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FAMILIES OF FINITE SETS SATISFYING UNION RESTRICTIONS

by

P. FRANKL

1. Introduction

Let X be a finite set of cardinality n. Let $\mathcal{F}$ be a family of subsets of X such that for $F_1, \ldots, F_k; G_1, \ldots, G_s \in \mathcal{F}$ we have

$$\left|\bigcup_{i=1}^{k} F_i\right| \leqq m \tag{1}$$

$$\left|\bigcap_{i=1}^{s} G_i\right| \geqq r \tag{2}$$

(k, s, m, r are integers, $m \geqq r$).

What is the maximal size of $\mathcal{F}$?

This problem was solved for $k=2$, $r=0$ by GY. KATONA [1] and for $k=2$, $s=2$, $r=1$ by P. FRANKL [2].

The aim of this paper is to solve this problem for $s=2$, k is arbitrary, m is small with respect to n.

Let $m=hk+t$ $r \leqq t < r+k$. Let us define the following function:

$$f(r,t) = \begin{cases} 2^t & \text{if } r=0 \\ \displaystyle\sum_{i=\frac{t+r}{2}}^{t} \binom{t}{i} & \text{if } r>0,\ t+r \text{ is even} \\ \displaystyle\sum_{i=\frac{t+r-1}{2}}^{t-1} 2\binom{t-1}{i} & \text{if } r>0,\ t+r \text{ is odd} \end{cases}.$$

We prove the following

THEOREM. *If $\mathcal{F}$ satisfies conditions* (1) *and* (2), $s=2$ *and* $n>n_0(m)$ *then*

$$|\mathcal{F}| \leqq \sum_{i=0}^{h} \binom{n-t}{i} f(r,t). \tag{3}$$

We determine the systems for which equality holds in (3). *These families are unique up to isomorphisms unless* $r=1$.

MAGYAR TUDOMÁNYOS AKADÉMIA

2. The proof of the theorem

We apply induction on k and m. Is $k=1$, the statement of the theorem follows from ERDŐS—KO—RADO [3]. It follows from [3], too, that if we have equality in (3) then there exists $Y\subset X$, $|Y|=r$ such that

$$\mathcal{F}=\{F\subset X\,|\,|F|\leqq m,\ Y\subset F\}.$$

Let us suppose that the theorem is proven for any pairs k', m' whenever $k'<k$ or $k'=k$ but $m'<m$. Let $\mathcal{F}$ be a family of subsets of X satisfying (1) and (2) and of maximal cardinality. Hence $|\mathcal{F}|$ is not less than the right-hand side of (3). As the cardinality of the extremal families is strictly increasing in m so there are sets $F_1^0,\ldots,F_k^0\in\mathcal{F}$ such that

$$\left|\bigcup_{i=1}^{k}F_i^0\right|=m.$$

If there are sets $F_1,\ldots,F_{k-1}\in\mathcal{F}$ such that $\left|\bigcup_{i=1}^{k-1}F_i\right|>m-h$, then by (1) for every set $F\in\mathcal{F}$

$$\left|F\cap\left(X-\left(\bigcup_{i=1}^{k-1}F_i\right)\right)\right|\leqq h-1 \tag{4}$$

holds. As a consequence we have

$$|\mathcal{F}|\leqq 2^{\left|\bigcup_{i=1}^{k-1}F_i\right|}\sum_{j=1}^{h-1}\binom{n-\left|\bigcup_{i=1}^{k-1}F_i\right|}{j}<2^m\sum_{j=0}^{h-1}\binom{n}{j}<\sum_{j=0}^{h}\binom{n-t}{j}f(r,t)$$

for sufficiently large values of n (the degree of the polynomial on the right hand side is greater than of the polynomial on the left hand side). Hence for $F_1,\ldots,F_{k-1}\in\mathcal{F}$ we have

$$\left|\bigcup_{i=1}^{k-1}F_i\right|\leqq m-h. \tag{5}$$

The statement of the theorem follows by induction from (5) unless $t=r+k-1$, what we may assume from now on. Let us set

$$Z_0=\bigcup_{i=1}^{k}F_i^0.$$

Then the sets $G_i=Z_0-\bigcup_{j\neq i}F_j^0$ are pairwise disjoint. It follows:

