a$. Then it follows from Lemma 2 and (iii) that the relation $\Theta(a) = \Theta(0_b, 1_b)$ is a required congruence of A.

Case 2. $1_a \nless 0_x$ and $1_x \nless 0_a$ for all $x \in A(a)$. We shall prove that all elements $x \in A_2$ satisfy (6) and (7). We have $1_a \vee 0_x$, $0_a \vee 1_x$, $1_a \vee 1_x \in A_2$ and $1_a \wedge 0_x$, $0_a \wedge 1_x$,

$0_a \wedge 0_x \in A_2$ because of Lemma 4. If $1_a \vee 0_x = 0_{a \vee x}$ then $1_a < 1_a \vee 0_x = 0_{a \vee x}$ and $a \vee \vee x \in A(a)$ — a contradiction. Hence $1_a \vee 0_x = 1_{a \vee x}$ and also $1_a \vee 1_x = 1_{a \vee x}$ since $1_a \vee 1_x \geqq 1_a \vee 0_x$. If $0_a \wedge 1_x = 1_{a \wedge x}$ then $1_{a \wedge x} < 0_a$ and $a \wedge x \in A(a)$ — a contradiction. So $0_a \wedge 1_x = 0_{a \wedge x}$ and also $0_a \wedge 0_x = 0_{a \wedge x}$. Finally, if $0_a \vee 1_x = 0_{a \vee x}$ then $0_a \vee 1_x = = 0_a \vee 0_x$ and $0_a \wedge 1_x = 0_a \wedge 0_x$ what by the distributivity of $\underset{\sim}{\bar{A}}$ gives $1_x = 0_x$ — a contradiction. Thus $0_a \vee 1_x = 1_{a \vee x}$. The proof of the equality $1_a \wedge 0_x = 0_{a \wedge x}$ is analogous.

Now we apply Lemma 5 what completes the proof.

THEOREM 2. *If a locally Boolean algebra $\underset{\sim}{A}$ is subdirectly irreducible then $\underset{\sim}{A}$ is a 1-element algebra or $\underset{\sim}{A}$ is a 2-element algebra with $x' = x$ or there exist lattices $\underset{\sim}{L}_1 \in CDS$ and $\underset{\sim}{L}_2 \in DS$ such that $\underset{\sim}{A} = \widetilde{\underset{\sim}{L}_1 \oplus \underset{\sim}{L}_2}$.*

PROOF. By Lemma 3 we have $A_3 = \emptyset$. If $|A_2| > 2$ then by Lemma 6 we can associate with any element $a \in A_2$ a nontrivial congruence $\Theta(a)$ of $\underset{\sim}{A}$ such that $0_a \not\equiv \not\equiv 1_a(\Theta(a))$. Let us take $\mathscr{D} = \{\Theta(a) \colon a \in A_2\} \cup \{R\}$. It is easy to see that $\bigcap \mathscr{D}$ is the trivial congruence of $\underset{\sim}{A}$ so $\underset{\sim}{A}$ is subdirectly reducible. Thus we have two possibilities: $A_2 = \emptyset$ or $|A_2| = 2$.

If $A_2 = \emptyset$ then $A = A_1$ and the bounding congruence R of $\underset{\sim}{A}$ is trivial. Hence $\underset{\sim}{A} \cong \underset{\sim}{A}/R$ and $\underset{\sim}{A}$ is subdirectly irreducible iff $\underset{\sim}{A}/R$ is. Thus $\underset{\sim}{A}$ has exactly one element or $\underset{\sim}{A}$ has exactly two elements satisfying $x' = x$.

Let us assume that $|A_2| = 2$. Then there exists $a \in A$ such that $A_2 = [a]_R = = \{0_a, 1_a\}$ and other congruence classes of R are 1-element. Hence R is the only one atom in Con $\underset{\sim}{A}$.

Let us denote by $L_1 = (0_a]$ the principal ideal of $\underset{\sim}{A}$ generated by 0_a and by $L_2 = = [1_a)$ the dual principal ideal of $\underset{\sim}{\bar{A}}$ generated by 1_a. Obviously, lattices $\underset{\sim}{L}_i = (L_i; \vee, \wedge)$, $i = 1, 2$ are distributive and $L_1 \cap L_2 = \emptyset$. To prove that $L_1 \cup L_2 = A$ let us take $x \in A$. Then $x \in A_1$ or $x \in A_2$. If $x \in A_2$ then $x \in [a]_R$ and consequently $x \in L_1 \cup L_2$. If $x \in A_1$ then by (b) of Lemma 4 we have $x \vee 0_a \in \{0_a, 1_a\}$ or $x \wedge 0_a \in \{0_a, 1_a\}$. If $x \vee 0_a \in \{0_a, 1_a\}$ then $x = 0_x \leqq 0_x \vee 0_a = 0_{x \vee a} = 0_a$ so $x \in L_1$. If $x \wedge 0_a \in \{0_a, 1_a\}$ then $x \wedge 1_a = 1_a$. Hence $1_a \leqq x$ and $x \in L_2$.

Now we shall prove that $\underset{\sim}{L}_1 \in$ CDS and $\underset{\sim}{L}_2 \in$ DS. If $|L_1| = 1$ then the lattice $\underset{\sim}{L}_1$ is codisjunctive. Let $|L_1| > 1$, $c, d \in L_1$ and $c < d$. Then the congruence $\Theta(c, d)$ of $\underset{\sim}{\bar{A}}$ does not belong to Con $\underset{\sim}{A}$ since otherwise we have $R \subseteq \Theta(c, d)$, so $0_a \equiv 1_a(\Theta(c, d))$ what together with (iii) gives $0_a = 1_a$ — a contradiction. Thus there exist two distinct elements $x_0, y_0 \in A$ such that

(v) $$x_0 \equiv y_0(\Theta(c, d))$$

and

(vi) $$x_0' \not\equiv y_0'(\Theta(c, d)).$$

By (v) and (ii) we have

(14) $$c \wedge x_0 = c \wedge y_0$$

and

$$d \vee x_0 = d \vee y_0. \tag{15}$$

If $x_0, y_0 \in A_1$ then $x_0' = x_0$ and $y_0' = y_0$ what contradicts (vi). If $x_0, y_0 \in A_2$ then $\{x_0, y_0\} = \{0_a, 1_a\}$ and $x_0' = y_0$, $y_0' = x_0$ so $x_0' \equiv y_0'(\Theta(c, d))$ — a contradiction. Thus $x_0 \in A_1$ and $y_0 \in \{0_a, 1_a\}$ or $x_0 \in \{0_a, 1_a\}$ and $y_0 \in A_1$. In the first case we have $c < d \leqq y_0$ what together with (14) and (15) gives $c \wedge x_0 = c$ and $y_0 = d \vee x_0 = 0_d \vee 0_{x_0} = 0_{d \vee x_0} = 0_{y_0} = 0_a$. Hence $x_0 \in L_1$ and the lattice $\underset{\times}{L}_1$ is codisjunctive. The same result we get for $x_0 \in \{0_a, 1_a\}$ and $y_0 \in A_1$.

The proof that $\underset{\times}{L}_2$ is a disjunctive lattice is similar. Finally observe that $\underset{\times}{A} = \widetilde{\underset{\times}{L}_1 \oplus \underset{\times}{L}_2}$. Q. E. D.

By Birkhoff's theorem (see [3] p. 124) we have

COROLLARY. *Any locally Boolean algebra is isomorphic to a subdirect product of some family of algebras being either 2-element chains with $x' = x$ or algebras of the form $\widetilde{\underset{\times}{L}_1 \oplus \underset{\times}{L}_2}$ where $\underset{\times}{L}_1 \in CDS$ and $\underset{\times}{L}_2 \in DS$.*

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(Received November 18, 1985)

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CONVEXITY OF MULTIVARIATE BERNSTEIN POLYNOMIALS AND BOX SPLINE SURFACES

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Abstract

We give sufficient conditions on the control points for a Bernstein—Bézier representation of a multivariate polynomial which guarantees that it is a convex surface. A similar result is given for box spline surfaces.

1. Introduction

Expansions of the form

$$S(x) = \sum_i c_i B_i(x), \quad x \in \Omega,\ c_i \in \mathbf{R}^m, \tag{1.1}$$

where Ω is some domain in $\mathbf{R}^s$ and the basis functions B_i satisfy

$$\begin{aligned} B_i(x) &\geqq 0, \qquad x \in \Omega, \\ \sum_i B_i(x) &= 1, \qquad x \in \Omega, \end{aligned} \tag{1.2}$$

are important both for theoretical and practical purposes. The control points c_i contain "visible" information about the geometrical features of the surface $S(x)$ which is useful in Computer-Aided Design. Typical examples of this type of surface representation are based on Bernstein polynomials or various spline surfaces.

The objective of this paper is to study the relationship between properties of the control points c_i and the convexity of the surface S. Our interest in this subject arose from the recent paper by Chang and Davis [2] which addressed this question for bivariate Bernstein polynomials. In Section 2 we study the s-variate Bernstein polynomial surfaces. As we shall show the bivariate case does not exhibit all the features of the general multivariate situation. Our approach also provides an alternative proof of some of the results in [2]. Analogous questions for linear combinations of translates of box splines are discussed in Section 3. Finally, in Section 4, we conclude with some remarks about interpolation of scattered data by convex functions suggested by the material in Sections 3 and 4.

This work was partially supported by NATO Grant No. DJ RG 639/84.

1980 *Mathematics Subject Classification.* Primary 41A15; Secondary 41A63, 26B25, 41A05.
Key words and phrases. Bernstein polynomials, box splines, conditions for convexity of polynomial and spline surfaces, monotone convergence, conditionally positive definite matrices, convex interpolation of multivariate data.

2. Bernstein polynomials

For $x=(x_1, \ldots, x_s)^T \in \mathbf{R}^s$ and any $s+1$ affinely independent points $v^i \in \mathbf{R}^s$, $i=1, \ldots, s+1$ the barycentric coordinates of x with respect to the s-simplex with vertices $v^1, \ldots, v^{s+1}$,

$$\sigma = [v^1, \ldots, v^{s+1}] = \text{convex hull } \{v^1, \ldots, v^{s+1}\}$$

are denoted by $\lambda=\lambda(\sigma; x)=(\lambda_1, \ldots, \lambda_{s+1})^T$. Therefore, we have

$$x = \sum_{j=1}^{s+1} \lambda_j v^j, \quad \sum_{j=1}^{s+1} \lambda_j = 1. \tag{2.1}$$

We will sometimes write

$$x = F_\sigma(\lambda) = \sum_{j=1}^{s+1} \lambda_j v^j$$

so that F maps the standard s-simplex $\Delta_s=\left\{(\lambda_1, \ldots, \lambda_{s+1}) \colon \sum_{i=1}^{s+1} \lambda_i=1,\ \lambda_i \geqq 0\right\}$ onto σ. The Bernstein polynomial basis functions of degree $\leqq k$ are given by

$$B_\beta^k(\lambda) = \frac{k!}{\beta!} \lambda^\beta, \quad \beta \in \mathbf{Z}_+^{s+1}, \quad |\beta| = k,$$

where $|\beta|=\beta_1+\ldots+\beta_{s+1}$, $\lambda^\beta=\lambda_1^{\beta_1}\ldots\lambda_{s+1}^{\beta_{s+1}}$ and $\beta!=\beta_1!\ldots\beta_{s+1}!$. With any set of scalars $\Phi=\{b_\beta\}_{|\beta|=k}$ we define the Bernstein polynomials

$$B_k[\Phi; \lambda] = \sum_{|\beta|=k} b_\beta B_\beta^k(\lambda).$$

Since

$$\sum_{|\beta|=k} B_\beta^k(\lambda) = 1, \quad B_\beta^k(\lambda) \geqq 0, \quad \lambda \in \Delta_s, \tag{2.2}$$

it follows that

$$B_k[\Phi; \lambda] \subseteq [\Phi], \quad \lambda \in \Delta_s, \tag{2.3}$$

where as above $[\Phi]$ denotes the convex hull of Φ.

It is sometimes convenient to regard Φ as a function on the "discrete simplex"

$$(\sigma)_k = \{F_\sigma(\beta/k) \colon \beta \in \mathbf{Z}_+^{s+1},\ |\beta| = k\} \subseteq \sigma.$$

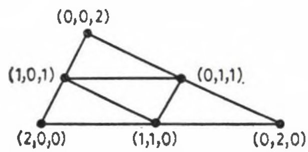

Fig. 1. Triangulation: $k=s=2$

Specifically, when $b_\beta=f\left(F_\sigma(\beta/k)\right)$ for some continuous function f on σ we use the shorthand notation

$$B_k[\Phi; \lambda] = B_k[f; \lambda].$$

We wish to discuss conditions on Φ which ensure the convexity of $B_k[\Phi; \lambda]$ over σ.

To this end, recall that $f\in C(\Omega)$, Ω a convex set, is called convex if for any points $x^1, \dots, x^l \in \Omega$ and any $\lambda \in \Delta_{l-1}$

(2.4)
$$\sum_{j=1}^{l} \lambda_j f(x) \geqq f\left(\sum_{j=1}^{l} \lambda_j x^j\right).$$

When f is in $C^2(\Omega)$ it is well-known that the convexity of f is equivalent to the positive semidefiniteness of the Hessian $\left(\frac{\partial^2 f(x)}{\partial x_j \partial x_k}\right)_{j,k=1}^{s}$ on Ω.

In terms of barycentric coordinates this becomes

PROPOSITION 2.1. *Suppose $f\in C^2(\sigma)$ and let*

$$g(\lambda) = f(F_\sigma(\lambda)), \quad \lambda \in \Delta_s.$$

Then f is convex on σ if and only if the $(s+1)\times(s+1)$ matrix

$$\left(\frac{\partial^2}{\partial \lambda_1 \partial \lambda_j} g(\lambda)\right)_{i,j=1}^{s+1}$$

is conditionally positive definite on Δ_s.

Recall that an $(s+1)\times(s+1)$ symmetric matrix $A=(a_{ij})$ is called *conditionally positive definite* if

$$(c, Ac) = \sum_{i=1}^{s+1} \sum_{j=1}^{s+1} c_i c_j a_{ij} \geqq 0$$

for all $c = (c_1, \dots, c_{s+1})^T \in \mathscr{P}_0 = \{c\in \mathbf{R}^{s+1}\colon c_1+\dots+c_{s+1} = 0\}$.

PROOF. Straightforward application of the chain rule yields

(2.5)
$$\frac{\partial^2 f}{\partial x_l \partial x_k} = \sum_{i=1}^{s+1} \sum_{j=1}^{s+1} \frac{\partial^2 g}{\partial \lambda_i \partial \lambda_j} \frac{\partial \lambda_i}{\partial x_l} \frac{\partial \lambda_j}{\partial x_k}.$$

Let V denote the matrix with columns, v^i, $i=1, \dots, s+1$ and set

$$\Lambda = \left(\frac{\partial \lambda_i}{\partial x_j}\right)_{i,j=1}^{s+1,s}.$$

Differentiation of (2.1) provides

(2.6)
$$I = V\Lambda; \quad \sum_{i=1}^{s+1} \frac{\partial \lambda_i}{\partial x_j} = 0, \quad j = 1, \dots, s.$$

Now (2.5) yields

(2.7)
$$\sum_{l=1}^{s} \sum_{k=1}^{s} c_l c_k \frac{\partial^2 f}{\partial x_l \partial x_k} = \sum_{i=1}^{s+1} \sum_{j=1}^{s+1} \frac{\partial^2 g}{\partial \lambda_i \partial \lambda_j} \left(\sum_{l=1}^{s} c_l \frac{\partial \lambda_i}{\partial x_l}\right)\left(\sum_{k=1}^{s} c_k \frac{\partial \lambda_j}{\partial x_k}\right).$$

Note that in view of (2.6) the coefficients

$$a_i = \sum_{l=1}^{s} c_l \frac{\partial \lambda_i}{\partial x_l}, \qquad i = 1, \dots, s+1,$$

sum to zero. Moreover, (2.6) says that $\mathbf{R}^s$ is mapped by A onto $\mathscr{P}_0$. Hence f is convex if and only if

$$\sum_{i=1}^{s+1}\sum_{j=1}^{s+1} a_i a_j \frac{\partial^2 g}{\partial\lambda_i\,\partial\lambda_j} \geqq 0, \quad a\in\mathscr{P}_0,$$

which completes the proof.

For Bernstein polynomials, we note that

$$\frac{\partial}{\partial\lambda_i} B_\beta^k(\lambda) = kB_{\beta-e^i}^{k-1}(\lambda)$$

where $e^i=(\delta_{ij})_{j=1}^{s+1}$ denotes the i-th coordinate vector in $\mathbf{R}^{s+1}$. Hence

$$\begin{aligned}\frac{\partial^2}{\partial\lambda_j\,\partial\lambda_i} B_k[\Phi;\,\lambda] &= k(k-1)\sum_{|\beta|=k} b_\beta\, B_{\beta-e^i-e^j}^{k-2}(\lambda)\\ &= k(k-1)\sum_{|\beta|=k-2} b_{\beta+e^i+e^j}\, B_\beta^{k-2}(\lambda).\end{aligned} \tag{2.8}$$

As an immediate consequence of (2.2), (2.8) and Proposition 2.1, we state

PROPOSITION 2.2. *Let* $\Phi=\{b_\beta\}_{|\beta|=k}$. *Suppose that for every* $\beta\in\mathbf{Z}_+^{s+1}$, $|\beta|=k-2$, *the matrix*

$$A=(a_{ij})_{i,j=1}^{s+1}=(b_{\beta+e^i+e^j})_{i,j=1}^{s+1}$$

is conditionally positive definite. Then $B_k[\Phi;\lambda]$ *is convex over* Δ_s.

Following [8] we note that any control net of the form $b_\beta=h(\|\beta\|_s)$ where $h'(t)=\int_0^\infty e^{t\sigma}\,d\mu(\sigma)$ for $t\geqq 0$, $d\mu\geqq 0$, and $\|\cdot\|^2$ is any quadratic norm on $\mathbf{R}^{s+1}$ satisfies the hypothesis of Proposition 2.2. This result comes from the identity

$$\sum_{i=1}^{s+1}\sum_{j=1}^{s+1} c_i c_j\,\|\beta+e^i+e^j\|^2 = 2(c,\,c)$$

when $c\in\mathscr{P}_0$. Consequently, for $t\geqq 0$ the matrix $(e^{t\|\beta+c^i+c^j\|^2})$ is positive semi-definite and we obtain

COROLLARY 2.1. *If* $b_\beta=h(\|\beta\|^2)$ *whenever* $h'(t)=\int_0^\infty e^{t\sigma}\,d\mu(\sigma)$, $t\geqq 0$, $d\mu\geqq 0$, *and* $\|\cdot\|^2$ *is any quadratic norm on* $\mathbf{R}^{s+1}$ *then* $B_k[\Phi;\,\lambda]$ *is convex.*

Of course, from the univariate case it is expected that under the hypothesis of Corollary 2.1, $B_k[\Phi,\lambda]$ has "higher order" convexity properties.

A simple sufficient condition for conditional positive definiteness for low order matrices is given by

THEOREM 2.1. *Suppose* $A=(a_{ij})_{i,j=1}^m$ *is a symmetric matrix which satisfies*

$$a_{kk}+a_{ij}\geqq a_{ik}+a_{kj} \tag{2.9}$$

for all i, j, k. If $m \leqq 4$ then A is conditionally positive definite. For $m > 4$ there exist matrices A satisfying (2.9) *which are not conditionally positive definite.*

It is well-known, c.f. [8], that A is conditionally positive if and only if the "difference" matrix.

$$B = (b_{ij})_{i,j=1}^{m-1}, \quad b_{ij} = a_{ij} - a_{im} - a_{jm} + a_{mm} \tag{2.10}$$

is positive definite. Using this fact, the assertion follows easily for $m=2, 3$. The case $m=3$ was established by Chang, Davis [2]. Before proving the remaining assertions let us state an immediate consequence of Theorem 2.1 and Proposition 2.2 on the convexity of Bernstein polynomials.

COROLLARY 2.2. *Let $s \leqq 3$ and suppose $\Phi = \{b_\beta\}_{|\beta|=k}$ satisfies*

$$b_{\beta+2e^i} + b_{\beta+e^i+e^j} \geqq b_{\beta+e^i+e^l} + b_{\beta+e^j+e^l}, \quad |\beta| = k-2, \tag{2.11}$$

then $B_k[\Phi; \lambda]$ is convex.

PROOF of Theorem 2.1.

Note that $A = (a_{ij})_{i,j=1}^m$ is conditionally positive definite if and only if the matrix

$$D = (d_{ij})_{i,j=1}^m, \quad d_{ij} = \frac{1}{2}(a_{ii} + a_{jj}) - a_{ij} \tag{2.12}$$

is conditionally *negative* definite. Clearly

$$d_{ii} = 0, \qquad i = 1, \ldots, m,$$

while (2.9) implies

$$d_{ij} \geqq 0, \qquad i, j = 1, \ldots, m.$$

Furthermore, note that

$$d_{ik} + d_{kj} - d_{ij} = a_{kk} + a_{ij} - (a_{ik} + a_{kj}) \geqq 0,$$

i.e. D is a *distance matrix*. In general, distance matrices are not conditionally negative definite. It is easy to check that the matrix.

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 0 & 1 \\ 2 & 1 & 2 & 1 & 0 \end{pmatrix}$$

is a distance matrix which is not conditionally negative definite. On the other hand, it is known that a matrix is conditionally negative definite if and only if

$$d_{ij} = \|x^i - x^j\|^2$$

for some points x^i in some Euclidean space where $\|\cdot\|$ is the Euclidean norm, c.f. [8]. In particular, by distributing four points at the endpoints of $[0, 1]$ we see that the symmetric matrices

$$D_1=\begin{pmatrix}0&1&1&1\\1&0&0&0\\1&0&0&0\\1&0&0&0\end{pmatrix},\quad D_2=\begin{pmatrix}0&1&0&0\\1&0&1&1\\0&1&0&0\\0&1&0&0\end{pmatrix},$$

$$D_3=\begin{pmatrix}0&0&1&0\\0&0&1&0\\1&1&0&1\\0&0&1&0\end{pmatrix},\quad D_4=\begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\1&1&1&0\end{pmatrix},$$

$$D_5=\begin{pmatrix}0&0&1&1\\0&0&1&1\\1&1&0&0\\1&1&0&0\end{pmatrix},\quad D_6=\begin{pmatrix}0&1&0&1\\1&0&1&0\\0&1&0&1\\1&0&1&0\end{pmatrix},$$

$$D_7=\begin{pmatrix}0&1&1&0\\1&0&0&1\\1&0&0&1\\0&1&1&0\end{pmatrix},$$

are conditionally negative definite distance matrices. Let D be any distance matrix and define

$$y_1=-y_7+\frac{1}{2}(d_{12}+d_{13}-d_{23})$$

$$y_2=-y_7+\frac{1}{2}(d_{12}+d_{24}-d_{14})$$

$$y_3=-y_7+\frac{1}{2}(d_{34}+d_{13}-d_{14})$$

$$y_4=-y_7+\frac{1}{2}(d_{34}+d_{24}-d_{23})$$

$$y_5=\quad y_7+\frac{1}{2}(d_{23}+d_{14}-d_{12}-d_{34})$$

$$y_6=\quad y_7+\frac{1}{2}(d_{23}+d_{14}-d_{13}-d_{24}).$$

Then $D=\sum_{i=1}^{7} y_iD_i$ and for some choice of $y_7\geqq 0$ we can insure that $y_i\geqq 0$, $i=1,\ldots,$ $\ldots, 6$. To see this we must show both $d_{34}+d_{12}-d_{23}-d_{14}$, $d_{24}+d_{13}-d_{23}-d_{14}\leqq$

$\leqq d_{ij}+d_{jk}-d_{ik}$, $i\neq k\neq j$. For instance, we have

$$d_{24}+d_{13}-d_{23}-d_{14} \leqq (d_{21}+d_{14})+d_{13}-d_{23}-d_{14} = d_{21}+d_{13}-d_{23}.$$

The other cases follow similarly.

Next we recall a geometric interpretation of the condition (2.11) which was the basic approach taken in [2] in the bivariate case.

To this end, we say a collection $\mathcal{T}$ of simplices is called a triangulation of Ω if $\Omega=\bigcup\{\delta\colon \delta\in\mathcal{T}\}$ and for any $\delta, \delta'\in\mathcal{T}$ the intersection $\delta\cap\delta'$ is either empty or a *common face* of δ and δ'. Here any (lower dimensional) simplex spanned by a subset of the vertices of a simplex δ is called a face of δ.

Now observe that for $s=2$, $(\sigma)_k$ induces a unique triangulation $\mathcal{T}_k(\sigma)$ of σ (see Fig. 1) with the following properties:

(2.13)
i) The vertices of the elements of $\mathcal{T}_k(\sigma)$ belong to $(\sigma)_k$.
ii) The elements of $\mathcal{T}_k(\sigma)$ have all equal volume and are congruent to σ.
iii) The union of any two simplices in $\mathcal{T}_k(\sigma)$ with a common face forms a parallelogram.

Let us denote by $S(\Phi)$ the unique continuous function on σ which interpolates Φ on $(\sigma)_k$ and which is linear on each $\delta\in\mathcal{T}_k(\sigma)$. Then we have

LEMMA. 2.1. *Let $s=2$. $S(\Phi)$ is convex if and only if Φ satisfies* (2.11).

PROOF. It was already pointed out in [2] that the convexity of $S(\Phi)$ implies (2.11). Let us briefly recall their reasoning which will be helpful to understand our subsequent discussion. If $S(\Phi)$ is convex on σ, then it is convex on any paralellogram formed by two adjacent triangles in $\mathcal{T}_k(\sigma)$. The barycentric coordinates of the vertices of two such triangles are $(\beta+2e^l)/k$, $(\beta+e^i+e^l)/k$, $(\beta+e^j+e^l)/k$ and $(\beta+e^i+e^l)/k$, $(\beta+e^j+e^l)/k$, $(\beta+e^i+e^j)/k$, respectively.

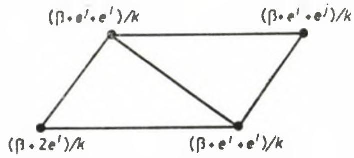

Fig. 2. Parallelogram

Thus

$$\begin{aligned} b_{\beta+2e^l}+b_{\beta+e^i+e^j} &= S(\Phi)\big(F_\sigma(\beta+2e^l)/k\big)+S(\Phi)\big(F_\sigma(\beta+e^i+e^j)/k\big) \\ &\geqq 2S(\Phi)\left(\frac{1}{2}\{F_\sigma(\beta+e^i+e^l)/k+F_\sigma(\beta+e^j+e^l)/k\}\right) \\ &= b_{\beta+e^i+e^l}+b_{\beta+e^j+e^l}, \end{aligned}$$

because the restriction of $S(\Phi)$ to the common edge of the two triangles is linear.

Conversely, since a function is convex if and only if its restriction to any line is a (univariate) convex function convexity of $S(\Phi)$ follows from (2.11) by using the well-known fact:

LEMMA 2.2. *The piecewise continuous linear interpolant to the points* $\{(t_i, f_i)\}_{i=1}^n$, *with breakpoints at* $t_1 < \ldots < t_n$ *is convex if and only if*

$$\frac{t_{i+2}-t_{i+1}}{t_{i+2}-t_i} f_i + \frac{t_{i+1}-t_i}{t_{i+2}-t_i} f_{i+2} \geqq f_{i+1}, \quad i = 1, \ldots, n-2, \tag{2.14}$$

that is, if and only if the broken line connecting any three consecutive points is convex.

Therefore, as a consequence of Corollary 2.2 and Lemma 2.1 we see that for $s=2$, $B_k[\Phi; \lambda]$ will be convex when $S(\Phi)$ is convex. This fact appears in Theorem 3 and Theorem 5 of [2]. In order to motivate our analysis of the general case, we will first give an alternative derivation of this fact which differs from the above reasoning used in [2]. The method below will not involve the Hessian of $B_k[\Phi, \lambda]$ at all but relies instead on Lemma 2.1 and a simple *degree raising* argument. We formulate this part of the argument for arbitrary $s \geqq 2$ for later purposes.

Given the control vertices $\Phi = \{b_\beta\}_{|\beta|=k}$, we introduce new control vertices $E\Phi = \{b_\beta^*\}_{|\beta|=k+1}$ by defining

$$b_\beta^* = \frac{1}{k+1} \sum_{j=1}^{s+1} \beta_j b_{\beta - e^j}. \tag{2.15}$$

Therefore, observing that for $\lambda \in \Delta_s$

$$B_k[\Phi; \lambda] = (\lambda_1 + \ldots + \lambda_{s+1}) B_k[\Phi; \lambda]$$

we can rewrite the right-hand side and obtain

$$B_k[\Phi; \lambda] = B_{k+1}[E\Phi; \lambda].$$

This process of representing the k-th Bernstein polynomial in terms of the $(k+1)$-st Bernstein polynomials is sometimes called *degree raising*. Repeating this procedure, $E^l\Phi = E^{l-1}(E\Phi)$ leads to the useful fact

$$\lim_{l \to \infty} \|B_k[\Phi; \cdot] - S(E^l\Phi)\|_\infty(\sigma) = 0 \tag{2.16}$$

observed in [7]. Here $\|f\|_\infty(\sigma)$ denotes the maximum norm on σ. To use this equation, we observe that the map $\Phi \to E\Phi$ preserves simple linear inequalities.

LEMMA 2.3. *Let s be arbitrary and let* $\Diamond$ *denote either one of the relations* $\geqq$, $\leqq$. *Suppose that the control net* $\Phi = \{b_\beta\}_{|\beta|=k}$ *satisfies the linear inequalities*

$$b_{\beta+e^{i_1}+e^{j_1}} + b_{\beta+e^{i_2}+e^{j_2}} \Diamond b_{\beta+e^{i_1}+e^{j_2}} + b_{\beta+e^{i_2}+e^{j_1}}$$

for $|\beta| = k-2$ *and* $j_1 < j_2$, $i_1 < i_2$. *Then*

$$b^*_{\beta+e^{i_1}+e^{j_1}} + b^*_{\beta+e^{i_2}+e^{j_2}} \Diamond b^*_{\beta+e^{i_1}+e^{j_2}} + b^*_{\beta+e^{i_2}+e^{j_1}}$$

also hold for $|\beta| = k-1$ *and* $j_1 < j_2$, $i_1 < i_2$.

PROOF. By (2.15) we get

$$(k+1)\big(b^*_{\beta+e^{i_1}+e^{j_1}}+b^*_{\beta+e^{i_2}+e^{j_2}}-(b^*_{\beta^{i_1}+e^{j_2}}+b^*_{\beta+e^{i_2}+e^{j_1}})\big)$$

$$=\sum_{j=1}^{s+1}\beta_j\big(b_{\beta+e^{i_1}+e^{j_1}-e^j}+b_{\beta+e^{i_2}+e^{j_2}-e^j}-(b_{\beta+e^{i_1}+e^{j_2}-e^j}+b_{\beta+e^{i_2}+e^{j_1}-e^j})\big)$$

$$=\sum_{j=1}^{s+1}\{(\delta_{i_1,j}+\delta_{j_1,j})b_{\beta+e^{i_1}+e^{j_1}-e^j}+(\delta_{i_2,j}+\delta_{j_2,j})b_{\beta+e^{i_2}+e^{j_2}-e^j}$$

$$-(\delta_{i_1,j}+\delta_{j_2,j})b_{\beta+e^{i_1}+e^{j_2}-e^j}-(\delta_{i_2,j}+\delta_{j_1,j})b_{\beta+e^{i_2}+e^{j_1}-e^j}\}=\Sigma_1+\Sigma_2.$$

Now Σ_1 has the sign determined by $\Diamond$ while

$$\Sigma_2=b_{\beta+e^{i_1}}-b_{\beta+c^{j_2}}+b_{\beta+e^{i_1}}-b_{\beta+e^{i_2}}+b_{\beta+e^{j_2}}-b_{\beta+e^{j_1}}+b_{\beta+e^{i_2}}-b_{\beta+e^{i_1}}=0$$

proving the assertion.

Thus for the bivariate case, the convexity of $B_k[\Phi;\lambda]$ when $S(\Phi)$ is convex follows from Lemma 2.1, Lemma 2.3 and (2.16).

Of course, the essential ingredient in this method of proof is that the convexity of the surface $S(\Phi)$ is preserved under "degree raising", that is, $S(E\Phi)$ is convex when $S(\Phi)$ is convex. To extend this reasoning to the general multivariate case one is confronted with the difficulty that there is no unique analogue of the triangulation $\mathscr{T}_k(\sigma)$, when $s\geq 2$. The following figure illustrates this for $s=3$ and $k=2$.

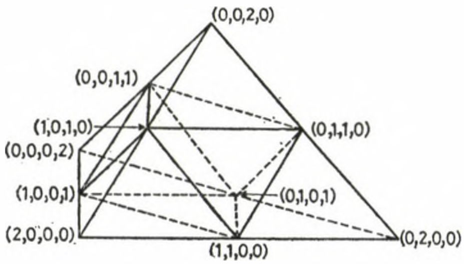

Fig. 3. Triangulation: $s=3$, $k=2$

Note that there are several possibilities for triangulating the interior polytope with vertices $(1,0,1,0)$, $(1,1,0,0)$, $(0,1,0,1)$, $(0,1,1,0)$, $(0,0,1,1)$, and $(1,0,0,1)$. For instance, either connecting $(1,0,1,0)$ to $(0,1,0,1)$ or $(0,0,1,1)$ to $(1,1,0,0)$ by an edge produces a triangulation.

Nevertheless there is a canonical way to construct triangulations for σ which will allow us to employ a degree raising argument even in the general s-variate case. To this end, let $\mathscr{P}_s$ denote the group of all permutations of $\{1,\dots,s\}$ and for $\pi\in\mathscr{P}_s$ define the simplex

$$\delta_\pi=\{u\in[0,1]^s:\ u_{\pi(1)}\geq\dots\geq u_{\pi(s)}\}$$

$$=[v^0,\dots,v^s]$$

where $v^0=0$, $v^j=v^{j-1}+e^{\pi(j)}$, $j=1, \ldots, s$. All these simplices are congruent and therefore have equal volume. Specifically, letting ι be the identitity in $\mathscr{P}_s$ we have

$$\delta_\iota = \{u\in[0, 1]^s\colon\ u_1 \geqq \ldots \geqq u_s\} = [0, e^1, e^1+e^2, \ldots, e^1+\ldots+e^s].$$

It is well-known [1, 4] that

$$\mathscr{K}_s = \{\delta_\pi+\alpha\colon\ \alpha\in\mathbf{Z}^s,\ \pi\in\mathscr{P}_s\}$$

forms a triangulation of $\mathbf{R}^s$. Moreover, for any non-negative integer k

$$\mathscr{C}_{s,k} = \{\delta/k\colon\ \delta\in\mathscr{K}_s,\ \delta\subseteq k\delta_\iota = [0, ke^1, \ldots, k(e^1+\ldots+e^s)]\}$$

is a triangulation of δ_ι. Thus for any affine map $A\colon \delta_\iota\to\sigma$ and any $k\in\mathbf{N}$, the collection

(2.17) $$\mathscr{T}_{k,A}(\sigma) = \{A(\delta)\colon\ \delta\in\mathscr{C}_{s,k}\}$$

is obviously a triangulation of σ.

Each mapping A is determined by a permutation of the vertices of σ. Therefore we may expect distinct triangulations as we vary A, see Fig. 3. However, when $s=2$, it can be verified that $\mathscr{T}_{k,A}(\sigma)$ is independent of A and agrees with the triangulation $\mathscr{T}_k(\sigma)$ described by (2.13), see Fig. 1.

The relevant properties of $\mathscr{T}_{k,A}(\sigma)$ are stated in

LEMMA 2.4. (i) *The vertices of the elements of $\mathscr{T}_{k,A}(\sigma)$ belong to $(\sigma)_k$.*

(ii) *For any two simplices $\delta=[u^1, u^2, \ldots, u^{s+1}]$ and $\tilde{\delta}=[\tilde{u}^1, u^2, \ldots, u^{s+1}]$ in $\mathscr{T}_{k,A}(\sigma)$ there exist vertices u^p, u^q in $\delta\cap\tilde{\delta}$ such that $u^1, \tilde{u}^1, u^p, u^q$ span a planar parallelogram.*

The property (i), is clear from the above construction of $\mathscr{T}_{k,A}(\sigma)$. Concerning (ii) let $\delta=[u^1, \ldots, u^{s+1}]$ be any s-simplex. To each $l\in\{1, \ldots, s+1\}$ assign the numbers

$$l^+ = \begin{cases} l+1, & \text{for } l\leqq s; \\ 1, & \text{for } l = s+1; \end{cases}$$

$$l^- = \begin{cases} l-1, & \text{for } l > 1; \\ s+1, & \text{for } l = 1; \end{cases}$$

and set

(2.18) $$\tilde{u}^l = u^{l^-}+u^{l^+}-u^l.$$

The simplex $\tilde{\delta}=[u^1, \ldots, \tilde{u}^l, \ldots, u^{s+1}]$ is said to be generated from pivoting $\delta=[u^1, \ldots, u^{s+1}]$ by *reflection* of u^l across the edge $[u^{l^-}, u^{l^+}]$. It was shown in [1] that whenever δ is in $\mathscr{K}_s$ and $\tilde{\delta}$ is obtained from pivoting any of the vertices of δ by reflection, then $\tilde{\delta}\in\mathscr{K}_s$. As a consequence of this result it follows that whenever $\delta=[u^1, u^2, \ldots, u^{s+1}]$ and $\tilde{\delta}=[\tilde{u}^1, u^1, \ldots, u^{s+1}]$ are simplices in $\mathscr{K}_s$ with a common $(s-1)$-dimensional face then there exist $i_1, i_2\neq 1$ such that

$$\tilde{u}^1+u^1 = u^{i_1}+u^{i_2}$$

which proves (ii).

Specializing the construction (2.17) to the situation shown in Fig. 2 let A be defined by setting $A((0,0,0))=v^1$, $A((1,0,0))=v^2$, $A((1,1,0))=v^3$, $A((1,1,1))=v^4$. This corresponds to connecting (1, 0, 1, 0) and (0, 1, 0, 1) in Fig. 2 while the mappings given by $A((0,0,0))=v^1$, $A((1,0,0))=v^3$, $A((1,1,0))=v^2$, $A((1,1,1))=v^4$ and $A((0,0,0,))=v^1$, $A((1,0,0))=v^2$, $A((1,1,0))=v^4$, $A((1,1,1))=v^3$ give rise to connecting (1, 1, 0, 0) to (0, 0, 1, 1) and (0, 1, 1, 0) to (1, 0, 0, 1), respectively.

Before turning to the general s-variate case let us briefly discuss Corollary 2.2 in this context. We begin noting that condition (2.9) or (2.11) for $s=3$ does *not* imply the convexity of a piecewise linear continuous interpolant $S_{k,A}(\Phi)$ of Φ with respect to $\mathcal{T}_{k,A}(\sigma)$, any A as above.

In fact, for the triangulation obtained by introducing an edge connecting (0, 0, 1, 1) and (1, 1, 0, 0) in Fig. 3 the convexity of the corresponding piecewise linear interpolant implies for $a_{ij}=b_{e^i+e^j}$,

$$a_{13}+a_{24} \geqq a_{34}+a_{12}, \quad a_{14}+a_{23} \geqq a_{34}+a_{12}. \tag{2.19}$$

However, if we connect (1, 0, 1, 0) and (0, 1, 0, 1) to form another triangulation the convexity of the piecewise linear interpolant implies that

$$a_{34}+a_{12} \geqq a_{13}+a_{24}, \qquad a_{14}+a_{23} \geqq a_{13}+a_{24}. \tag{2.20}$$

Finally, connecting (1, 0, 0, 1) and (0, 1, 1, 0) yields the inequalities

$$a_{13}+a_{24} \geqq a_{14}+a_{23}, \quad a_{34}+a_{12} \geqq a_{14}+a_{23}. \tag{2.21}$$

In terms of the entries d_{ij} of the corresponding distance matrix D (cf. (2.12)) these inequalities read respectively

$$d_{13}+d_{24} \leqq d_{34}+d_{12}, \qquad d_{14}+d_{23} \leqq d_{34}+d_{12}, \tag{2.22}$$

$$d_{34}+d_{12} \leqq d_{13}+d_{24}, \qquad d_{14}+d_{23} \leqq d_{13}+d_{24}, \tag{2.23}$$

and

$$d_{13}+d_{24} \leqq d_{14}+d_{23}, \qquad d_{34}+d_{12} \leqq d_{14}+d_{23}. \tag{2.24}$$

The matrices $D_1, \ldots, D_4$ satisfy these conditions (indeed equality holds throughout) while for each of the remaining D_5, D_6, D_7 at least one of the conditions (2.22)—(2.24) is violated. Nevertheless, it can be verified that every nonnegative combination of D_5, D_6 and D_7 satisfies at least one of the requirements (2.22)—(2.24). Thus, in this case, for every Φ satisfying (2.11) there exists an A such that the piecewise linear interpolant $S_{k,A}(\Phi)$ with respect to $\mathcal{T}_{k,A}(\sigma)$ is convex. In general, we do not know whether or not this remains valid. Nonetheless, this leads to the following general result for arbitrary spatial dimension s.

THEOREM 2.2. *Suppose there exists an affine map A such that the piecewise linear interpolant $S_{k,A}(\Phi)$ of Φ with respect to $\mathcal{T}_{k,A}(\sigma)$ is convex. Then $B_k[\Phi;\cdot]$ is convex.*

PROOF. The proof of the theorem follows the pattern of the bivariate case and is based upon

LEMMA 2.5. *Suppose $S_{k,A}(\Phi)$ is convex. Then $S_{k+1,A}(E\Phi)$ is convex, too, where $E\Phi$ is given by* (2.15).

PROOF. $S_{k,A}(\Phi)$ is convex on σ if and only if for any line L the piecewise linear function $S_{k,A}(\Phi)|_{L\cap\sigma}$ is convex. By continuity it is sufficient to consider only lines such that for any $\delta\in\mathscr{T}_{k,A}(\sigma)$, $L\cap\delta$ is not contained in any lower dimensional face of δ and $L\cap(\sigma)_k=\emptyset$. Therefore, by Lemma 2.2 $S_{k,A}(\Phi)$ is convex on σ if and only if for any two simplices $\delta=[u^1, \ldots, u^{s+1}]$, $\tilde{\delta}=[\tilde{u}^1, u^2, \ldots u^{s+1}]\in\mathscr{T}_{k,A}$

$$G = S_{k,A}(\Phi)|_{\delta\cup\tilde{\delta}}$$

is convex on $\delta\cup\tilde{\delta}$. Without loss of generality we may assume that G has the form

$$G(x) = \begin{cases} c\lambda_1(x, \delta), & x\in\delta, \\ 0, & x\in\tilde{\delta}. \end{cases}$$

Clearly, G is convex on $\delta\cup\tilde{\delta}$ if and only if $c\geqq 0$ or equivalently if and only if

$$G(u^1)+G(\tilde{u}^1) \geqq 2G\left(\frac{1}{2}(u^1+\tilde{u}^1)\right). \tag{2.25}$$

Now by Lemma 2.4 (ii) there must exist $p, q\in\{2, \ldots, s+1\}$ such that u^1, $\tilde{u}^1$, u^p, u^q form a plane parallelogram. Hence there must exist some $\alpha\in\mathbf{Z}_+^{s+1}$, $|\alpha|=k-2$, and $i_1, j_1, i_2, j_2\in\{1, \ldots, s+1\}$ such that $u^1=F_\sigma\left(\frac{\alpha+e^{i_1}+e^{j_1}}{k}\right)$, $\tilde{u}^1=F_\sigma\left(\frac{\alpha+e^{i_2}+e^{j_2}}{k}\right)$, $u^p=F_\sigma\left(\frac{\alpha+e^{i_1}+e^{j_2}}{k}\right)$, $u^q=F_\sigma\left(\frac{\alpha+e^{i_2}+e^{j_1}}{k}\right)$. In particular, G is then linear on $[u^p, u^q]$. Furthermore, since the diagonals of a parallelogram halve each other (2.25) reads

$$\begin{aligned} b_{\alpha+e^{i_1}+e^{j_1}}+b_{\alpha+e^{i_2}+e^{j_2}} &= G(u^1)+G(\tilde{u}^1) \\ &\geqq 2G\left(\frac{1}{2}(u^1+\tilde{u}^1)\right) = 2G\left(\frac{1}{2}(u^p+u^q)\right) = G(u^p)+G(u^q) \\ &= b_{\alpha+e^{i_1}+e^{j_2}}+b_{\alpha+e^{i_2}+e^{j_1}}. \end{aligned} \tag{2.26}$$

Since $\mathscr{T}_{s,A}(\sigma)$ is obtained by translating $\mathscr{T}_{2,A}(\sigma^k)$ where

$$\sigma^k = \left[F_\sigma(e^1), F_\sigma\left(\frac{(k-2)e^1+2e^2}{k}\right), \ldots, F_\sigma\left(\frac{(k-2)e^1+2e^{s+1}}{k}\right)\right]$$

(2.26) holds for any $|\alpha|=k-2$. Thus, $S_{k,A}(\Phi)$ is convex if and only if (2.26) holds for all $|\alpha|=k-2$ and certain quadruples (i_1, j_1, i_2, j_2) depending only on A. Since by Lemma 2.3 the inequalities (2.26) are preserved under degree raising the assertion follows.

The proof of Theorem 2.2 follows now from Lemma 2.3 and the following extension of (2.16).

PROPOSITION 2.3.

$$\lim_{l\to\infty} \|B_k[\Phi;\ \cdot]-S_{k+l,A}(E^l\Phi)\|_\infty(\sigma) = 0.$$

The proof of this result follows just as in the case $s=2$ (cf. [7]).

We conclude this section with a remark on the monotonicity of the Bernstein polynomials as a function of their degree. This remark also extends a similar result in [2] for the bivariate case to arbitrary spatial dimension s.

Since the Bernstein polynomials preserve linear functions it readily follows that

$$f(x) \leqq B_k[f;\ \lambda(x;\sigma)], \qquad x\in\sigma,$$

when f is convex. Moreover, using the degree raising relation (2.15) we get for $g(\lambda) = f(F_\sigma(\lambda))$

$$B_{k+1}[f;\lambda]-B_k[f,\lambda] =$$

$$= \sum_{|\beta|=k+1}\left(g\left(\frac{\beta}{k+1}\right)-\frac{1}{k+1}\sum_{j=1}^{s+1}\beta_j\, g\left(\frac{\beta-e^j}{k}\right)\right)B_\beta^{k+1}(\lambda).$$

Since

$$\frac{\beta}{k+1} = \frac{1}{k+1}\sum_{j=1}^{s+1}\frac{\beta_j}{k}(\beta-e^j)$$

and f is convex it follows that

$$g\left(\frac{\beta}{k+1}\right)-\frac{1}{k+1}\sum_{j=1}^{s+1}\beta_j\, g\left(\frac{\beta-e^j}{k}\right)\leqq 0$$

so that

(2.27) $$B_k[f;\lambda] \geqq B_{k+1}[f;\lambda], \quad \lambda\in\Delta_s.$$

3. Box spline surfaces

In this section, we wish to discuss similar questions for multivariate spline surfaces based on the notion of box spline. For this purpose, we recall the requisite definitions. Let $X=\{x^1, \dots, x^n\}$ denote a set of not necessarily distinct vectors in $\mathbf{Z}^s\diagdown\{0\}$. We also use X for the matrix whose columns are $x^1, \dots, x^n$. The box spline is defined by requiring that [4]

(3.1) $$\int_{\mathbf{R}^s} f(x)B(x|X)dx = \int_{[0,1]^n} f(Xu)\,du$$

holds for any continuous function f on $\mathbf{R}^s$. $B(\cdot|X)$ is known to be a nonnegative piecewise polynomial of degree $n-s$ supported on

$$Z(X) = \{Xu\colon\ u\in[0,1]^n\}.$$

$B(\cdot|X)$ is continuous if $\langle X\diagdown\{y\}\rangle = \operatorname{span}\{X\diagdown\{y\}\} = \mathbf{R}^s$ for all $y\in X$. In general, the smoothness properties and regions on which $B(\cdot|X)$ is a polynomial can be described in terms of properties of X. For further details the reader is referred to [4]. Our first problem is to determine under which circumstances the spline function

$$S(x|X) = \sum_{\alpha\in\mathbf{Z}^s} c_\alpha B(x-\alpha|X)$$

is convex. As with the Bernstein polynomials, it is not sufficient for $c_\alpha = f(\alpha)$, where f is convex. For instance, in two dimensions we choose $f(x, y) = (x+y)_+$, a convex function and $X = \{(1,0),(0,1),(1,1)\}$ or $X = \{(1,0),(0,1),(1,0),(0,1)\}$. $S(x|X)$ is not convex in either case because its support has a zig-zag lower boundary. However, a simple consequence of the definition (3.1) is

PROPOSITION 3.1. *If* $S(\cdot|X)$ *is convex then* $S(\cdot|X \cup V)$ *is convex for any* V.

PROOF. It is sufficient to assume that $V = \{y\}$, $y \in \mathbf{Z}^s$. Thus when $z = \sum_{i=1}^{l} \lambda_i x^i$, $\sum_{i=1}^{l} \lambda_i = 1$, we have

$$\sum_{j=1}^{l} \lambda_j S(x^j|X \cup \{y\}) = \int_0^1 \Big(\sum_{j=1}^{l} \lambda_j S(x^j - ty|X) \Big) dt$$

$$\geqq \int_0^1 S(z - ty|X) dt = S\big(z|X \cup \{y\}\big).$$

Thus, we see that if a box spline surface is convex it remains convex with the introduction of new vectors. In order to derive from this fact explicit conditions on the control coefficients c_α to ensure convexity of $S(x|X)$ we introduce the special set

$$X_0 = \{e^1, \ldots, e^s, e\}$$

where $e^i = (\delta_{ij})_{j=1}^{s}$ and $e = e^1 + \ldots + e^s$.

THEOREM 3.1. *Suppose* $X_0 \subseteq X$. *Then* $S(x|X) = \sum_{\alpha \in \mathbf{Z}^s} c_\alpha B(x - \alpha|X)$ *is convex if the control coefficients* c_α, $\alpha \in \mathbf{Z}^s$, *satisfy*

$$c_\alpha + c_{\alpha + e^j - e^i} \geqq c_{\alpha - e^i} + c_{\alpha + e^j} \tag{3.2}$$

for any $\alpha \in \mathbf{Z}^s$ *and any* $i, j \in \{1, 2, \ldots, s+1\}$, $i \neq j$, *where* $e^{s+1} = -e$.

PROOF. $B(\cdot|X_0)$ is a piecewise linear continuous function satisfying

$$B(\alpha|X_0) = \begin{cases} 1 & \text{if } \alpha = e, \\ 0 & \text{if } \alpha \in \mathbf{Z}^s \backslash \{e\}. \end{cases}$$

Thus $S(x+e|X_0)$ is a piecewise linear interpolant to the coefficients c_α, i.e. $S(\alpha + e|X_0) = c_\alpha$, $\alpha \in \mathbf{Z}^s$. Note that Proposition 3.1 readily establishes the following analogue to Theorem 2.2: If $S(\cdot|X_0)$ is convex then so, too, is $S(\cdot|X)$ whenever $X_0 \subseteq X$.

Hence it remains to confirm that the conditions (3.2) imply the convexity of $S(\cdot|X_0)$. To this end, we have to identify first the regions where $S(\cdot|X_0)$ and hence $S(\cdot + e|X_0)$ agrees with a linear function. It is well-known (see e.g. [4]) that these regions are bounded by but not intersected by the hyperplanes in

$$\mathscr{H} = \{\langle V \rangle + \alpha \colon \alpha \in \mathbf{Z}^s,\ V \subset X_0,\ |V| = s-1\}.$$

CLAIM. The partition of $\mathbf{R}^s$ induced by $\mathscr{H}$ is the triangulation $\mathscr{K}_s$ defined in Section 2.

To see this recall that $\mathscr{H}_s$ is composed of the multi-integer translates of the simplices δ_π, $\pi\in\mathscr{P}_s$, defined by

$$\delta_\pi = [0, e^{\pi(1)}, e^{\pi(1)}+e^{\pi(2)}, \dots, e^{\pi(1)}+\dots+e^{\pi(s-1)}, e].$$

The hyperplanes $\alpha+\langle e^1, \dots, e^{i-1}, e^{i+1}, \dots, e^s\rangle$, $\alpha\in\mathbf{Z}^s$, $i=1, \dots, s$, in $\mathscr{H}$ are partitioned by the translates of those $(s-1)$-dimensional faces of δ_π, $\pi\in\mathscr{P}_s$, which are contained in the $(s-1)$-dimensional faces of $[0, 1]^s$. Hence we only have to consider those faces which contain the edge $[0, e]$, i.e. faces of the form

$$[0, e^1, \dots, e^{j-1}, e^{j+1}, \dots, e^{s-1}, e]$$

for some $j\in\{1, \dots, s-1\}$, where $e_\pi^j=e^{\pi(1)}+\dots+e^{\pi(j)}$, $j=1, \dots, s$. A basis for the $(s-1)$-dimensional subspace of $\mathbf{R}^s$ spanned by this face is $e_\pi^1, \dots, e_\pi^{j-1}$, $e_\pi^{j+1}, \dots, \dots, e_\pi^s$. Subtracting e_π^i from e_π^{i+1} for $i=1, \dots, j-2$ and $i=j+1, \dots, s$, we obtain $\{e^{\pi(1)}, \dots, e^{\pi(j-1)}, e^{\pi(j+2)}, \dots, e\}$ as another basis which reveals that the subspace belongs to $\mathscr{H}$. Conversely, given $V=\langle e^1, \dots, e^{j-1}, e^{j+1}, \dots, e^{i-1}, e^{i+1}, \dots, e\rangle$ one may choose $\pi\in\mathscr{P}_s$ such that for some l, $j=\pi(l)$ and $i=\pi(l+1)$ which relates V to some face of δ_π. This confirms the above claim.

It was already pointed out in the proof of Lemma 2.5 that the convexity of a piecewise linear interpolant with respect to the triangulation $\mathscr{H}_s$ is equivalent to inequalities of the type given in (3.2) (see (2.26)). To utilize this fact in the present context we again let α be any point in $\mathbf{Z}^s$. Then, for some $\beta\in\mathbf{Z}^s$, $\pi\in\mathscr{P}_s$ α is a vertex of $\beta+$ $+\delta_\pi\in\mathscr{H}_s$. Recalling the definition of $\delta_\pi=[0, e_\pi^1, \dots, e_\pi^s]$ and the definition of l^-, l^+ in the proof of Lemma 2.4 the 'neighbours' α^-, α^+ of α with respect to $\beta+\delta_\pi$ have the form

$$\alpha^- = \alpha-e^i, \quad \alpha^+ = \alpha+e^j$$

for some $i, j\in\{1, \dots, s+1\}$, $i\neq j$, $e^{s+1}=-e$. Setting

$$\tilde{\alpha} = \alpha^-+\alpha^+-\alpha = \alpha+e^j-e^i$$

Lemma 2.4 states that α, $\tilde{\alpha}$, α^-, α^+ span a planar parallelogram and that α^-, α^+ are contained in the common $(s-1)$-dimensional face of the simplex $\beta+\delta_\pi=\delta$ and some adjacent simplex $\tilde{\delta}\in\mathscr{H}_s$ having $\tilde{\alpha}$ as a vertex. The same argument which led to (2.26) therefore implies that $S(\cdot+e|X_0)$ is convex on $\delta\cup\tilde{\delta}$ if and only if $S(\alpha+e|X_0)+S(\tilde{\alpha}+e|X_0)\geqq S(\alpha^+ +e|X_0)+S(\alpha^- +e|X_0)$ which, in view of the interpolation properties of $S(\cdot+e|X_0)$ is (3.2). On account of Lemma 2.4 ii), this completes the proof of Theorem 3.1.

Next we study the following box spline operator. Let $\Sigma(X)=\frac{1}{2}\sum_{i=1}^{n} x^i$ and define

$$Q(f|X)(x) = \sum_{\alpha\in\mathbf{Z}^s} f(\alpha+\Sigma(X))B(x-\alpha|X).$$

We also set $X_l=X\setminus\{x^l\}$ for each $x^l\in X$.

LEMMA 3.1. *Suppose for each* $x^l\in X\subset\mathbf{Z}^s\setminus\{0\}$, $\langle X_l\rangle=\mathbf{R}^s$, *then* $Q(f|X)=f$ *for all linear functions* f *on* $\mathbf{R}^s$.

PROOF. Using the Poisson's summation formula we get

$$Q(x_l|X)(x) = \sum_{\alpha\in \mathbf{Z}^s} (\alpha_l+(X)_l)B(x-\alpha|X) = \tag{3.3}$$

$$= ((-i\frac{\partial}{\partial x_l}+x_l+\Sigma(X)_l)\hat{B}(\cdot|X))(0)$$

where

$$\hat{B}(\cdot|X)(x) = \int_{\mathbf{R}^s} B(u|X)e^{-iu\cdot x}du = \prod_{j=1}^{n} \frac{1-e^{-ix\cdot x^j}}{ix\cdot x^j}$$

is the Fourier transform of $B(\cdot|X)$. Here we have used the fact that

$$\hat{B}(\cdot|X)(2\pi\alpha) = \frac{\partial}{\partial x_l}\hat{B}(\cdot|X)(2\pi\alpha) = 0, \quad \alpha\in\mathbf{Z}^s/\{0\}$$

which follows from our hypothesis. Since we also have

$$\hat{B}(\cdot|X)(0) = 1 \tag{3.4}$$

and

$$\frac{\partial}{\partial x_l}\hat{B}(\cdot|X)(0) = -i\Sigma(X)_l$$

the right-hand side of (3.3) reduces to x_l. From (3.4), we also obtain $Q(1|X)=1$, which together with our above calculation proves the result.

This lemma easily implies

$$f(x) \leqq Q(f|X)(x), \qquad x\in\mathbf{R}^s, \tag{3.5}$$

when f is convex on $\mathbf{R}^s$.

Next, we make a "change of scale", and introduce for $h^{-1}=m\in\mathbf{N}$,

$$Q_h(f|X)(x) = h^s \sum_{\alpha\in h\mathbf{Z}^s} f(\alpha+\Sigma(hX))B(x-\alpha|hX). \tag{3.6}$$

Then

$$Q_h(f|X)(x) = \sum_{\alpha\in\mathbf{Z}^s} f(h(\alpha+\Sigma(X)))B\left(\frac{x}{h}-\alpha|X\right), \tag{3.7}$$

so that, as before, we obtain

$$f(x) \leqq Q_h(f|X)(x), \qquad x\in\mathbf{R}^s, \tag{3.8}$$

when f is convex on $\mathbf{R}^s$. Clearly, $\lim_{h\to 0} Q_h f=f$, uniformly on compact sets when f is continuous on $\mathbf{R}^s$. Next we will show that this convergence is monotonic for convex f.

THEOREM 3.2. *Let f be convex and h^{-1}, k^{-1}, $h^{-1}k\in\mathbf{N}$. Suppose that for all $x^l\in X$. $\langle X_l\rangle=\mathbf{R}^s$ and $\{X_l\beta\colon \beta\in\mathbf{Z}^{n-1}\}=\mathbf{Z}^s$. Then*

$$Q_h(f|X) \leqq Q_k(f|X). \tag{3.9}$$

PROOF. It is sufficient to prove this inequality when $k=1$ by scaling the independent argument of f. For the proof of this case, we recall that the discrete box spline $b_h(\alpha|X)$, $\alpha\in h\mathbf{Z}^s$, is defined by requiring that

$$\sum_{\alpha\in h\mathbf{Z}^s} f(\alpha)b_h(\alpha|X) = h^n \sum_{0\le\beta<h^{-1}} f(hX\beta) \tag{3.10}$$

holds for all sequences $\{f(\alpha)\colon \alpha\in h\mathbf{Z}^s\}$ with at most a finite number of nonzero terms. We showed in [5] that

$$B(x|X) = \sum_{\alpha\in h\mathbf{Z}^s} b_h(\alpha|X)B(x-\alpha|hX). \tag{3.11}$$

Therefore we can write

$$Q(f|X)(x) = h^s \sum_{\alpha\in h\mathbf{Z}^s} f_\alpha(h;X)B(x-\alpha|hX) \tag{3.12}$$

where

$$f_\alpha(h;X) = h^{-s} \sum_{\beta\in\mathbf{Z}^s} f(\beta+\Sigma(X))b_h(\alpha-\beta|X) \tag{3.13}$$

and so

$$\begin{aligned} &Q(f|X)(x)-Q_h(f|X)(x) = \\ &= h^s \sum_{\alpha\in h\mathbf{Z}^s} (f_\alpha(h;X)-f(\alpha+\Sigma(hX)))B(x-\alpha|hX). \end{aligned} \tag{3.14}$$

To complete the proof we make use of

LEMMA 3.2. *Let $\{c_\alpha\}_{\alpha\in h\mathbf{Z}^s}$ be a sequence with a finite number of nonzero terms. Let $z\in\mathbf{C}^s$ and consider the function*

$$F(z) = \sum_{\alpha\in\mathbf{Z}^s} z^\alpha c_{h\alpha}.$$

For any $z\in\mathbf{C}^s\setminus\{0\}$, we have

$$h^s \sum_{0\le\beta<h^{-1}} F(ze^{2\pi ih\beta}) = \sum_{\alpha\in\mathbf{Z}^2} z^{h^{-1}\alpha} c_\alpha.$$

This lemma allows us to pass from sums over the "coarse grid", $\alpha\in\mathbf{Z}^s$ to the "fine" grid, $\alpha\in h\mathbf{Z}^s$. If we choose $c_\alpha=b_h(\gamma-\alpha|X)$, $\alpha,\gamma\in h\mathbf{Z}^s$ above and use (3.10) we get

$$\sum_{\alpha\in\mathbf{Z}^s} b_h(\gamma-\alpha|X)z^{h^{-1}\alpha} = z^{h^{-1}\gamma}h^{n+s} \sum_{0\le\beta<h^{-1}} G(ze^{-2\pi ih\beta}) \tag{3.15}$$

where

$$G(z) = \prod_{i=1}^{n} g(z^{-x^i}), \tag{3.16}$$

$$g(t) = \frac{1-t^{h^{-1}}}{1-t} = \sum_{k=0}^{h^{-1}-1} t^k. \tag{3.17}$$

Since $g(1)=h^{-1}$ and $g(e^{2\pi ih\mu})=0$ whenever μ is an integer which is not a multiple of h^{-1} it follows by the condition $\{X\beta\colon \beta\in\mathbf{Z}^n\}=\mathbf{Z}^s$ that $G(e^{-2\pi ih\beta})=h^{-n}\delta_{0\beta}$ for

$0 \leqq \beta < h^{-1}$. Thus by choosing $z=(1, \ldots, 1)$ in equation (3.15) we obtain

$$h^{-s} \sum_{\alpha \in \mathbf{Z}^s} b_h(\gamma-\alpha|X) = 1. \tag{3.18}$$

Therefore from the convexity of f it follows that

$$f_\alpha(h; X) \geqq f\big(h^{-s} \sum_{\beta \in \mathbf{Z}^s} (\beta+\Sigma(X))\, b_h(\alpha-\beta|X)\big). \tag{3.19}$$

To evaluate the sum on the right, we will differentiate (3.15) with respect to z_k, $1 \leqq k \leqq s$ and evaluate at $z=(1, \ldots, 1)$. Before doing so we observe that $g'(1)=h^{-1}(h^{-1}-1)/2$, so that by our hypothesis on X we have

$$\nabla G(e^{-2\pi i h\beta}) = (1-h^{-1})h^{-n}\Sigma(X)\delta_{0\beta}, \qquad 0 \leqq \beta < h^{-1}. \tag{3.20}$$

Now, after differentiating the formula (3.15) and simplifying we obtain

$$h^{-s} \sum_{\beta \in \mathbf{Z}^s} \beta b_h(\alpha-\beta|X) = \alpha+(h-1)\Sigma(X). \tag{3.21}$$

This equation with (3.14), (3.18) and (3.19) proves the result. When the translates of the box spline are linearly independent (3.21) was proved in [6] by other means; (3.18) appears in [5] with different hypotheses.

REMARK. When

$$Q(f|X) = Q_h(f|X), \tag{3.22}$$

and f is convex we obtain

$$\sum_{\beta \in \mathbf{Z}^s} f(\beta+\Sigma(X))B(x-\beta|X)$$

$$= \sum_{\beta \in \mathbf{Z}^s} f(h(\beta+\Sigma(X)))B\left(\frac{x}{h}-\beta|X\right).$$

From this equation and the Poisson summation formula conditions on the derivatives of f can be obtained which are equivalent to (3.22). We leave the details to the reader.

PROPOSITION 3.2. *Let $f \in C^2(\mathbf{R}^s)$ and $\langle X \setminus Y \rangle = \mathbf{R}^s$ for all $Y \subset X$, $|Y| \leqq 2$. Then*

$$\lim_{h \to 0} h^{-2}(Q_h f - f) = \frac{1}{24} \sum_{i=1}^{n} D^2_{x^i} f$$

uniformly on compact sets.

PROOF. Let y^1, y^2 be any vectors in $\mathbf{R}^s$. Then our hypothesis gives by a lengthy calculation

$$Q_h(H(\cdot - x))(x) = \frac{h^2}{12} \sum_{i=1}^{n} H(x^i) \tag{3.23}$$

where $H(x)=(y^1, x)(y^2, x)$. The result now follows by expanding f about x in a Taylor series with remainder and using Lemma 3.1.

PROPOSITION 3.3. *Given any positive definite symmetric matrix* $A=(a_{ij})$, $i, j=1, \ldots, s$, *and any* $f\in C^1(\mathbf{R}^s)$ *such that*

$$|((\nabla f(x)-\nabla f(y)), (x-y))| \leqq (A(x-y), (x-y)) \tag{3.24}$$

for all $x, y\in \mathbf{R}^s$, *we have when* $\langle X\setminus Y\rangle=\mathbf{R}^s$ *for all* $Y\subset X$, $|Y|\leqq 2$,

$$|(Q_h f)(x)-f(x)| \leqq \frac{h^2}{24}\sum_{i=1}^{n}(Ax^i, x^i).$$

PROOF. Using (3.24) and Taylor's formula with remainder applied to the function $f(y+t(x-y))$, $0\leqq t\leqq 1$, it follows that

$$|f(y)-f(x)-(\nabla f(y), (y-x))| \leqq \frac{1}{2}(A(x-y), (x-y)).$$

Using this inequality, (3.23), Lemma 3.1 proves the proposition.

4. Interpolation of data by a convex function

The purpose of this section is to record some simple observations about interpolating data on any given set of distinct points in Euclidean space by a convex function. This problem was suggested by the questions we studied in the previous sections. Through conversations we had with others interested in this question we have concluded that the facts recorded below are not common knowledge. Therefore, we have decided to present them here. We make no claims for originality of these results and, in fact expect that they are available elsewhere. In this regard, they are treated informally in [9] where computation of convex interpolating surfaces is discussed.

THE PROBLEM. Given any set $X=\{x^i\}_{i=1}^n$ of distinct points in $\mathbf{R}^s$ as well as associated data $\mathscr{F}=\{f_i\}_{i=1}^n\subseteq \mathbf{R}$ we wish to determine under which circumstances there exists a convex function f: $[X]\mapsto \mathbf{R}$ such that

$$f_i=f(x^i), \qquad i=1, \ldots, n.$$

In this case, we say $(X, \mathscr{F})$ is convex. We will show that the convexity of $(X, \mathscr{F})$ is equivalent to the existence of a *convex piecewise linear* function interpolating $\mathscr{F}$ at X. Moreover, we will give necessary and sufficient conditions on X and $\mathscr{F}$ which assure the convexity of $(X, \mathscr{F})$ and reduce in the univariate case, $s=1$, to the familiar condition that

$$f_i \leqq \frac{x_i-x_{i-1}}{x_{i+1}-x_{i-1}} f_{i+1}+\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-1}} f_{i-1} \tag{4.1}$$

for $i=2, \ldots, N-1$ provided, of course, that $x_1<\ldots<x_n$.

For $x\in[X]$, let

$$\bar{f}(x) = \min\left\{\sum_{i=1}^{n}\lambda_i f_i\colon \sum_{i=1}^{n}\lambda_i = 1, \lambda_i \geqq 0, \sum_{i=1}^{n}\lambda_i x^i = x\right\}. \tag{4.2}$$

LEMMA 4.1. *$\bar{f}$ is convex.*

PROOF. For $x, y \in [X]$, let $\bar{\lambda}_i, \bar{\mu}_i$, $i=1, \dots, n$, be chosen such that

$$x = \sum_{i=1}^{n} \bar{\lambda}_i x^i, \quad y = \sum_{i=1}^{n} \bar{\mu}_i x^i \bar{\lambda}_i, \quad \bar{\mu}_i \geqq 0, \quad \sum_{i=1}^{n} \bar{\lambda}_i = 1 = \sum_{i=1}^{n} \bar{\mu}_i,$$

$$\bar{f}(x) = \sum_{i=1}^{n} \bar{\lambda}_i f_i, \quad \bar{f}(y) = \sum_{i=1}^{n} \bar{\mu}_i f_i.$$

Then for $0 \leqq t \leqq 1$ we have

$$\bar{f}(tx+(1-t)y) \leqq \sum_{i=1}^{n} (t\bar{\lambda}_i + (1-t)\bar{\mu}_i) f_i$$

$$= t\bar{f}(x) + (1-t)\bar{f}(y)$$

which confirms the convexity of $\bar{f}$.

PROPOSITION 4.1. *If there exists a convex function f such that $f(x^i) = f_i$, i.e., if $(X, \mathscr{F})$ is convex, then*

$$\bar{f}(x^i) = f(x^i) = f_i, \qquad i = 1, \dots, n. \tag{4.3}$$

PROOF. By definition we have

$$\bar{f}(x^i) \leqq f_i, \qquad i = 1, \dots, n. \tag{4.4}$$

Conversely, if $\sum_{j=1}^{n} \lambda_j = 1$, $\lambda_j \geqq 0$, $j=1, \dots, n$ and $x^i = \sum_{j=1}^{n} \lambda_j x^j$ then

$$f_i = f(x^i) = f\left(\sum_{j=1}^{n} \lambda_j x^j\right) \leqq \sum_{j=1}^{n} \lambda_j f_j.$$

Hence

$$f_i \leqq \bar{f}(x^i)$$

proving Proposition 4.1.

Therefore, we see that $\bar{f}$ is the upper envelope of all convex functions interpolating the data.

PROPOSITION 4.2. *There exists a convex function f satisfying $f(x^i) = f_i$, $i=1, \dots, n$, i.e. $(X, \mathscr{F})$ is convex if and only if the following condition holds:*

For every $J \subseteq \{1, \dots, n\}$, $|J| = s+1$, and every $j \notin J$ with $x^j \in \sigma_J = [x^i \colon i \in J]$, one has

$$f_j \leqq \sum_{i \in J} \lambda_i f_i \tag{4.5}$$

where

$$x^j = \sum_{i \in J} \lambda_i x^i, \quad \sum_{i \in J} \lambda_i = 1.$$

PROOF. For $x \in [X]$ let

$$\mathscr{J}_x = \{J \subseteq \{1, \dots, n\}, |J| = s+1 \colon x \in \sigma_J = [x^i \colon i \in J]\}$$

and denote the barycentric coordinates of x with respect to $\sigma=\sigma_J$ by $\lambda_i(\sigma_J; x)$, $i\in J$, i.e.

$$x = \sum_{i\in J} \lambda_i(\sigma_J; x)x^i, \quad \sum_{i\in J} \lambda_i(\sigma_J; x) = 1. \tag{4.6}$$

Now observe that

$$\bar{f}(x) = \min_{J\in\mathcal{J}_x} \{\sum_{i\in J} \lambda_i(\sigma_J; x)f_i\}. \tag{4.7}$$

To see this suppose there exists $M\subseteq\{1, \dots, n\}$ such that

$$\sum_{k\in M} \varrho_k x^k = 0, \qquad \sum_{k\in M} \varrho_k = 0, \qquad \sum_{k\in M} \varrho_k^2 > 0, \tag{4.8}$$

while

$$x = \sum_{k\in M} \lambda_k x^k, \quad \sum_{k\in M} \lambda_k = 1, \quad \lambda_k > 0, \quad k\in M.$$

Then $\lambda_k\pm\varepsilon\varrho_k>0$ for ε sufficiently small showing that $\{\lambda_k : k\in M\}$ cannot be an extremum of (4.2). In view of (4.8), if λ is an extremum of (4.2) then it has to be the barycentric coordinates of x for some simplex σ_J, $J\in\mathcal{J}_x$ which proves (4.7).

Now, in view of (4.7) condition (4.5) implies $f_j\leqq\bar{f}(x^j)$ and because always $\bar{f}(x^j)\leqq f_j$ we conclude

$$f_j = \bar{f}(x^j).$$

Thus Lemma 4.1 yields that $(X, \mathcal{F})$ is convex. The converse is trivial.

We observe that the upper envelope induces a triangulation of $[X]$. To see this, we choose $x\in[X]$, suppose $J\in\mathcal{J}_x$ achieves the minimum in (4.7). Then $\bar{f}(x)$ and $\sum_{i\in J} \lambda_i(\sigma_J; x) f_i$ agree at x and the vertices of J, i.e. at $s+2$ points. Since $\bar{f}$ is convex it is easy to see that $\bar{f}(x) = \sum_{i\in J} \lambda_i(\sigma_J; x) f_i$ everywhere on σ_J.

Next we prove (4.5) can be replaced by the following *local* condition on $(X, \mathcal{F})$. For every $x^j\in X$ and every $J\in\mathcal{J}_{x^j}$ such that $j\notin J$ and

$$x^l\notin\sigma_J = [x^k \colon k\in J], \quad \forall l\notin J,$$

one has

$$f_j \leqq \sum_{i\in J} \lambda_i(\sigma; x^j)f_i. \tag{4.9}$$

Note that for $s=1$, (4.9) reduces to (4.1).

PROPOSITION 4.3. *$(X, \mathcal{F})$ satisfies (4.5) if and only if $(X, \mathcal{F})$ satisfies (4.9).*

PROOF. Clearly, (4.5) implies (4.9). The proof of the converse is based on the following observation.

LEMMA 4.2. *Suppose that x^i, $i=1\dots, s+1$, $\bar{x}^1, \bar{x}^{s+1}$ are distinct points in $\mathbf{R}^s$ such that*

$$\bar{x}^1, \bar{x}^{s+1}\in[x^1, \dots, x^{s+1}],$$

and moreover,

$$\bar{x}^1\in\sigma_1 = [x^1, \dots, x^s, \bar{x}^{s+1}], \quad \bar{x}^{s+1}\in\sigma_{s+1} = [\bar{x}^1, x^2, \dots, x^{s+1}].$$

Moreover, suppose the data f_i, $i=1, \ldots, s+1$, $\bar{f}_1, \bar{f}_{s+1}$ *satisfy*

$$\bar{f}_1 \leqq \sum_{j=1}^{s} \lambda_j(\sigma_1; \bar{x}^1) f_j + \lambda_{s+1}(\sigma_1; \bar{x}^1) \bar{f}_{s+1} \tag{4.10}$$

and

$$\bar{f}_{s+1} \leqq \lambda_1(\sigma_{s+1}; \bar{x}^{s+1}) \bar{f}_1 + \sum_{j=2}^{s+1} \lambda_j(\sigma_{s+1}; \bar{x}^{s+1}) f_j. \tag{4.11}$$

Then

$$\bar{f}_1 \leqq \sum_{j=1}^{s+1} \lambda_j(\sigma; \bar{x}^1) f_j, \quad \bar{f}_{s+1} \leqq \sum_{j=1}^{s+1} \lambda_j(\sigma; \bar{x}^{s+1}) f_j. \tag{4.12}$$

PROOF OF LEMMA 4.2. Without loss of generality we may assume that $f_i=0$, $i=1, \ldots, s+1$. Suppose (4.12) does not hold and assume $\bar{f}_1>0$. Then (4.10) assures

$$\bar{f}_1 \leqq \lambda_{s+1}(\sigma_1; \bar{x}^1) \bar{f}_{s+1} \tag{4.13}$$

which combined with (4.11) provides

$$\bar{f}_{s+1} \leqq \lambda_1(\sigma_{s+1}; \bar{x}^{s+1}) \lambda_{s+1}(\sigma_1; \bar{x}^1) \bar{f}_{s+1}$$

whence we conclude

$$\lambda_1(\sigma_{s+1}; \bar{x}^{s+1}) \lambda_{s+1}(\sigma; \bar{x}^1) = 1.$$

This in turn means $\lambda_1(\sigma_{s+1}; \bar{x}^{s+1})=1=\lambda_{s+1}(\sigma_1; \bar{x}^1)$ and therefore $\bar{x}^{s+1}=\bar{x}^1$, contradicting our hypothesis.

In order to prove now that (4.9) implies (4.5) we proceed by induction on the cardinality of X. For $|X| \leqq s+2$ there is nothing to show. For $|X|=s+3$ we observe that if any simplex σ_J, $J \subset \{1, \ldots, s+3\}$ contains at most one further element of X (different from the vertices) conditions (4.5) and (4.9) coincide. If on the other hand $x^{s+2}, x^{s+3} \in [x^1, \ldots, x^{s+1}]$ say, we may assume without loss of generality that $x^{s+3} \in [x^{s+2}, x^2, \ldots, x^{s+1}]$, $x^{s+2} \in [x^1, \ldots, x^s, x^{s+3}]$. We have to show that for $\sigma=[x^1, \ldots, x^{s+1}]$

$$f_{s+2} \leqq \sum_{j=1}^{s+1} \lambda_j(\sigma; x^{s+2}) f_j, \qquad f_{s+3} \leqq \sum_{j=1}^{s+1} \lambda_j(\sigma; x^{s+3}) f_j. \tag{4.14}$$

But setting $x^{s+3}=\bar{x}^{s+1}$, $x^{s+2}=\bar{x}^1$ (4.9) agrees with the hypothesis of Lemma 4.2 which readily implies (4.14) and hence (4.5) in this case. Now let $|X|>s+3$ and suppose that for some $x^j \in X$, $J \in \mathscr{J}_{x^j}$, $x^j \notin J$ is arbitrary. We have to show that (4.9) implies

$$f_j \leqq \sum_{i \in J} \lambda_i(\sigma; x^j) f_i. \tag{4.15}$$

Suppose there exists some $x^l \in X$ such that x^l is an extreme point of $[X]$ and $x^l \notin \sigma_J$. Clearly, $(X, \mathscr{F})$ satisfies (4.5), (4.9) implies that $(X \setminus \{x^l\}, \mathscr{F} \setminus \{f_l\})$ satisfies (4.5), (4.9), respectively. Thus (4.15) follows in this case by our induction hypothesis. Thus it remains to consider the case $[X]=\sigma_J$. Pick any $x^l \in X$, $l \notin J$, $l \neq j$. Without loss of generality call $J=\{1, \ldots, s+1\}$, $x^j=\bar{x}^1$, $x^l=\bar{x}^{s+1}$, $\sigma=\sigma_J$ where as in Lemma 4.2, $\bar{x}^1 \in [x^1, \ldots, x^s, \bar{x}^{s+1}]=\sigma_1$, $\bar{x}^{s+1} \in [\bar{x}^1, x^2, \ldots, x^{s+1}]=\sigma_{s+1}$. Now if $(X, \mathscr{F})$ satisfies (4.9) then also $(X \setminus \{x^1\}, \mathscr{F} \setminus \{f_1\})$, $(X \setminus \{x^{s+1}\}, \mathscr{F} \setminus \{f_{s+1}\})$ satisfy (4.9). Thus conditions (4.10), (4.11) follow from our induction assumption so that (4.15) follows by Lemma 4.2. This completes the proof of Proposition 4.3.

The above observations may be summarized in

THEOREM 4.1. *$(X, \mathscr{F})$ is convex, i.e. there exists a convex function f such that*

$$f(x^i) = f_i, \qquad i = 1, \ldots, n,$$

if and only if there exists a convex piecewise linear interpolant to the data $\mathscr{F}$ at X if and only if $(X, \mathscr{F})$ satisfies condition (4.9).

ACKNOWLEDGEMENTS. We wish to thank Alan Edelman and Alan Hoffman of IBM for stimulating conversations concerning distance matrices.

ADDENDUM. We have learned from Professor Geng-zhe Chang that the positive assertion of Theorem 2.1 was independently obtained by Jian-wei Zhou [10]. Also, the idea of using degree raising to prove the Chang and Davis result is also used in [3]. We wish to thank Professor Chang for providing us with these references.

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(Received November 19, 1985)

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ON A SET OF RELATIONS ARISING FROM THE TRIANGULATION PROBLEM

D. DORNINGER and H. LÄNGER

1. Introduction

For a number of applications (cf. e.g. [3], [4]) the solution of the following problem is of importance: Given a real-valued matrix (a_{ij}) $(i,j=1, \dots, n;\ n\geqq 1)$ find a permutation $\pi\in S_n$ (S_n: symmetric group of degree n) such that $\sum_{i<j} a_{\pi i, \pi j}= \max_{\sigma\in S_n} \sum_{i<j} a_{\sigma i, \sigma j}$. This problem is called the triangulation problem for input–output matrices. In the literature it is also referred to as acyclic subgraph problem or linear ordering problem. (For its solution very effective numerical methods exist though the problem is NP-hard (cf. e.g. [1]).) If $\sum_{i<j} a_{ij}\geqq \sum_{i<j} a_{\sigma i, \sigma j}$ for all $\sigma\in S_n$ then (a_{ij}) is called triangulated. Now, for any $\pi\in S_n$ put $F(\pi):=\{(i,j)|1\leqq i<j\leqq n;\ \pi i>\pi j\}$ and define $\mathscr{F}:=\{F(\pi)|\pi\in S_n\}$. Then the following theorem holds (cf. e.g. [2]):

(a_{ij}) is triangulated iff for all $F\in\mathscr{F}$ $\sum_{(i,j)\in F} (a_{ij}-a_{ji})\geqq 0.$

This optimality criterion is the starting point for our investigations: We characterize the elements of $\mathscr{F}$ as relations, order these relations in such a way that we obtain an ortholattice and study the subset $\mathscr{F}_0$ of all "indecomposable" elements of $\mathscr{F}$ among which there are the join-irreducibles of the lattice. $\mathscr{F}_0$ has the property that one can substitute $\mathscr{F}$ by $\mathscr{F}_0$ in the optimality criterion stated above. Since $\mathscr{F}_0$ is "much smaller" than $\mathscr{F}$ (e.g. for $n=5$ one has $|\mathscr{F}_0|=39$ and $|\mathscr{F}|=120$) the investigation of $\mathscr{F}_0$ or "small" subsets $\bar{\mathscr{F}}$ of $\mathscr{F}$ including $\mathscr{F}_0$ is also of interest with respect to numerical solutions of the triangulation problem.

2. The ortholattice

Let n denote an arbitrarily chosen fixed positive integer. For every $\pi\in S_n$ $F(\pi)$ is a 2-place relation on $M:=\{1, 2, \dots, n\}$. This relation can be considered as a directed graph $G(F(\pi))$ with vertex-set M and a directed edge going from a vertex i to a vertex j iff $(i,j)\in F(\pi)$. If d^-i and d^+i denote the out- and indegree of the vertex i of $G(F(\pi))$ then we observe for the defining permutation π:

$$\pi i = i+d^-i-d^+i$$

1980 *Mathematics Subject Classification.* Primary 04A05; Secondary 06C15.

Key words and phrases. Triangulation of input–output matrices, set of 2-place relations, ortholattice.

for every $i\in M$. To see this define $V:=\{(i,j)\in M^2|1\leqq i<j\leqq n\}$ and for an arbitrary subset $R\subset V$ consider the mapping π_R which adjoins to any $i\in M$ the element $i+|\{j\in M|(i,j)\in R\}|-|\{j\in M|(j,i)\in R\}|$ which obviously belongs to M. Then for $\sigma\in S_n$ we have $\pi_{F(\sigma)}i=i+|\{j\in M|(i,j)\in F(\sigma)\}|-|\{j\in M|(j,i)\in F(\sigma)\}|=1+i-1-$ $-|\{j\in M|j<i;\sigma j>\sigma i\}|+|\{j\in M|(i,j)\in F(\sigma)\}|=1+|\{j\in M|j<i;\sigma j<\sigma i\}|+|\{j\in M|j>i;$ $\sigma j<\sigma i\}|=1+|\{j\in M|\sigma j<\sigma i\}|=1+|\{j\in M|j<\sigma i\}|=\sigma i$, wherefrom the above identity follows.

Now we show that the elements $F(\pi)$ of $\mathscr{F}$ can be characterized in a purely relation-theoretical manner. Using the notations introduced before and setting $R':=V\setminus R$ for every $R\subseteq V$ we prove:

THEOREM 1. *$\mathscr{F}=\{R\subseteq V|R, R'$ are transitive$\}$.*

PROOF. Assume $S\subseteq V$ and both S and S' to be transitive. For $(p,q)\in S$ put

$$a:=|\{k\in M|k<p;\ (k,p)\in S;\ (k,q)\in S\}|,$$
$$b:=|\{k\in M|k<p;\ (k,p)\notin S;\ (k,q)\in S\}|,$$
$$c:=|\{k\in M|k<p;\ (k,p)\notin S;\ (k,q)\notin S\}|,$$
$$d:=|\{k\in M|p<k<q;\ (p,k)\in S;\ (k,q)\in S\}|,$$
$$e:=|\{k\in M|p<k<q;\ (p,k)\in S;\ (k,q)\notin S\}|,$$
$$f:=|\{k\in M|p<k<q;\ (p,k)\notin S;\ (k,q)\in S\}|,$$
$$g:=|\{k\in M|k>q;\ (p,k)\in S;\ (q,k)\in S\}|,$$
$$h:=|\{k\in M|k>q;\ (p,k)\in S;\ (q,k)\notin S\}|\quad\text{and}$$
$$l:=|\{k\in M|k>q;\ (p,k)\notin S;\ (q,k)\notin S\}|.$$

Then $d+e+f=q-p-1$ whence $p+1=q-d-e-f$. From this it follows that $\pi_S(p)=p+d+e+1+g+h-a=q-d-e-f+d+e+g+h-a=q-a-f+g+h>q+$ $+g-a-b-1-d-f=\pi_S(q)$. Therefore $\pi_S(p)>\pi_S(q)$ for $(p,q)\in S$ for any $S\subseteq V$ such that S and S' are transitive.

Now let $R\subseteq V$ with both R and R' being transitive relations and let $(i_0,j_0)\in V$. If $(i_0,j_0)\in R$ then $\pi_R i_0>\pi_R j_0$ as shown above. Conversely, if $(i_0,j_0)\notin R$ then $(i_0,j_0)\in R'$ which yields $\pi_{R'}i_0>\pi_{R'}j_0$ by the above argument and hence $\pi_R i_0=$ $=n+1-\pi_{R'}i_0<n+1-\pi_{R'}j_0=\pi_R j_0$. This shows that π_R is injective and thus $\pi_R\in S_n$. It immediately follows that $F(\pi_R)=R$ which implies $R\in\mathscr{F}$.

Obviously $F(\pi)$ and $(F(\pi))'$ are both transitive for given $\pi\in S_n$ which concludes the proof.

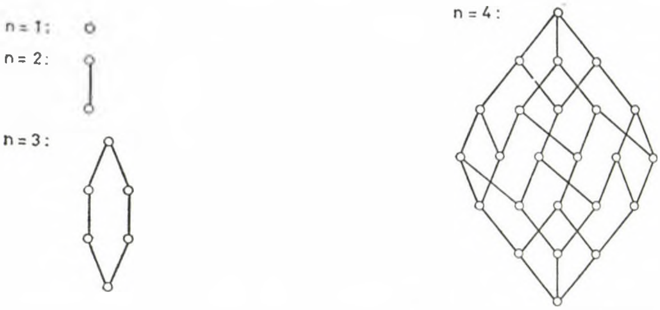
For $R\subseteq V$ let $\bar{R}$ denote the transitive hull of R.