$$|Z_0|=m\geqq\sum_{i=1}^{k}|G_i|+|F_1^0\cap F_2^0|. \tag{6}$$

Using (2) we obtain from (6) that for at least one value of j $|G_j|\leqq h$. By symmetry reasons we assume that

$$|G_k|\leqq h. \tag{7}$$

Let F be any set belonging to $\mathcal{F}$. Then we have

$$\left|\left(\bigcup_{i=1}^{k-1} F_i^0\right)\cup F\right| \geqq |Z_0-G_k|+|(X-Z_0)\cap F| \geqq m-h+|(X-Z_0)\cap F|.$$

Hence we have for any $F\in\mathcal{F}$

$$|F\cap(X-Z_0)| \leqq h. \tag{8}$$

Let Z be an arbitrary subset of X containing Z_0. By (8), if $F_1\in\mathcal{F}$ then $|F_1\cap(X-Z)|\leqq h$ and if $|F_1\cap(X-Z)|=h$ so we have $F_1\cap(Z-Z_0)=\Phi$. Let us further suppose

$$|Z| \leqq m+h^2k. \tag{9}$$

We pick out elements of $\mathcal{F}$ satisfying

(i) $\left|F_i\cap\left(X-\left(Z\cup\bigcup_{j<i}F_j\right)\right)\right| \geqq h$,

(ii) $\left|F_l\cap\left(Z_0-\bigcup_{j<i}F_j\right)\right|$ is maximal among all F_i's satisfying (i).

By (2) if F_1 exists then we have

$$|F_1\cap Z_0| \geqq r.$$

Hence if

$$\left|\bigcup_{i=1}^{k} F_i\cap Z_0\right| < t = r+k-1, \tag{10}$$

then by (ii)

$$\left|\bigcup_{i=1}^{k-1} F_i\cap Z_0\right| = \left|\bigcup_{i=1}^{k} F_i\cap Z_0\right|,$$

i.e. for every F either

$$\left|F\cap\left(X-\left(Z\cup\bigcup_{i=1}^{k-1}F_i\right)\right)\right| \leqq h-1 \tag{11a}$$

or

$$F\cap Z_0\subset\bigcup_{i=1}^{k-1}F_i\cap Z_0 = Y_0, \quad |Y_0| \leqq t-1. \tag{11b}$$

The number of F's satisfying (11a) is majorated by

$$\sum_{i=1}^{h}\binom{n-m}{h-i}\sum_{j=0}^{i}\binom{(h+1)hk}{j}2^m \tag{12}$$

as any set is determined by its intersection with Z_0, $\bigcup_{i=1}^{k-1}F_i\cup Z$, and $X-\left(Z\cup\bigcup_{i=1}^{k-1}F_i\right)$.

If we cannot pick out elements of $\mathcal{F}$ satisfying (i) and (ii) at least k times then we get in the same way that $|\mathcal{F}|$ is majorated by (12). But for large values of n (12), being a polynomial of degree $h-1$ in n, is much smaller than the right-hand side of (3).

We prove that (10) is impossible as well. For this purpose we estimate the number of F's satisfying (11b) and violating (11a). Let us denote the set of such F's by G.

For every $G\in\mathcal{G}$

$$\left|G\cap\left(X-\left(Z\cup\bigcup_{i=1}^{k-1}F_i\right)\right)\right|=h,\quad G\cap\left(Z\cup\bigcup_{i=1}^{k-1}F_i\right)\subset G\cap Z_0\subset\bigcup_{i=1}^{k-1}F_i\cap Z_0=Y_0$$

hold.

Let us choose $G_1\in\mathcal{G}$ in such a way that the number of G's satisfying

$$G_1\cap(X-Y_0)=G\cap(X-Y_0),\quad G\in\mathcal{G}$$

is maximal. We choose G_2 satisfying the same conditions and moreover $|G_2\cap(X-(Y_0\cup G_1))|=h$. The number of F's satisfying

$$\left|F\cap\left(X-\left(G_1\cup Z\cup\bigcup_{j=1}^{k-1}F_j\right)\right)\right|\leqq h-1$$

is majorated by

$$\sum_{i=1}^{h}\binom{n-m}{h-i}\sum_{j=0}^{i}\binom{(h+2)hk}{j}2^m. \tag{13}$$

For $i=1, 2$, let us define

$$\mathcal{A}_i=\{F\cap Y_0\,|\,G_i\cap(X-Y_0)=F\cap(X-Y_0)\}.$$

Now we make use of the following Theorem.