THEOREM 2. *$(\mathscr{F},\subseteq,')$ is an ortholattice and $F_1\vee F_2=\overline{F_1\cup F_2}$ for all $F_1, F_2\in\mathscr{F}$.*

PROOF. Clearly $(\mathscr{F},\subseteq)$ is a poset. Let $F_1, F_2\in\mathscr{F}$ and put $F:=\overline{F_1\cup F_2}$. By definition, F is transitive. Now let $(i,j),(j,k)\in F'$. Suppose $(i,k)\notin F'$. Then $(i,k)\in F$. Because of the construction of F there exists a chain $i=i_0<\ldots<i_m=k$ $(m\geqq 1)$ from i to k such that $(i_0,i_1),\ldots,(i_{m-1},i_m)\in F_1\cup F_2$. Since the elements of

the chain and j are all natural numbers there must exist an $s\in\{1, \dots, m\}$ such that $i_{s-1}\leqq j<i_s$. Because $(i, j)\notin F$ we have $i_{s-1}<j<i_s$. $(i_{s-1}, i_s)\in F_1\cup F_2$ means that there is some $\nu\in\{1, 2\}$ such that $(i_{s-1}, i_s)\in F_\nu$. Because of the transitivity of F'_ν (according to Theorem 1) we conclude that at least one of the two pairs (i_{s-1}, j), (j, i_s) is in F_ν. But this implies that at least one of the pairs (i, j), (j, k) belongs to F which is a contradiction. Therefore, $(i, k)\in F'$ which shows that F' is transitive. According to Theorem 1 $F\in\mathscr{F}$. Hence $(\mathscr{F}, \subseteq)$ is a join-semilattice and $F_1\vee F_2=\overline{F_1\cup F_2}$ for all $F_1, F_2\in\mathscr{F}$. Since $'$ is an involutorial order-antiautomorphism of $(\mathscr{F}, \subseteq)$ we obtain that for arbitrary $F_1, F_2\in\mathscr{F}$ $F_1\wedge F_2$ always exists and equals $(F_1'\vee F_2')'$. The rest of the proof is obvious.

For $n\in\{1, 2, 3, 4\}$ the Hasse-diagrams of the lattices $(\mathscr{F}_n, \subseteq)$ (here the index n indicates the degree of the underlying symmetric group) look as follows:



Remark 1. For $k\geqq 1$ and for $F_1, \dots, F_k\in\mathscr{F}$ $F_1\vee\dots\vee F_k=\{(i, j)\in V\,|$there exists some (ordered) chain $i=i_0<\dots<i_t=j\ (t\geqq 1)$ from i to j such that $(i_{s-1}, i_s)\in F_1\cup\dots\cup F_k$ for all $s\in\{1, \dots, t\}\}$, and as one can see easily $F_1\wedge\dots\wedge F_k=\{(i, j)\in V\,|$for any chain $i=i_0<\dots<i_t=j$ $(t\geqq 1)$ from i to j there exists some $s\in\{1, \dots, t\}$ such that $(i_{s-1}, i_s)\in F_1\cap\dots\cap F_k\}$.

Remark 2. If $n\leqq 2$ then $(\mathscr{F}, \subseteq, ')$ is a Boolean algebra; if $n>2$ then $(\mathscr{F}, \subseteq, ')$ is not even orthomodular, as e.g. the following argument shows: $F((12))=\{(1, 2)\}\subseteq\{(1, 2), (1, 3)\}=F((132))$ but $F((132))\wedge(F((12)))'\subseteq\{(1, 3)\}$ which implies $F((132))\wedge(F((12)))'=\emptyset$ since $\{(1, 3)\}'$ is not transitive (cf. Theorem 1). Hence $F((132))\neq F((12))\vee(F((132))\wedge(F((12)))')$.

Remark 3. $'$ is an antiautomorphism of $(\mathscr{F}, \vee, \wedge)$.

Remark 4. $|\mathscr{F}|=n!$. This follows from the fact (see above) that the mappings $\pi\mapsto F(\pi)$ and $R\mapsto\pi_R$ are mutually inverse mappings between S_n and $\mathscr{F}$.

Remark 5. If $1\leqq m\leqq n$ then $(\mathscr{F}_m, \vee, \wedge)$ is a sublattice of $(\mathscr{F}_n, \vee, \wedge)$ because of $F(\pi)=F(\pi\cup\{(i, i)|i\in M\})$ for all $\pi\in S_m$. (For the notation $\mathscr{F}_m$, $\mathscr{F}_n$ see the remark immediately before the Hasse-diagrams drawn above.)

3. Indecomposable elements

DEFINITION. Let $F\in\mathcal{F}$. G is called a part of F if both $G\in\mathcal{F}\setminus\{\emptyset\}$ and $G\subseteq F$. F is called decomposable if it can be written as the disjoint union of at least two parts of F. If this is not the case, F is called indecomposable. Let $\mathcal{F}_0$ denote the set of all indecomposable elements of $\mathcal{F}$.

We observe that if $F\in\mathcal{F}$ is decomposable into $F_1, \dots, F_k\in\mathcal{F}$ $(k\geqq 3)$ then in general $F_r\cup F_s\notin\mathcal{F}$ for $r, s\in\{1, \dots, k\}$, $r\neq s$, i.e. if F is decomposable into k parts then there need not exist a decomposition of F into a smaller number of parts of F. On the other hand a decomposition of an element of $\mathcal{F}$ may allow a refinement. If $F\in\mathcal{F}\setminus\mathcal{F}_0$ then it is obviously $\vee$-reducible within the lattice $(\mathcal{F}, \vee, \wedge)$. Therefore any $\vee$-irreducible element of the lattice $(\mathcal{F}, \vee, \wedge)$ belongs to $\mathcal{F}_0$. The converse is not true for $n>3$. This can be seen as follows: Because of $F((1243))=$ $=\{(1, 3), (2, 3), (2, 4)\}=F((123))\cup F((243))$ we have $F((1243))=F((123))\vee F((243))$ and hence $F((1243))$ is $\vee$-reducible. Because of Theorem 1 any part of $F((1243))$ containing $(1, 3)$ also must contain $(2, 3)$ and any part of $F((1243))$ containing $(2, 4)$ also must contain $(2, 3)$. Hence $F((1243))\in\mathcal{F}_0$.

By means of the criterion mentioned in the introduction one can see immediately

THEOREM 3. *The $n\times n$-matrix (a_{ij}) is triangulated iff for all $F\in\mathcal{F}_0$ $\sum_{(i,j)\in F}(a_{ij}-a_{ji})\geqq$ $\geqq 0$. Hence (a_{ij}) is triangulated iff $\sum_{(i,j)\in F}(a_{ij}-a_{ji})\geqq 0$ for all $F\in\tilde{\mathcal{F}}$ where $\tilde{\mathcal{F}}$ is an arbitrary subset of $\mathcal{F}$ including $\mathcal{F}_0$.*

For $n>2$ $|\mathcal{F}_0|<|\mathcal{F}|$. (For, $F((13))=F((123))\dot\cup F((12))$ and hence $F((13))\notin\mathcal{F}_0$; e.g. for $n=3, 4, 5$ one can calculate $|\mathcal{F}_0|=5, 13, 39$ whereas $|\mathcal{F}|=6, 24, 120$, respectively.) The number of $\vee$-irreducibles of $(\mathcal{F}, \vee, \wedge)$ is a lower bound for $|\mathcal{F}_0|$. (For $n=3, 4, 5$ the number of $\vee$-irreducibles of $(\mathcal{F}, \vee, \wedge)$ is 5, 12 and 26, respectively.)

If one cannot determine $\mathcal{F}_0$ a not "too big" subset $\tilde{\mathcal{F}}$ of $\mathcal{F}$ including $\mathcal{F}_0$ will be of advantage for verifying the condition of Theorem 3.

On the other hand subsets of $\mathcal{F}_0$ lead to sub-optimal solutions of the triangulation problem and are therefore of interest. Accordingly we will derive necessary and sufficient conditions for the indecomposability of $F\in\mathcal{F}$.

For a directed graph G let $\bar{G}$ denote the corresponding undirected graph (obtained from G by disregarding the directions of the edges of G).

THEOREM 4. *Let $F\in\mathcal{F}$ and assume $\overline{G(F)}$ to be disconnected. Then $F\notin\mathcal{F}_0$.*

PROOF. It is clear because of Theorem 1 that any of the connected components of F (within $\overline{G(F)}$) belongs to $\mathcal{F}$. This leads to a non-trivial decomposition of F.

THEOREM 5. *Let $F\in\mathcal{F}$. Further assume $\overline{G(F)}$ to be connected and to contain no 3-circle. Then $F\in\mathcal{F}_0$.*

PROOF. Since the theorem is trivial for $F=\emptyset$ we assume $F\neq\emptyset$. Let $(s, t), (u, v)\in F$. Then there exists a path $t=t_0, \dots, t_k=u$ from t to u in $\overline{G(F)}$ $(k\geqq 1)$

such that for all $i\in\{1, \ldots, k\}$ either (t_{i-1}, t_i) or (t_i, t_{i-1}) belongs to F. Since $\overline{G(F)}$ does not contain any 3-circle we have $s<t=t_0>t_1<t_2>t_3<>\ldots>t_k=u<v$. Theorem 1 and the property that $\overline{G(F)}$ does not contain any 3-circle imply that if $a<b<c$, $(a, b), (a, c)\in F$ and a part of F contains (a, c) then this part also must contain (a, b). This means that two edges of $\overline{G(F)}$ having one endpoint in common always must belong to the same part of F. Going along the path from t to u one sees that (s, t) and (u, v) must belong to the same part of F. Hence any element of F must belong to the same part of F as the element (s, t) does. This proves $F\in\mathscr{F}_0$.

Theorem 5 suggests the conjecture that if $F\in\mathscr{F}$ does contain a 3-circle then $F\notin\mathscr{F}_0$. The latter is true for all n up to $n=4$ but fails for $n>4$ as the following argument shows: Let $F:=F((1452))$. Then $\overline{G(F)}$ is connected and does contain the 3-circle $1\to3\to5\to1$. $F\in\mathscr{F}_0$ because any part of F containing $(1, 3)$ must also contain $(1, 2)$, any part of F containing $(3, 5)$ has to contain $(4, 5)$ and if a part of F contains $(1, 5)$ it must also contain $(1, 2)$ and $(4, 5)$.

THEOREM 6. *The following relations belong to* $\mathscr{F}_0$:

(a) $F^{(1)}_{stu}:=\{(i,j)|s\leqq i<t\leqq j<u\}$ *for* $1\leqq s<t<u\leqq n+1$.

(b) $F^{(2)}_{sk}:=\{(s+2i, s+2i+1)|0\leqq i\leqq k\}\cup\{(s+2i-2, s+2i+1)|1\leqq i\leqq k\}$

for $1\leqq s\leqq n$ *and* $1\leqq k\leqq[(n-s-1)/2]$.

(c) $F^{(3)}_{stu}:=\{(i, t+1)|s\leqq i<t\}\cup\{(t,j)|t<j\leqq u\}$ *for* $1\leqq s<t<u-1<n$.

PROOF. (a) $F^{(1)}_{stu}=F\left(\begin{pmatrix} s & \ldots t-1 & t\ldots & u-1\\ s+u-t\ldots u-1 & & s\ldots s+u-t-1\end{pmatrix}\right)\in\mathscr{F}$. Any part G of $F^{(1)}_{stu}$ containing $(s, u-1)$ must also contain all elements of the form (s,j), $t\leqq j<u-1$, and with any element of the form (s,j) $(t\leqq j<u)$ all elements of the form (i,j), $s<i<t$, have to belong to G. Hence $F^{(1)}_{stu}\in\mathscr{F}_0$.

(b) $F^{(2)}_{sk}=F((s\; s+2\; s+4\;\ldots\; s+2k\; s+2k+1\; s+2k-1\; s+2k-3\;\ldots\; s+1))$; therefore $F^{(2)}_{sk}\in\mathscr{F}$. Now any part of $F^{(2)}_{sk}$ containing $(s, s+3)$ must also contain $(s, s+1)$ and $(s+2, s+3)$, and with $(s+2, s+5)$ any part of $F^{(2)}_{sk}$ must also contain $(s+2, s+3)$ and $(s+4, s+5)$. Hence $(s, s+3)$ and $(s+2, s+5)$ cannot be in disjoint parts of $F^{(2)}_{sk}$. Going on that way one finally obtains $F^{(2)}_{sk}\in\mathscr{F}_0$.

(c) Obviously, $F^{(3)}_{stu}=F((s\; s+1\;\ldots\; t\; u\; u-1\;\ldots\; t+1))\in\mathscr{F}$. If a part G of $F^{(3)}_{stu}$ contains $(s, t+1)$, there must also be all the elements of the form $(i, t+1)$, $s<i\leqq t$, in G, and if G contains (t, u) it must also contain all elements of the form (t,j), $t<j<u$. Hence $F^{(3)}_{stu}\in\mathscr{F}_0$.

THEOREM 7. *For the following permutations* $\pi\in S_n$ $F(\pi)$ *does not belong to* $\mathscr{F}_0$:

(a) *Let* $1<i<n-1$. $\pi|\{1, \ldots, i\}$ *and* $\pi|\{i+1, \ldots, n\}$ *are non-trivial permutations on their respective domains.*

(b) *Let* $1\leqq s<t<u\leqq n+1$ *and* $1\leqq a\leqq n+s-u+1$.
$\pi(\{s, \ldots, t-1\})=\{a+u-t, \ldots, a+u-s-1\}$, $\pi(\{t, \ldots, u-1\})=\{a, \ldots, a+u-t-1\}$ *and*

$$\pi\neq\begin{pmatrix} s & \ldots t-1 & t\ldots & u-1\\ s+u-t\ldots & u-1 & s\ldots & s+u-t-1\end{pmatrix}.$$

PROOF. (a) $F(\pi)=F(\pi|\{1, \ldots, i\})\dot{\cup} F(\pi|\{i+1, \ldots, n\})$ is a non-trivial decomposition of $F(\pi)$ into parts.

(b) $F(\pi)=F^{(1)}_{stu}\dot{\cup}((F(\pi))\setminus F^{(1)}_{stu})$. That $F(\pi)\setminus F^{(1)}_{stu}\in\mathscr{F}$ follows according to Theorem 1 and by observing that $\pi(\{s, \ldots, u-1\})=\{a, \ldots, a+u-s-1\}$. (The latter equality implies that if $(i_0, j_0)\in F(\pi)$ for some $i_0\in\{s, \ldots, u-1\}$ and some $j_0\in\{u, \ldots, n\}$ then $(i, j_0)\in F(\pi)$ for all $i\in\{s, \ldots, u-1\}$.)

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(Received September 9, 1985)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 295—296

CORRECTION TO MY PAPER "PERFECT SINGLE LEE-ERROR CORRECTING CODES"

A. RACSMÁNY

In my paper mentioned in the title (Studia Sci. Math. Hungar. **9** (1974), 73—75) the following theorem was announced:

Let Z_q^n denote the set of all vectors of length n over the alphabet $q>2$. Then the necessary and sufficient condition for the existence of a perfect single Lee-error correcting code $C\subset Z_q^n$ is the fulfilling of the equation $n=\dfrac{\frac{q^r}{h}-1}{2}$, where $r\geqq 1$ and $h\geqq 1$ are integers.

Trying to reconstruct the proof of the sufficient condition I observed that the proof in the above paper was wrong. Here we present a correct proof.

Suppose that $n=\dfrac{\frac{q^r}{h}-1}{2}$, where $r\geqq 1$, $h\geqq 1$ are integers. We decompose $2n+1$ into prime factors: $2n+1=p_1^{i_1}\dots p_l^{i_l}\dots p_k^{i_k}$, $i_1\leqq\dots\leqq i_l\leqq\dots\leqq i_k$. We define the numbers q_j for $j=1,\dots,i_k$ so that $q_j=p_lp_{l+1}\dots p_k$ if $i_{l-1}<j\leqq i_l$ $(i_0=0)$. Then

$$(1)\quad \prod_{j=1}^{i_k} q_j=(p_1p_2\dots p_k)^{i_1}(p_2\dots p_k)^{i_2-i_1}\dots(p_lp_{l+1}\dots p_k)^{i_l-i_{l-1}}\dots p_k^{i_k-i_{k-1}}=2n+1.$$

Obviously, $q_1|q$. First we construct a perfect single Lee-error correcting code $C^*\subset Z_{q_1}^n$. We consider those vectors $h=\begin{pmatrix}h_1\\ \vdots\\ h_{i_k}\end{pmatrix}$ other than 0, for which $h_j\in Z_{q_j}$. Because of the decomposition (1), the number of these vectors is $2n$. From among these vectors we choose n vectors so that if $h=\begin{pmatrix}h_1\\ \vdots\\ h_{i_k}\end{pmatrix}$ and $h'=\begin{pmatrix}h'_1\\ \vdots\\ h'_{i_k}\end{pmatrix}$ are any two of them, then we cannot have $h_j+h'_j\equiv 0 \bmod q_j$ for all j. Without loss of generality

1980 *Mathematics Subject Classification.* Primary 94B25.
Key words and phrases. Perfect codes, Lee metric.

we may assume that among these vectors the vectors $\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\ \vdots\\0\end{pmatrix}, \dots, \begin{pmatrix}0\\0\\ \vdots\\1\end{pmatrix}$ occur.

We arrange the n vectors in the columns of a matrix H so that the first i_k columns of H should form the unit matrix. We consider the set M of those vectors $m=\begin{pmatrix}m_1\\ \vdots\\ m_j\\ \vdots\\ m_{i_k}\end{pmatrix}\in Z_{q_1}^{i_k}$ for which $m_j\equiv 0 \bmod q_j$. Let the code $C^*\subset Z_{q_1}^n$ consist of those vectors c for which $Hc=m\in M \bmod q_1$. We claim that C^* is a perfect single Lee-error correcting code.

First we show, that the Lee-distance of any two codewords of C^* is at least 3. For assume that the distance between $c_1\in C^*$ and $c_2\in C^*$ $(c_1\neq c_2)$ is at most 2. Then among the coordinates of $e=c_1-c_2$ either exactly one is 1 or -1, or exactly one is 2 or -2, or exactly two are equal to 1 or -1, and the rest of the coordinates are 0. Therefore He coincides either with the ± 1-fold of a column of H, or with the ± 2-fold of a column of H, or with the linear combination of two columns of H with the coefficients ± 1. Consequently we have, by the definition of H and M, $He\notin M$. On the other hand we have $He=Hc_1-Hc_2\in M$ which is a contradiction.

This implies that the unit balls centred at the points of C^* are disjoint, therefore $|C^*|\leqq\dfrac{q_1^n}{2n+1}$. In order to prove the perfectness we have only to show that $|C^*|=\dfrac{q_1^n}{2n+1}$. The number of the vectors c, for which Hc equals some fixed element m of M, is equal to $q_1^{n-i_k}$, because, choosing the last $n-i_k$ coordinates of c arbitrarily from Z_{q_1}, the first i_k coordinates are uniquely determined. Since $|M|=\prod_{j=1}^{i_k}\dfrac{q_1}{q_j}=\dfrac{q_1^k}{2n+1}$, therefore

$$|C^*| = q_1^{n-i_k}|M| = q_1^{n-i_k}\frac{q_1^{i_k}}{2n+1} = \frac{q_1^n}{2n+1}.$$

After these the existence of a perfect single Lee-error correcting code $C\subset Z_q^n$ is obvious, because writing $s=\dfrac{q}{q_1}$ such a code consists of the vectors $c+q_1x$, where $c\in C^*$ and $x\in Z_s^n$.

(Received June 28, 1985)

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HU ISSN 0081-6906 Index: 26.748

Studia Scientiarum Mathematicarum Hungarica

VOLUME 23
NUMBERS 3—4
1988

AKADÉMIAI KIADÓ, BUDAPEST

STUDIA SCIENTIARUM MATHEMATICARUM HUNGARICA

A QUARTERLY OF THE HUNGARIAN ACADEMY OF SCIENCES

Studia Scientiarum Mathematicarum Hungarica publishes original papers on mathematics mainly in English, but also in German, French and Russian.

Studia Scientiarum Mathematicarum Hungarica is published in yearly volumes of four issues (mostly double numbers published semiannually) by

AKADÉMIAI KIADÓ
Publishing House of the Hungarian Academy of Sciences
H—1054 Budapest, Alkotmány u. 21.

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 297—313

PROJECTIVE GENERATION OF PRESEMINORMED SPACES BY LINEAR RELATIONS

ÁRPÁD SZÁZ

Introduction

In this paper, and its immediate continuation [14], the fundamentals of the theory of projective and inductive generations of topological vector spaces [1] will be generalized and simplified. More precisely, projective and inductive generations of preseminormed spaces by linear relations will be defined and studied.

Linear relations [10] are natural generalizations of linear functions, and are mainly motivated by the fact that the inverse of a linear function is a linear relation. Preseminormed spaces [12] are natural generalizations of normed spaces, and are equivalent to topological vector spaces, but seem to be more convenient for several purposes.

Our main tools here, and in [14], are the notion of the infimum composition $q*S$ of a preseminorm q and a linear relation S, defined by $(q*S)(x)=\inf q(S(x))$, and a useful criterion which says that a linear relation S from a preseminormed space $X(\mathcal{P})$ into another $Y(\mathcal{Q})$, with $\mathcal{Q}$ being directed, is mildly uniformly continuous (lower semicontinuous) iff the preseminorm $q*S$ is continuous for all $q\in\mathcal{Q}$.

The necessary prerequisites, such as continuous and linear relations and preseminormed spaces, which are possibly unfamiliar to the reader, will be laid out in greater detail in the next preparatory section. This and the use of the terms "generation" and "coarsest" instead of "limit" and "weakest", respectively, have mainly been suggested to us by the referee whom we are therefore indebted to. Moreover, we are also indebted to the referee and Zsolt Páles for pointing out a serious mistake in the earlier version of Theorem 4.4.

0. Prerequisites

A relation from a set X into another Y is a subset S of $X\times Y$ such that $S(x)=\{y\colon (x,y)\in S\}$ is not empty for all $x\in X$. A relation S from a topological space $X(\mathcal{T})$ into another $Y(\mathcal{V})$ is called lower semicontinuous [9] if $S^{-1}(V)=\bigcup_{z\in V} S^{-1}(z)$ belongs to $\mathcal{T}$ for every $V\in\mathcal{V}$. In the sequel, we shall also need a localized form of this notion which says that S is lower semicontinuous at a point x of X if for each $V\in\mathcal{V}$ with $S(x)\cap V\neq\emptyset$ there exists $U\in\mathcal{T}$ with $x\in U\subset S^{-1}(V)$.

A relation S from a uniform space $X(\mathcal{U})$ into another $Y(\mathcal{V})$ will be called mildly uniformly continuous if $S^{-1}\circ V\circ S\in\mathcal{U}$ for every $V\in\mathcal{V}$. Note that for func-

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1980 *Mathematics Subject Classification.* Primary 46A99; Secondary 46M10.

Key words and phrases. Preseminormed spaces, continuities of linear relations, projective generation.

 Akadémiai Kiadó, Budapest

tions this notion reduces to the usual uniform continuity. But, in general, a mildly uniformly continuous relation need not even be lower semicontinuous. Therefore, we shall also say that S is lower semiperfectly mildly uniformly continuous if $V\in\mathscr{V}$ implies $S^{-1}\circ V\circ\Phi\in\mathscr{U}$ for every selection relation Φ for S. (A relation Φ from X into Y is a selection relation for S if $\Phi\subset S$.)

A relation S from a vector space X into another Y, over the same scalar field $\mathbf{K}=\mathbf{R}$ or $\mathbf{C}$, is called linear [10] if

$$S(x)+S(y)\subset S(x+y) \quad\text{and}\quad \lambda S(x)\subset S(\lambda x)$$

for all $x, y\in X$ and $\lambda\in\mathbf{K}$, where the linear operations for subsets of Y are to be understood in the usual elementwise sense. Concerning linear relations, we shall only need here the following assertions which were mostly proved in [10].

THEOREM 0.1. *If S is a linear relation from X into Y, then $S(0)$ is a subspace of Y. Moreover, S is a function if and only if $S(0)=\{0\}$.*

THEOREM 0.2. *If S is a linear relation from X onto Y, then S^{-1} is a linear relation from Y onto X.*

THEOREM 0.3. *If S is a linear relation from X into Y and Φ is a selection relation for S, then*

$$S(x) = \Phi(x)+S(0)$$

for all $x\in X$.

COROLLARY 0.4. *If S is a linear relation from X into Y, then S is nonmingled-valued* [7].

COROLLARY 0.5. *If S is a linear relation from X into Y, then $S^{-1}\circ S=S^{-1}\circ\Phi$ for any selection relation Φ for S.*

COROLLARY 0.6. *If S is a linear relation from X into Y, then $S\circ S^{-1}\circ S=S$.*

THEOREM 0.7. *If S is a linear relation from X into Y, then there exists a linear selection function φ for S.*

A subadditive real function p on a vector space X is called a preseminorm on X [12] if

$$p(\lambda x)\leqq p(x) \quad\text{and}\quad \lim_{\mu\to 0} p(\mu x) = 0$$

for all $|\lambda|\leqq 1$ and $x\in X$. An ordered pair $X(\mathscr{P})=(X,\mathscr{P})$ consisting of a vector space X and a nonvoid family $\mathscr{P}$ of preseminorms on X is called a preseminormed space.

A preseminormed space $X(\mathscr{P})$ can immediately be turned into a uniform space, and hence a topological space, with the help of the r-sized p-surroundings

$$B_p^r = \{(x, y)\colon p(x-y) < r\}$$

defined for all $p\in\mathscr{P}$ and $r>0$. More precisely, according to [5, Theorems 6.3 and 6.5], we can at once state

THEOREM 0.8. *If $X(\mathscr{P})$ is a preseminormed space, then the family of all surroundings B_p^r, where $p\in\mathscr{P}$ and $r>0$, is a subbase for a uniformity $\mathscr{U}_{\mathscr{P}}$ for X.*

THEOREM 0.9. *If $X(\mathcal{P})$ is a preseminormed space, then $\mathcal{U}_{\mathcal{P}}$ induces a topology $\mathcal{T}_{\mathcal{P}}$ on X such that, for each $x \in X$, the family of all balls $B_p^r(x)$, where $p \in \mathcal{P}$ and $r > 0$, is a subbase for the neighbourhood system of x.*

The topology $\mathcal{T}_{\mathcal{P}}$ can also be easily described directly in terms of the corresponding balls. Namely, we obviously have

THEOREM 0.10. *If $X(\mathcal{P})$ is a preseminormed space, then the family of all balls $B_p^r(x)$, where $p \in \mathcal{P}$, $r > 0$ and $x \in X$, is a subbase for $\mathcal{T}_{\mathcal{P}}$.*

REMARK 0.11. If the family $\mathcal{P}$ is directed with respect to its usual pointwise partial order, then the subbases given in the above theorems are actually bases.

In view of [12, Theorems 2.6 and 2.8], now we can also state

THEOREM 0.12. *If $X(\mathcal{P})$ is a preseminormed space, then $\mathcal{T}_{\mathcal{P}}$ is a vector topology on X.*

THEOREM 0.13. *If $X(\mathcal{P})$ is a preseminormed space, then $\mathcal{T}_{\mathcal{P}}$ is the coarsest translation-invariant topology on X for which each $p \in \mathcal{P}$ is continuous.*

Concerning continuities of preseminorms on preseminormed spaces, one can also easily prove

THEOREM 0.14. *If $X(\mathcal{P})$ is a preseminormed space and q is a preseminorm on X, then the following assertions are equivalent:*

(i) *q is uniformly continuous;* (ii) *q is continuous at* 0; (iii) $0 \in B_q^r(0)^0$ *for all* $r > 0$; (iv) $B_q^r \in \mathcal{U}_{\mathcal{P}}$ *for all* $r > 0$.

REMARK 0.15. Note that if (i) holds, then q is continuous and thus each B_q^r is open-valued [7].

If $X(\mathcal{P})$ is a preseminormed space, then the family of all preseminorms on X which are continuous for $\mathcal{T}_{\mathcal{P}}$ will be denoted by $\bar{\mathcal{P}}$.

The importance of this notation lies mainly in the next theorem which follows easily from Theorems 0.13 and 0.14.

THEOREM 0.16. *If $\mathcal{P}$ and $\mathcal{Q}$ are nonvoid families of preseminorms on X, then the following assertions are equivalent:*

(i) $\bar{\mathcal{P}} = \bar{\mathcal{Q}}$; (ii) $\mathcal{T}_{\mathcal{P}} = \mathcal{T}_{\mathcal{Q}}$; (iii) $\mathcal{U}_{\mathcal{P}} = \mathcal{U}_{\mathcal{Q}}$.

Because of this theorem, two nonvoid families $\mathcal{P}$ and $\mathcal{Q}$ of preseminorms on X may be called equivalent if $\bar{\mathcal{P}} = \bar{\mathcal{Q}}$.

As an immediate consequence of Theorems 0.13 and 0.14, now we can also state

THEOREM 0.17. *If $X(\mathcal{P})$ is a preseminormed space, then $\bar{\mathcal{P}}$ is the largest family of preseminorms on X which is still equivalent to $\mathcal{P}$.*

There are cases, when some proper subfamilies of $\bar{\mathcal{P}}$, containing $\mathcal{P}$, prove to be more suitable than $\bar{\mathcal{P}}$.

If $X(\mathcal{P})$ is a preseminormed space, then $\mathcal{P}^*$ ($\mathcal{P}_*$) will denote the family of all preseminorms q on X for which there exists $\{p_k\}_{k=1}^n \subset \mathcal{P}$ ($p \in \mathcal{P}$) such that $q = \max_{1 \leq k \leq n} p_k$ ($q \leq p$).

A nonvoid family $\mathcal{P}$ of preseminorms on X, or a preseminormed space $X(\mathcal{P})$, will be called saturated, descending and total if $\mathcal{P}^* = \mathcal{P}$, $\mathcal{P}_* = \mathcal{P}$ and $\overline{\mathcal{P}} = \mathcal{P}$, respectively.

On the other hand, we also say that a nonvoid family $\mathcal{P}$ of preseminorms on X is separating, or a preseminormed space $X(\mathcal{P})$ is separated, if for each $x \in X$ with $x \neq 0$ there exists $p \in \mathcal{P}$ such that $p(x) \neq 0$.

The importance of this latter notion lies mainly in the next obvious

Theorem 0.18. *If $X(\mathcal{P})$ is a preseminormed space, then the following assertions are equivalent:*

(i) *$\mathcal{P}$ is separating;* (ii) *$\mathcal{T}_{\mathcal{P}}$ is T_2;* (iii) *$\mathcal{T}_{\mathcal{P}}$ is T_0.*

Finally, we remark that if p is a preseminorm on X, then we simply write $X(p)$ instead of $X(\{p\})$. Moreover, if $\{p\}$ is separating, then p is called a prenorm.

1. The infimum composition

Definition 1.1. If S is a linear relation from X into Y and q is a preseminorm on Y, then the function $q * S$ defined on X by

$$(q * S)(x) = \inf q(S(x))$$

will be called the infimum composition of q and S.

Remark 1.2. The above definition has been motivated by that of a quotient seminorm [4, p. 105].

Therefore it is not surprising that we have

Theorem 1.3. *If S is a linear relation from X into Y and q is a preseminorm (seminorm) on Y, then $q * S$ is a preseminorm (seminorm) on X.*

Proof. If $y \in S(x)$ and $\varepsilon > 0$, then because of $\lim_{\lambda \to 0} q(\lambda y) = 0$, there exists $\delta > 0$ such that $q(\lambda y) < \varepsilon$ if $|\lambda| < \delta$. Hence, since $\lambda y \in \lambda S(x) \subset S(\lambda x)$, it is clear that

$$(q * S)(\lambda x) = \inf q(S(\lambda x)) \leq q(\lambda y) < \varepsilon,$$

whenever $|\lambda| < \delta$. Consequently, $\lim_{\lambda \to 0} (q * S)(\lambda x) = 0$.

To prove the remaining properties of $q * S$, one can apply similar arguments as in [4, p. 105].

Remark 1.4. Note that if q is a prenorm (norm), then $q * S$ need not be a prenorm (norm) even if S is a function.

The importance of the infimum composition lies mainly in the following simple theorem on balls.

THEOREM 1.5. *If S is a linear relation from X into Y, q is a preseminorm on Y and $r>0$, then*

$$B^r_{q*S}(x) = S^{-1}(B^r_q(y))$$

for any $x\in X$ and $y\in Y$ with $y\in S(x)$.

PROOF. If $z\in B^r_{q*S}(x)$, then $(q*S)(z-x)<r$. Thus, there exists $w\in S(z-x)$ such that $q(w)<r$. Hence, if $y\in S(x)$, it is clear that $w+y\in S(z)$ and $w+y\in B^r_q(y)$. Consequently, $z\in S^{-1}(B^r_q(y))$.

Conversely, if $z\in S^{-1}(B^r_q(y))$, then $z\in S^{-1}(w)$ for some $w\in B^r_q(y)$. Hence, if $x\in S^{-1}(y)$, it is clear that $y-w\in S(x-z)$ and $q(y-w)<r$. Consequently, $(q*S)(x-z)<r$, i.e., $z\in B^r_{q*S}(x)$.

From this theorem, we can at once derive two useful assertions for the corresponding surroundings.

COROLLARY 1.6. *If S is a linear relation from X into Y, q is a preseminorm on Y and $r>0$, then*

$$S^{-1}\circ B^r_q\circ\Phi = B^r_{q*S}$$

for any selection relation Φ for S.

PROOF. By Theorem 1.5, we have

$$(S^{-1}\circ B^r_q\circ\Phi)(x) = \bigcup_{y\in\Phi(x)} (S^{-1}\circ B^r_q)(y) = B^r_{q*S}(x)$$

for any $x\in X$, namely $y\in\Phi(x)$ implies $y\in S(x)$.

COROLLARY 1.7. *If S is a linear relation from X onto Y, q is a preseminorm on Y and $r>0$, then*

$$B^r_{q*S}\circ\Psi = S^{-1}\circ B^r_q$$

for any selection relation Ψ for S^{-1}.

PROOF. Again by Theorem 1.5, we have

$$(B^r_{q*S}\circ\Psi)(y) = \bigcup_{x\in\Psi(y)} B^r_{q*S}(x) = (S^{-1}\circ B^r_q)(y)$$

for any $y\in Y$, namely $x\in\Psi(y)$ implies $y\in S(x)$.

Our next theorem establishes an important, associativity-like property of the infimum composition.

THEOREM 1.8. *If T is a linear relation from Z into X, S is a linear relation from X into Y and q is a preseminorm on Y, then*

$$(q*S)*T = q*(S\circ T).$$

PROOF. If $x\in X$, then we have

$$\alpha = ((q*S)*T)(x) = \inf_{y\in T(x)} (q*S)(y) = \inf_{y\in T(x)}\inf_{z\in S(y)} q(z)$$

and

$$\beta = (q*(S\circ T))(x) = \inf_{z\in S(T(x))} q(z).$$

Thus, if $\alpha<\beta$, then there exists $y_0 \in T(x)$ such that $\inf_{z \in S(y_0)} q(z)<\beta$. And hence, there exists $z_0 \in S(y_0)$ such that $q(z_0)<\beta$. Hence, since $z_0 \in S(T(x))$, we get

$$\beta = \inf_{z \in S(T(x))} q(z) \leqq q(z_0) < \beta,$$

which is a contradiction.

The assumption $\beta<\alpha$ also leads to a similar contradiction. Consequently, we have $\alpha=\beta$, which proves the theorem.

By an immediate application of this theorem, we shall now prove a rather curious, but useful theorem.

THEOREM 1.9. *If S is a linear relation from X into Y and q is a preseminorm on Y, then*

(i) $(q*S)*\Psi \leqq q|S(X)$,

(ii) $(q*S)*\Psi = (q*S)*S^{-1}$,

(iii) $((q*S)*\Psi)*S = q*S$

for any linear selection relation Ψ for S^{-1}.

PROOF. We clearly have

$$\Delta_{S(X)} \subset S \circ \Psi.$$

Hence, by Theorem 1.8, it is clear that

$$(q*S)*\Psi = q*(S\circ\Psi) \leqq q*\Delta_{S(X)} = q|S(X).$$

On the other hand, by Theorem 0.2 and Corollaries 0.5 and 0.6, it is clear that

$$S\circ\Psi = S\circ S^{-1} \quad \text{and} \quad S\circ\Psi\circ S = S.$$

Hence, again by Theorem 1.8, the assertions (ii) and (iii) immediately follow.

REMARK 1.10. By defining the infimum composition of an extended real-valued relation with an arbitrary one, Theorem 1.8 and the first assertion of Theorem 1.9 can be significantly extended.

In this respect, it is also worth mentioning that by using nonmingled-valued relations instead of the linear ones, the second and the third assertions of Theorem 1.9 can also be significantly extended.

2. Continuities of linear relations

The following theorem greatly extends well-known standard results on continuities of linear functions.

THEOREM 2.1. *If S is a linear relation from a preseminormed space $X(\mathcal{P})$ into a directed one $Y(\mathcal{Q})$, then the following assertions are equivalent:*

(i) *S is lower semiperfectly mildly uniformly continuous;*
(ii) *S is mildly uniformly continuous;*
(iii) *S is lower semicontinuous;*
(iv) *S is lower semicontinuous at* 0;
(v) *$q*S$ is continuous for all $q\in\mathcal{Q}$.*

PROOF. The implications (i)$\Rightarrow$(ii) and (iii)$\Rightarrow$(iv) are quite obvious.

To prove the implication (ii)$\Rightarrow$(iii), assume that (ii) holds and $V\in\mathcal{T}_{\mathcal{Q}}$. If $x\in S^{-1}(V)$, then there exists $y\in V$ such that $x\in S^{-1}(y)$, i.e., $y\in S(x)$. Moreover, since $V\in\mathcal{T}_{\mathcal{Q}}$ and $\mathcal{Q}$ is directed, there exist $q\in\mathcal{Q}$ and $r>0$ such that $B^r_q(y)\subset V$. Hence, by Theorem 1.5, it follows that

$$B^r_{q*S}(x) = S^{-1}(B^r_q(y)) \subset S^{-1}(V).$$

On the other hand, by Corollary 1.6 and the assertion (ii), we also have

$$B^r_{q*S} = S^{-1}\circ B^r_q\circ S\in\mathcal{U}_{\mathcal{P}}.$$

Consequently, $S^{-1}(V)\in\mathcal{T}_{\mathcal{P}}$, and thus (iii) holds.

Next, we show that (iv) implies (v). For this, assume that (iv) is true and $q\in\mathcal{Q}$. If $r>0$, then again by Theorem 1.5, we have

$$B^r_{q*S}(0) = S^{-1}(B^r_q(0)).$$

Hence, using that $B^r_q(0)$ is a neighbourhood of 0 in $Y(\mathcal{T}_{\mathcal{Q}})$ and the assertion (iv), we can infer that $B^r_{q*S}(0)$ is a neighbourhood of 0 in $X(\mathcal{T}_{\mathcal{P}})$. And this already implies (v) by Theorem 0.14.

Finally, to prove that (v) also implies (i), assume now that (v) holds, $V\in\mathcal{U}_{\mathcal{Q}}$ and Φ is a selection relation for S. Then, because of $V\in\mathcal{U}_{\mathcal{Q}}$ and the directedness of $\mathcal{Q}$, there exist $q\in\mathcal{Q}$ and $r>0$ such that $B^r_q\subset V$. Hence, by Corollary 1.6, it follows that

$$B^r_{q*S} = S^{-1}\circ B^r_q\circ\Phi \subset S^{-1}\circ V\circ\Phi.$$

On the other hand, by the assertion (v) and Theorem 0.14, now we have $B^r_{q*S}\in\mathcal{U}_{\mathcal{P}}$. Consequently, $S^{-1}\circ V\circ\Phi\in\mathcal{U}_{\mathcal{P}}$, and thus (i) holds.

REMARK 2.2. If $X(\mathcal{P})$ and $Y(\mathcal{Q})$ are, in particular, seminormed spaces, then the next assertion is also equivalent to the former ones:

(vi) for each $q\in\mathcal{Q}$, there exist $p\in\mathcal{P}^*$ and $M>0$ such that $q*S\leqq Mp$.

Note that in this case by Theorem 1.3, $q*S$ is also a seminorm for all $q\in\mathcal{Q}$, and thus a standard argument on seminorms [4, p. 98] can be applied.

REMARK 2.3. If S is, in particular, a function, then $\mathcal{Q}$ need not be assumed to be directed in Theorem 2.1 and Remark 2.2.

Namely, in this case, we have

$$S^{-1}\Big(\bigcap_{i=1}^{n} B^r_{q_i}(y)\Big) = \bigcap_{i=1}^{n} S^{-1}(B^r_{q_i}(y)) \quad\text{and}\quad S^{-1}\circ\Big(\bigcap_{i=1}^{n} B^r_{q_i}\Big)\circ S = \bigcap_{i=1}^{n} S^{-1}\circ B^r_{q_i}\circ S$$

for any $\{q_i\}_{i=1}^n\subset\mathcal{Q}$, $r>0$ and $y\in Y$.

REMARK 2.4. The directedness of $\mathcal{Q}$ can also be dropped in Theorem 2.1 and Remark 2.2 if one writes $\mathcal{Q}^*$ instead of $\mathcal{Q}$ in (v) and (vi).

To check this, recall that $\mathcal{Q}^*$ is always directed and is equivalent to $\mathcal{Q}$, and thus the assertions (i)—(iv) will not change if we take $\mathcal{Q}^*$ instead of $\mathcal{Q}$.

By a simple application of Theorem 2.1 and Remark 2.4, we can now prove an improvement of the first part of our former Theorem 1 in [*Aequationes Math.* **22** (1981), 308].

THEOREM 2.5. *If S is a linear relation from a preseminormed space $X(\mathcal{P})$ into another $Y(\mathcal{Q})$ such that there exists a lower semicontinuous selection relation Φ for S, then S is lower semiperfectly mildly uniformly continuous.*

PROOF. By Theorem 2.1 and Remark 2.4, we need only show that S is also lower semicontinuous.

According to Theorem 0.3, now we have

$$S(x) = \Phi(x) + S(0)$$

for all $x \in X$. Hence, it is not hard to infer that

$$S^{-1}(V) = \Phi^{-1}(V + S(0))$$

for all $V \subset Y$. Thus, $S^{-1}(V) \in \mathcal{T}_{\mathcal{P}}$ whenever $V \in \mathcal{T}_{\mathcal{Q}}$.

REMARK 2.6. Note that if Φ had been linear, then because of $q * S \leqq q * \Phi$ we could have applied a simpler argument.

As an immediate consequence of Theorem 2.5, we have

COROLLARY 2.7. *If S is a linear equivalence relation on a preseminormed space $X(\mathcal{P})$, then S is lower semiperfectly mildly uniformly continuous.*

REMARK 2.8. In the light of the above results, it does not seem to be an easy task to construct preseminormed spaces $X(\mathcal{P})$ and $Y(\mathcal{Q})$ and a linear relation S from X into Y such that $q * S$ is continuous for all $q \in \mathcal{Q}$, but S is still not lower semicontinuous.

3. Projective generation

DEFINITION 3.1. If S_α is a linear relation from X into a preseminormed space $Y_\alpha(\mathcal{Q}_\alpha)$ for each α in a nonvoid set Γ and

$$\mathcal{P} = \bigcup_{\alpha \in \Gamma} \mathcal{Q}_\alpha * S_\alpha,$$

where $\mathcal{Q}_\alpha * S_\alpha = \{q * S_\alpha \colon q \in \mathcal{Q}_\alpha\}$, then we say that the preseminormed space $X(\mathcal{P})$ is projectively generated from the spaces $Y_\alpha(\mathcal{Q}_\alpha)$ by the relations S_α, and write

$$X(\mathcal{P}) = \operatorname*{proj\,gen}_{\alpha \in \Gamma} {}_{S_\alpha} Y_\alpha(\mathcal{Q}_\alpha).$$

REMARK 3.2. Note that if each $Y_\alpha(\mathcal{Q}_\alpha)$ is a seminormed space, then $X(\mathcal{P})$ is also a seminormed space.

The appropriateness of the above definition is apparent from the next

THEOREM 3.3. *If*

$$X(\mathscr{P}) = \operatorname*{proj\,gen}_{\alpha\in\Gamma}{}_{S_\alpha} Y_\alpha(\mathscr{Q}_\alpha)$$

with each $\mathscr{Q}_\alpha$ being directed, then $\mathscr{U}_{\mathscr{P}}(\mathscr{T}_{\mathscr{P}})$ is the coarsest uniformity (topology) on X for which each S_α is mildly uniformly continuous (lower semicontinuous).

PROOF. If $\alpha\in\Gamma$ and $q\in\mathscr{Q}_\alpha$, then by Definition 3.1, $q*S_\alpha\in\mathscr{P}$, and thus by Theorem 0.13, $q*S_\alpha$ is continuous for $\mathscr{T}_{\mathscr{P}}$. Hence, by Theorem 2.1, S_α is mildly uniformly continuous (lower semicontinuous) for $\mathscr{U}_{\mathscr{P}}(\mathscr{T}_{\mathscr{P}})$.

Suppose now that $\mathscr{U}(\mathscr{T})$ is a uniformity (topology) on X for which each S_α is mildly uniformly continuous (lower semicontinuous). If $p\in\mathscr{P}$, then again by Definition 3.1, $p=q*S_\alpha$ for some $\alpha\in\Gamma$ and $q\in\mathscr{Q}_\alpha$. Thus, if $r>0$, then by Corollary 1.6 (Theorem 1.5),

$$B_p^r = S_\alpha^{-1}\circ B_q^r\circ S_\alpha \qquad (B_p^r(x) = S_\alpha^{-1}(B_q^r(y)) \text{ if } y\in S_\alpha(x)).$$

Hence, by the mild uniform continuity (lower semicontinuity) of S_α for $\mathscr{U}(\mathscr{T})$,

$$B_p^r\in\mathscr{U} \quad (B_p^r(x)\in\mathscr{T} \text{ if } x\in X).$$

Consequently, by Theorem 0.8 (0.10), we have $\mathscr{U}_{\mathscr{P}}\subset\mathscr{U}(\mathscr{T}_{\mathscr{P}}\subset\mathscr{T})$.

From this theorem, using Theorem 0.16, we can at once derive

COROLLARY 3.4. *If*

$$X(\mathscr{P}) = \operatorname*{proj\,gen}_{\alpha\in\Gamma}{}_{S_\alpha} Y_\alpha(\mathscr{Q}_\alpha) \quad \textit{and} \quad X(\mathscr{P}') = \operatorname*{proj\,gen}_{\alpha\in\Gamma}{}_{S_\alpha} Y_\alpha(\mathscr{Q}'_\alpha)$$

and $\mathscr{Q}_\alpha$ and $\mathscr{Q}'_\alpha$ are directed and equivalent for all $\alpha\in\Gamma$, then $\mathscr{P}$ and $\mathscr{P}'$ are also equivalent.

REMARK 3.5. If each S_α is a function, then by Remark 2.3, the $\mathscr{Q}_\alpha$'s need not be assumed to be directed in the above assertions.

Moreover, using Theorem 3.3, we can also easily prove an extension of the first assertion of § 11 in [4, p. 149].

THEOREM 3.6. *If S_α is a linear relation from X into a topological vector space $Y_\alpha(\mathscr{T}_\alpha)$ for each α in a nonvoid set Γ, then the coarsest topology $\mathscr{T}$ on X for which each S_α is lower semicontinuous is a vector topology.*

PROOF. For each $\alpha\in\Gamma$, denote by $\mathscr{Q}_\alpha$ the family of all preseminorms q on Y_α which are continuous for $\mathscr{T}_\alpha$. Note that each $\mathscr{Q}_\alpha$ is nonvoid and directed. Moreover, define

$$X(\mathscr{P}) = \operatorname*{proj\,gen}_{\alpha\in\Gamma}{}_{S_\alpha} Y_\alpha(\mathscr{Q}_\alpha).$$

Then, by [15, Proposition 1.2] and Theorems 0.16 and 0.17, it is clear that $\mathscr{T}_\alpha=\mathscr{T}_{\mathscr{Q}_\alpha}$ for all $\alpha\in\Gamma$. Thus, by Theorem 3.3, $\mathscr{T}=\mathscr{T}_{\mathscr{P}}$, whence by Theorem 0.12, the assertion follows.

However, as an analogue of the assertions (7.1.4) and (7.3.4) in [2], from Theorem 3.3, now we can only get

THEOREM 3.7. *If T is a relation from a topological (uniform) space $Z(\mathcal{T})(Z(\mathcal{U}))$ into*

$$X(\mathcal{P}) = \operatorname*{proj\,gen}_{\alpha\in\Gamma} {}_{S_\alpha} Y_\alpha(\mathcal{Q}_\alpha)$$

and each $\mathcal{Q}_\alpha$ and $\mathcal{P}$ are directed, then the following assertions are equivalent:

(i) *T is lower semicontinuous (mildly uniformly continuous);*

(ii) *each $S_\alpha \circ T$ is lower semicontinuous (mildly uniformly continuous).*

REMARK 3.8. If each $S_\alpha(T)$ is a function, then the $\mathcal{Q}_\alpha$'s ($\mathcal{P}$) need not be assumed to be directed in the above theorem.

If Γ is not a singleton, then $\mathcal{P}$ is usually not directed even if each $\mathcal{Q}_\alpha$ is directed. But, of course, we can state

THEOREM 3.9. *If*

$$X(\mathcal{P}) = \operatorname{proj\,gen}_S Y(\mathcal{Q})$$

and $\mathcal{Q}$ is directed, then $\mathcal{P}$ is also directed.

PROOF. This is immediate from the simple fact that $q_1 \leqq q_2$ implies $q_1 * S \leqq q_2 * S$ for any $q_1, q_2 \in \mathcal{Q}$.

4. Separatedness for projective generation

The following theorem may be considered as an extension of the assertion 5.1 in [8, p. 51].

THEOREM 4.1. *If*

$$X(\mathcal{P}) = \operatorname*{proj\,gen}_{\alpha\in\Gamma} {}_{S_\alpha} Y_\alpha(\mathcal{Q}_\alpha),$$

then $\mathcal{P}$ is separating if and only if

$$\bigcap_{\alpha\in\Gamma} \bigcap_{q\in\mathcal{Q}_\alpha} \bigcap_{r>0} S_\alpha^{-1}(B_q^r(0)) = \{0\}.$$

PROOF. Clearly, $\mathcal{P}$ is separating if and only if

$$\bigcap_{p\in\mathcal{P}} \bigcap_{r>0} B_p^r(0) = \{0\}.$$

Hence, because of

$$\mathcal{P} = \bigcup_{\alpha\in\Gamma} \mathcal{Q}_\alpha * S_\alpha \quad \text{and} \quad B_{q*S_\alpha}(0) = S^{-1}(B_q^r(0)),$$

the stated condition immediately follows.

Using that intersections are preserved under the inverses of functions and $\bigcap_{q\in\mathcal{Q}_\alpha} \bigcap_{r>0} B_q^r(0)$ is the closure of $\{0\}$ in $Y_\alpha(\mathcal{Q}_\alpha)$, from Theorem 4.1 we can at once get

COROLLARY 4.2. *If*

$$X(\mathcal{P}) = \operatorname*{proj\,gen}_{\alpha\in\Gamma} {}_{S_\alpha} Y_\alpha(\mathcal{Q}_\alpha)$$

and each S_α is a function, then $\mathscr{P}$ is separating if and only if

$$\bigcap_{\alpha\in\Gamma} S_\alpha^{-1}(\overline{\{0\}}) = \{0\}.$$

REMARK 4.3. Note that the "only if part" of the above assertion does not require the S_α's to be functions.

Moreover, note also that the closure $\overline{\{0\}}$ of $\{0\}$ in $Y_\alpha(\mathscr{Q}_\alpha)$ is equal to $\{0\}$ if and only if $\mathscr{Q}_\alpha$ is separating.

The following striking theorem shows that there is another, quite different, important particular case when the difficult condition given in Theorem 4.1 can also be substantially simplified.

THEOREM 4.4. *If*

$$X(\mathscr{P}) = \operatorname*{proj\,gen}_{\alpha\in\Gamma}{}_{S_\alpha} Y_\alpha(\mathscr{Q}_\alpha)$$

such that each $\mathscr{Q}_\alpha$ is directed and

$$S_\alpha\Big(\bigcap_{\beta\in\Gamma\setminus\{\alpha\}} S_\beta^{-1}(\overline{\{0\}})\Big) = Y_\alpha$$

for all $\alpha\in\Gamma$, then the following assertions are equivalent:

(i) *$\mathscr{P}$ is separating;*

(ii) *each $S_\alpha(0)$ is closed and $\bigcap_{\alpha\in\Gamma} S_\alpha^{-1}(0)=\{0\}$.*

PROOF. If (i) holds, then by Theorem 4.1 or Remark 4.3, it is clear that $\bigcap_{\alpha\in\Gamma} S_\alpha^{-1}(0)=\{0\}$.

To prove the remaining part of (ii), assume now that $\alpha\in\Gamma$ and $y\in\overline{S_\alpha(0)}$. Then, for any $q\in\mathscr{Q}_\alpha$ and $r>0$, there exists $z\in S_\alpha(0)$ such that $z\in B_q^r(y)$, i.e., $y-z\in B_q^r(0)$. Moreover, because of our strange assumption on the S_α's, there exists $x\in X$ with $y\in S_\alpha(x)$ such that $x\in S_\beta^{-1}(\overline{\{0\}})$ for all $\beta\in\Gamma\setminus\{\alpha\}$. Hence, since $y-z\in S_\alpha(x)$, i.e., $x\in S_\alpha^{-1}(y-z)$, it is clear that $x\in S_\alpha^{-1}(B_q^r(0))$. Moreover, it is also clear that $x\in S_\beta^{-1}(B_k^s(0))$ for all $\beta\in\Gamma\setminus\{\alpha\}$, $k\in\mathscr{Q}_\beta$ and $s>0$. Therefore, if (i) holds, then by Theorem 4.1, we necessarily have $x=0$. Consequently, $y\in S_\alpha(0)$ holds, and thus $S_\alpha(0)$ is closed.

Next, we show that (ii) also implies (i). For this, assume that (ii) is true and $x\in S_\alpha^{-1}(B_q^r(0))$, i.e., $S_\alpha(x)\cap B_q^r(0)\neq\emptyset$ for all $\alpha\in\Gamma$, $q\in\mathscr{Q}_\alpha$ and $r>0$. Then, since each $\mathscr{Q}_\alpha$ is directed, we also have $0\in\overline{S_\alpha(x)}$ for all $\alpha\in\Gamma$. On the other hand, a simple application of Theorem 0.3 and 0.12 shows that now $S_\alpha(x)$ is closed for all $\alpha\in\Gamma$. Therefore, we actually have $0\in S_\alpha(x)$, i.e., $x\in S_\alpha^{-1}(0)$ for all $\alpha\in\Gamma$. Hence, it follows now that $x=0$. And thus, by Theorem 4.1, (i) is also true.

REMARK 4.5. Note that the implication (i)$\Rightarrow$(ii) ((ii)$\Rightarrow$(i)) does not need the extra condition on $\mathscr{Q}_\alpha$'s (S_α's).

Moreover, note also that if each S_α is a function, then the $\mathscr{Q}_\alpha$'s need not also be directed for the implication (ii)$\Rightarrow$(i), too.

REMARK 4.6. In this respect, it is also worth mentioning that if the extra condition on the S_α's were not assumed, then the assertion (ii) should be complicated by writing that each $S_\alpha(0)$ is closed in $S_\alpha\Big(\bigcap_{\beta\in\Gamma\setminus\{\alpha\}} S_\beta^{-1}(\overline{\{0\}})\Big)$.

REMARK 4.7. Finally, we remark that Theorem 4.4 can also be proved without using Theorem 4.1.

However, for this nets and the definition of the infimum composition have to be used instead of the B^r_{qn}'s.

By a very particular case of Theorem 4.4, we can at once state the following analogue of Exercise 4 in [4, p. 108].

COROLLARY 4.8. *If S is a linear relation from X onto a preseminormed (seminormed) space $Y(q)$, then the following assertions are equivalent:*
(i) *$q*S$ is a prenorm (norm);*
(ii) *$S(0)$ is closed and $S^{-1}(0)=\{0\}$.*

Hence, using Theorems 0.3 and 0.12, and also Theorems 0.2 and 0.1, one can easily derive

COROLLARY 4.9. *If S is a linear relation from X onto a preseminormed space $Y(q)$ such that $q*S$ is a prenorm, then S is closed-valued and S^{-1} is a function.*

REMARK 4.10. Note if S is not onto, then we can only state that S is relatively closed-valued in the sense that $S(x)$ is closed in $S(X)$ for all $x\in X$.

5. Reductions for projective generation

The following theorem may be considered as an extension of Proposition 1 in [4, p. 150].

THEOREM 5.1. *If*

$$X(\mathscr{P}) = \operatorname*{proj\,gen}_{\alpha\in\Gamma}{}_{S_\alpha} Y_\alpha(\mathscr{Q}_\alpha) \quad \text{and} \quad Y_\alpha(\mathscr{Q}_\alpha) = \operatorname*{proj\,gen}_{\beta\in\Gamma_\alpha}{}_{T_{\alpha\beta}} Z_{\alpha\beta}(\mathscr{R}_{\alpha\beta})$$

for all $\alpha\in\Gamma$, and

$$X(\mathscr{P}') = \operatorname*{proj\,gen}_{(\alpha,\beta)\in \bigcup_{\alpha\in\Gamma}\Gamma\times\Gamma_\alpha}{}_{T_{\alpha\beta}\circ S_\alpha} Z_{\alpha\beta}(\mathscr{R}_{\alpha\beta})$$

then $\mathscr{P}'=\mathscr{P}$.

PROOF. If $p\in\mathscr{P}$, then $p=q*S_\alpha$ for some $\alpha\in\Gamma$ and $q\in\mathscr{Q}_\alpha$. Moreover, $q=r*T_{\alpha\beta}$ for some $\beta\in\Gamma_\alpha$ and $r\in\mathscr{R}_{\alpha\beta}$. Thus, by Theorem 1.8

$$p = q*S_\alpha = (r*T_{\alpha\beta})*S_\alpha = r*(T_{\alpha\beta}\circ S_\alpha),$$

whence $p\in\mathscr{P}'$ follows. Consequently, $\mathscr{P}\subset\mathscr{P}'$.
The converse inclusion can be proved quite similarly.

Next, we shall prove some unusual statements which do really require relations.

THEOREM 5.2. *Let S be a linear relation from X onto a preseminormed space $Y(\mathscr{Q})$, and $(\Psi_\alpha)_{\alpha\in\Gamma}$ be a nonvoid family of linear selection relations for S^{-1}. Define*

$$X(\mathscr{P}) = \operatorname{proj\,gen}_S Y(\mathscr{Q})$$

and

$$Y(\mathcal{Q}') = \operatorname*{proj\,gen}_{\alpha\in\Gamma}{}_{\Psi_\alpha} X(\mathcal{P}) \quad \text{and} \quad Y(\mathcal{Q}'') = \operatorname{proj\,gen}_{S^{-1}} X(\mathcal{P}),$$

and

$$X(\mathcal{P}') = \operatorname{proj\,gen}_S Y(\mathcal{Q}').$$

Then $\mathcal{Q}'=\mathcal{Q}''\subset\mathcal{Q}^*$ *and* $\mathcal{P}'=\mathcal{P}$. *Moreover,* $\mathcal{Q}'=\mathcal{Q}$ *if* S *is a function.*

Proof. If $q'\in\mathcal{Q}'$, then there exist $\alpha\in\Gamma$ and $p\in\mathcal{P}$ such that $q'=p*\Psi_\alpha$. Moreover, there exists $q\in\mathcal{Q}$ such that $p=q*S$. Thus, by (i) in Theorem 1.9, we have

$$q' = p*\Psi_\alpha = (q*S)*\Psi_\alpha \leqq q$$

whence $q'\in\mathcal{Q}^*$ follows. Consequently, $\mathcal{Q}'\subset\mathcal{Q}^*$.

On the other hand, because of (ii) in Theorem 1.9, we may clearly write

$$\mathcal{Q}' = \bigcup_{\alpha\in\Gamma} \mathcal{P}*\Psi_\alpha = \bigcup_{\alpha\in\Gamma} (\mathcal{Q}*S)*\Psi_\alpha = \bigcup_{\alpha\in\Gamma} (\mathcal{Q}*S)*S^{-1} = \mathcal{P}*S^{-1} = \mathcal{Q}''.$$

The assertion $\mathcal{P}'=\mathcal{P}$ can be proved quite similarly by using (iii) in Theorem 1.9.

Finally, if S is in particular a function, then because of Theorem 1.8, we clearly have

$$\mathcal{Q}' = \mathcal{Q}'' = \mathcal{P}*S^{-1} = (\mathcal{Q}*S)*S^{-1} = \mathcal{Q}*(S\circ S^{-1}) = \mathcal{Q}*\Delta_Y = \mathcal{Q}\circ\Delta_Y = \mathcal{Q}.$$

By a simple application of a particular case of this theorem, we can prove

Theorem 5.3. *If S is a linear relation from a preseminormed space $X(\mathcal{P})$ onto another $Y(\mathcal{Q})$ such that*

$$X(\mathcal{P}) = \operatorname{proj\,gen}_S Y(\mathcal{Q}),$$

then each linear selection relation Ψ for S^{-1} is mildly uniformly continuous (lower semicontinuous).

Proof. If Ψ is as above, then by Theorem 0.7, there exists a linear selection function ψ for Ψ. Moreover, if $p\in\mathcal{P}$, then by Theorem 5.2, we necessarily have $p*\psi\in\mathcal{Q}^*$. Hence, it is clear that $p*\psi$ is continuous. Thus, by Theorem 2.1 and Remark 2.3, ψ is also continuous. Consequently, by Theorem 2.5, Ψ is lower semiperfectly mildly uniformly continuous.

Remark 5.4. If in particular $\mathcal{Q}$ is directed, or S is a function, then S is also mildly uniformly continuous.

In this latter particular case, we can also easily prove a certain converse to a useful reformulation of Theorem 5.3.

Theorem 5.5. *If f is a continuous linear function from a total preseminormed space $X(\mathcal{P})$ onto another total one $Y(\mathcal{Q})$ such that f^{-1} is lower semicontinuous, then*

$$Y(\mathcal{Q}) = \operatorname{proj\,gen}_{f^{-1}} X(\mathcal{P}).$$

Proof. This is also a consequence of Theorem 2.1. Namely, since f is continuous, $\mathcal{Q}*f\subset\bar{\mathcal{P}}=\mathcal{P}$, whence

$$\mathcal{Q} = \mathcal{Q}*\Delta_Y = \mathcal{Q}*(f\circ f^{-1}) = (\mathcal{Q}*f)*f^{-1} \subset \mathcal{P}*f^{-1}.$$

On the other hand, since f^{-1} is lower semicontinuous, $\mathscr{P}*f^{-1}\subset\bar{\mathscr{Q}}=\mathscr{Q}$. Consequently, $\mathscr{Q}=\mathscr{P}*f^{-1}$.

Combining Theorems 5.3 and 5.5, we get

Corollary 5.6. *If f is a linear function from a total preseminormed space $X(\mathscr{P})$ onto another total one $Y(\mathscr{Q})$, then any of the following assertions implies the subsequent one:*

(i) $X(\mathscr{P})=\operatorname{proj\,gen}_f Y(\mathscr{Q})$;

(ii) *f is continuous and each linear selection relation Ψ for f^{-1} is mildly uniformly continuous;*

(iii) *f is continuous and f^{-1} is lower semicontinuous;*

(iv) $Y(\mathscr{Q})=\operatorname{proj\,gen}_{f^{-1}} X(\mathscr{P})$.

Moreover, if in particular f is injective, then (iv) *also implies* (i).

6. Applications of projective generation

Definition 6.1. If $Y(\mathscr{Q})$ is a preseminormed space and X is a linear subspace of Y, then the preseminormed space

$$X(\mathscr{P}) = \operatorname{proj\,gen}_{\Delta_X} Y(\mathscr{Q}),$$

where Δ_X is the identity function of X, will be called the subspace of $Y(\mathscr{Q})$ generated by X.

As an immediate consequence of Definition 6.1, Theorem 3.3 and Remark 3.5, we have

Theorem 6.2. *If $X(\mathscr{P})$ is a subspace of a preseminormed space $Y(\mathscr{Q})$, then*

$$\mathscr{P} = \mathscr{Q}|X = \{q|X\colon q\in\mathscr{Q}\}$$

and

$$\mathscr{U}_{\mathscr{P}} = \mathscr{U}_{\mathscr{Q}}|X \quad \text{and} \quad \mathscr{T}_{\mathscr{P}} = \mathscr{T}_{\mathscr{Q}}|X.$$

Several further properties of subspaces can be easily derived from the results of Sections 3, 4 and 5. For instance, Theorem 5.1 yields

Theorem 6.3. *If*

$$X(\mathscr{P}) = \operatorname*{proj\,gen}_{\alpha\in\Gamma} {}_{S_\alpha} Y_\alpha(\mathscr{Q}_\alpha)$$

and $Y_\alpha(\mathscr{Q}_\alpha)$ is a subspace of $Z_\alpha(\mathscr{R}_\alpha)$ for each $\alpha\in\Gamma$, then

$$X(\mathscr{P}) = \operatorname*{proj\,gen}_{\alpha\in\Gamma} {}_{S_\alpha} Z_\alpha(\mathscr{R}_\alpha).$$

The next trivial theorem also offers an important application of projective generation.

Theorem 6.4. *If $\mathscr{P}_\alpha$ is a nonvoid family of preseminorms on X for each α in a nonvoid set Γ, and $\mathscr{P}=\bigcup_{\alpha\in\Gamma}\mathscr{P}_\alpha$, then*

$$X(\mathscr{P}) = \operatorname{proj\,gen}_{\Delta_X} X(\mathscr{P}_\alpha).$$

For instance, combining Theorem 6.4 with Theorem 3.3 and Remark 3.5, we can at once state

COROLLARY 6.5. *If $\mathscr{P}_\alpha$ is a nonvoid family of preseminorms on X for each α in a nonvoid set Γ and $\mathscr{P}=\bigcup_{\alpha\in\Gamma}\mathscr{P}_\alpha$, then*

$$\mathscr{U}_{\mathscr{P}}=\sup_{\alpha\in\Gamma}\mathscr{U}_{\mathscr{P}_\alpha} \quad \text{and} \quad \mathscr{T}_{\mathscr{P}}=\sup_{\alpha\in\Gamma}\mathscr{T}_{\mathscr{P}_\alpha}.$$

REMARK 6.6. An important particular case is when each $\mathscr{P}_\alpha$ is a singleton.

However, the most important applications of projective generation are product spaces.

DEFINITION 6.7. If $Y_\alpha(\mathscr{Q}_\alpha)$ is a preseminormed space for each α in a nonvoid set Γ and $X=\bigtimes_{\alpha\in\Gamma}Y_\alpha$, then the preseminormed space

$$X(\mathscr{P})=\operatorname*{proj\,gen}_{\alpha\in\Gamma}{}_{\pi_\alpha}Y_\alpha(\mathscr{Q}_\alpha),$$

where π_α is the projection of X onto Y_α, will be called the Cartesian product of the spaces $Y_\alpha(\mathscr{Q}_\alpha)$ and the notation

$$X(\mathscr{P})=\bigtimes_{\alpha\in\Gamma}Y_\alpha(\mathscr{Q}_\alpha)$$

will be used.

As an immediate consequence of Definition 6.7, Theorem 3.3 and Remark 3.5, we have

THEOREM 6.8. *If*

$$X(\mathscr{P})=\bigtimes_{\alpha\in\Gamma}Y_\alpha(\mathscr{Q}_\alpha)$$

and π_α is the projection of X onto Y_α, then

$$\mathscr{P}=\bigcup_{\alpha\in\Gamma}\mathscr{Q}_\alpha\circ\pi_\alpha,$$

and moreover

$$\mathscr{U}_{\mathscr{P}}=\bigtimes_{\alpha\in\Gamma}\mathscr{U}_{\mathscr{Q}_\alpha} \quad \text{and} \quad \mathscr{T}_{\mathscr{P}}=\bigtimes_{\alpha\in\Gamma}\mathscr{T}_{\mathscr{Q}_\alpha}.$$

Moreover, in addition to the corresponding particular case of Theorem 3.3, now we also have

THEOREM 6.9. *If*

$$X(\mathscr{P})=\bigtimes_{\alpha\in\Gamma}Y_\alpha(\mathscr{Q}_\alpha),$$

then the inverse π_α^{-1} of the projection π_α of X onto Y_α is lower semiperfectly mildly uniformly continuous for all $\alpha\in\Gamma$.

PROOF. From the third assertion of Theorem 6.8, by [5, Theorem 3.2], it follows that π_α is open, i.e., π_α^{-1} is lower semicontinuous. Thus, by Theorem 2.1 and Remark 2.4, π_α^{-1} is lower semiperfectly mildly uniformly continuous, too.

From the third assertion of Theorem 6.8, using the results of [5, pp. 92—93] and Theorem 0.18, one can also easily derive

THEOREM 6.10. *If*

$$X(\mathcal{P}) = \mathop{\times}_{\alpha\in\Gamma} Y_\alpha(\mathcal{Q}_\alpha)$$

then $\mathcal{P}$ *is separating if and only if each* $\mathcal{Q}_\alpha$ *is separating.*

However, this theorem can now be more naturally obtained from Theorem 4.4 and Remark 4.5, since now we obviously have

$$\pi_\alpha\Big(\bigcap_{\beta\in\Gamma\setminus\{\alpha\}} \pi_\beta^{-1}(0)\Big) = Y_\alpha \quad \text{for all } \alpha\in\Gamma \text{ and } \bigcap_{\alpha\in\Gamma} \pi_\alpha^{-1}(0) = \{0\}.$$

The importance of Cartesian products as particular projective generations lies mainly in the following reduction

THEOREM 6.11. *If*

$$X(\mathcal{P}) = \mathop{\mathrm{proj\,gen}_{S_\alpha}}_{\alpha\in\Gamma} Y_\alpha(\mathcal{Q}_\alpha) \quad \textit{and} \quad Y(\mathcal{Q}) = \mathop{\times}_{\alpha\in\Gamma} Y_\alpha(\mathcal{Q}_\alpha)$$

and S *is the relation from* X *into* Y *defined by*

$$S(x) = \mathop{\times}_{\alpha\in\Gamma} S_\alpha(x)$$

then

$$X(\mathcal{P}) = \mathrm{proj\,gen}_S\, Y(\mathcal{Q}).$$

PROOF. A simple computation shows that S is linear. Moreover, for $\alpha\in\Gamma$, we clearly have $S_\alpha=\pi_\alpha\circ S$, where π_α is again the projection of X onto Y_α. Thus, Theorem 5.1 can be applied to obtain the assertion.

From this theorem, using Theorems 6.3 and 5.3 and Remark 5.4, one can easily derive

COROLLARY 6.12. *If*

$$X(\mathcal{P}) = \mathop{\mathrm{proj\,gen}_{f_\alpha}}_{\alpha\in\Gamma} Y_\alpha(\mathcal{Q}_\alpha),$$

where each f_α *is a function, and* $\bigcap_{\alpha\in\Gamma} f_\alpha^{-1}(0)=\{0\}$, *then the function* f *defined on* X *by*

$$f(x) = (f_\alpha(x))_{\alpha\in\Gamma}$$

is an algebraic and uniform isomorphism of $X(\mathcal{P})$ *onto a subspace of* $Y(\mathcal{Q})=\mathop{\times}_{\alpha\in\Gamma} Y_\alpha(\mathcal{Q}_\alpha)$.

Moreover, applying Corollary 6.12 to the particular case of Theorem 6.4 mentioned in Remark 6.6, we get

COROLLARY 6.13. *If* $X(\mathcal{P})$ *is a preseminormed space, then the function* f *defined on* X *by*

$$f(x) = (x)_{p\in\mathcal{P}}$$

is an algebraic and uniform isomorphism of $X(\mathcal{P})$ *onto a subspace of* $\mathop{\times}_{p\in\mathcal{P}} X(p)$.

REMARK 6.14. This simple fact will play an important role in the completion of preseminormed spaces.

REMARK 6.15. Besides Definitions 6.1 and 6.7 and Theorem 6.4, there are some further important applications of projective generation.

For instance, if $X(\mathscr{P})$ is a preseminormed space over $\mathbf{K}$ and

$$X(\mathscr{P}_w) = \operatorname*{proj\,gen}_{f \in X^*} {}_f\, \mathbf{K}(|\,|),$$

where X^* is the usual dual space of $X(\mathscr{T}_{\mathscr{P}})$, then $\mathscr{T}_{\mathscr{P}_w}$ is the usual weak topology of $X(\mathscr{T}_{\mathscr{P}})$.

The weak*-topology of X^* can also be obtained in a similar way. The corresponding facts for locally convex spaces seem to be stressed only in [8, p. 52].

ADDED IN PROOF. In the meantime we learned that the continuity properties of linear relations had also been studied by L. Holá and I. Kupka [Closed graph and open mapping theorems for linear relations, *Acta Math. Univ. Comenianae* **46—47** (1985), 157—162].

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(Received February 8, 1983; in revised form March 5, 1987)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 315—318

APPLICATIONS OF THE THEORY OF DIRECTIONAL STRUCTURES II. AN EMBEDDING THEOREM

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We aim at proving a new characterization of the class of subspaces of a Euclidean space. (Such characterizations can be found in [1, 7, 4, 5, 6].) This paper being a sequel to [5], the terminology and notations of that paper will be used without explanation. In particular, we call attention to [5] 1:1—1:6.

§ 0. Pseudo-directions

0:1 Definition (E. Deák [2, 3]). Let X be a set. A collection $\mathscr{R}$ of ordered pairs of subsets of X is a *pseudo-direction* on X if it satisfies the following conditions:

(i) $G\subset F$ for each $(G, F)\in\mathscr{R}$;

(ii) if (G_1, F_1) and (G_2, F_2) are two distinct elements of $\mathscr{R}$ then $F_1\subset G_2$ or $F_2\subset G_1$.

Many of the notations, definitions and theorems listed in [5] § 0 can be applied to, or are valid for, pseudo-directions. For the sake of completeness, we shall run through the whole list, although parts of it are irrelevant to the subject of this paper.

0:2 to **0:15** (E. Deák [2, 3]). Read pseudo-direction(al) for direction(al), except that

a) a pseudo-direction may not be compact (0:5);

b) the part of 0:12 b) concerning orderability does not hold for pseudo-directions.

0:16 to **0:21** No corresponding definitions.

0:22 Theorem [6]. *If $\mathfrak{R}$ is a compatible pseudo-directional structure of a separable metrizable space X then X can be topologically embedded into* $\mathbf{R}^{|\mathfrak{R}|}$.

Erratum to the first part of this series. For the last sentence in [5] 4:2 a), read: "Now if $\mathfrak{R}$ is the natural directional structure of the cube then

$$\mathscr{S}=\{V\cap N\colon V\in\tilde{\mathcal{O}}(\mathfrak{R}),\ \bar{V}\neq[0,1]^n\}$$

is a comparable T_1-complementary subbase of N such that inc $\mathscr{S}=n$."

1980 *Mathematic Subject Classification.* Primary 54C25.

Key words and phrases. Pseudo-direction, pseudo-directional structure, subbase, comparable complementary, T_1-complementary, inc, separable metrizable space, Euclidean space, embedding.

§ 1. An embedding theorem

Generalizing an embedding theorem due to de Groot [7], we have proved

1:1 THEOREM ([5] 4:1). *A separable metrizable space can be topologically embedded into* $\mathbf{R}^n$ $(n=0, 1, \ldots)$ *iff it has a comparable T_1-complementary subbase* $\mathcal{S}$ *with* $\operatorname{inc}\mathcal{S} \leqq n$.

The more interesting part of the theorem (i.e. the sufficiency of the existence of such an $\mathcal{S}$) is strengthened by

1:2 THEOREM. *If a separable metrizable space X has a comparable complementary subbase* $\mathcal{S}$ *with* $\operatorname{inc}\mathcal{S} \leqq n$ *such that*

$$A \subsetneqq B \Rightarrow \bar{A} \subset B \qquad (A, B \in \mathcal{S}) \tag{1}$$

then X can be topologically embedded into $\mathbf{R}^n$.

This is indeed a generalization of 1:1 since a) a T_1-complementary system is evidently complementary; b) a comparable T_1-complementary system satisfies (1) [take $p \in B-A$ and $C \in \mathcal{S}$ with $B \cup C = X$, $p \notin C$; now $A\gamma C$ and $A\bar{\sigma}C$ (cf. [5] 2:2[1]) but $p \notin A \cup C$, so $A \cap C = \emptyset$ and $\bar{A} \subset X - C \subset B$].

PROOF. According to van Dalen and Wattel [8] 3.3, if $\mathcal{S}$ is comparable and complementary then σ is an equivalence relation on $\mathcal{S}$ and the σ equivalence classes can be paired off such that $A\gamma B\bar{\sigma}A$, $A, B \in \mathcal{S}$ imply that A and B belong to complementary equivalence classes. Let us denote the family of the σ equivalence classes by $\mathfrak{A}$; the element of $\mathfrak{A}$ containing an $A \in \mathcal{S}$ by $[A]$; the complementary class of an $\mathcal{A} \in \mathfrak{A}$ by $\mathcal{A}^*$. Thus:

$$A\gamma B\bar{\sigma}A \Rightarrow [A] = [B]^* \qquad (A, B \in \mathcal{S}). \tag{2}$$

Consider now the relation $A \sim B$ iff $\operatorname{inc}[A] \cup [B] = 1$ on $\mathcal{S}$. First of all, we have

$$A\sigma B \Rightarrow A \sim B \Rightarrow A\gamma B \tag{3}$$

[remember that for a system $\mathcal{C} \neq \emptyset$, $\operatorname{inc}\mathcal{C} = 1$ is equivalent to $C_1\gamma C_2$ $(C_1, C_2 \in \mathcal{C})$].

Assume $A \sim B \sim C$. By (3), we have $A\gamma B\gamma C$. If, in addition, $A\bar{\sigma}B\bar{\sigma}C$ then (2) implies $[A] = [B]^* = [C]$, thus $\operatorname{inc}[A] \cup [C] = \operatorname{inc}[A] = 1$, i.e. $A \sim C$. On the other hand, if $A\sigma B$ then $[A] = [B]$, so $\operatorname{inc}[A] \cup [C] = \operatorname{inc}[B] \cup [C] = 1$. The case $B\sigma C$ is analogous, so $\sim$ is transitive. Consequently, $\sim$ is an equivalence relation on $\mathcal{S}$.

The family of the $\sim$ equivalence classes will be denoted by $\mathfrak{E}$. According to (3), each element of $\mathfrak{E}$ is the union of some elements of $\mathfrak{A}$. In fact,

$$\mathcal{E} \in \mathfrak{E} - \mathfrak{A} \Rightarrow \exists \mathcal{A} \in \mathfrak{A}, \quad \mathcal{E} = \mathcal{A} \cup \mathcal{A}^* \tag{4}$$

and

$$\mathcal{E} \in \mathfrak{E} \cap \mathfrak{A} \Rightarrow \operatorname{inc}\mathcal{E} \cup \mathcal{E}^* > 1. \tag{5}$$

To prove (4), let $A \in \mathcal{E}$ be fixed. Then (3) gives $A\gamma B$ for any $B \in \mathcal{E}$, so (2) implies $[B] = [A]$ or $[B] = [A]^*$, thus $\mathcal{E} \subset [A] \cup [A]^*$; as $\mathcal{E} \notin \mathfrak{A}$, we have $\mathcal{E} = [A] \cup [A]^*$. To

[1] Only Case 8 from [5] 2:4 is really needed here.

prove (5), take an $\mathscr{A}\in\mathfrak{A}$ with $\operatorname{inc}\mathscr{A}\cup\mathscr{A}^*=1$; now for any $A\in\mathscr{A}$ and $B\in\mathscr{A}^*$, we have $[A]\cup[B]=\mathscr{A}\cup\mathscr{A}^*$, so $\operatorname{inc}[A]\cup[B]=1$, i.e. $A\sim B$ and $\mathscr{A}\notin\mathfrak{E}$.

Let us pick sets $S_{\mathscr{E}}\in\mathscr{E}$ for each $\mathscr{E}\in\mathfrak{E}$ such that

a) if $\mathscr{E}\in\mathfrak{A}$ then $S_{\mathscr{E}}\bar{\gamma}S_{\mathscr{E}^*}$ (according to (5), there are such sets);

b) if $\mathscr{E}\notin\mathfrak{A}$ then $S_{\mathscr{E}}$ is arbitrary.

It can be readily checked that if $\mathscr{D},\mathscr{E}\in\mathfrak{E}$, $\mathscr{D}\neq\mathscr{E}$ then $S_{\mathscr{D}}\bar{\gamma}S_{\mathscr{E}}$. Thus

(6)[2] $$|\mathfrak{E}|\leqq\operatorname{inc}\mathscr{S}.$$

We are going to construct for each $\mathscr{E}\in\mathfrak{E}$ a pseudo-direction $\mathscr{R}_{\mathscr{E}}$ of X such that $\mathscr{E}\subset\mathscr{O}(\mathscr{R}_{\mathscr{E}})$; then (6) and 0.22 imply that X can be embedded into $\mathbf{R}^{\operatorname{inc}\mathscr{S}}$.

If $\mathscr{E}\in\mathfrak{E}\cap\mathfrak{A}$, put

$$\mathscr{R}_{\mathscr{E}}=\{(E,\bar{E})\colon E\in\mathscr{E}\}.$$

(1) guarantees that $\mathscr{R}_{\mathscr{E}}$ is a pseudo-direction.

On the other hand, if $\mathscr{A}\cup\mathscr{A}^*=\mathscr{E}\in\mathfrak{E}$ for some $\mathscr{A}\in\mathfrak{A}$ then the family

$$\mathscr{C}=\mathscr{A}\cup\{X-B\colon B\in\mathscr{A}^*\}$$

in ordered by inclusion [cf. (3)]. Let

$$\mathscr{R}^1_{\mathscr{E}}=\{(G,F)\colon G\in\mathscr{A},\ X-F\in\mathscr{A}^*,\ G\subset F,\ (G\subsetneqq S\subsetneqq F\Rightarrow S\notin\mathscr{C})\};$$

$$\mathscr{R}^2_{\mathscr{E}}=\{(G,\bar{G})\colon G\in\mathscr{A}-\mathscr{G}(\mathscr{R}^1_{\mathscr{E}})\};$$

$$\mathscr{R}^3_{\mathscr{E}}=\{(\operatorname{int}F,F)\colon X-F\in\mathscr{A}^*,\ F\notin\mathscr{F}(\mathscr{R}^1_{\mathscr{E}})\}$$

and

$$\mathscr{R}_{\mathscr{E}}=\mathscr{R}^1_{\mathscr{E}}\cup\mathscr{R}^2_{\mathscr{E}}\cup\mathscr{R}^3_{\mathscr{E}}.$$

Now $\mathscr{R}_{\mathscr{E}}$ is a pseudo-direction (the straightforward proof is left to the reader: use (1) and the fact that $\mathscr{C}$ is ordered by inclusion).

Clearly, $\mathscr{E}\subset\mathscr{O}(\mathscr{R}_{\mathscr{E}})$ holds in both cases, thus the proof of the theorem is complete.

§ 2. Remarks

2:1 The example given in [6] between Theorems A and B shows that 1:2 is not only seemingly more general than 1:1.

2:2 If Theorem 1:2 could be proved without making use of 0:22, it would provide a new proof for Theorem 0:22:

Let $\{\mathscr{R}_1,\dots,\mathscr{R}_n\}$ be a compatible pseudo-directional structure of a separable metrizable space X, $n\geqq1$. Let Y be the set of the functions $\{1,\dots,n\}\to\{0,1\}$ endowed with the discrete topology, and denote the topological sum of X and Y by Z. Put

$$A_i=\{f\in Y\colon f(i)=0\},\quad B_i=Y-A_i\quad(1\leqq i\leqq n).$$

Now

$$\mathscr{S}=\{G\cup A_i\colon G\in\mathscr{G}(\mathscr{R}_i)\cup\{\emptyset\},\ 1\leqq i\leqq n\}\cup\{H\cup B_i\colon H\in\mathscr{H}(\mathscr{R}_i)\cup\{\emptyset\},\ 1\leqq i\leqq n\}$$

satisfies the conditions in 1:2, thus $Z\supset X$ can be embedded into $\mathbf{R}^n$.

2:3 We do not know if 1:2 holds without condition (1).

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[2] In fact, equality holds here, but the other (trivial) inequality will not be needed.

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(Received February 17, 1983)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 319—326

AN EXTREMAL PROBLEM FOR COMPLETE BIPARTITE GRAPHS

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Dedicated to the memory of Paul Turán

Abstract

Define $f(n, k)$ to be the largest integer q such that for every graph G of order n and size q, $\bar{G}$ contains every complete bipartite graph $K_{a,b}$ with $a+b=n-k$. We obtain (i) exact values for $f(n, 0)$ and $f(n, 1)$, (ii) upper and lower bounds for $f(n, k)$ when $k \geqq 2$ is fixed and n is large, and (iii) an upper bound for $f(n, \lfloor \varepsilon n \rfloor)$.

1. Introduction

Extremal graph theory, which was initiated by Turán in 1941 [4], is still the source of many interesting and difficult problems. The standard problem is to determine $f(n, G)$, the smallest integer q such that every graph with n vertices and q edges contains a subgraph isomorphic to G. It is striking that whereas Turán completely determined $f(n, K_m)$, there is much which is as yet unknown concerning $f(n, K_{a,b})$. In this paper, we consider a variant of the extremal problem for complete bipartite graphs. In this variant we ask how many edges must be deleted from K_n so that the resulting graph no longer contains $K_{a,b}$ for *some* pair (a, b) with $a+b=m$. Specifically, we seek to determine an extremal function $f(n, k)$ defined as follows. For $m>1$, let B_m denote the class of all graphs G such that $G \supset K_{a,b}$ for every pair (a, b) with $a+b=m$. Then for $n>k+1$, $f(n, k)$ is the largest integer q such that every graph G of order n and size $\binom{n}{2}-q$ is a member of B_{n-k}. In this paper we obtain exact values for $f(n, 0)$ and $f(n, 1)$, upper and lower bounds for $f(n, k)$ when $k>1$ is fixed and n is large, and an upper bound for $f(n, \lfloor \varepsilon n \rfloor)$.

2. Terminology and notation

All graphs considered in this paper will be ordinary graphs, i.e. finite, undirected graphs, without loops or multiple edges.

A graph with vertex set V and edge set E will be denoted $G(V, E)$. If $|V|=p$ and $|E|=q$, G is said to be of *order* p and *size* q. With $X, Y \subseteq V$, the set of edges in E of the form $\{x, y\}$ where $x \in X$ and $y \in Y$ will be denoted $E(X, Y)$. The complement of G will be denoted $\bar{G}$.

1980 *Mathematics Subject Classification*. Primary 05C35; Secondary 05C05.
Key words and phrases. Bipartite graphs, extremal graphs, trees.

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The *size* of G will be given by $q(G)$. The order of the *largest connected component* will be given by $\mu(G)$ and the order of the *smallest connected component* will be given by $\eta(G)$. In particular, $\eta(G)=1$ means that G contains an isolated vertex.

Let A be a finite set. Then A^k will denote the Cartesian product $A\times A\times\ldots\times A$ with k factors and $[A]^k$ will denote the collecton of k-element subsets of A.

Where x is a real number, $\lfloor x\rfloor$ and $\lceil x\rceil$ denote the greatest integer $\leqq x$ and the least integer $\geqq x$, respectively.

For any notation or terminology not explicitly mentioned in this section, we refer the reader to [1] or [2].

3. Calculation of $f(n, k)$ where k is fixed

Our starting point is the following simple observation. If G is of order n and $\mu(G)>\lceil n/2\rceil$, then $\bar{G}\not\supseteq K_{a,b}$ with $a=\lceil n/2\rceil$, $b=\lfloor n/2\rfloor$ and so $\bar{G}\notin B_n$. The opposite direction is described by the following useful lemma.

Lemma 1. *If $G(V, E)$ is a graph of order n such that* (i) $\mu(G)\leqq\{n/2\}$, (ii) $\eta(G)=1$, *and* (iii) $q(G)\leqq\lfloor 2n/3\rfloor-1$, *then $\bar{G}\in B_n$. This result is sharp.*

Proof. The proof is by induction on n. If $n=2$, then G is required to be empty and so the conclusion holds. Let $\mu(G)=k$. It is easy to see that the result holds if $k=1$ or 2, so we may assume that $k\geqq 3$. Let $H=G-X$, where X is a component of order k. Then H is a graph of order $n-k$ and $\eta(H)=1$. Now $q(H)\leqq\lfloor 2n/3\rfloor-k\leqq$ $\leqq\lfloor 2(n-k)/3\rfloor-1$, the second inequality being by virtue of the fact that $k\geqq 3$. Also, $\mu(H)\leqq\min(k, \lfloor 2n/3\rfloor-k+1)$. If $3k\leqq n$, then $k\leqq\lceil(n-k)/2\rceil$ and if $3k\geqq n+1$, then $\lfloor 2n/3\rfloor-k+1\leqq\lceil(n-k)/2\rceil$. Hence, in all cases H satisfies (i)—(iii) and so, by the induction hypothesis, $\bar{H}\in B_{n-k}$. Since $\bar{X}$ and $\bar{H}$ are completely joined in $\bar{G}$, it follows that $\bar{G}\in B_n$.