Let $\mathcal{A}_1$, $\mathcal{A}_2$ be families of subsets of the t-element set Y. Let us suppose that for $A_1\in\mathcal{A}_1$, $A_2\in\mathcal{A}_2$ $|A_1\cap A_2|\geqq r$.

Then either $|\mathcal{A}_1|\leqq f(r,t)$ or $|\mathcal{A}_2|<f(r,t)$. (It is an easy consequence of theorem 1 of R. AHLSWEDE—G. KATONA [4].)

By this we have

$$|\mathcal{A}_2|\leqq f(r,t-1)\leqq\frac{1}{2}f(r,t). \tag{14}$$

Hence we have

$$|\mathcal{F}|\leqq\sum_{i=1}^{h}\binom{n-m}{h-i}\sum_{j=0}^{i}\binom{(h+2)hk}{j}2^m+\binom{n-m}{h}\frac{1}{2}f(r,t). \tag{15}$$

For $n>(h+1)^3k2^{m+1}$ (15) implies $|\mathcal{F}|<\binom{n-t}{h}f(r,t)$ which contradicts the maximality of $|\mathcal{F}|$.

Hence (10) does not hold.

Let us apply the construction starting with $Z=Z_0$ and choosing the sets $F_1^1, F_2^1, \ldots, F_k^1$. If we have already chosen the sets $F_1^j, \ldots, F_k^j$ then let us set $Z_j=\bigcup_{i=0}^{j}\bigcup_{q=1}^{k}F_q^i$ and choose the sets F_q^{j+1} $q=1, \ldots, k$. So we construct F_q^j for $0\leqq j\leqq h+1$; $q=1, \ldots, k$. Let us define

$$Y_i=Z_0\cap\left(\bigcup_{j=1}^{k}F_j^i\right),\quad i=1, \ldots, h+1,$$

$$U_q=Z_0\cap F_q^1,\quad q=1, \ldots, k,$$

$$V=Z_0\cap\left(\bigcup_{j=1}^{k-1}F_j^1\right).$$

We want to show

$$Y_1 = Y_j \quad j = 2, \ldots, h+1.$$

If $|U_1|>r$ then $V=Y_1$, as otherwise we would have $U_1 \subsetneqq Z_0 \cap (F_1^1 \cup F_1^2) \subsetneqq \ldots \subset V \subset Y_1$ and as a consequence $|Y_1| \geqq |U_1|+k-1 > r+k-1 = t$ which is impossible because

$$\left| \bigcup_{j=1}^{k} F_j^1 \right| = |Y_1| + hk \leqq m.$$

If we have $V=Y_1$ but for some j $Y_j \neq Y_1$ then we can find an F_q^j such that $F_q^j \cap Z_0 \subsetneqq Y_1$ and consequently we should have chosen F_q^j instead of F_k^1.

If $|U_1|=r$ then for $q=2, \ldots, k$ $|U_q|=r$ holds as well because of the choice of F_1^1.

By (2) it follows $U_i \neq F_j^0$, $i, j=1, 2, \ldots, k$ which entails $\bigcup_{i=1}^{k} U_i = Y_1 \subset F_j^0$, $j=1, \ldots, k$ and by (5) $|G_j|=h$, $j=1, \ldots, k$.

Let F be an arbitrary member of $\mathscr{F}$. If $|F \cap Y_1| = r-p$ $(p>0)$ then by (2) $|F \cap G_j| \geqq p$ for $j=1, \ldots, k$ and consequently

$$|F \cap Z_0| \geqq r+(k-1)p > r. \tag{16}$$

By (16) we have $F_q^j \cap Z_0 \subseteq Y_1$, $j=1, \ldots, h+1$, $q=1, \ldots, k$, and $Y_j = Y_1$ follows from $|Y_j| = t = |Y_1|$ (for $j=2, \ldots, h+1$).