From the remark made earlier, we know that condition (i) cannot be weakened. To see that (ii) cannot be weakened, note that if $\eta(G)>1$, then $\bar{G}\not\supseteq K_{1,n-1}$. Finally, with $n\geqq 7$ set $m=\lfloor(n+1)/3\rfloor+1$, $k=\lfloor n/3\rfloor+1$, $l=n-m-k$ and consider the graph $G=T_m\cup T_k\cup\bar{K}_l$, where T_m and T_k denote arbitrary trees of orders m and k, respectively. In this case, we have $\mu(G)\leqq\lceil n/2\rceil$, $\eta(G)=1$ and $q(G)=\lfloor 2n/3\rfloor$. However, $\bar{G}\not\supseteq K_{a,b}$ with $a=\lfloor 2n/3\rfloor+1$, $b=\lceil n/3\rceil-1$. This example shows that condition (iii) cannot be weakened. □

With the aid of Lemma 1, we can obtain the exact value of $f(n, k)$ in case $k=0$ or 1.

Theorem 1. *For all $n\geqq 2, f(n, 0)=\lceil n/2\rceil-1$ and for all $n\geqq 3, f(n, 1)=\lceil(n+1)/2\rceil$.*

Proof. With $m=\lceil n/2\rceil+1$, let $G=T_m\cup\bar{K}_{n-m}$, where T_m denotes an arbitrary tree of order m. Thus, G is a graph of order n, $q(G)=\lceil n/2\rceil$ and $\mu(G)=\lceil n/2\rceil+1$. Since $\mu(G)>\lceil n/2\rceil$, it follows that $\bar{G}\notin B_n$ and this example shows that $f(n, 0)\leqq\lceil n/2\rceil-1$. To prove the inequality in the other sense, consider an arbitrary graph G of order n and size $q(G)\leqq\lceil n/2\rceil-1$. Note that such a graph must satisfy (i) $\mu(G)\leqq\lceil n/2\rceil$, (ii) $\eta(G)=1$, and (iii) $q(G)\leqq\lfloor 2n/3\rfloor-1$. Hence, by Lemma 1, $\bar{G}\in B_n$.

With $m=\lceil(n+1)/2\rceil+1$, let $G=C_m\cup\bar{K}_{n-m}$, where C_m denotes the cycle of order m. Thus, G is a graph of order n and size $q(G)=\lceil(n+1)/2\rceil+1$. Moreover, if x is an arbitrary vertex of G, then $\mu(G-x)\geqq\lceil(n+1)/2\rceil>\lceil(n-1)/2\rceil$. It follows that for each x $\overline{G-x}\not\supseteq K_{a,b}$ with $a=\lceil(n-1)/2\rceil$, $b=\lfloor(n-1)/2\rfloor$ and so this example shows that $f(n,1)\leqq\lceil(n+1)/2\rceil$. To prove the inequality in the other sense, consider an arbitrary graph G of order n and size $q(G)\leqq\lceil(n+1)/2\rceil$. Let x be a vertex of maximal degree in G, and let $H=G-x$. If x has degree $\geqq 2$, then $q(H)\leqq \leqq\lceil(n+1)/2\rceil-2=\lceil(n-1)/2\rceil-1$. If x has degree $\leqq 1$, then G is the union of a collection of disjoint edges and so in this case as well $q(H)\leqq\lceil(n-1)/2\rceil-1$. Therefore, by the first part of this theorem, $\overline{H}\in B_{n-1}$ and so $\overline{G}\in B_{n-1}$. □

Corollary. *Let $t(n)$ denote the largest integer q such that for every graph G of order n and size q, $\overline{G}$ contains every tree of order n. For all $n\geqq 2$, $t(n)=\lceil n/2\rceil-1$.*

Proof. Since each tree of order n is contained in an appropriate complete bipartite graph $K_{a,b}$ with $a+b=n$, it follows that $t(n)\geqq f(n,0)=\lceil n/2\rceil-1$. On the other hand, the graph $G=(n/2)P_2$ (n even) or $G=((n-3)/2)P_2\cup P_3$ (n odd) is a graph of order n and size $q(G)=\lceil n/2\rceil$ such that $\overline{G}\not\supseteq K_{1,n-1}$. (Here, mH is used to denote the graph with m components, each isomorphic to H.) This example shows that $t(n)\leqq\lceil n/2\rceil-1$. □

At this point, one may be tempted to conjecture that for each fixed value of k, $f(n,k)=n/2+O(1)$, perhaps even exactly calculable as in the case of $k=0$ or $k=1$. In fact, we find that for all $k\geqq 2$, $n/2+A\sqrt{n}<f(n,k)<n/2+B\sqrt{n}$, where the positive numbers A and B depend only on k. Thus, there is a very striking difference between the case of $k=1$ and that of $k=2$. In order to establish the facts concerning the behavior of $f(n,k)$ when $k\geqq 2$, we shall need several preliminary results.

The following lemma uses the term *suspended path.* A path $x_0, x_1, ..., x_k$ in a graph G will be called suspended if its interior vertices $x_1, ..., x_{k-1}$ are of degree 2 in G, whereas its end vertices (x_0 and x_k) have degree $\neq 2$.

Lemma 2. *Any tree having k vertices of degree 1 is the union of at most $2k-3$ edge-disjoint suspended paths.*

Proof. The proof is left to the reader.

Lemma 3. *Let T be a tree of order $n+1$ where $n\geqq 2$. There exists a vertex x such that $\mu(T-x)\leqq\lceil n/2\rceil$. Consequently, there is a partition of the components of $T-x$ into two parts such that each part has at least $\lceil n/3\rceil$ vertices.*

Proof. The proof is left to the reader.

Lemma 4. *Let $G(V,E)$ be a connected graph of order p and size $p+l-1$. With $k\geqq 2$, set $\delta=\min\left(\lfloor k/2\rfloor/(4l-3),\, 1/4\right)$. Then, there exists $X\in[V]^k$ such that $\mu(G-X)\leqq\lceil(1-\delta)p\rceil$.*

Proof. Delete l edges from G in such a way that the resulting graph H is still connected, i.e. so that H is a tree. The deleted edges determine a subtree T in the following way. First, we find those vertices which were incident in G with one of the deleted edges and so define a set A. Then, we define T to be the union of all paths in H which join pairs of vertices from A. Let A_1 denote the vertices of A which have degree 1 in T and set $A_2=A-A_1$. According to Lemma 2, T is the union of at most $2|A_1|-3$ edge-disjoint suspended paths. The vertices of A_2 now subdivide these suspended paths into what we shall call *elementary paths*. The elementary paths may be described in the following way. The end-vertices of the elementary paths are precisely those vertices x such that either (i) $x\in A$ or (ii) $\deg(x)>2$ in T. Suppose that there are r elementary paths $P_1, P_2, \ldots, P_r$. Since $|A|\leqq 2l$, it follows that $r\leqq 2|A_1|+|A_2|-3\leqq 4l-3$.

Note the following useful property of the construction described thus far. Suppose that x is a vertex of G and that it is not a vertex of T. Then, there is a unique path in G from x to T. If there were two such paths, then one of them would have to use one of the edges which were deleted in going from G to H. This would put x on a path in H joining two vertices from A and so force x to belong to T. In light of this property, we note that the collection of elementary paths $P_1, P_2, \ldots, P_r$ may be used to define a partition $V=(V_1, V_2, \ldots, V_r)$ of the vertices of G according to the following scheme. If x is an end-vertex of one or more elementary paths, it is identified with an arbitrarily chosen one of those paths. If x is an interior vertex of an elementary path, it is identified with that path. Finally, if x is a vertex of G which is not a vertex of T, let w be the other end-vertex of the unique path from x to T and identify x with the same elementary path as is w.

Now we are ready to describe and put to use the crucial properties of the elementary paths. Let u_i and v_i be the end-vertices of the i^{th} elementary path, P_i. Our construction insures that if x is any vertex of V_i other than u_i or v_i, every path from x to a vertex in $V-V_i$ contains either u_i or v_i. In other words, by deleting u_i and v_i from G, we completely disconnect the vertices of V_i from the remaining vertices of G. Without loss of generality, we may suppose that $|V_1|\geqq\ldots\geqq|V_r|$. Set $m=\min(\lceil r/4\rceil, \lfloor k/2\rfloor)$ and consider the graph $G-X$, where $X=\{u_i, v_i, i=1, \ldots, m\}$. Since $|V_1|+\ldots+|V_m|\geqq mp/r\geqq\delta p$, it follows that $\mu(G-X)$ satisfies the stated bound unless $|V_1|>\lceil(1-\delta)p\rceil$. In case $|V_1|>\lceil(1-\delta)p\rceil$, set $B=V_1\cup\{u_1, v_1\}$ and consider the tree T' spanned by the vertices of B. By Lemma 3, there exists a vertex x of this tree such that the components of $T'-x$ can be partitioned into two parts, each of cardinality at least $\lceil(|V_1|-1)/3\rceil$. Now we may delete x and either u_1 or v_1, whichever is appropriate, and so disconnect from G a set of at least $\lfloor p/4\rfloor$ vertices. In this case, for $X=\{x, u_1\}$ or $\{x, v_1\}$ we obtain $\mu(G-X)\leqq\lceil 3p/4\rceil$. $\square$

Now we are prepared to prove our theorem concerning $f(n, k)$ with $k\geqq 2$.

THEOREM 2. *Let $k>1$ be fixed and set $A=\sqrt{\lfloor k/2\rfloor/16}$ and $B=\sqrt{3k(k-1)/(k+1)}$. Then, for all sufficiently large n,*

$$n/2+A\sqrt{n}<f(n, k)<n/2+B\sqrt{n}.$$

PROOF. Let $G(V, E)$ be a graph of order n and size $q=n/2+\Delta$, where $\Delta=A\sqrt{n}$. We wish to prove that there exists $X\in[V]^k$ such that $G-X$ satisfies the conditions of Lemma 1. This will establish the lower bound for $f(n, k)$. Since $\Delta=o(n)$, it follows that the number of connected components of G is at least $n-q=n/2-o(n)$. Consequently, $\eta(G)\leqq 2$. On the other hand, if $\eta(G)=2$, then $\mu(G)=o(n)$ and so by deleting just one vertex from G we obtain a graph which satisfies the conditions of Lemma 1. Hence, we now assume that $\eta(G)=1$. Since this is the case, we may assume that $\mu(G)>\lceil(n-k)/2\rceil$, in fact $\mu(G)>\lceil(n+k)/2\rceil$ for, otherwise, we may simply delete any k vertices from the largest component. Suppose that the largest component is of order p and size $p+l-1$. Hence, we have the bounds $p \leqq q = n/2 + \Delta$ and $l \leqq q-p+1 \leqq \Delta$. With a view toward applying Lemma 4, note that if $\delta = \lfloor k/2 \rfloor/(4l-3)$ then $(1-\delta)p < (1-\lfloor k/2 \rfloor/4\Delta)(n/2+\Delta) < n/2 + (\Delta^2 - \lfloor k/2 \rfloor n/8)/\Delta$. Therefore, in this case and with our choice of Δ, we have $\lceil (1-\delta)p \rceil \leqq \lceil (n-k)/2 \rceil$. Certainly if $\delta = 1/4$, $\lceil (1-\delta)p \rceil \leqq \lceil (n-k)/2 \rceil$ and so the desired result follows from Lemma 4.

The upper bound is established by the following simple construction. With m chosen to be an even integer, let H be a graph of order m which is regular of degree $k+1$ and $(k+1)$-connected. An example of such a graph has vertices $0, 1, \ldots, m-1$ with two vertices i and j joined if $i - \lfloor (k+1)/2 \rfloor \leqq j \leqq i + \lfloor (k+1)/2 \rfloor \pmod m$ and, if $k+1$ is odd, i is joined to $i+m/2$ for $1 \leqq i \leqq m/2$. The fact that such a graph is, indeed, $(k+1)$-connected was proved by Harary in [3] and the proof is also given in [1, pp. 48—49]. Set $r = m(k+1)/2$ and let the edges of H be $e_1, e_2, \ldots, e_r$. For $i = 1, 2, \ldots, r$, insert a vertex y_i subdividing e_i and make y_i adjacent to $l_i - 1$ new vertices. Finally, add isolated vertices so that the resulting graph $G(V, E)$ is of order n. Thus, G is of size $q(G) = r + (l_1 + \ldots + l_r)$. Without loss of generality, we may assume that $l_1 \geqq l_2 \geqq \ldots \geqq l_r$. Now make the following choices for the parameters of G. Set $m = 2\lceil \sqrt{5kn/8(k^2-1)} \rceil$ and $l_1 = \ldots = l_k = \lceil \sqrt{5(k-1)n/8k(k+1)} \rceil \doteq l$. Then choose $l_{k+1}, \ldots, l_r$ so that $m + (l_{k+1} + \ldots + l_r) = \lceil (n-k)/2 \rceil + 1$. Let $Y = \{y_1, \ldots, y_k\}$. It is apparent that for every $X \in [V]^k$, we have $\mu(G-X) \geqq \mu(G-Y) = \lceil (n-k)/2 \rceil + 1$. Also, we have $q(G) = \lceil (n-k)/2 \rceil + 1 + kl + (k-1)m/2 < n/2 + B\sqrt{n}$ for every $\varepsilon > 0$. Since $\mu(G-X) > \{(n-k)/2\}$ for every $X \in [V]^k$, it follows that $\bar{G} \notin B_{n-k}$. This establishes the upper bound. □

4. An upper bound for $f(n, \lfloor \varepsilon n \rfloor)$

At present, very little is known about $f(n, k)$ when $k \to \infty$ with n. However, the results of the preceding section suggest that $f(n, \lfloor \varepsilon n \rfloor) < \lceil (1/2 + \delta)n \rceil$ where $\delta \downarrow 0$ with ε and this much can be proved without difficulty.

THEOREM 3. *Let* $0 < \varepsilon < e^{-4}$ *be fixed and set* $\delta = \sqrt{6\varepsilon \log(1/\varepsilon)}$. *For all sufficiently large values of n,*

$$f(n, \lfloor \varepsilon n \rfloor) < \lceil (1/2 + \delta)n \rceil.$$

PROOF. Set $p = \lceil 1/2(1+\delta)n \rceil$, $q = \lceil (1/2+\delta)n \rceil$, $k = \lfloor \varepsilon n \rfloor$, $r = q - p$, $a = \lceil (n-k)/2 \rceil$, $b = \lfloor (n-k)/2 \rfloor$, and $c = a + p - n$. Using the probabilistic method, we shall prove the existence of a graph G of order n and size $\leqq q$ such that $\bar{G} \not\supseteq K_{a,b}$. Let $V = \{1, 2, \ldots, n\}$, $X = \{1, 2, \ldots, p\}$ and $Y = [V]^2$. The probability space used to prove the existence of G may be described as follows. Let $\Omega = \Omega_1 \times \Omega_2$ where $\Omega_1 = X^p$ and $\Omega_2 = Y^r$. Each point in Ω is given probability $1/|\Omega|$. A typical point in Ω is $\omega = (\omega_1, \omega_2)$ where $\omega_1 = (x_1, \ldots, x_p)$ and $\omega_2 = (y_1, \ldots, y_r)$. Corresponding to ω there is a graph defined as follows: $\{i, j\}$ is an edge in the graph for each occurrence of $x_i = j$, $x_j = i$ or $y_k = \{i, j\}$, $k = 1, \ldots, r$. It is understood that any loops and/or extra edges which may be generated by the random method are simply not included in the graph so formed. If $\bar{G} \supseteq K_{a,b}$ then for some m, $c \leqq m \leqq a$, there are disjoint subsets of X, namely A and B with $|A|=m$ and $|B|=p-k-m$, such that $E(A,B)=\varphi$. Now for fixed A and B, consider the event $E(A,B)=\varphi$. The number of points of Ω_1 in this event is $(m+k)^m(p-m)^{p-k-m}p^k$ and the number of points of Ω_2 in this event is $\left(\binom{n}{2}-m(p-k-m)\right)^r$. Hence, we obtain the bound

$$\operatorname{Prob}(\bar{G} \supseteq K_{a,b}) \leqq \sum_{m=c}^{a} \binom{p}{m}\binom{p-m}{k} \frac{(m+k)^m(p-m)^{p-k-m}p^k}{p^p}\left(1-\frac{m(p-k-m)}{\binom{n}{2}}\right)^r.$$

Using Stirling's formula and some elementary bounds, we find that each term in the sum is bounded by

$$(1+2k/n)^n(p/k)^k\left(1-a(p-k-a)\Big/\binom{n}{2}\right)^r.$$

Substituting the values of a, k, p and r, we find that $\operatorname{Prob}(\bar{G}\supseteq K_{a,b})\to 0$ as $n\to\infty$ provided that $(1+2\varepsilon)((1+\delta)/2\varepsilon)^\varepsilon(1-(1-\varepsilon)(\delta-\varepsilon)/2)^{\delta/2}<1$. A simple calculation shows this to be the case when $0<\varepsilon<e^{-4}$ and $\delta=\sqrt{6\varepsilon\log(1/\varepsilon)}$. □

5. Additional problems and results

The bound for $f(n, \lfloor \varepsilon n \rfloor)$ provides a satisfying tie with the results for $f(n,k)$ where k is fixed; still, it leaves us with more questions than answers. Among other things, the result shows that if $F(\varepsilon) \overset{\text{def}}{=} \lim_{n\to\infty} f(n,\lfloor \varepsilon n\rfloor)/n$ exists, then $\lim_{\varepsilon\downarrow 0} F(\varepsilon)=1/2$. But, does $\lim_{n\to\infty} f(n,\lfloor \varepsilon n\rfloor)/n$ exist?

PROBLEM 1. *For* $0<x<1$, *does* $\lim_{n\to\infty} f(n,\lfloor xn\rfloor)/n$ *exist?*

By a variety of simple arguments, it is possible to prove bounds of the form $F_1(x)<f(n,\lfloor xn\rfloor)/n<F_2(x)$ which hold when $0<x<1$ is fixed and n is sufficiently large. Hence, it is at least plausible that $\lim_{n\to\infty} f(n,\lfloor xn\rfloor)/n$ exists. As an example of an upper bound for $f(n,\lfloor xn\rfloor)/n$, we give the following argument. Starting with the complete graph K_n, we wish to remove $q=\lfloor yn\rfloor$ edges $e_1, e_2, \dots, e_q$ in such a way that all $K_{m,m}$ subgraphs with $m=\lceil(1-x)n/2\rceil$ are destroyed. Having found such a number y, we are assured that $f(n,\lfloor xn\rfloor)/n<y$. Let X_i denote the set of $K_{m,m}$ subgraphs which remain after e_i has been removed. Clearly, $|X_0|=\binom{n}{m}\binom{n-m}{m}$. At the stage of removing the edge e_{i+1} there are $|X_i|$ remaining $K_{m,m}$ subgraphs and $\binom{n}{2}-i$ remaining edges. Counting multiplicity, the remaining $K_{m,m}$ subgraphs contain $|X_i|m^2$ edges. It follows that there is an edge whose removal destroys at least $|X_i|m^2\Big/\binom{n}{2}$ of the subgraphs in X_i. By choosing such an edge for e_{i+1}, we obtain $|X_{i+1}|\leqq|X_i|\left(1-m^2\Big/\binom{n}{2}\right)$. Following such a procedure for $i=1, 2, \dots, q$, we obtain

$|X_q| < \binom{n}{m}\binom{n-m}{m}\left(1 - m^2\Big/\binom{n}{2}\right)^q$. An easy calculation using Stirling's formula allows us to conclude that if y is chosen so that $(1-(1-x)^2/2)^y < ((1-x)/2)^{1-x}x^x$ and n is sufficiently large, then $|X_q|=0$. As $x \to 0$, the upper bound for $f(n, \lfloor xn \rfloor)/n$ that is obtained by this argument is quite inferior to the bound given in Theorem 3. The advantage of this argument is that it is applicable for all x satisfying $0<x<1$.

The second problem is not concerned with the calculation of $f(n, k)$, but is certainly related to the investigation described in this paper.

PROBLEM 2. *For all* $n \geqq 2$, *determine the largest integer* $m=f(n)$ *such that for every tree* T *of order* n, $\overline{T} \in B_m$.

We have obtained upper and lower bounds for $f(n)$ and these results may be published elsewhere.

Finally, we note the following generalization of the basic problem considered in this paper.

PROBLEM 3. *For* $r \geqq 2$ *and* $n \geqq k+r$, *let* $f_r(n, k)$ *denote the largest integer* q *such that for every graph* G *of order* n *and size* q, $\overline{G} \supseteq K(a_1, \ldots, a_r)$ *for every partition* $(a_1, \ldots, a_r)$ *of* $n-k$ *into* r *parts. Determine* $f_r(n, k)$.

The proofs given in this paper extend naturally and easily to the study of $f_r(n, k)$. For $r \geqq 3$, the induction argument used in the proof of Lemma 1 yields the following result.

LEMMA. *Let* $r \geqq 3$. *If* G *is a graph of order* n *such that* (i) $\mu(G) \leqq \lceil n/r \rceil$ *and* (ii) $q(G) \leqq \lfloor 2n/(r+1) \rfloor - 1$, *then* $\overline{G} \supseteq K(a_1, \ldots, a_r)$ *for every partition* $(a_1, \ldots, a_r)$ *of* n.

Now we can state the following generalizations of Theorems 1, 2 and 3. The reader will find that the proofs given earlier in the paper have been so structured that they readily yield the results now stated.

THEOREM. *For all* $r \geqq 2$ *and* $n \geqq r$, $f_r(n, 0) = \lceil n/r \rceil - 1$. *Except for certain exceptional cases,* $f_r(n, 1) = \lceil (n-1)/r \rceil + 1$ *holds for all* $r \geqq 2$ *and* $n \geqq r+1$. *The exceptional cases are* $f_3(4, 1)=1$, $f_3(6, 1)=2$, $f_3(8, 1)=3$ *and, for* $r \geqq 4$, $f_r(r+1, 1)=1$ *and* $f_r(r+2, 1)=f_r(r+3, 1)=2$.

THEOREM. *Let* $r, k > 1$ *be fixed and set* $A=\sqrt{\lfloor k/2 \rfloor/8r}$ *and* $B=\sqrt{6k(k-1)/((k+1)r)}$. *Then, for all sufficiently large* n,

$$n/r + A\sqrt{n} < f_r(n, k) < n/r + B\sqrt{n}.$$

THEOREM. *Let* $0<\varepsilon<e^{-4}$ *be fixed and set* $\delta=\sqrt{r(r+1)\varepsilon \log(1/\varepsilon)}$. *For all sufficiently large values of* n,

$$f_r(n, \lfloor \varepsilon n \rfloor) < \lceil (1/r+\delta)n \rceil.$$

Exactly as in the special case of $r=2$, the methods used in this paper provide an effective means of studying $f_r(n, k)$ only when $k \ll n$. Thus, for example, the generalization of Problem 1 to consider $f_r(n, \lfloor xn \rfloor)$, $0<x<1$, is an important problem about which little is known at present. With n and r fixed, $f_r(n, k)$ is defined for $0 \leqq k \leqq n-r$, and it is worth pointing out that in addition to the $k=0$ and $k=1$ cases, $f_r(n, k)$ is known exactly for $k=n-r$. We know that $f_r(n, n-r)= =(n-r+t+1)s/2-1$, where $n=(r-1)s+t$, $0 \leqq t < r-1$. This is Turán's theorem.

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(Received March 9, 1983)

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DURCH NORMEN DEFINIERTE IDEALKLASSENGRUPPEN

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1. Einleitung

Für einen algebraischen Zahlkörper K sei $\mathcal{J}_K$ die Gruppe der gebrochenen Ideale des Ganzheitsringes von K, und für jede Untergruppe $\mathcal{H}\subset\mathcal{J}_K$ ist $\mathcal{J}_K/\mathcal{H}$ eine „Idealklassengruppe" von K. Ist $\mathcal{H}_K$ die Gruppe der gebrochenen Hauptideale von K, so ist $\mathcal{C}_K=\mathcal{J}_K/\mathcal{H}_K$ die „gewöhnliche" Idealklassengruppe von K, eine der wichtigsten Invarianten der algebraischen Zahlentheorie. Ist $\mathcal{H}_K^+$ die Gruppe der von totalpositiven Körperelementen erzeugten Hauptideale, so ist $\mathcal{J}_K/\mathcal{H}_K^+$ die „engere" Idealklassengruppe von K, die bereits in der Gauss-schen Theorie der Geschlechter eine zentrale Rolle spielt. Für einen quadratischen Zahlkörper K ist $\mathcal{H}_K^+=\{(\alpha)|\alpha\gg 0\}=\{(\beta)|N\beta\in\mathbf{Q}^+\}$, also $\mathcal{H}_K^+$ mit Hilfe der Norm N von $K/\mathbf{Q}$ beschreibbar. In der vorliegenden Arbeit werden nun ganz allgemein solche durch Normen definierte Idealklassengruppen untersucht. Ist L/K eine endliche Erweiterung algebraischer Zahlkörper und F eine (multiplikative) Untergruppe von $K^\times$, so sei $\mathcal{H}(F)$ die Gruppe aller Hauptideale (α) von L mit $N_{L/K}\alpha\in F$. Die Faktorgruppe nach $\mathcal{H}(F)$ ergibt dann eine Idealklassengruppe $\mathcal{C}(F)$ von L. Im Gegensatz zu den von Kuroda [3] definierten Klassengruppen enthalten die Gruppen $\mathcal{C}(F)$ im allgemeinen keine Information über das Zerlegungsverhalten von Primidealen in Oberkörpern, da die Primidealdichten in den einzelnen Klassen in Abhängigkeit von F beliebig variieren können (vgl. Satz 3). Auch die Struktur und die Ordnung von $\mathcal{C}(F)$ hängen im allgemeinen von F ab. Andererseits ermöglicht es diese Variabilität, durch geeignete Wahl von F die Gruppe $\mathcal{C}(F)$ mit vorgegebenen Eigenschaften zu versehen.

Von besonderem Interesse ist der Fall, wenn F direkter Kofaktor der Einheitengruppe ist, also $K^\times=E_K\times F$. Dann liegen zwei Hauptideale mit gleicher relativer Idealnorm bezüglich K genau dann in derselben Klasse von $\mathcal{C}(F)$, wenn sie Erzeugende mit gleicher Relativnorm besitzen. Bumby [1] untersuchte, wann eine endliche, normale Erweiterung L/K von algebraischen Zahlkörpern die folgende Eigenschaft besitzt, die er (N) nannte: je zwei ganze Zahlen $\alpha,\beta\in L$ mit $N_{L/K}\alpha=$ $=N_{L/K}\beta$ sind entweder beide irreduzibel oder beide nicht. Ein vollständige Charakterisierung aller Erweiterungen L/K mit der Eigenschaft (N) ist unbekannt. Ist $G=\mathrm{Gal}\,(L/K)$, $K^\times=E_K\times F$ und enthält jede Klasse von $\mathcal{C}(F)$ Primideale, so zeigt sich, daß die Eigenschaft (N) nur von der Struktur von $\mathcal{C}_L$ und $\mathcal{C}(F)$ als G-Moduln abhängt.

In dieser Arbeit wird auf den Zusammenhang von (N) mit den Gruppen $\mathcal{C}(F)$ nicht näher eingegangen, sondern es werden ausschließlich Resultate über die Klas-

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1980 *Mathematics Subject Classification.* Primary 12A75; Secondary 12A45.
Key words and phrases. Ideal class group, prime ideal density.

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sengruppen $\mathscr{C}(F)$ hergeleitet. Für $L/\mathbf{Q}$ und $\mathbf{Q}^\times=\{1,-1\}\times F$ geben die Sätze 4 und 5 den genauen Zusammenhang zwischen den Klassengruppen $\mathscr{C}(F)$ und $\mathscr{C}_L$ an. Im allgemeinen Fall bleibt die Frage offen, welche F als Kofaktoren von E_K gewählt werden sollen, damit $\mathscr{C}(F)$ minimale Ordnung hat bzw. wann $\mathscr{C}(F)$ mit $\mathscr{C}_L$ übereinstimmt.

2. Die (F)-Idealklassengruppen

Für eine endliche Erweiterung algebraischer Zahlkörper L/K bezeichne N: $L^\times \to K^\times$ die Relativnorm, $\mathscr{H}_L$ die Gruppe der Hauptideale von L und $\mathscr{C}_L= =\mathscr{J}_L/\mathscr{H}_L$.

Definition. Es sei F eine Untergruppe von $K^\times$. Ein Hauptideal $(\alpha)\in\mathscr{J}_L$ heißt (F)-*Hauptideal,* wenn $N\alpha\in F$ ist. Bezeichnet $\mathscr{H}(F)$ die Gruppe aller (F)-Hauptideale von L, so heißt $\mathscr{C}(F)=\mathscr{J}_L/\mathscr{H}(F)$ die (F)-*Idealklassengruppe* von L.

Ist $\mathfrak{A}\in\mathscr{J}_L$, so bezeichnen wir mit $[\mathfrak{A}]$ bzw. $[\mathfrak{A}]_F$ die gewöhnliche Idealklasse bzw. die (F)-Idealklasse, welche $\mathfrak{A}$ enthält. Für $\alpha\in L^\times$ gilt $(\alpha)\in\mathscr{H}(F)$ genau dann, wenn $N\alpha\in F\cdot NE_L$ ist. Setzen wir $\Phi_F:=(F\cdot NL^\times)/(F\cdot NE_L)$, so gilt $\mathscr{H}_L/\mathscr{H}(F)\cong \cong\Phi_F\leqq K^\times/(F\cdot NE_L)$.

Lemma 1. a) *Ist* $[K^\times:(F\cdot E_K)]$ *endlich, so ist* $\mathscr{C}(F)$ *endlich.*
b) *Ist* $K^\times/(F\cdot E_K)$ *keine Torsionsgruppe, so ist* $\mathscr{C}(F)$ *unendlich.*

Beweis. a) Da $[(F\cdot E_K):(F\cdot NE_L)]\leqq[E_K:NE_L]<\infty$ ist, erhält man

$$[K^\times:(F\cdot NE_L)] = [K^\times:(F\cdot E_K)]\cdot[(F\cdot E_K):(F\cdot NE_L)]<\infty.$$

Nun ist aber $K^\times/(F\cdot NE_L)\cong\Phi_F\cong\mathscr{H}_L/\mathscr{H}(F)$ und

$$(1)\qquad 0\to\mathscr{H}_L/\mathscr{H}(F)\to\mathscr{C}(F)\to\mathscr{C}_L\to 0$$

eine exakte Sequenz, woraus sich die Endlichkeit von $\mathscr{C}(F)$ ergibt.

b) Nach Voraussetzung existiert ein $\lambda\in K^\times$ mit $\lambda^n\notin F\cdot E_K$ für alle $n\in\mathbf{Z}\setminus\{0\}$. Die Potenzen von λ erzeugen Hauptideale in L, deren (F)-Idealklassen $[(\lambda^n)]_F$ paarweise verschieden sind, also ist $\mathscr{C}(F)$ unendlich.

Satz 1. *Ist F direkter Kofaktor von E_K (d. h. $K^\times=E_K\times F$) und h_L die Klassenzahl von L, so gilt*

$$(2)\qquad [(E_K\cap NL^\times):NE_L]\cdot h_L\leqq\#\mathscr{C}(F)\leqq[E_K:NE_L]\cdot h_L.$$

Beweis. Wegen der exakten Sequenz (1) genügt es, $[(E_K\cap NL^\times):NE_L]\leqq \leqq\#(\mathscr{H}_L/\mathscr{H}(F))\leqq[E_K:NE_L]$ zu zeigen. Nun ist aber

$$\mathscr{H}_L/\mathscr{H}(F)\cong\Phi_F=(F\cdot NL^\times)/(F\cdot NE_L)\leqq K^\times/(F\cdot NE_L)=$$

$$=(E_K\times F)/(NE_L\times F)\cong E_K/NE_L,$$

andererseits gilt

$$\Phi_F\geqq(F\cdot(E_K\cap NL^\times))/(F\cdot NE_L)=(F\times(E_K\cap NL^\times))/(F\times NE_L)\cong(E_K\cap NL^\times)/NE_L.$$

Wir setzen nun voraus, daß L/K normal mit Galoisgruppe G ist. Da G die Gruppe $\mathscr{H}(F)$ invariant läßt, operiert G auf $\mathscr{C}(F)$. Wie üblich, schreiben wir Klassengruppen additiv und daher die Operation von G auf $\mathscr{C}_L$ bzw. $\mathscr{C}(F)$ in Präfixnotation, auf L bzw. $\mathscr{J}_L$ jedoch in Exponentennotation. Der folgende Satz zeigt, daß der Stabilisator einer (F)-Idealklasse von F unabhängig ist.

SATZ 2. *Es sei L/K eine endliche, normale Erweiterung algebraischer Zahlkörper mit Galoisgruppe G. Für $a \in \mathscr{C}_L$ sei $G_a \leqq G$ der Stabilisator von a.*

Dann existiert ein Homomorphismus $\gamma_a\colon G_a \to E_K/NE_L$, sodaß für jedes $F \leqq K^\times$ mit $E_K \cap F \subseteq NE_L$ gilt: $G'_a := \ker(\gamma_a)$ ist der Stabilisator für jede in a enthaltene (F)-Idealklasse.

BEWEIS. Wir wählen ein Ideal $\mathfrak{A} \in a$. Für $\sigma \in G_a$ sei $\alpha_\sigma \in L$ mit $\mathfrak{A}^{\sigma-1} = (\alpha_\sigma)$. Dann definieren wir $\gamma_a\colon G_a \to E_K/NE_L$ durch $\gamma_a(\sigma) := N\alpha_\sigma \cdot NE_L$. Zunächst zeigen wir, daß diese Definition von der Wahl von $\mathfrak{A}$ unabhängig ist. Ist $\mathfrak{B} \in a$, so gibt es zu jedem $\sigma \in G_a$ ein $\beta_\sigma \in L$ mit $\mathfrak{B}^{\sigma-1} = (\beta_\sigma)$. Weiters gibt es ein $\delta \in L$ mit $\mathfrak{B} = \mathfrak{A} \cdot (\delta)$, womit wir $(\beta_\sigma) = (\alpha_\sigma \cdot \delta^{\sigma-1})$ und wegen $N\delta^{\sigma-1} = 1$ $N\beta_\sigma \in N\alpha_\sigma \cdot NE_L$ erhalten.

Nun beweisen wir, daß γ_a ein Homomorphismus ist. Für $\sigma, \tau \in G_a$ seien $\alpha_\sigma, \alpha_\tau \in L$ mit $\mathfrak{A}^{\sigma-1} = (\alpha_\sigma)$ und $\mathfrak{A}^{\tau-1} = (\alpha_\tau)$. Wegen $\mathfrak{A}^{\sigma\tau-1} = (\mathfrak{A}^{\sigma-1})^\tau \cdot \mathfrak{A}^{\tau-1} = (\alpha_\sigma^\tau \cdot \alpha_\tau)$ ergibt sich $\gamma_a(\sigma\tau) = N(\alpha_\sigma^\tau \cdot \alpha_\tau) \cdot NE_L = N\alpha_\sigma \cdot N\alpha_\tau \cdot NE_L = \gamma_a(\sigma) \cdot \gamma_a(\tau)$.

Schließlich sei $F \leqq K^\times$ mit $E_K \cap F \subseteq NE_L$, $a' \in \mathscr{C}(F)$ mit $a' \subseteq a$ und $\mathfrak{A} \in a'$. Für $\sigma \in G_a$ sei wieder $\mathfrak{A}^{\sigma-1} = (\alpha_\sigma)$. Dann gilt $(\sigma a' = a') \Leftrightarrow ([\mathfrak{A}^\sigma]_F = [\mathfrak{A}]_F) \Leftrightarrow (N\alpha_\sigma \in F \cdot NE_L)$. Nun ist aber $N\alpha_\sigma \in E_K$, und die Voraussetzung über F ergibt $(F \cdot NE_L) \cap E_K = NE_L$, also gilt $(N\alpha_\sigma \in F \cdot NE_L) \Leftrightarrow (N\alpha_\sigma \in NE_L) \Leftrightarrow (\sigma \in G'_a)$.

3. Primidealdichten der (F)-Idealklassen

In diesem Abschnitt sei L/K eine endliche, normale Erweiterung algebraischer Zahlkörper mit Galoisgruppe G. Weiters sei F ein direkter Kofaktor der Einheitengruppe von K. Dann ist F eine freie abelsche Gruppe mit abzählbarer Basis (siehe z. B. Narkiewicz [4], S. 123). Wir werden zeigen, daß durch geeignete Wahl von F „beliebig" vorgegebene Primidealdichten der einzelnen (F)-Idealklassen erreicht werden können. Da (F)-Idealklassen, die unter G konjugiert sind, gleiche Primidealdichte haben, muß dies bei der „beliebigen" Vorgabe der Dichten ebenso berücksichtigt werden wie die Tatsache, daß die Dichte der Primideale in einer gewöhnlichen Idealklasse $1/h_L$ ist.

Nach Skolem [5] läßt sich ein direkter Kofaktor F_0 zu E_K folgendermaßen konstruieren. Die Menge aller Primideale[1] von K sei $\{\mathfrak{p}_i | i \in \mathbf{N}\}$, wobei die Reihenfolge so gewählt wird, daß für ein $n_0 \in \mathbf{N} \cup \{0\}$ die Menge $\{[\mathfrak{p}_i] | 1 \leqq i \leqq n_0\}$ eine Basis für $\mathscr{C}_K$ ist. Für $n \in \mathbf{N}$ und $1 \leqq i \leqq n_0$ existieren eindeutig bestimmte Zahlen $h_{i,n} \in \mathbf{Z}$ mit $0 \leqq h_{i,n} < \operatorname{ord}[\mathfrak{p}_i]$, sodaß $\mathfrak{p}_n \prod_{i=1}^{n_0} \mathfrak{p}_i^{h_{i,n}} = (\pi_n)$ ein Hauptideal ist. Dann ist $F_0 = \prod_{n \in \mathbf{N}} \langle \pi_n \rangle$ eine freie Gruppe und $E_K \times F_0 = K^\times$. Ist v_n die zu $\mathfrak{p}_n$ gehörige

[1] Primideale seien stets ungleich (0).

und auf 1 normierte Exponentenbewertung, so gilt für jedes $\lambda\in K^\times$

(3) $$\lambda = \varepsilon \prod_{n=1}^{n_0} \pi_n^{c_n} \prod_{n=n_0+1}^{\infty} \pi_n^{v_n(\lambda)},$$

wobei $\varepsilon\in E_K$ und $c_n\in \mathbf{Z}$ durch λ eindeutig bestimmt sind. Das folgende Lemma zeigt, daß sich jeder direkte Kofaktor F zu E_K in der Form $F=\prod_{n\in\mathbf{N}} \langle\varepsilon_n\pi_n\rangle$ mit eindeutig bestimmten $\varepsilon_n\in E_K$ schreiben läßt.

Lemma 2. *Es seien A eine multiplikative, abelsche Gruppe, B eine Untergruppe von A, F_0 und F freie Untergruppen von A mit $A=B\times F_0=B\times F$. Ist $\{f_i|i\in I\}$ eine Basis von F_0, so existieren eindeutig bestimmte $b_i\in B$, sodaß $\{b_if_i|i\in I\}$ eine Basis von F ist.*

Beweis. Ist $\{g_j|\, j\in J\}$ eine Basis von F, so existieren für $i\in I$, $j\in J$ eindeutig bestimmte $b_i\in B$ und $\varepsilon_{i,j}\in\mathbf{Z}$ mit $f_i=b_i^{-1}\prod_{j\in J} g_j^{\varepsilon_{i,j}}$. Für $F'=\prod_{i\in I}\langle b_if_i\rangle$ gilt $F'\subseteq F$. Ist $a\in F$, so existieren $b\in B$ und $\gamma_i\in\mathbf{Z}$ mit

$$a = b\prod_{i\in I} f_i^{\gamma_i} = b\prod_{i\in I}\Big(b_i^{-\gamma_i}\prod_{j\in J} g_j^{\varepsilon_{i,j}\gamma_i}\Big).$$

Daraus ergibt sich $b=\prod_{i\in I} b_i^{\gamma_i}$ und $a=\prod_{i\in I}(b_if_i)^{\gamma_i}$, womit wir $F\subseteq F'$ und somit $F=F'$ bewiesen haben.

Es sei nun L/K normal mit Galoisgruppe G und $K^\times=E_K\times F$. Jede Idealklasse $a\in\mathscr{C}_L$ enthält wegen (2) höchstens $[E_K:NE_L]$ (F)-Idealklassen. G_a, γ_a und G'_a seien wie in Satz 2 definiert und $\Gamma_a:=\operatorname{im}(\gamma_a)\leqq E_K/NE_L$. Auf der Menge der in a enthaltenen (F)-Idealklassen operiert G_a, wodurch diese in höchstens $m(a)=[E_K:NE_L]/\#\Gamma_a$ Bahnen der Mächtigkeit $i(a)=[G_a:G'_a]=\#\Gamma_a$ zerfällt. Da G'_a und Γ_a nur von a, nicht aber von F abhängen, gilt dies auch für $i(a)$ und $m(a)$. Mit $\mathscr{N}\colon \mathscr{I}_L\to\mathscr{I}_K$ bzw. $\mathscr{N}_{L/\mathbf{Q}}\colon \mathscr{I}_L\to\mathbf{Q}$ bezeichnen wir die relative bzw. absolute Idealnorm. Ist $M\subseteq\mathscr{I}_L$, so ist die Primidealdichte von M durch

$$\delta(M) = \lim_{n\to\infty}\frac{\#\{\mathfrak{P}\in M|\mathfrak{P}\text{ Primideal und }\mathscr{N}_{L/\mathbf{Q}}(\mathfrak{P})\leqq n\}}{\#\{\mathfrak{P}\in\mathscr{I}_L|\mathfrak{P}\text{ Primideal und }\mathscr{N}_{L/\mathbf{Q}}(\mathfrak{P})\leqq n\}}$$

definiert, falls dieser Grenzwert existiert. Bekanntlich hängt $\delta(M)$ nur von den Primidealen mit Restklassengrad 1, also auch nur von den Primidealen mit Relativgrad $f_{L/K}=1$ ab. Jede zu M unter G konjugierte Menge hat dieselbe Primidealdichte, also können nur solche (F)-Idealklassen verschiedene Dichten haben, die unter G nicht konjugiert sind.

Satz 3. *Es sei L/K eine normale Erweiterung algebraischer Zahlkörper mit Galoisgruppe G. Für $1\leqq j\leqq l$ seien $a_j\in\mathscr{C}_L$ Repräsentanten für die verschiedenen Bahnen, in die $\mathscr{C}_L$ unter G zerfällt. Weiters seien $m_j\in\mathbf{N}$ mit $m_j\leqq m(a_j)$ und $\varepsilon_{i,j}\in\mathbf{R}$ mit $0\leqq\varepsilon_{i,j}\leqq 1$ und $\sum_{i=1}^{m_j}\varepsilon_{i,j}=1$. Dann existieren eine Gruppe F mit $K^\times=E_K\times F$ und für $1\leqq j\leqq l$, $1\leqq i\leqq m_j$ paarweise verschiedene (F)-Idealklassen $b_{i,j}$ mit $b_{i,j}\subseteq a_j$ und Primidealdichten $\delta(b_{i,j})=\varepsilon_{i,j}/(h_L\cdot i(a_j))$.*

Beweis. Wir gehen von einer Zerlegung $K^\times = E_K \times F_0$ aus, wie sie zu Beginn des Kapitels beschrieben wurde, d. h. $F_0 = \coprod_{n \in \mathbf{N}} \langle \pi_n \rangle$, $\{[\mathfrak{p}_n] \mid 1 \leqq n \leqq n_0\}$ ist eine Basis von $\mathscr{C}_K$, und für jedes $\lambda \in K^\times$ gilt (3). Für $1 \leqq j \leqq l$ sei $\mathfrak{A}_j \in a_j$ fest gewählt. Die endliche Menge $S \subseteq \mathscr{J}_L$ enthalte genau die Primideale von L, welche über den Idealen $\mathfrak{p}_n$ mit $1 \leqq n \leqq n_0$ oder mit $\mathfrak{p}_n | \mathscr{N}\mathfrak{A}_j$ für $j \in \{1, \ldots, l\}$ liegen. Mit geeigneten $\varepsilon_n \in E_K$ werden wir $F = \coprod_{n \in \mathbf{N}} \langle \varepsilon_n \pi_n \rangle$ bilden und damit alle Behauptungen des Satzes verifizieren.

Es sei nun $j \in \{1, \ldots, l\}$. Wir wählen $\eta_1, \ldots, \eta_{m_j} \in E_K$ so, daß $\eta_1 NE_L, \ldots, \eta_{m_j} NE_L$ m_j verschiedene Nebenklassen von $(E_K/NE_L)/\Gamma_{a_j}$ repräsentieren.

μ: $\mathbf{N} \to \{1, 2, \ldots, m_j\}$ sei eine Funktion mit $\mu(k) = k$ für $1 \leqq k \leqq m_j$ und $\lim_{n \to \infty} \# \{k | \mu(k) = i$ und $k \leqq n\}/n = \varepsilon_{i,j}$ für $1 \leqq i \leqq m_j$.

$\{\mathfrak{P}_i | i \in \mathbf{N}\}$ sei eine maximale Menge von unverzweigten Primidealen aus a_j mit Relativgrad $f_{L/K} = 1$, die nicht in S enthalten sind und paarweise nicht konjugiert unter G sind. Außerdem sei ihre Reihenfolge so gewählt, daß $\mathscr{N}_{L/Q}\mathfrak{P}_i \leqq \mathscr{N}_{L/Q}\mathfrak{P}_{i+1}$ gilt. Für $i \in \mathbf{N}$ ist dann $\mathscr{N}\mathfrak{P}_i = \mathfrak{p}_{n_i}$ mit $n_i > n_0$ und $\mathfrak{P}_i \mathfrak{A}_j^{-1} = (\alpha_i)$ mit $v_{n_i}(N\alpha_i) = 1$. Wir erhalten daher $N\alpha_i = \varepsilon \pi_{n_i} \prod_{n \in \mathbf{N} \setminus \{n_i\}} \pi_n^{c_n}$ mit $\varepsilon \in E_K$, $c_n \in \mathbf{Z}$ und setzen $\varepsilon_{n_i} := \eta_{\mu(i)}^{-1} \varepsilon$.

In dieser Darstellung ist $c_n \neq 0$ nur möglich, wenn $\mathfrak{p}_n | \mathscr{N}\mathfrak{A}_j$ oder $n \leqq n_0$. Auf diese Weise konstruieren wir ε_{n_i} für alle $i \in \mathbf{N}$ und analog für jede Klasse a_k $(1 \leqq k \leqq l)$. Für die von dieser Konstruktion nicht erfaßten Indizes $n \in \mathbf{N}$ (das sind genau die, wo über $\mathfrak{p}_n$ Ideale aus S, Primideale mit Relativgrad $f_{L/K} > 1$ oder verzweigte Primideale liegen) definieren wir $\varepsilon_n := 1$ und setzen

$$F := \coprod_{n \in \mathbf{N}} \langle \varepsilon_n \pi_n \rangle.$$

Wir kehren nun zu der oben betrachteten Idealklasse a_j zurück und behaupten, daß für $1 \leqq i \leqq m_j$, $b_{i,j} = \{\mathfrak{B} \in a_j | \mathfrak{B}\mathfrak{A}_j^{-1} = (\beta)$ mit $N\beta \in \eta_i NE_L \times F\}$ alle Behauptungen des Satzes erfüllt.

Man prüft leicht nach, daß $b_{i,j}$ eine (F)-Idealklasse ist, die $\mathfrak{P}_k$ genau dann enthält, wenn $\mu(k) = i$ ist. Insbesondere ist $b_{i,j}$ wegen $\mathfrak{P}_i \in b_{i,j}$ nicht leer. Aus der Wahl der η_i folgt, daß die (F)-Idealklassen $b_{i,j}$ paarweise nicht konjugiert unter G sind. Für die Primidealdichte von $b_{i,j}$ erhalten wir:

$$\delta(b_{i,j}) = \lim_{n \to \infty} \frac{\# \{\mathfrak{P} \in b_{i,j} \setminus S | \mathfrak{P} \text{ prim}, f_{L/K}(\mathfrak{P}) = 1, \mathscr{N}_{L/Q}\mathfrak{P} \leqq n\}}{h_L \# \{\mathfrak{P} \in a_j \setminus S | \mathfrak{P} \text{ prim}, f_{L/K}(\mathfrak{P}) = 1, \mathscr{N}_{L/Q}\mathfrak{P} \leqq n\}} =$$

$$= \frac{1}{h_L} \lim_{n \to \infty} \frac{\# \{\mathfrak{P}_k^\sigma | k \leqq n, \sigma \in G'_{a_j}, \mu(k) = i\}}{\# \{\mathfrak{P}_k^\sigma | k \leqq n, \sigma \in G_{a_j}\}} = \frac{1}{h_L} \lim_{n \to \infty} \frac{\# \{k | k \leqq n, \mu(k) = i\} \# G'_{a_j}}{n \# G_{a_j}} =$$

$$= \varepsilon_{i,j}/(h_L i(a_j)).$$

Bemerkung. Durch geeignete Wahl der Funktion μ im Beweis kann erreicht werden, daß für einige oder für alle (F)-Idealklassen die Primidealdichten nicht existieren. Es kann auch (F)-Idealklassen geben, die keine Primideale enthalten, wie das folgende Beispiel zeigt: Für $K = \mathbf{Q}$ und $L = \mathbf{Q}(\sqrt{34})$ ist $NE_L = \{1\}$, aber

$E_K \cap NL^\times = \{1, -1\}$, da $N((3+\sqrt{34})/5) = -1$ ist. Nach Satz 1 gilt $\# \mathscr{C}(F) = 2h_L = 4$ für jedes F mit $\mathbf{Q}^\times = \{1, -1\} \times F$. Wählt man nun F so, daß die Normen aller Primelemente von L in F liegen, so enthält die (F)-Idealklasse $\{(\alpha) | N\alpha \in (-F)\}$ kein Primideal.

4. (F)-Idealklassengruppen von Erweiterungen über Q

In diesem Kapitel betrachten wir algebraische Zahlkörper L über $K = \mathbf{Q}$ und untersuchen, für welche $F \leqq \mathbf{Q}^\times$ mit $\mathbf{Q}^\times = \{1, -1\} \times F$, $\mathscr{C}(F) = \mathscr{C}_L$ gelten kann. Außerdem werden wir zeigen, daß die exakte Sequenz (1) unabhängig von der Wahl von F als Kofaktor zu $\{1, -1\}$ spaltet bzw. nicht spaltet. Lemma 2 zeigt, daß für jedes solche F die mit geeigneten Vorzeichen versehenen Primzahlen eine Basis bilden. Ist $NE_L = \{1, -1\}$ (z. B. wenn $[L: \mathbf{Q}]$ ungerade ist), so ist $\mathscr{C}(F) = \mathscr{C}_L$ für jedes F mit $\mathbf{Q}^\times = \{1, -1\} \times F$.

Satz 4. *Ist $L/\mathbf{Q}$ ein algebraischer Zahlkörper und $NE_L = \{1\}$, so sind folgende Aussagen äquivalent:*
(i) *Es existiert ein F_0 mit $\mathbf{Q}^\times = \{1, -1\} \times F_0$ und $\mathscr{C}(F_0) = \mathscr{C}_L$.*
(ii) *Es gibt kein $\alpha \in L^\times$ mit $N\alpha = -r^2$, $r \in \mathbf{Q}$.*

Zum Beweis dieses Satzes benötigen wir das folgende Lemma.

Lemma 3. *Für $n \in \mathbf{N}$ sei $V_n = \mathbf{F}_2^n$ der n-dimensionale Vektorraum über $\mathbf{F}_2 = \{0, 1\}$. Es seien $M_n = \{(\alpha_1, \ldots, \alpha_n) \in V_n | \forall\ 1 \leqq i \leqq n\colon \alpha_i = 0$ oder $\# \{i | \alpha_i = 0\} = \# \{i | \alpha_i = 1\}\}$ und $\pi_i\colon V_n \to \mathbf{F}_2$ die Projektion auf die i-te Komponente $(1 \leqq i \leqq n)$. Ist A eine Untergruppe von V_n mit $A \subseteq M_n$, so existiert ein $i_0 \in \{1, \ldots, n\}$ mit $\pi_{i_0}(A) = \{0\}$.*

Beweis. Ist n ungerade oder A trivial, ergibt sich die Behauptung unmittelbar. Es sei nun $m \in \mathbf{N}$ und $n = 2m$. Nehmen wir an, es gäbe eine Untergruppe $A = \{e_1, \ldots, e_{2^d}\} \subseteq M_n$ mit $d \geqq 1$ und für alle i sei $\pi_i(A) \neq 0$. Ist $\varepsilon(A)$ die Anzahl der „1", die als Komponenten in den Elementen von A auftreten, so erhält man $\varepsilon(A) = (2^d - 1)m$. Ist aber $\pi_i(A) \neq \{0\}$, so ist $\pi_i(e_j) = 1$ für genau 2^{d-1} Indizes $j \in \{1, 2, \ldots, 2^d\}$, womit sich $\varepsilon(A) = 2^{d-1}n = 2^d m$ ergibt, was wegen $m \geqq 1$ einen Widerspruch darstellt.

Beweis von Satz 4. (i)$\Rightarrow$(ii) ist klar, denn $\mathscr{C}(F_0) = \mathscr{C}_L$ und $NE_L = \{1\}$ ergeben $N\alpha \in F_0$ für alle $\alpha \in L^\times$, und es ist $F_0 \cap \{-r^2 | r \in \mathbf{Q}^\times\} = \emptyset$.

(ii)$\Rightarrow$(i). $\mathbf{P}$ bezeichne die Menge aller rationalen Primzahlen. Für $p \in \mathbf{P}$ sei $v_p\colon \mathbf{Q}^\times \to \mathbf{Z}$ die p-adische Exponentenbewertung und

$$m(p) = \min \{v_p(N\alpha) | \alpha \in L^\times \text{ und } v_p(N\alpha) > 0\}.$$

$m(p)$ ist der größte gemeinsame Teiler der Restklassengrade aller Primideale von L, die über p liegen, und $m(p) | v_p(N\beta)$ für alle $\beta \in L^\times$. Es sei $\mathbf{P}_1 = \{p \in \mathbf{P} | m(p) \equiv 0 \bmod (2)\}$ und $\mathbf{P} \setminus \mathbf{P}_1 = \{p_1, p_2, \ldots\}$. Wir beweisen zunächst folgende Behauptung:

Ist $n \in \mathbf{N}$ und

$$L_n = \{\alpha \in L^\times | N\alpha \in \{1, -1\} \times \prod_{p \in \mathbf{P}_1} \langle p \rangle \times \prod_{i=1}^{n} \langle p_i \rangle\},$$

(4)

so existieren $\varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\}$, *sodaß*

$$NL_n \subseteq \prod_{p\in \mathbf{P}_1} \langle p\rangle \times \prod_{i=1}^{n} \langle \varepsilon_i p_i\rangle \text{ gilt.}$$

Für $1 \leqq j \leqq 2^n$ sei $F_j = \prod_{p\in \mathbf{P}_1} \langle p\rangle \times \prod_{i=1}^{n} \langle \varepsilon_i p_i\rangle$, wobei $(\varepsilon_1, \ldots, \varepsilon_n)$ die Menge $\{1, -1\}^n$ durchläuft. Wir definieren die Abbildung $\varphi\colon L_n \to \mathbf{F}_2^{2^n}$ durch $\varphi(\alpha) = (e_1, \ldots, e_{2^n})$ und $e_j = \begin{cases} 0, & \text{wenn } N\alpha \in F_j \\ 1, & \text{wenn } N\alpha \notin F_j \end{cases}$. Man prüft leicht nach, daß φ ein Gruppenhomomorphismus ist. Für $\alpha \in L_n$ ist $N\alpha = \pm \prod_{p\in \mathbf{P}_1} p^{a(p)} \prod_{i=1}^{n} p_i^{a_i}$. Aus der Definition von $\mathbf{P}_1$ folgt, daß für alle $p \in \mathbf{P}_1$ $a(p) \equiv 0 \bmod (2)$ ist. Sind alle a_i gerade, so gilt wegen (ii) das positive Vorzeichen für $N\alpha$, und es ist $\varphi(\alpha) = (0, \ldots, 0)$. Ist hingegen a_{i_0} ungerade und sind $\varepsilon_i \in \{1, -1\}$ für alle $i \neq i_0$ gewählt, so ist je nach der Wahl von $\varepsilon_{i_0} \in \{1, -1\}$ $N\alpha$ in der zu $(\varepsilon_1, \ldots, \varepsilon_n)$ gehörigen Menge F_j enthalten oder nicht. Es folgt, daß in diesem Fall $N\alpha$ in genau 2^{n-1} der Mengen F_j enthalten ist und $\varphi(\alpha)$ gleich viele „0" wie „1" als Komponenten besitzt. $\varphi(L_n)$ erfüllt somit die Voraussetzungen von Lemma 3. Es existieren daher ein $j \in \{1, \ldots, 2^n\}$, sodaß für $\varphi(L_n)$ die j-te Komponente 0 ist. Das heißt aber $NL_n \subseteq F_j$, womit (4) bewiesen ist.

Wollen wir mit (4) durch Induktion eine Vorzeichenfolge $(\varepsilon_i)_{i\in \mathbf{N}}$ konstruieren, sodaß $NL \subseteq F_0 = \prod_{p\in \mathbf{P}_1} \langle p\rangle \times \prod_{i\in \mathbf{N}} \langle \varepsilon_i p_i\rangle$ gilt, müssen wir noch zeigen, daß für ein $n_0 \in \mathbf{N}$ und für alle $k \in \mathbf{N}$ mit $k \geqq n_0$ gilt:

Sind $\varepsilon_1, \ldots, \varepsilon_k \in \{1, -1\}$ *und*

$$NL_k \subseteq \prod_{p\in \mathbf{P}_1} \langle p\rangle \times \prod_{i=1}^{k} \langle \varepsilon_i p_i\rangle,$$

(5) *so existiert ein* $\varepsilon_{k+1} \in \{1, -1\}$ *mit*

$$NL_{k+1} \subseteq \prod_{p\in \mathbf{P}_1} \langle p\rangle \times \prod_{i=1}^{k+1} \langle \varepsilon_i p_i\rangle.$$

Wählen wir dazu $n_0 \in \mathbf{N}$ so, daß die Idealklassen der über $\mathbf{P}_1 \cup \{p_1, \ldots, p_{n_0}\}$ liegenden Primideale von L die Klassengruppe $\mathscr{C}_L$ erzeugen, und $k \geqq n_0$. Nach (4) existieren $\varepsilon_1, \ldots, \varepsilon_k \in \{1, -1\}$ mit $NL_k \subseteq \prod_{p\in \mathbf{P}_1} \langle p\rangle \times \prod_{i=1}^{k} \langle \varepsilon_i p_i\rangle$. Wegen der Wahl von n_0 existiert ein $\alpha \in L_{k+1}$ mit $v_{p_{k+1}}(N\alpha) = m(p_{k+1}) \equiv 1 \bmod (2)$, also

$$N\alpha = \varepsilon p_{k+1}^{m(p_{k+1})} \prod_{p\in \mathbf{P}_1} p^{a(p)} \prod_{i=1}^{k} (\varepsilon_i p_i)^{a_i}$$

mit $\varepsilon \in \{1, -1\}$. Wir behaupten, daß $\varepsilon_{k+1} := \varepsilon$ die gewünschte Eigenschaft besitzt. Gäbe es nämlich ein $\beta \in L_{k+1}$ mit

$$N\beta = -\prod_{p\in \mathbf{P}_1} p^{b(p)} \prod_{i=1}^{k} (\varepsilon_i p_i)^{b_i} (\varepsilon_{k+1} p_{k+1})^{bm(p_{k+1})}$$

mit $b \in \mathbf{Z}$, so ist

$$N(\beta\alpha^{-b}) = -\prod_{p \in P_1} p^{b(p)-ba(p)} \prod_{i=1}^{k} (\varepsilon_i p_i)^{b_i - ba_i}.$$

Wegen $\beta\alpha^{-b} \in L_k$ ist dies aber ein Widerspruch zur Voraussetzung von (5). Mit dem Beweis von (5) ist aber auch der Beweis von Satz 4 abgeschlossen.

Abschließend bringen wir noch ein Resultat über die exakte Sequenz (1).

SATZ 5. *Ist L ein algebraischer Zahlkörper und* $NE_L = \{1\}$, *so sind folgende Aussagen äquivalent:*
(a) *Es existiert ein F_0 mit* $\mathbf{Q}^\times = \{1, -1\} \times F_0$, *sodaß die Sequenz* (1) *spaltet.*
(b) *Für jedes F mit* $\mathbf{Q}^\times = \{1, -1\} \times F$ *spaltet die Sequenz* (1).
(c) *Es existiert kein Hauptideal* $(\alpha) \in \mathscr{J}_L^2$ *mit* $N\alpha = -r^2$, $r \in \mathbf{Q}$.

BEWEIS. Eine exakte Sequenz von abelschen Gruppen $0 \to A \to B \to C \to 0$ spaltet genau dann, wenn A eine reine Untergruppe von B ist. (1) spaltet in unserem Fall daher genau dann, wenn für alle $a' \in \mathscr{C}(F)$ gilt: ist $2a' \in \mathscr{H}_L/\mathscr{H}(F)$, so ist $2a' = \mathscr{H}(F)$.

(a)$\Rightarrow$(c). Es spalte $0 \to \mathscr{H}_L/\mathscr{H}(F_0) \to \mathscr{C}(F_0) \to \mathscr{C}_L \to 0$. Es seien $\mathfrak{A} \in \mathscr{J}_L$ und $\alpha \in L$ mit $\mathfrak{A}^2 = (\alpha)$ und $N\alpha = \pm r^2$, $r \in \mathbf{Q}$. Weiters sei $a' = [\mathfrak{A}]_{F_0} \in \mathscr{C}(F_0)$. Dann ist aber $(\alpha) \in 2a' = \mathscr{H}(F_0)$ und somit $N\alpha = r^2$.

(c)$\Rightarrow$(b). Es sei $\mathbf{Q}^\times = \{1, -1\} \times F$ und $a' \in \mathscr{C}(F)$ mit $2a' \in \mathscr{H}_L/\mathscr{H}(F)$. Wählen wir $\mathfrak{A} \in a'$, so ist $\mathfrak{A}^2 = (\alpha)$ ein Hauptideal. (c) ergibt $N\alpha = r^2 \in F$, also $2a' = \mathscr{H}(F)$. Daher spaltet die Sequenz (1).

(b)$\Rightarrow$(a). Klar.

Ich möchte Herrn Professor F. Halter-Koch für viele anregende Diskussionen und für seine Ratschläge beim Verfassen des Manuskripts an dieser Stelle herzlichst danken.

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(Received February 3, 1984)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 335—338

LACUNARY INTERPOLATION BY SPLINES (0, 2, 3) CASE

TH. FAWZY

1. Introduction

R. S. Mishra and K. K. Mathur [1] and Győrváry [2] continued the study of lacunary interpolation by splines. They used spline methods for solving the (0, 2, 3) and (0, 2, 4)-interpolation problems for functions $f\in C^5$. In both methods, the end conditions $S'(x_0)=y_0'$ and $S'(x_n)=y_n'$ are imposed.

We first mention, two alternatives for the spline interpolant of J. Győrvári [2] in the intervals $[x_0, x_1]$ and $[x_{m-1}, x_m]$ such that the end conditions are not needed and the convergence is faster.

For the case (0, 2, 3) in [2], the spline interpolants $S_0(x)$ and $S_{m-1}(x)$ are replaced by:

$$S_j(x) = y_j+a_{j,1}(x-x_j)+\frac{1}{2}y_j''(x-x_j)^2+\frac{1}{3!}y_j'''(x-x_j)^3+ \tag{1.1}$$
$$+\frac{1}{4!}a_{j,4}(x-x_j)^4+\frac{1}{5!}a_{j,5}(x-x_j)^5$$

where $x_j\leqq x\leqq x_{j+1}$ and $j=0, m-1$.

The coefficients $a_{j,k}$ are obtained from the condition

$$S_j^{(q)}(x_{j+1}) = y_{j+1}^{(q)}, \quad q = 0, 2, 3 \quad \text{and} \quad j = 0, m-1. \tag{1.2}$$

The convergence of (1.1)—(1.2) is given by

THEOREM 1.1. *Let* $y=f(x)$ *where* $f\in C^5[0, 1]$ *and let S be the spline interpolant given in* (1.1)—(1.2). *Then for all* $q=0, 1, \dots, 5$ *we have*

$$\|D^q(S-f)\|_{L_\infty[x_j, x_{j+1}]} \leqq c_{j,q}h^{5-q}\omega(D^5f; h)$$

where $j=0, m-1$ *and* $c_{j,q}$*'s are given constants.*

For the (0, 2, 4) case in [2], the spline interpolants $G_0(x)$ and $G_{m-1}(x)$ are replaced by

$$G_k(x) = y_k+a_{k,1}(x-x_k)+\frac{1}{2}y_k''(x-x_k)^2+\frac{1}{3!}a_{k,3}(x-x_k)^3+ \tag{1.6}$$
$$+\frac{1}{4!}y_k^{(4)}(x-x_k)^4+\frac{1}{5!}a_{k,5}(x-x_k)^5$$

where $x_k\leqq x\leqq x_{k+1}$ and $k=0, m-1$.

1980 *Mathematics Subject Classification.* Primary 65D07.
Key words and phrases. Lacunary interpolation, splines.

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The coefficients $a_{k,p}$ are obtained from the conditions

$$G_k^{(q)}(x_{k+1}) = y_{k+1}^{(q)}, \quad q = 0, 2, 4 \quad \text{and} \quad k = 0, m-1. \tag{1.7}$$

This construction leads easily to the following convergence theorem:

THEOREM 1.2. *Let $y=f(x)$ where $f\in C^5[0, 1]$ and let G be the spline interpolant given in* (1.6)—(1.10). *Then for all $q=0, 1, \ldots, 5$ we have*

$$\|D^q(f-G)\|_{L_\infty[x_k, x_{k+1}]} \leqq c_{k,q}^* h^{5-q}\omega(D^5 f; h)$$

where $k=0, m-1$ and $c_{k,q}^*$'s are given constants.

In this paper we study the following (0, 2, 3)-interpolation problem:

PROBLEM 1. Given Δ: $\{x_i=ih\}_{i=0}^n$ and real numbers $\{f_i^{(q)}\}_{i=0}^n$ where $q=0, 2, 3$. Find S such that

$$S^{(q)}(x_i) = f_i^{(q)}, \qquad q = 0, 2, 3 \quad \text{and} \quad i = 0, 1, \ldots, n. \tag{1.11}$$

The purpose of this paper is to construct a spline for solving Problem 1 using piecewise polynomials of degree 6, such that for all functions $f\in C^6$, the order of approximation is the same as the best approximation with splines of degree 6.

2. Construction of the spline interpolant

We shall construct a solution S of Problem 1 in the form:

$$S_\Delta(x) = S_k(x) = \sum_{j=0}^{6} \frac{S_k^{(j)}}{j!}(x-x_k)^j, \quad x_k \leqq x \leqq x_{k+1}, \quad k = 0, 1, \ldots, n-1. \tag{2.1}$$

We shall define each of the $S_k^{(j)}$ explicitly in terms of the data. In particular we choose

$$S_k^{(q)} = f^{(q)}(x_k) = f_k^{(q)}, \quad q = 0, 2, 3 \quad \text{and} \quad k = 0, 1, \ldots, n-1. \tag{2.2}$$

For $k=1, 2, \ldots, n-2$ we take

$$S_k^{(6)} = \frac{1}{h^3}\{f_{k+1}^{(3)} - 3f_{k+1}^{(3)} + 3f_k^{(3)} - f_{k-1}^{(3)}\}, \tag{2.3}$$

$$S_k^{(5)} = \frac{12}{h^3}\left\{\frac{h}{2}\left(f_{k+1}^{(3)} - f_k^{(3)} - \frac{h^3}{3!}S_k^{(6)}\right) - \left(f_{k+1}^{(2)} - f_k^{(2)} - hf_k^{(3)} - \frac{h^4}{4!}S_k^{(6)}\right)\right\}, \tag{2.4}$$

$$S_k^{(4)} = \frac{1}{h}\left\{f_{k+1}^{(3)} - f_k^{(3)} - \frac{h^2}{2}S_k^{(5)} - \frac{h^3}{3!}S_k^{(6)}\right\} \tag{2.5}$$

and

$$S_k^{(1)} = \frac{1}{6}\left\{f_{k+1} - f_k - \frac{h^2}{2}f_k'' - \frac{h^3}{3!}f_k^{(3)} - \frac{h^4}{4!}S_k^{(4)} - \frac{h^5}{5!}S_k^{(5)} - \frac{h^6}{6!}S_k^{(6)}\right\}. \tag{2.6}$$

For $k=0$ we take

(2.7) $$S_0^{(6)} = S_1^{(6)}(x_1),$$

(2.8) $$S_0^{(5)} = \frac{12}{h^3}\left\{\frac{h}{2}\left(f_1^{(3)}-f_0^{(3)}-\frac{h^3}{3!}S_0^{(6)}\right)-\left(f_1''-f_0''-hf_0^{(3)}-\frac{h^4}{4!}S_0^{(6)}\right)\right\},$$

(2.9) $$S_0^{(4)} = \frac{1}{h}\left\{f_1^{(3)}-f_0^{(3)}-\frac{h^2}{2}S_0^{(5)}-\frac{h^3}{3!}S_0^{(6)}\right\}$$

and

(2.10) $$S_0^{(1)} = \frac{1}{h}\left\{f_1-f_0-\frac{h^2}{2}f_0''-\frac{h^3}{3!}f_0^{(3)}-\frac{h^4}{4!}S_0^{(4)}-\frac{h^5}{5!}S_0^{(5)}-\frac{h^6}{6!}S_0^{(6)}\right\}.$$

Finally, for $k=n-1$ we take

(2.11) $$S_{n-1}^{(6)} = S_{n-2}^{(6)}(x_{n-1}),$$

(2.12) $$S_{n-1}^{(5)} = \frac{12}{h^3}\left\{\frac{h}{2}\left(f_n^{(3)}-f_{n-1}^{(3)}-\frac{h^3}{3!}S_{n-1}^{(6)}\right)-\left(f_n''-f_{n-1}''-hf_{n-1}^{(3)}-\frac{h^4}{4!}S_{n-1}^{(6)}\right)\right\},$$

(2.13) $$S_{n-1}^{(4)} = \frac{1}{h}\left\{f_n^{(3)}-f_0^{(3)}-\frac{h^2}{2}S_{n-1}^{(5)}-\frac{h^3}{3!}S_{n-1}^{(6)}\right\}$$

and

(2.14) $$S_{n-1}^{(1)} = \frac{1}{h}\left\{f_n-f_{n-1}-\frac{h^2}{2}f_{n-1}''-\frac{h^3}{3!}f_{n-1}^{(3)}-\frac{h^4}{4!}S_{n-1}^{(4)}-\frac{h^5}{5!}S_{n-1}^{(5)}-\frac{h^6}{6!}S_{n-1}^{(6)}\right\}.$$

Clearly, the function S defined in (2.1)—(2.14) solves the (0, 2, 3)-interpolation Problem 1. Moreover, by the construction, it is clear that S is a piecewise polynomial of degree 6.

3. Error bounds

THEOREM 3.1. *Let $f\in C^6[x_0, x_n]$ and let S_Δ be the lacunary-spline constructed in* (2.1)—(2.14). *Then for all $0\leqq j\leqq 6$ and all $1\leqq k\leqq n-2$, we have*

$$\|D^j(f-S_\Delta)\|_{L_\infty[x_k, x_{k+1}]} \leqq c_{j,k}^* h^{6-j}\omega(D^6 f; h)$$

where the constants $c_{j,k}^*$ are given by

$$c_{0,k}^* = \frac{13}{40},\quad c_{1,k}^* = \frac{619}{720},\quad c_{2,k}^* = \frac{11}{2},\quad c_{3,k}^* = \frac{17}{6},\quad c_{4,k}^* = 6,\quad c_{5,k}^* = 7$$

and $$c_{6,k}^* = 2.$$

THEOREM 3.2. *Let $f\in C^6[x_0, x_n]$ and let S_Δ be the lacunary-spline constructed in* (2.1)—(2.14). *Then for all $0\leqq j\leqq 6$ and $k=0, n-1$, we have*

$$\|D^j(f-S_\Delta)\|_{L_\infty[x_k, x_{k+1}]} \leqq c_{j,k}^{**} h^{6-j}\omega(D^6 f; h)$$

where the constants $c_{j,k}^{**}$ *are given by*

$$c_{0,k}^{**}=\frac{5}{16},\quad c_{1,k}^{**}=\frac{397}{480},\quad c_{2,k}^{**}=\frac{9}{4},\quad c_{4,k}^{**}=\frac{35}{4},\quad c_{3,k}^{**}=\frac{11}{2},\quad c_{5,k}^{**}=\frac{15}{2}$$

and

$$c_{6,k}^{**}=3.$$

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(Received August 8, 1984)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 339—342

ARCHIMEDEAN DECOMPOSITION OF COMMUTATIVE SEMIGROUPS WITH OPERATORS

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§ 1. Introduction

Let $\underset{\sim}{A}=(A; +)$ be a commutative semigroup and F be a domain of operations on the carrier A of $\underset{\sim}{A}$ that may or may not be empty. We write $\underset{\sim}{A}=(A; +, F)$ and call it an *F-commutative semigroup*. This F-commutative semigroup is, by definition, an F-semilattice iff $(A; +)$ is a semilattice (see [1]). We say that the F-commutative semigroup $\underset{\sim}{A}$ is F-distributive iff

$$\begin{aligned} &f_s(a_1, \dots, a_{i-1}, a_i+b_i, a_{i+1}, \dots, a_s) = \\ &= f_s(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_s)+f_s(a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_s) \end{aligned} \tag{1}$$

for all $s\in N$ (being N the set of the natural numbers excluding the zero), for all $i\in N$, $i\leqq s$, for all $f_s\in F_s$ (where F_s is the subclass of F formed by all the operations of F of arity s) (see [2]), and for all $a_1, \dots, a_{i-1}, a_i, b_i, a_{i+1}, \dots, a_s$ elements of A. $\underset{\sim}{A}=(A; +, F)$ will be called *F-idempotent* iff

$$f_s(a, \overset{s}{\dots}, a) = a, \quad \text{for all} \quad s\in N, f_s\in F_s \quad \text{and for all } a \text{ of } A. \tag{2}$$

As it is well-known, a congruence C on $\underset{\sim}{A}$ is an equivalence relation C on A compatible with all the operations defined on $\underset{\sim}{A}$. C will be named *additive idempotent* iff $(a+a)Ca$, for all $a\in A$; *F-idempotent* iff $f_m(a, \overset{m}{\dots}, a)Ca$, for all $m\in N, f_m\in F_m$, $a\in A$; *idempotent* iff it is both additive idempotent and F-idempotent.

Proposition 1 (The Tamura—Kimura congruence). *Let $\underset{\sim}{A}=(A; +, F)$ be an F-distributive commutative semigroup. The binary relation T on A:*

aTb iff $m, n\in N$; $x, y\in A$ exist so that $a+x=mb$ and $b+y=na$;

(where $mb==b+\overset{m}{\dots}+b$) is the smallest additive idempotent congruence on $\underset{\sim}{A}$. It will be named the Tamura—Kimura congruence on $\underset{\sim}{A}$.

Proof. In [1] and [3] was demonstrated T is the smallest idempotent congruence on $(A; +)$. We must now show that T is compatible with all $f_s\in F_s$ for all $s\in N$.

1980 *Mathematics Subject Classification.* Primary 20M14; Secondary 16A78, 08B99.

Key words and phrases. F-distributive commutative semigroup, species E of an algebraic variety of V, E-congruence on an algebra, E-maximal homomorphic image, Archimedean decomposition.

Assume $a_i T b_i$, for $i=1, 2, ..., s$. Then

$$a_i+x_i = m_i b_i \quad \text{and} \quad b_i+y_i = n_i a_i, \quad i = 1, 2, ..., s.$$

Being

$$x_i, y_i \in A; \ m_i, n_i \in N, \quad \text{for all} \quad i = 1, 2, ..., s.$$

So

$$(3) \qquad f_s(a_1+x_1, ..., a_s+x_s) = f_s(a_1, a_2, ..., a_s)+w_1 = f_s(m_1 b_1, ..., m_s, b_s) = \\ = m_i m_1 ... m_s f_s(b_1, ..., b_s)$$

for some $w_1 \in A$.

Similarly,

$$f_s(b_1, ..., b_s)+w_2 = n_1 n_2 ... n_s f_s(a_1, ..., a_s), \qquad w_2 \in A.$$

This means $f_s(a_1, ..., a_s) T f_s(b_1, ..., b_s)$. □

A commutative semigroup $(A; +)$ is called *archimedean* [1] iff the Tamura—Kimura congruence on it coincides with the universal congruence.

§ 2. Species E of an algebraic variety V

We introduce here a new concept of *"species"* E of an algebraic variety V of the similarity class $K(t)$ of non-void type t (see [2]). In [2] *species* means also variety, but not here. In this paper all subvarieties of V are species of V, but there are species of V which are not subvarieties of V. Let E be a non empty subclass of V; the congruence C on the algebra $\underset{\sim}{B} \in V$ will be named *E-congruence* on $\underset{\sim}{B}$ iff $\underset{\sim}{B}/C \in E$. We say that $\underset{\sim}{B} \in V$ has an *E-maximal homomorphic (or epimorphic) image* if and only if an E-congruence D on $\underset{\sim}{B}$ exists such that every epimorphic image of $\underset{\sim}{B}$ that belongs to E is epimorphic image of $\underset{\sim}{B}/D$. Then $\underset{\sim}{B}/D$ is called the *E-maximal homomorphic (or epimorphic) image of* $\underset{\sim}{B}$ and it is unique, up to isomorphisms.

Definition 1. A non-void subclass E of the algebraic variety V will be called in this paper "species" of V iff

(1) Every isomorphic copy of any member of E belongs to E.
(2) An E-congruence on $\underset{\sim}{B}$, at least, exists for all $\underset{\sim}{B} \in V$.
(3) The intersection of all E-congruences on $\underset{\sim}{B}$ is also an E-congruence on $\underset{\sim}{B}$ for all $\underset{\sim}{B} \in V$. We note it $C(\underset{\sim}{B}, E, V)$.

The proof given by Clifford and Preston [1] of the Proposition 1.7, called by them "The Principle of the Maximal Homomorphic Image of given Type" (where by them *type* means a different concept of what it means in [2]), can be easily generalized to every species E of V in this form:

Proposition 2. *If E is a species of the algebraic variety $V \subseteq K(t)$, then every member $\underset{\sim}{B}$ of V has a unique E-maximal homomorphic image: $\underset{\sim}{B}/C(\underset{\sim}{B}, E, V)$, up to isomorphisms.*

§ 3. Archimedean decompositions

Let F be a fixed operator domain, so that V_F denotes the variety of all the F-distributive commutative semigroups and E_F the subclass of V_F formed by all the F-distributive semilattices. Obviously, $E_F \neq \emptyset$, and every isomorphic copy of every member of E_F belongs to E_F. Thanks Definition 1 and Proposition 1, E_F is a species of V_F. For every $\underset{\sim}{A} \in V_F$ the Tamura—Kimura congruence T is just $C(\underset{\sim}{A}, E_F, V_F)$.

We generalize here the Tamura—Kimura—Thierrin decomposition theorem (see [3], [4] and Theorem 4.13 of [1]) for commutative semigroups and the (archimedean) decomposition Theorems 1 and 2 of [5] for left or right semimodules over semirings and for semirings, respectively, to F-distributive commutative semigroups.

THEOREM. *Every F-distributive commutative semigroup* $\underset{\sim}{A}=(A; +, F)$ *is expressible as an F-distributive semilattice* $\underset{\sim}{X}=(X; +, F)$ *of disjoint components* S_α, $\alpha \in X$, $A = \bigcup_{\alpha \in X} S_\alpha$.

Each S_α, *endowed with the restricted addition of* $\underset{\sim}{A}$, $\underset{\sim}{S}_\alpha=(S_\alpha; +)$ *is an archimedean commutative semigroup and* $\underset{\sim}{X} \cong \underset{\sim}{A}/T = \underset{\sim}{A}/C(\underset{\sim}{A}, E_F, V_F)$ *is the maximal F-distributive semilattice homomorphic image of* $\underset{\sim}{A}$, *unique up to isomorphisms; being* S_α *also a class of* $\underset{\sim}{A}$ *modulo the Tamura—Kimura congruence* T *on* $\underset{\sim}{A}$, *for every* $\alpha \in X$.

PROOF. This theorem follows from the foregoing definitions, from Propositions 1 and 2 and the previous observations. □

REMARK. In the Tamura—Kimura—Thierrin Theorem (in Theorem 1 of [5]; in Theorem 2 of [5]) $\underset{\sim}{X}$ is the maximal semilattice (additive idempotent left semimodule over the semirings; additive idempotent semirings, respectively) homomorphic image of $\underset{\sim}{A}$. For this reason we prefer to call $A = \bigcup_{\alpha \in X} S_\alpha$ the *archimedean* decomposition of $\underset{\sim}{A}$, in all cases, rather the *F-distributive semilattice* decomposition of $\underset{\sim}{A}$.

COROLLARY. *The necessary and sufficient condition in order to be every component* S_α, $\alpha \in X$ *of the archimedean decomposition of an F-distributive commutative semigroup* $\underset{\sim}{A}=(A; +, F)$ *an archimedean F-subsemigroup* $\underset{\sim}{S}_\alpha=(S_\alpha; +; F)$ *of* $\underset{\sim}{A}$ *is that the Tamura—Kimura congruence* T *on* A *is idempotent.*

In this case, $\underset{\sim}{X}=(X; +, F) \cong \underset{\sim}{A}/T = \underset{\sim}{A}/C(\underset{\sim}{A}, E_F, V_F)$ *is the unique maximal F-distributive F-idempotent semilattice epimorphic image of* $\underset{\sim}{A}$.

PROOF. Obvious □

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(Received November 2, 1984)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 343—368

LEBESGUE COVER AND LEBESGUEAN EXTENSION

V. K. ZAHAROV

Contents

Preface

It is known in measure theory that the most extensive class of functions, on which all Radon measures can be extended, consists of the universally measurable functions (in the terminology of Bourbaki). We shall call them *Lebesgue functions.* Lebesgue functions constitute one of the most remarkable classes of discontinuous functions.

In spite of the fact that the class of measurable functions is extensively investigated, the natural question, what properties distinguish *the Lebesgue extension* $L^*(T)$ among all the other extensions of the set $C^*(T)$ of all bounded continuous functions on a space T, had no answer.

The paper consists of three paragraphs on three different topics, which are connected by the fact that the proofs of the consequent results are based on the previous ones.

The first paragraph is devoted to *the Lebesguean cover* K of a completely regular space T. Briefly it can be defined as such a "good" preimage of the space T that discontinuous Lebesgue functions, lifted on K, become "almost" continuous on K. This cover was considered by Gordon [1], Sentilles [2], [3], Graves [3], [4] and others, however no characterization of it was known. The first characterization of the Lebesguean cover was given by the author in the paper [5] with the help of the notion of *perfect preimages lifting Kelley covering.*

1980 *Mathematics Subject Classification.* Primary 54G05; Secondary 46E30, 16A08, 16A86.

Key words and phrases. Universally measurable functions, Kelley ideal, inheritance of Kelley covering of completely regular space, inheritance of Lebesgue decomposition of vector lattice or lattice ring of continuous functions.

On this base the notion of *extensions of $C^*(T)$ inheriting Lebesgue decomposition* was introduced. With the help of this notion in the second and the third paragraphs characterization of the Lebesguean extension $L^*(T)$ are presented.

In the paper we shall adhere to the terminology accepted in the books [6]—[11].

The author expresses his profound gratitude to Á. Császár and E. Makai for their help during fulfilling this work at the time of the author's staying in the Budapest University in 1983—84.

1. Kelley ideals

Let T be a completely regular space and $\mathcal{P}(T)$ be the field of all subsets of T.

1.1. A subset E of T will be called a *K_σ-set* if $E=\bigcup F_k$ for some sequence of compact subsets F_k. The set of all the K_σ-subsets of T will be denoted by $\mathcal{K}_\sigma(T)$.

A σ-ideal N in $\mathcal{P}(T)$ will be called *regular* (or more exactly *compactly regular*) if the following conditions are fulfilled:

(a) for any $P\in N$ there exists a sequence of open sets G_k such that $P\subset \bigcap G_k\in N$;
(b) for any open set G in T there exists a K_σ-set $E\subset G$ such that $G\setminus E\in N$;
(c) for any K_σ-set E there exists a disjoint K_σ-set E' such that $T\setminus(E\cup E')\in N$.

1.2. Let $\mathcal{B}$ be some subset of $\mathcal{P}(T)$ and $\{B_p|p\leqq m\}$ be a finite sequence of elements of $\mathcal{B}$. The number $i_N\{B_p\}\equiv\max\left\{\frac{l}{m}\,\middle|\,\exists 1\leqq p_1<\ldots<p_l\leqq m\ (B_{p_1}\cap\ldots\cap B_{p_l}\notin N)\right\}$ will be called *the intersection number of the sequence $\{B_p\}$ with respect to the ideal N*. We shall say that $\mathcal{B}$ has a *nonzero intersection number with respect to N* if $i_N\{B_p\}\geqq\frac{1}{r}$ for some natural number r and any finite sequence $\{B_p\}$ in $\mathcal{B}$.

A regular σ-ideal N will be called a *Kelley ideal* if $\mathcal{K}_\sigma(T)\setminus N$ is the union of a sequence of subsets $\mathcal{K}_k$, which satisfy the following conditions:

(a) if $E\in\mathcal{K}_k$ and $E'\in\mathcal{K}_\sigma(T)$ is equivalent to E with respect to N, then $E'\in\mathcal{K}_k$;
(b) every $\mathcal{K}_k$ has a nonzero intersection number with respect to N;
(c) if $\{E_k|k<+\infty\}$ is an increasing sequence in $\mathcal{K}_\sigma(T)$ and $\bigcup E_k\in\mathcal{K}_m$ then $E_{k_0}\in\mathcal{K}_m$ for some k_0.

The set of all the Kelley ideals in $\mathcal{P}(T)$ will be denoted by $\mathcal{N}(T)$.

1.3. Let $\mathcal{B}(T)$ be the σ-field of all Borel subsets of T. Let ν be a Radon measure on T, i.e. a bounded countably additive real-valued function ν on the field $\mathcal{B}(T)$ such that $\nu B=\sup\{\nu K|K\subset B\ \&\ K \text{ is compact}\}$ for any Borel set B. The set of all Radon measures on T will be denoted by $M(T)$. Let $n\equiv\{\mu\in M(T)|\mu\ll\nu\ll\mu\}$ be the class of all measures coabsolutely continuous with the measure ν and let $\mathcal{M}(T)$ be the set of all such classes.

LEMMA. *For any Kelley ideal N there exists a measure $\nu\in M(T)$ such that $N=\{P\in\mathcal{P}(T)|\exists B\in\mathcal{B}(T)\ (P\subset B\ \&\ \nu B=0)\}$. The mapping $\zeta\colon N\mapsto n$ is a bijection between $\mathcal{N}(T)$ and $\mathcal{M}(T)$.*

PROOF. Let ν be a measure, $N_0 \equiv \{B \in \mathscr{B}(T) | \nu B = 0\}$ be the corresponding ideal and $\bar{\nu}$ be the corresponding strictly positive σ-additive measure on the Boolean algebra $\mathscr{B}_N(T) \equiv \mathscr{B}(T)/N_0$. According to the Kelley's criterion (see [12] or [9], § 42) $\mathscr{B}_N(T) \setminus 0$ is a union of some sequence of subsets $\mathscr{E}_k$ satisfying the Kelley conditions:

a) any $\mathscr{E}_k$ has a nonzero intersection number, i.e.

$$i\{\bar{B}_p\} \equiv \max\left\{\frac{l}{m}\middle| \exists 1 \leqq p_1 < \ldots < p_l \leqq m(\bar{B}_{p_1} \wedge \ldots \wedge \bar{B}_{p_l} \neq 0)\right\} \geqq \frac{1}{r}$$

for some natural number $r = r(k)$ and any finite sequence $\{\bar{B}_p | p \leqq m\}$ in $\mathscr{E}_k$;

b) if $\{\bar{B}_p | p < +\infty\}$ is an increasing sequence in $\mathscr{B}_N(T)$ and $\sup \bar{B}_p \in \mathscr{E}_k$ then $\bar{B}_{p_0} \in \mathscr{E}_k$ for some p_0.

Consider the σ-ideal $N \equiv \{P \in \mathscr{P}(T) | \exists B \in N_0 (P \subset B)\}$. It is clear that it is regular. Let $\mathscr{K}_k \equiv \{E \in \mathscr{K}_\sigma(T) | \bar{E} \in \mathscr{E}_k\}$. Then $\mathscr{K}_\sigma(T) \setminus N = \bigcup \mathscr{K}_k$ and every $\mathscr{K}_k$ satisfies the conditions from the definition of Kelley ideals. Hence N is a Kelley ideal.

Conversely, let N be a Kelley ideal. Then $\mathscr{K}_\sigma(T) \setminus N = \bigcup \mathscr{K}_k$ and $\mathscr{K}_k$ satisfy the conditions from the definition of Kelley ideals. Let

$$\mathscr{B}_k \equiv \{B \in \mathscr{B}(T) | \exists E \in \mathscr{K}_k (E \subset B \ \& \ B \setminus E \in N)\}.$$

Consider the set $\mathscr{M}(T, N) \equiv \{P \cup E | P \in N \ \& \ E \in \mathscr{K}_\sigma(T)\}$. Then $\mathscr{M}(T, N)$ is a σ-field. In fact for E there is a disjoint K_σ-set E' such that $P' \equiv T \setminus (E \cup E') \in N$. Let $E' = \bigcup F_k$ and $P \subset \bigcap G_k \in N$. Consider the sets $F_{ki} \equiv F_k \setminus G_i$,

$$Q \equiv (P' \setminus P) \cup ((\cap G_k \setminus P) \cap E)$$

and $B' \equiv \bigcup\limits_k \bigcup\limits_i F_{ki} \cup Q$. Then $B' \cup B = T$ and $B' \cap B = \emptyset$, i.e. B' is the complement to the set $B \equiv E \cup P$. Besides $\mathscr{M}(T, N)$ is closed under countable unions. Since $\mathscr{M}(T, N)$ contains all open sets and is a σ-field we conclude that $\mathscr{B}(T) \subset \mathscr{M}(T, N)$. Therefore for any $B \in \mathscr{B}(T)$ there exists a K_σ-set $E \subset B$ such that $B \setminus E \in N$. Consequently $\bigcup \mathscr{B}_k = \mathscr{B}(T) \setminus N$ and $\mathscr{B}_k$ contains together with any of its elements all its class of N-equivalence. Let $N_0 \equiv N \cap \mathscr{B}(T)$, $\mathscr{B}_N(T) \equiv \mathscr{B}(T)/N_0$ and $\mathscr{E}_k \equiv \{\bar{B} | B \in \mathscr{B}_k\} = \{\bar{E} | E \in \mathscr{K}_k\}$. Then $\mathscr{B}_N(T) \setminus 0 = \bigcup \mathscr{E}_k$. Let $\{\bar{E}_p\}$ be a finite sequence in $\mathscr{E}_k$. Then $\{E_p\} \subset \mathscr{K}_k$ and $i\{\bar{E}_p\} = i_N\{E_p\} \geqq \frac{1}{r}$. Consequently, $\mathscr{E}_k$ has a nonzero intersection number. Let $\{\bar{E}_p\}$ be an increasing sequence in $\mathscr{B}_N(T)$ and $\sup \bar{E}_p \in \mathscr{E}_k$. Then we can choose E_p such that they are increasing. As $\overline{\bigcup E_p} \in \mathscr{E}_k$ then $\bigcup E_p \in \mathscr{K}_k$. Hence $E_{p_0} \in \mathscr{K}_k$ and therefore $\bar{E}_{p_0} \in \mathscr{E}_k$. Thus $\mathscr{E}_k$ satisfy the above mentioned Kelley conditions (a) and (b). By virtue of the Kelley's criterion there exists a finite strictly positive σ-additive measure $\bar{\nu}$ on the Boolean algebra $\mathscr{B}_N(T)$. Extend this measure on $\mathscr{B}(T)$ by setting $\nu B \equiv \bar{\nu}\bar{B}$. Then $N_0 = \{B \in \mathscr{B}(T) | \nu B = 0\}$ and consequently N has the form given in the lemma. As for any $B \in \mathscr{B}(T)$ there exists a K_σ-set $E = \bigcup F_k \subset B$ such that $B \setminus E \in N_0$ thus $\nu B = \nu \bigcup F_k = \sup \nu F_k$, i.e. the measure ν is compactly regular. The lemma is proved.

2. Perfect preimages lifting Kelley covering

Let T and K be completely regular spaces and $\varkappa\colon K\to T$ a surjective perfect mapping.

2.1. The family of all cozero-sets in K will be denoted by $\mathscr{C}(K)$. Consider some base in K consisting of a subset $\mathscr{C}_0(K)$ of $\mathscr{C}(K)$ and containing K and the empty set. Let $\mathscr{Z}_0(K)$ denote the set of all complements to the elements of $\mathscr{C}_0(K)$ and $\Delta_0(K)$ denote the set of all open-closed elements of $\mathscr{C}_0(K)\cap\mathscr{Z}_0(K)$.

The base $\mathscr{C}_0(K)$ will be called *completely normal* if:

1) $\mathscr{C}_0(K)$ is closed under countable unions and finite intersections;

2) any two disjoint zero-sets from $\mathscr{Z}_0(K)$ are contained in two disjoint cozero-sets from $\mathscr{C}_0(K)$;

3) any cozero-set in $\mathscr{C}_0(K)$ is a countable union of zero-sets from $\mathscr{Z}_0(K)$.

Note that if $\mathscr{C}_0(K)$ is a completely normal base then $(K, \mathscr{Z}_0(K))$ is a completely normal Alexandrov space ([13]). Further we shall assume that $\varkappa^{-1}C\in\mathscr{C}_0(K)$ for any $C\in\mathscr{C}(T)$.

2.2. Let K be a perfect preimage of T with a completely normal base $\mathscr{C}_0(K)$. The preimage K will be called *lower extremally disconnected* if $\operatorname{cl}\varkappa^{-1}G\in\Delta_0(K)$ for any open set G in T.

2.3. Let T_N denote the support of the Kelley ideal N, i.e. the complement to the union of all open elements from N. The covering $\{T_N|N\in\mathscr{N}(T)\}$ will be called *the Kelley covering of the space T.*

The preimage K will be called *lifting Kelley covering* if K has a family of closed subsets $\{K_N|N\in\mathscr{N}(T)\}$ such that $\cup K_N$ is dense in K, $\varkappa K_N=T_N$ and $N_1\subset N_2$ implies $K_{N_2}\subset K_{N_1}$. The mapping $T_N\mapsto K_N$ will be called *the lifting of the Kelley covering.*

Let $\{K, \varkappa\colon K\to T, T_N\mapsto K_N\}$ and $\{\hat{K}, \hat{\varkappa}, \hat{K}\to T, T_N\mapsto\hat{K}_N\}$ be preimages lifting Kelley covering. The preimage K will be called *larger* than the preimage $\hat{K}$ if there exists a surjective perfect mapping $\gamma\colon K\to\hat{K}$ such that $\varkappa=\hat{\varkappa}\circ\gamma$ and $\gamma K_N=\hat{K}_N$. The preimage K will be called *isomorphic* to the preimage $\hat{K}$ if there exist mutually inverse homeomorphisms $\gamma\colon K\to\hat{K}$ and $\delta\colon\hat{K}\to K$ such that K is larger than $\hat{K}$ relative to γ and $\hat{K}$ is larger than K relative to δ.

2.4. Let K be a perfect preimage of T lifting Kelley covering.

The preimage K will be called *saturated* if for any K_N and any open set G intersecting K_N there exists a K_M such that $\emptyset\neq K_M\subset K_N\cap G$ and $M\supset N$.

The preimage K will be called *filled* if $\cup K_{N_k}$ is dense in K_N for any sequence of ideals N_k such that $\cap N_k=N$. Any saturated preimage is filled.

The preimage K will be called *lower disjoined* if $\varkappa^{-1}G\cap K_N=\emptyset$ implies $\operatorname{cl}\varkappa^{-1}G\cap\cap K_N=\emptyset$ for an open set G in T.

2.5. Let K be a perfect preimage of T lifting Kelley covering and having a completely normal base $\mathscr{C}_0(K)$.

The preimage K will be called *collectively σ-separated* if for any family $\{C_N, Z_N | N\in\mathcal{N}(T)\}$ from $(\mathscr{C}_0(K), \mathscr{Z}_0(K))$, such that $\bigcup C_N\subset\bigcap Z_N$ and $Z_N\setminus C_N\not\supset K_M$ for any $M\supset N$, there exists an open-closed set $U\in\Delta_0(K)$ such that $\bigcup C_N\subset U\subset\subset\bigcap Z_N$.

3. Vector-lattice extensions inheriting Lebesgue decomposition

We shall suppose that all considered vector lattices are Archimedean, have fixed strong units and are uniformly complete with respect to their units and that all considered vector-lattice homomorphisms preserve these units. Also we shall suppose that all considered vector-lattice ideals are uniformly closed.

Let T be a completely regular space and $C^*(T)$ be the vector lattice of all bounded continuous functions on T. Let X be a vector lattice and $u\colon C^*(T)\to X$ be an injective vector-lattice homomorphism. We shall say that X is *an extension of* $C^*(T)$ and shall identify $C^*(T)$ with its image in X.

3.1. The extension X will be called *lower Dedekind complete* if for any set from $C^*(T)$, which is bounded above, there exists the supremum of this set in X.

Let Y be an ideal in X. The ideal Y is called a *component of* X if $y_\xi\in Y$, $x\in X$ and $x=\sup y_\xi$ imply $x\in Y$. The ideal Y will be called a *lower component of* X if $y_\xi\in C^*(T)\cap Y$, $x\in X$ and $x=\sup y_\xi$ imply $x\in Y$.

Let Y and Z be ideals in X. The ideal $\{x\in X | \exists y\in Y \exists z\in Z\ (|x|\leq|y|+|z|)\}$ which is the supremum of Y and Z in the lattice of all ideals of X, will be denoted by $Y\vee Z$.

3.2. For any Kelley ideal $N\in\mathcal{N}(T)$ consider the ideal

$$C^*_N(T)\equiv\{f\in C^*(T)| f(T_N)=0\}$$

in $C^*(T)$. The family $\{C^*(T)|N\in\mathcal{N}(T)\}$ will be called *the Lebesgue decomposition of the vector lattice* $C^*(T)$.

The extension X of $C^*(T)$ will be called *inheriting Lebesgue decomposition* if X has a family of ideals $\{X_N|N\in\mathcal{N}(T)\}$ such that $\bigcap X_N=\{0\}$, $uf\in X_N$ iff $f\in C^*_N(T)$ and $N_1\subset N_2$ implies $X_{N_1}\subset X_{N_2}$. The mapping $C^*_N(T)\mapsto X_N$ will be called *the inheritance of Lebesgue decomposition.*

Let $\{X, u\colon C^*(T)\to X, C^*_N(T)\mapsto X_N\}$ and $\{\hat{X}, \hat{u}\colon C^*(T)\to\hat{X}, C^*_N(T)\mapsto\hat{X}_N\}$ be extensions inheriting Lebesgue decomposition. The extension X will be called *larger* than the extension $\hat{X}$ if there exists an injective vector-lattice homomorphism $v\colon\hat{X}\to X$ such that $v\circ\hat{u}=u$ and $v\hat{X}_N\subset X_N$. The extension X will be called *isomorphic* to the extension $\hat{X}$ if there exist mutually inverse vector-lattice homomorphisms $v\colon\hat{X}\to X$ and $w\colon X\to\hat{X}$ such that X is larger than $\hat{X}$ relative to v and $\hat{X}$ is larger than X relative to w.

3.3. Let X be an extension of $C^*(T)$ inheriting Lebesgue decomposition.

The extension X will be called *saturated* if for any X_N and any proper component Y such that $Y_\alpha\equiv\{x\in X|\forall_y\in Y\ (|x|\cap|y|=0)\}\not\subset X_N$ there exists an X_M such that $X_N\cup Y\subset X_M$ and $M\supset N$.

The extension X will be called *filled* if $\bigcap X_{N_k}=X_N$ for any sequence of ideals N_k such that $\bigcap N_k=N$.

The extension X will be called *lower component* if every ideal X_N is a lower component of X.

The extension X will be called *collectively σ-complete* if for any family $\{y_N^k, Z_N^k | N\in\mathcal{N}(T), k\in\mathbf{N}\}$ from X, such that $y_M^j \leqq z_N^k$ for any indexes and $0\leqq u\leqq \leqq z_N^k - y_N^j$ for all j, k implies $u\in X_N$, there exists an $x\in X$ such that $y_M^j \leqq x \leqq z_N^k$.

LEMMA. *Any saturated extension X is filled.*

PROOF. On the strength of Yosida's theorem ([14]) there is a compact K such that the vector lattice X is isomorphic to the vector lattice $C(K)$. Consider the non-empty closed subsets $K_N \equiv \{s\in K | \forall x\in X_N\ (x(s)=0)\}$. Let $\bigcap N_k = N$. Then $\bigcup K_{N_k}$ is dense in K_N. In fact assume that there exists an open set G such that $G\cap(K_N\setminus \bigcup K_{N_k}) \neq \emptyset$. Take a regular closed set $F\subset G$ such that $(\operatorname{int} F)\cap K_N \neq \emptyset$. Consider the proper component $Y\equiv\{y\in X | y(F)=0\}$. Then there exists an $M\supset N$ such that $X_N \vee Y \subset X_M \neq X$. So $K_M\subset G$.

Assume that for any k there exists a set $P_k\in M$ such that $T\setminus P_k \in N_k$. Then $P\equiv \bigcup P_k \in M$ and $T\setminus P\in N$ imply $\mathbf{1}\in C_M^*(T)$. But this is impossible because X_M is a proper ideal. Therefore there exists a number k such that $T\setminus P\notin N_k$ for any $P\in M$.

Consider the bijection $\zeta\colon N \mapsto n$ from 1.3. Let $n_k \equiv \zeta N_k$ and $m\equiv \zeta M$. Then $m_k \equiv n_k \wedge m \neq 0$. Take the proper ideal $M_k \equiv \zeta^{-1} m_k$. Then $\emptyset \neq K_{M_k} \subset K_{N_k} \cap K_M = \emptyset$. From this contradiction we conclude that such a set G does not exist.

Now take an $0\leqq x\in \bigcap X_{N_k}$. Then $x(K_N)=0$. Consider the functions

$$x_p \equiv \left(x - \frac{1}{y}\mathbf{1}\right) \vee 0.$$

From the property $K_N \cap \operatorname{cl}\operatorname{coz} x_p = \emptyset$ we conclude that $x_p \in X_N$. As this ideal is uniformly closed we get $x\in X_N$. The lemma is proved.

4. Lattice-ring extensions inheriting Lebesgue decomposition

We shall suppose that all considered f-rings are commutative, Archimedean, have the strong units[1] and are uniformly complete with respect to their units and that all considered f-ring homomorphisms are unitary. Also we shall suppose that all considered f-ring ideals are uniformly closed.

Let T be a completely regular space and $C^*(T)$ be the f-ring of all bounded continuous functions on T. Let X be an f-ring and $u\colon C^*(T)\to X$ be an injective f-ring homomorphism. We shall say that X is an *f-ring extension of* $C^*(T)$ and shall identify $C^*(T)$ with its image in X.

4.1. If Y and Z are modules over the f-ring X then the set of all module homomorphisms from Y into Z is denoted by $\operatorname{Hom}_X(Y, Z)$. Let Y and Z be ring ideals in the f-ring X. A homomorphism $g\in \operatorname{Hom}_X(Y, Z)$ will be called *bounded* if there is a natural number n such that $|gy| \leqq n|y|$ for any $y\in Y$. The subset of $\operatorname{Hom}_X(Y, Z)$ consisting of all bounded homomorphisms will be denoted by $\operatorname{Hom}_X^*(Y, Z)$.

[1] The unit $\mathbf{1}$ of an f-ring X is called *the strong unit*, if for any $x\in X$ there exists a natural number $n=n(x)$ such that $|x|\leqq n\mathbf{1}$.

The second annihilator of a subset Y of X will be denoted as usual by Y^{**}.

The extension X will be called *lower continuing* if for any ring ideal Y of the ring $C^*(T)$ and for any homomorphism $g\in \mathrm{Hom}^*_{C^*(T)}(Y, C^*(T)\cap Y^{**})$ there exists a homomorphism $h\in \mathrm{Hom}^*_X(X, Y^{**})$ extending g.

An f-ring ideal Z in X will be called a *lower segment of* X if for any ring ideal Y of the ring $C^*(T)$ and for any pair of homomorphisms $g\in \mathrm{Hom}^*_{C^*(T)}(Y, C^*(T)\cap T^{**})$ and $h\in \mathrm{Hom}^*_X(X, Y^{**})$ such that h extends g the condition $gY\subset Z$ implies $hX\subset Z$.

4.2. For any Kelley ideal $N\in\mathcal{N}(T)$ consider the f-ring ideal

$$C^*_N(T)\equiv\{f\in C^*(T)|\, f(T_N)=0\}$$

in the f-ring $C^*(T)$. The family $\{C^*_N(T)|N\in\mathcal{N}(T)\}$ will be called the *Lebesgue decomposition of the f-ring* $C^*(T)$.

The extension X of $C^*(T)$ will be called *inheriting Lebesgue decomposition* if X has a family of f-ring ideals $\{X_N|N\in\mathcal{N}(T)\}$ such that $\bigcap X_N=\{0\}$, $uf\in X_N$ iff $f\in C^*_N(T)$ and $N_1\subset N_2$ implies $X_{N_1}\subset X_{N_2}$. The mapping $C^*_N(T)\mapsto X_N$ will be called the *inheritance of Lebesgue decomposition.*

Let $\{X, u\colon C^*(T)\to X, C^*_N(T)\mapsto X_N\}$ and $\{\hat{X}, \hat{u}\colon C^*(T)\to \hat{X}, C^*(T)\mapsto \hat{X}_N\}$ be f-ring extensions inheriting Lebesgue decomposition. The extension X will be called *larger* than the extension $\hat{X}$ if there exists an injective f-ring homomorphisms $v\colon \hat{X}\to X$ such that $v\circ\hat{u}=u$ and $v\hat{X}_N\subset X_N$. The extension X will be called *isomorphic* to the extension $\hat{X}$ if there exist mutually inverse f-ring homomorphisms $v\colon \hat{X}\to X$ and $w\colon X\to\hat{X}$ such that X is larger than $\hat{X}$ relative to v and $\hat{X}$ is larger than X relative to w.

4.3. Let X be an f-ring extension of $C^*(T)$ inheriting Lebesgue decomposition.

An f-ring ideal Y in X is called an *annihilator f-ring ideal* if Y coincides with its own second annihilator Y^{**}. If Y and Z are f-ring ideals in X then the f-ring ideal generated by Y and Z will be denoted by $Y\vee Z$.

The annihilator of a subset Y of X is denoted as usual by Y^*.

The extension X will be called *saturated* if for any X_N and any proper annihilator f-ring ideal Y such that $Y^*\not\subset X_N$ there exists an X_M such that $X_N\vee Y\subset X_M$ and $M\supset N$.

The extension X will be called *filled* if $\bigcap X_{N_k}=X_N$ for any sequence of ideals N_k such that $\bigcap N_k=N$. Any saturated extension is filled.

The extension X will be called *lower segment* if any X_N is a lower segment of X.

If Y and Z are ring ideals in X then the homomorphisms $g\in\mathrm{Hom}^*_X(Y, X)$ and $h\in\mathrm{Hom}^*_X(Z, X)$ will be called *consistent* if g and h coincide on the intersection $Y\cap Z$. If $\{Y_\alpha\}$ is a family of ring ideals in X then a family of homomorphisms $\{h_\alpha\in\mathrm{Hom}^*_X(Y_\alpha, X)\}$ will be called *uniformly bounded,* if there is a natural number n such that $|h_\alpha y|\leqq n|y|$ for any $y\in Y_\alpha$ and any α. The extension X will be called *collectively σ-continuing* if for any family $\{Y_N|N\in\mathcal{N}(T)\}$ of countably generated ring ideals of X and for any uniformly bounded family of consistent homomorphisms $\{h_N\in\mathrm{Hom}^*_X(Y_N, X)\}$ such that $Y^*_N\subset X_N$ there exists a homomorphism $h\in\mathrm{Hom}^*_X(X, X)$ extending all h_N.

§ 1. Lebesguean cover

Let T be a completely regular space. An open subset G of T will be called *cocompact* if the complement of G is compact. A subset H of T will be called a co K_σ-*set if* $H=\bigcap G_k$ for some sequence of cocompact subsets G_k. The set of all the co K_σ-subsets of T will be denoted by co $\mathscr{K}_\sigma(T)$.

A subset L of T will be called a *Lebesgue subset* (or universally measurable [6]) if for every Kelley ideal N there exist a K_σ-set E and co K_σ-set H (depending on N) such that $E\subset L\subset H$ and $H\setminus E\in N$. The set of all Lebesgue subsets of T will be denoted by $\mathscr{L}(T)$. This set is a σ-field containing the Borel field $\mathscr{B}(T)$.

Consider the Stone compact K_0 of all ultrafilters in $\mathscr{L}(T)$. For any point $s\in K_0$ let P_s denote the set $\bigcap\{\mathrm{cl}\, L|L\in\Theta_s\}$ where s corresponds to the ultrafilter Θ_s. Consider the subspace $K\equiv\{s\in K_0|P_s\neq\emptyset\}$ and define the surjective continuous mapping $\varkappa\colon K\to T$ such that $\varkappa s\equiv P_s$. The space K with the mapping $\varkappa$ will be called the *Lebesguean cover of T.*

Let i_0 be the Stone isomorphism between $\mathscr{L}(T)$ and the Boolean algebra $\Delta(K_0)$ of all open-closed subsets of K_0. Let $i\colon \mathscr{L}(T)\to\Delta(K)$ be the corresponding homomorphism of Boolean algebras such that $iL\equiv K\cap i_0L$.

It can be checked that the subspace K is dense in K_0, the homomorphism i is injective and the mapping $\varkappa$ is perfect.

Associate with a Kelley ideal N the closed subspace K_N of all ultrafilters from K not containing elements of N. If N is a point ideal $N_t\equiv\{P\in\mathscr{P}(T)|t\notin P\}$ for some point t then $K_{N_t}=it$. Then K_{N_t} is an isolated point in K for every $t\in T$, $\bigcup\{K_{N_t}|t\in T\}$ is dense in K and there are no other isolated points in K.

Let $\mathscr{L}_N(T)$ denote the Boolean algebra of all classes of N-equivalence $\bar{L}$ of elements L from $\mathscr{L}(T)$. Let $L\triangle L'\in N$. Let $s\in iL\cap K_N$ and s correspond to an ultrafilter Θ_s. Assume $L'\notin\Theta_s$. Then $L\triangle L'\supset L\setminus L'\in\Theta_s$ but this is impossible. Hence $s\in iL'\cap K_N$. Thus $iL\cap K_N=iL'\cap K_N$. So we can define correctly the homomorphism of Boolean algebras $i_N\colon \mathscr{L}_N(T)\to\Delta(K_N)$ such that $i_N\bar{L}\equiv iL\cap K_N$.

LEMMA 1. *The space K_N is extremally disconnected, i_N is an isomorphism and $\varkappa K_N=T_N$.*

PROOF. Let Q_N denote the Stone compact of all ultrafilters of the Boolean algebra $\mathscr{L}_N(T)$. As this Boolean algebra is complete the space Q_N is extremally disconnected. Denote by K_{0_N} the subspace of K_0 consisting of all ultrafilters not containing elements of N. Consider the Boolean homomorphism $h_N\colon \mathscr{L}(T)\to\mathscr{L}_N(T)$ such that $h_N L\equiv\bar{L}$. Let Θ' be an ultrafilter in $\mathscr{L}_N(T)$. Then $\Theta\equiv h_N^{-1}\Theta'$ is an ultrafilter in $\mathscr{L}(T)$, $\Theta\in K_{0_N}$ and the mapping $\gamma_N\colon \Theta'\to\Theta$ is an injective continuous mapping from Q_N onto K_{0_N}. This implies that the space K_{0_N} is extremally disconnected.

The homomorphism i_N is injective. In fact let $i_N\bar{L}=\emptyset$ and assume $L\notin N$. Consider a compact set $F\subset L$ such that $F\notin N$. Consider the proper filter base Θ_0 in $\mathscr{L}(T)$ consisting of the set F and all open sets G containing F. Then $\Theta_0'\equiv\{\bar{L}|L\in\Theta_0\}$ is a proper filter base in $\mathscr{L}_N(T)$. Imbed Θ_0' in some ultrafilter Θ' and consider the ultrafilter $\Theta\equiv h_N^{-1}\Theta'\in K_{0_N}$. If $L'\in\Theta$ then $F\cap\mathrm{cl}\, L'\neq\emptyset$. Otherwise $\mathrm{cl}\, L'\cap G=\emptyset$ for some $G\in\Theta_0\subset\Theta$. It implies $L'\cap G\in\Theta$. Hence $\emptyset=L'\cap G\notin N$. Thus $\bigcap\{F\cap \mathrm{cl}\, L'|L'\in\Theta\}\neq\emptyset$. Therefore $\Theta\in K_N$. Besides $\bar{F}\in\Theta'$ implies $\Theta\in i_N\bar{L}=\emptyset$. Thus $\bar{L}=0$.

Define the homomorphism $i_{0_N}: \mathscr{L}_N(T) \to \Delta(K_{0_N})$ by setting $i_{0_N}\bar{L} \equiv i_0 L \cap K_{0_N}$. Let U be an arbitrary open-closed subset of K_{0_N}. Consider $V \equiv \gamma_N^{-1} U$. Then $V = \{\Theta' \in Q_N | \bar{L} \in \Theta'\}$ for some $\bar{L} \neq 0$. Take $\Theta \in i_{0_N}\bar{L}$ and $\Theta' = h_N \Theta \in V$. Since $\Theta \in h_N^{-1}\Theta'$ we have $\Theta \in U$. On the other hand let $\Theta \in U$ and $\Theta' \equiv \gamma_N^{-1}\Theta$. Then $\bar{L} \in \Theta'$ implies $\Theta \in i_{0_N}\bar{L}$. Thus $U = i_{0_N}\bar{L}$. So $U \cap K_N = i_N \bar{L} \neq \emptyset$. It means that K_N is dense in K_{0_N}. Therefore the space K_N is extremally disconnected.

Let U be an open-closed subset of the space K_N, V be its complement, U' and V' be the closures of U and V in the space K_{0_N}. As $K_{0_N} = \beta K_N$ we have $U' \cap V' = \emptyset$ and $K_{0_N} = U' \cup V'$. So U' is open-closed. It follows from above that $U' = i_{0_N}\bar{L}$. Hence $U = i_N \bar{L}$. Thus the homomorphism i_N is surjective.

Let $t \in T_N$. Consider the proper filter base Θ_0 consisting of all open sets G containing t. Imbed $h_N \Theta_0$ in some ultrafilter Θ' and consider the ultrafilter $\Theta \equiv h_N^{-1}\Theta' \in K_{0_N}$. As $t \in \bigcap \{\mathrm{cl}\, L | L \in \Theta\}$ we have $\Theta \in K_N$ and $\varkappa\Theta = t$. Further choose a $\Theta \in K_N$ and G be a neighborhood of $\varkappa\Theta$. As $G \cap L \neq \emptyset$ for any $L \in \Theta$ we obtain $G \in \Theta$. So $G \cap T_N \neq \emptyset$. It means $\varkappa\Theta \in T_N$. Thus $\varkappa K_N = T_N$. The lemma is proved.

Corollary. *The space K_N satisfies the Souslin condition, i.e. there are at most countably many disjoint open subsets in K_N.*

Lemma 2. *In K_N any meager subset is nowhere dense.*

Proof. Let F_k be a sequence of closed nowhere dense subsets in K_N. Assume that $\mathrm{cl} \bigcup F_k$ is not nowhere dense, i.e. there is an open-closed set U in K_N such that $U \subset \mathrm{cl} \bigcup F_k$. We can suppose that $F_k \subset U$ and so $U = \mathrm{cl} \bigcup F_k$. According to the previous corollary for every k there exists in U a sequence $\{V_{kj}\}$ of decreasing open-closed subsets with nowhere dense intersection, containing the set F_k. Let $V_{kj} = i_N \bar{L}_{kj}$ and $U = i_N \bar{L}$. We can suppose that $L \supset L_{kj} \supset L_{kj+1}$. Take $v \in \zeta N$. Then $\inf vL_{kj} = v(\bigcap L_{kj}) = 0$ for every k. Take some j_k such that $vL_{kj_k} \leqq vL/2^{k+1}$. Consider $L_0 \equiv \bigcup L_{kj_k} \subset L$. We have $vL_0 \leqq \sum vL_{kj_k} < vL$. Denote $L_1 \equiv L \setminus L_0$. Then $V \equiv i_N \bar{L}_1 \neq \emptyset$. So we get $\bigcup F_k \cap V \subset \bigcup (V_{kj_k} \cap V) = \emptyset$ and $V \subset U$, but this is impossible.

Corollary. *The space K_N is Baire.*

Denote by $\varkappa_N$ the restriction of $\varkappa$ to K_N.

Lemma 3. *For every Lebesgue set L the set $i_N \bar{L} \triangle \varkappa_N^{-1} L$ is nowhere dense in the space K_N.*

Proof. Let G be an open set. Then $i_N \bar{G} = \mathrm{cl}\, \varkappa_N^{-1} G$. In fact assume that there exists an $s \in \varkappa_N^{-1} G \cap i_N(\overline{T \setminus G})$. Then $\varkappa_N s \in T \setminus G$ but this is false. On the other hand assume that there exists $\emptyset \neq U \equiv i_N \bar{L}$ such that $U \subset i_N \bar{G} \setminus \mathrm{cl}\, \varkappa_N^{-1} G$ and $L \subset G$. Consider a compact set $F \subset L$ such that $F \notin N$ and let $V \equiv i_N \bar{F}$. Then $\emptyset \neq \varkappa_N V \subset F$. But $V \subset U \subset \varkappa_N^{-1}(T \setminus G)$ that is impossible.

If F is a closed set then $i_N \bar{F} = \mathrm{int}\, \varkappa_N^{-1} F$.

Let L be a Lebesgue set. Then $E \equiv \bigcup F_j \subset L \subset \bigcap G_k \equiv H$ and $H \setminus E \in N$. Consider the open-closed sets $U \equiv \mathrm{cl} \bigcup i_N \bar{F}_j$ and $V \equiv \mathrm{int} \bigcap i_N \bar{G}_k$. Then $U = i_N \bar{E}$ and $V = i_N \bar{H}$. Therefore $\varkappa_N^{-1} E \sim U = i_N \bar{L} = V \sim \varkappa_N^{-1} H$ with respect to the ideal of meager subsets.

COROLLARY 1. *Let* $L \in \mathscr{L}(T)$. *Then* $L \in N$ *iff* $\varkappa_N^{-1} L$ *is nowhere dense in* K_N.

COROLLARY 2. *Let L be a Lebesgue set. Then* $iL \triangle \varkappa^{-1} L \not\supset K_N$ *for any N.*

LEMMA 4. *For every* K_N *and every open-closed set U in K there exists a Kelley ideal* $M \supset N$ *such that* $K_N \cap U = K_M$.

PROOF. Consider $\emptyset \neq V \equiv K_N \cap U$. Then $V = i_N \bar{L}$ for some L. Consider the Kelley ideal $M \equiv \{P \in \mathscr{P}(T) | P \cap L \in N\}$. Then $K_M \subset K_N$. Let $\Theta \in V$. Then $L \in \Theta$. If $L' \in \Theta$ then $L \cap L' \notin N$ implies $L' \notin M$. Hence $\Theta \in K_M$. Conversely, let $\Theta \in K_M$. For any $L' \in \Theta$ we have $L' \cap L \notin N$. This implies that $L \in \Theta$ and so $\Theta \in V$.

LEMMA 5. *If* $N_1 \subset N_2$ *then* K_{N_2} *is an open-closed subset in the space* K_{N_1}.

PROOF. Consider the bijection $\zeta \colon N \mapsto n$ from 1.3. As $n_1 \geqq n_2$ there exists an n such that $n_1 = n \vee n_2$ and $n \wedge n_2 = 0$. Then there exists a Borel set $B \in N$ such that $T \setminus B \in N_2$. Assume that there exists an ultrafilter $\Theta \in K_N \cap K_{N_2}$. Then $B \notin \Theta$ implies $T \setminus B \in \Theta$ but this is false. Hence $K_N \cap K_{N_2} = \emptyset$. Further $K_N \cup K_{N_2} \subset K_{N_1}$. Assume $E \equiv K_{N_1} \setminus (K_N \cup K_{N_2}) \neq \emptyset$. Then by the previous lemma there exists an ideal $M \supset N_1$ such that $K_M \subset E$. Assume $m_1 \equiv m \wedge n \neq 0$. Then $\emptyset \neq K_{M_1} \subset K_M \cap K_N = \emptyset$. This means $m \wedge n = 0$. Similarly $m \wedge n_2 = 0$. Therefore $m \wedge n_1 = 0$. But this fact contradicts the inequality $m \leqq n_1$. Thus $K_N \cup K_{N_2} = K_{N_1}$. The lemma is proved.

Consider in the space K the completely normal base $\mathscr{C}_0(K)$ consisting of all cozero sets C which can be represented in the form $C = \bigcup U_k$ for some sequence of open-closed subsets $U_k \in i\mathscr{L}(T)$. Then $\Delta_0(K) = i\mathscr{L}(T)$.

LEMMA 6. *Let G be an open subset of T. Then* $\operatorname{cl} \varkappa^{-1} G = iG$.

PROOF. Assume that there exists a point $s \in \varkappa^{-1} G \cap i(T \setminus G)$. Then $\varkappa s \in G \cap \cap (T \setminus G)$ but this is impossible. On the other hand assume that there exists a non-empty set $U \equiv iL$ such that $U \subset iG \setminus \operatorname{cl} \varkappa^{-1} G$. We can suppose that $L \subset G$. Take $t \in L$ and $V \equiv it \neq \emptyset$. Then the inclusion $\emptyset \neq \varkappa V = t \in G$ contradicts the inclusion $V \subset \varkappa^{-1}(T \setminus G)$.

COROLLARY 1. *The preimage K is lower extremally disconnected.*

COROLLARY 2. *The preimage K is lower disjoined.*

PROOF. Let $\varkappa^{-1} G \cap K_N = \emptyset$. By the corollary to Lemma 3 $G \in N$. This implies $iG \cap K_N = \emptyset$.

LEMMA 7. *The preimage K is collectively σ-separated.*

PROOF. Consider a family $\{C_N, Z_N\}$ from the definition of the σ-separation. As $C_N = \bigcup iL_N^j$ we can consider the set $L_N' \equiv \bigcup L_N^j$. Similarly as $Z_N = \bigcap iL_N^k$ we can consider the set $L_N'' \equiv \bigcap L_N^k$. For L_N' there exists a K_σ-set $E_N \subset L_N'$ such that $L_N' \setminus E_N \in N$ and for L_N'' there exists a co K_σ-set $H_N \supset L_N''$ such that $H_N \setminus L_N'' \in N$.

Consider the sets $L' \equiv \bigcup E_N$ and $L'' \equiv \bigcap H_N$. Assume that there exists a point $t \in L'' \setminus L'$. Take the point ideal $N \equiv \{P \in \mathscr{P}(T) | P \not\ni t\}$. Then $E_N \in N$ and $H_N \notin N$ imply $L_N' \in N$ and $L_N'' \notin N$. Therefore $C_N \cap K_N = \emptyset$. By Lemma 1 $iL_N'' \cap K_N \neq \emptyset$. As K_N consists of only one point we have $K_N \subset Z_N \setminus C_N$ but this is impossible. Thus we can consider the set $L \equiv L' = L''$.

Assume that $H_N \setminus E_N \notin N$. Then there exists a closed set $F \subset L''_N \setminus L'_N$ such that $F \notin N$. Consider the Kelley ideal $M \equiv \{P \in \mathcal{P}(T) | P \cap F \in N\} \supset N$. Then $C_N \cap K_M = \emptyset$. As the preimage K is lower disjoined and lower extremally disconnected we have $K_M \subset \operatorname{int} \varkappa^{-1} T_M \subset iF \subset iL''_N \subset Z_N$. Thus $K_M \subset Z_N \setminus C_N$ but this is impossible.

This has as a consequence that the set L is Lebesgue. Consider the set $U \equiv iL$. From $C_M \subset Z_N$ we get $L'_M \subset L''_N$ for any M and N. This implies $L = L' \subset \bigcup L'_M \subset \subset \bigcap L''_N \subset L'' = L$. Therefore $\bigcup L'_M = L = \bigcap L''_N$. From this equality we get $C_N \subset \subset iL'_M \subset U \subset iL''_N \subset Z_N$. The lemma is proved.

Let K be an arbitrary perfect preimage of T lifting Kelley covering and having a completely normal base $\mathscr{C}_0(K)$. The preimage K will be called *Lebesgue determined* if for any cozero-set $C \in \mathscr{C}_0(K)$ and for any Kelley ideal $N \in \mathcal{N}(T)$ there exist a K_σ-set E_N and a $\operatorname{co} K_\sigma$-set H_N such that

$$\bigcup E_N \subset \bigcap H_N, \quad H_N \setminus E_N \in N, \quad \varkappa^{-1} E_N \setminus C \not\supset K_M \quad \text{and} \quad C \setminus \varkappa^{-1} H_N \not\supset K_M$$

for any $M \supset N$.

LEMMA 8. *Let K be the Lebesguean cover of T. Then the preimage K is Lebesgue determined.*

PROOF. Let $C = \bigcup iL_k$. Consider the sets $L \equiv \bigcup L_k$ and $U \equiv iL$. It is clear that $U = \operatorname{cl} C$. For the set L there exist a K_σ-set E_N and a $\operatorname{co} K_\sigma$-set H_N such that $E_N \subset L \subset H_N$ and $H_N \setminus E_N \in N$. Assume that $\varkappa^{-1} E_N \setminus C \supset K_M$ for some $M \supset N$. Then according to Lemma 1 we get $L \in M$. On the other hand the inclusion $L \supset T_M$ means that $L \notin M$. It follows from this contradiction that $\varkappa^{-1} E_N \setminus C \not\supset K_M$ for any M. Now assume that $C \setminus \varkappa^{-1} H_N \supset K_M$ for some $M \supset N$. Then $iL_k \cap K_M \neq \emptyset$ for some k implies by Lemma 1 that $L_k \notin M$. So $L \notin M$. On the other hand $H_N \cap T_M = \emptyset$ means that $L \subset H_N \in M$. Thus $C \setminus \varkappa^{-1} H_N \not\supset K_M$ for any $M \supset N$. The lemma is proved.

Further uniqueness is understood up to isomorphism.

THEOREM 1. *Let K be the Lebesguean cover of T. Then*

(1) *K is the unique largest of all the perfect saturated Lebesgue determined preimages of T lifting Kelley covering;*

(2) *K is the unique smallest of all the perfect filled lower extremally disconnected lower disjoined collectively σ-separated preimages of T lifting Kelley covering and moreover K is the unique universal (in the sense of Bourbaki) among all such preimages;*

(3) *K is the unique perfect saturated Lebesgue determined lower extremally disconnected lower disjoined collectively σ-separated preimage of T.*

PROOF. Let $\{\hat{K}, \hat{\varkappa}\colon \hat{K} \to T,\ T_N \mapsto \hat{K}_N,\ \mathscr{C}_0(\hat{K})\}$ be a preimage of T having the properties from (1). Then for any C there are E_N and H_N such that $L \equiv \bigcup E_N \subset \bigcap H_N$, $H_N \setminus E_N \in N$, $\hat{\varkappa}^{-1} E_N \setminus C \not\supset \hat{K}_M$ and $C \setminus \hat{\varkappa}^{-1} H_N \not\supset \hat{K}_M$ for any $M \supset N$. The set L is Lebesgue. Assume that $\hat{K}_N \subset C \triangle \hat{\varkappa}^{-1} L$ for some N. If $\hat{K}_N \cap (C \setminus \hat{\varkappa}^{-1} L) \neq \emptyset$ then there exists $\hat{K}_M \subset \hat{K}_N \cap C$ for some $M \supset N$. So $\hat{K}_M \subset C \setminus \hat{\varkappa}^{-1} L \subset C \setminus \hat{\varkappa}^{-1} E_N$. This implies $T_M \subset T \setminus E_N$. Therefore $T \setminus H_N \notin M$. Consider a closed set $F \subset T \setminus H_N$, such that $F \notin M$, and the Kelley ideal $M_1 \equiv \{P \in \mathcal{P}(T) | P \cap F \in M\} \supset M$. Then $\hat{K}_{M_1} \subset (\hat{K} \setminus \hat{\varkappa}^{-1} H_N) \cap \hat{K}_M \subset C \setminus \hat{\varkappa}^{-1} H_N$ but this is false. It means that $\hat{K}_N \subset \hat{\varkappa}^{-1} L \setminus C$. So $T_N \subset L$ implies $E_N \notin N$. That is why there exists an ideal $M \supset N$ such that

$T_M \subset E_N$. So we have $\hat{K}_M \subset \hat{\varkappa}^{-1}E_N \cap \hat{K}_N \subset \hat{\varkappa}^{-1}E_N \setminus C$ but this is false. Thus $C \triangle \hat{\varkappa}^{-1}L \not\supset \hat{K}_N$ for any N. Now assume that there exists another Lebesgue set L_1 having this property. Assume that there exists a point $t \in L \setminus L_1$ and consider the point ideal $N_t \equiv \{P \in \mathscr{P}(T) | t \notin P\}$. Then $\hat{K}_{N_t} \subset \hat{\varkappa}^{-1}L \cap (\hat{K} \setminus \hat{\varkappa}^{-1}L_1)$. By the given property $\hat{K}_{N_t} \cap C \neq \emptyset$. Therefore $\hat{K}_M \subset \hat{K}_{N_t} \cap C \subset C \setminus \hat{\varkappa}^{-1}L_1$ for some ideal M, but this is false. That is why $L = L_1$. So we can define correctly the mapping $k\colon \mathscr{C}_0(\hat{K}) \to \mathscr{L}(T)$ such that $kC \equiv L$.

Verify that k is a lattice homomorphism. Let $kC_1 = L_1$ and $kC_2 = L_2$. Assume that $\hat{K}_N \subset (C_1 \cup C_2) \triangle \hat{\varkappa}^{-1}(L_1 \cup L_2)$. If $\hat{K}_N \cap ((C_1 \cup C_2) \setminus \hat{\varkappa}^{-1}(L_1 \cup L_2)) \neq \emptyset$ then there exists a $\hat{K}_M \subset \hat{K}_N \cap (C_1 \cup C_2)$. Let $\hat{K}_M \cap C_1 \neq \emptyset$. Then there exists $\hat{K}_{M_1} \subset \hat{K}_M \cap C_1 \subset C_1 \setminus \hat{\varkappa}^{-1}L_1$ but this is impossible. Hence $\hat{K}_N \subset \hat{\varkappa}^{-1}(L_1 \cup L_2) \setminus (C_1 \cup C_2)$. Then the inclusion $T_N \subset L_1 \cup L_2$ means that $L_1 \notin N$ for example. From this fact we conclude that there exists an ideal $M \supset N$ such that $\emptyset \neq T_M \subset L_1$. Therefore $\hat{K}_M \subset \hat{\varkappa}^{-1}L_1 \cap \hat{K}_N \subset \hat{\varkappa}^{-1}L_1 \setminus C_1$ but this is false. As a result we get $k(C_1 \cup C_2) = L_1 \cup L_2$.

Assume that $\hat{K}_N \subset (C_1 \cap C_2) \triangle \hat{\varkappa}^{-1}(L_1 \cap L_2)$. If $\hat{K}_N \cap (C_1 \cap C_2 \setminus \hat{\varkappa}^{-1}(L_1 \cap L_2)) \neq \emptyset$ then there exists a $\hat{K}_M \subset \hat{K}_N \cap (C_1 \cap C_2)$. This implies $\hat{K}_M \cap \hat{\varkappa}^{-1}L_1 \neq \emptyset$. Assume $L_1 \in M$. Then there exists an ideal $M_1 \supset M$ such that $\emptyset \neq T_{M_1} \subset T \setminus L_1$. So $\hat{K}_{M_1} \subset \hat{K}_M \cap (\hat{K} \setminus \hat{\varkappa}^{-1}L_1) \subset C_1 \setminus \hat{\varkappa}^{-1}L_1$ but this is false. Hence $L_1 \notin M$. But then similarly $\emptyset \neq T_{M_2} \subset L_1$ for some $M_2 \supset M$. So $\hat{K}_{M_2} \subset \hat{K}_M \cap \hat{\varkappa}^{-1}L_1$. As $\hat{K}_M \cap \hat{\varkappa}^{-1}(L_1 \cap L_2) = \emptyset$ we get $\hat{K}_{M_2} \subset C_2 \setminus \hat{\varkappa}^{-1}L_2$. It follows from this contradiction that

$$\hat{K}_N \subset \hat{\varkappa}^{-1}(L_1 \cap L_2) \setminus (C_1 \cap C_2).$$

Then $\hat{K}_N \cap C_1 \neq \emptyset$. So there exists a $\hat{K}_M \subset \hat{K}_N \cap C_1$. But then $\hat{K}_M \subset \hat{\varkappa}^{-1}L_2 \setminus C_2$. As this is impossible we get as a result that $k(C_1 \cap C_2) = L_1 \cap L_2$. It is clear that k preserves the unit.

Let $C \neq \emptyset$. Then $C \cap \hat{K}_N \neq \emptyset$ for some ideal N. So there exists a $\hat{K}_M \subset \hat{K}_N \cap C$. This implies $kC \neq \emptyset$. Conversely, let $kC \neq \emptyset$. Take a point $t \in kC$ and the point ideal N_t. Then $\hat{K}_{N_t} \subset \hat{\varkappa}^{-1}kC$ implies $C \neq \emptyset$.

Check that $\mathrm{cl}\, \hat{\varkappa}C = \mathrm{cl}\, kC$. Denote the left-hand set by P and the right-hand set by Q. Let $s \in C$ and $t \equiv \hat{\varkappa}s \notin Q$. Then there exists a cozero-set G such that $t \in G \subset T \setminus Q$. Denote $C_1 \equiv \hat{\varkappa}^{-1}G$. We have $\emptyset = kC \cap G = k(C \cap C_1) \neq \emptyset$. From this contradiction we get $\hat{\varkappa}C \subset Q$. Now assume that there exists a cozero-set G such that $G \cap P = \emptyset$ and $G \cap Q \neq \emptyset$. Denote $C_1 \equiv \hat{\varkappa}^{-1}G$. We get $C \cap C_1 \neq \emptyset$ because of $k(C_1 \cap C) = G \cap kC \neq \emptyset$. But this is false.

Now let $\{K, \varkappa\colon K \to T,\ T_N \mapsto K_N,\ \mathscr{C}_0(K)\}$ be a preimage of T with the properties from (2). Let L be a Lebesgue set. Then there exist K_σ-sets $E_N \equiv \bigcup F_j \subset L$ and co K_σ-sets $H_N \equiv \bigcap G_k \supset L$ such that $H_N \setminus E_N \in N$. Consider the cozero-sets $C_N \equiv \bigcup \mathrm{int}\, \varkappa^{-1}F_j \in \mathscr{C}_0(K)$ and the zero-sets $Z_N \equiv \bigcap \mathrm{cl}\, \varkappa^{-1}G_k \in \mathscr{Z}_0(K)$. It is evident that $\bigcup C_N \subset \bigcap Z_N$. Assume that $K_M \subset \varkappa^{-1}E_N \setminus C_N$ for some M. Then $T_M \subset \bigcup F_j$ means that $F_j \notin M$ for some j. So there exists $\emptyset \neq T_{M_1} \subset F_j$ for some $M_1 \supset M$. Therefore $K_{M_1} \subset \varkappa^{-1}F_j \setminus \mathrm{int}\, \varkappa^{-1}F_j$ but this contradicts to the lower disjointness of K. Thus $K_M \not\subset \varkappa^{-1}E_N \setminus C_N$ for any M. Similarly $K_M \not\subset Z_N \setminus \varkappa^{-1}H_N$ for any M.

Assume that $K_M \subset Z_N \setminus C_N$ for some $M \supset N$. If $E_N \notin M$ then there exists an ideal $M_1 \supset M$ such that $T_{M_1} \subset E_N$. So $K_M \subset \varkappa^{-1}E_N \setminus C_N$ but this is false. Thus $E_N \in M$. Since $H_N \setminus E_N \in M$ we get $T \setminus H_N \notin M$. Then there exists an ideal $M_2 \supset M$ such that $T_{M_2} \subset T \setminus H_N$. So $K_{M_2} \subset Z_N \setminus \varkappa^{-1}H_N$ but this is false. Thus we get $K_M \not\subset Z_N \setminus C_N$ for any $M \supset N$.

By the property of the collectively σ-separation there exists a set $U\in\Delta_0(K)$ such that $\bigcup C_N\subset U\subset\bigcap Z_N$. For a given ideal N consider the Kelley ideals $M_j\equiv\equiv\{P\in\mathscr{P}(T)|P\cap F_j\in N\}$ and $M_k\equiv\{P\in\mathscr{P}(T)|P\cap(T\setminus G_k)\in N\}$. By virtue of the lower disjointness of the preimage K we get $K_j\equiv K_{M_j}\subset\operatorname{int}\varkappa^{-1}F_j$ and $K_k\equiv K_{M_k}\subset\subset K\setminus\operatorname{cl}\varkappa^{-1}G_k$. Consequently, $\bigcup K_j\subset U\subset K\setminus\bigcup K_k$.

Assume that for L there exist another K_σ-set $E'_N\equiv\bigcup F_p\subset L$ and co K_σ-set $H'_N\equiv\bigcap G_q\supset L$ such that $H'_N\setminus E'_N\in N$. Let M_p, M_q, K_p, K_q and U' be the corresponding sets. Consider the ideals $M_{jp}\equiv\{P\in\mathscr{P}(T)|P\cap(F_j\cap F_p)\in N\}$ and $M_{kq}\equiv\equiv\{P\in\mathscr{P}(T)|P\cap((T\setminus G_k)\cap(T\setminus G_q))\in N\}$. Denote the sets $K_{M_{rs}}$ by K_{rs}. Then $K_{rs}\subset\subset K_r\cap K_s$. Therefore $K_{jp}\subset U\subset K\setminus\bigcup K_{kq}$ and similarly $K_{jp}\subset U'\subset K\setminus\bigcup K_{kq}$. Hence $U\triangle U'\subset K\setminus((\bigcup K_{jp})\cup(\bigcup K_{kq}))$. As $(\bigcap M_{jp})\cap(\bigcap M_{kq})=N$ and the preimage K is filled the set $(\bigcup K_{jp})\cup(\bigcup K_{kq})$ is dense in K_N. That is why $(U\triangle U')\cap K_N=\emptyset$. As this condition is fulfilled for any N we get $U=U'$.

Thus we can define correctly the mapping i: $\mathscr{L}(T)\to\Delta(K)$ by setting $iL\equiv U$. Check that this mapping is a homomorphism of Boolean algebras. Let $iL_1=U_1$, $iL_2=U_2$ and $i(L_1\cup L_2)=U$. Take some representations $\bigcup F_j\subset L_1\subset\bigcap G_k$ and $\bigcup F_p\subset L_2\subset\bigcap G_q$. Consider the sets $F_{jp}\equiv F_j\cap F_p$, $F_{jq}\equiv F_j\cap(T\setminus G_q)$, $F_{kp}=(T\setminus G_k)\cap\cap F_p$ and $F_{kq}\equiv(T\setminus G_k)\cap(T\setminus G_q)$. Consider the ideals $M_{rs}\equiv\{P\in\mathscr{P}(T)|P\cap F_{rs}\in N\}$ for $r,s\in\{j,k,p,q\}$. Denote the sets $K_{M_{rs}}$ by K_{rs} and the set $(\bigcup K_{jp})\cup(\bigcup K_{jq})\cup\cup(\bigcup K_{kp})\cup(\bigcup K_{kq})$ by Q_N. Then by the definition of the mapping i we have

$$(\bigcup K_{jp})\cup(\bigcup K_{jq})\subset U_1\subset K\setminus((\bigcup K_{kp})\cup(\bigcup K_{kq})),$$

$$(\bigcup K_{jp})\cup(\bigcup K_{kp})\subset U_2\subset K\setminus((\bigcup K_{jq})\cup(\bigcup K_{kq}))$$

and

$$(\bigcup K_{jp})\cup(\bigcup K_{jq})\cup(\bigcup K_{kp})\subset U\subset K\setminus\bigcup K_{kq}.$$

This implies

$$P\equiv U\triangle(U_1\cup U_2)\subset K\setminus Q_N.$$

As

$$(\bigcap M_{jp})\cap(\bigcap M_{jq})\cap(\bigcap M_{kp})\cap(\bigcap M_{kq})=N$$

the set Q_N is dense in K_N. Then $P\cap K_N=\emptyset$ and as a result $P=\emptyset$. Hence i preserves the supremum.

Let $U=iT$. Then $\bigcup K_j\subset U\subset K\subset K\setminus\bigcup K_k$ implies that $U\cap K_N=K_N$ for any N. So $U=K$. Hence i preserves the unit. It is evident that i preserves the complement.

If $L\neq\emptyset$ then for some point $t\in L$ we consider the point ideal $N\equiv\{P\in\mathscr{P}(T)|t\notin P\}$. In this case $M_j=N$ implies that $K_j\neq\emptyset$. So $U\neq\emptyset$. Thus i is injective.

Consider the unit preserving lattice homomorphism α: $\mathscr{C}_0(\hat{K})\to\Delta_0(K)$ such that $\alpha\equiv i\circ k$. Let $t\in K$. Consider the sets $\Gamma\equiv\{C\in\mathscr{C}_0(\hat{K})|t\in\alpha C\}$ and $P\equiv\mathfrak{K}^{-1}\varkappa t$. Assume that $P\cap\operatorname{cl}C=\emptyset$ for some $C\in\Gamma$. Then $\varkappa t\in G\equiv T\setminus\operatorname{cl}\mathfrak{K}C$ implies $t\in\varkappa^{-1}G\subset iG$. As $G\cap\operatorname{cl}kC=\emptyset$ we have $iG\cap\alpha C=\emptyset$ but this is false. Hence $P\cap\operatorname{cl}C\neq\emptyset$ for any $C\in\Gamma$. Let $C_1,C_2\in\Gamma$. Then $C_1\cap C_2\in\Gamma$ implies $P\cap\operatorname{cl}C_1\cap\operatorname{cl}C_2\neq\emptyset$. Therefore $P_t\equiv\bigcap\{\operatorname{cl}C\cap P|C\in\Gamma\}\neq\emptyset$ because of the compactness of P. Assume that there exist points $s_1,s_2\in P_t$. As the base $\mathscr{C}_0(\hat{K})$ is completely normal we can deduce that there exist cozero-sets C_1 and C_2 from this base such that $s_1\in C_1$, $s_2\in C_2$, $s_1\notin\operatorname{cl}C_2$, $s_2\notin\operatorname{cl}C_1$ and $C_1\cup C_2=\hat{K}$. Then $K=\alpha\hat{K}=\alpha C_1\cup\alpha C_2$. Assume that $t\in\alpha C_1$. Then $s_2\in\operatorname{cl}C_1$ but this is false. This means that the set P_t consists of only one point. So we can define correctly the mapping γ: $K\to\hat{K}$ by means of the equality $\gamma t\equiv P_t$.

This mapping is continuous. In fact let G be a neighbourhood of the point $s \equiv \gamma t$. Consider a cozero-set C from our base such that $s \in C \subset \mathrm{cl}\, C \subset G$. We can deduce that there exists a cozero-set C_1 from the base such that $s \notin \mathrm{cl}\, C_1$ and $C \cup C_1 = \hat{K}$. This implies $K = \alpha C \cup \alpha C_1$. The assumption $t \in \alpha C_1$ implies $s \in \mathrm{cl}\, C_1$ but this is false. Hence $t \in \alpha C$. Let $t_1 \in \alpha C$. Then $\gamma t_1 \in \mathrm{cl}\, C \subset G$. As the set αC is open we obtain the continuity of γ.

This mapping is surjective. In fact consider a point $s \in \hat{K}$, the set

$$\Gamma \equiv \{C \in \mathscr{C}_0(\hat{K}) | s \in C\}$$

and the set $P \equiv \varkappa^{-1} \hat{\varkappa} s$. Assume that $\alpha C \cap P = \emptyset$ for some $C \in \Gamma$. Then there exists a cozero-set G such that $\hat{\varkappa} s \in G \subset T \setminus \varkappa \alpha C$. So $s \in \hat{\varkappa}^{-1} G \equiv C_1$. On the other hand $\varkappa^{-1} G \cap \alpha C = \emptyset$ implies $\alpha(C_1 \cap C) = ikC_1 \cap \alpha C = iG \cap \alpha C = \emptyset$. This has as a consequence $C_1 \cap C = \emptyset$ but this is false. It follows from this contradiction that $\alpha C \cap P \neq \emptyset$ for any $C \in \Gamma$. Therefore there exists a point $t \in \bigcap \{\alpha C \cap P | C \in \Gamma\}$. Consequently, $\gamma t \in \bigcap \{\mathrm{cl}\, C | C \in \Gamma\} = s$.

From the definition of the mapping γ we conclude that $\hat{\varkappa} \circ \gamma = \varkappa$. This has as a consequence that this mapping is perfect ([8], VI, § 2, 56).

Prove that $\gamma K_N = \hat{K}_N$. Assume that there exists a cozero-set $C \in \mathscr{C}_0(\hat{K})$ such that $C \cap \hat{K}_N \neq \emptyset$ and $\mathrm{cl}\, C \cap \gamma K_N = \emptyset$. Assume that there exists a point $t \in \alpha C \cap K_N$. Then $\gamma t \in \mathrm{cl}\, C \cap \gamma K_N = \emptyset$ but this is impossible. Therefore $\alpha C \cap K_N = \emptyset$. Consider the set $L \equiv kC$. Consider for L the corresponding sets F_N^j and $K_j \subset K_N$ defined above. Then $\bigcup K_j \subset iL$ implies $K_j = \emptyset$. So $T_{M_j} = \emptyset$ means $F_N^j \in N$. Therefore $L \in N$. On the other hand there exists an ideal $M \supset N$ such that $\hat{K}_M \subset C \cap \hat{K}_N$. As $L \in M$ there exists an ideal $M_1 \supset M$ such that $T_{M_1} \subset T \setminus L$. That is why $\hat{K}_{M_1} \subset (\hat{K} \setminus \hat{\varkappa}^{-1} L) \cap \cap \hat{K}_M \subset C \setminus \hat{\varkappa}^{-1} L$ but this is impossible. We conclude from this contradiction that $\hat{K}_N \subset \gamma K_N$.

Conversely assume that there exists a cozero-set C such that $C \cap \gamma K_N \neq \emptyset$ and $C \cap \hat{K}_N = \emptyset$. Consider the set $L \equiv kC$. Assume that $L \notin N$. Then there exists an ideal $M \supset N$ such that $T_M \subset L$. This implies $\hat{K}_M \subset \hat{\varkappa}^{-1} L \cap \hat{K}_N \subset \hat{\varkappa}^{-1} L \setminus C$ but this is impossible. So $L \in N$. Consider for L the corresponding sets E_N, C_N and Z_N defined above. We can suppose that $E_N = \emptyset$. Then $C_N = \emptyset$ means that $Z_N \cap K_N$ is nowhere dense in K_N. Therefore $iL \cap K_N = \emptyset$. Let t be a point of $\gamma^{-1} C$. Then there exists a cozero-set C_1 from the base such that $\gamma t \notin \mathrm{cl}\, C_1$ and $C \cup C_1 = \hat{K}$. Then $\alpha C \cup \alpha C_1 = K$ shows that $t \in \alpha C$. That is why $\gamma^{-1} C \subset \alpha C$. So we get $C \cap \gamma K_N = \emptyset$ but this contradicts to our assumption.

Thus K is larger than $\hat{K}$. Now let K denote the Lebesguean cover of T. As the Lebesguean cover has the properties from (1) and (2) simultaneously we get as a result that the Lebesguean cover is the largest of all the preimages with the properties from (1) and the smallest of all the preimages with the properties from (2).

Let $\hat{K}$ be some other largest preimage of T. Then there are mappings $\gamma\colon K \to \hat{K}$ and $\delta\colon \hat{K} \to K$ such that K is larger than $\hat{K}$ relative to γ and $\hat{K}$ is larger than K relative to δ. Let $t \in K_N$. Then $t = \bigcap \{K_M\}$. This implies $\gamma t \in \bigcap \{\hat{K}_M\}$ and $\delta \gamma t \in \bigcap \{K_M\} = t$. As $\bigcup K_N$ is dense we conclude that $\delta \circ \gamma = \mathrm{id}$. It means that γ and δ are mutually inverse homeomorphisms and so the preimages K and $\hat{K}$ are isomorphic.

The uniqueness of the smallest preimage and assertion (3) are checked in a similar manner. The theorem is proved.

This Theorem will be used for the proof of the following Theorems 2 and 3.

Now with the help of this Theorem we shall give a functional characterization of the Lebesguean cover.

Let $\{K, \varkappa\colon K\to T, T_N\mapsto K_N, \mathscr{C}_0(K)\}$ be a perfect saturated preimage of T lifting Kelley covering and having a completely normal base.

Let P and Q be subsets of K. If $P\setminus Q\not\supset K_N$ for any N we shall say that P is *almost contained in* Q and write $P\underset{\sim}{\subset} Q$.

Let $\{C_k\}\subset\mathscr{C}_0(K)$ be a finite covering of the space K and $\{L_k\}\subset\mathscr{L}(T)$ be a finite covering of the space T. The family $\{C_k, \varkappa^{-1}L_k\}$ will be called *a cohesive covering of the space* K if $\varkappa^{-1}L_k\underset{\sim}{\subset} C_k$ for any k. Note that for any cohesive covering $\{C_k, \varkappa^{-1}L_k\}$ the set $\bigcup(\varkappa^{-1}L_k\cap C_k)$ is dense in K.

LEMMA 9. *Let* $\{C_j, \varkappa^{-1}L_j\}$ *and* $\{C_k, \varkappa^{-1}L_k\}$ *be cohesive coverings. Then the family* $\{C_j\cap C_k, \varkappa^{-1}(L_j\cap L_k)\}$ *is a cohesive covering, too.*

PROOF. Assume that there exists an ideal N such that $K_N\subset\varkappa^{-1}(L_j\cap L_k)\setminus(C_j\cap C_k)$. As $K_N\cap C_j\neq\emptyset$ there exists an ideal $M\supset N$ such that $K_M\subset K_N\cap C_j$. This implies $K_M\subset\varkappa^{-1}L_k\setminus C_k$ but this is impossible. It follows from this contradiction that $\varkappa^{-1}(L_j\cap L_k)\underset{\sim}{\subset} C_j\cap C_k$.

Further for a natural number n the number $\dfrac{1}{n}$ will be denoted by u_n.

Let f and g be functions on K. The functions f and g will be called *equivalent* if for any n there exists a cohesive covering $\{C_k, \varkappa^{-1}L_k\}$ of K such that $|f(s)-g(s)|<u_n$ for any $s\notin\bigcup(\varkappa^{-1}L_k\setminus C_k)$. In this case we shall write $f\sim g$. It follows from the previous lemma that this relation "$\sim$" is indeed an equivalence relation.

Consider on K the set $C_0^*(K)$ of all functions $f\in C^*(K)$ such that

$$f^{-1}(]a, b[)\in\mathscr{C}_0(K)$$

for any open interval $]a, b[$. It follows from the theorem of Alexandrov ([13]) that $C_0^*(K)$ is a uniformly complete vector lattice and $\mathscr{C}_0(K)=\{\operatorname{coz} f\,|\, f\in C_0^*(K)\}$.

A function x on T will be called *a Lebesgue function* (or universally measurable [6]) if $x^{-1}(]a, b[)\in\mathscr{L}(T)$ for any open interval $]a, b[$. The set of all bounded Lebesgue functions on T will be denoted by $L^*(T)$.

There holds the following functional description of the Lebesgue determinedness.

LEMMA 10. *The following assertions are equivalent.*

(a) *K is Lebesgue determined;*

(b) *for any function* $f\in C_0^*(K)$ *there is a (unique) Lebesgue function* $x\in L^*(T)$ *such that* $f\sim x\circ\varkappa$.

PROOF. Denote $L^*(T)$ by X and $C_0^*(K)$ by Φ. Let K be Lebesgue determined. Consider a function $0\equiv f\in\Phi$. Divide an interval, containing the range of the function f, by points a_{mj} so that $a_{mj+1}-a_{mj}=u_m/6$. Then for any $C_{mj}\equiv f^{-1}(]a_{mj-1}, a_{mj+1}[)$ there exists a Lebesgue set L'_{mj} such that $\varkappa^{-1}L_{mj}\triangle C_{mj}\not\supset K_N$ for any N. Assume that there exists a point $t\notin\bigcup L'_{mj}$. As $K_{N_t}\cap C_{mj}\neq\emptyset$ for some j and for the point ideal N_t we have $K_t\equiv K_{N_t}\subset C_{mj}$. So $K_t\subset C_{mj}\setminus\varkappa^{-1}L'_{mj}$ but this is impossible. This means that $\{L'_{mj}\}$ is a covering of T for any m. Consider the sets $L_{mj}\equiv L'_{mj}\setminus\bigcup$

$\cup\{L'_{mi}|i<j\}$. Then $\{C_{mj}, \varkappa^{-1}L_{mj}\}$ is a cohesive covering. Consider the Lebesgue step function $x_m \equiv \sum a_{mj}\chi(L_{mj})$. Let $x_n \equiv \sum a_{nk}\chi(L_{nk})$. Take a point $t \in T$. Then $t \in L_{mj} \cap L_{nk}$ for some j and k. So $K_t \subset \varkappa^{-1}L_{mj}$ implies $K_t \subset C_{mj}$. Similarly $K_t \subset C_{nk}$. Take a point $s \in K_t$. Then we have $|x_m(t)-x_n(t)| = |a_{mj}-a_{nk}| \leq |a_{mj}-f(s)| + |f(s)-a_{nk}| < u_m/3$ for $n \geq m$. That is why there exists a Lebesgue function x such that $|x(t)-x_n(t)| < 2u_n/3$. Let $s \notin P_n \equiv \cup(\varkappa^{-1}L_{nk} \setminus C_{nk})$. Then $|f(s)-x\circ\varkappa(s)| < u_n$. Thus $f \sim x\circ\varkappa$.

Assume that there exists another function $x' \in X$ having this property. Then for any n there exists a cohesive covering $\{C_l, B_l\}$ such that $|f(s)-x'\circ\varkappa(s)| < u_n$ for any $s \notin Q_n \equiv \cup(\varkappa^{-1}L_l \setminus C_l)$. Take a point $t \in T$. Then $t \in L_{nk} \cap L_l$ for some indices. Therefore $K_t \subset \varkappa^{-1}(L_{nk} \cap L_l)$. By virtue of Lemma 9 there exists a point $s \in K_t \cap \varkappa^{-1}(L_{nk} \cap L_l) \cap (C_{nk} \cap C_l)$. So $\varkappa s = t$. Besides $s \notin P_n \cup Q_n$. Hence we get $|x(t)-x'(t)| < 2u_n$. Consequently, $x = x'$.

Now let C be a cozero-set from the base. Then $C = \operatorname{coz} f$ for some function $0 \leqq f \in \Phi$. Consider the function f the corresponding Lebesgue function $x \geqq 0$ such that $f \sim x\circ\varkappa$. Consider the Lebesgue set $L \equiv \operatorname{coz} x$. Assume that $K_N \subset \varkappa^{-1}L \triangle C \equiv Q$. If $K_N \cap (C \setminus \varkappa^{-1}L) \neq \emptyset$ there exists an ideal $M \supset N$ such that $K_M \subset C \setminus \varkappa^{-1}L$. Consider the sets $C_n \equiv \{s \in K \mid f(s) > u_n\}$. For some n there exists an ideal $M_1 \supset M$ such that $K_{M_1} \subset C_n \cap K_M \subset C_n \setminus \varkappa^{-1}L$. But for this n there exists a cohesive covering $\{C_k, \varkappa^{-1}L_k\}$ such that $|f(s)-x\circ\varkappa(s)| < u_n$ for any $s \notin P_n \equiv \cup(\varkappa^{-1}L_k \setminus C_k)$. It is clear that $K_{M_1} \subset P_n$. As $L_k \notin M_1$ for some k there exists an ideal $M_2 \supset M_1$ such that $T_{M_2} \subset L_k$. So $K_{M_2} \subset \varkappa^{-1}L_k \setminus C_k$ but this is impossible. Consequently $K_N \subset \varkappa^{-1}L \setminus C$. Consider the sets $L_n \equiv \{t \in T \mid x(t) > u_n\}$. Then $L_n \notin N$ for some n. Hence there exists an ideal $M \supset N$ such that $T_M \subset L_n$. Therefore $K_M \subset \varkappa^{-1}L_n \setminus C$. This implies $K_M \subset P_n$. But as it was shown above this is impossible. From this contradiction we get that $\varkappa^{-1}L \triangle C \not\supset K_N$ for any N. So K is Lebesgue determined. The lemma is proved.

PROPOSITION 1. *Let K be the Lebesguean cover of T. Then*

(a) $\{K, \varkappa\colon K \to T,\ T_N \mapsto K_N,\ \mathscr{C}_0(K)\}$ *is a perfect saturated preimage of T lifting Kelley covering and having a completely normal base;*

(b) *there is a bijection $r\colon x \mapsto f$ between the family $L^*(T)$ and the family $C_0^*(K)$ such that $f \sim x\circ\varkappa$;*

(c) *K as a preimage of T lifting Kelley covering is completely determined (up to isomorphism) by the properties* (a)—(b).

PROOF. Denote $L^*(T)$ by X and $C_0^*(K)$ by Φ. Let $0 \leqq x \in X$. Then there exist step functions $x_n \in X$ such that $|x(t)-x_n(t)| < u_n$ for any t and $x_n \equiv \sum a_k\chi(L_k)$ for some Lebesgue partitions $\{L_k\}$ of T. Denote the set iL_k by U_k. Consider the functions $f_n \equiv \sum a_k\chi(U_k) \in \Phi$. Let $0 \leqq f \in \Phi$ be a uniform limit of the sequence f_n. By virtue of Corollary 2 of Lemma 3 the family $\{U_k, \varkappa^{-1}L_k\}$ is a cohesive covering. Let $s \notin P_n \equiv \cup(\varkappa^{-1}L_k \setminus U_k)$. Then $|f(s)-x\circ\varkappa(s)| < 3u_n$. Hence $f \sim x\circ\varkappa$.

Assume that there is another function $f' \in \Phi$ satisfying this condition, i.e. for any n there exists a cohesive covering $\{C_l, \varkappa^{-1}L_l\}$ such that $|f'(s)-x\circ\varkappa(s)| < u_n$ for any $s \notin Q_n \equiv \cup(\varkappa^{-1}L_l \setminus C_l)$. We can suppose that $\{L_l\}$ is a partition. Take a point $s \in \cup(\varkappa^{-1}(L_k \cap L_l) \cap (U_k \cap C_l)) \equiv R_n$. Then $s \notin P_n \cup Q_n$ implies that $|f(s)-f'(s)| < 4u_n$. By virtue of Lemma 9 and the density of the set R_n we conclude that this inequality is valid for any $s \in K$. As a result we get $f = f'$.

Thus the mapping $r\colon x \to f$ is defined correctly. By virtue of Lemma 10 we get that this mapping is bijective.

Now let $\{\hat{K}, \hat{\varkappa}\colon \hat{K} \to T, T_N \mapsto \hat{K}_N, \mathscr{C}_0(\hat{K})\}$ be a preimage of T with the properties from (a) and (b). Denote the vector lattice $C_0^*(\hat{K})$ by Φ. It is easy to check that the mapping $\hat{r}\colon X \to \Phi$ is an isomorphism of vector lattices. By virtue of Lemma 10 $\hat{K}$ is Lebesgue determined.

Let G be an open set from T and $x \equiv \chi(G)$. Then $x = \sup\{f_r \in C^*(T) | f_r \leqq x\}$ in X. Consider the function $f \equiv \hat{r}x$. Then $f = \sup\{\hat{r}f_r\} = \sup\{f_r \circ \hat{\varkappa}\} = \chi(U)$, where $U \equiv \mathrm{cl}\, \hat{\varkappa}^{-1}G$. Hence $U \in \Delta_0(\hat{K})$. This means that $\hat{K}$ is lower extremally disconnected. Let $\hat{\varkappa}^{-1}G \cap \hat{K}_N = \emptyset$. Assume $U \cap \hat{K}_N \neq \emptyset$. By virtue of the saturatedness there exists an ideal $M \supset N$ such that $\hat{K}_M \subset U \cap \hat{K}_N \subset U \setminus \hat{\varkappa}^{-1}G$. But it follows from the proof of Lemma 10 that $U \triangle \hat{\varkappa}^{-1}G \not\supset \hat{K}_M$ for any M. From this contradiction we conclude that the preimage $\hat{K}$ is lower disjoined.

Take a $C \in \mathscr{C}_0(K)$. Then $C = \mathrm{coz}\, f$ for some function $f = \hat{r}x$. Consider the sets $L \equiv \mathrm{coz}\, x$, $U \equiv \mathrm{cl}\, C$ and the functions $y \equiv \sup\{nx \wedge 1 | n\}$, $g \equiv \hat{r}y$. Then $y = \chi(L)$, $g = \sup\{nf \wedge 1\} = \chi(U)$ and $U \in \Delta_0(\hat{K})$. It follows from the proof of Lemma 10 that $U \triangle \hat{\varkappa}^{-1}L \not\supset \hat{K}_M$ for any M.

Take a $Z \in \mathscr{Z}_0(\hat{K})$. It follows from above that $V \equiv \mathrm{int}\, Z \in \Delta_0(\hat{K})$ and there are some Lebesgue set L' and some functions h, z such that $z = \chi(L')$, $h = \chi(V)$, $h = \hat{r}z$ and $V \triangle \hat{\varkappa}^{-1}L' \not\supset \hat{K}_M$ for any M.

Now let $\{C_N, Z_N\}$ be a family of cozero and zero-sets such that $\bigcup C_N \subset \bigcap Z_N$ and $Z_N \setminus C_N \not\supset \hat{K}_M$ for any $M \supset N$. Consider the open-closed sets $U_N \equiv \mathrm{cl}\, C_N$ and $V_N \equiv \mathrm{int}\, Z_N$. As it was shown above there are some sets L_N, L'_N and some functions g_N, h_N, y_N, z_N such that $y_N = \chi(L_N)$, $z_N = X(L'_N)$, $g_N = \chi(U_N)$, $h_N = \chi(V_N)$, $g_N = \hat{r}y_N$, $h_N = \hat{r}z_N$, $U_N \triangle \hat{\varkappa}^{-1}L_N \not\supset \hat{K}_M$ and $V_N \triangle \hat{\varkappa}^{-1}L'_N \not\supset \hat{K}_M$ for any M. Assume that $P \equiv L'_N \setminus L_N \notin N$. Then there exists an ideal $M \supset N$ such that $T_M \subset P$. So $\hat{K}_M \subset (\hat{\varkappa}^{-1}L'_N \setminus \hat{\varkappa}^{-1}L_N) \cap \hat{K}_N$. Therefore $\hat{K}_M \cap V_N \neq \emptyset$ and $\hat{K}_M \not\subset U_N$. By virtue of the saturatedness there exists an ideal $M_1 \supset M$ such that $\hat{K}_{M_1} \subset V_N \setminus U_N \subset Z_N \setminus C_N$ but this is impossible. From this contradiction we conclude $P \in N$. So $(iL'_N \setminus iL_N) \cap \cap K_N = \emptyset$. This implies $iL'_N \setminus iL_N \not\supset K_M$ for any $M \supset N$. As $\bigcup iL_N \subset \bigcap iL'_N$, by virtue of Lemma 7 and the remark before Lemma 6 there is a set $iL \in \Delta_0(K)$ which is situated between these sets. Consider the functions $x \equiv \chi(L)$ and $f \equiv \hat{r}x$. As the homomorphism i is injective we get $g_M \leqq f \leqq h_N$ for any M and N. It is evident that $U \equiv \mathrm{coz}\, f \in \Delta_0(\hat{K})$. Therefore $\bigcup U_M \subset U \subset \bigcap V_N$. Thus the preimage $\hat{K}$ is collectively σ-separated.

On the strength of Theorem 1 we conclude that the preimages K and $\hat{K}$ are isomorphic. The proposition is proved.

This proposition also will be used in the sequel.

§ 2. Vector lattice of Lebesgue functions

Let T be a completely regular space and $L^*(T)$ be the vector lattice of all bounded Lebesgue functions on T introduced in § 1. Let $u\colon C^*(T) \to L^*(T)$ be the canonical imbedding.

For a Kelley ideal N consider the ideal $L_N^*(T) \equiv \{x \in L^*(T) | \mathrm{coz}\, x \in N\}$. Then $\{L^*(T), u\colon C^*(T) \to L^*(T), C_N^*(T) \mapsto L_N^*(T)\}$ is an extension of $C^*(T)$ inheriting

Lebesgue decomposition. This extension will be called *the Lebesguean extension of* $C^*(T)$.

Let K be the Lebesguean cover of T and $\varkappa$: $K\to T$ be the canonical mapping. Let $\Phi\equiv C_0^*(K)$ be the vector lattice of functions on K defined in § 1. Consider the injective vector-lattice homomorphism φ: $C^*(T)\to\Phi$ such that $\varphi f\equiv f\circ\varkappa$. For a Kelley ideal N consider the ideal $\Phi_N\equiv\{f\in\Phi|\,f(K_N)=0\}$. Then

$$\{\Phi, \varphi\colon C^*(T)\to\Phi,\ C_N^*(T)\mapsto\Phi_N\}$$

is an extension of $C^*(T)$ inheriting Lebesgue decomposition.

Now let $\{X, u\colon C^*(T)\to X,\ C_N^*(T)\mapsto X_N\}$ be a vector-lattice extension of $C^*(T)$ inheriting Lebesgue decomposition. Identify $C^*(T)$ with its image in X.

Let $x\in X$ and $\{x_\xi\}\subset X$. The element x will be called *the d-supremum of the set* $\{x_\xi\}$ if $x\geqq x_\xi$ and for any X_N we have $\bar{x}=\sup\bar{x}_\xi$ in X/X_N. In this case we shall write $x=d-\sup x_\xi$. The element x will be called *the* d_N*-supremum of the set* $\{x_\xi\}$ if $x\geqq x_\xi$ and $\bar{x}=\sup\bar{x}_\xi$ in X/X_M for any $M\supset N$. In this case we shall write $x=d_N-\sup x_\xi$. In a similar way *the d-infimum* and *the* d_N*-infimum of the set* $\{x_\xi\}$ is defined.

Consider the sets $S_l(C^*(T), X)\equiv\{x\in X|\exists f_\xi\in C^*(T)\ (x=d-\sup f_\xi)\}$ and $S_u(C^*(T), X)\equiv\{x\in X|\exists f_\xi\in C^*(T)\ (x=d-\inf f_\xi)\}$.

The extension X will be called *Lebesgue generated* if for any $x\in X$ and for any Kelley ideal $N\in\mathcal{N}(T)$ there exist sequences $y_N^k\in S_u(C^*(T), X)$ and $z_N^k\in S_l(C^*(T), X)$ such that $y_M^j\leqq x\leqq z_N^k$ for any indices and $x=d_N-\sup y_N^k=$ $=d_N-\inf z_N^k$.

For the extensions $L^*(T)$ and Φ consider the mapping r: $L^*(T)\to\Phi$ from Proposition 1 of the previous paragraph.

Proposition 2. *With respect to the mapping r the extensions $L^*(T)$ and Φ are isomorphic saturated Lebesgue generated lower Dedekind complete lower component collectively σ-complete extensions of $C^*(T)$ inheriting Lebesgue decomposition.*

Proof. Denote $L^*(T)$ by X and $L_N^*(T)$ by X_N. It can be verified that $r\circ u=\varphi$ and r is a vector-lattice isomorphism. Let $x\in X$ and $f\equiv rx$. Consider the sets $L\equiv\operatorname{coz}x$ and $C\equiv\operatorname{coz}f$. As it was established in the proof of Lemma 10 $C\triangle\varkappa^{-1}L\not\supset$ $\not\supset K_M$ for any M. Let $x\in X_N$ and assume that $f\notin\Phi_N$. By virtue of the saturatedness there exists an ideal $M\supset N$ such that $K_M\subset C\cap K_N$. As $T\setminus L\notin M$ there exists a Kelley ideal $M_1\supset M$ such that $T_{M_1}\subset T\setminus L$. Then $K_{M_1}\subset K_M\cap(K\setminus\varkappa^{-1}L)\subset$ $\subset C\setminus\varkappa^{-1}L$ but this is impossible. Thus our assumption is false. Conversely let $f\in\Phi_N$ and assume that $x\notin X_N$. Take an ideal $M\supset N$ such that $T_M\subset L$. Then $K_M\subset\varkappa^{-1}L\setminus C$ but this is impossible, too. Consequently, $x\in X_N$. So the extensions X and Φ are isomorphic.

Let Y be a proper component of X such that $Y^d\not\subset X_N$. Consider the nonempty set $P\equiv\{t\in T|\forall y\in Y(y(t)=0)\}$. Then $Y=\{x\in X|x(P)=0\}$. Consequently $P\notin N$. Therefore there exists a proper ideal $M\supset N$ such that $T_M\subset P$. This implies $X_N\vee Y\subset X_M$. This means that X is saturated.

Now verify that X is Lebesgue generated. Let $S_l^*(T)$ and $S_u^*(T)$ denote the sets of all bounded lower semicontinuous and upper semicontinuous functions on T, respectively. Let $x\in S_l^*(T)$. Then $x(t)=\sup\{f_\xi(t)\}$ for some family $f_\xi\in C^*(T)$. Prove that $x=d-\sup f_\xi$. Consider the functions $g\equiv rx$ and $g_\xi\equiv rf_\xi=f_\xi\circ\varkappa$. Let

$\bar{f}\in\Phi/\Phi_N$ and $\bar{f}\geqq\bar{g}_\xi$ for any index. Then $f(s)\geqq(x\circ\varkappa)(s)$ for any $s\in K_N$. Assume that there exists a point $s\in K_N$ such that $a\equiv g(s)-f(s)>0$. Take a natural number k such that $a>1/k\equiv u_k$. Consider the open set $G\equiv\{r\in K\,|\,g(r)-f(r)>u_k\}$. Then $s\in G\cap K_N\neq\emptyset$. By virtue of Proposition 1 there exists a cohesive covering $\{C_j, \varkappa^{-1}L_j\}$ of K such that $|g(s)-(x\circ\varkappa)(s)|<u_k$ for any $s\notin P\equiv\bigcup(\varkappa^{-1}L_j\setminus C_j)$. We can suppose that $\{L_j\}$ is a partition. Then $G\cap K_N\subset P$. By virtue of the saturatedness of K there exists an ideal $M\supset N$ such that $K_M\subset G\cap K_N$. As $L_j\notin M$ for some j there exists an ideal $M_1\supset M$ such that $Z_{M_1}\subset L_j$. Then $K_{M_1}\subset\varkappa^{-1}L_j\cap P=\varkappa^{-1}L_j\setminus C_j$. But this contradicts to the definition of a cohesive covering. Thus $f(s)\geqq g(s)$ for any $s\in K_N$ means that $\bar{f}\geqq\bar{g}$. So $\bar{g}=\sup\bar{g}_\xi$ implies $\bar{x}=\sup\bar{f}_\xi$ in X/X_N.

As a result we get $S_l^*(T)\subset S_l(C^*(T), X)$. The similar inclusion is valid if we substitute l by u.

Now let $0\leqq x\in X$. Divide an interval, containing the range of x, by points $\equiv_{mj}|\,j\in J_m\}$ so that $a_{mj+1}-a_{mj}=u_m$. Fix an ideal N. Then for any set $L_{mj}\equiv\{ax^{-1}([a_{mj}, a_{mj+1}[)$ there exist an increasing sequence of compact sets F_{mjk} and a decreasing sequence of cocompact sets G_{mjk} such that $E_{mj}\equiv\bigcup_k F_{mjk}\subset L_{mj}$, $H_{mj}\equiv\equiv\bigcap G_{mjk}\supset L_{mj}$ and $P_{mj}\equiv H_{mj}\setminus E_{mj}\in N$. Consider the functions $y_{mjk}\equiv\chi(F_{mjk})$, $z_{mjk}\equiv\chi(G_{mjk})$. $y_{mk}\equiv\sup\{a_{mj}y_{mjk}\,|\,j\in J_m\}\leqq x$ and $z_{mk}\equiv\sup\{a_{mj+1}z_{mjk}\,|\,j\in J_m\}\geqq x$. Consider the set $P_m\equiv\bigcup P_{mj}$. Take a point $t\notin P_m$. Then $t\in L_{mj}$ for some j. Therefore $t\notin H_{mi}$ for any $i\neq j$. Hence $t\notin G_{mik_i}$ for some k_i. Take the number $k\equiv\sup\{k_i|i\in J_m\}$. Then $s\notin G_{mik}$ for any $i\neq j$ and for the given k. Therefore $z_{mk}(t)=a_{mj+1}$. That is why $0\leqq z_{mk}(t)-x(t)\leqq u_m$.

Let $\bar{y}\leqq\overline{z_{mk}}$ in X/X_M for $M\supset N$. Then $y(t)\leqq z_{mk}(t)$ for any $t\notin Q_{mk}$, where $Q_{mk}\in M$. Consider the set $P\equiv(\bigcup P_m)\cup(\bigcup Q_{mk})\in M$. Take a point $t\notin P$. As it was shown above for any m there is a function z_{mk} such that $0\leqq z_{mk}(t)-x(t)\leqq u_m$. Therefore $y(t)-x(t)\leqq u_m$ implies $y(t)\leqq x(t)$. So $\bar{y}\leqq\bar{x}$ means that $\bar{x}=\inf\bar{z}_{mk}$. It is clear that $\bar{x}=\sup\bar{y}_{mk}$. Thus $x=d_N-\sup y_{mk}=d_N-\inf z_{mk}$. This means that X is Lebesgue generated.

Now check that X is collectively σ-complete. Let $\{y_N^k, z_N^k\}$ be a corresponding family in X. We can suppose that the sequence $\{y_N^k\}$ increases and the sequence $\{z_N^k\}$ decreases. Define the function x by setting $x(t)\equiv\sup\{y_N^k(t)\}$. Fix an interval $]a, b[$ and an ideal N. Consider the set $L\equiv x^{-1}(]a, b[)$. Consider the Lebesgue functions $y\equiv\sup\{y_N^k|k\}$ and $z\equiv\inf\{z_N^k|k\}$. Then $P\equiv\mathrm{coz}\,(z-y)\in N$. Consider the Lebesgue sets $L'\equiv y^{-1}(]a, +\infty[)\cap z^{-1}(]-\infty, b[)\subset L$ and $L''\equiv y^{-1}(]-\infty, b[)\cap\cap z^{-1}(]a, +\infty[)\supset L$. It is clear that $(L\setminus L')\cup(L''\setminus L)\subset P$. Take some K_σ-set $E\subset L'$ and some co K_σ-set $H\supset L''$ such that $P'\equiv L'\setminus E\in N$ and $P''\equiv H\setminus L''\in N$. Then $H\setminus E\subset P\cup P'\cup P''\in N$. This means that L is a Lebesgue set and consequently x is a Lebesgue function. As the rest of the properties of X are well-known the proposition is proved.

Further uniqueness is understood up to isomorphism.

THEOREM 2. (1) *$L^*(T)$ is the unique largest of all the saturated Lebesgue generated extensions of $C^*(T)$ inheriting Lebesgue decomposition;*

(2) *$L^*(T)$ is the unique smallest of all the filled lower Dedekind complete lower component collectively σ-complete extensions of $C^*(T)$ inheriting Lebesgue decomposition and moreover $L^*(T)$ is the unique universal among all such extensions;*

(3) *$L^*(T)$ is the unique saturated Lebesgue generated lower Dedekind complete*

lower component collectively σ-complete extension of $C^*(T)$ inheriting Lebesgue decomposition.

PROOF. Let $\{X, u\colon C^*(T)\to X, C_N^*(T)\mapsto X_N\}$ be an extension having the properties from (1). On the strength of Yosida's theorem ([14]) there is a unique compact K_0 such that the vector lattice X is isomorphic to the vector lattice $C(K_0)$ relative to an isomorphism r_0. Then the mapping u generates a unique surjective continuous mapping $\varkappa_0\colon K_0\to\beta T$ such that $r_0u_f=f'\circ\varkappa_0$, where f' denotes the extension of a function $f\in C^*(T)$ on βT.

Consider the space $K\equiv\varkappa_0^{-1}T$ and the perfect mapping $\varkappa\colon K\to T$ which is the restriction of $\varkappa_0$. Consider the vector lattice Φ, consisting of the restrictions on K of all functions from $C(K_0)$, the homomorphism $r\colon X\to\Phi$, such that $rx\equiv r_0x|K$, and the homomorphism $\varphi\colon C^*(T)\to\Phi$ such that $\varphi f\equiv f\circ\varkappa$.