We prove now that for every $F \in \mathscr{F}$ we have

$$|F \cap (X-Y_1)| \leqq h. \tag{17}$$

Let us suppose that for some F (17) does not hold. Then either there is a j, $1 \leqq j \leqq h+1$ such that $|F \cap (X - \bigcup_{q=1} F_q^j)| \geqq h+1$ or for $j=1, \ldots, h+1$, $F \cap \left(\bigcup_{q=1}^{k} F_q^j - Y_1 \right) \neq \emptyset$ implying $\left| F \cap \left(X - \bigcup_{q=1}^{k} F_q^0 \right) \right| \geqq h+1$.

Then with the appropriate j we have $\left| \bigcup_{q=1}^{k-1} F_q^j \cup F \right| > m$, contradicting (1) whence (17) follows.

For $G \subset X - Y_1$ let us define

$$\mathscr{A}_G = \{A \subset Y_1 \mid G \cup A \in \mathscr{F}\}.$$

Let us first suppose that for some G_1 $|\mathscr{A}_{G_1}| > f(r, t)$. Then by a theorem of AHLSWEDE—KATONA [4] for every $G_2 \subset (X-(G_1 \cup Y_1))$ we have

$$|\mathscr{A}_{G_2}| \leqq f(r, t)-1. \tag{18}$$

Using (18) we obtain

$$|\mathscr{F}| \leqq \sum_{i=1}^{h} \binom{n-t}{i} (f(r, t)-1) + 2^{t+2h} \sum_{i=1}^{h-1} \binom{n-t}{i} < \sum_{i=1}^{h} \binom{n-t}{i} f(r, t)$$

for $n > (h+1)^3 k 2^{m+1}$. Hence for any $G \subset (X-Y_1)$ we have

$$|\mathscr{A}_G| \leqq f(r, t). \tag{19}$$

(17) and (19) imply (3). If we have equality in (3) then we have equality in (19) for every subset G of $X-Y_1$ of cardinality less than or equal to h. Let us suppose $h\geqq 1$, then we have equality in (19) for the 1-element subsets of $X-Y_1$. We wish to prove

$$\left|\bigcup_{x\in X-Y_1}\mathcal{A}_{\{x\}}\right| = f(r,t).$$

Otherwise as $n\gg f(r,t)$ we could find elements $x_1, x_2, \ldots, x_{f(r,t)+1}\in X-Y_1$ and distinct sets $A_1, \ldots, A_{f(r,t)+1}$; $A_i\in\mathcal{A}_{x_i}$. But by (2) $|A_i\cap A_j|\geqq r$ contradicting Theorem 4 of Gy. Katona [1], which implies that even in the case of equality for $r\geqq 2$ we must have

$$\mathcal{A} = \mathcal{A}_{\{x\}} = \ldots = \mathcal{A}_{\{x_{f(r,t)+1}\}} = \left\{A\subset Y_1 \middle| |A| \geqq \frac{r+t}{2}\right\}$$

for $r+t$ is even and

$$\mathcal{A} = \mathcal{A}_{\{x\}} = \ldots = \mathcal{A}_{\{x_{f(r,t)+1}\}} = \left\{A\subset Y_1 \middle| |A\cap(Y_1-y)| \geqq \frac{r+t-1}{2}\right\}$$

for some $y\in Y_1$ for $r+t$ is odd.

Let G be an arbitrary subset of $X-Y_1$ then it follows easily from (2) that $\mathcal{A}_G\subset\mathcal{A}_{\{x_1\}}$.

Hence in the case of equality we have

$$\mathcal{A}_G = \begin{cases}\emptyset & \text{if } |G|>h,\\ \mathcal{A}_{\{x_1\}} & \text{if } |G|\leqq h.\end{cases}$$

If $h=0$ then $\mathcal{F}=\mathcal{A}_0$ and Katona's result directly applies to yield

$$\mathcal{F} = \mathcal{A}.$$

In the case $r=0$ it follows directly from (17) that

$$|\mathcal{F}| < \sum_{i=0}^{h}\binom{n-t}{i}2^t$$

unless

$$\mathcal{F} = \left\{F\subset X \middle| |F\cap(X-Y_1)| \leqq h_1\right\}$$

q. e. d.