For a Kelley ideal N consider the ideals $\Phi_{0_N}\equiv r_0X_N$ and $\Phi_N\equiv rX_N$ and the closed subsets $K_{0_N}\equiv\{s\in K_0|\forall f\in\Phi_{0_N}(f(s)=0)\}\neq\emptyset$ and $K_N\equiv K_{0_N}\cap K$. Then $\cup K_{0_N}$ is dense in K_0 and $\varkappa_0K_{0_N}=\mathrm{cl}\,T_N$. It is clear that $N_1\subset N_2$ implies $K_{N_1}\supset K_{N_2}$. Take for the ideal N a proper ideal $M\supset N$ such that T_M is compact. Then $K_M=K_{0_M}$. It follows from this fact that $K_N\neq\emptyset$.

Let $f\geqq 0$ be a function from $C(K_0)$ and $f(K_{0_N})=0$. Consider the functions $f_k\equiv(f-u_k1)\vee 0$. From the property $K_{0_N}\cap\mathrm{cl}\,\mathrm{coz}\,f_k=\emptyset$ we conclude that $f_k\in\Phi_{0_N}$. This implies that f belongs to this set also. Thus $\Phi_{0_N}=\{f\in C(K_0)|\,f(K_{0_N})=0\}$.

Let C be the cozero-set of a function $f\in C(K_0)$ such that $C\cap K_{0_N}\neq\emptyset$. Take a sequence of compact sets F_k such that $F_k\notin N$ and $T\setminus\cup F_k\in N$. Consider the proper ideals $N_k\equiv\{P\in\mathscr{P}(T)|P\cap F_k\in N\}$. Then $\cap N_k=N$. As X is filled we have $f\notin\Phi_{0_{N_k}}$ for some k. Therefore $C\cap K_N\supset C\cap K_{0_{N_k}}\neq\emptyset$. This means that K_N is dense in K_{0_N}. As a consequence we get $\Phi_N=\{f\in\Phi|\,f(K_N)=0\}$ and $\varkappa K_N=T_N$.

Besides we established that K is dense in K_0. Hence the triplet

$$\{\Phi, \varphi\colon C^*(T)\to\Phi, C_N^*(T)\mapsto\Phi_N\}$$

is an extension isomorphic to the initial one.

In addition we get that $\cup K_N$ is dense in K. Consequently, K is the preimage of T lifting Kelley covering and having the completely normal base $\mathscr{C}_0(K)\equiv\{\mathrm{coz}\,f|\,f\in\Phi\}$.

Let G be an open set in K and $G\cap K_N\neq\emptyset$. Take a proper regular closed set $F\subset G$ such that $(\mathrm{int}\,F)\cap K_N\neq\emptyset$. Consider the proper component $Y\equiv\{f\in\Phi|\,f(F)=0\}$. As $Y^d\not\subset\Phi_N$ we get by virtue of the saturatedness that there exists an ideal Φ_M containing the set $\Phi_N\vee Y$. This means that $K_M\subset K_N\cap G$. So K is a saturated preimage.

Now let $0\leqq f\in\Phi$ and $C\equiv\mathrm{coz}\,f$. Then there are sequences $g_N^k\in S_u(C^*(T),\Phi)$ and $h_N^k\in S_l(C^*(T),\Phi)$ such that $0\leqq g_N^k\leqq f\leqq h_N^k$ and $f=d_N-\sup_k g_N^k=d_N-\inf_k h_N^k$. Also there are some families of continuous functions such that $g_N^k=d-\inf\{f_{N\xi}^k\circ\varkappa\}$ and $h_N^k=d-\sup\{f_{N\eta}^k\circ\varkappa\}$. Consider the semicontinuous functions y_N^k and z_N^k such that $y_N^k(t)\equiv\inf\{f_{N\xi}^k(t)\}$ and $z_N^k(t)\equiv\sup\{f_{N\eta}^k(t)\}$. Fix indexes M, N, j and k. Assume that there exists a point t such that $a\equiv y_M^j(t)-z_N^k(t)>0$. Then $f_{M\xi}^j(t)-f_{N\eta}^k(t)\geqq a$ for all Greek indices. Take for the point t the point ideal $N_t\equiv\{P\in\mathscr{P}(T)|t\notin P\}$. Then $\overline{f_{M\xi}^j\circ\varkappa-f_{N\eta}^k\circ\varkappa}\geqq\overline{a1}$ implies $\overline{g_M^j}-\overline{h_N^k}\geqq\overline{a1}$ in Φ/Φ_{N_t}. But this inequality contradicts to the initial one. Consequently $y_M^j\leqq z_N^k$ for any indices.

Consider the Lebesgue functions $y_M \equiv \sup y_M^j$ and $z_N \equiv \inf z_N^k$. Take K_σ-sets $E_M \subset \operatorname{coz} y_M \equiv Y_M$ and co K_σ-sets $H_N \supset \operatorname{coz} z_N \equiv Z_N$ such that $Y_M \setminus E_M \in M$ and $H_N \setminus Z_N \in N$. Then $E_M \subset H_N$. Assume that $H_N \setminus E_N \notin N$. Then there exists an ideal $M \supset N$ such that $T_M \subset Z_N \setminus Y_N$. Assume that $g_N^k \notin \Phi_M$ for some k. Then there exist a number $a > 0$ and a point $s \in K_M$ such that $g_N^k(s) > a$. Therefore $f_{N\xi}^n(t) > a$ for all ξ where $t \equiv \varkappa s \in T_M$. But this is false because of $y_N^k(t) = 0$. Thus $g_N^k \in \Phi_M$ for any k. On the other hand there exists a number $a > 0$ such that

$$A \equiv \{t \in T | z_N(t) > a\} \notin M.$$

That is why there exists a proper ideal $M_1 \supset M$ such that $T_{M_1} \subset A$. Therefore for any k and for any point $t \in T_{M_1}$ there exists an index η such that $f_{N\eta}^k(t) > a$. This means that $h_N^k(s) > a$ for any $s \in K_{M_1}$, i.e. $\overline{h_N^k} \geqq a\bar{1}$ in Φ/Φ_{M_1}. At the same time $\overline{g_N^k} = \bar{0}$. But this is impossible. Thus our assumption is false and in fact $H_N \setminus E_N \in N$.

Further assume that $K_M \subset \varkappa^{-1} E_N \setminus C$ for some $M \supset N$. Then $Y_N \notin M$ implies that $\operatorname{coz} y_N^k \notin M$ for some k. Therefore there exist a number $a > 0$ and a proper ideal $M_1 \supset M$ such that $T_{M_1} \subset \{t \in T | y_N^k(t) > a\}$. This implies that $\overline{f_{N\xi}^k \circ \varkappa} \geqq a\bar{1}$ in Φ/Φ_{M_1}. Consequently, $g_N^k \notin \Phi_{M_1}$. But this is impossible because of $K_{M_1} \cap \operatorname{coz} g_N^k = \emptyset$. So such an M does not exist.

Now assume that $K_M \subset C \setminus \varkappa^{-1} H_N$ for some $M \supset N$. Then $T_M \cap \operatorname{coz} y_N^k = \emptyset$ for any k. Hence $\inf \{(f_{N\xi}^k \circ \varkappa)(s) | \xi\} = 0$ for any $s \in K_M$ implies $g_N^k \in \Phi_M$. Therefore $f \in \Phi_M$. But this contradicts to the inclusion $K_M \subset C$. Thus such an M does not exist.

As a result we obtain that the preimage K is Lebesgue determined.

Now let $\{\hat{X}, \hat{u}: C^*(T) \to \hat{X}, C_N^*(T) \mapsto \hat{X}_N\}$ be an extension having the properties from (2). Consider as it was done above the isomorphic extension $\{\hat{\Phi}, \hat{\varphi}: C^*(T) \to \hat{\Phi}, C_N^*(T) \mapsto \hat{\Phi}_N\}$ for the corresponding preimage

$$\{\hat{K}, \hat{\varkappa}: \hat{K} \to T, T_N \mapsto \hat{K}_N, \mathscr{C}_0(\hat{K})\}.$$

Let $\cap N_k = N$. Then $\hat{\Phi}_N = \cap \hat{\Phi}_{N_k}$ implies that $\cup \hat{K}_{N_k}$ is dense in $\hat{K}_N$. This means that the preimage $\hat{K}$ is filled. Let G be an open set from T. Consider the family $\{f_\xi\}$ consisting of all continuous functions which are smaller than the characteristic function of G. Consider the function $f \equiv \sup \{\hat{\varphi} f_\xi\} \in \hat{\Phi}$. Then $f(s) = 1$ for any $s \in \hat{\varkappa}^{-1} G$ and $f(s) = 0$ for any $s \notin U \equiv \operatorname{cl} \hat{\varkappa}^{-1} G$. From the continuity of f we conclude that $f = \chi(U)$. So $U \in \Delta_0(\hat{K})$. Thus the preimage $\hat{K}$ is lower extremally disconnected.

Let $\hat{\varkappa}^{-1} G \cap \hat{K}_N = \emptyset$. Then $f_\xi \in C_N^*(T)$ implies that $f \in \hat{\Phi}_N$. Therefore $U \cap \hat{K}_N = \emptyset$. This means that $\hat{K}$ is lower disjoined.

Let L be a Lebesgue set. Then there exist K_σ-sets $E_N \equiv \cup F_N^k \subset L$ and co K_σ-sets $H_N \equiv \cap G_N^k \supset L$ such that $H_N \setminus E_N \in N$. Consider the sets $V_N^k \equiv \operatorname{int} \hat{\varkappa}^{-1} F_N^k \in \Delta_0(\hat{K})$, $W_N^k \equiv \operatorname{cl} \hat{\varkappa}^{-1} G_N^k \in \Delta_0(\hat{K})$, $C_N \equiv \cup V_N^k$ and $Z_N \equiv \cap W_N^k$. It is evident that $\cup C_N \subset \subset \cap Z_N$. Consider the functions $g_N^k \equiv \chi(V_N^k)$ and $h_N^k \equiv \chi(W_N^k)$ from $\hat{\Phi}$. Then $g_M^j \leqq h_N^k$ for any indices. Let $0 \leqq u \leqq h_N^k - g_N^j$ for all j, k. Consider the Kelley ideals $M_j \equiv \equiv \{P \in \mathscr{P}(T) | P \cap F_N^j \in N\}$ and $M_k \equiv \{P \in \mathscr{P}(T) | P \cap (T \setminus G_N^k) \in N\}$. By virtue of the lower disjointness of the preimage $\hat{K}$ we get $\hat{K}_j \equiv \hat{K}_{M_j} \subset V_N^j$ and $\hat{K}_k \equiv \hat{K}_{M_k} \subset \hat{K} \setminus W_N^k$. Consequently $u(s) = 0$ for any $s \in (\cup \hat{K}_j) \cup (\cup \hat{K}_k) \equiv R_N$. As $(\cap M_j) \cap (\cap M_k) = N$ and the preimage $\hat{K}$ is filled the set R_N is dense in $\hat{K}_N$. Therefore $u \in \hat{\Phi}_N$.

By the property of the collectively σ-completeness there is a function $f\in\hat{\Phi}$ such that $g^j_M\leqq f\leqq h^k_N$. Since $f(R_N)\subset\{0,1\}$ the function f takes only these two values on the whole set $\hat{K}_N$. Hence $f=\chi(U)$ for some set $U\in\Delta_0(\hat{K})$.

So for the set L we have found the set U such that $\bigcup C_N\subset U\subset\bigcap Z_N$. The fact now established gives the possibility to repeat further the arguments from the proof of Theorem 1. In this way we get a mapping $\gamma\colon \hat{K}\to K$ such that $\hat{K}$ is larger than K relative to γ.

Let C be a cozero-set from the base on K. Check that $\operatorname{cl}\gamma^{-1}C=\alpha C$ where $\alpha\colon \mathscr{C}_0(K)\to\Delta_0(\hat{K})$ is the lattice homomorphism from the proof of Theorem 1. Take an $s\in\gamma^{-1}C$. Take a set C_1 from the base such that $\gamma s\notin\operatorname{cl}C_1$ and $C\cup C_1=K$. Then $\hat{K}=\alpha C\cup\alpha C_1$. If we assume that $s\in\alpha C_1$ then we get $\gamma s\in\operatorname{cl}C_1$. As this is false we have $s\in\alpha C$. Thus $\gamma^{-1}C\subset\alpha C$. Further consider cozero-sets C_m from the base such that $C=\bigcup C_m$ and $\operatorname{cl}C_m\subset C$. Consider the Lebesgue sets $L_m\equiv kC_m$ and the set $L\equiv\bigcup L_m$. Assume that $C\triangle\varkappa^{-1}L\supset K_N$ for some N. If $K_N\cap(C\setminus\varkappa^{-1}L)\neq\emptyset$ then there exists an ideal $M\supset N$ such that $K_M\subset C_m\setminus\varkappa^{-1}L$ for some m, but this is impossible. Therefore $K_N\subset\varkappa^{-1}L\setminus C$. This implies that there exists an ideal $M\supset N$ such that $T_M\subset L_m$ for some m. Then $K_M\subset\varkappa^{-1}L_m\setminus C$, but this is impossible also. From this contradiction we conclude that such a set N does not exist. Hence $L=kC$. Let $U_m\equiv iL_m$ and $U\equiv iL$. Fix an ideal N. Take some K_σ-sets $E^m_N\equiv\bigcup F^{mj}_N\subset L_m$ such that $L_m\setminus E^m_N\in N$ and a K_σ-set $E_N\equiv\bigcup F^r_N\subset T\setminus L$ such that $(T\setminus L)\setminus E_N\in N$. Also take K_σ-sets $D^m_N\equiv\bigcup F^{mk}_N\subset L\setminus L_m$ such that $(L\setminus L_m)\setminus D^m_N\in N$. Then $E^m_N\subset L_m\subset(T\setminus E_N)\cup(T\setminus D^m_N)$ implies that $\bigcup\hat{K}^m_j\subset U_m\subset\hat{K}\setminus((\bigcup\hat{K}_r)\cup(\bigcup\hat{K}^m_k))\subset\subset\hat{K}\setminus\bigcup\hat{K}_r$. Therefore $\bigcup\bigcup\hat{K}^m_j\subset\bigcup U_m\subset\hat{K}\setminus\bigcup\hat{K}_r$. On the other hand $\bigcup\bigcup\hat{K}^m_j\subset\subset U\subset\hat{K}\setminus\bigcup\hat{K}_r$. As $(\bigcup\bigcup\hat{K}^m_j)\cup(\bigcup\hat{K}_r)$ is dense in $\hat{K}_N$ we get that $(\bigcup U_m)\cap\hat{K}_N$ is contained in $U\cap\hat{K}_N$ and moreover the first set is dense in the second one. Hence $U=\operatorname{cl}\bigcup U_m$. As a result we get $\alpha C=\operatorname{cl}\bigcup\alpha C_m$. If $s\in\alpha C_m$ then $\gamma s\in\operatorname{cl}C_m\subset C$. So $\alpha C=\operatorname{cl}\gamma^{-1}C$.

Now let $f\geqq 0$ be a function from Φ. Divide an interval, containing the range of f, by points a_j so that $a_{j+1}-a_j=u_m/2$. Consider the sets $C_j\equiv f^{-1}(]a_{j-1},a_{j+1}[)$ and $U_j\equiv\operatorname{cl}\gamma^{-1}C_j=\alpha C_j$. It was established above that there exist functions $g_j\in\hat{\Phi}$ which are the characteristic functions of the sets αC_j. Consider the step function $g_m\equiv\sup\{a_{j-1}g_j\}$. It is clear that $0\leqq f\circ\gamma(s)-g_m(s)\leqq u_m$ for any s. Therefore $f\circ\gamma\in\hat{\Phi}$.

This means that we can define correctly the injective vector-lattice homomorphism $v\colon \Phi\to\hat{\Phi}$ by setting $vf\equiv f\circ\gamma$. Then $\hat{\varphi}=v\circ\varphi$. Let $f\in\Phi_N$. Then $(vf)(\hat{K}_N)=0$ implies $vf\in\hat{\Phi}_N$. Thus the extension $\hat{\Phi}$ is larger than the extension Φ. This fact is valid for the initial extensions $\hat{X}$ and X, too.

Now let Φ be the extension from Proposition 2 isomorphic to the Lebesguean extension $L^*(T)$. As Φ has the properties from (1) and (2) simultaneously we get as a result that Φ is the largest of all the extensions with the properties from (1) and the smallest of all the extensions with the properties from (2).

Let $\hat{X}$ be some other largest extension of $C^*(T)$. Consider as it was done above the isomorphic extension $\{\hat{\Phi},\hat{\varphi}\colon C^*(T)\to\hat{\Phi},\ C^*_N\mapsto\hat{\Phi}_N\}$ for the preimage $\{\hat{K},\hat{\varkappa}\colon \hat{K}\mapsto T,\ T_N\mapsto\hat{K}_N,\ \mathscr{C}_0(\hat{K})\}$. Take some mapping $w\colon \Phi\to\hat{\Phi}$ such that $\hat{\Phi}$ is larger than Φ relative to w. Define the surjective perfect mapping $\delta\colon \hat{K}\to K$ by setting $\delta s\equiv\bigcap\{\operatorname{cl}\operatorname{coz}f\cap\varkappa^{-1}\hat{\varkappa}s\mid s\in\operatorname{coz}wf\}$. Then $\varkappa\circ\delta=\hat{\varkappa}$. Check that $wf=f\circ\delta$ for any function $0\leqq f\in\Phi$. Assume that there exists a point s such that $(wf)(s)\neq(f\circ\delta)(s)$. If $(wf)(s)>(f\circ\delta)(s)$ then we shall consider the function $g\equiv f$ otherwise $g\equiv -f$.

Denote the number $((wg)(s)+(g\circ\delta)(s))/2$ by a. Consider the function $h\equiv(g-a1)\vee 0$. Take a neighbourhood G of s such that $(wg)(t)>a$ for any $t\in G$. Also take a neighbourhood U of the point δs such that $g(r)<a$ for any $r\in U$. Then $U\subset K\setminus \operatorname{coz} h$ and $G\cap\delta^{-1}U\subset \operatorname{coz} wh$. Therefore $\delta s\notin \operatorname{cl}\operatorname{coz} h$ and $\delta s\in \operatorname{cl}\operatorname{coz} h$ but this is impossible. From this contradiction we conclude that such a point s does not exist.

Check that $\delta\hat{K}_N\subset K_N$. Assume that there exists a point $s\in\delta\hat{K}_N\setminus K_N$. Take a function $f\in\Phi_N$ such that $s\in\operatorname{coz} f$. Then for some point $t\in\hat{K}_N$ such that $s=\delta t$ we get $(wf)(t)\neq 0$. But on the other hand $wf\in\hat{\Phi}_N$ implies $(wf)(t)=0$. It follows from this contradiction that this inclusion is valid.

Now take the mapping $v\colon \hat{\Phi}\to\Phi$ defined above. Let $s\in K_N$. By virtue of the saturatedness of the Lebesguean cover we have $s=\bigcap\{K_M\}$. Then $\delta\gamma s\in\bigcap\{K_M\}=s$. From this fact we conclude that $\delta\gamma s=s$ for any point $s\in K$. Therefore $(wvf)(s)=$ $=f(s)$. Thus v and w are mutually inverse isomorphisms of vector lattices. So the extensions Φ and $\hat{\Phi}$ are isomorphic.

The uniqueness of the smallest extension and assertion (3) are checked in a similar way. The theorem is proved.

§ 3. Lattice ring of Lebesgue functions

Let T be a completely regular space and $L^*(T)$ be the f-ring of all bounded Lebesgue functions on T introduced in § 1. Let $u\colon C^*(T)\to L^*(T)$ be the canonical imbedding.

For a Kelley ideal N consider the f-ring ideal $L_N^*(T)\equiv\{x\in L^*(T)|\operatorname{coz} x\in N\}$. Then $\{L^*(T), u\colon C^*(T)\to L^*(T), C_N^*(T)\mapsto L_N^*(T)\}$ is an extension of $C^*(T)$ inheriting Lebesgue decomposition. This extension will be called *the Lebesguean extension of* $C^*(T)$.

Let K be the Lebesguean cover of T and $\varkappa\colon K\to T$ be the canonical mapping. Let $\Phi\equiv C_0^*(K)$ be the f-ring of functions on K defined in § 1. Consider the injective f-ring homomorphism $\varphi\colon C^*(T)\to\Phi$ such that $\varphi f\equiv f\circ\varkappa$. For a Kelley ideal N consider the f-ring ideal $\Phi_N\equiv\{f\in\Phi| f(K_N)=0\}$. Then $\{\Phi, \varphi\colon C^*(T)\to\Phi,$ $C_N^*(T)\mapsto\Phi_N\}$ is an extension of $C^*(T)$ inheriting Lebesgue decomposition.

Now let $\{X, u\colon C^*(T)\to X, C_N^*(T)\mapsto X_N\}$ be an f-ring extension of $C^*(T)$ inheriting Lebesgue decomposition. Identify $C^*(T)$ with its image in X. The notion for X to be *Lebesgue generated* is defined as in § 2.

For the extensions $L^*(T)$ and Φ consider the mapping $r\colon L^*(T)\to\Phi$ from Proposition 1 of § 1.

PROPOSITION 3. *Relative to the mapping r the extensions $L^*(T)$ and Φ are isomorphic saturated Lebesgue generated lower continuing lower segment collectively σ-continuing extensions of $C^*(T)$ inheriting Lebesgue decomposition.*

PROOF. Denote $L^*(T)$ by X and $L_N^*(T)$ by X_N. It can be verified that $r\circ u=\varphi$ and r is an isomorphism of f-rings. It has been checked in the proof of Proposition 2 that $x\in X_N$ iff $rx\in\Phi_N$. Therefore the extensions X and Φ are isomorphic.

In just the same way as in the proof of Proposition 2 it is checked that X is saturated and Lebesgue generated.

Let Y be a ring ideal in the ring $C^*(T)$ and $g\in\operatorname{Hom}_{C^*(T)}^*(Y, C^*(T)\cap Y^{**})$. Let $y_1, y_2\in Y$ and $t\in\operatorname{coz} y_1\cap\operatorname{coz} y_2$. Then $(gy_1)(t)/y_1(t)=(gy_2)(t)/y_2(t)$. Consequently, we can define correctly the Lebesgue function $z\in X$ by setting $z(t)\equiv\equiv(gy)(t)/y(t)$ for any $y\in Y$ and any $t\in\operatorname{coz} y$ and $z(t)\equiv 0$ for any $t\notin G\equiv\cup\cup\{\operatorname{coz} y|y\in Y\}$.

As $z\in Y^{**}$ we can define correctly the homomorphism $h\in\operatorname{Hom}_X^*(X, Y^{**})$ by setting $hx\equiv xz$. Let $y\in Y$ and $t\in G$. Then there exists a $y_1\in Y$ such that $t\in\operatorname{coz} y_1$. Therefore $y_1(t)(hy)(t)=y(t)(gy_1)(t)=y_1(t)(gy)(t)$ implies $(hy)(t)=(gy)(t)$. Since hy and gy belong to Y^{**} we have $(hy)(t)=0=(gy)(t)$ for any $t\notin G$. This means that $hy=gy$. Thus X is lower continuing.

Now let g and h be the homomorphisms from the definition of the lower segment and $gY\subset X_N$. Let $x\in X$ and $t\in T_N\cap G$. Then $t\in\operatorname{coz} y$ for some $y\in Y$ implies $y(t)(hx)(t)=x(t)(gy)(t)=0$ and hence $(hx)(t)=0$. If $t\in T_N\setminus G$ then $(hx)(t)=0$ because of $hx\in Y^{**}$. Consequently, $hx\in X_N$. This means that X_N is a lower segment of X.

Now we check that X is collectively σ-continuing. Let $\{Y_N\}$ and $\{h_N\}$ be the families from the definition of this property in 4.3. Let Y_N be generated by a countable set $\{y_N^k\}$. Consider the Lebesgue sets $L_N^k\equiv\operatorname{coz} y_N^k$, $L_N\equiv\cup L_N^k$ and $P_N\equiv T\setminus L_N$. Define the Lebesgue function $z_N\in X$ by setting $z_N(t)\equiv(h_N y_N^k)(t)/y_N^k(t)$ for any k and any $t\in L_N^k$ and $z_N(t)\equiv 0$ for any $t\notin L_N$. Let $t\in L_M\cap L_N$. Then $t\in L_M^j\cap L_N^k$ for some j, k implies $z_M(t)=z_N(t)$. This means that we can define correctly the function z by setting $z(t)\equiv z_N(t)$ for any $t\in L_N$ and any N and $z(t)\equiv 0$ for any $t\notin\cup L_N$. Consider the Lebesgue functions $u_N\equiv\chi(P_N)$. Since an arbitrary element of the ideal y_N has a form $\sum x_{k_1\dots k_m}y_N^{k_1}\dots y_N^{k_m}$ we conclude that $u_N\in Y_N^\perp$. Therefore $u_N\in X_N$ implies $P_N\in N$. Fix an interval $]a, b[$ and an ideal N. Consider the set $A\equiv z^{-1}(]a, b[)$ and the Lebesgue set $B\equiv z_N^{-1}(]a, b[)$. Take some K_σ-set $E\subset B\setminus P_N$ and some co K_σ-set $H\supset B\cup P_N$ such that $(B\setminus P_N)\setminus E\in N$ and $H\setminus(B\cup P_N)\in N$. Then $H\setminus E\in N$. Besides $A\cap L_N=B\cap L_N$ implies $E\subset A\subset H$. This means that A is a Lebesgue set and consequently z is a Lebesgue function. As the family $\{h_N\}$ is uniformly bounded we conclude that $z\in X$. That is why we can define correctly the homomorphism $h\in\operatorname{Hom}_X^*(X, X)$ by setting $hx\equiv xz$. Let $y\in Y_N$ and $t\in L_N$. Then $t\in L_N^k$ for some k. Hence $y_N^k(t)(hy)(t)=y(t)(h_N y_N^k)(t)=y_N^k(t)(h_N y)(t)$ implies $(hy)(t)=(h_N y)(t)$. Further $u_N hy=u_N h_N y$ implies that the functions hy and $h_N y$ coincide. As a result we get that h extends h_N. The proposition is proved.

THEOREM 3. 1) *$L^*(T)$ is the unique largest of all the saturated Lebesgue generated extensions of $C^*(T)$ inheriting Lebesgue decomposition.*

2) *$L^*(T)$ is the unique smallest of all the filled lower continuing lower segment collectively σ-continuing extensions of $C^*(T)$ inheriting Lebesgue decomposition and moreover $L^*(T)$ is the unique universal among all such extensions.*

3) *$L^*(T)$ is the unique saturated Lebesgue generated lower continuing lower segment collectively σ-continuing extension of $C^*(T)$ inheriting Lebesgue decomposition.*

PROOF. Let $\{X, u\colon C^*(T)\to X,\ C_N^*(T)\mapsto X_N\}$ be an extension having the properties from 1). On the strength of Johnson's theorem ([15]) there is a unique compact K_0 such that the f-ring X is isomorphic to the f-ring $C(K_0)$ relative to an isomorphism r_0. Further by completely the same arguments as in the proof of Theorem 2 we obtain the preimage $\{K, \varkappa\colon K\to T,\ T_N\mapsto K_N,\ \mathscr{C}_0(K)\}$ of T and the corresponding extension $\{\Phi, \varphi\colon C^*(T)\to\Phi,\ C_N^*(T)\mapsto\Phi_N\}$ isomorphic to the initial one.

In just the same way as in the proof of Theorem 2 it is established that the preimage K is saturated and Lebesgue determined.

Now let $\{\hat{X}, \hat{u}\colon C^*(T)\to\hat{X},\ C_N^*(T)\mapsto\hat{X}_N\}$ be an extension having the properties from (2). Consider the isomorphic extension $\{\hat{\Phi}, \hat{\varphi}\colon C^*(T)\to\hat{\Phi},\ C_N^*(T)\to\hat{\Phi}_N\}$ for the corresponding preimage $\{\hat{R}, \hat{\varkappa}\colon \hat{R}\to T,\ T_N\mapsto\hat{R}_N,\ \mathscr{C}_0(\hat{R})\}$. Then the preimage $\hat{R}$ is filled.

Let G be an open set from T. Denote the set $\hat{\varkappa}^{-1}G$ by V. Consider the ring $R\equiv\hat{\varphi}C^*(T)$ and the ring ideal $Y\equiv\{y\in R|\operatorname{coz} y\subset V\}$ of the ring R. Define the homomorphism $g\in\operatorname{Hom}_R^*(Y, R\cap Y^{**})$ by setting $gy\equiv y$. Then there exists a bounded $\hat{\Phi}$-module homomorphism $h\colon\hat{\Phi}\to Y^{**}$ extending g. Consider the function $u\equiv h1\in\hat{\Phi}$ and the set $U\equiv\operatorname{cl} V$. It is clear that $u(\hat{R}\setminus U)=0$. Let $s\in V$. Then $s\in\operatorname{coz} y$ for some $y\in Y$. Therefore $y(s)u(s)= =(gy)(s)=y(s)$ implies $u(s)=1$. Since the function u is continuous we conclude that $u=\chi(U)$ and $U\in\Delta_0(\hat{R})$. So the preimage $\hat{R}$ is lower extremally disconnected.

Let $V\cap\hat{R}_N=\emptyset$. Then $gY\subset\hat{\Phi}_N$ implies $u\in\hat{\Phi}_N$. Therefore $U\cap\hat{R}_N=\emptyset$. Thus the preimage $\hat{R}$ is lower disjoined.

Let L be a Lebesgue set. Then there exist K_σ-sets $E_N\equiv\bigcup F_N^k\subset L$ and co K_σ-sets $H_N\equiv\bigcap G_N^k\supset L$ such that $H_N\setminus E_N\in N$. Consider the sets $V_N^k\equiv\operatorname{int}\hat{\varkappa}^{-1}F_N^k\in\Delta_0(\hat{R})$, $W_N^k\equiv\operatorname{cl}\hat{\varkappa}^{-1}G_N^k\in\Delta_0(\hat{R})$, $C_N\equiv\bigcup V_N^k$ and $Z_N\equiv\bigcap W_N^k$. It is evident that $\bigcup C_N\subset\bigcap Z_N$. Consider the functions $g_N^k\equiv\chi(V_N^k)$ and $h_N^k\equiv\chi(\hat{R}\setminus W_N^k)$ from $\hat{\Phi}$. Consider the ring ideal Y_N' generated by the family $\{g_N^k\}$, the ring ideal Y_N'' generated by the family $\{h_N^k\}$ and the ring ideal $Y_N\equiv\{y'+y''|y'\in Y_N',\ y''\in Y_N''\}$. Let $u\in Y_N^*$. Consider the Kelley ideals $M_j\equiv\{P\in\mathscr{P}(T)|P\cap F_N^j\in N\}$ and $M_k\equiv\{P\in\mathscr{P}(T)|P\cap(T\setminus G_N^k)\in N\}$. By virtue of the lower disjointness of the preimage $\hat{R}$ we get $\hat{R}_j\equiv\hat{R}_{M_j}\subset V_N^j$ and $\hat{R}_\kappa\equiv \equiv\hat{R}_{M_k}\subset\hat{R}\setminus W_N^k$. Consequently, $u(s)=0$ for any $s\in(\bigcup\hat{R}_j)\cup(\bigcup\hat{R}_k)\equiv R_N$. As $(\bigcap M_j)\cap(\bigcap M_k)=N$ and the preimage $\hat{R}$ is filled the set R_N is dense in $\hat{R}_N$. Therefore $u\in\hat{\Phi}_N$. Define the bounded $\hat{\Phi}$-module homomorphism $h_N\colon Y_N\to\hat{\Phi}$ by setting $h_N(y'+y'')\equiv y'$. Then there is a bounded $\hat{\Phi}$-module homomorphism $h\colon\hat{\Phi}\to\hat{\Phi}$ extending all h_N. Consider the function $u\equiv h1\in\hat{\Phi}$ and the set $U\equiv\operatorname{coz} u$. Let $s\in C_N$. Then $s\in\operatorname{coz} y$ for some $y\in Y_N'$ implies $y(s)u(s)=y(s)$, i.e. $u(s)=1$. Let $s\in\hat{R}\setminus Z_N$. Then $s\in\operatorname{coz} y$ for some $y\in Y_N''$ implies $y(s)u(s)=0$, i.e. $u(s)=0$. Since $u(R_N)\subset \subset\{0, 1\}$ the function u takes only these two values on the whole set $\hat{R}_N$. Hence $u=\chi(U)$ and $U\in\Delta_0(K)$.

So for the set L we have found the set U such that $\bigcup C_N\subset U\subset\bigcap Z_N$. Further the proof is led in exactly the same way as the proof of Theorem 2.

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(Received April 11, 1985)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 369—373

ON LINEAR p-GROUPS OF DEGREE p

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In this paper we shall investigate the properties of finite p-groups having a faithful irreducible representation of degree p over an arbitrary field. The class of these p-groups will be denoted by $\mathcal{P}$. We shall give some equivalent characterizations of the class $\mathcal{P}$. We show that the maximal abelian subgroups of these groups have index p in their normalizers. p-groups containing such subgroups have been investigated in [2] where these abelian subgroups were called soft subgroups.

We shall see that groups belonging to $\mathcal{P}$ are closely related to p-groups of maximal class, in fact their nonabelian factors over normal subgroups containing the centre are all of maximal class. Special attention will be given to the regular p-groups in the class $\mathcal{P}$.

Finally, we shall generalize some properties of $\mathcal{P}$ to a wider class of p-groups.

We shall need two simple lemmas.

LEMMA 1. *Let P be a finite p-group such that $Z(P)$ is cyclic. Let $N \geqq Z(P)$ be a normal subgroup of P with $|N:Z(P)|=p$. Then $|P:C_P(N)|=p$.*

PROOF. Fix an element $x \in N \setminus Z(P)$. If $g \in P$ is an arbitrary element then $g^{-1}xg=xz$ for some $z \in Z(P)$. Now $x^p \in Z(P)$, hence $x^p=g^{-1}x^pg=(xz)^p=x^pz^p$, so $z \in \Omega_1(Z(P))$. Since $Z(P)$ is cyclic it follows that the group of automorphisms of N induced by P has order p. Hence $|P:C_P(N)|=p$.

LEMMA 2. *Let P be a finite nonabelian p-group containing an abelian maximal subgroup A. Then $|A|=|P'||Z(P)|$.*

PROOF. See [4], p. 204.

THEOREM 1. *For a nonabelian finite p-group P the following are equivalent:*

(i) *P has a faithful irreducible representation of degree p over some field.*

(ii) *P has a faithful irreducible representation of degree p over the complex numbers.*

(iii) *$Z(P)$ is cyclic and there exists an abelian maximal subgroup A in P.*

(iv) *$Z(P)$ is cyclic and $C_P(x)$ is abelian for any $x \in P \setminus Z(P)$.*

(v) *P is isomorphic to a finite nonabelian subgroup of the wreath product $Z_{p^\infty} \operatorname{wr} Z_p$.*

1980 *Mathematics Subject Classification.* Primary 20C99; Secondary 20D15.

Key words and phrases. Representation, linear group, p-group, regular p-group, p-groups of maximal class.

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PROOF. (i)$\Rightarrow$(iii) Let φ be a representation of P guaranteed by (i). Because of page 199 in [1] we can suppose that P has a noncyclic abelian normal subgroup N such that $|N|=p^2$. By page 64 in [1] $Z(P)$ is cyclic. Then there exists an $M\leqq P$ maximal subgroup such that $Z(M)$ is noncyclic.

By page 70 in [1] $\varphi|_M$ is not irreducible and thus its constituents are linear. But then M is abelian.

(iii)$\Rightarrow$(ii) See [4], p. 29 and p. 84.

(ii)$\Rightarrow$(i) Obvious.

(iii)$\Rightarrow$(iv) If $x\in A\setminus Z(P)$ then $C_P(x)=A$. If $x\notin A$ then $C_P(x)=\langle x, C_P(x)\cap A\rangle$ is abelian since $x\in Z(C_P(x))$.

(iv)$\Rightarrow$(iii) Choose N so that $Z(P)\leqq N\lhd P$ and $|N:Z(P)|=p$. Fix an element $x\in N\setminus Z(P)$. Then by Lemma 1 it follows that $C_P(N)=C_P(x)$ is a maximal subgroup of P.

(ii)$\Leftrightarrow$(v) See [5], p. 277.

In the following statements P will always belong to $\mathscr{P}$.

STATEMENT 1. *$P/Z(P)$ is of maximal class, generated by two elements of order p.*

PROOF. As for every $x\in P\setminus A$ $x^p\in Z(P)$, the second part of the statement is obvious. Now we prove that $P/Z(P)$ is of maximal class by induction on $\mathrm{cl}\,(P)$ and on $|P|$. Let us suppose first that $\mathrm{cl}\,(P)=2$. It is easy to see that $\exp(P')=p$. And hence, as $P'\leqq Z(P)$ is cyclic, $|P'|=p$. Now by $|A|=|P'||Z(P)|$ we have $|P/Z(P)|=p^2$. So we can suppose that $\mathrm{cl}\,(P)>2$.

If $Z(P)\not\leqq \Phi(P)$ then there exists a maximal subgroup $M\leqq P$ such that $Z(P)\not\leqq M$. Then $P/Z(P)=M/Z(M)$ is of maximal class by the induction hypothesis. So we may assume that $Z(P)\leqq\Phi(P)$. Take a maximal subgroup $M\leqq P$, $M\neq A$. If $Z(M)\not\leqq Z(P)$ then $Z(M)\not\leqq A$ and so $M=(A\cap M)\cdot Z(M)$ is abelian. It implies that $\mathrm{cl}\,(P)=2$, and we are done in this case.

So we may assume that for each maximal subgroup $M\leqq P$, $M\neq A$ we have $Z(M)=Z(P)$. Consider the factor group $\bar{P}=P/Z(P)$. If $Z(\bar{P})\not\leqq\bar{A}$ then $\bar{P}=\bar{A}\cdot Z(\bar{P})$ is abelian, $\mathrm{cl}\,(P)=2$, so we may suppose that $Z(\bar{P})\leqq\bar{A}$.

If $\bar{A}$ is the only maximal subgroup of $\bar{P}$ containing $Z(\bar{P})$, then $\bar{P}/Z(\bar{P})$ is cyclic and we have again $\mathrm{cl}\,(P)=2$. So there is a maximal subgroup $\bar{M}\leqq\bar{P}$, $\bar{M}\neq\bar{A}$ such that $Z(\bar{P})\leqq\bar{M}$. By induction $\bar{M}$ is of maximal class.

If $\bar{M}$ is abelian then $|\bar{P}|\leqq p^3$ and hence either $\bar{P}$ is abelian and $\mathrm{cl}\,(P)=2$ or $\bar{P}$ is of maximal class. If $\bar{M}$ is nonabelian then $|Z(\bar{P})|=|Z(\bar{M})|=p$. Hence $\bar{P}$ is nonabelian, contains an abelian maximal subgroup $\bar{A}$ and $Z(\bar{P})$ is cyclic. Hence by induction $\bar{P}/Z(\bar{P})$ and also $P/Z(P)$ is of maximal class.

STATEMENT 2. $|\gamma_i(P)/\gamma_{i+1}(P)|=p$ *for* $i=2, 3, \ldots, \mathrm{cl}\,(P)$.

PROOF. As

$$\prod_{i=2}^{\mathrm{cl}(P)} |\gamma_i(P)/\gamma_{i+1}(P)| = |\gamma_2(P)| = |P'| = |A: Z(P)| =$$

$$= |P: Z(P)|p^{-1} = p^{\mathrm{cl}(P/Z(P))} = p^{\mathrm{cl}(P)-1}$$

since $P/Z(P)$ is of maximal class by Statement 1.

STATEMENT 3. *If* $B \in SC(P)$, $B \neq A$ *then for any* $x \in B \setminus Z(P)$ *we have* $B = \langle x \rangle \cdot Z(P)$ *where* $x^p \in Z(P)$. *If B is cyclic then* $B = \langle x \rangle$, *otherwise there is an* $x \in B \setminus Z(P)$ *such that* $B = \langle x \rangle \times Z(P)$.

PROOF. Take an $x \in B \setminus A$ then we have $B = \langle x \rangle (B \cap A)$ and $x^p \in B \cap A$. Obviously, $B \cap A = Z(P)$.

STATEMENT 4. *If* $B \in SC(P)$ *then* $|N_P(B):B| = p$.

PROOF. It obviously holds for $B = A$. Otherwise apply Lemma 1 for $Z(P) \leqq B \lhd N_P(B)$, then it follows that $|N_P(B):B| = |N_P(B):C_{N_P(B)}(B)| = p$.

STATEMENT 5. *If* $B \in SC(P)$, $B \neq A$, $|P:B| = p^k$ *then* $\mathrm{cl}\,(P) = k+1$, $|P'| = p^k$.

PROOF. $|B| = p \cdot |Z(P)| = p|A:P'| = |P:P'|$.

STATEMENT 6. *If* $B \in SC(P)$ *then there is a unique maximal subgroup of P which contains B.*

PROOF. See [2].

STATEMENT 7. *If* $N \lhd P$ *and* $N \nleqq A$ *then* $\mathrm{cl}\,(N) \geqq \mathrm{cl}\,(P) - 1$.

PROOF. $P = N \cdot A$ hence $\mathrm{cl}\,(P) \leqq \mathrm{cl}\,(N) + \mathrm{cl}\,(A)$.

COROLLARY 1. *If* $\mathrm{cl}\,(P) \geqq 4$ *then just like in groups of maximal class the critical subgroup of P is the unique maximal abelian normal subgroup of P.*

PROOF. Let C denote the critical subgroup of P. Since $\mathrm{cl}\,(C) \leqq 2$ if $\mathrm{cl}\,(P) \geqq 4$ then Statement 7 implies that $C \leqq A$. Since $C_P(C) = Z(C) = C$ it follows that $C = A$.

COROLLARY 2. *If* $\mathrm{cl}\,(P) \geqq 4$, $\alpha \in \mathrm{Aut}\,(P)$, $p \nmid o(\alpha)$ *and* $\alpha | \Omega_1(A) = 1 | \Omega_1(A)$ *then* $\alpha = 1_P$.

THEOREM 2. *P satisfies exactly one of the following:*

(i) $P = T \cdot Z(P)$ *where T is of maximal class generated by two elements of order p and* $|Z(P)| > p$.

(ii) *P is of maximal class.*

(iii) *There exists a cyclic* $B \in SC(P)$, $B \neq A$, $|B| \geqq p^3$.

PROOF. If $|Z(P)| = p$ then we have (ii) by Statement 1. So let $|Z(P)| > p$. If there exists a cyclic $B \in SC(P)$, $B \neq A$ then $|B| = p \cdot |Z(P)| \geqq p^3$ by Statement 3, and we have (iii). Suppose now that no $B \in SC(P)$, $B \neq A$ is cyclic. Since $P/Z(P)$ is of maximal class, there are $x, y \in P \setminus A$ such that $\langle x \cdot Z(P), y \cdot Z(P) \rangle = P/Z(P)$. By our assumption we can even choose x, y of order p.

Now let $T = \langle x, y \rangle$, then $T/T' = \langle xT', yT' \rangle$, hence $|T/T'| = p^2$. Moreover $P = T \cdot Z(P)$ and so $\gamma_i(T) = \gamma_i(P)$ for $i \geqq 2$. Now it follows from Statement 2 that T is of maximal class. Notice also that in case (i) $Z(P) \nleqq \Phi(P)$, hence no $B \in SC(P)$ is cyclic.

REMARK 1. It is clear from the proof that in case (iii) there is at most one maximal subgroup of P different from A which may contain elements of order p from $P \setminus A$.

STATEMENT 8. *$\Omega_1(P)$ is either elementary abelian or belongs to $\mathscr{P}$.*

PROOF. It is easy to see that nonabelian subgroups of P are members of $\mathscr{P}$.

STATEMENT 9. *If $\Omega_1(P)=P$ then P is of maximal class or of type* (i) *in Theorem 2 with $|Z(P)|=p^2$.*

PROOF. If $Z(P)\not\leqq\Phi(P)$ then P is of type (i) and since $|Z(T)|=p$ and P/T is cyclic $|Z(P)|=p^2$ obviously follows.

If $Z(P)\leqq\Phi(P)$, $|P/\Phi(P)|=p^2$ by Statement 1 hence $\Omega_1(P)=P$ implies that P is generated by two elements of order p, therefore $|P/P|=p^2$ and P is of maximal class.

STATEMENT 10. *We have $|\Omega_1(A)|\leqq p^p$. If $|\Omega_1(A)|\leqq p^{p-2}$ then P is a regular p-group.*

PROOF. The first claim follows from the fact that P has a faithful representation of degree p in which A is represented by diagonal matrices (see [5] p. 277). As for the second claim we make use of the following sufficient condition for regularity: if P has no normal subgroup N with $\exp(N)=p$, $|N|\geqq p^{p-1}$ then P is regular (see [3], p. 332).

Suppose that P contains such a normal subgroup N. Then $|N|=p^{p-1}$ and $N\cap A=\Omega_1(A)$ has order p^{p-2}. Hence $P/\Omega_1(A)=N/\Omega_1(A)\times A/\Omega_1(A)$ is abelian, $P'\in\Omega_1(A)$, $|P'|\leqq p^{p-2}$ whence $\mathrm{cl}(P)\leqq p-1$ and P is regular, see [3], p. 332.

THEOREM 3. *If P is regular then we have*

(a) $\mathrm{cl}(P)\leqq p-1$;
(b) $|P/Z(P)|\leqq p^{p-1}$ *and* $\exp(P/Z(P))=p$;
(c) *P' is elementary abelian;*
(d) *$\Omega_1(P)$ is either elementary abelian or of maximal class.*

PROOF. (a) See [3], p. 330.

(b) Since $P/Z(P)$ is regular and of maximal class generated by two elements of order p by Statement 1, it follows that $|P/Z(P)|=p^{1+\mathrm{cl}(P/Z(P))}=p^{\mathrm{cl}(P)}\leqq p^{p-1}$ and $\exp(P/Z(P))=p$.

(c) Now let $x\in P\setminus A$, then $P'=\langle[x,a]|a\in A\rangle$ and $[x,a]^p=[x,a^p]=1$ since $a^p\in Z(P)$.

(d) As P is regular $\exp(\Omega_1(P))=p$. Let us suppose that $\Omega_1(P)$ is not abelian. Then $\Omega_1(P)\not\leqq A$ and $A\cap\Omega_1(P)$ is maximal in $\Omega_1(P)$. Choose an $x\in\Omega_1(P)\setminus A$. Then $\langle A,x\rangle=P$ and $C_{\Omega_1(A)}(x)\leqq Z(P)$. As $Z(P)$ is cyclic $|C_{\Omega_1(A)}(x)|=p$ and $|C_{\Omega_1(P)}(x)|=p^2$. Now (d) follows, see [3], p. 375.

It is easy to see that if a p-group P belongs to $\mathscr{P}$ then all of its nonlinear irreducible representations have degree p. As it is well-known (see [4], p. 203) a nonabelian p-group P has this property if and only if either it has an abelian maximal subgroup or $|P:Z(P)|=p^3$.

We shall generalize some of our observations on linear p-groups of degree p for such groups. As the following results are trivial when $|P:Z(P)|=p^3$ we shall formulate them in the case when P has an abelian maximal subgroup.

We introduce the following notation: let $SC_1(P)=SC(P)$ and $SC_k(P)$ denotes the set of subgroups of P which are maximal with the property that their class is k.

STATEMENT 11. *Let P be a finite nonabelian p-group. Let us suppose that there exists an abelian maximal subgroup A of P. Then for every $x\in P\setminus Z(P)$ $C_P(x)$ is abelian. Each $B\in SC(P)$, $B\neq A$ has the same order and $B=C_P(x)$ for some $x\in P\setminus Z(P)$ and $\mathrm{cl}\,(N_P(B))=2$. $N_P(B)\in SC_2(P)$ and every $H\in SC_2(P)$ is the normalizer of some $B\in SC(P)$. Generally, if $K\in SC_k(P)$, $(k=2, ..., \mathrm{cl}\,(P)-1)$ then $N_P(K)\in SC_{k+1}P$ and every $H\in SC_{k+1}(P)$ is the normalizer of some $K\in SC_k(P)$. All members of $SC_k(P)$ $(k=2, ..., \mathrm{cl}\,(P)-1)$ have the same order.*

PROOF. If $\mathrm{cl}\,(P)=2$ then the claims are trivially true. So we can suppose that $\mathrm{cl}\,(P)>2$. Let $B\in SC(P)$, $B\neq A$. Let $N=N_P(B)$. It is easy to see that $\mathrm{cl}\,(N)=2$ and $Z(N)=Z(P)$. Hence $N/Z(P)\leqq\bar{T}\in SC(\bar{P})$ where $\bar{P}=P/Z(P)$. It is easy to see that $B/Z(P)\lhd\bar{T}$ and thus $N/Z(P)=\bar{T}$. If $N\leqq K$ and $\mathrm{cl}\,(K)=2$ then $Z(K)=Z(N)=Z(P)$ and so $B\lhd K$ which is not the case.

Conversely, let $K\in SC_2(P)$. Let $B\in SC(K)$, $B\neq K\cap A$. As $\mathrm{cl}\,(K)=2$, $B\lhd K$. It is enough to prove that $B\in SC(P)$. Let $B_1\in SC(P)$ be such that $B\leqq B_1$. As $K\leqq N_P(B)$, $K\leqq N_P(C_P(B))$. Since for every $x\in P\setminus A$ $C_P(x)$ is abelian, we have $C_P(B)=B_1$ thus $K\leqq N_P(B_1)$. We know that $N_P(B_1)\in SC_2(P)$ so $K=N_P(B_1)$ and thus $B=B_1$.

If $H\in SC_2(P)$ it follows similarly as above that $H\cap A=Z_2(P)$, hence $|H|=p|Z_2(P)|$, so all subgroups in $SC_2(P)$ have equal orders. Now all claims follow by induction as if $H\in SC_{k+1}(P)$ then $H/Z(P)\in SC_k(P/Z(P))$.

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(Received May 8, 1985)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 375—380

AN EQUICONVERGENCE THEOREM WITH EXACT ORDER FOR FUNCTIONS FROM THE CLASS $W_1^1(0, 1)$

A. BOGMÉR

1. Let G be an arbitrary open interval on the real line, $q \in L^1_{\mathrm{loc}}(G)$ an arbitrary complex function and consider the formal Schrödinger operator $Lu = -u'' + qu$. Given a complex number λ, the function $u\colon G \to \mathbf{C}$, $u \equiv 0$ is called an eigenfunction of order -1 of the operator L with the eigenvalue λ. A function $u\colon G \to \mathbf{C}$, $u \not\equiv 0$ is called an eigenfunction of order m $(m=0, 1, \dots)$ of the operator L with the eigenvalue λ if u and u' are locally absolutely continuous on G and $Lu = u - u^*$ almost everywhere on G, where u^* is an eigenfunction of order $m-1$ of the operator L with the same eigenvalue λ.

Let $q, \hat{q} \in L^1(G)$ arbitrary complex functions. Let (u_k) (resp. $(\hat{u}_k)$) be a Riesz basis in $L^2(G)$ consisting of eigenfunctions of the operator $Lu = -u'' + qu$ (resp. $\hat{L}u = -u'' + \hat{q}u$) and having the following properties

$$\sup o_k < \infty, \quad (\text{resp. } \sup \hat{o}_k < \infty), \tag{1}$$

$$\text{in case } o_k > 0 \quad (\text{resp. } \hat{o}_k > 0), \tag{2}$$

$$\lambda_k u_k - Lu_k = u_{k-1} \quad (\text{resp. } \hat{\lambda}_k \hat{u}_k - \hat{L}\hat{u}_k = \hat{u}_{k-1}),$$

where λ_k and o_k (resp. $\hat{\lambda}_k$ and $\hat{o}_k$) denotes the eigenvalue and the order of u_k (resp. $\hat{u}_k$).

Introduce for any $f \in L^1(G)$ the notations

$$\sigma_\mu(f, x) := \sum_{|\varrho_k| < \mu} \langle f, v_k \rangle u_k.$$

According to (20) below $\sigma_\mu(f, x)$ has sense.

$$\hat{\sigma}_\mu(f, x) := \sum_{|\hat{\varrho}_k| < \mu} \langle f, \hat{v}_k \rangle \hat{u}_k,$$

$$\left(x \in G,\ \mu > 0,\ \varrho_k := \operatorname{Re}\sqrt{\lambda_k},\ \hat{\varrho}_k := \operatorname{Re}\sqrt{\hat{\lambda}_k}\right),$$

where (v_k) (resp. $(\hat{v}_k)$) is the dual system of (u_k) (resp. $(\hat{u}_k)$), i.e., $(v_k), (\hat{v}_k) \subset L^2(G)$ and $\langle v_k, u_j \rangle = \langle \hat{v}_k, \hat{u}_j \rangle = \delta_{k,j}$.

Assume (v_k) (resp. $(\hat{v}_k)$) consists of the eigenfunctions of the operator $L^* v := -v'' + \bar{q}v$ (resp. $\hat{L}^* v := -v'' + \bar{\hat{q}}v$) with eigenvalues $(\bar{\lambda}_k)$ (resp. $(\bar{\hat{\lambda}}_k)$). Denote l_k (resp. $\hat{l}_k$) the lenghts of the chain containing u_k (resp. $\hat{u}_k$). Assume v_k (resp. $\hat{v}_k$) has the order $l_k - o_k$ (resp. $\hat{l}_k - \hat{o}_k$). (For concrete situations this is the case.)

The aim of the present paper is to prove the

1980 *Mathematics Subject Classification* (1985 Revision). Primary 34B25; Secondary 42C15.
Key words and phrases. Eigenfunction, expansion, Schrödinger operator.

THEOREM. *Given any compact interval* $K\subset G$, $q,\hat{q}\in L^1(G)$, *for any* $f\in W_1^1(G)$ *we have*

$$\sup_{x\in K}|\sigma_\mu(f,x)-\hat{\sigma}_\mu(f,x)|\leqq C(K)\|f\|_{W_1^1}\frac{1}{\mu},\quad(\mu\geqq 1),\tag{3}$$

if $q,\hat{q}\in L^p_{\mathrm{loc}}(G)$ *for some* $p>1$.

REMARK 1. It is easy to see that (3) is not refinable.

REMARK 2. The proof of (3) is based on the ideas and results of the papers [3]—[6].

REMARK 3. For $f\in L^1_{\mathrm{loc}}(G)$ I. Joó [6] proved: for any $q,\hat{q}\in L^1_{\mathrm{loc}}(G)$ and almost every $x\in G$

$$\sigma_\mu(f,x)-\hat{\sigma}_\mu(f,x)=\bar{o}_x(1),\qquad(\mu\to\infty)$$

holds and hence a generalization of Kolmogorov's divergence theorem follows, e.g., for the classical orthogonal expansions (Jacobi, Laguerre, Hermite, Bessel, etc. expansions).

REMARK 4. N. H. Loi [7] generalized I. Joó's theorem for Riesz summation, giving the estimate $\bar{o}(\mu^{-s})$ in place of $\bar{o}(1)$ (it denotes the order of the Riesz summation). (He used the additional assumption $|\mathrm{Im}\sqrt{\lambda_n}|\leqq|\sqrt{\lambda_n}|^{-s}$.)

REMARK 5. Our theorem for $i=0$, $\lambda_n\geqq 0$ reduces to that of Theorem 1 of [8].

2. PROOF of the Theorem. For brevity, let us denote by μ_k a square root of λ_k, and put $\varrho_k:=|\mathrm{Re}\,\mu_k|$, $\nu_k:=|\mathrm{Im}\,\mu_k|$. Let $K=[a,b]\subset G$, $R>0$ such that $K_R:=[a-R,b+R]\subset G$

$$w(x,t):=\begin{cases}\dfrac{\sin\mu t}{\pi t}, & \text{if}\quad |t|<R\\ 0 & \text{otherwise,}\end{cases}$$

$$S_\mu^R(f,x):=\int_{x-R}^{x+R}\frac{\sin\mu(x-y)}{\pi(x-y)}f(y)\,dy,$$

$$S_{R_0}[g(R)]:=\frac{2}{R_0}\int_{R_0/2}^{R_0}g(R)\,dR.$$

In [6] the equality:

$$\begin{aligned}&S_{R_0}[S_\mu^R(f,x)-\sigma_\mu(f,x)]=\\ &=\int_G f(y)\Big[\frac{1}{2}\sum u_k(x)\overline{v_k(y)}+\sum c_k(x)\overline{v_k(y)}\Big]dy\end{aligned}\tag{4}$$

has been proved, further

$$\begin{aligned}c_k(x)&=\sum S_{R_0}\Big[\int_0^R\frac{\sin\mu t}{\pi t}f_i(\mu_k,t)\,dt\Big]u_{k-i}(x)+\\ &+\sum S_{R_0}\Big[\int_0^R\frac{\sin\mu t}{\pi t}g_i(u_{k-i},\mu_k,x,t)\,dt\Big]+\Delta_k u_k(x),\end{aligned}\tag{5}$$

and also the estimates

$$|\Delta_k| \leq C(R_0)\frac{1+\nu_k^2}{1+|\mu-\varrho_k|^2}\operatorname{ch}\nu_k R_0, \tag{6}$$

$$\left|S_{R_0}\left[\int_0^R \frac{\sin \mu t}{\pi t} f_i(\mu_k, t)\,dt\right]\right| \leq \frac{c_7(R_0)(1+|\mu_k|)^{-i}e^{\nu_k R_0}}{1+|\mu-\varrho_k|^2}, \tag{7}$$

$$\begin{aligned}&\left|S_{R_0}\left[\int_0^R \frac{\sin \mu t}{\pi t} g_i(u_{k-i}, \mu_k, x, t)\right]\right| \leq \\ &\leq C_\varepsilon(R_0, o_k)\min\left\{\frac{1}{\mu(1+|\mu_k|)^\varepsilon}+\frac{|\mu_k|}{\mu^2}, \frac{\mu}{(1+|\mu_k|)^2}\right\}\times \\ &\times\left|\int_{x-R_0}^{x+R_0}|q(\xi)|\ln\frac{1}{|x-\xi|}\,d\xi\right|(1+|\mu_k|)^{-i}e^{\nu_k R_0}\|u_{k-i}\|_{L^\infty_{(x-R_0,x+R_0)}},\end{aligned} \tag{8}$$

$$(i = 0, 1, \dots, o_k;\ \mu \geq 1;\ \mu_k \in \mathbf{C};\ 0 < \varepsilon < 1).$$

We need the following

Lemma. *Let $G=(a, b)$, $b-a<\infty$, $q\in L^1(G)$, $\lambda\in\mathbf{C}$. Then there exists a constant $c>0$ such that for any eigenfunction u_i of order i of the operator $Lu=-u''+qu$ with eigenvalue λ the following estimate holds:*

$$\left|\int_a^x u_i(t)\,dt\right| \leq \frac{c}{1+|\sqrt{\lambda}|}(1+|\operatorname{Im}\sqrt{\lambda}|)^i\|u_i\|_\infty, \qquad (a \leq x \leq b) \tag{9}$$

where $\|\cdot\|_p := \|\cdot\|_{L^p(G)}$ also in below.

Proof. Use induction on i. We may suppose for brevity that $G=(0, 1)$. Use the Titchmarsh formula for u in the form

$$\begin{aligned}&u_i(x+t)+u_i(x-t)-2u_i(x)\cos\sqrt{\lambda}t = \\ &= \int_{x+t}^{x-t}|u_i(\xi)q(\xi)-u_{i-1}(\xi)|\frac{\sin\sqrt{\lambda}(t\mp|x-\xi|)}{\sqrt{\lambda}}\,d\xi.\end{aligned} \tag{T}$$

$$(x\pm t\in G;\ -\ \text{if}\ t>0,\ +\ \text{if}\ t<0).$$

Integrating for t from $-x$ to x we get $\left(\text{suppose } 0\leq x\leq\frac{1}{2}\right)$,

$$-4u_i(x)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}+2\int_0^{2x}u_i(t)\,dt =$$

$$= \left(\int_{-x}^{0}+\int_0^x\right)[u_i(\xi)q(\xi)-u_{i-1}(\xi)]\frac{\sin\sqrt{\lambda}(t\pm|x-\xi|)}{\sqrt{\lambda}}\,d\xi\,dt,$$

hence

$$\left|\int_0^{2x} u_i(t)\,dt\right| \leqq 2\,|u_i(x)| \left|\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\right| +$$

$$(10) \qquad +\frac{1}{2}\Big(\int_{-x}^{0} + \int_0^x\Big) \left|u_i(\xi)q(\xi)\frac{\sin\sqrt{\lambda}(t\pm|x-\xi|)}{\sqrt{\lambda}}\right| d\xi\,dt +$$

$$+\frac{1}{2}\left|\Big(\int_{-x}^{0} + \int_0^x\Big) u_{i-1}(\xi)\frac{\sin\sqrt{\lambda}(t\pm|x-\xi|)}{\sqrt{\lambda}}\,d\xi\,dt\right| =$$

$$= M_1+M_2+M_3.$$

We need

$$(11) \qquad \left|\frac{\sin z}{e^{|\operatorname{Im} z|}}\right| \leqq 1, \qquad (z\in\mathbf{C}),$$

$$(12) \qquad |\sin z|,\ |\cos z| \leqq 2\,|z|, \quad (|\operatorname{Im} z| \leqq 1),$$

$$(13) \qquad |u_i(x)|\, e^{|\operatorname{Im}\sqrt{\lambda}|\,d(x)} \leqq C_i(G, \|q\|_1)\big(1+\big|\operatorname{Im}\sqrt{\lambda}\big|\big)^i\, \|u_i\|_\infty,$$

$\big(d(x){:=}\min\{x, 1-x\}$; cf. V. Komornik [4]$\big)$,

$$(14) \qquad \|u_{i-1}\|_\infty \leqq C_i(G, \|q\|_1)\big(1+\big|\sqrt{\lambda}\big|\big)\big(1+|\operatorname{Im}\sqrt{\lambda}|\big)\|u_i\|_\infty,$$

$(i=1, 2, \ldots;$ cf. I. Joó [3]).

Obviously,

$$(15) \qquad 0 < t-|x-\xi| \leqq d(\xi) \quad \text{if} \quad t\in(0, x),$$

$$(16) \qquad \big|t+|x-\xi|\big| \leqq d(\xi) \quad \text{if} \quad t\in(-x, 0).$$

Using (11), (12), and (13) we get

$$M_1 \leqq \text{const}\,\|u_i\|_\infty\big(1+\big|\operatorname{Im}\sqrt{\lambda}\big|\big)^i\frac{1}{1+|\sqrt{\lambda}|}.$$

Taking into account (11), (12), (13), (15) and (16) we have

$$M_2 \leqq \text{const}\,\|u_i\|_\infty\,\|q\|_1\big(1+\big|\operatorname{Im}\sqrt{\lambda}\big|\big)^i\frac{1}{1+|\sqrt{\lambda}|}.$$

If $i=0$, then $u_{i-1}=u_{-1}\equiv 0$ and the Lemma is proved for $i=0$. Now suppose it is true for $0, 1, \ldots, i-1$ and prove it for i, i.e. suppose we know (9) for $i-1$

in place of i, and estimate M_3. Integrating by parts we get:

$$M_3 \leqq \Bigg| \Big[-\frac{\cos\sqrt{\lambda}t}{\sqrt{\lambda}} \int_{x-t}^{x+t} u_{i-1}(\xi)\frac{\cos\sqrt{\lambda}|x-\xi|}{\sqrt{\lambda}}\,d\xi \Big]_{t=-x}^{t=x} +$$

$$+\int_{-x}^{x} \frac{\cos\sqrt{\lambda}t}{\lambda}\,[u_{i-1}(x+t)\cos\sqrt{\lambda}t - u_{i-1}(x-t)\cos\sqrt{\lambda}t]\,dt -$$

$$-\Big[\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}} \int_{x-t}^{x+t} u_{i-1}(\xi)\frac{\sin\sqrt{\lambda}|x-\xi|}{\sqrt{\lambda}}\,d\xi \Big]_{t=-x}^{t=x} +$$

$$+\int_{-x}^{x} \frac{\sin\sqrt{\lambda}t}{\lambda}\,[u_{i-1}(x+t)\sin\sqrt{\lambda}t - u_{i-1}(x-t)\sin\sqrt{\lambda}t]\,dt \Bigg| \leqq$$

$$\leqq \left|\frac{1}{\lambda}\int_{0}^{2x} u_{i-1}(\xi)\cos\sqrt{\lambda}(x+|x-\xi|)\,d\xi\right| +$$

$$+\left|\frac{1}{\lambda}\int_{-2x}^{0} u_{i-1}(\xi)\cos\sqrt{\lambda}(x-|x-\xi|)\,d\xi\right| +$$

$$+\left|\frac{1}{2\lambda}\int_{-x}^{x} u_{i-1}(x+t)\,dt\right| + \left|\frac{1}{2\lambda}\int_{-x}^{x} u_{i-1}(x-t)\,dt\right| =$$

$$= M_3^1 + M_3^2 + M_3^3 + M_3^4.$$

Using (13) and (14) we get

$$M_3^j \leqq \frac{1}{1+|\sqrt{\lambda}|}\,(1+|\operatorname{Im}\sqrt{\lambda}|)^i\,\|u_i\|_\infty, \quad (j = 1, 2, 3, 4).$$

The Lemma is proved.

Now return to the proof of the Theorem. Applying the Lemma for the functions (v_k), it is enough to prove the estimates:

$$\sum_{k=1}^{\infty}\sum_{i=1}^{o_k} \frac{1}{1+|\mu_k|}\,\frac{(1+\nu_k)^i(1+|\mu_k|)^{-i}e^{\nu_k R_0}}{1+|\mu-\varrho_k|^2}\,\|u_{k-i}\|_\infty\|v_k\|_\infty = O\left(\frac{1}{\mu}\right), \tag{17}$$

$$\sum_{k=1}^{\infty}\sum_{i=0}^{o_k} C_\varepsilon(R_0, o_k)\min\left\{\frac{1}{\mu(1+|\mu_k|)^\varepsilon} + \frac{|\mu_k|}{\mu^2},\ \frac{\mu}{(1+|\mu_k|)^2}\right\}\times \tag{18}$$

$$\times\left|\int_{x-R_0}^{x+R_0} |q(\xi)|\ln\frac{1}{|x-\xi|}\,d\xi\right|(1+|\mu_k|)^{-i}e^{\nu_k R_0}\|u_{k-i}\|_\infty\|v_k\|_\infty\frac{1}{1+|\mu_k|}(1+\nu_k)^i = O\left(\frac{1}{\mu}\right),$$

$$\sum_{k=1}^{\infty}\frac{1+\nu_k^2}{1+|\mu-\varrho_k|^2}\,\mathrm{ch}\,\nu_k R_0\,\|u_k\|_\infty\|v_k\|_\infty\frac{1}{1+|\mu_k|}(1+\nu_k)^{o_k} = O\left(\frac{1}{\mu}\right).$$

Using (14) and the estimate

$$\sup_{\mu>0} \sum_{|\mu-\varrho_k|\le 1} \begin{pmatrix} \|u_k\|_\infty \\ \|v_k\|_\infty \end{pmatrix}^2 < \infty \tag{20}$$

(cf. I. Joó [6]) the estimates (17), (18), (19) follow by the method of [6].

The Theorem is proved.

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(Received June 20, 1985)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 381—399

SPECIAL CLASSES OF PROJECTIVE FINSLER CONNECTION TRANSFORMATIONS

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Dedicated to Professor Radu Miron on his 60th birthday

Summary

The parallelism of the curves relative to a Finsler connection $F\Gamma=(N, F, C)$ and the notion of coparallel Finsler connections are defined by R. Miron [3]. In this paper the notions of H-autoparallel and of V-autoparallel curves relative to a Finsler connection are defined and some classes of transformations which preserve the parallelism of the tangent directions are given. For any of these classes of transformations also the characteristic invariants are established. Finally, the notion of the enveloping directions of a curve is given and the transformation group, which preserves the enveloping directions is established. The notions and notations of M. Matsumoto [1], [2] and R. Miron [3] are used.

§ 1. Preliminaries

Let M be an n-dimensional differentiable manifold of class C^∞ and let (x^i, y^i) be the canonical coordinates of a point $y\in T(M)$. The tangent bundle $T(M)$ is a vector bundle having the base M, the canonical projection π, the fibre type R^n, and the structural group $GL(n, R)$. The natural basis of $T(M)$ with respect to canonical coordinates is $\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right)$ and the mapping N: $y\in T(M)\to N_y\subset T(M)_y$ is a regular distribution on $T(M)$ such that

$$T(M)_y = N_y\oplus T(M)^v_y.$$

Let

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^k_i \frac{\partial}{\partial y^k}$$

be a local basis of the n-dimensional local distribution N, where $N^i_k(x, y)$ are called the coefficients of non-linear connection defined by N.

Let $F\Gamma=(N, F, C)$ be a Finsler connection with the coefficients $(N^i_j, F^i_{jk}, C^i_{jk})$, we denote by $\mathcal{T}$ the group of Finsler connection transformations t: $(N, F, C)\to\to(\bar N, \bar F, \bar C)$ and by $\mathcal{T}_N$ the group of Finsler connection transformations t: $(N, F, C)\to(\bar N, \bar F, \bar C)$, which preserves the non-linear connection N. The transformations from $\mathcal{T}_N$ have the form

$$\bar N^i_j = N^i_j;\quad \bar F^i_{jk} = F^i_{jk} - B^i_{jk};\quad \bar C^i_{jk} = C^i_{jk} - D^i_{jk} \tag{1.1}$$

1980 *Mathematics Subject Classification* (1985 Revision). Primary 53B40; Secondary 53B10.

Key words and phrases. Finsler connection transformations, H-projective Finsler connection transformation, V-projective Finsler connection transformations, enveloping Finsler directions.

Akadémiai Kiadó, Budapest

where $B, D \in Z_2^1(M)$ are arbitrary Finsler tensor fields. In the following we denote this set by $\mathcal{T}_N = \{t \mid t \in \mathcal{T};\ t = t(0, B, D)\}$ and by $_{|}, |$ and $_{||}, \|$ the h- and v-covariant derivatives relative to $F\Gamma = (N, F, C)$ and to $F\bar{\Gamma} = (N, \bar{F}, \bar{C})$, respectively.

Let $C(x^i(t))$ be a curve on M and let $\tilde{C}(x^i(t), y^i(t))$ be a curve on $T(M)$ such that $\pi(\tilde{C}) = C$. In general, for a Finsler vector $u^i(x, y) \neq 0$ on $\pi^{-1}(U_\alpha)$ and for an arbitrary Finsler function $f(x, y) \neq 0$ we have the Finsler direction fu^i defined by u^i. The Finsler direction fu^i is displaced by parallelism along the curve C relative to the curve $\tilde{C}, \pi(\tilde{C}) = C$ and relative to the Finsler connection $F\Gamma = (N, F, C)$ if and only if

$$\frac{D}{dt}(fu^i) = 0; \quad \forall f \neq 0 \quad \text{or} \quad \frac{Du^i}{dt} = \frac{\omega}{dt}u^i \tag{1.2}$$

where $\omega = \alpha_k dx^k + \beta_k \delta y^k$ is an 1-form $\omega \in \Lambda^1(T(M))$.

This definition given by R. Miron [3] is a direct generalization of the parallel displacement from the linear connection theory. If the first relations (1.2) are verified for any $f \neq 0$ and for any C and $\tilde{C}$ with $\pi(\tilde{C}) = C$, we obtain the Finsler connection transformations, which preserve this condition called by R. Miron [3] coparallel connection transformations and given by

$$\bar{N}^i_j = N^i_j; \quad \bar{F}^i_{jk} = F^i_{jk} + \delta^i_j \varrho_k; \quad \bar{C}^i_{jk} = C^i_{jk} + \delta^i_j \sigma_k \tag{1.3}$$

where ϱ_k and σ_k are arbitrary Finsler covectors.

An extension of this transformations for a changed non-linear connection N is given in [10]. We consider

$$X = X^i \frac{\delta}{\delta x^i} + \dot{X}^i \frac{\partial}{\partial y^i}, \quad \text{where} \quad X^i = \frac{dx^i}{dt} \quad \text{and} \quad \dot{X}^i = \frac{\delta y^i}{dt}$$

are Finsler vectors and consider also the direction fX ($\forall f \neq 0$) tangent to the curve $\tilde{C};\ \pi(\tilde{C}) = C$.

§ 2. *H*-projective Finsler connection transformations ($t \in \mathcal{T}^H_{N(p)}$)

Definition 2.1. If the Finsler direction fX^i with $X^i = \dfrac{dx^i}{dt} \neq 0$ and with an arbitrary function $f(x, y) \neq 0$ is displaced by parallelism along the curve C relative to $\tilde{C}$, where $\pi(\tilde{C}) = C$ and relative to the Finsler connection $F\Gamma = (N, F, C)$, for any tangent direction φX ($\varphi \neq 0$ being an arbitrary Finsler function), then C is called an H-autoparallel curve relative to the Finsler connection $F\Gamma$.

This definition is equivalent to the following:

The field of Finsler direction fX^i is parallel on C with respect to $\tilde{C}(\pi(\tilde{C}) = C)$, and relative to the Finsler connection $F\Gamma = (N, F, C)$ if and only if the covariant differential of the vector field X^i on C satisfies the condition

$$\frac{DX^i}{dt} = \frac{\omega}{dt}X^i; \qquad X^i = \frac{dx^i}{dt}; \qquad \forall \varphi X \tag{2.1}$$

where $\omega = \tau_k dx^k + \omega_k \delta y^k$ is an 1-form, $\omega \in \Lambda^1(T(M))$.

DEFINITION 2.2. If the curve C is an H-autoparallel curve relative to the Finsler connection $F\Gamma=(N, F, C)$ and:

$$\frac{\delta y^i}{dt} \neq hX^i \qquad (\forall h \neq 0) \tag{2.2}$$

then the curve C is called an $1H$-autoparallel curve relative to the Finsler connection $F\Gamma$.

DEFINITION 2.3. The Finsler connection transformations $t\in\mathscr{T}_N$ which preserve the H-autoparallel curves are called H-projective Finsler connection transformations. The set of these transformations will be denoted by $\mathscr{T}^H_{N(p)}$.

DEFINITION 2.4. The Finsler connection transformations $t\in\mathscr{T}_N$ which preserve the $1H$-autoparallel curves are called $1H$-projective Finsler connection transformations. The set of these transformations will be denoted by $\mathscr{T}^{1H}_{N(p)}$.

LEMMA 2.1. *The Finsler connection transformation t is a H-projective Finsler connection transformation* ($t\in\mathscr{T}^H_{N(p)}$) *if and only if from the relation* (2.1) *it follows the relation*

$$\frac{\bar{D}X^i}{dt} = \frac{\bar{\omega}}{dt} X^i \quad \textit{for any } \varphi X \tag{2.3}$$

and reciprocally from (2.3) *it follows* (2.1).

LEMMA 2.2. *If we denote*

$$X^{ji} \overset{\text{def}}{=} X^j\left(\frac{DX^i}{dt} - \frac{\bar{D}X^i}{dt}\right) - X^i\left(\frac{DX^j}{dt} - \frac{\bar{D}X^j}{dt}\right) \tag{2.4}$$

then we have

$$X^{ji} = (\delta^j_h B^i_{sk} - \delta^i_h B^j_{sk}) X^s X^k X^h + (\delta^j_h D^i_{sk} - \delta^i_h D^j_{sk}) X^h X^s \frac{\delta y^k}{dt}. \tag{2.5}$$

LEMMA 2.3. *The Finsler connection transformation t is a H-projective Finsler connection transformation* ($t\in\mathscr{T}^H_{N(p)}$) *if and only if $X^{ji}=0$, $\forall\varphi X$ and t is an $1H$-projective Finsler connection transformation* ($t\in\mathscr{T}^{1H}_{N(p)}$) *if and only if: $X^{ji}=0$ and $\dfrac{\delta y^i}{dt}\neq hX^i$ ($\forall\varphi X$; $h\neq 0$).*

THEOREM 2.1. *The Finsler connection transformations $t(0, B, 0)\in\mathscr{T}_{NC}$ of the form* (1.1), *with*

$$-B^i_{sk} = \delta^i_s \varrho_k + \delta^i_k \varrho_s + \Omega^i_{sk}; \quad D^i_{jk} = 0, \tag{2.6}$$

where ϱ_k is an arbitrary Finsler covector, and Ω^i_{jk} is an arbitrary Finsler tensor of type (1.2) *such that $\Omega^i_{jk} = -\Omega^i_{kj}$, are H-projective Finsler connection transformations* ($t\in\mathscr{T}^H_{N(p)}$) *and simultaneously $1H$-projective Finsler connection transformations* ($t\in\mathscr{T}^{1H}_{N(p)}$) *and reciprocally.*

PROOF. If $t(0, B, 0)$ is of the form (1.1) with the conditions (2.6), since $D^i_{jk}=0$, $\Omega^i_{sk}+\Omega^i_{ks}=0$ and $T^{ji}_{sk}+T^{ji}_{ks}=0$, where:

$$T^{ji}_{sk} = \delta^j_s \delta^i_k - \delta^j_k \delta^i_s, \tag{2.7}$$

substituting (2.6) in (2.5) we obtain $X^{ji}=0$. If $\frac{\delta y^i}{dt} \neq hX^i$, then we have $\frac{\bar{\delta} y^i}{dt} = \frac{\delta y^i}{dt} \neq hX^i$; $\forall \varphi X$. Using the Lemma 2.3 it follows that the Finsler connection transformations $t(0, B, 0)$ given by (1.1) with (2.6) are H-projective and simultaneously $1H$-projective.

Reciprocally, from the condition $t(0, B, 0) \in \mathscr{T}^{1H}_{N(p)}$ or $t(0, B, 0) \in \mathscr{T}^{H}_{N(p)}$ we obtain:

$$(\delta^j_e B^i_{sk} - \delta^i_e B^j_{sk}) X^s X^k X^e = 0; \quad \forall \varphi X. \tag{2.8}$$

We can write:

$$\begin{cases} B^i_{sk} = \dfrac{1}{2}(B^i_{sk}+B^i_{ks}) + \dfrac{1}{2}(B^i_{sk}-B^i_{ks}) = U^i_{sk}+V^i_{sk}; \\ U^i_{sk} \overset{\text{def}}{=\!=} \dfrac{1}{2}(B^i_{sk}+B^i_{ks}); \quad V^i_{sk} \overset{\text{def}}{=\!=} \dfrac{1}{2}(B^i_{sk}-B^i_{ks}); \quad U^i_{sk} = U^i_{ks}; \quad V^i_{sk} = -V^i_{ks}. \end{cases} \tag{2.9}$$

From (2.8) and (2.9) we have:

$$(\delta^j_h U^i_{sk} - \delta^i_h U^j_{sk}) X^s X^k X^h = 0; \quad \forall \varphi X. \tag{2.10}$$

From $U^i_{sk} = U^i_{ks}$ it follows:

$$U^i_{sk} = -(\delta^i_s \varrho_k + \delta^i_k \varrho_s); \quad \varrho_k = -\frac{1}{n+1} U^i_{ik}; \quad (U^i_{ik} = U^i_{ki} \overset{\text{def}}{=\!=} -(n+1)\varrho_k). \tag{2.11}$$

Putting $\Omega^i_{jk} = -V^i_{jk}$, from (2.9) and (2.11) we have (2.6). It follows that $t(0, B, 0) \in \mathscr{T}^H_{N(p)}$ or $t(0, B, 0) \in \mathscr{T}^{1H}_{N(p)}$ has the form (1.1) with (2.6).

THEOREM 2.2. *The Finsler connection transformations* $t(0, 0, D) \in \mathscr{T}_{NF}$ *of the form* (1.1), *with*

$$B^i_{jk} = 0; \quad -D^i_{jk} = \delta^i_j \sigma_k \tag{2.12}$$

where σ_k *is an arbitrary Finsler covector, are H-projective Finsler connection transformations* ($t \in \mathscr{T}^H_{N(p)}$) *and reciprocally. These transformations are simultaneously* $1H$*-projective Finsler connection transformations* ($t \in \mathscr{T}^{1H}_{N(p)}$). *The transformations* $t(0, 0, D) \in \mathscr{T}^{1H}_{N(p)}$ *with* $\frac{\delta y^i}{dt} \neq 0$ *are of the form* (1.1) *with* (2.12). *The transformations* $t(0, 0, D)$ *with arbitrary* D^i_{jk} *are* $1H$*-projective, if* $\frac{\delta y^i}{dt} = 0$ *and reciprocally.*

PROOF. Using the relations (2.12) from X^{ji} given by (2.4) we have

$$X^{ji} = (-\delta^j_h \delta^i_s + \delta^i_h \delta^j_s) X^s X^h \sigma_k \frac{\delta y^k}{dt} = 0; \quad \forall \varphi X. \tag{2.13}$$

It follows that the Finsler connection transformations $t(0, 0, D)$ given by (1.1) with (2.12) are H-projective and simultaneously $1H$-projective.

Reciprocally, if $t(0, 0, D) \in \mathcal{T}^H_{N(p)}$ we have:

$$X^{ji} = (\delta^j_h D^i_{sk} - \delta^i_h D^j_{sk}) X^s X^h \frac{\delta y^k}{dt} = 0; \quad \forall \varphi X. \tag{2.14}$$

Consequently,

$$(\delta^j_h D^i_{sk} - \delta^i_h D^j_{sk}) X^s X^h = 0; \quad \forall \varphi X. \tag{2.15}$$

It follows immediately the relation (2.12).

The second part of the Theorem is obvious.

Since $\mathcal{T}_N$ is an abelian group relative to the composition of the transformations, from $t(0, B, 0) \in \mathcal{T}^H_{N(p)}$ and $t(0, 0, D) \in \mathcal{T}^H_{N(p)}$ it follows $t(0, B, D) = t(0, 0, D) \circ t(0, B, 0) \in \mathcal{T}^H_{N(p)}$. We have

THEOREM 2.3. *The Finsler connection transformations* $t(0, B, D) \in \mathcal{T}_N$ *of the form*

$$\bar{N}^i_j = N^i_j; \quad \bar{F}^i_{jk} = F^i_{jk} + \delta^i_j \varrho_k + \delta^i_k \varrho_j + \Omega^i_{jk}; \quad \bar{C}^i_{jk} = C^i_{jk} + \delta^i_j \sigma_k \tag{2.16}$$

where ϱ_k, σ_k *and* Ω^i_{jk} *are given as above, are* $1H$*-projective Finsler connection transformations and reciprocally.*

It follows that the most general Finsler connection transformations $t(0, B, D) \in \mathcal{T}^{1H}_{N(p)}$ are of the form (2.16) and it coincide with the most general Finsler connection transformations $t(0, B, D) \in \mathcal{T}^H_{N(p)}$ with $\dfrac{\delta y^i}{dt} \neq 0$.

THEOREM 2.4. *The Finsler connection transformations* $(t, 0, B, D) \in \mathcal{T}_N$ *of the form*

$$\bar{N}^i_j = N^i_j; \quad \bar{F}^i_{jk} = F^i_{jk} + \delta^i_j \varrho_k + \delta^i_k \varrho_j + \Omega^i_{jk}; \quad \bar{C}^i_{jk} = C^i_{jk} - D^i_{kj} \tag{2.17}$$

where $D \in \mathcal{T}^1_2(M)$, *are* H*-projective Finsler connection transformations if* $\dfrac{\delta y^i}{dt} = 0$, *and reciprocally.*

Particularly, for $\Omega^i_{jk} = -\delta^i_k \varrho_j + \delta^i_j \varrho_k$, we obtain the coparallel Finsler connection transformations [3]:

$$\bar{N}^i_j = N^i_j; \quad \bar{F}^i_{jk} = F^i_{jk} + \delta^i_j \varrho_k; \quad \bar{C}^i_{jk} = C^i_{jk} + \delta^i_j \sigma_k. \tag{2.18}$$

For $\dfrac{\delta y^i}{dt} = 0$ we obtain: $X = X^i \dfrac{\delta}{\delta x^i} \left(X^i = \dfrac{dx^i}{dt} \right)$. But from $\bar{N}^i_j = N^i_j$ we have: $\dfrac{\bar{\delta} y^i}{dt} = \dfrac{\delta y^i}{dt}$ and $\dfrac{\bar{\delta}}{\bar{\delta} x^i} = \dfrac{\delta}{\delta x^i}$. Therefore the most general transformations $t(0, B, D)$, which preserve the H-autoparallel curves C with $\dfrac{\delta y^i}{dt} = 0$; $\pi(\bar{C}) = C$, are characterized by (2.17).

Let l_y: $M_x \to N_y(\pi(y)=x)$ be an isomorphism, with the property:

(2.19) $$\pi'_y \circ l_y = 1_{M_x}$$

(2.20) $$l_y\colon Y = y^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(M) \to l_y(Y) = y^i \frac{\delta}{\delta x^i} \overset{\text{def}}{=} X^H$$

where X^H is the horizontal lift of X.

One integral curve of X^H is given by:

(2.21) $$X^i = \frac{dx^i}{dt} = y^i \neq 0; \qquad \frac{\delta y^i}{dt} = \frac{dy^i}{dt} + N^i_s \frac{dx^s}{dt} = 0;$$
$$N^i_s = N^i_s(x(t)y(t))$$

and it is called on N-intrinsic curve [3]. His orthogonal projection is called the autoparallel curve and is given by: $\frac{\delta}{\delta t}\left(\frac{dx^i}{dt}\right)=0$. If the Finsler direction fX^i or fY^i is parallel, then the most general Finsler connection transformations, which preserve this property, are given by (2.17).

Definition 2.5. If the curve C is an H-autoparallel curve with: $\frac{\delta y^i}{dt} = hX^i$, $h \neq 0$ (2.22), then C is called an $2H$-autoparallel curve, relative to the connection $F\Gamma = (N, F, C)$.

Definition 2.6. The Finsler connection transformations $t \in \mathscr{T}_N$, which preserve the $2H$-autoparallel curves are called $2H$-projective Finsler connection transformations ($t \in \mathscr{T}^{2H}_{N(p)}$).

Theorem 2.5. *A necessary and sufficient condition for a Finsler connection transformation $t(0, B, 0) \in \mathscr{T}_N$ to be a $2H$-projective transformation is that this transformation be of the form*

(2.23) $$\bar{N}^i_j = N^i_j; \quad \bar{F}^i_{jk} = F^i_{jk} + \delta^i_j \varrho_k + \delta^i_k \varrho_j + \Omega^i_{jk}; \quad \bar{C}^i_{jk} = C^i_{jk}.$$

Proof. From the relations (2.22), (2.5) and (2.23), where $-B^i_{jk} = \delta^i_j \varrho_k + \delta^i_k \varrho_j + \Omega^i_{jk}$; ϱ_k is an arbitrary Finsler covector and $\Omega^i_{jk} = -\Omega^i_{kj}$ is an arbitrary Finsler tensor, we obtain $X^{ji}=0$ for any $\varphi \neq 0$ from φX. Therefore $t(0, B, 0)$ is an H-projective Finsler connection transformation ($t \in \mathscr{T}^H_{N(p)}$) and simultaneously a $2H$-projective Finsler connection transformation ($t \in \mathscr{T}^{2H}_{N(p)}$), since $\bar{\delta} y^k = \delta y^k$. The proof of the reciprocal parts is analogously to the proof from Theorem 2.1.

Theorem 2.6. *A necessary and sufficient condition for a Finsler connection transformation $t(0, 0, D) \in \mathscr{T}_N$ to be a $2H$-projective transformation is that this transformation be of the form*

(2.24) $$\bar{N}^i_j = N^i_j; \quad \bar{F}^i_{jk} = F^i_{jk}; \quad \bar{C}^i_{jk} = C^i_{jk} + \delta^i_j \sigma_k + \delta^i_k \sigma_j + \theta^i_{jk}$$

where σ_k is an arbitrary Finsler covector and $\theta^i_{jk} = -\theta^i_{kj}$ is an arbitrary Finsler tensor.

PROOF. Putting

$$(2.25)\qquad -D^i_{jk}=\delta^i_j\sigma_k+\delta^i_k\sigma_j+\theta^i_{jk};\qquad B^i_{jk}=0$$

in (2.5) we obtain $X^{ij}=0$ for any $\varphi\neq 0$ from φX. Having also $\bar{\delta} y^k=\delta y^k$, from (2.22) it follows $\frac{\delta y^k}{dt}=hX^i$; $h\neq 0$, therefore the transformation (2.24) is $2H$-projective. Reciprocally, from (2.22), (2.5) and (2.24) we obtain

$$(2.26)\qquad h(\delta^j_e D^i_{sk}-\delta^i_e D^j_{sk})X^s X^e X^k=0\qquad \forall\varphi X\quad (\forall\varphi X^k),\quad h\neq 0.$$

We denote

$$(2.27)\qquad D^i_{sk}=\overline{U}^i_{sk}+\overline{V}^i_{sk},$$

where $\overline{U}^i_{sk}=\frac{1}{2}(D^i_{sk}+D^i_{ks})=\overline{U}^i_{ks}$; $\overline{V}^i_{sk}=\frac{1}{2}(D^i_{sk}-D^i_{ks})=-\overline{V}^i_{ks}$. It follows $\overline{V}^i_{sk}X^sX^k=0$ and from (2.26), (2.27) we obtain (2.25).

Since $t(0,B,D)=t(0,B,0)\circ t(0,0,D)$, it follows

THEOREM 2.7. *A necessary and sufficient condition for a Finsler connection transformation $t\in\mathcal{T}_N$ to be a $2H$-projective transformation is that this transformation be of the form*

$$(2.28)\quad \overline{N}^i_j=N^i_j;\quad \overline{F}^i_{jk}=F^i_{jk}+\delta^i_j\varrho_k+\delta^i_k\varrho_j+\Omega^i_{jk};\quad \overline{C}^i_{jk}=C^i_{jk}+\delta^i_j\sigma_k+\delta^i_k\sigma_j+\theta^i_{jk}$$

where ϱ_k, σ_k are arbitrary Finsler covectors and $\theta^i_{jk}=-\theta^i_{kj}$; $\Omega^i_{jk}=-\Omega^i_{kj}$ are arbitrary Finsler tensors.

Consequently, the H-projective Finsler connection transformations, i.e. the Finsler connection transformations $t\in\mathcal{T}_N$ which preserve both the $1H$-projective curves and the $2H$-projective curves, are obtained from (2.28) for

$$(2.29)\qquad \theta^i_{jk}=-\delta^i_k\sigma_j+\delta^i_j\sigma_k.$$

THEOREM 2.8. *The $\mathcal{T}^H_{N(p)}$ group of the H-projective Finsler connection transformations is a subgroup of the $2H$-projective Finsler connection transformations group $\mathcal{T}^{2H}_{N(p)}$.*

We have in this way also a geometrical interpretation of the Finsler connection transformations considered algebraical in [7]. In [7] are introduced the invariants:

$$(2.30)\qquad \underset{1}{I}{}^i_{jk}=T^i_{jk}-\frac{1}{(n-1)}(\delta^i_jT_k-\delta^i_kT_j);\qquad \underset{2}{I}{}^i_{jk}=S^i_{jk}-\frac{1}{n-1}(\delta^i_jS_k-\delta^i_kS_j)$$

with: $T_k=T^i_{ik}$; $S_k=S^i_{ik}$, where T^i_{jk} and S^i_{jk} are the h- and the v-torsion tensor, respectively.

THEOREM 2.9 [7]. *The most general Finsler connection transformations $t\in\mathcal{T}_N$, which have the invariants $\underset{1}{I}{}^i_{jk}$ and $\underset{2}{I}{}^i_{jk}$, are given by*

$$(2.31)\quad \overline{N}^i_j=N^i_j;\quad \overline{F}^i_{jk}=F^i_{jk}+\delta^i_k\alpha_j-\delta^i_j\alpha_k-U^i_{jk};\quad \overline{C}^i_{jk}=C^i_{jk}+\delta^i_k\beta_j-\delta^i_j\beta_k-V^i_{jk}$$

where α_k, β_k *are arbitrary Finsler covectors and* $U^i_{jk}=U^i_{kj}$; $V^i_{jk}=V^i_{kj}$ *are arbitrary Finsler tensors.*

Imposing the condition that the Finsler connection transformations (2.31) preserve the $2H$-autoparallel curves, it follows

THEOREM 2.10. *A necessary and sufficient condition for the Finsler connection transformations* $t\in\mathcal{T}_N$ *which preserve the* $2H$*-autoparallel curves to have the invariants* $\underset{1}{I}$ *and* $\underset{2}{I}$ *is that* t *be of the form*

$$\text{(2.32)}\qquad \begin{cases} \bar{N}^i_j = N^i_j; \quad \bar{F}^i_{jk} = F^i_{jk}+\delta^i_j\varrho_k+\delta^i_k\varrho_j+\dfrac{1}{n-1}(\delta^i_j U_k-\delta^i_k U_j) \\ \bar{C}^i_{jk} = C^i_{jk}+\delta^i_j\sigma_k+\delta^i_k\sigma_j+\dfrac{1}{n-1}(\delta^i_j V_k-\delta^i_k V_j) \end{cases}$$

where ϱ_k, σ_k, U_k, V_k *are arbitrary Finsler covectors, or that* t *be of the form*

$$\text{(2.33)}\qquad \bar{N}^i_j = N^i_j; \quad \bar{F}^i_{jk} = F^i_{jk}+\delta^i_j\xi_k+\delta^i_k\eta_j; \quad \bar{C}^i_{jk} = C^i_{jk}+\delta^i_j\varphi_k+\delta^i_k\theta_j$$

where ξ_k, η_k, φ_k, θ_k *are arbitrary Finsler covectors.*

THEOREM 2.11. *The most general Finsler connection transformations* $t\in\mathcal{T}_N$ *which have the invariants* $\underset{1}{I}$, $\underset{2}{I}$ *and preserve the* H*-autoparallel curves have the form*

$$\text{(2.34)}\qquad \bar{N}^i_j = N^i_j; \quad \bar{F}^i_{jk} = F^i_{jk}+\delta^i_j\xi_k+\delta^i_k\eta_j; \quad \bar{C}^i_{jk} = C^i_{jk}+\delta^i_j\varphi_k$$

where ξ_k, η_k *and* φ_k *are arbitrary Finsler covectors.*

THEOREM 2.12. *If* $F\Gamma=(N, F, C)$ *is a fixed, semi-symmetric Finsler connection transformation* ($\underset{1}{I}=0$; $\underset{2}{I}=0$), *then the transformations* (2.33) *are the most general semi-symmetric Finsler connection transformations, which preserve the* $2H$*-autoparallel curves.*

THEOREM 2.13. *If* $F\Gamma=(N, F, C)$ *is a fixed, semi-symmetric Finsler connection transformation* ($I_1=0$, $I_2=0$), *then the transformations* (2.34) *are the most general semi-symmetric Finsler connection transformations* ($\underset{1}{\bar{I}}=0$, $\underset{2}{\bar{I}}=0$), *which preserve the* H*-autoparallel curves.*

If we consider in (2.33) $\xi_k=\eta_k$, $\varphi_k=\theta_k$ and $\underset{1}{I}=0$, $\underset{2}{I}=0$, it follows

THEOREM 2.14. *The Finsler connection transformations* $t\in\mathcal{T}_N$ *of the form*

$$\text{(2.35)}\qquad \bar{N}^i_j = N^i_j; \quad \bar{F}^i_{jk} = F^i_{jk}+\delta^i_j\varrho_k+\delta^i_k\varrho_j; \quad \bar{C}^i_{jk} = C^i_{jk}+\delta^i_j\sigma_k+\delta^i_k\sigma_j$$

with ϱ_k, σ_k *arbitrary Finsler covectors and* $\underset{1}{I}=0$, $\underset{2}{I}=0$, *are the semi-symmetric Finsler connection transformations, which preserve the* $2H$*-autoparallel curves, and have the general invariants of Weyl type* [6]:

$$\text{(2.36)}\qquad \underset{1}{\bar{W}}{}^i_{jkh} = \underset{1}{W}{}^i_{jkh}; \quad \underset{2}{\bar{W}}{}^i_{jkh} = \underset{2}{W}{}^i_{jkh}; \quad \underset{3}{\bar{W}}{}^i_{jkh} = \underset{3}{W}{}^i_{jkh}$$

where

$$W^i_{1\,jkh} = K^i_{jkh} + \frac{1}{n+1}\delta^i_j B_{nk} + \delta^i_k\left(\frac{K_{jh}}{n-1} + \frac{B_{jh}}{n^2-1}\right) - \delta^i_h\left(\frac{K_{jk}}{n-1} + \frac{B_{jk}}{n^2-1}\right) \tag{2.37}$$

$$W^i_{2\,jkh} = S^i_{jkh} + \frac{1}{n+1}\delta^i_j s_{hk} + \delta^i_k\left(\frac{S_{jh}}{n-1} - \frac{s_{jh}}{n^2-1}\right) - \delta^i_h\left(\frac{S_{jk}}{n-1} + \frac{s_{jk}}{n^2-1}\right) \tag{2.38}$$

$$W^i_{3\,jkh} = \mathscr{P}^i_{jkh} + \frac{1}{n+1}\delta^i_j \pi_{hk} + \delta^i_k\left(\frac{\mathscr{P}_{jh}}{n-1} + \frac{\pi_{jh}}{n^2-1}\right) - \delta^i_h\left(\frac{\mathscr{P}_{jk}}{n-1} + \frac{\pi_{jk}}{n^2-1}\right) \tag{2.39}$$

with

$$K^i_{jkh} = R^i_{jkh} - C^i_{jr}R^r_{kh}; \quad K_{jk} = K^i_{jki}; \quad B_{kh} = K^i_{ikh} \tag{2.40}$$

$$\mathscr{P}^i_{jkh} = \left(P^i_{jkh} - C^i_{jr}\frac{\partial N^r_k}{\partial y^h}\right) - \left(P^i_{jhk} - C^i_{jr}\frac{\partial N^r_h}{\partial y^k}\right); \quad \mathscr{P}_{jk} = \mathscr{P}^i_{jki}; \quad \pi_{kh} = \mathscr{P}^i_{ikh} \tag{2.41}$$

$$S_{jk} = S^i_{jki}, \quad s_{kh} = S^i_{ikh}. \tag{2.42}$$

R^i_{jkh}, S^i_{jkh}, P^i_{jkh} are the curvature tensors of the Finsler connection $F\Gamma = (N^i_j, F^i_{jk}, C^i_{jk})$ and R^r_{kh} are the torsion tensor of the non-linear connection N. From [5] we have

THEOREM 2.15. *The most general Finsler connection transformations* $t \in \mathscr{T}_N$, *which have the invariants* $J^i_{1\,jk}$, $J^i_{2\,jk}$ *where*

$$J^i_{1\,jk} \stackrel{\text{def}}{=} F^i_{jk} - \frac{1}{n+1}\left(\frac{1}{n-1}T_k + F_k\right)\delta^i_j + \frac{1}{n+1}\left(\frac{n}{n-1}T_j - F_j\right)\delta^i_k \tag{2.43}$$

$$J^i_{2\,jk} \stackrel{\text{def}}{=} C^i_{jk} - \frac{1}{n+1}\left(\frac{1}{n-1}S_k + C_k\right)\delta^i_j + \frac{1}{n+1}\left(\frac{n}{n-1}S_j - C_j\right)\delta^i_k \tag{2.44}$$

are of the form (2.32) *or* (2.33).

We have the following geometrical interpretation:

THEOREM 2.16. *The most general Finsler connection transformations* $t \in \mathscr{T}_N$, *which have the invariants* $\underset{1}{J}$ *and* $\underset{2}{J}$, *are the Finsler connection transformations which have the invariants* $\underset{1}{I}$ *and* $\underset{2}{I}$ *and preserve the 2H-autoparallel curves.*

A generalization of these Finsler connection transformations for a changed non-linear connection $N^i_j \to \overline{N}^i_j \neq N^i_j$ is given in [9], presented by the Romanian—Japanese Colloquium on Finsler Geometry, Iaşi—Braşov, 1984; here is given a geometrical interpretation of these transformations.

Since

$$J^i_{1\,jk} - J^i_{1\,kj} = I^i_{1\,jk}; \qquad J^i_{2\,jk} - J^i_{2\,kj} = I^i_{2\,jk}$$

we have

THEOREM 2.17. *The most general semi-symmetric Finsler connection transformations which preserve the 2H-autoparallel curves and have the invariants* $\underset{1}{J}$ *and* $\underset{2}{J}$ *are given by* (2.33) *with* $\underset{1}{J}{}^i_{jk}=\underset{1}{J}{}^i_{kj}$; $\underset{2}{J}{}^i_{jk}=\underset{2}{J}{}^i_{kj}$.

The invariants $\underset{2}{J}{}^i_{jk}$ are generalization of the projective connection coefficients of Thomas from the theory of symmetric linear connections [11], [12].

§ 3. V-projective Finsler connection transformations ($t\in\mathscr{T}^V_{N(P)}$)

DEFINITION 3.1. If the Finsler direction determined by $\dot{X}^i\left(\dot{X}^i=\frac{\delta y^i}{\delta t}\right)\neq 0\Big)$ is displaced by parallelism along the curve C relative to $\tilde{C}$, $\pi(\tilde{C})=C$, and relative to the Finsler connection $F\Gamma=(N, F, C)$, for any tangent direction φX $(\forall\varphi\neq 0)$, then C is called V-autoparallel curve relative to the Finsler connection $F\Gamma$.

From this definition it follows, that a field of Finsler directions $f\dot{X}^i$ is parallel on C with respect to $\tilde{C}$, $\pi(\tilde{C})=C$, and relative to the Finsler connection $F\Gamma=(N, F, C)$ if and only if the covariant differential of the vector field $\dot{X}^i$ on C satisfies the condition:

$$\frac{D\dot{X}^i}{dt}=\frac{\alpha}{dt}\dot{X}^i; \qquad \forall\varphi; \quad \forall\varphi X \quad (\dot{X}^i\neq 0) \tag{3.1}$$

where $\alpha=\alpha_k dx^k+\beta_k\delta y^k$ is a Finsler 1-form on $T(M)$.

DEFINITION 3.2. If the curve C is a V-autoparallel curve relative to the Finsler connection $F\Gamma=(N, F, C)$ and

$$\dot{X}^i\neq hX^i; \quad X^i=\frac{dx^i}{dt}, \quad \forall h\neq 0 \tag{3.2}$$

then the curve C is called an $1V$-autoparallel curve relative to the Finsler connection $F\Gamma$.

DEFINITION 3.3. The Finsler connection transformations $t\in\mathscr{T}_N$, which preserve the V-autoparallel curves are called V-projective Finsler connection transformations. The set of these transformations will be denoted by $\mathscr{T}^V_{N(p)}$.

DEFINITION 3.4. The Finsler connection transformations $t\in\mathscr{T}_N$ which preserve the $1V$-autoparallel curves are called $1V$-projective Finsler connection transformations. The set of these transformations will be denoted by $\mathscr{T}^{1V}_{N(p)}$.

It follows, that a Finsler connection transformation t: $F\Gamma=(N, F, C)\to F\bar{\Gamma}=(\bar{N}=N, \bar{F}, \bar{C})$ is V-projective if we have

$$\frac{\bar{D}\dot{X}^i}{dt}=\frac{\bar{\alpha}}{dt}\dot{X}^i\Leftrightarrow\frac{D\dot{X}^i}{dt}=\frac{\alpha}{dt}\dot{X}^i; \quad \forall\varphi X \tag{3.3}$$

since $\delta y^k=\delta y^k$.

LEMMA 3.1. *If we denote*

$$V^{ij} \overset{\text{def}}{=} \dot{X}^j\left(\frac{D\dot{X}^i}{dt}-\frac{\bar{D}\dot{X}^i}{dt}\right)-\dot{X}^i\left(\frac{D\dot{X}^j}{dt}-\frac{\bar{D}\dot{X}^j}{dt}\right) \tag{3.4}$$

then we have

$$V^{ij} = (\delta^j_h B^i_{sk}-\delta^i_h B^j_{sk})\dot{X}^s\dot{X}^h X^k+(\delta^j_h D^i_{sk}-\delta^i_h D^j_{sk})\dot{X}^s\dot{X}^h\dot{X}^k. \tag{3.5}$$

LEMMA 3.2. *The Finsler connection transformation t is a V-projective Finsler connection transformation ($t\in\mathcal{T}^V_{N(p)}$) if and only if $V^{ij}=0$, $\forall\varphi X$ and t is an 1V-projective Finsler connection transformation ($t\in\mathcal{T}^{1V}_{N(p)}$) if and only if $V^{ij}=0$ and $\dot{X}^i\neq hX^i$, $\forall h$.*

THEOREM 3.1. *A necessary and sufficient condition that the Finsler connection transformation $t(0, 0, D)\in\mathcal{T}_N$ to be an 1V-projective Finsler connection transformation is that this transformation to be of the form:*

$$\bar{N}^i_j = N^i_j;\quad \bar{F}^i_{jk} = F^i_{jk};\quad \bar{C}^i_{jk} = C^i_{jk}+\delta^i_j\sigma_k+\delta^i_k\sigma_j+\theta^i_{jk} \tag{3.6}$$

where σ_k is an arbitrary Finsler covector and $\theta^i_{jk}=-\theta^i_{kj}$ is and arbitrary Finsler tensor ($\theta\in\mathcal{T}^1_2(M)$).

Evidently, $t(0, 0, D)$ given by (3.6) is also a V-projective Finsler connection transformation. The proof is analogous with the proof for the H-projective Finsler connection transformations in § 2.

THEOREM 3.2. *A necessary and sufficient condition for a Finsler connection transformation $t(0, B, 0)\in\mathcal{T}_N$ to be an 1V-projective Finsler connection transformation is that this transformation be of the form*

$$\bar{N}^i_j = N^i_j;\quad \bar{F}^i_{jk} = F^i_{jk}+\delta^i_j\varrho_k;\quad \bar{C}^i_{jk} = C^i_{jk} \tag{3.7}$$

where ϱ_k is an arbitrary Finsler covector.

It follows easily that $t(0, B, 0)$ given by (3.7) is also a V-projective Finsler connection transformation.

Since $\mathcal{T}_N$ is an abelian group relative to the composition of the transformations, it follows:

$$t(0, B, D) = t(0, B, 0)\cdot t(0, 0, D)\in\mathcal{T}_N \tag{3.8}$$

and we have

THEOREM 3.3. *A necessary and sufficient condition for a Finsler connection transformation $t(0, B, D)\in\mathcal{T}_N$ to be an 1V-projective Finsler connection transformation is that this transformation be of the form*

$$\bar{N}^i_j = N^i_j;\quad \bar{F}^i_{jk} = F^i_{jk}+\delta^i_j\varrho_k;\quad \bar{C}^i_{jk} = C^i_{jk}+\delta^i_j\sigma_k+\delta^i_k\sigma_j+\theta^i_{jk} \tag{3.9}$$

where ϱ_k, σ_k are arbitrary Finsler covectors and $\theta^i_{jk}=-\theta^i_{kj}$ is an arbitrary Finsler tensor.

THEOREM 3.4. *A necessary and sufficient condition for a Finsler connection transformation* $t(0, B, D) \in \mathcal{T}_N$ *to be an* 1*V-projective Finsler connection transformation if* $X^i = 0$, *is that this transformation be of the form*

$$\bar{N}^i_j = N^i_j; \quad \bar{F}^i_{jk} = F^i_{jk} - B^i_{jk}; \quad \bar{C}^i_{jk} = C^i_{jk} + \delta^i_j \sigma_k + \delta^i_k \sigma_j + \theta^i_{jk} \tag{3.10}$$

where σ_k *is an arbitrary Finsler covector, and* B^i_{jk}, θ^i_{jk} *are arbitrary Finsler tensors, with* $\theta^i_{jk} = -\theta^i_{kj}$.

For $X^i = 0$ we have: $X^i = \frac{\delta y^i}{dt}\frac{\partial}{\partial y^i}$; $\dot{X}^i = \frac{\delta y^i}{dt}$ and the transformation (3.10) preserve this property.

Let l_v: $M_x \to T(M)^v_y$; $\pi(y) = x$ be the canonical isomorphism. It follows that the vertical lift of $Y = y^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(M)$ is of the form

$$X^v = l_v(Y) = y^i \frac{\partial}{\partial y^i} \qquad i = \overline{1, n}. \tag{3.11}$$

One integral curve of X^v is given by [3]:

$$\frac{dx^i}{dt} = 0; \quad \frac{dy^i}{dt} = y^i. \tag{3.12}$$

Consequently

$$X^i = 0; \quad \dot{X}^i = \frac{\delta y^i}{dt} = \frac{dy^i}{dt} \neq 0. \tag{3.13}$$

If $\pi(\tilde{C}) = C$ is a V-autoparallel curve, it follows easily that the transformation (3.10) preserve this curve.

DEFINITION 3.5. A V-autoparallel curve C with

$$\dot{X}^i = hX^i \qquad (h \neq 0) \tag{3.14}$$

is called a $2V$-autoparallel curve, relative to the connection $F\Gamma = (N, F, C)$.

DEFINITION 3.6. The Finsler connection transformations $t \in \mathcal{T}_N$, which preserve the $2V$-autoparallel curves are called $2V$-projective Finsler connection transformations. The set of these transformations will be denoted by $\mathcal{T}^{2v}_{N(p)}$.

It follows

THEOREM 3.5. *The Finsler connection transformations* $t(0, 0, D)$ *given by* (3.6) *are* $2V$*-projective Finsler connection transformations and reciprocally.*

THEOREM 3.6. *The Finsler connection transformations* $t(0, B, 0)$ *given by* (3.7) *are* $2V$*-projective Finsler connection transformations and reciprocally.*

We have also

THEOREM 3.7. *A necessary and sufficient condition for a Finsler connection transformation* $t(0, B, 0)$ *to be a* $2V$*-projective Finsler connection transformation is that this transformation be of the form* (2.28).

THEOREM 3.8. *The Finsler connection transformations* $t\in\mathcal{T}_N$ *which preserve the* $1V$*-autoparallel curves, preserve also the* $2V$*-autoparallel curves.*

The set $\mathcal{T}^{1v}_{N(p)}$ is a subgroup of $\mathcal{T}^{2v}_{N(p)}$, any $t\in\mathcal{T}^{1v}_{N(p)}$ is obtained from $t\in\mathcal{T}^{2V}_{N(p)}$ considering the particular case

$$\Omega^i_{jk}=-\delta^i_k\varrho_j+\delta^i_j\varrho_k. \tag{3.15}$$

§ 4. 1-projective ($t\in\mathcal{T}^1_{N(p)}$) and 2-projective ($t\in\mathcal{T}^2_{N(p)}$) Finsler connection transformations

DEFINITION 4.1. The Finsler connection transformations $t\in\mathcal{T}_N$ which preserve both the $1H$-autoparallel curves and the $1V$-autoparallel curves are called 1-projective Finsler connection transformations. We denote the set of these transformations with $\mathcal{T}^1_{N(p)}$.

It follows immediately

THEOREM 4.1. *The coparallel Finsler connection transformations are the most general* 1*-projective Finsler connection transformations.*

THEOREM 4.2. *The Finsler connection transformations*

$$\bar N^i_j=N^i_j;\quad \bar F^i_{jk}=F^i_{jk}+\delta^i_j\varrho_k+\delta^i_k\varrho_j;\quad \bar C^i_{jk}=C^i_{jk}+\delta^i_j\sigma_k+\delta^i_k\sigma_j \tag{4.1}$$

are 1*-projective Finsler connection transformations if the Finsler covectors* ϱ_k *and* σ_k *have the properties*

$$\varrho_kX^k=0;\quad \sigma_k\dot X^k=0. \tag{4.2}$$

DEFINITION 4.2. The tangent direction φX $(\forall\varphi\neq0)$ is displaced by parallelism along the curve C relative to $\tilde C$, $\pi(\tilde C)=C$, and relative to the Finsler connection $F\Gamma=(N, F, C)$, if the Finsler directions determined by X^i and $\dot X^i$ have this property. In this case, the curve C is called proper 1-autoparallel curve relative to the connection $F\Gamma$.

DEFINITION 4.3. The Finsler connection transformations $t\in\mathcal{T}_N$ which preserve the proper 1-autoparallel curves are called proper 1-projective Finsler connection transformations.

THEOREM 4.3. *The* 1*-projective Finsler connection transformations are proper* 1*-projective Finsler connection transformations.*

DEFINITION 4.4. A proper 1-autoparallel curve with $\dot X^i=hX^i(h\neq0)$ is called a proper 2-autoparallel curve.

DEFINITION 4.5. The Finsler connection transformations $t\in\mathcal{T}_N$ which preserve the proper 2-autoparallel curves are called proper 2-projective Finsler connection transformations. This set will be denoted by $\mathcal{T}^2_{N(p)}$.

It follows easily

THEOREM 4.4. *The 2H-autoparallel curves and the 2V-autoparallel curves are proper 2-autoparallel curves.*

THEOREM 4.5. *The most general proper 2-projective Finsler connection transformations have the form:*

$$\bar{N}^i_j = N^i_j;\quad \bar{F}^i_{jk} = F^i_{jk} + \delta^i_j \varrho_k + \delta^i_k \varrho_j + \Omega^i_{jk};\quad \bar{C}^i_{jk} = C^i_{jk} + \delta^i_j \sigma_k + \delta^i_k \sigma_j + \theta^i_{jk}. \tag{4.3}$$

THEOREM 4.6. *The most general proper 2-projective Finsler connection transformations ($t \in \mathscr{T}^2_{N(p)}$) which have the invariants $\underset{1}{I}$, $\underset{2}{I}$, are the most general Finsler connection transformations $t \in \mathscr{T}_N$, which have the invariants $\underset{1}{J}$, $\underset{2}{J}$. Their set is given by* (2.33).

§ 5. Enveloping directions

Let $\Sigma = (fU^i)$ a set of Finsler directions determined by the Finsler vector set $U^i(t)$ along the curve C relative to $\tilde{C}$ with $\pi(\tilde{C}) = C$.

DEFINITION 5.1. The set of Finsler directions Σ is an H-enveloping Finsler direction set if between $X^i = \dfrac{dx^i}{dt}$, $\dfrac{DU^i}{dt}$, U^i exists a linear dependence relation.

From the Definition 5.1 we have equivalently:

$$DU^{[i}U^j dx^{k]} = 0. \tag{5.1}$$

DEFINITION 5.2. A Finsler connection transformation $t \in \mathscr{T}_N$ which preserve the H-enveloping Finsler directions is called H-enveloping Finsler connection transformation. The set of these transformations will be denoted by $\mathscr{T}_{NH}$.

By the Definition 5.2, the relation (5.1) is equivalent to

$$\bar{D}U^{[i}U^j dx^{k]} = 0;\quad \forall t \in \mathscr{T}_{NH} \tag{5.2}$$

and it follows easily

LEMMA 5.1. *For a Finsler connection transformation $t \in \mathscr{T}_N$ we have*

$$\bar{D}U^{[i}U^j dx^{k]} = DU^{[i}U^j dx^{k]} - B^{[i}_{rs}U^j dx^{k]} U^r dx^s - D^{[i}_{rs}U^j dx^{k]} U^r y^s \tag{5.3}$$

$$\bar{D}U^{[i}U^j dx^{k]} = DU^{[i}U^j dx^{k]} - B^{[i}_{rs}\delta^j_h \delta^{k]}_p U^r U^h dx^s dx^p - D^{[i}_{rs}\delta^j_h \delta^{k]}_p U^r U^h dx^p \delta y^s. \tag{5.4}$$

THEOREM 5.1. *A necessary and sufficient condition for a Finsler connection transformation $t(0, 0, D) \in \mathscr{T}_N$ of the form*

$$\bar{N}^i_j = N^i_j;\quad \bar{F}^i_{jk} = F^i_{jk};\quad \bar{C}^i_{jk} = C^i_{jk} + D^i_{jk};\quad D^i_{jk} \in \mathscr{T}^1_2(M) \tag{5.5}$$

to be an H-enveloping Finsler transformation is that this transformation be of the form

$$\bar{N}^i_j = N^i_j;\quad \bar{F}^i_{jk} = F^i_{jk};\quad \bar{C}^i_{jk} = C^i_{jk} + \delta^i_j \sigma_k \tag{5.6}$$

where σ_k is an arbitrary Finsler covector.

PROOF. From (5.1) and (5.6) it follows:

$$\bar{D}U^{[i}U^j dx^{k]} = \delta^{[i}_r \delta^j_h \delta^{k]}_p U^r U^h dx^p \sigma_s \delta y^s = 0. \tag{5.7}$$

Consequently, (5.6) is an H-enveloping Finsler transformation. Reciprocally: If $t(0, 0, D)$ is an H-enveloping Finsler transformation, then from (5.1), (5.2), (5.3) and (5.5) it follows

$$D^{[i}_{rs}\delta^j_h\delta^{k]}_p U^r U^h dx^p \delta y^s = 0. \tag{5.8}$$

Writing detailed and contracting for $j=h$ and $k=p$, we obtain (5.6).

THEOREM 5.2. *A necessary and sufficient condition for a Finsler connection transformation* $t(0, B, 0)\in\mathcal{T}_N$ *of the form*

$$\bar{N}^i_j = N^i_j;\quad \bar{F}^i_{jk} = F^i_{jk} + B^i_{jk};\quad \bar{C}^i_{jk} = C^i_{jk};\quad B^i_{jk}\in Z^1_2(M) \tag{5.9}$$

to be an H-enveloping Finsler transformation is that this transformation be of the form

$$\bar{N}^i_j = N^i_j;\quad \bar{F}^i_{jk} = F^i_{jk} + \delta^i_j\alpha_k + \delta^i_k\beta_j;\quad \bar{C}^i_{jk} = C^i_{jk} \tag{5.10}$$

where α_k, β_k *are arbitrary Finsler covectors.*

PROOF. From (5.1), (5.3) and (5.10) it follows:

$$\bar{D}U^{[i}U^j dx^{k]} = \alpha_s\delta^{[i}_r\delta^j_h\delta^{k]}_p U^r U^h dx^p dx^s + \beta_r\delta^{[i}_s\delta^j_h\delta^{k]}_p U^r U^h dx^p dx^s. \tag{5.11}$$

Cancelling, we obtain (5.2), consequently (5.10) is an H-enveloping Finsler connection transformation. Reciprocally: from (5.1), (5.2), (5.3) and (5.8) it follows

$$B^{[i}_{rs}\delta^j_h\delta^{k]}_p U^r U^h dx^p dx^s = 0. \tag{5.12}$$

Making equal to zero the coefficients of this relations, we obtain

$$B^{[i}_{rs}\delta^j_h\delta^{k]}_p + B^{[i}_{hs}\delta^j_r\delta^{k]}_p + B^{[i}_{rp}\delta^j_h\delta^{k]}_s + B^{[i}_{hp}\delta^j_r\delta^{k]}_s = 0 \tag{5.13}$$

and contracting for $j=h$, $k=p$ we obtain (5.10).

Using the relation (3.8) we have

THEOREM 5.3. *A necessary and sufficient condition for a Finsler connection transformation* $t\in\mathcal{T}_N$ *to be an H-enveloping Finsler connection transformation* ($t\in\mathcal{T}_{NH}$) *is that t be of the form*

$$\bar{N}^i_j = N^i_j;\quad \bar{F}^i_{jk} = F^i_{jk} + \delta^i_j\alpha_k + \delta^i_k\beta_j;\quad \bar{C}^i_{jk} = C^i_{jk} + \delta^i_j\sigma_k \tag{5.14}$$

where α_k, β_k, σ_k *are arbitrary Finsler covectors.*

THEOREM 5.4. *The set* $\mathcal{T}_{NH}$ *is a subgroup of the* 1*H-projective Finsler connection transformation group* $\mathcal{T}^{1H}_{N(p)}$ *which is obtained for*

$$\Omega^i_{jk} = \delta^i_j U_k - \delta^i_k V_j \tag{5.15}$$

and characterized by the invariance of $\underset{1}{I}$, $\underset{2}{I}$ *(and of* $\underset{1}{J}$, $\underset{2}{J}$*), respectively.*

DEFINITION 5.3. The set Σ of Finsler directions is a set of V-enveloping Finsler direction if among $\dot{X}^i$, $\dfrac{DU^i}{dt}$, U^i exists a linear dependence relation.

From the Definition 5.3, we have equivalently

$$DU^{[i}U^j\delta y^{k]} = 0. \tag{5.16}$$

DEFINITION 5.4. A Finsler connection transformation $t \in \mathscr{T}_N$ which preserve the set of V-enveloping Finsler directions is called V-enveloping Finsler connection transformation. The set of these transformations will be denoted by $\mathscr{T}_{NV}$.

From (5.16) it follows equivalently:

$$\bar{D}U^{[i}U^j\,\bar{\delta}y^{k]} = 0 \quad \forall t \in \mathscr{T}_{NV}. \tag{5.17}$$

But for $t \in \mathscr{T}_N$ we have $\bar{\delta}y^k = \delta y^k$. It follows easily

LEMMA 5.2. *For a Finsler connection transformation $t \in \mathscr{T}_N$ we have*

$$\bar{D}U^{[i}U^j\,\delta y^{k]} = DU^{[i}U^j\,\delta y^{k]} - B^{[i}_{rs}\,\delta^j_h\,\delta^{k]}_p\,U^r\,U^h\,dx^s\,dx^p - D^{[i}_{rs}\,\delta^j_h\,\delta^{k]}_p\,U^r\,U^h\,\delta y^s\,\delta y^p. \tag{5.18}$$

THEOREM 5.5. *A necessary and sufficient condition for a Finsler connection transformation $t(0, B, 0) \in \mathscr{T}_N$ to be a V-enveloping Finsler connection transformation ($t \in \mathscr{T}_{NV}$) is that t be of the form*

$$\bar{N}^i_j = N^i_j; \quad \bar{F}^i_{jk} = F^i_{jk} + \delta^i_j\,\varrho_k; \quad \bar{C}^i_{jk} = C^i_{jk} \tag{5.19}$$

where ϱ_k is an arbitrary Finsler covector.

PROOF. From (5.16), (5.17), and (5.19) it follows

$$\bar{D}U^{[i}U^j\,\bar{\delta}y^{k]} = \varrho_s\,\delta^{[i}_r\,\delta^j_h\,\delta^{k]}_p\,U^r\,U^h\,dx^s\,\delta y^p = 0. \tag{5.20}$$

Reciprocally, from (5.16), (5.17), (5.18) and $t(0, B, 0)$ given by (5.9) it follows:

$$B^{[i}_{rs}\,\delta^j_h\,\delta^{k]}_p\,U^r\,U^h\,dx^s\,\delta y^p = 0. \tag{5.21}$$

Contracting for $j=h$, $k=p$ the calculus lead to (5.19).

THEOREM 5.6. *Any Finsler connection transformation $t(0, 0, D) \in \mathscr{T}_N$ is a V-enveloping Finsler connection transformation ($t \in \mathscr{T}_{NV}$) if and only if it has the form*

$$\begin{aligned} \bar{N}^i_j &= N^i_j; \quad \bar{F}^i_{jk} = F^i_{jk}, \\ \bar{C}^i_{jk} &= C^i_{jk} + \delta^i_j\,U_k + \delta^i_k\,V_j \end{aligned} \tag{5.22}$$

where U_k, V_k are arbitrary Finsler covectors.

PROOF. From (5.16), (5.18) and (5.22) it follows:

$$\bar{D}U^{[i}U^j\,\delta^{y]}_k = -\delta^{[i}_r\,\delta^j_h\,\delta^{k]}_p\,U^r\,U^h\,\delta y^p\,U_s\,\delta y^s - \delta^{[i}_r\,\delta^j_h\,\delta^{k]}_p\,U^h\,U^r\,V_r\,\delta y^s\,\delta y^p = 0. \tag{5.23}$$

consequently the transformation $t(0, 0, D)$ given by (5.22) is a V-enveloping Finsler connection transformation. Reciprocally, from (5.16), (5.17), (5.18) and (5.5) it follows:

$$D^{[i}_{rs}\,\delta^j_h\,\delta^{k]}_p\,U^r\,U^h\,\delta y^s\,\delta y^p = 0. \tag{5.24}$$

Making equal to zero the coefficients of this relation, we obtain:

$$D^{[i}_{rs}\,\delta^j_h\,\delta^{k]}_p + D^{[i}_{hs}\,\delta^j_r\,\delta^{k]}_p + D^{[i}_{sp}\,\delta^j_h\,\delta^{k]}_s + D^{[i}_{hp}\,\delta^j_r\,\delta^{k]}_s = 0. \tag{5.25}$$

Contracting for $j=h$, $k=p$, the calculus lead to (5.22).

Since $\mathscr{T}_N$ is an abelian group relative to the composition of the transformations, it follows $t(0, B, D) = t(0, B, 0) \circ t(0, 0, D) \in \mathscr{T}_N$ and we have

THEOREM 5.7. *A necessary and sufficient condition for a Finsler connection transformation $t \in \mathcal{T}_N$ to be a V-enveloping Finsler connection transformation ($t \in \mathcal{T}_{NV}$) is that t be of the form*

$$\bar{N}^i_j = N^i_j; \quad \bar{F}^i_{jk} = F^i_{jk} + \delta^i_j \varrho_k; \quad \bar{C}^i_{jk} = C^i_{jk} + \delta^i_j U_k + \delta^i_k V_j \tag{5.26}$$

where ϱ_k, U_k, V_k are arbitrary Finsler covectors.

THEOREM 5.8. *The set $\mathcal{T}_{NV}$ is a subgroup of the 1V-projective Finsler connection transformation group $\mathcal{T}^{1V}_{N(p)}$, characterized by the invariance of $\underset{1}{I}$, $\underset{2}{I}$ and $\underset{1}{J}$, $\underset{2}{J}$, respectively.*

DEFINITION 5.5. *A set Σ of Finsler directions which is both H-enveloping and V-enveloping is called a Σ-enveloping Finsler directions set.*

DEFINITION 5.6. A Finsler connection transformation $t \in \mathcal{T}_N$, which preserve the set of Σ-enveloping Finsler directions is called a Σ-enveloping Finsler connection transformation. The group of these transformations is denoted by $\mathcal{T}_{N\Sigma}$.

From Definition 5.5 it follows:

$$dx^k + bDU^k + cU^k = 0 \qquad (c = k\,dt,\ b \neq 0,\ c \neq 0) \tag{5.27}$$

$$\delta y^k + b' DU^k + c' U^k = 0 \qquad (c' = k' dt,\ b' \neq 0,\ c' \neq 0) \tag{5.28}$$

or

$$b' dx^k - b\delta y^k + (b'c - bc')U^k = 0 \tag{5.29}$$

$$U^{[i} dx^j \delta y^{k]} = 0. \tag{5.30}$$

Since c and c' are arbitrary and, in general, $b' dx^k - b\delta y^k \neq 0$ it follows

$$\varrho \overset{\text{def}}{=} b'c - bc' \neq 0. \tag{5.31}$$

From (5.27) and (5.28) we have

$$c' dx^k - c\delta y^k + (bc' - b'c)DU^k = 0, \quad \varrho \neq 0, \quad c' \neq 0, \quad c \neq 0. \tag{5.32}$$

Consequently,

$$DU^{[i} dx^j \delta y^{k]} = 0. \tag{5.33}$$

We observe that the pair of relations (5.27)—(5.28), (5.31)—(5.33) and (5.30)—(5.33) are equivalent. Consequently, the set Σ is a set of Σ-enveloping Finsler directions, if Σ is defined by U^i which satisfies the relations (5.30) and (5.33). This can be proved by calculus of $DU^{[i} U^j dx^{k]} = 0$, $DU^{[i} U^j \delta y^{k]} = 0$ using also (5.29).

Since $\bar{\delta} y^k = \delta y^k$, the condition (5.30) is invariant by a transformation $t \in \mathcal{T}_N$, the conditions (5.33) are necessary and

$$\bar{D}U^{[i} dx^j \delta y^{k]} = 0. \tag{5.34}$$

LEMMA 5.3. *For a Finsler connection transformation* $t\in\mathcal{T}_N$ *and a set of Finsler directions* Σ *determined by* U^i *which satisfies* (5.29) *we have*

$$\bar{D}U^{[i}dx^j\delta y^{k]} = DU^{[i}U^j\delta y^{k]} - \varphi B^{[i}_{rs}\delta^j_h\delta^{k]}_p dx^r dx^s dx^h \delta y^p - \tag{5.35}$$
$$-\psi D^{[i}_{rs}d^j_h\delta^{k]}_p \delta y^r \delta y^s \delta y^p dx^h$$

where $\varphi\neq 0$ *and* $\psi\neq 0$ *are arbitrary Finsler functions.*

THEOREM 5.9. *A necessary and sufficient condition for a Finsler connection transformation* $t\in\mathcal{T}_N$ *to be a* Σ*-enveloping Finsler connection transformation* $(t\in\mathcal{T}_{N\Sigma})$ *is that* t *be of the form*

$$\begin{aligned} \bar{N}^i_j &= N^i_j \\ \bar{F}^i_{jk} &= F^i_{jk} + \delta^i_j\alpha_k + \delta^i_k\beta_j \\ \bar{C}^i_{jk} &= C^i_{jk} + \delta^i_j\sigma_k + \delta^i_k\theta_j \end{aligned} \tag{5.36}$$

where α_k, β_k, σ_k, θ_k *are arbitrary Finsler covectors.*

PROOF. Let $t\in\mathcal{T}_N$ be a Finsler connection transformation of the form (5.36). From (5.30), (5.35) and (5.36) it follows (5.34), consequently $t\in\mathcal{T}_{N\Sigma}$. Reciprocally, from (5.30), (5.34), and (5.35) we obtain for $\varphi\neq 0$, $\psi\neq 0$ the relation (5.36).

The set $\mathcal{T}_{NH}$ and the set $\mathcal{T}_{NV}$ are evidently subgroups of the set $\mathcal{T}_{N\Sigma}$.

THEOREM 5.10. *The* Σ*-enveloping Finsler connection transformations* $t\in\mathcal{T}_{N\Sigma}$ *are the most general Finsler connection transformations* $t\in\mathcal{T}_N$, *which have the invariants* $\underset{1}{J}$ *and* $\underset{2}{J}$ *or are the most general Finsler connection transformations* $t\in\mathcal{T}_N$, *which have the invariants* $\underset{1}{I}$, $\underset{2}{I}$ *and are also 2-projective transformations.*

In the Definition 5.1 is imposed the condition that among $X^i=\dfrac{dx^i}{dt}$, U^i, $\dfrac{DU^i}{dt}$ be a linear dependence relation for any curves C and $\tilde{C}$ with $\pi(\tilde{C})=C$. If we consider this condition only for the curves C, $\tilde{C}$, $\pi(\tilde{C})=C$ for which we have $\dot{X}^i=hX^i$, $h\neq 0$, $X^i\neq 0$, then these curves will be called Σ-curves. It follows

THEOREM 5.11. *The most general Finsler connection transformations which preserve the H-enveloping directions to a* Σ*-curve, are given by* (5.36), *and reciprocally.*

PROOF. From (5.1), (5.4), (5.36) and (5.35) it follows

$$\bar{D}U^{[i}U^j dx^{k]} = \alpha_s\delta^{[i}_r\delta^j_h\delta^{k]}_p U^r U^h dx^p dx^s + \beta_r\delta^{[i}_s\delta^j_h\delta^{k]}_p U^r U^h dx^p dx^s + \tag{5.38}$$
$$+\delta^{[i}_r\delta^j_h\delta^{k]}_p U^r U^h dx^p\sigma_s\delta y^s + h\delta^{[i}_r\delta^j_h\delta^{k]}_p U^r U^h\theta_s dx^s dx^p = 0.$$

The proof of the reciprocal part is analogous to the proofs used in the above paragraphes.

We observe that these transformations are also V-enveloping Finsler connection transformations for a Σ-curve and have the invariants $\underset{1}{I}$, $\underset{2}{I}$, $\underset{1}{J}$, $\underset{2}{J}$.

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(Received June 24, 1985)

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TORUSFLÄCHEN DES GALILEISCHEN RAUMES $\mathbf{G}_3$

OTTO RÖSCHEL

1. Einleitung

Werden im reellen dreidimensionalen projektiven Raum $\mathbf{P}_3(\mathbf{R})$[1] — in dem wir Punkte wie üblich durch reelle projektive Koordinaten $x_0:x_1:x_2:x_3 \neq 0:0:0:0$ beschreiben — eine reelle Ebene $\omega\,(x_0=0)$ und eine reelle Gerade $f(x_0=x_1=0)$ als *absolute Ebene* beziehungsweise *absolute Gerade* angesprochen, so bilden alle ω und f einzeln festlassenden Projektivitäten des $\mathbf{P}_3(\mathbf{R})$, die mit der auf f operierenden *elliptischen Involution*

$$J\colon (0:0:x_2:x_3) \to (0:0:x_3:-x_2) \tag{1}$$

kommutieren, eine achtgliedrige Gruppe

$$\begin{aligned} x_0' &= x_0 \\ x_1' &= ax_0+\alpha x_1 \\ x_2' &= bx_0+cx_1+\varrho\cos\varphi x_2+\varrho\sin\varphi x_3 \\ x_3' &= dx_0+ex_1-\varrho\sin\varphi x_2+\varrho\cos\varphi x_3 \end{aligned} \tag{2}$$

($a, b, c, d, e, \varphi, \varrho \in \mathbf{R}$, $\alpha\varrho \neq 0$), die gemäß [9] *Ähnlichkeitsgruppe eines galileischen Raumes* $\mathbf{G}_3$ genannt wird. Die Geometrie dieser Ähnlichkeitsgruppe war Gegenstand der Untersuchungen von Р. Г. Бухараев [1] und А. И. Сирота [2]. Die in (2) durch $\alpha=\varrho=1$ ausgezeichnete sechsgliedrige Untergruppe, die in den üblichen affinen Koordinaten $1:x:y:z=x_0:x_1:x_2:x_3$ $(x_0\neq 0)$ durch

$$\begin{aligned} x' &= a+x \\ y' &= b+cx+y\cos\varphi+z\sin\varphi \\ z' &= d+ex-y\sin\varphi+z\cos\varphi \end{aligned} \tag{3}$$

beschrieben wird, wurde in [9] vom Autor als *Bewegungsgruppe* B_6 des galileischen Raumes $\mathbf{G}_3$ bezeichnet und liegt den Untersuchungen in [9] zugrunde.

[1] Fallweise werden wir auch die komplexe Erweiterung $\mathbf{P}_3(\mathbf{C})$ benützen.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 53A35.
Key words and phrases. Torus in 3-dimensional Cayley—Klein spaces.

2. Kreise und Drehungen des galileischen Raumes G_3

In [9] wurden *Kreise des* $\mathbf{G}_3$ ausführlich untersucht: Es existieren zwei verschiedene Typen von Kreisen. Die einen liegen in Ebenen, die die absolute Gerade f enthalten, die anderen nicht. Im ersten Fall schneiden die Kreise die absolute Gerade f in den konjugiert komplexen Doppelpunkten $(0:0:1:\pm i)$ der elliptischen Involution $J(1)$, im zweiten Fall handelt es sich um Parabeln, deren Fernpunkt im Schnittpunkt der Trägerebene mit der absoluten Geraden f liegt. Kreise des ersten Typs wurden in [9] als *euklidische Kreise*, solche des zweiten Typs als *isotrope Kreise* bezeichnet. Analog existieren im $\mathbf{G}_3$ nach [9, S. 15f.] zwei verschiedene Klassen von *Drehungen*.

(A) *Euklidische Drehungen:* Sie werden in der Normalform durch

$$\begin{aligned} x(t) &= x_0 \\ y(t) &= y_0 \cos t + z_0 \sin t \qquad t \in [0, 2\pi] \\ z(t) &= -y_0 \sin t + z_0 \cos t \end{aligned} \tag{4}$$

beschrieben und besitzen die x-Achse als Fixpunktgerade, die wir im folgenden als *Drehachse a* bezeichnen; die *Bahnkurven* sind *euklidische Kreise* in Ebenen des Büschels um die absolute Gerade f. Diese Drehungen können auch als Drehungen in einem geeignet gewählten euklidischen Raum aufgefaßt werden.

(B) *Isotrope Drehungen:* Sie werden in der Normalform durch

$$\begin{aligned} x(t) &= x_0 + bt \\ y(t) &= y_0 + x_0 t + b\frac{t^2}{2} \qquad (b \in \mathbf{R}/\{0\},\ t \in \mathbf{R} \cup \{\infty\}) \\ z(t) &= z_0 \end{aligned} \tag{5}$$

beschrieben. Die Bahnkurven sind *isotrope Kreise* mit dem isotropen *Radius* b[2]; die absolute Gerade f bleibt punktweise fest. Als *Meridianschnitte* von Drehflächen des Typs B sind damit die Schnitte mit Ebenen des Büschels um f anzusehen. Diese isotropen Drehungen lassen sich als Drehungen in einem geeignet gewählten Flaggenraum $I_3^{(2)}$ auffassen und wurden in diesem Zusammenhang bereits von H. Sachs [10] angegeben. Die entstehenden Drehflächen lassen sich nach [10] auch durch *Schiebung* der Meridiankurven längs der kongruenten isotropen Drehkreise erzeugen.

3. Normalformen der Torusflächen des G_3

Analog einer möglichen euklidischen Definition wollen wir jene *Drehflächen* des $\mathbf{G}_3$ als *Torusflächen* ansprechen, deren *Meridiankurven Kreise* sind. Wir werden dabei die beiden Drehungstypen getrennt untersuchen und gelangen so zu zwei verschiedenen Klassen von Torusflächen:

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[2] Dieser Radius ist im Sinne der ebenen isotropen Geometrie (vgl. K. Strubecker [11]) zu verstehen. Die Bewegungsgruppe B_6 (3) induziert nämlich in jeder f nicht enthaltenden Ebene eine ebene isotrope Geometrie mit der Ferngeraden und dem Schnittpunkt der Ebene mit f als Absolutgebilde.

(A) *Die Meridiankurve m ist ein isotroper Kreis,* der oBdA durch $(v, 2pv^2-A, 0)$ $(v\in \mathbf{R}\cup\{\infty\}, p\in \mathbf{R}/\{0\}, A\in \mathbf{R})$ beschrieben wird. Bei der Drehung (4) überstreicht m die Torusfläche vom Typ A Φ_A

$$(v,\ (2pv^2-A)\cos t,\ -(2pv^2-A)\sin t), \tag{6}$$

deren Gleichung

$$y^2+z^2 = (2px^2-A)^2 \tag{7}$$

lautet. Die Dreh- und Meridiankreise bilden nach [9] die isotropen Flächenkurven und die Krümmungslinien der Fläche Φ_A. Φ_A ist eine *algebraische Fläche vierter Ordnung,* deren Doppelkurve aus der doppelt zu zählenden absoluten Geraden f besteht. Die absoluten Punkte $(0:0:1:\pm i)$ sind *uniplanare Knoten* von Φ_A; die entsprechenden Tangentialebenen werden durch $y^2+z^2=0$ beschrieben. Je nach dem Schnitt der Meridiankurve m mit der Drehachse a stellen sich drei Typen von Torusflächen Φ_A ein:

(A1) $\dfrac{A}{p}>0: \Phi_A$ werde als *Spindeltorus* bezeichnet; es existieren *zwei reelle konische Knoten* auf der Drehachse.

(A2) $A=0: \Phi_A$ werde als *Dorntorus* bezeichnet; die beiden konischen Knoten sind zusammengefallen.

(A3) $\dfrac{A}{p}<0: \Phi_A$ werde als *Ringtorus* bezeichnet; es existieren zwei konjugiert komplexe konische Knoten.

Keine Fläche dieser drei Typen besitzt Plattkegelschnitte; die einzigen auf Φ_A gelegenen Geraden sind neben f die Schnittgeraden von Φ_A mit den beiden konjugiert komplexen Ebenen $x^2+y^2=0$. Diese Geraden verbinden die absoluten Punkte $(0:0:1:\pm i)$ mit den konischen Knoten $(2p:\pm\sqrt{A}:0:0)$ auf der Drehachse a. Wir fassen zusammen in

SATZ 1. *Wird ein isotroper Kreis m des galileischen Raumes* $\mathbf{G}_3$ *einer euklidischen Drehung des* $\mathbf{G}_3$ *unterworfen, deren Drehachse a in derselben Ebene wie m liegt, so entsteht eine algebraische Fläche vierter Ordnung* Φ_A, *die als Torusfläche vom Typ A im* $\mathbf{G}_3$ *anzusprechen ist.* Φ_A *trägt im algebraischen Sinn zwei konische Knoten auf der Drehachse und zwei konjugiert komplexe uniplanare Knoten in den absoluten Punkten* $(0:0:1:\pm i)$. *Neben der absoluten Geraden f sind die einzigen auf* Φ_A *gelegenen Geraden die paarweise konjugiert komplexen Verbindungsgeraden der konischen Knoten mit den absoluten Punkten* $(0:0:1:\pm i)$.

(B) *Die Meridiankurve m ist ein euklidischer Kreis,* der oBdA durch $(0, R\cos u, R\sin u)$ $(u\in[0, 2\pi], R\in\mathbf{R}/\{0\})$ beschrieben werden kann. m beschreibt bei der isotropen Drehung (5) die Fläche Φ_B

$$\left(bt,\ R\cos u+b\frac{t^2}{2},\ R\sin u\right), \tag{8}$$

die durch die algebraische Gleichung

$$\left(y - \frac{x^2}{2b}\right)^2 = R^2 - z^2 \tag{9}$$

erfaßt wird. Es handelt sich dabei wegen $R \neq 0$ um eine *Fläche vierter Ordnung*, deren Doppelkurve aus der doppelt zu zählenden absoluten Geraden f besteht. Die absoluten Punkte $(0:0:1:\pm i)$ sind konische *Knoten* der Fläche, während der Fernpunkt $F(0:0:1:0)$ der isotropen Drehkreise als *Zwickpunkt* aufzufassen ist. Φ_B ist nach obigem auch durch Schiebung von m längs der isotropen Drehkreise erzeugbar und besitzt daher *zwei isotrope Plattkreise* p_1 und p_2 in den Ebenen $z = \pm R$, die in Abbildung 1 eingetragen sind. Die Abbildung 1 zeigt Φ_B in einem galileischen Normalriß in die Ebene $x = 0$. Wieder sind die isotropen Drehkreise und die Meridiankreise Krümmungslinien bzw. isotrope Flächenkurven von Φ_B (vgl. [9]).

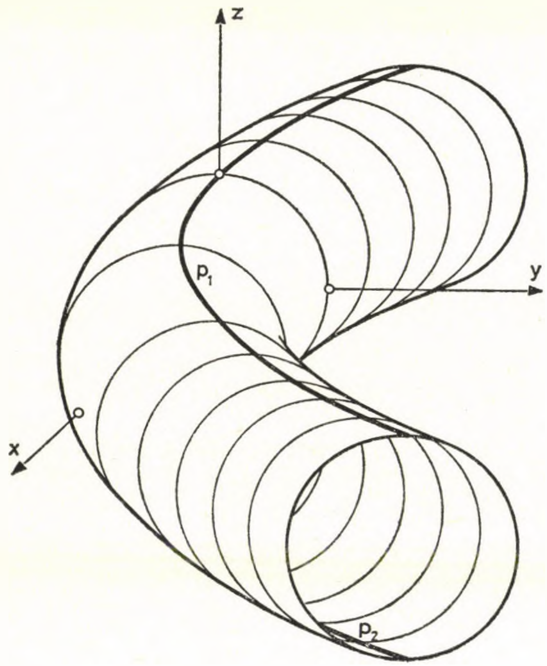

Abb. 1

Wir notieren den

SATZ 2. *Wird ein euklidischer Kreis m des galileischen Raumes G_3 einer isotropen Drehung des G_3 unterworfen, so entsteht eine algebraische Fläche vierter Ordnung Φ_B, die als Torusfläche vom Typ B des G_3 anzusprechen ist. Φ_B (9) besitzt konische Knoten in den absoluten Punkten $(0:0:1:\pm i)$ und einen Zwickpunkt in $(0:0:1:0)$ sowie zwei isotrope Plattkreise und läßt sich auch als Schiebfläche erzeugen.*

4. Villarceau-Kreise auf dem Ringtorus des G_3

Analog zu den auf dem euklidischen Ringtorus existierenden Villarceau-Kreisen (vgl. [14, S. 154]) gibt es auch auf den Ringtorusflächen Φ_A ($pA<0$) im G_3 eine solche reelle Kreisschar: Die Ebenen

$$(10) \qquad y = \pm Bx \quad \text{mit} \quad B = \sqrt{-8Ap}$$

sind wegen $Ap<0$ reelle *Doppeltangentialebenen* des Ringtorus (7). Sie schneiden den Torus nach

$$(11) \qquad y = \pm Bx, \quad z = \pm\left(\frac{y^2}{4A} - A\right);$$

der Schnitt zerfällt damit in je ein Paar isotroper Kreise, die wir als *Villarceau-Kreise des galileischen Ringtorus* Φ_A bezeichnen.[3] Durch Drehung um die Achse a entsteht daraus eine stetige Schar solcher Kreise. Wir fassen zusammen im

SATZ 3. *Ringtorusflächen des galileischen Raumes* G_3 *werden von ihren Doppeltangentialebenen nach isotropen Kreispaaren geschnitten; diese Ebenen umhüllen einen Drehkegel mit der Torusachse als Achse.*

Einer der in (11) angegebenen Villarceau-Kreise k kann in (6) durch $\cos t = \frac{Bv}{2pv^2 - A}$ bzw. $\sin t = \frac{2pv^2 + A}{2pv^2 - A}$ erfaßt werden; wir erhalten damit als Parameterdarstellung dieses isotropen Kreises

$$(12) \qquad (v, Bv, -2pv^2 - A) \qquad (v \in \mathbf{R} \cup \{\infty\}).$$

Die Villarceau-Kreise des euklidischen Ringtorus schneiden die Meridiankreise unter festem Winkel (vgl. [14, S. 155]). Wir zeigen, daß dies auch auf dem galileischen Ringtorus gilt — die galileischen Villarceau-Kreise sind somit galileische *Loxodromenkreise* der isotropen Meridiankreise: In den Punkten des Villarceau-Kreises (12) besitzen die Meridiankreise die Tangentenvektoren $\left(1, 4pv\frac{Bv}{2pv^2 - A}, -4pv\frac{4pv^2 + A}{4pv^2 - A}\right)$; die Tangenten an den isotropen Kreis k (12) haben die Richtungsvektoren $(1, B, -4pv)$. Der Winkel zwischen diesen Vektoren kann im galileischen Raum G_3 nach [9] als Abstand der zugehörigen Fernpunkte gemessen werden. Dieser Abstand ist ein *euklidischer Abstand*, da von der Bewegungsgruppe B_6 in der Fernebene genau jene *ebene euklidische Metrik* induziert wird, für die f die Ferngerade und $J(1)$ die Rechtwinkelinvolution ist. Mit [9, S. 9] erhalten wir für unseren Winkel

$$(13) \qquad \Delta = B = \text{konst} \qquad (\forall v \in \mathbf{R} \cup \{\infty\}).$$

Wir haben damit den

[3] Für den Spindeltorus fallen diese Doppeltangentialebenen und auch die Villarceau-Kreise komplex aus; für den Dorntorus werden die Ebenen zu den Meridianebenen, die Kreise zu den Meridiankreisen. Es sei auch bemerkt, daß Torusflächen vom Typ B (9) keine derartige reelle stetige Kreisschar tragen.

SATZ 4. *Die isotropen Villarceau-Kreise der Ringtorusflächen des galileischen Raumes* $\mathbf{G}_3$ *schneiden die isotropen Meridiankreise unter konstantem galileischen Winkel und sind damit als Loxodromenkreise des galileischen Ringtorus anzusprechen.*

Längs k (12) besitzt der Ringtorus *Flächennormalen* (vgl. [9, S. 97]) mit dem Richtungsvektor

$$\mathfrak{n} = \big(0,\ Bv,\ -(2pv^2+A)\big); \tag{14}$$

die Flächennormalen erfüllen die Regelfläche Ψ

$$\big(v,\ Bv(1+\lambda),\ -(2pv^2+A)(1+\lambda)\big) \qquad \big(\lambda\in\mathbf{R}\cup\{\infty\}\big) \tag{15}$$

mit der algebraischen Gleichung

$$Bxz+y(2px^2+A) = 0. \tag{16}$$

Ψ ist eine *Regelfläche dritten Grades* mit der absoluten Geraden f als Doppelgerade und der Torusachse a (x-Achse) als einfache Leitgerade; das konjugiert komplexe Ebenenpaar

$$y^2+z^2 = 0 \tag{17}$$

bildet die beiden Ψ berührenden Torsalebenen. Ψ ist daher nach [4, S. 176] eine *Regelfläche dritten Grades II. Typs.* Im $\mathbf{G}_3$ liegt ein *gerades Konoid* vor, das nach [9, S. 72] dem *Typ C* zuzuordnen ist. Wir haben damit den[4]

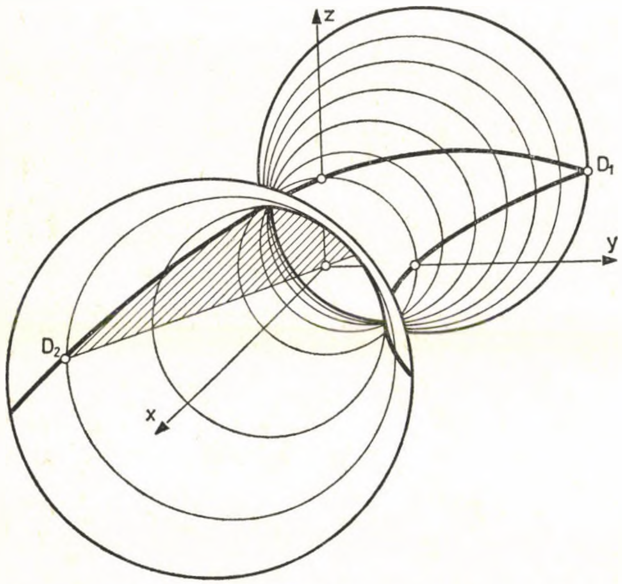

Abb. 2

[4] Längs eines Villarceau'schen Kreises eines euklidischen Torus stellt sich bekanntlich eine Regelfläche 4. Grades VII. Sturmscher Art als Normalenfläche ein (vgl. [4, S. 283 f.]).

SATZ 5. *Die von den Flächennormalen in den Punkten eines Villarceau'schen Kreises des galileischen Ringtorus gebildete Regelfläche Ψ ist von drittem Grad und II. Typ; im galileischen Raum $\mathbf{G}_3$ liegt ein gerades Konoid vor, das dem Typ C zuzuordnen ist. Ψ besitzt die absolute Gerade f als Doppelgerade und die Drehachse des Torus als einfache Leitgerade.*

In Abbildung 2 ist ein Ringtorus mit einem Villarceau'schen Kreis in einer galileischen normalen Axonometrie mit der Bildebene $x=0$ dargestellt.

5. Loxodromen auf den Torusflächen vom Typ A

Jene Flächenkurven $v=v(t)$ $(v\in C^1[0, 2\pi])$ der Torusflächen (6), die sämtliche *Meridiankreise* ($v=$konst.) unter *konstantem galileischen Winkel a* schneiden, bezeichnen wir als *Loxodromen.*[5] Sie sind mit [9, S. 9] durch

$$\frac{dv}{dt}=\frac{2pv^2-A}{a} \qquad a\in\mathbf{R}-\{0\} \tag{18}$$

gekennzeichnet. Je nach Typ der Torusfläche ergeben sich die folgenden Lösungskurven der Differentialgleichung (18):

$$v(t)=-\sqrt{\frac{A}{2p}}\tanh\left(\frac{t+K}{a}\sqrt{2Ap}\right) \quad \text{für den } \textit{Spindeltorus},$$

$$v(t)=-\frac{a}{2p(t+K)} \quad \text{für den } \textit{Dorntorus} \text{ und} \tag{19}$$

$$v(t)=\sqrt{\frac{-A}{2p}}\tan\left(\frac{t+K}{a}\sqrt{-2Ap}\right) \quad \text{für den } \textit{Ringtorus}.$$

K stellt dabei eine reelle Integrationskonstante dar. Für $a=\sqrt{-8Ap}$ erhalten wir auf dem Ringtorus die in Abschnitt 4 untersuchten Loxodromenkreise.[6]

Die Tangenten aller zu einem festen Winkel $a\in\mathbf{R}-\{0\}$ gehörenden Torusloxodromen erfüllen die *Geradenkongruenz S* mit der Parameterdarstellung

$$\begin{aligned}\mathfrak{x}(t,u,v)=(v+u,\ &[2pv^2-A]\cos t+u[4pv\cos t-a\sin t],\\ &-[2pv^2-A]\sin t-u[4pv\sin t+a\cos t]).\end{aligned} \tag{20}$$

Diese Kongruenz kann in eine einparametrige Schar kongruenter Regelscharen zweiten Grades zerlegt werden, die von den längs den Meridianen auftretenden Loxodromentangenten erfüllt werden. Ihre Trägerflächen sind *hyperbolische Paraboloide;* so stellt sich für $t=0$ das hyperbolische Paraboloid mit der Gleichung

$$F:=2p(a^2x^2-z^2)-a^2(y+A)=0 \tag{21}$$

[5] Für den euklidischen Raum gibt W. Wunderlich in [13, S. 313 f.] die analogen Resultate an.
[6] Für $K=-\pi$ stellt sich der Loxodromenkreis (12) ein.

ein. Das *Brennflächenpaar* der Kongruenz S ist somit das Hüllflächenpaar jener durch Drehung um die Torusachse aus (21) hervorgehenden Schar von Paraboloiden. Dabei ist die Charakteristik von (21) durch

$$\frac{dF}{dy}:\frac{dF}{dz} = a^2:4pz = y:z \tag{22}$$

gekennzeichnet. Sie besteht damit aus dem *Torusmeridian* m $\left(z=0\right.$ in (21) bzw. $\left.t=u=0\right.$ in (20)$\left.\right)$ und der in der Ebene $y=\dfrac{a^2}{4p}$ gelegenen *Hyperbel* mit der Parameterdarstellung

(23)
$$(x, y, z) = \left(\frac{C}{a}\cosh s, \frac{a^2}{4p}, C\sinh s\right) \quad \left(s\in[0, 2\pi], \ C^2 = \frac{a^2}{2p}\left(A+\frac{a^2}{4p}\right)\right).$$

Durch euklidische Drehung der beiden Teile dieser Charakteristik entsteht einerseits der *Ausgangstorus* und andererseits das *Hyperboloid* mit der Parameterdarstellung

$$\left(\frac{C}{a}\cosh s, \frac{a^2}{4p}\cos t + C\sinh s\sin t, -\frac{a^2}{4p}\sin t + C\sinh s\cos t\right) \tag{24}$$

und der algebraischen Gleichung

$$a^2x^2 - y^2 - z^2 = \frac{a^2}{2p}\left(A+\frac{a^2}{8p}\right). \tag{25}$$

Genau für die *Loxodromenkreise* $\left(a=\sqrt{-8Ap}\right)$ wird dieses Hyperboloid zu einem *Drehkegel.* Wir haben damit den

SATZ 6. *Auf den galileischen Torusflächen Φ_A vom Typ A erfüllen die Tangenten der zu einem festen Kurswinkel $a\in\mathbb{R}/\{0\}$ gehörenden Torusloxodromen die in* (20) *beschriebene Strahlkongruenz S, deren Brennflächen vom Torus Φ_A selbst und im allgemeinen von einem zum Torus koaxialen Drehhyperboloid gebildet werden.*

6. Loxodromen auf den Torusflächen vom Typ B

Da die Meridiankreise auf der Torusfläche Φ_B (9) in euklidischen Ebenen liegen, werden wir zur Definition der Loxodromen die in der Parameterdarstellung (8) durch $u=$konst. erfaßten Drehkreise heranziehen und jene Flächenkurven $t=t(u)$ $\left(t\in C^1[0, 2\pi]\right)$ als *Loxodromen* bezeichnen, die sämtliche *Drehkreise unter konstantem galileischen Winkel* $a\neq 0$ *schneiden.* Sie sind durch

$$\frac{dt}{du} = \frac{R}{ba} \tag{26}$$

gekennzeichnet und werden daher durch

(27) $$t(u) = \frac{R}{ba}(u+K) \qquad (K = \text{konst} \in \mathbf{R})$$

erfaßt. Als Parameterdarstellung der Loxodromen erhalten wir somit

(28) $$\left(\frac{R}{a}(u+K),\ \frac{R^2}{2ba^2}(u+K)^2 + R\cos u,\ R\sin u\right).$$

Werden diese Loxodromen aus dem Zwickpunkt $F(0:0:1:0)$ der Fläche in die $[x, z]$-Ebene projiziert, so entstehen *Sinuslinien.*[7] Wir haben damit den

SATZ 7. *Die Loxodromen der Torusflächen vom Typ B erscheinen bei Normalprojektion aus dem absoluten Punkt der Drehkreise auf eine dazu galileisch orthogonale Ebene als Sinuslinien.*

Die Tangenten aller zu einem festen Kurswinkel $a \in \mathbf{R} - \{0\}$ gehörenden Torusloxodromen erfüllen hier die *Geradenkongruenz* T mit der Parameterdarstellung

(29) $$\mathfrak{x}(t, u, v) = \left(bt,\ R\cos u + \frac{bt^2}{2},\ R\sin u\right) + \\ + v(1,\ -a\sin u + t,\ a\cos u) \qquad (v \in \mathbf{R} \cup \{\infty\}).$$

Längs dem Meridiankreis m $(t=0)$ bilden die Loxodromentangenten den *Regulus*

(30) $$\mathfrak{x}(u, v) = (0,\ R\cos u,\ R\sin u) + v(1,\ -a\sin u,\ a\cos u)$$

auf dem *galileischen Drehhyperboloid* H mit der Gleichung

(31) $$a^2x^2 + R^2 = y^2 + z^2.$$

Bei der den Torus erzeugenden isotropen Drehung (5) besitzt dieses Drehhyperboloid H die *Brennflächen der Geradenkongruenz* T als Hüllflächen. Die Punkte der Charakteristik c von H sind dabei in (30) durch

(32) $$\det\left(\frac{d\mathfrak{x}}{du},\ \frac{d\mathfrak{x}}{dv},\ \mathfrak{t}\right) = 0 \quad \text{mit} \quad \mathfrak{t} = (b, v, 0)$$

gekennzeichnet.[8] Nach kurzer Rechnung erkennen wir, daß c aus dem Torusmeridiankreis m $(v=0$ in (30)) und einer *Hyperbel* h in der Ebene $y=a^2b$ besteht, wobei letztere bei der isotropen Drehung (5) das *hyperbolische Paraboloid*

(33) $$a^2(2by - x^2) + z^2 = a^4b^2 + R^2$$

überstreicht.

Wir haben damit den

[7] Das euklidische Analogon hat W. Wunderlich in [14, S. 321] bewiesen.

[8] $\mathfrak{t}$ stellt den Tangentenvektor der Punkte $\mathfrak{x}(u, v)$ (30) bei der isotropen Drehung (5) zum Zeitpunkt $t=0$ dar.

SATZ 8. *Auf den galileischen Torusflächen Φ_B vom Typ B erfüllen die Tangenten der zu einem festen Kurswinkel $a \in \mathbf{R}-\{0\}$ gehörenden Torusloxodromen die in* (29) *beschriebene Strahlkongruenz T. Die Brennflächen dieser Kongruenz sind der Ausgangstorus und ein hyperbolisches Paraboloid, das die den Torus erzeugende isotrope Drehung gestattet.*

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(Eingegangen am 20. November 1985)

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HEREDITARY CONDITIONS ON CLASSES OF NEAR-RINGS

STEFAN VELDSMAN

Betsch and Kaarli [2] have shown that any radical class of near-rings (in the sense of Kurosh—Amitsur) with hereditary semisimple class must be supernilpotent, i.e. it must contain all the zero near-rings. The converse need not be true. This result gives rise to a few questions. We will mainly be concerned with the next two problems:

(1) Can the hereditariness of the semisimple class be replaced with a weaker type of hereditariness with the same end result?

(2) Can the supernilpotent radicals be characterized by some hereditariness condition on the corresponding semisimple class?

Concerning the first question, we show that a condition dual to "weakly homomorphically closed", namely weakly hereditary, has the desired property. Concerning the second, we only have a partial answer. Introducing the condition "hereditary on annihilating ideals" on a semisimple class, we show that this condition is necessary and sufficient to ensure that the radical is either supernilpotent or subidempotent. Both the above results relies on a construction due to Betsch and Kaarli [2]. We might also mention that it has been proved elsewhere [5] that a radical is supernilpotent if and only if the corresponding semisimple class is weakly homomorphically closed.

1. Preliminaries

All considerations below will be in the class of all, not necessarily 0-symmetric, (right) near-rings. We refer to Pilz [3] for more information and notation on near-rings. For the near-ring N, $I \lhd N$ will mean I is an ideal in N, N^+ will denote the underlying group of N and N^0 will be the zero near-ring on N (i.e. all products are zero). $\mathscr{Z}$ will denote the class of all zero near-rings. Furthermore, $N^{(c)}$ will be the 0-symmetric near-ring on N^+ with multiplication given by: $ab=a$ and $a0=0$ for all $a, b \in N$, $b \neq 0$. If $B \subseteq N$, then $(B)_N$ (or just (B)) will denote the ideal in N generated by B and B^2 is the set $\{ab | a, b \in B\}$. A class of near-rings $\mathscr{R}$ is a *radical class* (in the sense of Kurosh—Amitsur) if:

(1) $\mathscr{R}$ is homomorphically closed;

(2) For all N, $\mathscr{R}(N) := \sum (I \lhd N | I \in \mathscr{R}) \in \mathscr{R}$;

(3) For all N, $\mathscr{R}(N/\mathscr{R}(N))=0$.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 16A76; Secondary 16A21.

Key words and phrases. Radical class, semisimple class, weakly hereditary, hereditary on annihilating ideals.

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The *semisimple class* of $\mathcal{R}$, denoted by $\mathcal{S}\mathcal{R}$ is given by $\mathcal{S}\mathcal{R}:=\{N|\mathcal{R}(N)=0\}$. More details on radical and semisimple classes can be found in [4]. A fact which is often used, is: If $\mathcal{R}$ is a radical class and $K\lhd N$ such that $N/K\in\mathcal{S}\mathcal{R}$, then $\mathcal{R}(N)\subseteq\mathcal{R}(K)$.

If $\mathcal{M}$ is a regular class (i.e. if $0\neq I\lhd N\in\mathcal{M}$, then I has a non-zero homomorphic image in $\mathcal{M}$) then the class

$$\mathcal{U}\mathcal{M}:=\{N|\text{if } I\lhd N \text{ such that } N/I\in\mathcal{M}, \text{ then } N=I\}$$

is a radical class, the *upper radical* determined by $\mathcal{M}$. A radical class $\mathcal{R}$ is *supernilpotent* if $\mathcal{Z}\subseteq\mathcal{R}$ (note that we do not require $\mathcal{R}$ to be hereditary as is sometimes the case) and *subidempotent* if $\mathcal{R}\subseteq\mathcal{U}\mathcal{Z}$. Because $\mathcal{Z}$ is a regular class, $\mathcal{U}\mathcal{Z}$ is a radical class and $\mathcal{U}\mathcal{Z}=\{N|(N^2)=N\}$.

For our main results, we need two constructions due to Betsch and Kaarli [2]:

(1) Let N and M be near-rings. Define a near-ring Σ $(\Sigma=\Sigma(N, M))$ by:

$$\Sigma^+ = N^+\oplus N^+\oplus M^+ \quad \text{with multiplication given by}$$

$$(a, b, c,)(x, y, z) = \begin{cases}(b, 0, 0) & \text{if} \quad z\neq 0\\ (0, 0, 0) & \text{if} \quad z=0.\end{cases}$$

Then Σ is a 0-symmetric near-ring. If

$$K=\{(a, 0, m)|a\in N, m\in M\} \quad \text{and}$$

$$L=\{(0, 0, m)|m\in M\}, \quad \text{then}$$

$$L\lhd K\lhd\Sigma \quad (\text{but } L \text{ is not necessarily an ideal in } \Sigma),$$

$$\Sigma/K\cong N^0, \quad K/L\cong N^0 \quad \text{and} \quad L\cong M^0.$$

(2) Let N be a 0-symmetric near-ring. Define a near-ring φ $(\varphi=\varphi(N, N^*)$ where $*$ is some multiplication on N^+ such that $N^*=(N, +, *)$ is a near-ring) by:

$$\varphi^+ = N^+\oplus N^+\oplus N^+$$

with multiplication given by

$$(a, b, c)(x, y, z) = (a*x, czx, 0).$$

Then φ is a near-ring and if

$$K=\{(a, b, 0)|a, b\in N\} \quad \text{and}$$

$$L=\{(a, 0, 0)|a\in N\} \quad \text{we have}$$

$$L\lhd K\lhd\varphi \quad (\text{but } L \text{ is not necessarily an ideal in } \varphi),$$

$$K\cong N^*\oplus N^0, \varphi/K\cong N^0 \quad \text{and} \quad L\cong N^*.$$

2. Weakly hereditary classes

The condition "weakly homomorphically closed class" has been introduced by Anderson and Wiegandt [1] to investigate semisimple classes of supernilpotent radicals of associative rings. For near-rings it has been considered in [5]. In this section we will be concerned with the dual condition namely weakly hereditary.

Let $\mathcal{M}$ be a class of near-rings. $\mathcal{M}$ is said to be:

(1) *weakly homomorphically closed* if $I \triangleleft N \in \mathcal{M}$, $I^2=0$ implies $N/I \in \mathcal{M}$;
(2) *weakly hereditary* if $I \triangleleft N \in \mathcal{M}$, $N^2 \subseteq I$ implies $I \in \mathcal{M}$;
(3) *hereditary* if $I \triangleleft N \in \mathcal{M}$ implies $I \in \mathcal{M}$.

Clearly, if a class is hereditary, then it is also weakly hereditary. The converse is not true, not even for radical classes. In fact, every subidempotent radical is weakly hereditary but it need not be hereditary. I do not know if every weakly hereditary semisimple class must be hereditary or not.

The usual conditions for hereditariness of radical and semisimple classes holds for weakly hereditary with respect to certain ideals. Indeed, for a radical class $\mathcal{R}$ we have:

(1) $\mathcal{R}$ is weakly hereditary iff $\mathcal{R}(N) \cap I \subseteq \mathcal{R}(I)$ for all $I \triangleleft N$ with $N^2 \subseteq I$.
(2) $\mathcal{S}\mathcal{R}$ is weakly hereditary iff $\mathcal{R}(I) \subseteq \mathcal{R}(N) \cap I$ for all $I \triangleleft N$ with $N^2 \subseteq I$.
Furthermore, if $\mathcal{M}$ is weakly hereditary, then $N \in \mathcal{M}$ implies $N_k \in \mathcal{M}$ for all $k=1, 2, 3, \ldots$ where $N_1=(N^2)$, the ideal generated by N^2 in N and if N_n has been defined for all $n<k$, then N_k is the ideal in N_{k-1} generated by N_{k-1}^2.

PROPOSITION 2.1. *Let $\mathcal{R}$ be a radical class.*
(1) *$\mathcal{R}$ is supernilpotent if and only if $N^2 \subseteq \mathcal{R}(N)$ implies $N \in \mathcal{R}$.*
(2) *If $\mathcal{R}$ is supernilpotent, then the following are equivalent:*
(i) *$\mathcal{R}$ is weakly hereditary*
(ii) *$N \in \mathcal{R}$ iff $(N^2) \in \mathcal{R}$.*

PROOF. (1) Suppose $\mathcal{R}$ is supernilpotent and let N be a near-ring such that $N^2 \subseteq \mathcal{R}(N)$. Because $N/\mathcal{R}(N) \in \mathcal{Z} \cap \mathcal{S}\mathcal{R} \subseteq \mathcal{R} \cap \mathcal{S}\mathcal{R}=0$, we have $N=\mathcal{R}(N) \in \mathcal{R}$. Conversely, if the condition is satisfied, let $N \in \mathcal{Z}$. Then $N^2=0 \subseteq \mathcal{R}(N)$, hence $N \in \mathcal{R}$.

(2) Let $\mathcal{R}$ be supernilpotent and firstly, suppose $\mathcal{R}$ is weakly hereditary. Clearly, if $N \in \mathcal{R}$, then $(N^2) \in \mathcal{R}$. On the other hand, if N is a near-ring such that $(N^2) \in \mathcal{R}$, we have $N \in \mathcal{R}$ from (1) above. Conversely, suppose (ii) is satisfied. Let $I \triangleleft N \in \mathcal{R}$ with $N^2 \subseteq I$. Then, also $(N^2) \in \mathcal{R}$ by (ii) and $I/(N^2) \in \mathcal{Z} \subseteq \mathcal{R}$. By the extension property which is satisfied by any radical class, $I \in \mathcal{R}$ follows.

We now turn our attention to weakly hereditary semisimple classes.

THEOREM 2.2. *Let $\mathcal{R} \neq 0$ be a radical class such that $\mathcal{S}\mathcal{R}$ is weakly hereditary. Then $\mathcal{Z} \subseteq \mathcal{R}$.*

PROOF. The proof corresponds to the proof given by Betsch and Kaarli [2] for the case when $\mathcal{S}\mathcal{R}$ is hereditary; the ideals in the constructions Σ and φ have the right properties. For completeness we provide the proof which will be divided into three sections:

(1) If $\mathcal{S}\mathcal{R}\cap\mathcal{Z}\neq\varphi$, then $\mathcal{Z}\subseteq\mathcal{S}\mathcal{R}$.
(2) If $0\neq N$ is a 0-symmetric near-ring with $N^*\in\mathcal{R}$ and $N^0\in\mathcal{S}\mathcal{R}$, then $N^3=0$ where N^* is any near-ring on N^+.
(3) $Z\subseteq\mathcal{R}$.

(1) Let $0\neq N\in\mathcal{S}\mathcal{R}\cap\mathcal{Z}$ and let $M\in\mathcal{Z}$. Then $N=N^0$ and $M=M^0$. Construct $\Sigma=\Sigma(N, M)$. Then $L\lhd K\lhd\Sigma$ with $\Sigma/K\cong N\in\mathcal{S}\mathcal{R}$ and $K/L\cong N\in\mathcal{S}\mathcal{R}$, hence $\mathcal{R}(\Sigma)\subseteq\mathcal{R}(K)\subseteq\mathcal{R}(L)\subseteq L$. Now $\mathcal{R}(\Sigma)=0$, for if $\mathcal{R}(\Sigma)\neq 0$, let $o\neq m\in M$ with $(o, o, m)\in\mathcal{R}(\Sigma)$. Because $N\neq 0$, let $0\neq n\in N$. Since Σ is 0-symmetric, we have

$$(n, o, u) = (o, n, o)(o, o, m)\in\mathcal{R}(\Sigma)\subseteq L = \{(o, o, z)|z\in M).$$

Hence $n=0$, a contradiction.

From $K\lhd\Sigma\in\mathcal{S}\mathcal{R}$ and $\Sigma^2\subseteq K$ we have $K\in\mathcal{S}\mathcal{R}$ because $\mathcal{S}\mathcal{R}$ is weakly hereditary. Using this again with $L\lhd K\in\mathcal{S}\mathcal{R}$ and $K^2\subseteq L$, we have $M\cong L\in\mathcal{S}\mathcal{R}$. Hence $\mathcal{Z}\subseteq\mathcal{S}\mathcal{R}$.

(2) Let $0\neq N$ be a 0-symmetric near-ring with $N^*\in\mathcal{R}$ and $N^0\in\mathcal{S}\mathcal{R}$. By (1), we have $\mathcal{Z}\subseteq\mathcal{S}\mathcal{R}$. Construct $\varphi=\varphi(N, N^*)$. Then $\mathcal{R}(K)=\mathcal{R}(N^*\oplus N^0)=\mathcal{R}(N^*)= =N^*\cong L$. From $\varphi/K\cong N^0\in\mathcal{S}\mathcal{R}$, we have $\mathcal{R}(\varphi)\subseteq\mathcal{R}(K)\cong L$. But $\varphi^2\subseteq K\lhd\varphi$, hence $L\cong\mathcal{R}(K)\subseteq\mathcal{R}(\varphi)$ as $\mathcal{S}\mathcal{R}$ is weakly hereditary. We conclude that $L\cong\mathcal{R}(\varphi)\lhd\varphi$. Let $x, y, z\in N$. Then $(z, o, o)\in L$, hence

$$(o, xyz, o) = (o, o, x)\big((o, o, y)+(z, o, o)\big)-(o, o, x)(o, o, y)$$

is in $L=\{(a, o, o)|a\in N\}$.

Hence $xyz=0$ and $N^3=0$ follows.

(3) Suppose $\mathcal{Z}\nsubseteq\mathcal{R}$. Then there is an $M\in\mathcal{Z}$, $M\notin\mathcal{R}$. Then

$$0\neq M/\mathcal{R}(M)\in\mathcal{Z}\cap\mathcal{S}\mathcal{R}$$

and from (1) we have $\mathcal{Z}\subseteq\mathcal{S}\mathcal{R}$. Because $\mathcal{R}\neq 0$, choose $0\neq N^*\in\mathcal{R}$. Let $N=N^{(c)}$. Then N is a 0-symmetric near-ring with $N^*\in\mathcal{R}$ and $N^0\in\mathcal{S}\mathcal{R}$. From (2) above, we have $(N^{(c)})^3=N^3=0$. This, however, is only possible if $N=0$, i.e. $N^*=0$ which contradicts our choice of N^*.

COROLLARY 2.3. *If $0\neq\mathcal{R}$ is not a supernilpotent radical, then $\mathcal{S}\mathcal{R}$ is not weakly hereditary. Hence any subidempotent radical ($\neq 0$) cannot have a weakly hereditary semisimple class.*

It is not known whether the converse of Theorem 2.2 is true or not.

3. Semisimple classes hereditary on annihilating ideals

Motivated by: a radical class $\mathcal{R}$ is supernilpotent if and only if $I\lhd N\in\mathcal{S}\mathcal{R}$, $I^2=0$ implies $I=0$; we introduce the following notions: Let $\mathcal{M}$ be a class of near-rings. $\mathcal{M}$ is called:

(1) *hereditary on zero ideals* (or just zero-hereditary) if

$$I\lhd N\in\mathcal{M},\quad I^2=0 \quad \text{implies} \quad I\in\mathcal{M},$$

(2) *hereditary on annihilating ideals* (or just annihilating-hereditary) if

$$I \lhd N \in \mathcal{M}, \quad IN = 0 \quad \text{implies} \quad I \in \mathcal{M}.$$

Clearly, zero-hereditary implies annihilating-hereditary. If $\mathcal{R}$ is a radical class such that $\mathcal{Z} \subseteqq \mathcal{R}$ or $\mathcal{R} \subseteqq \mathcal{U}\mathcal{Z}$, then $\mathcal{S}\mathcal{R}$ is zero-hereditary. Hence, if $\mathcal{S}\mathcal{R}$ is weakly hereditary, then $\mathcal{S}\mathcal{R}$ is zero-hereditary from Theorem 2.2.

THEOREM 3.1. *Let $\mathcal{R}$ be a radical class such that $\mathcal{S}\mathcal{R}$ is annihilating-hereditary. If $\mathcal{Z} \cap \mathcal{S}\mathcal{R} \neq 0$, then $\mathcal{Z} \subseteqq \mathcal{S}\mathcal{R}$.*

PROOF. Let $0 \neq N \in \mathcal{Z} \cap \mathcal{S}\mathcal{R}$ and let $M \in \mathcal{Z}$. Using the construction $\Sigma = \Sigma(N, M)$ again, we have as in Theorem 2.2 that $\mathcal{R}(\Sigma) = 0$. By our condition on $\mathcal{S}\mathcal{R}$ and $K \lhd \Sigma$ with $K\Sigma = 0$ and $L \lhd K$ with $LK = 0$, we have

$$M \cong L \in \mathcal{S}\mathcal{R}.$$

Hence $\mathcal{Z} \subseteqq \mathcal{S}\mathcal{R}$.

COROLLARY 3.2. *Let $\mathcal{R}$ be a radical class. Then $\mathcal{S}\mathcal{R}$ is annihilating-hereditary if and only if $\mathcal{R}$ is supernilpotent or subidempotent.*

From the above and [5], we also have

COROLLARY 3.3. *Let $\mathcal{R}$ be a radical class such that $\mathcal{R} \cap \mathcal{Z} \neq 0$. Then the following are equivalent:*

(1) *$\mathcal{R}$ is supernilpotent;*
(2) *$\mathcal{S}\mathcal{R}$ is weakly homomorphically closed;*
(3) *$\mathcal{S}\mathcal{R}$ is zero-hereditary;*
(4) *$\mathcal{S}\mathcal{R}$ is annihilating-hereditary.*

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(Received November 25, 1985)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 417—433

UNIVALENCY AS TAUBERIAN CONDITION

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1. Introduction

Let

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

be univalent in $|z|<1$, and suppose that the series is "fast" Abel summable at $z=1$, i.e.

$$f(r)-A = O\big((1-r)^\alpha\big) \quad \text{for some} \quad 0 < \alpha \leqq 2. \tag{1.2}$$

We also include the case of ordinary Abel summability

$$f(r)-A = o(1) \quad \text{if} \quad \alpha = 0. \tag{1.3}$$

In this paper we investigate under what conditions the power series (1.1) is "fast" convergent at $z=1$, i.e.

$$\sum_{i=0}^{n} a_i - A = O(n^{-\beta}) \quad \text{for some} \quad \beta > 0. \tag{1.4}$$

Our condition will be geometric and it will mean that the length of the longest arc in the intersection of $|z-A|=R$ with the image of $|z|<1$ under the mapping $f(z)$ is smaller than $R\pi(1-\beta)$, for many values of R.