REMARK. The same method applies to the case $s\geqq 3$, and yields that for n sufficiently large and $\mathcal{F}$ optimal we can find a subset Y_1 of X such that (17) holds for every $F\in\mathcal{F}$.

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A TOPOLOGICAL APPROACH TO MINIMIZING $\int_a^b f(x, y, y')\,dx$ IN SUBSPACES OF THE PLANE

by

H. W. MILNES and S. K. HILDEBRAND

In this paper we depart from the conventional methods of the differential calculus in the theory of the calculus of variations, and we apply instead techniques which are topological and depend upon measure theory. This has permitted us to treat minimizing problems in which the minimizing arc is not necessarily an element of $C^{(3)}$, or even $C^{(1)}$. It is no longer essential that the minimizing arc be imbedded in an open region, and the subspace of $E^{(2)}$ over which the minimizing problem occurs can be quite arbitrary. In addition we have been able to relax the requirement that the defining functional to be minimized necessarily involves a piecewise continuous derivative of the minimizing arc. We have succeeded in obtaining sufficient conditions for the existence of a minimizing arc which are entirely topological in nature.

Let H denote an arbitrary subset of the Euclidean plane $E^{(2)}$. The fixed endpoint problem will be considered for which $p_1 \equiv (x_1, y_1)$ and $p_2 \equiv (x_2, y_2) \in H$ are the endpoints and $-\infty < x_1 < x_2 < \infty$. Let $A \equiv \{y = y(x)\}$ denote the class of functions for which the graph lies in H, which are of class $C^{(1)}$, and for which $y(x_1) = y_1$, $y(x_2) = y_2$. A function $f \equiv f(\alpha, \beta, \gamma)$ of three variables will be considered with the properties:

(i) $f(x, y(x), \gamma) \geqq 1$ for $y = y(x) \in A$ and for γ satisfying $y'(x) - a < \gamma < y'(x) + a$ for some constant $a > 0$;

(ii) $f(x, y(x), \gamma)$ is uniformly continuous for values of x satisfying $x_1 \leqq x \leqq x_2$, $y = y(x) \in A$ and for γ satisfying $y'(x) - a < \gamma < y'(x) + a$ for some constant $a > 0$.

Initially, the problem to be considered is that of minimizing the functional:

$$J[y] = \int_{x_1}^{x_2} f(x, y, y')\,dx \tag{1}$$

among that subclass of A for which this integral exists. This subclass will be referred to as the admissible class of this functional. Subsequently, the differentiability condition on $y(x)$ will be relaxed and the minimum sought in the class of all continuous functions joining p_1 to p_2. Since $J[y]$ is not defined for this wider admissible class an appropriate generalization of (1) will be introduced at that time.

It may be remarked that the condition (i) on f can be met by all functions f which have a finite lower bound m on this set, simply by replacing f with $1 - m + f$ which has the same minimizing arcs as the original function.

The following definitions are basic to the subsequent analysis. If $X \subseteq H$ is any subset, $p \equiv (r, s)$, $q \equiv (x, y) \in X$ then $S_0(p; X) \equiv p$, and for $a > 0$:

$$S_a(p; X) \equiv p \cup \left\{ q \mid f\left(r, s, \frac{y-s}{x-r}\right)(x-r) < a,\ x > r \right\}. \tag{2}$$

The set $T_a(p; X)$ is the arcwise connected component of p in $S_a(p; X)$; the set

$$T_a^{-1}(p; X) \equiv \{q \mid p \in T_a(q; X)\}. \tag{3}$$

Let $c > 0$ be any given number and suppose that c is finitely partitioned as $c = a_1 + a_2 + \ldots + a_k$, $a_i > 0$, $\max_{1 \leqq i \leqq k} a_i < \eta$ where η is said to be the norm of the partition. A recursive definition is made according to the relation:

$$T_{a_1, a_2, \ldots, a_{i+1}}(p; X) = T_{a_1, a_2, \ldots, a_i}(p, X) \cup \Big[\bigcup_{q \in T_{a_1, a_2, \ldots, a_i}(p; X)} T_{a_{i+1}}(q; X) \Big] \tag{4}$$

$i = 1, 2, \ldots, (k-1)$, with this set being simply $T_{a_1}(p; X)$ when $i = 0$. Similarly, $T^{-1}_{a_1, a_2, \ldots, a_{i+1}}(p; X)$ is defined by:

$$T^{-1}_{a_1, a_2, \ldots, a_{i+1}}(p; X) = T^{-1}_{a_1, a_2, \ldots, a_i}(p; X) \cup \Big[\bigcup_{q \in T^{-1}_{a_1, a_2, \ldots, a_i}(p; X)} T^{-1}_{a_{i+1}}(q; X) \Big] \tag{5}$$

$i = 1, 2, \ldots, (k-1)$, with this set being simply $T_{a_1}^{-1}(p; X)$ when $i = 0$.

DEFINITION. A *c-vicinity* of the point p, written $V_c(p; X)$, is defined as follows:
(i) when $c = 0$, as $V_0(p; X) \equiv p$;
(ii) when $c > 0$, as the set of points r such that for every $\varepsilon > 0$ there exists a finite partition: $a_1 + a_2 + \ldots + a_k = c$ of norm ε, and a finite set of positive numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ such that $\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n \leqq \varepsilon$ with the property that:

$$T_{a_1, a_2, \ldots, a_k}(p; X) \cap T^{-1}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n}(r; X) \neq \varnothing. \tag{6}$$

DEFINITION. The *inverse c-vicinity,* written $V_c^{-1}(p; X)$ is the set of all points q such that $p \in V_c(q; X)$.

When the set X is the space H, we shall subsequently drop its designation, and write simply $S_a(p)$, $T_a(p)$, $V_c(p)$, etc.

As an example, one may consider $J(y) = \int_{x_1}^{x_2} (1 + y'^2)\,dx$ with $p_1 \equiv (-1, 0)$, $p_2 \equiv (1, 0)$ and H the plane E^2 with the line segment $x = 0$, $-1 < y < 1$ removed. Some elementary calculations show that for $p \equiv (r, s)$, the sets $S_a(p)$ are the point p together with the points of H which are interior to a circle of radius $a/2$ centered at $(r + a/2, s)$. The sets $T_a(p)$ are identical with $S_a(p)$ except when the deleted line segment completely cuts away a portion of the circle, in which case the circular segment connected to p is $T_a(p)$. If the deleted line segment only partially enters $S_a(p)$, then $S_a(p)$ and $T_a(p)$ are the same. Remote from the deleted line segment, the sets $V_c(p)$ are identical with $S_a(p)$ and $T_a(p)$, but close to the deleted line segment some pertinent cases are illustrated below.

As a second example, if the function $f=y'^2$ is considered, which does not satisfy the criterion $f\geqq 1$, then the c-vicinity will become unbounded in the x-direction in certain instances.

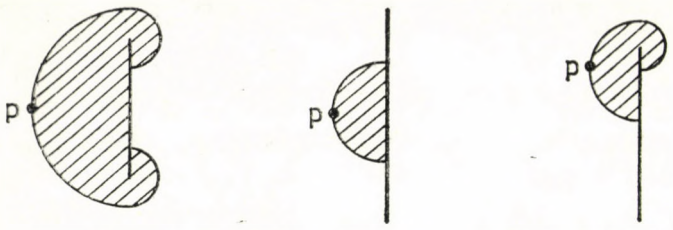

Figure 1

LEMMA 1. $V_c(p; X)\subset V_{c+x}(p; X)$ *and* $V_c^{-1}(p; X)\subset V_{c+x}^{-1}(p; X)$, $x\geqq 0$.

PROOF. Follows immediately from the definitions.

LEMMA 2. *Suppose that:* $\{t_i|i=1, 2, \ldots\}$ *is a monotonic non-increasing sequence tending to the limit* t_0, *and that* $q\in V_{t_i}(p; X)$ *for all* i *then* $q\in V_{t_0}(p; X)$. *Also if* $q\in V_{t_i}^{-1}(p; X)$, *then* $q\in V_{t_0}^{-1}(p; X)$.