G. Halász [1] and W. K. Hayman [2] investigated under what conditions the power series (1.1) is convergent at $z=1$, if we only know the existence of the Abel limit. In this case our condition also will imply the convergence.

G. Freud [8] and J. Korevaar [9] investigated under what conditions $\sum a_n$ is summable by the first Cesàro means, if we know the existence of the "fast" Abel limit.

Based on the above-mentioned results P. L. Duren [10] managed to obtain an estimation of coefficients of univalent functions.

I would like to thank Gábor Halász for his advice while working on these questions.

1980 *Mathematics Subject Classification*. Primary 30C50; Secondary 40E05.
Key words and phrases. Univalent functions, Tauberian theorems, integral means.

Akadémiai Kiadó, Budapest

2. Definitions and results

Suppose that $f(z)$ is univalent in $|z|<1$ and $f(r)\to A$ as $r\to 1-0$. We set $D:=f(|z|<1)$ and $D':=f(\{|z|<1\}\cap\{|z-1|<\delta\})$ if $\delta>0$. Let β be any number $0\leqq\beta<1$, and let B be any fixed point of D'. We define S to be the set of all $R>0$ such that there is an arc in the intersection of $|w-A|=R$ with D' which does not separate A and B in D' and is longer than $R\pi(1-\beta)$, or, if $R>1$, there is an arc in the intersection of $|w-A|=R$ with D which is longer than $R\pi(1-\beta)$.

If for some positive δ the logarithmic length of S is finite, we will say that $f(z)$ is β-admissible. We define the logarithmic length by

$$L=\int_S 1\,d\log r=\int_S\frac{dr}{r}=\int_{\log S}1. \tag{2.1}$$

We note that if $f(z)$ is β-admissible, then it is β'-admissible for all $\beta'\leqq\beta$, too.

It is convenient to define $G_\alpha(N)$ and s_N as follows:

$$G_\alpha(N)=\begin{cases} o(1) & \text{if } \alpha=0\\ N^{-\alpha} & \text{if } 0<\alpha<\dfrac{1}{2}\\ N^{-\left(\frac{1}{2}+\frac{2\alpha-1}{322}\right)} & \text{if } \dfrac{1}{2}<\alpha<1\\ N^{-\frac{1}{2}+o(1)} & \text{if } \alpha=\dfrac{1}{2},\end{cases} \tag{2.2}$$

$$s_N=\sum_{n=0}^{N}a_n. \tag{2.3}$$

THEOREM 1. *Let $f(z)$ be given by* (1.1) *and suppose that it is "fast" Abel summable at $z=1$ with $0<\alpha<1$ and β-admissible, $\beta>0$. Then we have for $\lambda=\min\{\alpha,\beta\}$*

$$|s_N-A|=O(G_\lambda(N)\log N), \tag{2.4}$$

where A means the Abel limit of $f(z)$ at $z=1$.

If $\beta>\alpha$, then we can dispose of the log factor.

THEOREM 2. *If $0\leqq\alpha<\dfrac{1}{2}$ and $\beta>\alpha$ in Theorem 1, then we have*

$$|s_N-A|=O(G_\alpha(N)). \tag{2.5}$$

If $\alpha\geqq 1$, we cannot prove anything better than what follows from Theorem 1.

3. Auxiliary results

We note, following G. Halász [1], that (1.4) implies

$$|f(z)-A| = O\left(|1-z|^{\beta}\left(\frac{|1-z|}{1-|z|}\right)^{1-\beta}\right), \tag{3.1}$$

as $|z|\to 1-0$ uniformly.

First we prove at least this necessary condition. We will need the next four lemmas.

LEMMA 1. *If $f(z)$ is given by* (1.1) *and* $K>0$, *then we have for* $|1-z|<K(1-|z|)$

$$|f(z)-A| < C(K)\big|f(|z|)-A\big|, \tag{3.2}$$

where $C(K)$ depends only on K.

PROOF. We repeat the argument of [1, p. 424]. Let $0<r<1$ be fixed and $\{z_n\}$ be a sequence, such that $z_0=r$, $|z_n|=r$, and $|z_n-z_{n-1}|\leqq(1-r)/2$. Of course, $A\notin f(|z|<1)$, so we can apply the "1/4 theorem of Koebe" to the circle $|z-z_0|<1-r$ and we get

$$|f(z_0)-A| \geqq \frac{1}{4}|f'(z_0)|(1-r). \tag{3.3}$$

Koebe's inequality for this circle gives

$$|f(z_0)-f(z_1)| \leqq |f'(z_0)|(1-r)\frac{\dfrac{|z_1-z_0|}{1-r}}{\left(1-\dfrac{|z_1-z_0|}{1-r}\right)^2} \leqq 2|f'(z_0)|(1-r), \tag{3.4}$$

and we have from (3.3) and (3.4)

$$|f(z_1)-A| \leqq |f(z_1)-f(z_0)|+|f(z_0)-A| \leqq 9|f(z_0)-A|. \tag{3.5}$$

Next, we can apply the same argument to the circles $|z_i-z|<1-r$, $i=1, 2, \ldots$ and we get

$$|f(z_n)-A| \leqq 9^n|f(z_0)-A|,$$

which proves the Lemma.

We will apply the length-area principle [3, Th. 2.1]. Let $f(z)$ be regular in an open set Δ and let $n(w)$ be the number of roots in Δ of the equation $f(z)=w$. We write, following Hayman,

$$p(R):=\frac{1}{2\pi}\int_0^{2\pi} n(Re^{i\varphi})\,d\varphi. \tag{3.6}$$

If $f(z)$ is a univalent function, then the length of the intersection of $|w|=R$ with the image equals

$$2\pi Rp(R) \leqq 2\pi R. \tag{3.7}$$

The length-area principle asserts

$$\int_0^\infty \frac{l(R)^2}{2\pi R p(R)}\,dR \leqq \text{Area}\,(\Delta), \tag{3.8}$$

where $l(R)$ is the total length of the curves in Δ, on which $|f(z)|=R$, and Area (Δ) is the area of Δ.

Let $f(z)$ be univalent in $|z|<1$, $0\leqq\beta<1$ and $\frac{3}{4}<p<r<1$, and let $0<\varphi\leqq 2\pi$ be fixed. Let γ be the image of the segment $[pe^{i\varphi}, re^{i\varphi}]$ under the mapping $f(z)$. Suppose that $|f(re^{i\varphi})|>|f(pe^{i\varphi})|$ and define $\tilde{S}$, similarly to S, to be the set of all R, $|f(pe^{i\varphi})|<R<|f(re^{i\varphi})|$, such that the length of the longest arc in the intersection of $|w|=R$ with $f(|z|<1)$ which intersects γ is bigger than $R\pi(1-\beta)$. If L denotes the logarithmic length of $\tilde{S}$ defined by (2.1) then we can state:

LEMMA 2. *If we use the above notations we have*

$$\frac{|f(re^{i\varphi})|}{|f(pe^{i\varphi})|} < C(L)\left(\frac{1-p}{1-r}\right)^{1-\beta},$$

where $C(L)$ depends only on L.

PROOF. We assume, as we may, that $\varphi=0$. Let $D\gamma$ be the union of all arcs of the circles $|w|=R$, $|f(p)|<R<|f(r)|$, which intersects γ and are in $f(|z|<1)$.

It is clear from the regularity of γ that

$$\text{int}\, D_\gamma \cap \{|z| = R\} \neq \emptyset, \qquad |f(p)| < R < |f(r)|. \tag{3.9}$$

(It would be enough to have (3.9) for a.e.)

We set, following Hayman [3],

$$s(z) := \log\frac{1+z}{1-z} \quad \text{and} \quad g(z) := f\circ s^{-1}(z). \tag{3.10}$$

Now, $g(z)$ maps the strip $|\text{Im}\,(z)|<\frac{\pi}{2}$ onto $f(|z|<1)$.

We write

$$Q := \left\{z\colon -\frac{\pi}{2}+\log\frac{1+p}{1-p} < \text{Re}\,(z) < \frac{\pi}{2}+\log\frac{1+r}{1-r} \quad \text{and} \quad |\text{Im}\,(z)| < \frac{\pi}{2}\right\}.$$

It is a rectangle. Now, let us restrict $g(z)$ to $\tilde{Q} := Q\cap g^{-1}(\text{int}\, D_\gamma)$, and let us apply the length-area principle to $g|_{\tilde{Q}}$. With the above-mentioned notations, we set $T=[|f(p)|, |f(r)|]-\tilde{S}$, and suppose that $R\in T$. We will show that

$$\frac{l(R)^2}{p(R)} \geqq \frac{2\pi^2}{1-\beta}. \tag{3.11}$$

Let i be a component of the intersection of $|w|=R$ with $\text{int}\, D_\gamma$. It is clear from the definition of D_γ that i intersects γ, so $g^{-1}(i)$ intersects the segment

$\left[\log\frac{1+p}{1-p}, \log\frac{1+r}{1-r}\right]$ and the end points of $g^{-1}(i)$ lie on the straight lines $\mathrm{Im}\,(z)=$ $=\pm\frac{\pi}{2}$. Denote by $i_1, i_2, \dots$ the components of the intersection of $|w|=R$ with $g(\tilde{Q})$ which are in i. Since $g^{-1}(i_n)$ is a component of the intersection of $g^{-1}(i)$ with Q, $n=1, 2, \dots$, the end points of $g^{-1}(i_n)$ lie on the edge of Q, and there is an i_m that $g^{-1}(i_m)$ intersects the segment $\left[\log\frac{1+p}{1-p}, \log\frac{1+r}{1-r}\right]$. It shows that the total length of $g^{-1}(i_1), g^{-1}(i_2), \dots$ is not smaller than π. On the other hand the total length of $i_1, i_2, \dots$ is not greater than the length of i which is smaller than $R\pi(1-\beta)$, $(R\in T)$.

Suppose that the intersection of $|w|=R$ with D_γ consists of k components. Using the above-mentioned notations for the $g|_Q$ we have

$$l(R) \geqq k\pi \quad \text{and} \quad R2\pi p(R) \leqq kR\pi(1-\beta),$$

and this yields (3.11).

Thus, by the length-area principle

$$\frac{\pi}{1-\beta}\int_T \frac{dR}{R} \leqq \int_0^\infty \frac{l(R)^2}{2\pi R p(R)}\,dR \leqq \mathrm{Area}\,(\tilde{Q}) \leqq$$

$$\leqq \mathrm{Area}\,(Q) = \pi\left(\log\frac{1-p}{1-r}+\log\frac{1+r}{1+p}+\pi\right),$$

and so

$$\int_{|f(p)|}^{|f(r)|}\frac{dR}{R} \leqq \log\left(\frac{1-p}{1-r}\right)^{1-\beta}+C+\int_S\frac{dR}{R}.$$

Then we have

$$\log\left|\frac{f(r)}{f(p)}\right| \leqq \log\left(\frac{1-p}{1-r}\right)^{1-\beta}+C(L)$$

and the proof is completed.

Remark 1. From the proof it is clear that it would be enough to define $\tilde{S}$ to be the set of all R, $|f(pe^{i\varphi})|<R<|f(re^{i\varphi})|$, such that the longest arc in the intersection of $|w|=R$ with

$$f(\{|z|<1\}\cap\{|z-e^{i\varphi}|<10(1-p)\})$$

which intersects γ is bigger than $R\pi(1-\beta)$.

Now, we investigate the case of separating arcs.

We write $\xi=pe^{i\varphi}$ and $\eta=re^{i\varphi}$, and suppose that $\frac{3}{4}<p<r<1$, $|1-\xi|$, $|1-\eta|<$ $<\frac{1}{4}$, and that ξ and η lie outside a Stolz angle, i.e. $|1-\xi|>2(1-|\xi|)$ and $|1-\eta|>$ $>2(1-|\eta|)$. Let $f(z)$ be given by (1.1) and denote by γ the image of the segment $[\xi, \eta]$ and by Γ the image of the segment $[0, 1]$ under $f(z)$.

Suppose that $|f(\xi)-A|<|f(\eta)-A|$ and for every R, $|f(\xi)-A|<R<|f(\eta)-A|$, there is a component of the intersection of $|w-A|=R$ with $f(|z|<1)$ which intersects both γ and Γ. Now, we get

LEMMA 3. *By the above-mentioned assumptions, we have*

$$\left|\frac{f(\eta)-A}{f(\xi)-A}\right| < e^{200}. \tag{3.12}$$

PROOF. We assume that $A=0$. Let $s(z)$, $g(z)$ be given by (3.10). It is easy to see that

$$\left|\log\left|\frac{1+\xi}{1-\xi}\right|-\log\left|\frac{1+\eta}{1-\eta}\right|\right| < 2\log 2 < 2.$$

We define Q to be the rectangle

$$Q:=\left\{z\colon \log\left|\frac{1+\xi}{1-\xi}\right|-3 < \mathrm{Re}(z) < \log\left|\frac{1+\eta}{1-\eta}\right|+3,\quad |\mathrm{Im}(z)| < \frac{\pi}{2}\right\}. \tag{3.13}$$

It is clear that $\mathrm{Area}(Q)<\pi 8$.

Suppose that $|f(\xi)|<R<|f(\eta)|$, then, by the assumptions of Lemma 3, there is a component i of the intersection of $|w|=R$ with $f(|z|<1)$ which intersects both γ and Γ. Thus, $g^{-1}(i)$ intersects the real axis and its endpoints lie on the straight lines $\mathrm{Im}(z)=\pm\frac{\pi}{2}$. So it is clear that the total length of $g^{-1}(i)\cap Q$ is greater than 1.

Now, let us apply the length-area principle to $g|_Q$ with its notations, then we have $l(R)\geqq 1$ if $|f(\xi)|<R<|f(\eta)|$, and since (3.7) implies that $p(R)\leqq 1$, we obtain

$$\frac{1}{2\pi}\int_{|f(\xi)|}^{|f(\eta)|}\frac{dR}{R} \leqq \mathrm{Area}(Q) < \pi 8,$$

and this yields Lemma 3.

Now, we can prove the necessary condition.

LEMMA 4. *By the assumptions of Theorem 1, we have*

$$|f(\eta)-A| = O\left(|1-\eta|^{\alpha}\left(\frac{|1-\eta|}{1-|\eta|}\right)^{1-\beta}\right) \quad \textit{if} \quad \alpha>0, \tag{3.14}$$

$$|f(\eta)-A| = o(1)\left(\frac{|1-\eta|}{1-|\eta|}\right)^{1-\beta'} \quad \textit{for all} \quad 0<\beta'<\beta \quad \textit{if} \quad \alpha=0, \tag{3.15}$$

as $|\eta|\to 1-0$ *uniformly.*

PROOF. We assume, as we may, that $A=0$, $\delta<\frac{1}{4}$. If $|1-\eta|>\delta/2$, then we apply Lemma 2 with $\eta = re^{i\varphi}$ and $p=\frac{3}{4}$. This yields (3.14), and (3.15) follows from

$$\left(\frac{|1-\eta|}{1-|\eta|}\right)^{1-\beta} = \left(\frac{1-|\eta|}{|1-\eta|}\right)^{\beta-\beta'} \left(\frac{|1-\eta|}{1-|\eta|}\right)^{1-\beta'}.$$

Suppose now that $|1-\eta| < \delta/2$, and let K be large enough and fixed. If η belongs to the Stolz angle, i.e. $|1-\eta| \leqq K(1-|\eta|)$, then we apply Lemma 1 and this implies (3.14) and (3.15).

If η lies outside the Stolz angle, then we write $\eta = re^{i\varphi}$ and $\xi = pe^{i\varphi}$ where $|1-\xi| = K(1-|\xi|)$. We will estimate the growth of $|f(z)|$ in the segment $[\xi, \eta]$. By notation of Theorem 1, it is clear that for sufficiently small or large R, an arc i in the intersection of $|w|=R$ with D' separates O and B, if and only if it separates O and $f(1-\delta)$. Hence we may assume that $B=f(1-\delta)$.

If $|f(\eta)| \leqq |f(\xi)|$ then in the sense of previous notation it implies (3.14) and (3.15), so we may assume that $|f(\xi)| < |f(\eta)|$. Denote by γ the map of the segment $[\xi, \eta]$ and by Γ the map of the segment $[1-\delta, 1]$ under $f(z)$. We define U to be the set of all R, $|f(\xi)| < R < |f(\eta)|$, such that there exists an arc i in the intersection of $|w|=R$ with D' which intersects both γ and Γ.

We will show that U is an interval. In fact, suppose that $R_1, R_2 \in U$ and $R_1 < R_2$. Denote the arcs by i_1 and i_2 which belong to R_1 and R_2, respectively, then $i_m \subset \{|w|=R_m\} \cap D'$, $m=1, 2$ and they connect γ and Γ. Denote by γ' (and Γ') a minimal subarc of γ (Γ resp.) that connects i_1 and i_2, and denote by i_1' (and i_2') a minimal subarc of i_1 (i_2 resp.) that connects γ' and Γ'. By the univalency of $f(z)$, γ does not intersect Γ, so $i_1', \gamma', i_2', \Gamma'$ form a simple closed curve Ω in D'. Since D' is simply connected, the interior of Ω is also contained in D' and for all $R_1 < R < R_2$ there is an arc in $|w|=R$ which belongs to int Ω and connects γ' and Γ', i.e. $R \in U$ as stated. The empty set is also considered to be an "interval".

We remark that if for some R, $|f(\xi)| < R < |f(\eta)|$, there is an arc in the intersection of $|w|=R$ with D' which separates O and $B=f(1-\delta)$ and intersects γ then $R \in U$.

Let $R_1 \leqq R_2$ be the endpoints of U (if $U \neq \emptyset$) and because $|f(\xi)| \leqq R_1 \leqq R_2 \leqq$ $\leqq |f(\eta)|$ it is easy to see that there are two points $\mu_1 = q_1 e^{i\varphi}$, $\mu_2 = q_2 e^{i\varphi}$ on the segment $[\xi, \eta]$ such that $|f(\mu_1)| = R_1$, $|f(\mu_2)| = R_2$ and $q_1 \leqq q_2$.

It is clear that if $K > 2$ we can apply Lemma 3 to the segment $[\mu_1, \mu_2]$ and we have

$$\left|\frac{f(\mu_2)}{f(\mu_1)}\right| < e^{200}. \tag{3.16}$$

Moreover, if K is large enough then the set $\{|z-e^{i\varphi}| < 10(1-p)\} \cap \{|z|<1\}$ is contained in the set $\{|1-z|<\delta\} \cap \{|z|<1\}$ and if for some R, $|f(\xi)| < R < |f(\eta)|$ and $R \notin U$, we take an arc in the intersection of $|w|=R$ with

$$f(\{|z| < 1\} \cap \{|z-e^{i\varphi}| < 10(1-p)\})$$

which is longer than $R\pi(1-\beta)$ and intersects γ then this arc also belongs to the intersection of $|w|=R$ with D' and does not separate O and B in D', i.e. $R \in S$, where S is defined in the second section.

Now, it is clear that $\tilde{S}\subset S$, where $\tilde{S}$ is defined in Remark 1, and by Remark 1 we can apply Lemma 2 to the segments $[\xi, \mu_1]$, $[\mu_2, \eta]$ and we have

$$(3.17)\qquad \left|\frac{f(\mu_1)}{f(\xi)}\right| < C(L)\left(\frac{1-p}{1-q_1}\right)^{1-\beta} \quad\text{and}\quad \left|\frac{f(\eta)}{f(\mu_2)}\right| < C(L)\left(\frac{1-q_2}{1-r}\right)^{1-\beta},$$

where L means the logarithmic length of S.

If $U=\emptyset$ then we can immediately apply Lemma 2 to the segment $[\xi, \eta]$.

Thus it follows from (3.16) and (3.17) that

$$(3.18)\qquad \left|\frac{f(\eta)}{f(\xi)}\right| < e^{200}C(L)^2\left(\frac{1-p}{1-r}\right)^{1-\beta}.$$

Since ξ belongs to the Stolz angle, Lemma 1 yields

$$(3.19)\qquad |f(\xi)| < C(K)|f(\xi)|.$$

On the other hand $1-|\xi|<|1-\eta|$ (if $K>2$) and thus if $\alpha>0$, (3.18) and (3.19) imply (3.14).

If $\alpha=0$, then we write by (3.18) and (3.19)

$$|f(\eta)| \leqq C(L, K)f(|\xi|)\left(\frac{1-|\eta|}{|1-\eta|}\right)^{\beta-\beta'}\left(\frac{|1-\eta|}{1-|\eta|}\right)^{1-\beta'}.$$

Now, if $\eta\to 1$ then $f(|\xi|)\to 0$, otherwise $\left(\frac{1-|\eta|}{|1-\eta|}\right)^{\beta-\beta'}\to 0$ and this yields (3.15) and also Lemma 4.

4. Proof of Theorem 1 and 2

We remark that if $\alpha=0$, $\beta>0$, then Lemma 4 and a result of Hayman [2, Theorem 1] also imply the convergence but for the sake of completeness we will investigate this case, too.

In this section we will refer to a result of G. Halász which has not been published yet. With his permission, we will give his proof at the end of the paper.

LEMMA 5. *Let $f(z)$ be univalent in $|z|<1$ and suppose that $f(z)\neq 0$ everywhere. Let $\varepsilon, \delta>0$ then we have*

$$\int_0^{2\pi}\left|\frac{f'}{f}(re^{i\varphi})\right|^{2+\varepsilon}\frac{1}{|1-re^{i\varphi}|^{\delta}}\,d\varphi = O\left(\frac{1}{(1-r)^{1+\varepsilon+\delta}}\right), \quad \textit{as} \quad r\to 1-0.$$

In the remaining part of the article we shall denote by $C(X)$ a constant which only depends on X.

We assume that $\beta\geqq\alpha$ in Theorem 1 since, otherwise, we can put β instead of α in (1.2).

Let $f(z)$ be given by (1.1) and write $s_n=\sum_{i=0}^{n} a_i$ then we have

$$z\left(\frac{f(z)-A}{1-z}\right)' = z\Big(\sum_{n=0}^{\infty}(s_n-A)z^n\Big)' = \sum_{n=0}^{\infty} n(s_n-A)z^n.$$

Thus, by Cauchy's formula,

$$n(s_n - A) = \frac{1}{2\pi i} \int_{|z|=r} \frac{1}{z^n}\left(\frac{f'(z)}{1-z} + \frac{f(z)-A}{(1-z)^2}\right) dz.$$

Let us choose $r = 1 - \frac{1}{n}$ and so $(z = re^{i\varphi})$

$$n|s_n - A| \leqq C\left[\int_0^{2\pi}\left|\frac{f'(z)}{1-z}\right| d\varphi + \int_0^{2\pi}\left|\frac{f(z)-A}{(1-z)^2}\right| d\varphi\right]. \tag{4.1}$$

We apply Hölder's inequality $\left(\frac{1}{r}+\frac{1}{q}=1,\ z=re^{i\varphi},\ 0 \leqq \varepsilon \leqq \beta-\alpha\right)$

$$\int_0^{2\pi}\left|\frac{f'(z)}{1-z}\right| d\varphi \leqq \left[\int_0^{2\pi}\left|\frac{f(z)-A}{(1-z)^{1-\varepsilon}}\right|^q d\varphi\right]^{\frac{1}{q}}\left[\int_0^{2\pi}\left|\frac{f'(z)}{f(z)-A}\right|^p \frac{d\varphi}{|1-z|^{\varepsilon p}}\right]^{\frac{1}{p}}. \tag{4.2}$$

To prove the cases of $\alpha = 0$ and $\alpha > 0$ together, we set

$$Q_\alpha(r) = \begin{cases} 1 & \text{if} \quad \alpha > 0 \\ o(1) & \text{if} \quad \alpha = 0 \end{cases} \quad \text{as} \quad r \to 1-0. \tag{4.3}$$

To estimate the above integrals, we use a differential inequality method which has been applied by G. Halász [1]. Let $F(z)$ be a regular and non-zero function in $|z|<1$. We write

$$U(r) := \int_0^{2\pi} |F(re^{i\varphi})|\, d\varphi,$$

and so

$$U'(r) = \int_0^{2\pi} \frac{d}{dr}|F(re^{i\varphi})|\, d\varphi \leqq \int_0^{2\pi} |F'(re^{i\varphi})|\, d\varphi.$$

We define

$$\Omega(r) := \left\{\varphi : \left|\frac{F'}{F}(re^{i\varphi})\right| > \frac{\delta}{1-r}\right\} \quad \text{for some} \quad \delta > 0,$$

and

$$V(r) := \int_{\Omega(r)} |F'(re^{i\varphi})|\, d\varphi.$$

If we denote by $\bar{\Omega}(r)$ the complement of $\Omega(r)$, we have

$$\int_{\bar{\Omega}(r)} |F'(re^{i\varphi})|\, d\varphi = \int_{\bar{\Omega}(r)} |F(re^{i\varphi})|\left|\frac{F'}{F}(re^{i\varphi})\right| d\varphi \leqq \frac{\delta}{1-r} U(r),$$

thus

$$U'(r) \leqq \frac{\delta}{1-r} U(r) + V(r).$$

Hence

$$(U(r)(1-r)^\delta)' \leqq (1-r)^\delta V(r).$$

Let $0<a<1$ be fixed. Integrating from a to r $(a<r<1)$ we have

$$U(r) \leqq \frac{1}{(1-r)^\delta} \int_a^r (1-x)^\delta V(x)\,dx + \frac{U(a)(1-a)^\delta}{(1-r)^\delta}. \tag{4.4}$$

Now, to estimate the second integral on the right-hand side of (4.1) we set $F(z)=\frac{f(z)-A}{(1-z)^2}$. We have

$$V(r) = \int_{\Omega(r)} \left|\frac{F'}{F}(re^{i\varphi})\right| |F(re^{i\varphi})|\,d\varphi,$$

so, by the definition of $\Omega(r)$, we get $(s>2,\ z=re^{i\varphi})$

(4.5)

$$V(r) \leqq \int_{\Omega(r)} \left|\frac{F'}{F}(z)\right| \left|\frac{F'}{F}(z)\frac{1-r}{\delta}\right|^{s-1} |F(z)|\,d\varphi \leqq \left(\frac{1-r}{\delta}\right)^{s-1} \int_0^{2\pi} \left|\frac{F'}{F}(z)\right|^s |F(z)|\,d\varphi.$$

Using Lemma 4 we know $(z=re^{i\varphi})$

$$|F(z)| = \left|\frac{f(z)-A}{(1-z)^2}\right| = O\left(Q_\alpha(r)\frac{1}{(1-r)^{1-\beta}}\frac{1}{|1-z|^{1+\beta-\alpha}}\right),$$

where if $\alpha=0$ then β means β' in Lemma 4, but we will see that it does not cause any difficulty. It yields $(z=re^{i\varphi})$

$$V(r) \leqq C(f)Q_\alpha(r)\left(\frac{1-r}{\delta}\right)^{s-1}\frac{1}{(1-r)^{1-\beta}}\int_0^{2\pi}\left|\frac{F'}{F}(z)\right|^s\frac{1}{|1-z|^{1+\beta-\alpha}}\,d\varphi.$$

We know

$$\left|\frac{F'}{F}(z)\right|^s = \left|\frac{f'(z)}{f(z)-A}+\frac{2}{1-z}\right|^s \leqq 2^{s-1}\left(\left|\frac{f'(z)}{f(z)-A}\right|^s + \left|\frac{2}{1-z}\right|^s\right),$$

and

$$\int_0^{2\pi}\frac{d\varphi}{|1-re^{i\varphi}|^{1+\varepsilon}} = \begin{cases} O\left(\dfrac{1}{(1-r)^\varepsilon}\right) & \text{if } \varepsilon>0 \\ O\left(\log\dfrac{1}{1-r}\right) & \text{if } \varepsilon=0 \end{cases} \tag{4.6}$$

is well-known [6, Lemma 2] as well. Hence it follows from Lemma 5 that

$$V(r) \leqq O(1)Q_\alpha(r)(1-r)^{s-2+\beta}\frac{1}{(1-r)^{s+\beta-\alpha}} = O(1)\frac{Q_\alpha(r)}{(1-r)^{2-\alpha}}.$$

Using (4.4) we have $(z=re^{i\varphi},\ \delta<1-\alpha)$.

$$\begin{aligned} \int_0^{2\pi}\left|\frac{f(z)-A}{(1-z)^2}\right|d\varphi &\leqq O(1)\frac{Q_\alpha(a)}{(1-r)^\delta}\int_a^r\frac{dx}{(1-x)^{2-\alpha-\delta}} + \frac{U(a)(1-a)^\delta}{(1-r)^\delta} \leqq \\ &\leqq O(1)\frac{Q_\alpha(a)}{(1-r)^{1-\alpha}} + \frac{U(a)(1-a)^\delta}{(1-r)^\delta} \leqq O(1)\frac{Q_\alpha(a)}{(1-r)^{1-\alpha}} \end{aligned} \tag{4.7}$$

if r is sufficiently near 1.

To estimate the first integral on the right-hand side of (4.2), we use the same method. We choose ε and q in such a way that $\varepsilon=0$ if $\beta=\alpha$ and otherwise $0<\varepsilon<<\beta-\alpha$ and $q(1-\alpha-\varepsilon)>1$ and set

$$F(z)=\left(\frac{f(z)-A}{(1-z)^{1-\varepsilon}}\right)^q.$$

Using Lemma 4, we know $(z=re^{i\varphi})$

$$|F(z)|\leqq C(f)\left(Q_\alpha(r)\frac{1}{(1-r)^{1-\beta}}\frac{1}{|1-z|^{\beta-\alpha-\varepsilon}}\right)^q,$$

where if $\alpha=0$ then β means β' in Lemma 4. It follows from (4.5) that $(s>2, z=re^{i\varphi})$

$$V(r)\leqq C(f)\left(\frac{1-r}{\delta}\right)^{s-1}\frac{Q_\alpha(r)^q}{(1-r)^{q(1-\beta)}}\int_0^{2\pi}\left|\frac{F'}{F}(z)\right|^s\frac{d\varphi}{|1-z|^{q(\beta-\alpha-\varepsilon)}}.$$

It is easy to show that

$$\left|\frac{F'}{F}(z)\right|^s=\left|q\frac{f'(z)}{f(z)-A}+\frac{q(1-\varepsilon)}{1-z}\right|^s\leqq 2^{s-1}\left(\left|q\frac{f'(z)}{f(z)-A}\right|^s+\left|\frac{q(1-\varepsilon)}{1-z}\right|^s\right),$$

and so if $\beta>\alpha$, then Lemma 5 and (4.6) imply

$$V(r)\leqq O(1)\frac{Q_\alpha(r)^q}{(1-r)^{q(1-\alpha-\varepsilon)}}.\tag{4.8}$$

But if $\beta=\alpha$ and so $\varepsilon=0$, then we have to apply the well-known result of Biernacki [4] (see also [5]), $(z=re^{i\varphi})$

$$\int_0^{2\pi}\left|\frac{f'(z)}{f(z)-A}\right|^2d\varphi=O\left(\frac{1}{1-r}\log\frac{1}{1-r}\right)\tag{4.9}$$

instead of Lemma 5, hence (with $s=2$)

$$V(r)\leqq O(1)\frac{Q_\alpha(r)^q}{(1-r)^{q(1-\alpha)}}\log\frac{1}{1-r}.\tag{4.10}$$

Using (4.4), if $\beta>\alpha$, we have $(z=re^{i\varphi}, \delta<q(1-\alpha-\varepsilon)-1)$

$$\begin{aligned}\int_0^{2\pi}\left|\frac{f(z)-A}{(1-z)^{1-\varepsilon}}\right|^q d\varphi &\leqq O(1)\frac{Q_\alpha(a)^q}{(1-r)^\delta}\int_a^r\frac{(1-x)^\delta}{(1-x)^{q(1-\alpha-\varepsilon)}}dx+\frac{U(a)(1-a)^\delta}{(1-r)^\delta}\leqq\\ &\leqq O(1)\frac{Q_\alpha(a)^q}{(1-r)^{q(1-\alpha-\varepsilon)-1}}+\frac{U(a)(1-a)^\delta}{(1-r)^\delta}\leqq O(1)\frac{Q_\alpha(a)^q}{(1-r)^{q(1-\alpha-\varepsilon)-1}},\end{aligned}\tag{4.11}$$

if r is sufficiently near 1. In the same way if $\alpha=\beta$, we get $\left(z=re^{i\varphi},\ \delta<q(1-\alpha)-1\right)$

$$\int_0^{2\pi}\left|\frac{f(z)-A}{(1-z)}\right|^q d\varphi \leqq O(1)\frac{Q_\alpha(a)^q}{(1-r)^{q(1-\alpha)-1}}\log\frac{1}{1-r}, \tag{4.12}$$

if r is sufficiently near 1.

Now, suppose that $\alpha<\frac{1}{2}$. Then we can choose $\delta>0$, $\varepsilon\geqq 0$, $1<q<2$, such that $0<\delta<q(1-\alpha-\varepsilon)-1$ and set $p=q/(q-1)>2$. If $\beta>\alpha$ then (4.7), (4.11) and Lemma 5 imply by (4.1) and (4.2)

$$n|s_n-A|\leqq O(1)\left(\frac{Q_\alpha(a)}{(1-r)^{1-\alpha}}+\frac{Q_\alpha(a)}{(1-r)^{1-\alpha-\varepsilon-\frac{1}{q}}}\frac{1}{(1-r)^{1+\varepsilon-\frac{1}{p}}}\right)=$$

$$=O(1)Q_\alpha(a)\frac{1}{(1-r)^{1-\alpha}},$$

and regarding the choice of r we have

$$n|s_n-A|\leqq O(1)Q_\alpha(a)n^{1-\alpha},$$

if n is large enough. This implies immediately Theorem 2 when $\alpha>0$, and if $\alpha=0$, then we know from (4.3) that $Q_\alpha(a)\to 0$ as $a\to 1$, and this yields Theorem 2.

If $\beta=\alpha$, then by choosing $q=p=2$, (4.7), (4.12) and (4.9) imply by (4.1) and (4.2)

$$n|s_n-A|=O(1)Q_\alpha(a)\left[\frac{1}{(1-r)^{1-\alpha}}+\frac{\left(\log\frac{1}{1-r}\right)^{\frac{1}{q}}}{(1-r)^{1-\alpha-\frac{1}{q}}}\,\frac{\left(\log\frac{1}{1-r}\right)^{\frac{1}{p}}}{(1-r)^{\frac{1}{p}}}\right]=$$

$$=O(1)Q_\alpha(a)\frac{1}{(1-r)^{1-\alpha}}\log\frac{1}{1-r},$$

and regarding the choice of r we have

$$n|s_n-A|=O(1)n^{1-\alpha}\log n.$$

This implies Theorem 1 when $0<\alpha<1/2$.

It remains to prove the case of $\alpha\geqq 1/2$. To apply the above method, we have to estimate the integral

$$\int\left|\frac{f'}{f-A}\right|^p,$$

when $1<p<2$.

Let S be the set of all univalent functions $g(z)$, such that $g(0)=0$ and $g'(0)=1$. Following Pommerenke [7], we prove a Lemma which generalizes [7, Theorem 5.2].

LEMMA 6. *If* $g(z)\in S$ *and* $1\leqq p<2$, *then* $(z=re^{i\varphi})$

$$\int_0^{2\pi}\left|\frac{g'}{g}(re^{i\varphi})\right|^p d\varphi \leqq \frac{C}{(1-r)^{\frac{p}{2}-\frac{2-p}{321}}}.$$

If we set $p=1$, *we get Pommerenke's Theorem.*

For the proof we have to argue in exactly the same way as Pommerenke, so we leave it to the reader.

It is easy to see that Lemma 6 remains true when $g(z)\notin S$ but $cg(z)\in S$.

In our case we would like to apply the estimate of Lemma 6 to a non-zero univalent function. A short calculation shows that Lemma 6 remains true for every non-zero univalent functions.

In fact, let $g(z)$ be a non-zero univalent function in $|z|<1$; then $h(z):=\frac{1}{g(z)}$ has the same property, too. We write $(z=re^{i\varphi})$

$$\int_0^{2\pi}\left|\frac{g'}{g}(z)\right|^p d\varphi = \int_{|g|\geqq 1}\left|\frac{g'}{g}(z)\right|^p d\varphi+\int_{|h|>1}\left|\frac{h'}{h}(z)\right|^p d\varphi.$$

Dealing only with the first integral, we write $(D=g(0))$

$$|g(z)-D|\leqq|g(z)|+|D|\leqq\big||D|+1\big||g(z)|\quad\text{if}\quad|g(z)|\geqq 1$$

and so

$$\int_{|g|\geqq 1}\left|\frac{g'}{g}(z)\right|^p d\varphi\leqq\big||D|+1\big|^p\int_{|g|\geqq 1}\left|\frac{g'(z)}{g(z)-D}\right|^p d\varphi\leqq\big||D|+1\big|^p\int_0^{2\pi}\left|\frac{g'(z)}{g(z)-D}\right|^p d\varphi.$$

Now, we can apply Lemma 6 to $g(z)-D$. We estimate the integral of $\frac{h'}{h}$ in the same way and this proves the statement.

Returning to the proof of Theorem 1, we choose ε, q, δ, p such that $\varepsilon=0$, $0<\delta<q(1-\alpha)-1$, and $\frac{1}{q}+\frac{1}{p}=1$. Since $\alpha\geqq 1/2$, $q>2$ and $1<p<2$. Now, by Lemma 6, we can estimate the second integral on the right-hand side of (4.2) even if $1<p<2$. Thus (4.7), (4.12) and Lemma 6 imply by (4.1) and (4.2)

$$n|s_n-A| = O(1)Q_\alpha(a)\left[\frac{1}{(1-r)^{1-\alpha}}+\frac{\left(\log\frac{1}{1-r}\right)^{\frac{1}{q}}}{(1-r)^{1-\alpha-\frac{1}{q}}}\,\frac{1}{(1-r)^{\frac{1}{2}-\frac{2-p}{321p}}}\right].$$

Noting that $\alpha\geqq 1/2$, for q sufficiently near $1/(1-\alpha)$, and so p sufficiently near $1/\alpha$. we obtain

$$n|s_n-A| = O(1)\frac{1}{(1-r)^{\frac{1}{2}-\frac{2-\frac{1}{\alpha}}{322}\alpha}}\quad\text{if}\quad\alpha>\frac{1}{2}$$

and

$$n|s_n - A| = O(1)\frac{1}{(1-r)^{\frac{1}{2}+\mu}} \qquad \text{if} \quad \alpha = \frac{1}{2}$$

for arbitrarily small $\mu > 0$.

Regarding the choice of r this completes the proof of Theorem 1.

5. Proof of Lemma 5

In the remaining part of this paper, we will prove Lemma 5. I learned this proof from G. Halász at a seminar in complex analysis at the Eötvös Loránd University, in 1981.

The function $s(z) := \log(f(z))$ is also analytic and univalent in $|z| < 1$. The image of $|z| < 1$ under the mapping $s(z)$ is strip-like which is not wider than 2π.

Let $0 < T < 1$ be fixed. We set

$$\Delta := \{\varphi\colon \varphi_0 < \varphi < \varphi_0 + \delta\} \subset [0, 2\pi]$$

and let, $\eta > 0$,

$$\Omega_\eta := \left\{re^{i\varphi}\colon \varphi \in \Delta, \ \left|\frac{f'}{f}(re^{i\varphi})\right| \geqq \frac{\eta}{1-r}\right\}.$$

Our aim is to estimate the measure of Ω_η. We write

$$a := \min_{\varphi \in \Delta} \operatorname{Re}(s(re^{i\varphi})), \quad b := \max_{\varphi \in \Delta} \operatorname{Re}(s(re^{i\varphi})),$$

and let φ_a and φ_b be the values in Δ where

$$a = \operatorname{Re}(s(re^{i\varphi_a})) \quad \text{and} \quad b = \operatorname{Re}(s(re^{i\varphi_b})).$$

We may assume that $\varphi_a < \varphi_b$.

Let us draw around each point of Ω_η a disc of radius $\frac{1-r}{2}$. Now, we choose a maximal number of disjoint discs from among these and denote by N the number of the chosen discs. Now, if we put a disc of radius $1-r$ instead of each chosen disc with the same center, it is easy to see that these new discs cover the whole Ω_η, so

$$|\Omega_\eta| \leqq 2N(1-r)$$

where $|\Omega_\eta|$ means the measure of Ω_η.

On the other hand, by the "1/4 theorem of Koebe", we know that each image of a chosen disc under the mapping $s(z)$, contains a disc of radius $\frac{\eta}{8}$, so by the definition of a and b we have

$$N\left(\frac{\eta}{8}\right)^2 \frac{\pi}{2} \leqq 2\pi(b-a),$$

and thus if $L := b - a$,

$$\Omega_\eta \leqq C(1-r)\frac{L}{\eta^2}. \tag{5.1}$$

To estimate L, we choose $r_1 < r$, so $\frac{L}{3} = C_1 \log \frac{1-r_1}{1-r}$, where C_1 is defined below.

Denote by γ_p the arcs $\{pe^{i\varphi}\colon \varphi \in [\varphi_a, \varphi_b]\}$. It is clear that we can apply the Koebe distortion theorem to a non-zero univalent function, i.e.

$$\left|\frac{f'}{f}(ue^{i\varphi})\right| < C_1 \frac{1}{1-u}$$

where C_1 depends on $f(z)$. Now, we can estimate the length of the image of the segments $[pe^{i\varphi_a}, re^{i\varphi_a}]$, $[pe^{i\varphi_b}, re^{i\varphi_b}]$ under the mapping $s(z)$, where $r_1 \leqq p \leqq r$, and it is equal:

$$\int_p^r \left|\frac{f'}{f}(ue^{i\psi})\right| du \leqq \int_p^r \frac{C_1}{1-u}\, du = C_1 \log \frac{1-p}{1-r} \leqq \frac{L}{3}, \tag{5.2}$$

where $\psi \in [\varphi_a, \varphi_b]$. Since the distance of the endpoints of the image of γ_r under the mapping $s(z)$ is L, the length of the image of γ_p, $r_1 \leqq p \leqq r$, under the mapping $s(z)$ is bigger than $\frac{L}{3}$.

Let us apply the length-area principle to the domain

$$E := \{ue^{i\varphi}\colon r_1 < u < r \quad \text{and} \quad \varphi_a < \varphi < \varphi_b\}$$

and the mapping $s^{-1}(z)$. If we use the introduced notations, we have $(r_1 < R < r)$

$$2\pi p(R) \leqq |\Delta|$$

where $|\Delta|$ means the measure of Δ, and

$$l(R) \geqq \frac{L}{3}$$

where $l(R)$ means the length of the image of γ_R under the mapping $s(z)$. Now, we know from (3.9)

$$\int_{r_1}^r \frac{\left(\frac{L}{3}\right)^2}{|\Delta|}\, dR \leqq \int_0^\infty \frac{l(R)^2}{2\pi R p(R)}\, dR \leqq \text{Area}\,(s(E)).$$

Knowing from (5.2) that the length of the image of $[Ue^{i\varphi}\colon r_1 < U < r]$, $\varphi \in [\varphi_a, \varphi_b]$ is smaller than $L/3$ we get that

$$s(E) \subset \{\omega\colon a - L/3 < \text{Re}\,(\omega) < b + L/3\},$$

and so

$$\operatorname{Area}\bigl(s(E)\bigr) \leqq 2\pi \frac{5}{3}$$

Thus

$$(r-r_1)L \leqq C_2|\Delta|.$$

Regarding the choice of r_1 we obtain

$$L \leqq C_3 \log \frac{C_4|\Delta|}{1-r},$$

and it yields from (5.1)

$$|\Omega_\eta| \leqq C_5 \frac{1}{\eta^2}(1-r)\log\frac{C_4|\Delta|}{1-r}.$$

Now, we estimate

$$\int_\Delta \left|\frac{f'}{f}(re^{i\varphi})\right|^{2+\varepsilon} d\varphi.$$

Let

$$\Omega_l := \left\{\varphi\in\Delta:\ C_1\frac{2^{-l}}{1-r} \leqq \left|\frac{f'}{f}(re^{i\varphi})\right| < C_1\frac{2^{-l+1}}{1-r}\right\}$$

and so

$$\int_\Delta \left|\frac{f'}{f}(re^{i\varphi})\right|^{2+\varepsilon} d\varphi = \sum_{l=1}^{\infty}\int_{\Omega_l}\left|\frac{f'}{f}(re^{i\varphi})\right|^{2+\varepsilon} d\varphi \leqq \sum_{l=1}^{\infty}|\Omega_l|\left[C_1\frac{2^{-l+1}}{1-r}\right]^{2+\varepsilon} \leqq$$

$$\leqq C_6 \sum_{l=1}^{\infty}\frac{1-r}{2^{-2l}}\left(\frac{2^{-l+1}}{1-r}\right)^{2+\varepsilon}\log\frac{C_4|\Delta|}{1-r} = \tag{5.3}$$

$$= \frac{C_6}{(1-r)^{1+\varepsilon}}\log\frac{C_4|\Delta|}{1-r}\sum_{l=1}^{\infty}2^{-l\varepsilon} \leqq \frac{C(\varepsilon)}{(1-r)^{1+\varepsilon}}\log\frac{C_4|\Delta|}{1-r},$$

where $C(\varepsilon)$ depends only on ε and, of course, on $f(z)$.

Now we can complete the proof. It is clear that $|1-re^{i\varphi}| > \dfrac{\varphi}{4}$ and we set

$$\Delta_l := [2^l(1-r),\, 2^{l+1}(1-r)], \quad l = 0, 1, \dots, K$$

where $K = \left[\overset{2}{\log}\dfrac{\pi}{1-r}\right]+1$, and so

$$I = \int_0^\pi \left|\frac{f'}{f}(re^{i\varphi})\right|^{2+\varepsilon}\frac{d\varphi}{|1-re^{i\varphi}|^\delta} \leqq \int_0^{1-r}\left|\frac{f'}{f}(re^{i\varphi})\right|^{2+\varepsilon}\frac{d\varphi}{(1-r)^\delta} +$$

$$+\sum_{l=1}^{K}\int_{\Delta_l}\left|\frac{f'}{f}(re^{i\varphi})\right|^{2+\varepsilon}\left(\frac{4}{2^l(1-r)}\right)^{\delta} d\varphi.$$

Thus, from (5.3) we have

$$I \leqq \frac{C_1}{(1-r)^{1+\varepsilon+\delta}} + \sum_{l=1}^{K} \frac{C(\varepsilon)}{(1-r)^{1+\varepsilon}} \log(C_4 2^l) \left(\frac{4}{2^l(1-r)}\right)^{\delta} \leqq$$

$$\leqq \frac{C(\varepsilon)}{(1-r)^{1+\varepsilon+\delta}} \sum_{l=1}^{\infty} C_7 l 2^{-l\delta} = \frac{C(\varepsilon, \delta)}{(1-r)^{1+\varepsilon+\delta}},$$

where $C(\varepsilon, \delta)$ depends on ε, δ and $f(z)$. In the same way we can estimate the integral from 0 to $-\pi$, and so this completes Lemma 5.

The fact that $C(\varepsilon, \delta)$ also depends on $f(z)$ follows from $f(z) \notin S$, so the constant of the Koebe distortion theorem depends on $f(z)$.

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(Received November 29, 1985)

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ON SUPER LUCAS AND SUPER LEHMER PSEUDOPRIMES

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1. Introduction and results

Let P and Q be non-zero integers such that $D=P^2-4Q\neq 0$. A Lucas sequence $R=\{R_n\}_{n=0}^{\infty}$ is defined by the initial terms $R_0=0$, $R_1=1$ and by the recursion

$$R_n = PR_{n-1}-QR_{n-2}$$

for $n>1$. We shall write $R(P, Q)$ for R when it is necessary to show the dependence on P and Q. It is well-known that

$$R_n = (\alpha^n-\beta^n)/(\alpha-\beta), \tag{1}$$

for any $n\geqq 0$, where α and β are the distinct roots of the equation $x^2-Px+Q=0$. In the following we say that $R(P, Q)$ is a non-degenerate sequence if $(P, Q)=1$ and α/β is not a root of unity.

For odd primes n with $(n, QD)=1$, as it is well-known, we have

$$R_{n-(D/n)} \equiv 0 \pmod{n}, \tag{2}$$

where (D/n) is the Jacobi symbol. If n is composite, but (2) still holds, then n is called a Lucas pseudoprime with parameters P, Q (or lpsp (P, Q)). We say n is a super Lucas pseudoprime with parameters P, Q (or sulpsp (P, Q)) if n is a lpsp (P, Q) and each divisor of it is a prime or a lpsp (P, Q).

Lucas and super Lucas pseudoprimes are generalizations of pseudoprimes and super pseudoprimes to base an integer $c>1$, respectively, namely a composite n is called a pseudoprime to base c (or psp (c)) if $(n, c)=1$ and

$$c^{n-1} \equiv 1 \pmod{n},$$

and we say n is a super pseudoprime to base c (or supsp (c)) if each divisor of it is a prime or a psp (c). In case $c=2$ we only say n is a pseudoprime or a super pseudoprime.

The properties of pseudoprimes and their generalizations have been studied intensively, since they can be used for primality tests (e.g. see [1], [6]). We list some results which are in connection with ours. K. Szymiczek [11] showed that $F_n F_{n+1}$ is a supsp (2) for any $n>1$, where

$$F_n = 2^{2^n}+1$$

1980 *Mathematics Subject Classification.* Primary 10A35; Secondary 10A10.

Key words and phrases. Pseudoprime, Lehmer sequence, Jacobi symbol, primitive divisor.

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is the n-th Fermat number. In [12] K. Szymiczek proved that there exist infinitely many supsp (2)'s which are products of exactly three primes. This result was extended by J. Fehér and P. Kiss [3] for supsp (c), where $c>1$ is an integer with $4\nmid c$. In [7] A. Rotkiewicz has obtained another generalization of Szymiczek's result, he proved that for infinitely many primes p of the form $ax+b$, where $(a, b)=1$, there exist primes q and r such that pqr is a supsp (2). In [2] we extended the result of Rotkiewicz and the result of Fehér and Kiss mentioned above proving that for every integers $a>1$ and $c>1$ there are infinitely many triplets of distinct primes p, q and r of the form $ax+1$ such that pqr is a supsp (c). We also showed that if the square-free kernel of the base c is congruent to ± 1 modulo 4, then the series

$$\sum \frac{1}{\log n}$$

is divergent, where n runs through all supsp (c)'s which are products of exactly three distinct primes.

P. Kiss [4] studied the sulpsp (P, Q)'s for non-degenerate Lucas sequences $R(P, Q)$ and proved that R_{2p}/P is a sulpsp (P, Q) for every large prime p, furthermore he showed that the series

$$\sum \frac{1}{\log n},$$

where n runs through all sulpsp (P, Q)'s, is divergent.

The Lehmer sequences are much more general sequences than Lucas sequences.
Let L and M be non-zero integers such that $K=L-4M\neq 0$. A Lehmer sequence $U=\{U_n\}_{n=0}^{\infty}$ is defined by initial terms $U_0=0$, $U_1=1$ and by recursion

$$U_n = \begin{cases} LU_{n-1}-MU_{n-2} & \text{for } n \text{ odd} \\ U_{n-1}-MU_{n-2} & \text{for } n \text{ even.} \end{cases}$$

We also shall use the notation $U(L, M)$ for the sequence U when it is necessary to show the dependence on L and M. For $n\geqq 0$, we have

$$U_n = \begin{cases} (\alpha^n-\beta^n)/(\alpha-\beta) & \text{for } n \text{ odd} \\ (\alpha^n-\beta^n)/(\alpha^2-\beta^2) & \text{for } n \text{ even,} \end{cases} \tag{3}$$

where α and β are the distinct roots of the equation $z^2-\sqrt{L}z+M=0$. We note that in case $L=P^2$ and $M=Q$ by (1) and (3) we have

$$R_n(P, Q) = \begin{cases} U_n(P^2, Q) & \text{if } n \text{ odd} \\ PU_n(P^2, Q) & \text{if } n \text{ even,} \end{cases} \tag{4}$$

which is a connection between the Lucas and Lehmer sequences. In the case of Lehmer sequences we can assume, without any essential loss of generality, that $(L, M)=1$ (see [5]). It is not true for Lucas sequences. In the following we also say that Lehmer sequence $U(L, M)$ is a non-degenerate one if α/β is not a root of unity.

A. Rotkiewicz [8] gave a proper generalization of pseudoprimes for Lehmer sequences. A composite number n is called a Lehmer pseudoprime with parameters L, M if $(n, LMK)=1$ and

$$U_{n-(LK/n)} \equiv 0 \pmod{n},$$

where (LK/n) is the Jacobi symbol and $K=L-4M$. A number n is called super Lehmer pseudoprime with parameters L, M if each divisor of it is a prime or a Lehmer pseudoprime with parameters L, M.

A. Rotkiewicz [8] proved that if $U(L, M)$ is a non-degenerate Lehmer sequence with $L>0$ and $K=L-4M>0$, then every arithmetic progression $ax+b$, where a and b are relatively prime positive integers, contains an infinite number of odd Lehmer pseudoprimes with parameters L, M.

The aim of this paper is to extend the results mentioned above for the super Lehmer pseudoprimes. We shall prove the following three theorems.

THEOREM 1. *Let $U(L, M)$ be a non-degenerate Lehmer sequence. Then there exists a positive integer w_0 such that for infinitely many primes p of the form $ax+b$, where $(a, b)=1$ and $b\equiv 1 \pmod{(a, w_0)}$, there are primes q and r such that pqr is a super Lehmer pseudoprime with parameters L, M. The constant w_0 is effectively computable in terms of L and M.*

THEOREM 2. *Let $U(L, M)$ be a non-degenerate Lehmer sequence with condition $LK=L(L-4M)>0$ and let $a>1$ be an integer. Then there are infinitely many triplets of distinct primes p, q and r of the form $ax+1$ such that pqr is a super Lehmer pseudoprime with parameters L, M.*

THEOREM 3. *Let S_1 and S_2 denote the set of all super Lehmer pseudoprimes with parameters L, M which are determined in Theorem 1 and Theorem 2, respectively. Then the series*

$$\sum_{n\in S_1} \frac{1}{\log n} \quad \text{and} \quad \sum_{n\in S_2} \frac{1}{\log n}$$

are divergent.

We note that the conditions of Theorem 1 are satisfied for any integer $a>1$ if $b=1$ and for every pairs a, b if $(a, bw_0)=1$. Furthermore by (4) our results remain valid if we replace the super Lehmer pseudoprimes with super Lucas pseudoprimes. For example, from Theorem 3 we get

COROLLARY. *For every integers $a, c>1$ the series*

$$\sum \frac{1}{\log n}$$

where n runs through all supsp (c)'s *which are products of three distinct primes of the form $ax+1$, is divergent.*

2. Known results and lemmas

First we recall some results on Lehmer sequences and prove two lemmas which will be used at the proofs of our theorems.

Let $U(L, M)$ be a non-degenerate Lehmer sequence defined by integers L and M for which $LM \neq 0$, $(L, M)=1$, $K=L-4M \neq 0$ and α/β is not a root of unity, where α, β are roots of $z^2-\sqrt{L}z+M=0$. It is known that for any non-zero integer n with $(n, M)=1$ there are terms in $U(L, M)$ divisible by n. The least positive integer u, for which $n|U_u$ is called the rank of apparation of n in the sequence $U(L, M)$ and we shall denote it by $u(n)$. If a prime p is a divisor of U_n but $p \nmid MLKU_1 \dots U_{n-1}$ then p is called a primitive prime divisor of U_n. It is well-known that there is an absolute constant n_0 such that U_n has at least one primitive prime divisor for every $n > n_0$ (see A. Schinzel [9] or C. L. Stewart [10]).

Let m and n be positive integers with $(mn, MK)=1$ and let p be a prime for which $(p, 2LMK)=1$. Using the notations defined above, we have

(i) $n|U_m$ if and only if $u(n)|m$,

(ii) $u(p)|(p-(LK/p))$,

(iii) $u(p)|(p-(LK/p))/2$ if and only if $(LM/p)=1$,

(iv) $u([m, n])=[u(m), u(n)]$,

where $[x, y]$ denotes the least common multiple of integers x, y and (LK/p), (LM/p) are Jacobi symbols. For these properties of Lehmer sequences we refer to D. H. Lehmer [5].

Let α and β be the roots of the polynomial $z^2-\sqrt{L}z+M$ and let $K=L-4M$. J. Wójcik [13] showed that there exists a maximal positive integer T such that

$$\frac{\alpha}{\beta} = \zeta_\omega^t \Xi^T, \tag{5}$$

where ω is the number of the roots of unity in $F=\mathbf{Q}(\sqrt{LK})$, $\Xi \in F$, Ξ is quotient of two conjugate integers of F if F is quadratic, ζ_ω is an ω-th root of unity and t is an integer. In [14] Wójcik proved that for T in (5) there exists a positive integer e such that

$$M = \pm e^T. \tag{6}$$

Let $k_v(n)$ be the v-th power-free kernel of n, $k_2(n)=k(n)$, n^* be the product of all distinct prime factors of n. With these notations J. Wójcik [13, 14] proved the following result:

(v) Let $U(L, M)$ be a non-degenerate Lehmer sequence and let $w>0$ be an arbitrary common multiple of the numbers ω^2T and $8k(LK)k_\omega^*(e)$. For any positive integers a and b, where $(a, b)=1$ and $b \equiv 1 \pmod{(a, w)}$, there exist infinitely many primes p satisfying the conditions

$$p \equiv b \pmod{a}, \quad p \equiv 1 \pmod{w} \quad \text{and} \quad p|U_{(p-1)/w}.$$

(vi) The Dirichlet density of this set of primes is equal to $\omega T/w\varphi([w, a])$, where ω, T, e are given by (5) and (6) and φ denotes the Euler function.

We prove the following property of Lehmer sequences.

LEMMA 1. *Let* $U(L, M)$ *be a non-degenerate Lehmer sequence and let* p *be a prime number with* $(p, 2LMK)=1$. *If* $4k(L)k(K)k(M)|u(p)$ *then* $(LM/p)=1$ *and* $u(p)|(p-(LK/p))/2$; *furthermore* $(LK/p)=1$ *if* $LK>0$.

PROOF. First we note that $k(mn)=k(m)k(n)$ if $(m, n)=1$ and so $k(LK)|k(L)k(K)$, $k(LM)=k(L)k(M)$, because $(L, M)=1$ and $K=L-4M$.

Since $LMK\neq 0$ we can write $k(LK)=\pm 2^a t$ and $k(LM)=\pm 2^b h$, where $0\leqq a, b\leqq 1$ and t, h are odd positive integers. It can be easily shown that if $a=1$ or $b=1$ then $2|k(L)k(K)k(M)$. If $4k(L)k(K)k(M)|u(p)$ for a prime p with $(p, 2LMK)=1$, then by (ii) p is of the form

$$p = 4k(L)k(K)k(M)x+(LK/p), \tag{7}$$

where x is an integer and so $(2^a/p)=(2^b/p)=1$ and $(-1/p)=(LK/p)$.

Let first $LK>0$. In this case

$$(LK/p) = \big(k(LK)/p\big) = (2^a t/p) = (t/p) = (-1)^{\frac{t-1}{2}\frac{p-1}{2}}(p/t) =$$

$$= (-1/p)^{\frac{t-1}{2}}\big((LK/p)/t\big) = (LK/p)^{\frac{t-1}{2}}(LK/p)^{\frac{t-1}{2}} = 1.$$

Thus p has the form $4y+1$, where $h|y$, and so

$$(LM/p) = (h/p) = (p/h) = (1/h) = 1. \tag{8}$$

Now let $LK<0$, then $LK=L^2-4LM<0$ and $LM>L^2/4>0$. Similarly as above, by (7) we have

$$\begin{aligned}(LM/p) &= \big(k(LM)/p\big) = (2^b h/p) = (h/p) = (-1)^{\frac{h-1}{2}\frac{p-1}{2}}(p/h) = \\ &= (-1/p)^{\frac{h-1}{2}}\big((LK/p)/h\big) = (LK/p)^{\frac{h-1}{2}}(LK/p)^{\frac{h-1}{2}} = 1.\end{aligned} \tag{9}$$

Thus, by (8) and (9), $(LM/p)=1$ for any $LK\neq 0$. So (iii) implies that

$$u(p)|(p-(LK/p))/2,$$

furthermore, as we have seen above, $(LK/p)=1$ in case $LK>0$.

LEMMA 2. *Let* $U(L, M)$ *be a non-degenerate Lehmer sequence and let* p, q *and* r *be distinct primes. If*

$$u(pqr)\big|\big(p-(LK/p), q-(LK/q), r-(LK/r)\big), \tag{10}$$

where $(x, y, \ldots)$ *denotes the greatest common divisor of integers* $x, y, \ldots$ *then the number* pqr *is a super Lehmer pseudoprime with parameters* L *and* M.

PROOF. It can be easily shown by (iv) that if p, q, r are distinct primes, then

$$u(pqr) = [u(p), u(q), u(r)].$$

Suppose that (10) holds. Let m be a divisor of the number pqr. Since $m|pqr$ and $pqr|U_{u(pqr)}$, by (i) we have $u(m)|u(pqr)$. Thus by (10) we get

$$p \equiv (LK/p), \quad q \equiv (LK/q), \quad r \equiv (LK/r) \pmod{u(m)},$$

and so

$$m \equiv (LK/m) \pmod{u(m)}.$$

Hence

$$m|U_{u(m)}|U_{m-(LK/m)},$$

thus m is really a Lehmer pseudoprime with parameters L, M, which proves Lemma 2.

3. Proofs of the results

PROOF OF THEOREM 1. Let $U(L, M)$ be a non-degenerate Lehmer sequence. We put

(11) $$w_0 = [\omega^2 T, 8k(L)k(K)k(M)k_\omega^*(e)],$$

where ω, T, e are given by (5) and (6); $k(L)$, $k(K)$, $k(M)$, $k_\omega^*(e)$ are the notations defined in Section 2.

Let a and b be positive integers for which $(a, b)=1$ and $b \equiv 1 \pmod{(a, w_0)}$. By the result of J. Wójcik (v), it follows that there exist infinitely many primes p of the form $ax+b$ such that

(12) $$p \equiv 1 \pmod{w_0} \quad \text{and} \quad p|U_{(p-1)/w_0},$$

because w_0 is a common multiple of the numbers $\omega^2 T$ and $8k(LK)k_\omega^*(e)$.

Let p be a prime of the form $ax+b$ satisfying the condition (12) and $(p-1)/2 > |LMK|n_0$ (n_0 is the constant defined in Section 2). As we have seen above, there exist distinct primes q and r for which $u(q)=(p-1)/2$ and $u(r)=p-1$. By (ii) we have

(13) $$u(q) = (p-1)/2|q-(LK/q) \quad \text{and} \quad u(r) = (p-1)|r-(LK/r).$$

On the other hand, by (11) and (12), we have

(14) $$4k(L)k(K)k(M)|w_0/2 \quad \text{and} \quad w_0/2|(p-1)/2 = u(q).$$

By Lemma 1, using (14), we get

(15) $$u(q) = (p-1)/2\big|(q-(LK/q))/2 \quad \text{and} \quad (p-1)|q-(LK/q).$$

Since $w_0>2$, it can be easily seen that p, q and r are distinct primes, and so by (12), (13) and (15) we have

$$u(pqr) = [u(p), u(q), u(r)] = (p-1)\big|(p-1, q-(LK/q), r-(LK/r)),$$

and so by Lemma 2 Theorem 1 is proved.

PROOF OF THEOREM 2. Let $U(L, M)$ be a non-degenerate Lehmer sequence with condition $LK=L(L-4M)>0$. Let $a \geqq 2$ be an integer. Let w_0 be as in (11).

By (v) there exist infinitely many primes p of the form $ax+1$ such that

(16) $$p \equiv 1 \pmod{aw_0} \quad \text{and} \quad p|U_{(p-1)/aw_0}.$$

Let p be a prime of the form $ax+1$ satisfying the condition (16) and $(p-1)/2>$ $>|LMK|n_0$. Then there exist primes q and r such that $u(q)=(p-1)/2$ and $u(r)=$ $=p-1$. By (ii), as we have seen in the proof of Theorem 1, we get

$$u(q) = (p-1)/2|(q-(LK/q))/2$$

and since $LK>0$, by Lemma 1 we have $(LK/q)=1$, $(LK/r)=1$. Thus

$$(p-1)|q-1 \quad \text{and} \quad (p-1)|r-1. \tag{17}$$

Since $p\equiv 1 \pmod a$, by (17) it follows that q and r are primes of the form $ax+1$. Hence

$$u(pqr) = (p-1)|(p-1, q-1, r-1)$$

and so by Lemma 2, pqr is a super Lehmer pseudoprime with parameters L, M; where p, q and r are primes of the form $ax+1$. It completes the proof of the Theorem 2.

Proof of Theorem 3. Let S_1 and S_2 denote the set of all super Lehmer pseudoprimes with parameters L, M which are determined in Theorem 1 and Theorem 2, respectively.

As we have seen in the proofs of Theorem 1 and Theorem 2, for each prime p satisfying the condition (12) or (16), respectively, there are primes q and r such that pqr is a super Lehmer pseudoprime and $u(pqr)=p-1$. It is well-known that for any integer $n\geqq 0$ we have

$$|U_n| < |\alpha|^{Cn},$$

where C is a positive number which is effectively computable in terms of L and M. Thus we get

$$pqr \leqq |U_{p-1}| < |\alpha|^{C(p-1)} < |\alpha|^{Cp}$$

and so

$$\frac{1}{\log(pqr)} > \frac{1}{C\log|\alpha|}\frac{1}{p}.$$

Hence

$$\sum_{n\in S_i}\frac{1}{\log n} \geqq \sum\frac{1}{\log(pqr)} > \frac{1}{C\log|\alpha|}\sum\frac{1}{p}, \tag{18}$$

where $i=1$ or $i=2$ and p runs through all primes satisfying (12) or (16), respectively. By (vi) and (18) Theorem 3 is proved.

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(Received December 1, 1985)

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ON THE LINEAR DIOPHANTINE PROBLEM OF FROBENIUS

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1. Introduction

Throughout this paper, we use lower case letters to denote integers and we use $[\alpha]$ to denote the integral part of any real number α.

In § 2, we consider the minimization problem:

$$\min_{x \geqq \xi}\left(\delta x - \gamma\left[\frac{\beta x}{\alpha}\right]\right), \tag{1}$$

where α, β, γ and δ are positive integers and ξ is any integer. We are able to express (1) in terms of the negative continued fraction of β/α. This is then applied in § 3 to prove some theorems concerning a problem of Frobenius.

Given relatively prime positive integers $a_1, a_2, \ldots, a_m$, we say that an integer N is dependent on these a_i's if there exist non-negative integers $x_1, x_2, \ldots, x_m$ such that

$$N = a_1 x_1 + a_2 x_2 + \ldots + a_m x_m.$$

The problem of Frobenius consists in determining the largest integer $g_m = g(a_1, a_2, \ldots, a_m)$ which is not dependent on $a_1, a_2, \ldots, a_m$. An interesting question related to this is the determination of the number $n_m = n(a_1, a_2, \ldots, a_m)$ of positive integers which are not dependent on $a_1, a_2, \ldots, a_m$.

When $m=2$, we have the classical result that

$$g(a, b) = (a-1)(b-1)-1$$

and

$$n(a, b) = \frac{1}{2}(a-1)(b-1).$$

For $m \geqq 3$, the problem is much more difficult. Although there have been some individual results concerning g_m and n_m in which the a_i's satisfy extra conditions (see Selmer [4] for a survey of the problem), the first formulas for g_3 and n_3 were obtained in 1978 by Selmer and Beyer at Bergen [5] and, almost simultaneously, by Rödseth [2] at Stavanger. The formulas of Rödseth, however, are much simpler.

Let L be a complete residue system modulo a_1 and let t_l be the smallest integer $\equiv l \pmod{a_1}$ which is dependent on $a_1, a_2, \ldots, a_m$. Rödseth derived his results

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1980 *Mathematics Subject Classification.* Primary 10B05; Secondary 10B35.
Key words and phrases. Linear diophantine equations, Frobenius problem.

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from the formulas:

$$g_m = -a_1 + \max_{t \in L} t_l$$

of Brauer and Shockley [1], and

$$n_m = -\frac{1}{2}(a_1 - 1) + \frac{1}{a_1} \sum_{l \in L} t_l$$

of Selmer [4]. His proof was short and elegant and was built on a counting argument. About a year later, he managed to extend his idea to $g(a_1, \ldots, a_m)$ and $n(a_1, \ldots, a_m)$ when $a_1, a_2, \ldots, a_m$ is an almost arithmetic sequence, that is, when $m-1$ of the a_i's form an arithmetic sequence.

Our solution to (1) enables us to prove all the theorems of Rödseth [2, 3] via a different route. Unlike his proofs, we actually obtain explicit formulas for each t_l and we do not employ the type of combinatorial argument he used. We shall illustrate the ideas and the techniques by proving his formula for $g(a, a+d, \ldots, a+kd, c)$.

2. A minimization problem

The solution to the minimization in (1) is given by

THEOREM 1. *Let s_{-1}, s_0, Q_{-1} and Q_0 be positive integers satisfying*

$$s_0 Q_{-1} > s_{-1} Q_0. \tag{2}$$

We define the sequences $q_1, q_2, \ldots$ and $s_1, s_2, \ldots$ by the following Euclidean algorithm:

$$\begin{aligned}
s_{-1} &= q_1 s_0 - s_1, & 0 &< s_1 < s_0,\\
s_0 &= q_2 s_1 - s_2, & 0 &< s_2 < s_1,\\
&\ \vdots\\
s_{r-1} &= q_{r+1} s_r - s_{r+1}, & 0 &< s_{r+1} < s_r,\\
&\ \vdots\\
s_{p-1} &= q_{p+1} s_p, & 0 &= s_{p+1} < s_p.
\end{aligned} \tag{3}$$

Let $Z_0 = Z_0(\xi) = \xi$ be some integer and, for $r \geqq 0$, define

$$Z_{r+1} = Z_{r+1}(\xi) = -\left[-\frac{s_{r+1}}{s_r} Z_r(\xi)\right]$$

and

$$Q_{r+1} = q_{r+1} Q_r - Q_{r-1}. \tag{4}$$

Then there exists a unique integer $v \geqq 0$ such that

$$Q_0 > Q_1 > Q_2 > \ldots > Q_v > 0 \geqq Q_{v+1} > \ldots > Q_{p+1}. \tag{5}$$

Moreover, if

$$M = \min_{x \geqq \xi} \left(Q_{-1} x - Q_0 \left[\frac{s_{-1}}{s_0} x \right] \right),$$

we have

$$M = Z_{\nu+1}(\xi) Q_\nu - Z_\nu(\xi) Q_{\nu+1}.$$

REMARK. In case that (2) is not satisfied, we have

$$M = \begin{cases} 0, & \text{when } s_0 Q_{-1} = s_{-1} Q_0, \\ -\infty, & \text{when } s_0 Q_{-1} < s_{-1} Q_0. \end{cases}$$

PROOF. First of all, we notice that for $-1 \leqq r \leqq p$ we have, by (2),

$$s_{r+1} Q_r - s_r Q_{r+1} = s_r Q_{r-1} - s_{r-1} Q_r = \ldots = s_0 Q_{-1} - s_{-1} Q_0 > 0.$$

Therefore,

$$Q_{r+1} < \frac{s_{r+1}}{s_r} Q_r, \quad -1 \leqq r \leqq p-1$$

and

$$Q_{p+1} < \frac{s_{p+1}}{s_p} Q_p = 0.$$

So, there exists a unique $\nu \geqq 0$ such that

$$Q_0 > Q_1 > \ldots > Q_\nu > 0 \geqq Q_{\nu+1}, Q_{\nu+2}, \ldots, Q_{p+1}.$$

For $r \geqq \nu+1$, since $q_{r+1} \geqq 2$ and $Q_r \leqq 0$, we have

$$Q_{r+1} = (q_{r+1} - 1) Q_r + (Q_r - Q_{r-1}) \leqq Q_r + (Q_r - Q_{r-1}).$$

This gives, inductively,

$$Q_{\nu+1} > Q_{\nu+2} > \ldots > Q_{p+1},$$

and (5) then follows.

Now, using (3) and (4), we can write

$$M = \min_{x \geqq \xi} \left((Q_{-1} - q_1 Q_0) x - Q_0 \left[-\frac{s_1}{s_0} x \right] \right)$$

$$= \min_{x \geqq Z_0(\xi)} \left(-Q_1 x - Q_0 \left[-\frac{s_1}{s_0} x \right] \right).$$

If $\nu = 0$, then $Q_1 \leqq 0$ and

$$M = -Q_1 Z_0(\xi) - Q_0 \left[-\frac{s_1}{s_0} Z_0(\xi) \right] = Z_1 Q_0 - Z_0 Q_1.$$

If $\nu \geqq 1$ so that $Q_1 > 0$, we let

$$y = -\left[-\frac{s_1}{s_0} x \right].$$

The condition $x \geqq \xi$ is translated into $y \geqq -[-s_1\xi/s_0]=Z_1$ and, for each $y \geqq Z_1$, we have

$$\frac{s_0}{s_1}(y-1) < x \leqq \frac{s_0}{s_1}y.$$

Since $-Q_1<0$, the minimum occurs at the upper end, that is

$$x = \left[\frac{s_0}{s_1}y\right].$$

Notice that for each $y \geqq Z_1$, this choice of x satisfies $x \geqq \xi$. Thus

$$M = \min_{y \geqq Z_1}\left(Q_0 y - Q_1\left[\frac{s_0}{s_1}y\right]\right).$$

This reduction procedure can be repeated until

$$M = \min_{x \geqq Z_\nu}\left(-Q_{\nu+1}x - Q_\nu\left[-\frac{s_{\nu+1}}{s_\nu}x\right]\right).$$

Now, $-Q_{\nu+1} \geqq 0$. Therefore, the minimum is attained at the lower limit of x, that is

$$M = -Q_{\nu+1}Z_\nu - Q_\nu\left[-\frac{s_{\nu+1}}{s_\nu}Z_\nu\right] = Z_{\nu+1}Q_\nu - Z_\nu Q_{\nu+1}.$$

This proves our theorem.

3. Formula for $g(a, a+d, \ldots, a+kd, c)$

Our Theorem 1 enables us to give an alternate proof of Rödseth's formulas in [2, 3]. For the purpose of illustration, we shall prove his formula for $g(a_1, a_2, \ldots, a_m)$ when $\{a_i\}$ is an almost arithmetic sequence.

Let a, d, c, k be given positive integers such that $(a, d)=1$. Let

$$a_k = a+kd.$$

Suppose

$$(a, c) = h \geqq 1.$$

There exist positive integers u, v, m and n such that

$$au - cv = h$$

and

$$am - dn = 1. \tag{6}$$

Writing $a=ha'$ and $c=hc'$, we have

$$d'u - c'v = 1. \tag{7}$$

We shall use these equations from time to time, but sometimes without explicit references.

Let

$$s_{-1} = nc, \quad s_0 = a \tag{8}$$

and define the sequences $s_1, s_2, \ldots$ and $q_1, q_2, \ldots$ according to (3). Similarly, we put

$$Q_{-1} = c(n+km), \quad Q_0 = a_k \tag{9}$$

and define $Q_1, Q_2, \ldots$ by (4).

We shall encounter in the sequel several sequences generated by the recurrence relation (4). It is easy to see that for any two such sequences $\{\alpha_r\}$, $\{\beta_r\}$, that is, $\alpha_{r+1} = q_{r+1}\alpha_r - \alpha_{r-1}$ and $\beta_{r+1} = q_{r+1}\beta_r - \beta_{r-1}$ for $r \geqq 0$, we have

$$\alpha_{r+1}\beta_r - \alpha_r\beta_{r+1} = \alpha_0\beta_{-1} - \alpha_{-1}\beta_0.$$

Following Rödseth, we use the formula:

$$g(a, a+d, \ldots, a+kd, c) = -a + \max_{l \in L} t_l. \tag{10}$$

For each $l \in L$, t_l is the smallest integer $\equiv l \pmod a$ such that

$$t_l = ax_0 + (a+d)x_1 + \ldots + (a+kd)x_k + cz$$

for some non-negative integers $x_0, x_1, \ldots, x_k$ and z. Using Lemma 1 of Rödseth [3], this is equivalent to

$$t_l = ax + dy + cz, \quad dy + cz \equiv l \pmod a,$$

with $0 \leqq y \leqq kx$ and $z \geqq 0$. By the minimality of t_l, x must be the smallest integer subject to the constraint $y \leqq kx$, that is $x = -[-y/k]$. Thus

$$\begin{aligned} t_l &= -a\left[-\frac{y}{k}\right] + dy + cz \\ &= l - a\left[-\frac{y}{k} - \frac{dy+cz-l}{a}\right] \\ &= l - a\left[\frac{kl - \tau_l}{ak}\right], \end{aligned} \tag{11}$$

where τ_l is the smallest integer with a representation

$$\tau_l = a_k y + kcz, \quad dy + cz \equiv l \pmod a, \quad y, z \geqq 0. \tag{12}$$

Suppose $z \geqq 0$ is given. Equations (12) and (6) imply that $y \equiv n(cz-l) \pmod a$. The smallest such non-negative y is given by

$$y = n(cz-l) - a\left[\frac{n}{a}(cz-l)\right].$$

Thus,

$$\tau_l = \min_{z \geqq 0}\left\{c(k+na_k)z - aa_k\left[\frac{1}{d'}\left(nc'z - \frac{nl}{h}\right)\right]\right\} - a_k nl. \tag{13}$$

Let

$$l' = \left[-\frac{n}{h}l\right].$$

It follows easily from (6) and (7) that

$$c(k+na_k) = ac(n+km) = aQ_{-1}$$

and

$$nc'dv - a'(vcm-u) = 1.$$

Therefore,

$$\left[\frac{1}{a'}\left(nc'z-\frac{n}{h}l\right)\right] = \left[\frac{1}{a'}(nc'z+l')\right]$$

$$= \left[\frac{nc'}{a'}(z+dvl')\right]-(vcm-u)l'$$

and from (13), (9) we have

$$\tau_l = \min_{z\geqq 0}\left\{aQ_{-1}z - aQ_0\left[\frac{nc'}{a'}(z+dvl')\right]\right\} + aa_k(vcm-u)l' - a_k nl$$

$$= a\min_{z\geqq dvl'}\left\{Q_{-1}z - Q_0\left[\frac{nc'}{a'}z\right]\right\} + a\{a_k(vcm-u) - Q_{-1}dv\}l' - a_k nl$$

$$= a\min_{z\geqq dvl'}\left\{Q_{-1}z - Q_0\left[\frac{s_{-1}}{s_0}z\right]\right\} - a(h+kud)l' - a_k nl.$$

We now have a minimization problem to which Theorem 1 applies. Thus, if $Z_0(\xi)=\xi$ and, for $r\geqq 0$,

$$Z_{r+1}(\xi) = -\left[-\frac{s_{r+1}}{s_r}Z_r(\xi)\right], \tag{14}$$

we have

$$\tau_l = a(Z_{v+1}(dvl')Q_v - Z_v(dvl')Q_{v+1}) - a(h+kud)l' - a_k nl,$$

where $v \geqq 0$ is defined by $Q_v > 0 \geqq Q_{v+1}$. Substituting this into (11), we obtain

$$t_l = l - a\left[\frac{1}{ak}\{(k+na_k)l + a(h+kud)l' + aZ_v(dvl')Q_{v+1} - aZ_{v+1}(dvl')Q_v\}\right]$$

$$= l - a\left[\frac{1}{k}\{(n+km)l + (h+kud)l' + Z_v(dvl')Q_{v+1} - Z_{v+1}(dvl')Q_v\}\right] \tag{15}$$

$$= -ndl - audl' - a\left[\frac{1}{k}\{nl + hl' + Z_v(dvl')Q_{v+1} - Z_{v+1}(dvl')Q_v\}\right].$$

Let A be a complete residue system modulo a'. The set $\{-cx+dy | x\in A, 0\leqq y<h\}$ contains a elements which are pairwise incongruent modulo a. Therefore,

$$L = \{-cx+dy | x\in A, 0\leqq y < h\}.$$

When l is replaced by $-cx+dy \in L$, l' becomes

$$\left[-\frac{n}{h}(-cx+dy)\right] = \left[\frac{ncx-amy+y}{h}\right] = nc'x-md'y$$

and dvl' becomes

$$\begin{aligned} dv(nc'x-md'y) &= (am-1)(a'u-1)x-a'dvmy \\ &= x-amx+a'u(am-1)x-a'dvmy. \end{aligned}$$

It is easy to see that for any $r \geqq 0$ and any integers e, f, we have

$$Z_r(e+fa') = Z_r(e)+fs'_r$$

where $s'_r = s_r/h$. Thus, $Z_r(dvl')$ becomes

$$Z_r(x)-s_r mx+s'_r\{u(am-1)x-dvmy\}.$$

Substituting all these into (15) and utilizing the fact that

$$s_{v+1}Q_v - s_v Q_{v+1} = s_0 Q_{-1} - s_{-1} Q_0 = ck$$

and

$$s'_{v+1}Q_v - s'_v Q_{v+1} = c'k,$$

we have

$$t_{-cx+dy} = -cx+dy-a\left[\frac{1}{k}(-y+Q_{v+1}Z_v(x)-Q_v Z_{v+1}(x))\right].$$

Thus

$$\max_{l \in L} t_l = \max_{\substack{x \in A \\ 0 \leqq y < h}} t_{-cx+dy}$$

(16)

$$= d(h-1)+\max_{x \in A}\left\{-cx-a\left[\frac{1}{k}(1-h+Q_{v+1}Z_v(x)-Q_v Z_{v+1}(x))\right]\right\}.$$

LEMMA. *Let* $P_{-1} = -1$, $P_0 = 0$ *and, for* $r \geqq 0$,

$$P_{r+1} = q_{r+1}P_r - P_{r-1}.$$

Define the functions

$$\lambda_r(z) = P_{r+1}z + P_r\left[-\frac{s_{r+1}}{s_r}z\right], \quad r = 0, 1, 2, \ldots.$$

Then, for any given integer y, the set of integers x such that $Z_r(x) = y$ consists of all those integers in the interval $(\lambda_r(y-1), \lambda_r(y)]$.

PROOF. We use induction on r. For $r=0$, the lemma is true because $\lambda_0(z)=z$ and $Z_0(x)=x$. Suppose $r \geqq 0$, then, by (14), $Z_{r+1}(x)=y$ if and only if

$$\frac{s_r}{s_{r+1}}(y-1) < Z_r(x) \leqq \frac{s_r}{s_{r+1}}y.$$

By the induction assumption, the largest such x is equal to $\lambda_r([s_r y/s_{r+1}])$ and the smallest such x is equal to $\lambda_r([s_r(y-1)/s_{r+1}])+1$. Hence, it suffices to show that

$$\lambda_{r+1}(y) = \lambda_r\left(\left[\frac{s_r}{s_{r+1}}y\right]\right).$$

Using the induction assumption again,

$$\lambda_r\left(\left[\frac{s_r}{s_{r+1}}y\right]\right) = P_{r+1}\left[\frac{s_r}{s_{r+1}}y\right]+P_r\left[-\frac{s_{r+1}}{s_r}\left[\frac{s_r}{s_{r+1}}y\right]\right]$$

$$= P_{r+1}\left[q_{r+2}y-\frac{s_{r+2}}{s_{r+1}}y\right]+P_r\left[-y+\frac{s_{r+1}}{s_r}\theta\right],$$

where θ is the fractional part of $s_r y/s_{r+1}$. Since $s_{r+1}\theta/s_r\in[0, 1)$,

$$\lambda_r\left(\left[\frac{s_r}{s_{r+1}}y\right]\right) = (P_{r+1}q_{r+2}-P_r)y+P_{r+1}\left[-\frac{s_{r+2}}{s_{r+1}}y\right]$$

$$= P_{r+2}y+P_{r+1}\left[-\frac{s_{r+2}}{s_{r+1}}y\right]$$

$$= \lambda_{r+1}(y).$$

This proves the lemma.

Applying this lemma to (16), we have

$$\max_{l\in L} t_l = d(h-1)+\max_{y\in A}\left\{-c(\lambda_v(y-1)+1)-a\left[\frac{1}{k}\left(1-h+yQ_{v+1}+Q_v\left[-\frac{s_{v+1}}{s_v}y\right]\right)\right]\right\}$$

(17) $$= d(h-1)+c(P_{v+1}-1)+\max_{y\in A} E(y),$$

say, where

$$E(y) = -cP_{v+1}y-cP_v\left[-\frac{s_{v+1}}{s_v}(y-1)\right]-a\left[\frac{1}{k}\left(1-h+yQ_{v+1}+Q_v\left[-\frac{s_{v+1}}{s_v}y\right]\right)\right].$$

It is easy to see that $E(y\pm s_v')=E(y)$, that is, $E(y)$ is a function of the residue classes modulo s_v'.

Let $\varrho_{-1}=mh-ndu$, $\varrho_0=-dv$ and, for $r\geqq 0$,

$$\varrho_{r+1} = q_{r+1}\varrho_r-\varrho_{r-1}.$$

It follows from (6), (7) and (8) that

(18) $$\varrho_v s_{v+1}'-\varrho_{v+1}s_v' = \varrho_{-1}s_0'-\varrho_0 s_{-1}' = 1.$$

Thus, $(\varrho_\nu, s'_\nu)=1$ and so,

$$\max_y E(y) = \max_x E(-\varrho_\nu x)$$

$$= \max_x \left\{ c\varrho_\nu P_{\nu+1} x - cP_\nu \left[\frac{s'_{\nu+1}+\varrho_\nu s'_{\nu+1} x}{s'_\nu}\right]\right.$$

$$\left. -a\left[\frac{1}{k}\left(1-h-x\varrho_\nu Q_{\nu+1}+Q_\nu\left[\frac{s'_{\nu+1}\varrho_\nu x}{s'_\nu}\right]\right)\right]\right\}.$$

By (18), this is equal to

$$\max_x \left\{ c(\varrho_\nu P_{\nu+1}-\varrho_{\nu+1}P_\nu)x - cP_\nu\left[\frac{s'_{\nu+1}+x}{s'_\nu}\right]\right.$$

$$\left. -a\left[\frac{1}{k}\left(1-h-(\varrho_\nu Q_{\nu+1}-\varrho_{\nu+1}Q_\nu)x+Q_\nu\left[\frac{x}{s'_\nu}\right]\right)\right]\right\}$$

$$= \max_x \left\{-cdvx - cP_\nu\left[\frac{s'_{\nu+1}+x}{s'_\nu}\right]-a\left[\frac{1}{k}\left(1-h-(udk+h)x+Q_\nu\left[\frac{x}{s'_\nu}\right]\right)\right]\right\},$$

since $\varrho_{-1}P_0-\varrho_0P_{-1}=-dv$ and $\varrho_{-1}Q_0-\varrho_0Q_{-1}=h+udk$. Thus

$$\max_y E(y) = \max_x\left\{dhx - cP_\nu\left[\frac{s'_{\nu+1}+x}{s'_\nu}\right]-a\left[\frac{1}{k}\left(1-h-hx+Q_\nu\left[\frac{x}{s'_\nu}\right]\right)\right]\right\}$$

$$= \max_{0\leqq x<s'_\nu}\left\{dhx-cP_\nu\left[\frac{s'_{\nu+1}+x}{s'_\nu}\right]-a\left[\frac{1-h-hx}{k}\right]\right\}$$

$$= \max\left\{dh(s'_\nu-s'_{\nu+1}-1)-a\left[\frac{1}{k}(1-h-h(s'_\nu-s'_{\nu+1}-1))\right],\right.$$

$$\left. dh(s'_\nu-1)-cP_\nu-a\left[\frac{1}{k}(1-h-h(s'_\nu-1))\right]\right\}$$

$$= dh(s'_\nu-1)-\min\left\{dhs'_{\nu+1}+a\left[\frac{s_{\nu+1}-s_\nu+1}{k}\right],\ cP_\nu+a\left[\frac{1-s_\nu}{k}\right]\right\}.$$

Substituting this into (17), we get

$$\max_{l\in L} t_l = d(h-1)+c(P_{\nu+1}-1)+d(s_\nu-h)$$

$$-\min\left\{ds_{\nu+1}+a\left[\frac{s_{\nu+1}-s_\nu+1}{k}\right],\ cP_\nu+a\left[\frac{1-s_\nu}{k}\right]\right\}.$$

In view of (10), we have proved

THEOREM 2. *If* $(a, d)=1$ *and* $k, c>0$, *then*

$$g(a, a+d, \ldots, a+kd, c) = -a+c(P_{\nu+1}-1)+d(s_\nu-1)$$
$$-\min\left\{ds_{\nu+1}+a\left[\frac{s_{\nu+1}-s_\nu+1}{k}\right], cP_\nu+a\left[\frac{1-s_\nu}{k}\right]\right\}.$$

The other results of Rödseth contained in [2, 3] can be proved along similar lines.