PROOF. Let $\varepsilon>0$ be given, and $\varepsilon'=\varepsilon/4$. By lemma 1, $q\in V_t(p; X)$ if $t>t_0$. Using ε' as the norm, there exist partitions $a_1+a_2+\ldots+a_m=t$, $\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_n\leqq\varepsilon'$ for which

$$T_{a_1, a_2, \ldots, a_m}(p; X)\cap T^{-1}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n}(q; X)\neq\varnothing. \tag{7}$$

From the definitions of the sets T and T^{-1} it then follows that

$$T_{a_1, a_2, \ldots, a_{m-1}}(p; X)\cap T^{-1}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, a_m}(q; X)\neq\varnothing \tag{8}$$

and by repetition:

$$T_{a_1, a_2, \ldots, a_k}(p; X)\cap T^{-1}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, a_m, \ldots, a_{k+1}}(q; X)\neq\varnothing \tag{9}$$

for any $k\geqq 0$. Let $t_0<t<t_0+\varepsilon'$ and choose k so that $a_1+\ldots+a_k=\tau$ where $t_0-\varepsilon'<\tau\leqq t_0$. Noting that $t_0-\tau<\varepsilon'$ and $t-t_0<\varepsilon'$, so that $t-\tau=(t-t_0)+(t_0-\tau)\leqq 2\varepsilon'=\varepsilon/2$ and $(\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_n)+(a_m+\ldots+a_{k+1})\leqq\varepsilon/2+\varepsilon'\leqq\varepsilon$, it is seen from (9) that:

$$T_{a_1, a_2, \ldots, a_k, b}(p; X)\cap T^{-1}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, a_m, \ldots, a_{k+1}}(q; X)\neq\varnothing \tag{10}$$

where $b=t_0-\tau$. Thus, $q\in V_{t_0}(p; X)$.

COROLLARY. $V_c(p; X)=\bigcap_{\varepsilon>0} V_{c+\varepsilon}(p; X)$.

The proof follows immediately from lemmas 1 and 2.

Theorems 1 to 6 that follow indicate relationships between $J[y]$, taken in the Riemann sense, on the finite $I\equiv[x_1, x_2]$ and the c-vicinities and other associated sets we have just defined. In these theorems the function $y=y(x)$ is assumed to be a differentiable arc between its endpoints and to have finite right- and left-hand derivatives at p_1 and p_2 respectively.

THEOREM 1. *If $y(x)$ is an arc of class $C^{(1)}$ connecting p_1 to p_2 and $J[y]$ is defined as a finite quantity: c, then $p_2 \in V_c(p_1)$.*

PROOF. If $c=0$ the theorem is trivial since $f(\alpha, \beta, \gamma) \geqq 1$. If $c>0$, since $J[y]=c$ exists, for each $j=1, 2, \ldots$ there exist points $x_0, x_1, x_2, \ldots, x_j$ such that $c/j = \int_{x_i}^{x_{i+1}} f(x, y(x), y'(x))dx$ for $i=0, 1, \ldots, j-1$. We note $f(\alpha, \beta, \gamma) \geqq 1$ on $y=y(x)$. For each j, let $z_1, z_2, \ldots, z_j$ satisfy $x_i < z_{i+1} \leqq x_{i+1}$. By the uniform continuity of f and the mean value theorem for integrals, there exist points $z_0^*, z_1^*, \ldots, z_{j-1}^*$ such that:

$$c/j = \int_{x_i}^{x_{i+1}} f(x, y(x), y'(x))\, dx \geqq \int_{x_i}^{z_{i+1}} f(x, y(x), y'(x))\, dx$$

$$= f(z_i^*, y(z_i^*), y'(z_i^*))(z_{i+1}-x_i) \geqq (z_{i+1}-x_i) \tag{11}$$

where $x_i < z_i^* < z_{i+1} \leqq x_{i+1}$. Therefore $(z_{i+1}-x_i) \leqq c/j$.