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(Received December 10, 1985)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 453—469

SPECTRAL THEOREM FOR NORMAL ELEMENTS OF GW^*-ALGEBRAS

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1. Introduction

First of all we give the broad outlines of the spectral problem for normal elements of C^*-algebras, however, we will not enter into the details of the extended historical background of the spectral theorem. The evolution of the classical spectral problem is detailed e.g. in [5].

If L and L' are both σ-complete orthomodular lattices then a mapping $u: L \to L'$ is called a *σ-orthohomomorphism* between L and L' if it satisfies

(i) $u(e^\perp) = u(e)^\perp$ for all $e \in L$,

(ii) $u\left(\bigvee_{n\in\mathbf{N}} e_n\right) = \bigvee_{n\in\mathbf{N}} u(e_n)$ for every sequence $(e_n)_{n\in\mathbf{N}}$ in L.

Then the axioms of ortholattices imply that u is an order and unit preserving map, such that $u\left(\bigwedge_{n\in\mathbf{N}} e_n\right) = \bigwedge_{n\in\mathbf{N}} u(e_n)$ for every sequence $(e_n)_{n\in\mathbf{N}}$ in L.

If $\mathscr{B}$ is a σ-algebra of subsets of the set T then $\mathscr{B}$, equipped with the inclusion relation is a σ-complete Boolean lattice admitting a unique orthocomplementation, thus $\mathscr{B}$ is a σ-complete Boolean ortholattice. In the sequel every σ-algebra will be treated as a σ-complete Boolean ortholattice, whose structure is that of described above.

Let A be a unital C^*-algebra and suppose that the orthocomplemented partially ordered set $L(A)$ of self-adjoint idempotents (i.e., projections) of A is a σ-complete lattice. By a *projection valued measure* in A we mean a σ-orthohomomorphism defined on a σ-algebra taking values in the σ-complete orthomodular lattice $L(A)$. As it will be clarified in Section III (in a more general context), given a σ-algebra $\mathscr{B}$ of subsets of the set T and an arbitrary projection valued measure $u: \mathscr{B} \to L(A)$ in A, the mapping u can be lifted uniquely to a unit preserving $*$-algebra morphism $\hat{u}$ between the unital C^*-algebra of measurable bounded complex functions on T and A. The morphism $\hat{u}$ will be referred to as the *integral* defined by u.

Then the *general spectral problem* for normal elements of C^*-algebras can be formulated as follows. Given a unital C^*-algebra A such that $L(A)$ is a σ-complete lattice and a normal element x in A; is there a uniquely determined projection valued measure $u: \mathscr{B}(\mathrm{Sp}_A(x)) \to L(A)$ defined on the σ-algebra of Borel subsets of the spectrum $\mathrm{Sp}_A(x)$, such that $\hat{u}(\mathrm{id}_{\mathrm{Sp}_A(x)}) = x$?

1980 *Mathematics Subject Classification.* Primary 46L05, 46H30; Secondary 46G12.
Key words and phrases. Spectral resolution, vector integral.

It is obvious that the answer is negative, in general. For instance, if A is the unital C^*-algebra of continuous complex functions defined on a connected compact Hausdorff space then $L(A)$ is a complete ortholattice since it contains exactly two elements. Consequently, every element of A lying in the range of the integral defined by a projection valued measure in A is necessarily equal to a constant multiple of the unit of A, though A can have other normal elements.

This example shows that the unital C^*-algebra examined must contain a sufficiently large number of projections in order that we might give an affirmative answer to the spectral problem for its normal elements.

In this paper the spectral theorem will be proved for the normal elements of GW^*-algebras (see [7]). It is worth mentioning that our proof will not (and cannot) follow the well-known classical way of reasoning, i.e. the solution of the problem will not (and cannot) be reduced to the commutative case, although we also have an independent spectral theorem for commutative GW^*-algebras (see [7] Th. 4). Moreover, our result, applied to the normal elements of the complete GW^*-algebra of the continuous linear operators in a Hilbert space furnishes a new proof of the classical spectral theorem.

2. Some results concerning GW^*-algebras

The notion of weak GW^*-algebras was introduced in [7], however, for the sake of completeness and to fix the terminology, here we repeat the basic notions and give a short summary of the most important results concerning weak GW^*-algebras.

If A is a $*$-algebra then A^* denotes the vector space of linear forms on A and the weak $\sigma(A, A^*)$ and $\sigma(A^*, A)$ topologies relate to the canonical duality between A and A^*. Further, if A has a unit (denoted by $\mathbf{1}$ throughout this paper) and P is a set of positive linear forms on A then $P(1)$ stands for the set $\{f\in P \mid f(\mathbf{1})\leqq 1\}$. Besides, if $P(1)$ is non-void and $\sigma(A^*, A)$-bounded then $\|\cdot\|_P$ denotes the map from A into $\mathbf{R}_+$ defined by

$$\|x\|_P := \sup_{f\in P(1)} \sqrt{f(x^*x)}$$

for all $x\in A$. It is easy to see that $\|\cdot\|_P$ is a seminorm on A; the dual semnormi is denoted by $\|\cdot\|_P'$.

Given a unital $*$-algebra A and a subset S of A^*, the linear subspace of A^* spanned by S and the convex hull of S is denoted by $\mathrm{sp}(S)$ and $\mathrm{co}(S)$, respectively, while the $\sigma(A^*, A)$-closed linear subspace of A^* spanned by S and the $\sigma(A^*, A)$-closed convex hull of S is denoted by $\widetilde{\mathrm{sp}}(S)$ and $\widetilde{\mathrm{co}}(S)$, respectively. If P is a set of positive linear forms on A such that $P(1)$ is non-void and $\sigma(A^*, A)$-bounded, then the $\|\cdot\|_P'$-closed linear subspace of A^* spanned by S and the $\|\cdot\|_P'$-closed convex hull of S will be denoted by $\overline{\mathrm{sp}}(S)$ and $\overline{\mathrm{co}}(S)$, respectively, provided no confusion arises as for P.

If f is a linear form on the $*$-algebra A then for every $x\in A$ we define the linear forms $x\cdot f$ and $f\cdot x$ on A as the mappings $y\mapsto f(xy)$ and $y\mapsto f(yx)$, respectively. If $f\in A^*$ and $x, y\in A$ then $x\cdot f\cdot y$ stands for $(x\cdot f)\cdot y$.

The pair (A, P) is called a *weak GW^*-algebra* if A is a unital $*$-algebra and P is a separating set of positive linear forms on A satisfying

(I) $P(1)$ is non-void and $\sigma(A^*, A)$-bounded.

(II$_w$) $\lambda f \in P$ and $x^* \cdot f \cdot x \in \widetilde{\overline{\mathrm{co}}}(P)$ for every $\lambda \in \mathbf{R}_+$, $f \in P$ and $x \in A$.

(III) $x \cdot f \in \overline{\mathrm{sp}}(P)$ for every $x \in A$ and $f \in P$.

(IV) A is sequentially complete with respect to the uniform structure defined by the $\sigma(A, \mathrm{sp}(P))$ topology.

The weak GW^*-algebra (A, P) is referred to as a *GW^*-algebra* if it satisfies

(II) $\lambda f \in P$ and $x^* \cdot f \cdot x \in \overline{\mathrm{co}}(P)$ for every $\lambda \in \mathbf{R}_+$, $f \in P$ and $x \in A$.

If besides (A, P) satisfies

(IV$_s$) A is quasi complete with respect to the uniform structure defined by the $\sigma(A, \mathrm{sp}(P))$ topology,

then (A, P) is called a *complete GW^*-algebra* (see [7]).

The spectrum of an element x in the unital algebra A will be denoted by $\mathrm{Sp}_A(x)$, or if no danger of confusion as for A, the letter A will be omitted.

If T is a compact Hausdorff space then $\mathscr{C}_{\mathbf{C}}(T)$ denotes the commutative unital C^*-algebra of complex continuous functions defined on T. Further, $\mathscr{B}(T)$ and $\mathscr{B}_0(T)$ denotes the σ-algebra of Borel and Baire subsets of T, respectively.

Given a σ-algebra $\mathscr{B}$ of subsets of the set X, we choose to write $\mathscr{F}^b_{\mathbf{C}}(X, \mathscr{B})$ for the commutative unital C^*-algebra of complex bounded measurable functions defined on X. Here the C^*-norm of $\mathscr{F}^b_{\mathbf{C}}(X, \mathscr{B})$ equals the sup-norm $|||\cdot|||_X$ on X.

If (A, P) is a weak GW^*-algebra then

— A is a C^*-algebra whose C^*-norm equals $\|\cdot\|_P$ (cf. [6] Th. 2 and [7] Prop. 1),

— the $\sigma(A, \mathrm{sp}(P))$ and $\sigma(A, \overline{\mathrm{sp}}(P))$ topologies in every C^*-norm bounded subset of A (cf. [6] Lemma 1),

— the multiplication in A is left and right continuous on C^*-norm bounded subsets of A with respect to the $\sigma(A, \mathrm{sp}(P))$ topology (cf. [6] Lemma 2) and the involution of A is continuous in the same topology,

— the order in A defined as $x \leqq y$ iff $f(y-x) \in \mathbf{R}_+$ for all $f \in P$, coincides with the algebraic order of the $*$-algebra A (cf. [7] Prop. 1),

— the orthocomplemented partially ordered set $L(A)$ of projections of A is a σ-complete orthomodular lattice admitting a separating set of σ-additive states (cf. [7] Th. 1), moreover, if (A, P) is a complete weak GW^*-algebra then $L(A)$ is a complete orthomodular lattice admitting a separating set of completely additive states (cf. [7] Th. 2),

— the partial isometries are countably summable in A and, consequently, the equivalence of projections is countably additive (cf. [7] Prop. 2).

If (A, P) is a commutative GW^*-algebra then

— the $*$-algebra A is a Rickart $*$-algebra and the set $L(A)$ is total in A with respect to the C^*-norm topology (cf. [8] Th. 2),

— if $x \in A$ and θ_x denotes the unique unit preserving $*$-homomorphism from $\mathscr{C}_{\mathbf{C}}(\mathrm{Sp}(x))$ into A such that $\theta_x(\mathrm{id}_{\mathrm{Sp}(x)})=x$ then θ_x has a unique extension $\theta_x^P\colon \mathscr{F}^b_{\mathbf{C}}(\mathrm{Sp}(x), \mathscr{B}(\mathrm{Sp}(x)))\to A$ which is a unit preserving $*$-homomorphism with the property

$$f(\theta_x^P(\varphi)) = \int_{\mathrm{Sp}(x)} \varphi\, d(f\circ\theta_x)$$

for every $f\in P$ and $\varphi\in\mathscr{F}^b_{\mathbf{C}}(\mathrm{Sp}(x), \mathscr{B}(\mathrm{Sp}(x)))$ (cf. [8] Th. 4).

3. Vector integrals

In order to give a concise formulation of the general spectral problem for normal elements of unital C^*-algebras, we have to define the integral generated by a projection valued measure. In this section such a definition will be done in a very general context.

If $\mathscr{R}$ is a ring of subsets of the set T then $\mathscr{E}_{\mathbf{C}}(T, \mathscr{R})$ denotes the $*$-algebra of complex $\mathscr{R}$-step functions defined on T. We write $\bar{\mathscr{E}}_{\mathbf{C}}(T, \mathscr{R})$ for the closure of $\mathscr{E}_{\mathbf{C}}(T, \mathscr{R})$ in the sup-norm topology. Obviously, $\bar{\mathscr{E}}_{\mathbf{C}}(T, \mathscr{R})$ is a commutative C^*-algebra the C^*-norm of which coincides with the sup-norm $|||\cdot|||_T$ on T. It can be shown easily that a subset E of T belongs to $\mathscr{R}$ if and only if $\chi_E\in\bar{\mathscr{E}}_{\mathbf{C}}(T, \mathscr{R})$. Moreover, if $\mathscr{R}$ is a σ-algebra of subsets of the set T then $\bar{\mathscr{E}}_{\mathbf{C}}(T, \mathscr{R})=\mathscr{F}^b_{\mathbf{C}}(T, \mathscr{R})$.

It is well-known that given a ring $\mathscr{R}$ of subsets of the set T and a vector space F over the field of complex numbers, there is a canonical linear isomorphism between the vector space of additive set functions $\mathscr{R}\to F$ and that of the linear operators $\mathscr{E}_{\mathbf{C}}(T, \mathscr{R})\to F$. Consequently, we may write the same symbol for an additive set function $\mathscr{R}\to F$ and for the associated linear operator. If $u\colon \mathscr{R}\to F$ is an additive set function then the corresponding linear operator will be called the simple integral generated by u.

Proposition 1. *Let $\mathscr{R}$ be a ring of subsets of the set T, F a sequentially complete locally convex Hausdorff space over the field of complex numbers and $u\colon \mathscr{R}\to F$ is an additive set function. Then the following statements are equivalent:*

(i) *There is a unique sup-norm continuous linear operator $\bar{u}\colon \bar{\mathscr{E}}_{\mathbf{C}}(T, \mathscr{R})\to F$ with the property that $\bar{u}(\chi_E)=u(E)$ for every $E\in\mathscr{R}$.*

(ii) *The simple integral generated by u is sup-norm continuous.*

(iii) *The range of u is a bounded set in the topological vector space F.*

Proof. If $\bar{u}$ is a linear operator satisfying the conditions of (i) then the restriction of $\bar{u}$ to $\mathscr{E}_{\mathbf{C}}(T, \mathscr{R})$ coincides with the simple integral generated by u, thus (i) implies (ii). On the other hand, since F is a sequentially complete Hausdorff topological vector space, if the simple integral generated by u is continuous then it can be extended uniquely to $\bar{\mathscr{E}}_{\mathbf{C}}(T, \mathscr{R})$ as a continuous linear operator, thus (ii) implies (i).

It is evident that (iii) is an immediate consequence of (ii). It remained to show that the implication (iii)$\Rightarrow$(ii) holds. Let F' denote the complex vector space of continuous linear forms on F. If p is a continuous seminorm on F then the Hahn—

Banach theorem yields $p(z)=\sup_{f\in A'_p}|f(z)|$ for all $z\in F$, where

$$A'_p := \{f\in F' \mid \forall z\in F\colon |f(z)| \leqq p(z)\}.$$

Given an element $f\in F'$, the mapping $f\circ u\colon \mathcal{R}\to\mathbf{C}$ is a bounded additive set function and the simple integral generated by $f\circ u$ coincides with the composition of f and the simple integral generated by u. In the subsequent inequalities we shall use the well-known fact, that if $\mu\colon \mathcal{R}\to\mathbf{C}$ is a bounded additive set function then the total variation $|\mu|$ of μ satisfies the inequalities $|\mu(\varphi)|\leqq|\mu|(|\varphi|)$ for $\varphi\in\mathcal{E}_{\mathbf{C}}(T,\mathcal{R})$ and $|\mu|(E)\leqq 4\sup_{\substack{E'\in\mathcal{R}\\ E'\subset E}}|\mu(E')|$ for $E\in\mathcal{R}$.

Let p be an arbitrary continuous seminorm on F and φ a fixed element of $\mathcal{E}_{\mathbf{C}}(T,\mathcal{R})$. Then we have

$$p(u(\varphi)) = \sup_{f\in A'_p}\left|f(u(\varphi))\right| \leqq \sup_{f\in A'_p}|f\circ u|(|\varphi|) \leqq |||\varphi|||_T \sup_{f\in A'_p}|f\circ u|([\varphi\neq 0])$$

$$\leqq 4|||\varphi|||_T \sup_{f\in A'_p}\sup_{E\in\mathcal{R}}|(f\circ u)(E)| = 4\big(\sup_{E\in\mathcal{R}} p(u(E))\big)|||\varphi|||_T$$

and the number $\sup_{E\in\mathcal{R}} p(u(E))$ is finite, since the range of u is bounded in F. This means that the simple integral generated by u is continuous in the sup-norm topology, thus (iii) implies (ii).

There are two widely used corollaries of this proposition.

Corollary 1. *Let $\mathcal{R}$ be a ring of subsets of the set T, A be a Banach algebra and $u\colon \mathcal{R}\to A$ is an additive and multiplicative set function (i.e. $u(E\cap E')=u(E)u(E')$ for every $E, E'\in\mathcal{R}$). Then the following statements are equivalent:*

(i) *There is a unique continuous morphism $\bar{u}$ between the Banach algebras $\bar{\mathcal{E}}_{\mathbf{C}}(T,\mathcal{R})$ and A such that $\bar{u}(\chi_E)=u(E)$ for every $E\in\mathcal{R}$.*

(ii) *The range of u is a bounded subset of A.*

Proof. The multiplicativity of u assures that the simple integral generated by u be a morphism between the algebras $\mathcal{E}_{\mathbf{C}}(T,\mathcal{R})$ and A. Then our proposition is a simple consequence of Proposition 1.

Corollary 2. *Let $\mathcal{R}$ be a ring of subsets of the set T, A a C^*-algebra and $u\colon \mathcal{R}\to A$ is an additive and multiplicative set function. Then the following statements are equivalent:*

(i) *There is a unique morphism $\bar{u}$ between the $*$-algebras $\bar{\mathcal{E}}_{\mathbf{C}}(T,\mathcal{R})$ and A such that $\bar{u}(\chi_E)=u(E)$ for all $E\in\mathcal{R}$.*

(ii) *The range of u consists of self-adjoint idempotents (i.e. projections) of A.*

Proof. It is easy to see that the simple integral generated by u is a morphism between the $*$-algebras $\mathcal{E}_{\mathbf{C}}(T,\mathcal{R})$ and A if and only if u is multiplicative and $u(E)^*=$ $=u(E)$ for all $E\in\mathcal{R}$. Since the projections of a C^*-algebra are in the unit ball and a morphism between C^*-algebras is necessarily continuous, our assertion is an immediate consequence of Proposition 1.

DEFINITION 1. If $\mathscr{R}$ is a ring of subsets of the set T, F a sequentially complete complex locally convex Hausdorff space and $u\colon \mathscr{R}\to F$ is an additive set function the range of which is bounded in F, then the linear operator $\bar{u}$ defined in Proposition 1 (i) will be referred to as the *F-integral* generated by u.

REMARK 1. The existence of the F-integral generated by an additive set function taking values in F depends merely on the duality between F and F', since the bounded sets in F coincide with the weakly bounded sets.

REMARK 2. If G is another sequentially complete complex locally convex Hausdorff space and $L\colon F\to G$ is a continuous linear operator and the F-integral generated by the additive set function $u\colon \mathscr{R}\to F$ exists then the G-integral generated by the additive set function $L\circ u\colon \mathscr{R}\to G$ also exists and $\overline{L\circ u}=L\circ\bar{u}$.

REMARK 3. Let $M_F^b(T,\mathscr{R})$ denote the vector space of the additive set functions $\mathscr{R}\to F$ with bounded range and let $\mathscr{L}(\bar{\mathscr{E}}_{\mathbf{C}}(T,\mathscr{R});F)$ be the vector space of the sup-norm continuous linear operators from $\bar{\mathscr{E}}_{\mathbf{C}}(T,\mathscr{R})$ into F. By virtue of Proposition 1, the map

$$M_F^b(T,\mathscr{R})\to\mathscr{L}(\bar{\mathscr{E}}_{\mathbf{C}}(T,\mathscr{R});\ F);\quad u\mapsto\bar{u}$$

is a linear isomorphism, provided the complex locally convex Hausdorff space F is sequentially complete. The inverse isomorphism is the mapping

$$\mathscr{L}(\bar{\mathscr{E}}_{\mathbf{C}}(T,\mathscr{R});F)\to M_F^b(T,\mathscr{R});\quad \theta\mapsto u_\theta,$$

where $u_\theta(E):=\theta(\chi_E)$ for all $E\in\mathscr{R}$ and $\theta\in\mathscr{L}(\bar{\mathscr{E}}_{\mathbf{C}}(T,\mathscr{R});F)$.

Now we are going to select and characterize an important subspace of $\mathscr{L}(\bar{\mathscr{E}}_{\mathbf{C}}(T,\mathscr{R});F)$.

PROPOSITION 2. *Let $\mathscr{R}$ be a ring of subsets in the set T, F a complex locally convex Hausdorff space and $\theta\in\mathscr{L}(\bar{\mathscr{E}}_{\mathbf{C}}(T,\mathscr{R});F)$. Then the following statements are equivalent:*

(i) *For every uniformly bounded sequence $(\varphi_n)_{n\in\mathbf{N}}$ in $\bar{\mathscr{E}}_{\mathbf{C}}(T,\mathscr{R})$, if $\varphi_n\to 0$ pointwise in T then $\theta(\varphi_n)\to 0$ in the weak $\sigma(F,F')$-topology.*

(ii) *For every uniformly bounded sequence $(\varphi_n)_{n\in\mathbf{N}}$ in $\bar{\mathscr{E}}_{\mathbf{C}}(T,\mathscr{R})$, if $\varphi_n\to 0$ pointwise in T then $\theta(\varphi_n)\to 0$ in the weak $\sigma(F,F')$-topology.*

(iii) *For every sequence $(E_n)_{n\in\mathbf{N}}$ of pairwise disjoint sets in $\mathscr{R}$, if $\bigcup_{n\in\mathbf{N}}E_n\in\mathscr{R}$ then the series $\sum_{n\in\mathbf{N}}u_\theta(E_n)$ is summable in the $\sigma(F,F')$-topology and its sum equals $u_\theta(\bigcup_{n\in\mathbf{N}}E_n)$.*

(iv) *For every $f\in F'$, the function $f\circ u_\theta\colon \mathscr{R}\to\mathbf{C}$ is σ-additive.*

PROOF. Obviously we have (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv).

In order to prove that (iv) implies (i), assume that $(\varphi_n)_{n\in\mathbf{N}}$ is a uniformly bounded sequence in $\bar{\mathscr{E}}_{\mathbf{C}}(T,\mathscr{R})$ such that $\varphi_n\to 0$ pointwise in T. Let f be a fixed element of F'. We have to show that $f(\theta(\varphi_n))\to 0$. Since by (iv) the map $f\circ u_\theta\colon \mathscr{R}\to\mathbf{C}$ is a bounded σ-additive set function and the elements of $\bar{\mathscr{E}}_{\mathbf{C}}(T,\mathscr{R})$ are bounded, we have $\varphi_n \in \mathscr{L}^1_{\mathbf{C}}(T, \mathscr{R}, f\circ u_\theta)$ for $n \in \mathbf{N}$. Further, the set $E := \bigcup_{n\in\mathbf{N}} [\varphi_n \neq 0]$ is a countable union of the elements of $\mathscr{R}$, thus the function $\varphi := (\sup_{n\in\mathbf{N}} |||\varphi_n|||_T)\chi_E$ is integrable with respect to the bounded complex measure $f\circ u_\theta$. Then the theorem of Lebesgue applied to $f\circ u_\theta$ gives that $\int_T \varphi_n \, \mathrm{d}(f\circ u_\theta) \to 0$. Since the $\mathbf{C}$-integral generated by the additive set function $f\circ u_\theta$ coincides with the Lebesgue extension of $f\circ u_\theta$ restricted to $\bar{\mathscr{E}}_{\mathbf{C}}(T, \mathscr{R})$, we finally obtain that $f(\theta(\varphi_n)) = \int_T \varphi_n \, \mathrm{d}(f\circ u_\theta) \to 0$.

DEFINITION. Given a ring $\mathscr{R}$ of subsets of the set T and a locally convex Hausdorff space F over the field of complex numbers, the linear operator $\theta \in \mathscr{L}(\bar{\mathscr{E}}_{\mathbf{C}}(T, \mathscr{R}); F)$ is called of *Lebesgue type* if θ satisfies the conditions (i)—(iv) in Proposition 2. Further, an additive set function $u \in M^b_F(T, \mathscr{R})$ is called *weakly σ-additive* if u — in lieu of u_θ — satisfies (iv) in Proposition 2.

We shall denote by $\mathscr{L}_L(\bar{\mathscr{E}}_{\mathbf{C}}(T, \mathscr{R}); F)$ and $\mathfrak{M}^b_F(T, \mathscr{R})$ the vector space of Lebesgue type operators in $\mathscr{L}(\bar{\mathscr{E}}_{\mathbf{C}}(T, \mathscr{R}); F)$ and the vector space of weakly σ-additive set functions in $M^b_F(T, \mathscr{R})$, respectively. In fact, Proposition 2 combined with Proposition 1 yields that the mapping

$$\mathscr{L}_L(\bar{\mathscr{E}}_{\mathbf{C}}(T, \mathscr{R}); F) \to \mathfrak{M}^b_F(T, \mathscr{R}); \quad \theta \mapsto u_\theta$$

is a linear isomorphism, provided F is sequentially complete.

To end this section we prove an alternative form of Proposition 6 in [3] Ch. VI, § 2, n° 3. In order to do this we need some notions and notations.

Let T be a compact Hausdorff space and F a complex locally convex Hausdorff space. Then a sup-norm continuous linear operator from $\mathscr{C}_{\mathbf{C}}(T)$ into F is called an *F-valued Radon integral* on T. The vector space of F-valued Radon integrals on T is denoted by $\mathscr{L}(\mathscr{C}_{\mathbf{C}}(T); F)$. If $\theta \in \mathscr{L}(\mathscr{C}_{\mathbf{C}}(T); F)$ then

$$\mathscr{L}^1_{\mathbf{C}}(T, \theta) := \bigcap_{f\in F'} \mathscr{L}^1_{\mathbf{C}}(T, f\circ\theta)$$

where F' stands for the topological dual of F. Then the *weak integral* generated by θ is the linear operator θ'^* from $\mathscr{L}^1_{\mathbf{C}}(T, \theta)$ into the algebraic dual F'^* of F' defined as

$$\theta'^*(\varphi)(f) := \int_T \varphi \, d(f\circ\theta)$$

for every $\varphi \in \mathscr{L}^1_{\mathbf{C}}(T, \theta)$ and $f \in F'$. With regard to the Hahn—Banach theorem, the complex vector space F is embedded into F'^* algebraically by the map $z \mapsto \bar{z}$, where $\bar{z}(f) := f(z)$ for all $f \in F'$. That is why we identify F with the corresponding linear subspace of F'^*.

If T is a compact Hausdorff space then for every ordinal number $\alpha < \omega_1$ we define by ω_1-induction the function space $\mathscr{C}^\alpha_{\mathbf{C}}(T)$ over T as follows

(i) $\mathscr{C}^0_{\mathbf{C}}(T) := \mathscr{C}_{\mathbf{C}}(T)$.

(ii) If $0<\alpha<\omega_1$ and for every ordinal number $\beta<\alpha$ the function space $\mathscr{C}^\beta_{\mathbf{C}}(T)$ is defined then $\varphi\in\mathscr{C}^\alpha_{\mathbf{C}}(T)$ if and only if φ is a function $T\to\mathbf{C}$ and there is a sequence $(\varphi_n)_{n\in\mathbf{N}}$ of functions in $\bigcup\limits_{\beta<\alpha}\mathscr{C}^\beta_{\mathbf{C}}(T)$ which is uniformly bounded and converges to φ pointwise in T.

Then we define $\mathscr{C}^\infty_{\mathbf{C}}(T) := \bigcup\limits_{\alpha<\omega_1}\mathscr{C}^\alpha_{\mathbf{C}}(T)$. Simple topological considerations lead to the result that $\mathscr{C}^\infty_{\mathbf{C}}(T)$ coincides with the set $\mathscr{F}^b_{\mathbf{C}}(T, \mathscr{B}_0(T))$ of complex bounded Baire functions on T. The sequence of function spaces $(\mathscr{C}^\alpha_{\mathbf{C}}(T))_{\alpha<\omega_1}$ will be referred to as the *standard graduation* of $\mathscr{C}^\infty_{\mathbf{C}}(T)$.

Finally, if T is a compact Hausdorff space then $\mathscr{L}^1_{\mathbf{C}}(T)$ will denote the vector space of complex universally integrable functions defined on T. It is easy to show that $\mathscr{L}^1_{\mathbf{C}}(T)$ consists precisely of the complex bounded universally measurable functions defined on T. Applying the theorem of Lebesgue, by ω_1-induction, we obtain that $\mathscr{C}^\alpha_{\mathbf{C}}(T)\subset\mathscr{L}^1_{\mathbf{C}}(T)$ for every ordinal number $\alpha<\omega_1$, thus $\mathscr{C}_{\mathbf{C}}(T)\subset\mathscr{C}^\infty_{\mathbf{C}}(T)\subset\mathscr{L}^1_{\mathbf{C}}(T)$.

A detailed study of the theory of integration with respect to vector valued Radon integrals can be found in [3] Ch. VI.

PROPOSITION 3. *Let T be a compact Hausdorff space, F a complex locally convex Hausdorff space and θ an F-valued Radon integral on T.*

(i) *If F is sequentially complete in the $\sigma(F, F')$-topology then $\theta'^*(\mathscr{C}^\infty_{\mathbf{C}}(T))\subset F$.*

(ii) *If F is quasi complete in the $\sigma(F, F')$-topology then $\theta'^*(\mathscr{L}^1_{\mathbf{C}}(T,\theta))\subset F$.*

PROOF. In order to prove (i), by ω_1-induction, we show that $\theta'^*(\mathscr{C}^\alpha_{\mathbf{C}}(T))\subset F$ for every ordinal number $\alpha<\omega_1$. Since the restriction of θ'^* to $\mathscr{C}_{\mathbf{C}}(T)$ coincides with θ, the assertion holds for $\alpha=0$, obviously.

Assume that $0<\alpha<\omega_1$ and $\theta'^*(\mathscr{C}^\beta_{\mathbf{C}}(T))\subset F$ for every $\beta<\alpha$. Let φ be an element of $\mathscr{C}^\alpha_{\mathbf{C}}(T)$ and choose a sequence $(\varphi_n)_{n\in\mathbf{N}}$ in $\bigcup\limits_{\beta<\alpha}\mathscr{C}^\beta_{\mathbf{C}}(T)$ such that $\varphi_n\to\varphi$ pointwise in T and $\sup\limits_{n\in\mathbf{N}}|||\varphi_n|||_T<+\infty$. If $f\in F'$ then the theorem of Lebesgue applied to the sequence $(\varphi_n)_{n\in\mathbf{N}}$ and to the complex Radon integral $f\circ\theta$ provides that

$$\theta'^*(\varphi_n)(f) = \int_T \varphi_n\, d(f\circ\theta) \to \int_T \varphi\, d(f\circ\theta) = \theta'^*(\varphi)(f)$$

so $\theta'^*(\varphi_n)\to\theta'^*(\varphi)$ in F'^* in the $\sigma(F'^*, F')$-topology. Consequently, $(\theta'^*(\varphi_n))_{n\in\mathbf{N}}$ is a Cauchy sequence in F'^* with respect to $\sigma(F'^*, F')$. Since, by the induction hypothesis, we have $\theta'^*(\varphi_n)\in F$ $(n\in\mathbf{N})$ and the restriction of $\sigma(F'^*, F')$ to F equals $\sigma(F, F')$; we deduce that $(\theta'^*(\varphi_n))_{n\in\mathbf{N}}$ is a Cauchy sequence in F with respect to the $\sigma(F, F')$-topology. Then the sequentially completeness of F with respect to the $\sigma(F, F')$-topology implies the existence of an element z in F such that $\theta'^*(\varphi_n)\to z$ in F in the $\sigma(F, F')$-topology. Then $\theta'^*(\varphi_n)\to z$ in F'^* in the $\sigma(F'^*, F')$-topology also holds, thus $\theta'^*(\varphi)=z\in F$.

In order to prove (ii), first we denote by $\mathscr{L}^b_{\mathbf{C}}(T,\theta)$ the vector space of complex bounded θ-integrable functions defined on T and show that $\theta'^*(\mathscr{L}^b_{\mathbf{C}}(T,\theta))\subset F$.

For every $f\in F'$ define the mapping $\|\cdot\|_{\theta,f}$ from $\mathscr{L}^b_{\mathbf{C}}(T,\theta)$ into $\mathbf{R}_+$ as follows

$$\|\varphi\|_{\theta,f} := \int_T |\varphi|\, d|f\circ\theta|$$

for every $\varphi\in\mathcal{L}^b_{\mathbf{C}}(T,\theta)$. Then $(\|\cdot\|_{\theta,f})_{f\in F'}$ is a family of seminorms on $\mathcal{L}^b_{\mathbf{C}}(T,\theta)$ determining thus a locally convex topology $\mathcal{T}$ on $\mathcal{L}^b_{\mathbf{C}}(T,\theta)$ (which is not a Hausdorff topology, in general).

Since $|(f\circ\theta)(\varphi)|\leqq\|\varphi\|_{\theta,f}$ for $\varphi\in\mathcal{C}_{\mathbf{C}}(T)$ and $f\in F'$, the linear operator $\theta:\mathcal{C}_{\mathbf{C}}(T)\to F$ is continuous in the $\mathcal{T}|_{\mathcal{C}_{\mathbf{C}}(T)}$ and $\sigma(F,F')$-topologies in $\mathcal{C}_{\mathbf{C}}(T)$ and F, respectively.

Let B denote the closed unit ball of the C^*-algebra $\mathcal{C}_{\mathbf{C}}(T)$. Clearly, if $c\in\mathbf{R}_+$ then cB is a bounded set in $\mathcal{L}^b_{\mathbf{C}}(T,\theta)$ in the topology $\mathcal{P}$. Indeed, $\sup\limits_{\varphi\in cB}\|\varphi\|_{\theta,f}\leqq$ $\leqq c|f\circ\theta|(T)<+\infty$ for every $f\in F'$.

Next we show that each element φ of $\mathcal{L}^b_{\mathbf{C}}(T,\theta)$ belongs to the closure in the topology $\mathcal{T}$ of the set $\sqrt{2}\,|||\varphi|||_T B$. Indeed, let $\varphi\in\mathcal{L}^b_{\mathbf{C}}(T,\theta)$ be a fixed function and V an arbitrary neighbourhood in the topology $\mathcal{T}$ of 0. Then there are finite sequences $(\varepsilon_k)_{1\leqq k\leqq n}$ and $(f_k)_{1\leqq k\leqq n}$ in $\mathbf{R}^*_+$ and F', respectively, such that $V\supset\bigcap\limits_{k=1}^{n}[\|\cdot\|_{\theta,f_k}\leqq\varepsilon_k]$. The function φ is measurable with respect to the complex Radon integral $f_k\circ\theta$ for $k\in\{1,\dots,n\}$. Since T is a compact Hausdorff space, to every $\varepsilon>0$ and $k\in\{1,\dots,n\}$ there is a compact subset $T_k(\varepsilon)$ of T with the property that $\varphi|_{T_k(\varepsilon)}\in\mathcal{C}_{\mathbf{C}}(T_k(\varepsilon))$ and $|f_k\circ\theta|(T\setminus T_k(\varepsilon))\leqq\varepsilon$ (cf. [3] Ch. IV, § 5, n° 1, Prop. 1). For every $\varepsilon>0$ take $T(\varepsilon):=\bigcup\limits_{k=1}^{n}T_k(\varepsilon)$. Then by virtue of simple topological considerations we obtain that for every $\varepsilon>0$, $\varphi|_{T(\varepsilon)}\in\mathcal{C}_{\mathbf{C}}(T(\varepsilon))$ thus by the theorem of Tietze there is a function $\varphi_\varepsilon\in\mathcal{C}_{\mathbf{C}}(T)$ such that $\varphi_\varepsilon|_{T(\varepsilon)}=\varphi|_{T(\varepsilon)}$ and $|\varphi_\varepsilon|\leqq\sqrt{2}\,|||\varphi|||_T$ (cf. [1] Ch. IX, § 4, n° 2, cor. de Th. 2). Consequently, $\varphi_\varepsilon\in\sqrt{2}\,|||\varphi|||_T B$ for every $\varepsilon>0$. We claim that for sufficiently small $\varepsilon>0$; $\varphi_\varepsilon\in\varphi+V$. Indeed, we have for $k\in\{1,\dots,n\}$

$$\|\varphi_\varepsilon-\varphi\|_{\theta,f_k}:=\int\limits_T|\varphi_\varepsilon-\varphi|\,d|f_k\circ\theta|=\int\limits_{T(\varepsilon)}|\varphi_\varepsilon-\varphi|\,d|f_k\circ\theta|+$$
$$+\int\limits_{T\setminus T(\varepsilon)}|\varphi_\varepsilon-\varphi|\,d|f_k\circ\theta|=\int\limits_{T\setminus T(\varepsilon)}|\varphi_\varepsilon-\varphi|\,d|f_k\circ\theta|\leqq$$
$$\leqq(|||\varphi_\varepsilon|||_T+|||\varphi|||_T)|f_k\circ\theta|(T\setminus T(\varepsilon))\leqq(1+\sqrt{2})|||\varphi|||_T|f_k\circ\theta|(T\setminus T_k(\varepsilon))\leqq$$
$$\leqq(1+\sqrt{2})|||\varphi|||_T\varepsilon,$$

thus for $0<\varepsilon<\min\limits_{1\leqq k\leqq n}\varepsilon_k/(1+(1+\sqrt{2})|||\varphi|||_T)$ we obtain that $\|\varphi_\varepsilon-\varphi\|_{\theta,f_k}\leqq\varepsilon$ for every $k\in\{1,\dots,n\}$, i.e. $\varphi_\varepsilon\in\varphi+V$. Summing up the above considerations; the complex vector space $\mathcal{L}^b_{\mathbf{C}}(T,\theta)$, equipped with the topology $\mathcal{T}$ is a locally convex space and $\mathcal{C}_{\mathbf{C}}(T)$ is a linear subspace of $\mathcal{L}^b_{\mathbf{C}}(T,\theta)$ with the property that every element of $\mathcal{L}^b_{\mathbf{C}}(T,\theta)$ belongs to the $\mathcal{T}$-closure of a $\mathcal{T}$-bounded subset of $\mathcal{C}_{\mathbf{C}}(T)$. Furthermore; the linear operator θ: $\mathcal{C}_{\mathbf{C}}(T)\to F$ is continuous in the $\mathcal{T}|\mathcal{C}_{\mathbf{C}}(T)$ and $\sigma(F,F')$-topologies in $\mathcal{C}_{\mathbf{C}}(T)$ and F, respectively.

Then, assuming that F is quasi complete in the $\sigma(F,F')$-topology, by [2], Ch. III, § 2, n° 5, Prop. 8, we conclude that there is a unique linear operator $\bar\theta$: $\mathcal{L}^b_{\mathbf{C}}(T,\theta)\to F$ which is $\mathcal{T}-\sigma(F,F')$-continuous and satisfies $\bar\theta|_{\mathcal{C}_{\mathbf{C}}(T)}=\theta$. We intend to show that $\bar\theta=\theta'^*|_{\mathcal{L}^b_{\mathbf{C}}(T,\theta)}$. In fact, both the linear operators $\bar\theta$ and $\theta'^*|_{\mathcal{L}^b_{\mathbf{C}}(T,\theta)}$

take values in F'^* and they are $\mathcal{T}-\sigma(F'^*, F')$ continuous (by the definition of $\mathcal{T}$). Since they coincide in $\mathcal{C}_{\mathbf{C}}(T)$ and $\mathcal{C}_{\mathbf{C}}(T)$ is a dense subspace of $\mathcal{L}^b_{\mathbf{C}}(T, \theta)$ in the topology $\mathcal{T}$; we infer that they coincide in $\mathcal{L}^b_{\mathbf{C}}(T, \theta)$, as well.

This proves that $\theta'^*(\mathcal{L}^b_{\mathbf{C}}(T, \theta))\subset F$, provided F is quasi complete in the $\sigma(F, F')$-topology.

Finally, we show that $\theta'^*(\mathcal{L}^1_{\mathbf{C}}(T, \theta))\subset F$, if the presumption of (ii) is satisfied.

Let φ be an element of $\mathcal{L}^1_{\mathbf{C}}(T, \theta)$ and define a sequence $(E_n)_{n\in\mathbf{N}}$ of subsets of T as follows

$$E_n := \{t\in T \,|\, |\varphi(t)| \leqq n\}.$$

Take $\varphi_n := \chi_{E_n}\varphi$ $(n\in\mathbf{N})$; then, of course, $\varphi_n\to\varphi$ pointwise in T and $|\varphi_n|\leqq|\varphi|$ and $\varphi_n\in\mathcal{L}^b_{\mathbf{C}}(T, \theta)$ $(n\in\mathbf{N})$. Then the above considerations include that $\theta'^*(\varphi_n)\in F$ $(n\in\mathbf{N})$ and for every $f\in F'$, applying the theorem of Lebesgue to the sequence $(\varphi_n)_{n\in\mathbf{N}}$ and to the complex Radon integral $f\circ\theta$, we obtain

$$f(\theta'^*(\varphi_n)) = \int_T \varphi_n\, d(f\circ\theta) \to \int_T \varphi\, d(f\circ\theta) = \theta'^*(\varphi)(f),$$

i.e. $\theta'^*(\varphi_n)\to\theta'^*(\varphi)$ in F'^* in the $\sigma(F'^*, F')$-topology. From this it follows that $(\theta'^*(\varphi_n))_{n\in\mathbf{N}}$ is a Cauchy sequence in F with respect to $\sigma(F, F')$ thus it converges to an element z of F in the same topology. Since $\theta'^*(\varphi_n)\to\theta'^*(\varphi)$ and $\theta'^*(\varphi_n)\to z$ in F'^* in the $\sigma(F'^*, F')$-topology, we finally obtain that $\theta'^*(\varphi)=z\in F$.

Remark 4. If T is a compact Hausdorff space and F a complex locally convex Hausdorff space then, according to our former definition, $\theta\in\mathcal{L}_L(\mathcal{C}^\infty_{\mathbf{C}}(T); F)$ means that θ is a sup-norm continuous linear operator from $\mathcal{C}^\infty_{\mathbf{C}}(T)$ into F and it has all the following properties

(i) For every uniformly bounded sequence $(\varphi_n)_{n\in\mathbf{N}}$ in $\mathcal{C}^\infty_{\mathbf{C}}(T)$, if $\varphi_n\to 0$ pointwise in T then $\theta(\varphi_n)\to 0$ in the $\sigma(F, F')$-topology.

(ii) For every uniformly bounded sequence $(\varphi_n)_{n\in\mathbf{N}}$ in $\mathcal{E}_{\mathbf{C}}(T, \mathcal{B}_0(T))$, if $\varphi_n\to 0$ pointwise in T then $\theta(\varphi_n)\to 0$ in the $\sigma(F, F')$-topology.

(iii) The mapping $u_\theta\colon \mathcal{B}_0(T)\to F;\ E\mapsto\theta(\chi_E)$ is a σ-additive set function with respect to the $\sigma(F, F')$-topology.

(iv) For every $f\in F'$, the function $f\circ u_\theta\colon \mathcal{B}_0(T)\to\mathbf{C}$ is a complex σ-additive Baire measure on T.

The equivalence of these properties is a simple consequence of Proposition 2 and the fact that $\mathcal{C}^\infty_{\mathbf{C}}(T)=\bar{\mathcal{E}}_{\mathbf{C}}(T, \mathcal{B}_0(T))$.

Lemma 1. *Let T be a compact Hausdorff space, F a complex locally convex Hausdorff space and $\theta\in\mathcal{L}_L(\mathcal{C}^\infty_{\mathbf{C}}(T); F)$. Then $\theta=0$ if and only if $\theta=0$ in $\mathcal{C}_{\mathbf{C}}(T)$.*

Proof. Assume that $\theta=0$ in $\mathcal{C}_{\mathbf{C}}(T)$. By ω_1-induction we show that $\theta=0$ in $\mathcal{C}^\alpha_{\mathbf{C}}(T)$ for every ordinal number $\alpha<\omega_1$. According to our hypothesis, the assertion is true for $\alpha=0$. If $0<\alpha<\omega_1$ and for every $\beta<\alpha$ we have $\theta=0$ in $\mathcal{C}^\beta_{\mathbf{C}}(T)$, then take a function φ in $\mathcal{C}^\alpha_{\mathbf{C}}(T)$ and choose a sequence $(\varphi_n)_{n\in\mathbf{N}}$ in $\bigcup_{\beta<\alpha}\mathcal{C}^\beta_{\mathbf{C}}(T)$ which is uniformly bounded and pointwise convergent to φ in T. Since θ is a Lebesgue type operator $\theta(\varphi_n)\to\theta(\varphi)$ in the $\sigma(F, F')$-topology and our induction hypothesis yields that $\theta(\varphi)=0$.

PROPOSITION 4. *If T is a compact Hausdorff space and F a complex locally convex Hausdorff space then the map*

$$\mathscr{L}_L(\mathscr{C}_{\mathbf{C}}^{\infty}(T);\ F) \to \mathscr{L}(\mathscr{C}_{\mathbf{C}}(T);\ F); \quad \theta \mapsto \theta|_{\mathscr{C}_{\mathbf{C}}(T)} \tag{1}$$

is a one-to-one linear operator. If besides F is sequentially complete in the $\sigma(F, F')$-topology then this mapping is a linear isomorphism the inverse of which is

$$\mathscr{L}(\mathscr{C}_{\mathbf{C}}(T);\ F) \to \mathscr{L}_L(\mathscr{C}_{\mathbf{C}}^{\infty}(T);\ F); \quad \theta \mapsto \theta'^*|_{\mathscr{C}_{\mathbf{C}}^{\infty}(T)} \tag{2}$$

further, the map

$$\mathscr{L}_L(\mathscr{C}_{\mathbf{C}}^{\infty}(T);\ F) \to \mathfrak{M}_F^b(T, \mathscr{B}_0(T)); \quad \theta \mapsto u_\theta \tag{3}$$

is also a linear isomorphism.

PROOF. The fact that the mapping defined by (1) is one-to-one, is a simple reformulation of Lemma 1. Assume now that F is sequentially complete in the $\sigma(F, F')$-topology. First we show that for every F-valued Radon integral θ on T, the weak integral θ'^* generated by θ is continuous with respect to the sup-norm topology in $\mathscr{C}_{\mathbf{C}}^{\infty}(T)$ and the initial topology in F. Indeed, if p is a continuous seminorm on F then by the Hahn—Banach theorem, the set $A'_p := \{f \in F' | \forall z \in F\colon\ |f(z)| \leqq p(z)\}$ has the property that $p(z) = \sup\limits_{f \in A'_p} |f(z)|$ for all $z \in F$. Then, given a function $\varphi \in \mathscr{C}_{\mathbf{C}}^{\infty}(T)$, we have

$$p(\theta'^*(\varphi)) = \sup_{f \in A'_p} |f(\theta'^*(\varphi))| = \sup_{f \in A'_p} \Big|\int_T \varphi\, d(f \circ \theta)\Big| \leqq$$

$$\leqq \sup_{f \in A'_p} \int_T |\varphi|\, d|f \circ \theta| \leqq |||\varphi|||_T \sup_{f \in A'_p} |f \circ \theta|(T) =$$

$$= |||\varphi|||_T \sup_{f \in A'_p} \sup_{\substack{\psi \in \mathscr{C}_{\mathbf{C}}(T) \\ |||\psi|||_T \leqq 1}} |(f \circ \theta)(\psi)| = |||\varphi|||_T \sup_{\substack{\psi \in \mathscr{C}_{\mathbf{C}}(T) \\ |||\psi|||_T \leqq 1}} \sup_{f \in A'_p} |f(\theta(\psi))| =$$

$$= \Big(\sup_{\substack{\psi \in \mathscr{C}_{\mathbf{C}}(T) \\ |||\psi|||_T \leqq 1}} p(\theta(\psi))\Big) |||\varphi|||_T$$

and here the multiplier of $|||\varphi|||_T$ is independent of φ and finite since the image of the unit ball of $\mathscr{C}_{\mathbf{C}}(T)$ established by θ is bounded in F with respect to the initial topology and p is a continuous seminorm on F. Consequently, for every $\theta \in \mathscr{L}(\mathscr{C}_{\mathbf{C}}(T); F)$ we obtain $\theta'^*|_{\mathscr{C}_{\mathbf{C}}^{\infty}(T)} \in \mathscr{L}(\mathscr{C}_{\mathbf{C}}^{\infty}(T); F)$. (Note that the range of the latter operator is in F, by Proposition 3.) Next we show that $\theta'^*|_{\mathscr{C}_{\mathbf{C}}^{\infty}(T)}$ is a Lebesgue type operator for $\theta \in \mathscr{L}(\mathscr{C}_{\mathbf{C}}(T); F)$. Indeed, if $(\varphi_n)_{n \in \mathbf{N}}$ is a uniformly bounded sequence in $\mathscr{C}_{\mathbf{C}}^{\infty}(T)$ such that $\varphi_n \to 0$ pointwise in T then for every $f \in F$ the theorem of Lebesgue applied to the complex Radon integral $f \circ \theta$ provides that

$$f(\theta'^*(\varphi_n)) = \theta'^*(\varphi_n)(f) = \int_T \varphi_n\, d(f \circ \theta) \to 0,$$

i.e. $\theta'^*(\varphi_n) \to 0$ in F with respect to the $\sigma(F, F')$-topology. This shows that the map defined by (2), in fact, takes its values in $\mathscr{L}_L(\mathscr{C}_{\mathbf{C}}^{\infty}(T); F)$.

Since $(\theta'^*|_{\mathscr{C}_{\mathbf{C}}^{\infty}(T)})|_{\mathscr{C}_{\mathbf{C}}(T)}=\theta'^*|_{\mathscr{C}_{\mathbf{C}}(T)}=\theta$ for $\theta\in\mathscr{L}(\mathscr{C}_{\mathbf{C}}(T);F)$; the mapping defined by (2) is the right inverse of the map defined by (1).

Conversely, if $\theta\in\mathscr{L}_L(\mathscr{C}_{\mathbf{C}}^{\infty}(T);F)$ then θ and $(\theta|_{\mathscr{C}_{\mathbf{C}}(T)})'^*|_{\mathscr{C}_{\mathbf{C}}^{\infty}(T)}$ are both Lebesgue type operators (as we have seen before) and the coincide in $\mathscr{C}_{\mathbf{C}}(T)$. Consequently, by Lemma 1, they are equal, thus the map defined by (2) is the left inverse of that defined by (1).

Finally, our last assertion is a simple consequence of Proposition 2 and of the well-known fact that the sequentially completeness of a locally convex Hausdorff space in its weak topology implies that in the initial topology, as well (cf. [2] Ch. I., § 1, n° 5, Proposition 8).

4. Spectral theorem for normal elements of GW^*-algebras

Let (A, P) be a GW^*-algebra and $\mathscr{B}$ a σ-algebra of subsets of the set T. If u is a projection valued measure in A defined on $\mathscr{B}$ then, by Corollary 2 of Proposition, there is a unique morphism between the $*$-algebras $\bar{\mathscr{E}}_{\mathbf{C}}(T,\mathscr{B})=\mathscr{F}_{\mathbf{C}}^b(T,\mathscr{B})$ and A which sends the characteristic function χ_E to $u(E)$, for all $E\in\mathscr{B}$. Since $u(T)=1$, this morphism is unit preserving automatically. It will be called the *integral* generated by the projection valued measure u and we will denote it by the symbol $\hat{u}$. Sometimes we write $\int_T \varphi\, du$ instead of $\hat{u}(\varphi)$ for $\varphi\in\mathscr{F}_{\mathbf{C}}^b(T,\mathscr{B})$. Properly speaking, $\hat{u}$ is the A-integral generated by u (cf. Corollary 2 of Proposition 1).

If F is a complex locally convex Hausdorff space the underlying vector space of which is that of A and whose topology is less fine than the C^*-norm topology of A then the F-integral generated by u also exists (namely, every C^*-norm bounded subset of A, including the range of u, is bounded in F) and Remark 2 in § 3 provides that it is equal to $\hat{u}$.

Now we are in position to prove the main theorem of this paper.

THEOREM 1. *Let (A, P) be a GW^*-algebra and x a normal element in A. Then there exists a unique projection valued measure u in A defined on the Borel σ-algebra of the spectrum* $\mathrm{Sp}(x)$ *of x, such that* $x=\int_{\mathrm{Sp}(x)} \mathrm{id}_{\mathrm{Sp}(x)}\, du$.

PROOF. Existence.

Let F denote the complex vector space A, equipped with the $\sigma(A, \mathrm{sp}(P))$-topology. Since $\sigma(F, F')=\sigma(A, \mathrm{sp}(P))$, F is a locally convex Hausdorff space which is sequentially complete with respect to the $\sigma(F, F')$ topology. (By our choice, here the initial topology of F coincides with the weak topology.) Then there exists a unique unit preserving morphism θ_x between the $*$-algebras $\mathscr{C}_{\mathbf{C}}(\mathrm{Sp}(x))$ and A such that $x=\theta_x(\mathrm{id}_{\mathrm{Sp}(x)})$ (cf. [4] Ch. I, § 6, n° 6, Proposition 5). Since θ_x is an A-valued Radon integral on the compact space $\mathrm{Sp}(x)$ and the C^*-norm topology of A is finer than $\sigma(A, \mathrm{sp}(P))$, we obtain that $\theta_x\in\mathscr{L}(\mathscr{C}_{\mathbf{C}}(\mathrm{Sp}(x));F)$. Consequently, by Proposition 4, there is a unique operator $\bar{\theta}_x\in\mathscr{L}_L(\mathscr{C}_{\mathbf{C}}^{\infty}(\mathrm{Sp}(x));F)$ extending θ_x. As it can be read out of the proof of Proposition 4, $\bar{\theta}_x$ satisfies

$$f(\bar{\theta}_x(\varphi))=\int_{\mathrm{Sp}(x)} \varphi\, d(f\circ\theta_x) \qquad (f\in\mathrm{sp}(P)) \tag{4}$$

for every $\varphi\in\mathscr{C}_{\mathbf{C}}^{\infty}(\mathrm{Sp}(x))$.

According to Proposition 2, the map u: $\mathscr{B}_0(\mathrm{Sp}(x)) \to F$; $E \mapsto \bar{\theta}_x(\chi_E)$ is a weakly σ-additive set function. We claim that u is the projection valued measure looked for.

First we show that (4) holds for every $f \in \overline{\mathrm{sp}}(P)$ and $\varphi \in \mathscr{C}_{\mathbf{C}}^{\infty}(\mathrm{Sp}(x))$. Indeed, if $f \in \overline{\mathrm{sp}}(P)$ then there is a sequence $(f_n)_{n\in\mathbf{N}}$ in $\mathrm{sp}(P)$ such that $\|f_n - f\|' \to 0$ (where $\|\cdot\|'$ denotes the dual norm associated with the C^*-norm of A). Then we have for $n \in \mathbf{N}$ $\|f_n \circ \theta_x - f \circ \theta_x\| \leqq \|f_n - f\|' \|\theta_x\| \leqq \|f_n - f\|'$ thus the sequence $(f_n \circ \theta_x)_{n\in\mathbf{N}}$ of complex Radon integrals on $\mathrm{Sp}(x)$ converges to $f \circ \theta_x$ in the measure norm topology. Consequently, for every $\varphi \in \mathscr{C}_{\mathbf{C}}^{\infty}(\mathrm{Sp}(x))$ we have

$$\Big| \int_{\mathrm{Sp}(x)} \varphi \, d(f_n \circ \theta_x) - \int_{\mathrm{Sp}(x)} \varphi \, d(f \circ \theta_x) \Big| \leqq |||\varphi|||_{\mathrm{Sp}(x)} \|f_n - f\|'$$

so the equality (4) yields

$$f_n(\bar{\theta}_x(\varphi)) = \int_{\mathrm{Sp}(x)} \varphi \, d(f_n \circ \theta_x) \to \int_{\mathrm{Sp}(x)} \varphi \, d(f \circ \theta_x).$$

On the other hand, $f_n \to f$ also in the $\sigma(\overline{\mathrm{sp}}(P), A)$ topology, thus $f_n(\bar{\theta}_x(\varphi)) \to f(\bar{\theta}_x(\varphi))$. After all we arrive at the equality

$$f(\bar{\theta}_x(\varphi)) = \int_{\mathrm{Sp}(x)} \varphi \, d(f \circ \theta_x) \qquad (f \in \overline{\mathrm{sp}}(P)) \tag{5}$$

for all $\varphi \in \mathscr{C}_{\mathbf{C}}^{\infty}(\mathrm{Sp}(x))$.

Next we show that $\bar{\theta}_x$ is a morphism between the $*$-algebras $\mathscr{C}_{\mathbf{C}}^{\infty}(\mathrm{Sp}(x))$ and F. (Note that F is a unital $*$-algebra and a locally convex Hausdorff space at the same time, however, F is not a locally convex $*$-algebra since the multiplication of F is not continuous even separately or sequentially.) By ω_1-induction we show that given a function $\varphi \in \mathscr{C}_{\mathbf{C}}(\mathrm{Sp}(x))$, for every ordinal number $\alpha < \omega_1$ and $\psi \in \mathscr{C}_{\mathbf{C}}^{\alpha}(\mathrm{Sp}(x))$ we have $\bar{\theta}_x(\varphi\psi) = \bar{\theta}_x(\varphi)\bar{\theta}_x(\psi)$. Since the restriction of $\bar{\theta}_x$ to $\mathscr{C}_{\mathbf{C}}(\mathrm{Sp}(x))$ coincides with θ_x, our assertion is true for $\alpha = 0$.

Suppose that $0 < \alpha < \omega_1$ and for every $\beta < \alpha$ and $\psi \in \mathscr{C}_{\mathbf{C}}^{\beta}(\mathrm{Sp}(x))$ we have $\bar{\theta}_x(\varphi\psi) = \bar{\theta}_x(\varphi)\bar{\theta}_x(\psi)$. Let $\psi \in \mathscr{C}_{\mathbf{C}}^{\alpha}(\mathrm{Sp}(x))$ and choose a sequence $(\psi_n)_{n\in\mathbf{N}}$ in $\bigcup_{\beta<\alpha} \mathscr{C}_{\mathbf{C}}^{\beta}(\mathrm{Sp}(x))$ such that $\psi_n \to \psi$ pointwise in $\mathrm{Sp}(x)$ and $\sup_{n\in\mathbf{N}} |||\psi_n|||_{\mathrm{Sp}(x)} < +\infty$. Since $(\varphi\psi_n)_{n\in\mathbf{N}}$ is a uniformly bounded sequence in $\mathscr{C}_{\mathbf{C}}^{\infty}(\mathrm{Sp}(x))$ converging to $\varphi\psi$ pointwise in $\mathrm{Sp}(x)$, we obtain that $\bar{\theta}_x(\varphi\psi_n) \to \bar{\theta}_x(\varphi\psi)$ in the $\sigma(F, F') = \sigma(A, \mathrm{sp}(P))$-topology (namely $\bar{\theta}_x$ is a Lebesgue type operator). If $f \in \mathrm{sp}(P)$ then $\bar{\theta}_x(\varphi) \cdot f \in \overline{\mathrm{sp}}(P)$ thus by (5) we deduce

$$f(\bar{\theta}_x(\varphi)\bar{\theta}_x(\psi_n)) = (\bar{\theta}_x(\varphi) \cdot f)(\bar{\theta}_x(\psi_n)) = \int_{Sp(x)} \psi_n \, d((\bar{\theta}_x(\varphi) \cdot f) \circ \theta_x) \to$$

$$\to \int_{Sp(x)} \psi \, d((\bar{\theta}_x(\varphi) \cdot f) \circ \theta_x) = (\bar{\theta}_x(\varphi) \cdot f)(\bar{\theta}_x(\psi)) = f(\bar{\theta}_x(\varphi)\bar{\theta}_x(\psi))$$

where we have applied the theorem of Lebesgue to the complex Radon integral $(\bar{\theta}_x(\varphi) \cdot f) \circ \theta_x$ and to the sequence $(\psi_n)_{n\in\mathbf{N}}$. This means that $\bar{\theta}_x(\varphi)\bar{\theta}_x(\psi_n) \to \bar{\theta}_x(\varphi)\bar{\theta}_x(\psi)$ in the $\sigma(A, \mathrm{sp}(P)) = \sigma(F, F')$-topology. According to our induction hypothesis $\bar{\theta}_x(\varphi\psi_n) = \bar{\theta}_x(\varphi)\bar{\theta}_x(\psi_n)$ for $n \in \mathbf{N}$, showing that $\bar{\theta}_x(\varphi\psi) = \bar{\theta}_x(\varphi)\bar{\theta}_x(\psi)$. Now we prove that given a function $\psi \in \mathscr{C}_{\mathbf{C}}^{\infty}(\mathrm{Sp}(x))$, for every $\alpha < \omega_1$ and $\varphi \in \mathscr{C}_{\mathbf{C}}^{\alpha}(\mathrm{Sp}(x))$ we have $\bar{\theta}_x(\varphi\psi) = \bar{\theta}_x(\varphi)\bar{\theta}_x(\psi)$. Our previous result shows that the assertion is true for $\alpha=0$. If $0<\alpha<\omega_1$ and $\varphi\in\mathscr{C}_{\mathbf{C}}^{\alpha}(\mathrm{Sp}(x))$ then there is a uniformly bounded sequence $(\varphi_n)_{n\in\mathbf{N}}$ in $\bigcup\limits_{\beta<\alpha}\mathscr{C}_{\mathbf{C}}^{\beta}(\mathrm{Sp}(x))$ such that $\varphi_n\to\varphi$ pointwise in $\mathrm{Sp}(x)$. Since $(\varphi_n\psi)_{n\in\mathbf{N}}$ is a uniformly bounded sequence in $\mathscr{C}_{\mathbf{C}}^{\infty}(\mathrm{Sp}(x))$ converging to $\varphi\psi$ pointwise in $\mathrm{Sp}(x)$, we obtain that $\bar\theta_x(\varphi_n\psi)\to\bar\theta_x(\varphi\psi)$ in the $\sigma(F, F')$-topology (namely $\bar\theta_x$ is a Lebesgue type operator). Further, if $f\in\mathrm{sp}(P)$ then $f\cdot\bar\theta_x(\psi)\in\overline{\mathrm{sp}}(P)$ thus by (5) we deduce

$$f(\bar\theta_x(\varphi_n)\bar\theta_x(\psi))=(f\cdot\bar\theta_x(\psi))(\bar\theta_x(\varphi_n))=\int\limits_{\mathrm{Sp}(x)}\varphi_n\,d((f\cdot\bar\theta_x(\psi))\circ\theta_x)\to$$

$$\to\int\limits_{\mathrm{Sp}(x)}\varphi\,d((f\cdot\bar\theta_x(\psi))\circ\theta_x)=(f\cdot\bar\theta_x(\psi))(\bar\theta_x(\varphi))=f(\bar\theta_x(\varphi)\bar\theta_x(\psi))$$

where we have applied the theorem of Lebesgue to the complex Radon integral $(f\cdot\bar\theta_x(\psi))\circ\theta_x$ and to the sequence $(\varphi_n)_{n\in\mathbf{N}}$. This means that $\bar\theta_x(\varphi_n)\bar\theta_x(\psi)\to\bar\theta_x(\varphi)\bar\theta_x(\psi)$ in the $\sigma(A,\mathrm{sp}(x))=\sigma(F, F')$-topology. With regard to our induction hypothesis $\bar\theta_x(\varphi_n\psi)=\bar\theta_x(\varphi_n)\bar\theta_x(\psi)$ $(n\in\mathbf{N})$ showing that $\bar\theta_x(\varphi\psi)=\bar\theta_x(\varphi)\bar\theta_x(\psi)$. Summarizing, we have proved the multiplicativity of the linear operator $\bar\theta_x$.

In order to prove that $\bar\theta_x$ is an involution preserving map, we show by ω_1-induction that $\bar\theta_x(\bar\varphi)=(\bar\theta_x(\varphi))^*$ for all $\alpha<\omega_1$ and $\varphi\in\mathscr{C}_{\mathbf{C}}^{\alpha}(\mathrm{Sp}(x))$.

The assertion is true for $\alpha=0$, obviously. If $0<\alpha<\omega_1$ and $\varphi\in\mathscr{C}_{\mathbf{C}}^{\alpha}(\mathrm{Sp}(x))$ then choose a uniformly bounded sequence $(\varphi_n)_{n\in\mathbf{N}}$ in $\bigcup\limits_{\beta<\alpha}\mathscr{C}_{\mathbf{C}}^{\beta}(\mathrm{Sp}(x))$ such that $\varphi_n\to\varphi$ pointwise in $\mathrm{Sp}(x)$. Then $(\bar\varphi_n)_{n\in\mathbf{N}}$ is also a uniformly bounded sequence in $\mathscr{C}_{\mathbf{C}}^{\infty}(\mathrm{Sp}(x))$ converging to $\bar\varphi$ pointwise in $\mathrm{Sp}(x)$. Consequently, $\bar\theta_x(\varphi_n)\to\bar\theta_x(\varphi)$ and $\bar\theta_x(\bar\varphi_n)\to\bar\theta_x(\bar\varphi)$ in the $\sigma(F, F')=\sigma(A,\mathrm{sp}(P))$-topology. Then we have $(\bar\theta_x(\varphi_n))^*\to(\bar\theta_x(\varphi))^*$ also in the $\sigma(A,\mathrm{sp}(x))$-topology, since the involution of A is continuous with respect to the $\sigma(A,\mathrm{sp}(P))$-topology. By our induction hypothesis $\bar\theta_x(\bar\varphi_n)=(\bar\theta_x(\varphi_n))^*$ $(n\in\mathbf{N})$, so $\bar\theta_x(\bar\varphi)=(\bar\theta_x(\varphi))^*$.

After all we arrive at the result that $\bar\theta_x$ is a unit preserving morphism between the $*$-algebras $\mathscr{C}_{\mathbf{C}}^{\infty}(\mathrm{Sp}(x))$ and A satisfying (5).

As a simple consequence of the above fact we obtain that the map u: $\mathscr{B}_0(\mathrm{Sp}(x))\to A$; $E\mapsto\bar\theta_x(\chi_E)$ is an additive and multiplicative set function taking values in the σ-complete lattice $I(A)$ of projections of A. Moreover, since the operator $\bar\theta_x$ is in $\mathscr{L}_L(\mathscr{C}_{\mathbf{C}}^{\infty}(\mathrm{Sp}(x)); F)$, by Proposition 2 we deduce that the function $f\circ u$: $\mathscr{B}_0(\mathrm{Sp}(x))\to\mathbf{C}$ is σ-additive for all $f\in F'=\mathrm{sp}\,(P)$. Combining this result with Theorem 1 in [7], we obtain that u is a projection valued measure in A defined on the Baire σ-algebra of $\mathrm{Sp}(x)$. With regard to Proposition 4, the F-integral generated by u coincides with $\bar\theta_x$. On the other hand, the remark preceding the present theorem tells us that the F-integral generated by u equals the integral (i.e. the A-integral) generated by u. Consequently, the integral $\hat u$ generated by u restricted to $\mathscr{C}_{\mathbf{C}}(\mathrm{Sp}(x))$ coincides with θ_x thus $x=\theta_x(\mathrm{id}_{\mathrm{Sp}(x)})=\bar\theta_x(\mathrm{id}_{\mathrm{Sp}(x)})=\hat u(\mathrm{id}_{\mathrm{Sp}(x)})=$ $=\int\limits_{\mathrm{Sp}(x)}\mathrm{id}_{\mathrm{Sp}(x)}\,du$. Since $\mathrm{Sp}(x)$ is a metrizable compact topological space, we have $\mathscr{B}_0(\mathrm{Sp}(x))=\mathscr{B}(\mathrm{Sp}(x))$, so we see that the projection valued measure u complies with our requirement.

Uniqueness.

Let F denote the same locally convex Hausdorff space introduced above and suppose that u, u' are both projection valued measures in A defined on $\mathscr{B}(\mathrm{Sp}(x))$ satisfying the equality

$$\int\limits_{\mathrm{Sp}(x)} \mathrm{id}_{\mathrm{Sp}(x)}\, du = x = \int\limits_{\mathrm{Sp}(x)} \mathrm{id}_{\mathrm{Sp}(x)}\, du'.$$

Then by Theorem 1 in [7] we conclude that $u, u' \in \mathfrak{M}_F^b(\mathrm{Sp}(x), \mathscr{B}_0(\mathrm{Sp}(x)))$ thus Proposition 4 implies that the F-integrals $\bar{u}$ and $\bar{u}'$ generated by u and u', respectively, belong to $\mathscr{L}_L(\mathscr{C}_{\mathrm{C}}^{\infty}(\mathrm{Sp}(x)); F)$.

On the other hand, our remark preceding the theorem results in the fact that the integrals generated by u and u' coincide with the F-integrals generated by them, respectively. Then Corollary 2 of Proposition 1 provides that the F-integrals $\bar{u}$ and $\bar{u}'$ are both unit preserving morphisms between the $*$-algebras $\bar{\mathscr{E}}_{\mathrm{C}}(\mathrm{Sp}(x), \mathscr{B}(\mathrm{Sp}(x))) = \mathscr{C}_{\mathrm{C}}^{\infty}(\mathrm{Sp}(x))$ and A. Consequently, the uniqueness of the operator θ_x involves that $\bar{u}$ and $\bar{u}'$ coincide in $\mathscr{C}_{\mathrm{C}}(\mathrm{Sp}(x))$. Then Lemma 1 yields $\bar{u} = \bar{u}'$ and, 'a fortiori', $u = u'$.

Definition. If (A, P) is a GW^*-algebra and x is a normal element in A then the projection valued measure u defined on the Borel σ-algebra of $\mathrm{Sp}(x)$ taking values in $L(A)$ and satisfying $x = \int\limits_{\mathrm{Sp}(x)} \mathrm{id}_{\mathrm{Sp}(x)}\, du$ will be referred to as the *spectral resolution* of x.

Our next theorem is a direct generalization of Theorem 4 proved in [8] for the case of commutative GW^*-algebras. Note that our spectral theorem for the elements of commutative GW^*-algebras had been obtained in a perfectly different way compared to that presented here.

Theorem 2. *Let (A, P) be a GW^*-algebra and x a normal element in A. Then there exists a unique unit preserving $*$-homomorphism $\theta_x^P\colon \mathscr{F}_{\mathrm{C}}^b(\mathrm{Sp}(x), \mathscr{B}(\mathrm{Sp}(x))) \to A$ which is an extension of θ_x and satisfies for every bounded complex valued Borel function φ defined on* $\mathrm{Sp}(x)$

$$f(\theta_x^P(\varphi)) = \int\limits_{\mathrm{Sp}(x)} \varphi\, d(f \circ \theta_x) \qquad (f \in P) \tag{6}$$

where θ_x is the unique unit preserving morphism between the $*$-algebras $\mathscr{C}_{\mathrm{C}}(\mathrm{Sp}(x))$ and A satisfying $x = \theta_x(\mathrm{id}_{\mathrm{Sp}(x)})$.

Proof. Applying the notions and notations introduced in the proof of Theorem 1, we see that the map $\theta_x^P := \bar{\theta}_x$ complies with all our requirements. If θ' is another map satisfying our conditions then substituting θ' instead of θ_x^P, we infer that the equality (6) holds for every $f \in \mathrm{sp}(P) = F'$. Then the theorem of Lebesgue results in $\theta' \in \mathscr{L}_L(\mathscr{C}_{\mathrm{C}}^{\infty}(\mathrm{Sp}(x)); F)$ and we have already seen that $\bar{\theta}_x \in \mathscr{L}_L(\mathscr{C}_{\mathrm{C}}^{\infty}(\mathrm{Sp}(x)); F)$ and we have already seen that $\bar{\theta}_x \in \mathscr{L}_L(\mathscr{C}_{\mathrm{C}}^{\infty}(\mathrm{Sp}(x)); F)$. Since the maps θ' and $\bar{\theta}_x$ coincide with θ_x in $\mathscr{C}_{\mathrm{C}}(\mathrm{Sp}(x))$, by virtue of Lemma 1, we obtain that $\theta' = \bar{\theta}_x$.

Now we are going to justify our terminology "spectral resolution" showing that the spectral resolution of a normal element in a GW^*-algebra possesses all the properties of a spectral resolution in the sense of [5].

PROPOSITION 5. *Let (A, P) be a GW^*-algebra and x a normal element in A. If $u\colon \mathscr{B}(\mathrm{Sp}_A(x)) \to L(A)$ is the spectral resolution of x then $u(E)x = xu(E)$ and $\mathrm{Sp}_{u(E)Au(E)}(xu(E)) \subset \bar{E}$ for every $E \in \mathscr{B}(\mathrm{Sp}_A(x))$, where $\bar{E}$ denotes the closure of E.*

PROOF. It is obvious that x commutes with the projections taken from the range of u.

Let E be a Borel subset of $\mathrm{Sp}_A(x)$ and $\lambda \in \mathbf{C} \setminus \bar{E}$. Let ε be a positive real number such that $|\lambda - \lambda'| > \varepsilon$ if $\lambda' \in E$. Then we define the function $\varphi_\varepsilon\colon \mathrm{Sp}_A(x) \to \mathbf{C}$ as follows

$$\lambda' \mapsto \begin{cases} 1/(\lambda' - \lambda) & \text{if } \lambda' \in E \\ 0 & \text{if } \lambda' \in Sp_A(x) \setminus E. \end{cases}$$

It is clear that $\varphi_\varepsilon \in \mathscr{F}_{\mathbf{C}}^b(\mathrm{Sp}_A(x), \mathscr{B}(\mathrm{Sp}_A(x)))$, hence the element $y_\varepsilon := \int\limits_{\mathrm{Sp}_A(x)} \varphi_\varepsilon \, du$ is well defined. Since the range of the integral generated by u is a commutative subalgebra of A, and $\varphi_\varepsilon \chi_E = \varphi_\varepsilon$, we have $y_\varepsilon u(E) = y_\varepsilon = u(E) y_\varepsilon$. Thus $y_\varepsilon \in u(E)Au(E)$ and the equality $\chi_E = \varphi_\varepsilon(\mathrm{id}_{\mathrm{Sp}_A(x)} - \lambda)\chi_E$ yields that $u(E) = y_\varepsilon(x - \lambda 1)u(E) = y_\varepsilon(xu(E) - \lambda u(E))$. Since $u(E)$ is the unit element of the algebra $u(E)Au(E)$, this means that y_ε is the inverse of $xu(E) - \lambda u(E)$ in $u(E)Au(E)$, i.e. $\lambda \notin \mathrm{Sp}_{u(E)Au(E)}(xu(E))$.

Our last assertion shows that the spectral resolution of a normal element x in a GW^*-algebra lives on $\mathrm{Sp}(x)$.

PROPOSITION 6. *Let (A, P) be a GW^*-algebra and x a normal element in A. If $u\colon \mathscr{B}(\mathrm{Sp}(x)) \to L(A)$ is the spectral resolution of x and Ω is a non-void subset of $\mathrm{Sp}(x)$ which is open with respect to the relative topology then $u(\Omega) \neq 0$.*

PROOF. Since the topological space $\mathrm{Sp}(x)$ is compact, hence completely regular, given a non-void open subset Ω of $\mathrm{Sp}(x)$ there is a function $\varphi \in \mathscr{C}_{\mathbf{C}}(\mathrm{Sp}(x))$ such that $0 \leqq \varphi \leqq \chi_\Omega$ and $\varphi \neq 0$. Then we have

$$f\Big(\int\limits_{\mathrm{Sp}(x)} \varphi \, du\Big) = \int\limits_{\mathrm{Sp}(x)} \varphi \, d(f \circ u) \leqq \int\limits_{\mathrm{Sp}(x)} \chi_\Omega \, d(f \circ u) = f(u(\Omega))$$

for every $f \in P$. Since, by Theorem 2, the integral defined by u is isometric on $\mathscr{C}_{\mathbf{C}}(\mathrm{Sp}(x))$, we have $\Big\| \int\limits_{\mathrm{Sp}(x)} \varphi \, du \Big\| = |||\varphi|||_{\mathrm{Sp}(x)} > 0$, thus $\int\limits_{\mathrm{Sp}(x)} \varphi \, du \neq 0$. The set P separates the points of A thus there is a positive linear form f in P such that $f\big(\int\limits_{\mathrm{Sp}(x)} \varphi \, du\big) \neq 0$. Then our former inequality yields $f(u(\Omega)) \neq 0$, i.e. $u(\Omega) \neq 0$.

To finish the paper I give expression to the conviction that the abstract spectral theorem established above throws new light upon the classical spectral problem revealing the proper reasons why it can have an affirmative solution.

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(Received December 10, 1985)

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Studia Scientiarum Mathematicarum Hungarica 23 (1988), 471—475

A NOTE ON Γ-NEAR-RINGS

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Abstract

In this note, we define the left and right operator near-rings, L and R, respectively, of a Γ-near-ring M. These provide a generalization of the equivalent concepts for Γ-rings. It is shown that if the addition operation on M is commutative, then R is a ring. Finally, we show that if M has a strong left unity and a right unity, then the lattices of right (resp. two-sided) ideals of L and M are isomorphic.

1. Preliminaries

We recall that a (right) near-ring is a triple $(N, +, \cdot)$ where

(i) $(N, +)$ is a (not necessarily abelian) group;

(ii) $(N, \cdot)$ is a semigroup;

(iii) $(x+y)z=xz+yz$ for all $x, y, z\in N$.

If (iii) is replaced with

(iii′) $x(y+z)=xy+xz$ for all $x, y, z\in N$,

then N is called a left near-ring. For all concepts relative to near-rings, we refer to [3].

A Γ-near-ring is a triple $(M, +, \Gamma)$ where

(iv) $(M, +)$ is a group;

(v) Γ is a nonempty set of binary operators on M such that for each $\gamma\in\Gamma$, $(M, +, \gamma)$ is a (right) near-ring.

(vi) For all $x, y, z\in M$, $\gamma, \mu\in\Gamma$, $x\gamma(y\mu z)=(x\gamma y)\mu z$.

This definition is due to Satyanarayana [4]. A subset A of M is called a left (right) ideal of M if:

(a) A is a normal divisor of $(M, +)$;

(b) For all $x\in A$, $u, v\in M$, $\gamma\in\Gamma$

$$u\gamma(x+v)-u\gamma v\in A \qquad (x\gamma u\in A).$$

I is called a (two-sided) ideal of M if I is both a left and a right ideal.

2. The operator near-rings

For all details concerning Γ-rings and their operator rings, we refer to [2]. Throughout this paper, let M be a Γ-near-ring. Let $\mathcal{L}$ be the set of all mappings of M into itself which act on the left. Then $\mathcal{L}$ is a (right) near-ring with the operations pointwise addition and composition of mappings. Let $x\in M$ and $\gamma\in\Gamma$. We

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1980 *Mathematics Subject Classification*. Primary 16A76; Secondary 16A78.
Key words and phrases. Γ-ring, Γ-near-ring, operator near-rings (left and right).

define the mapping $[x, \gamma]$ by $[x, \gamma]y = x\gamma y$ $(y \in M)$. Let L be the sub-near-ring of $\mathscr{L}$ generated by the set $\{[x, \gamma]: x \in M, \gamma \in \Gamma\}$. L is called the left operator near-ring of M. This is a generalization of the concept for Γ-rings, but L does not, in general, consist exclusively of elements of the form $\sum_i [x_i, \gamma_i]$, $x_i \in M$, $\gamma_i \in \Gamma$, as is the case for Γ-rings.

The right operator near-ring R of M is similarly defined. Let $\mathscr{R}$ be the left near-ring of mappings of M into itself which act on the right. Then R is defined to be the sub-near-ring of $\mathscr{R}$ generated by the set $\{[\gamma, x]: \gamma \in \Gamma, x \in M\}$, where $y[\gamma, x] = y\gamma x$ for all $y \in M$. We will now provide a characterization for the elements of R.

PROPOSITION 1. *R is the set of all elements of the form*

$$\sum_i n_i[\gamma_i, x_i] \qquad (n_i \in \mathbf{Z}, \gamma_i \in \Gamma, x_i \in M).$$

PROOF. It is sufficient to show that the set S of all elements of the stated form is a sub-near-ring of $\mathscr{R}$. Clearly, S is a subgroup of the additive group of $\mathscr{R}$. Hence, we need only show that S is closed under multiplication. Firstly, it is easily verified that, if $m \in \mathbf{Z}$, $\gamma \in \Gamma$ $x, y \in M$, then

$$m(x\gamma y) = (mx)\gamma y. \tag{1}$$

Now let $\sum_i p_i[\alpha_i, a_i]$, $\sum_j q_j[\beta_j, b_j] \in S$. Then, if $x \in M$,

$$x(\sum_i p_i[\alpha_i, a_i])(\sum_j q_j[\beta_j, b_j]) = \sum_i p_i(x\alpha_i a_i)(\sum_j q_j[\beta_j, b_j]) =$$

$$= \sum_i (p_i x)\alpha_i a_i(\sum_j q_j[\beta_j, b_j]) = \qquad \text{by (1)}$$

$$= \sum_j q_j((\sum_i (p_i x)\alpha_i a_i)\beta_j b_j) =$$

$$= \sum_j (q_j(\sum_i (p_i x)\alpha_i a_i \beta_j b_j)) = \qquad \text{by (v) of the definition of } M$$

$$= \sum_j (q_j \sum_i p_i(x\alpha_i a_i \beta_j b_j)).$$

It follows that

$$(\sum_i p_i[\alpha_i, a_i])(\sum_j q_j[\beta_j, b_j]) = \sum_j q_j(\sum_i p_i[\alpha_i, a_i\beta_j b_j]) \tag{2}$$

which can be written in the form $\sum_i n_i[\gamma_i, x_i]$ as required.

PROPOSITION 2. *If $(M, +)$ is an abelian group, then R is an associative ring.*

PROOF. In this case, it is easily seen that the addition defined on R is commutative. Hence it remains only to show that the right distributive law holds for R. Let

$$\sum_i p_i[\alpha_i, a_i], \quad \sum_i q_i[\beta_i, b_i], \quad \sum_j n_j[\gamma_j, x_j] \in R.$$

Then

$$\Big(\sum_i p_i[\alpha_i, a_i]+\sum_i q_i[\beta_i, b_i]\Big)\sum_j n_j[\gamma_j, x_j] = \sum_j n_j\Big(\sum_i p_i[\alpha_i, a_i\gamma_j x_j]+\sum_i q_i[\beta_i, b_i\gamma_j x_j]\Big)$$

by (2) in Proposition 1. By the commutativity of addition in R, this last expression may be written in the form:

$$\sum_j n_j\Big(\sum_i p_i[\alpha_i, a_i\gamma_j x_j]\Big)+\sum_j n_j\Big(\sum_i q_i[\beta_i, b_i\gamma_j x_j]\Big) =$$
$$= \sum_i p_i[\alpha_i, a_i]\sum_j n_j[\gamma_j, x_j]+\sum_i q_i[\beta_i, b_i]\sum_j n_j[\gamma_j, x_j],$$

by (2), as required.

3. Unities of Γ-near-rings

Throughout this section, let R and L denote, respectively, the right and left operator near-rings of M.

M said to have a right (left) unity if there exist $d_1, \dots, d_n \in M$ and $\delta_1, \dots, \delta_n \in \Gamma$ such that, for all $x\in M$, $\sum_i x\delta_i d_i = x$ $(\sum d_i\delta_i x = x)$. M is said to have a strong right (left) unity if there exist $d\in M$, $\delta\in\Gamma$ such that $x\delta d = x$ $(d\delta x = x)$ for all $x\in M$. Suppose that $A\subseteq M$. Then we define

$$A^{*\prime} = \{r\in R : Mr \subseteq A\}$$

and

$$A^{+\prime} = \{l\in L : lM \subseteq A\}.$$

Furthermore, if $B\subseteq R$ and $C\subseteq L$, then

$$B^{*} = \{x\in M : [\gamma, x]\in B \quad \text{for all} \quad \gamma\in\Gamma\}$$

and

$$C^{+} = \{x\in M : [x, \gamma]\in C \quad \text{for all} \quad \gamma\in\Gamma\}.$$

PROPOSITION 3. *Suppose that M has a right unity and a strong left unity. Then the mapping $I\to I^{+\prime}$ defines an isomorphism of the lattice of right ideals of M onto the lattice of right ideals of L.*

PROOF. Suppose that I is a right ideal of M. Let $i, j\in I^{+\prime}$. If $x\in M$, $(i-j)x = ix-jx\in I$. Furthermore, if $l\in L$, then $(l+i-l)x = lx+ix-lx\in I$, since $ix\in I$ and I is a normal divisor of M. Hence $I^{+\prime}$ is a normal divisor of L. Moreover, $(il)x = i(lx)\in I$ since $i\in I^{+\prime}$. Hence, $I^{+\prime}$ is a right ideal of L. Now it is easily verified that

$$(I^{+\prime})^{+} = \{x\in M : x\gamma y\in I \quad \text{for all} \quad y\in M, \gamma\in\Gamma\}.$$

Since I is a right ideal of M, it is clear that $I\subseteq (I^{+\prime})^{+}$. Now suppose that $\delta_i\in\Gamma$ and $d_i\in M$ are such that $\sum_i x\delta_i d_i = x$ for all $x\in M$. Then, if $x\in(I^{+\prime})^{+}$, $\sum_i x\delta_i d_i\in I$, i.e. $x\in I$. Hence, $I=(I^{+\prime})^{+}$.

Now suppose that J is a right ideal of L. Let $x, y\in J^{+}$, $\gamma\in\Gamma$. Then if $z\in M$, $[x-y, \gamma]z = (x-y)\gamma z = x\gamma z - y\gamma z = ([x, \gamma]-[y, \gamma])z$. Hence $[x-y, \gamma] = [x, \gamma]-[y, \gamma]\in J$, whence $x-y\in J^{+}$. Now suppose $u\in M$ and $x\in J^{+}$. Then $[u+x-u, \gamma] = [u, \gamma]+$

$+[x, \gamma]-[u, \gamma]\in J$, since J is a normal divisor of M. Hence J^+ is a normal divisor of M. For $x\in J^+$, $z\in M$, $\gamma, \mu\in\Gamma$, $[x\gamma z, \mu]=[x, \gamma][z, \mu]\in J$. It follows that J^+ is a right ideal of M. It is easily verified that $(J^+)^{+\prime}=\{l\in L\colon l[x, \gamma]\in J$ for all $x\in M$, $\gamma\in\Gamma\}$. Since J is a right ideal of L, it is clear that $J\subseteq(J^+)^{+\prime}$. Now let $e\in M$ and $\varepsilon\in\Gamma$ be such that $e\varepsilon x=x$ for all $x\in M$. Then if $l\in(J^+)^{+\prime}$, $l[e, \varepsilon]\in J$. But if $z\in M$, $l[e, \varepsilon]z=$ $=l(e\varepsilon z)=lz$. Hence, $l=l[e, \varepsilon]\in(J^+)^{+\prime}$. Thus, $J=(J^+)^{+\prime}$, and the proof is complete.

Lemma 4. *If I is a left ideal of M, $l\in L$ and $x, y\in M$ are such that $x+I=y+I$, then $lx+I=ly+I$.*

Proof. Let $z\in M$, $\gamma\in\Gamma$. Then

$$[z, \gamma]x-[z, \gamma]y = z\gamma x-z\gamma y = z\gamma((x-y)+y)-z\gamma y\in I,$$

since I is a left ideal of M and $x-y\in I$. Hence

$$[z, \gamma]x+I = [z, \gamma]y+I.$$

Since L is the near-ring generated by elements of the form $[z, \gamma]$, it follows that $lx+I=ly+I$ for all $l\in L$.

Proposition 5. *Suppose that M has a right unity and a strong left unity. Then the mapping $A\to A^{+\prime}$ defines an isomorphism between the lattices of two-sided ideals of M and L.*

Proof. Let A be an ideal of M. Then by Proposition 3, $A^{+\prime}$ is a right ideal of L and $(A^{+\prime})^+=A$. Let $l, l'\in L$ and $a\in A^{+\prime}$. If $x\in M$, $(a+l')x=ax+l'x=l'x+A$, since $ax\in A$. By Lemma 4, $l(a+l')x+A=ll'x+A$, i.e. $(l(a+l')-ll')x\in A$. Hence $l(a+l')-ll'\in A^{+\prime}$, so $A^{+\prime}$ is a two-sided ideal of L.

Suppose that B is a two-sided ideal of L. Then, by Proposition 3, B^+ is a left ideal of M, and $(B^+)^{+\prime}=B$. Let $b\in B^+$, $x, y\in M$, $\gamma, \mu\in\Gamma$.

$$[x\gamma(b+y)-x\gamma y, \mu] = [x\gamma(b+y), \mu]-[x\gamma y, \mu] = [x, \gamma]([b, \mu]+[y, \mu])-[x, \gamma][y, \mu]\in B,$$

since B is a left ideal of L and $[b, \mu]\in B$. Hence, $x\gamma(b+y)-x\gamma y\in B^+$, so B^+ is a two-sided ideal of M. This completes the proof.

Remark. Clearly, Proposition 3 and 5 extend Theorems 1 and 2, respectively, of [1] to Γ-near-rings. The proofs of both Propositions make use of the identity $[x+y, \gamma]=[x, \gamma]+[y, \gamma]$, which is a consequence of the right distributivity of M. Because of the lack of left distributivity of M, relationships between M and R corresponding to Propositions 3 and 5 are unlikely to hold without imposing additional conditions on M.

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(Received December 12, 1985)

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HU ISSN 0081-6906

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