There exist points $\bar{z}_0, \bar{z}_1, \ldots, \bar{z}_{j-1}$ such that

$$y'(\bar{z}_i) = \frac{y(z_{i+1})-y(x_i)}{z_{i+1}-x_i}, \quad i = 0, 1, \ldots, (j-1) \tag{12}$$

where $x_i < \bar{z}_i < z_{i+1} \leqq x_{i+1}$, since $y(x)$ is of class $C^{(1)}$. Therefore

$$f\left(x_i, y(x_i), \frac{y(z_{i+1})-y(x_i)}{z_{i+1}-x_i}\right) = f(x_i, y(x_i), y'(\bar{z}_i)). \tag{13}$$

Let $\varepsilon>0$ be given, then by property (ii) of f there exists a $\delta>0$ such that if $|z_{i+1}-x_i|<\delta$, $i=0, 1, 2, \ldots$ then;

$$f(x_i, y(x_i), y'(\bar{z}_i)) - \varepsilon/c \leqq f(z_i^*, y(z_i^*), y'(z_i^*)) \tag{14}$$

since $\bar{z}_i$ and z_i^* are elements of (x_i, z_{i+1}). By (11), it follows that $\lim_{j\to\infty} |z_{i+1}-x_i| = 0$, even for $z_{i+1}=x_{i+1}$; hence, select j so that $|x_{i+1}-x_i|<\delta$ as well as $|z_{i+1}-x_i|<\delta$. By (11) and (14)

$$c/j \geqq [f(x_i, y(x_i), y'(\bar{z}_i)) - \varepsilon/c](z_{i+1}-x_i) \tag{15}$$

$$\geqq f(x_i, y(x_i), y'(\bar{z}_i))(z_{i+1}-x_i) - \frac{\varepsilon}{c}\cdot\frac{c}{j}$$

$$= f(x_i, y(x_i), y'(\bar{z}_i))(z_{i+1}-x_i) - \varepsilon/j.$$

Therefore: $(c+\varepsilon)/j \geqq f(x_i, y(x_i), y'(\bar{z}_i))(z_{i+1}-x_i)$ and by (12) $(z_{i+1}, y(z_{i+1})) \in S_{\frac{c+\varepsilon}{j}}(x_i, y(x_i))$ for all z_{i+1} where $x_i < z_{i+1} \leqq x_{i+1}$. Since $y=y(x)$ is a connected set, it follows that $(x_{i+1}, y(x_{i+1})) \in T_{\frac{c+\varepsilon}{j}}(x_i, y(x_i))$. Thus, by (4) $(x_1, y(x_1)) \in T_{\frac{c+\varepsilon}{j}}(p_1)$, $(x_2, y(x_2)) \in T_{\frac{c+\varepsilon}{j}, \frac{c+\varepsilon}{j}}(p_1)$, etc., and finally the endpoint $p_2 \in T_{\frac{c+\varepsilon}{j}, \ldots, \frac{c+\varepsilon}{j}}(p_1)$, for j repetitions of $(c+\varepsilon)/j$. Since this is true for any value of j, $p_2 \in V_{c+\varepsilon}(p_1)$ according to the definition of this set, and therefore by lemma 2, $p_2 \in V_c(p_1)$.

COROLLARY (1): *With the hypotheses of the above theorem* $p_1 \in V_c^{-1}(p_2)$.

PROOF. This follows directly from the definition of $V_c^{-1}(p_2)$.

COROLLARY (2): *If* Y *is the graph of* $y=y(x)$ *then with the hypotheses of the above theorem* $p_2 \in V_c(p_1; Y)$ *and* $p_1 \in V_c^{-1}(p_2; Y)$.

PROOF. It needs only be remarked that properties (i) and (ii) of f apply on Y when they apply in H. The proof is then identical with that of the main theorem and corollary (1) with the appropriate restrictions of the sets S, T and V.

REMARK 1. The use of condition (ii) on f in the proof of this theorem points out an analogy which exists between condition (ii) and the Weierstrass condition in the classical calculus of variations.

THEOREM 2. *If* $y=y(x)$ *is a differentiable arc connecting* p_1 *and* p_2 *with* $J[y]=c$, *and if* $q \equiv (x_3, y(x_3))$ *is any point of the arc with* $\int_{x_1}^{x_3} f(x, y, y')dx = a$, *then* $q \in V_a(p_1) \cap \cap V_{c-a}^{-1}(p_2)$.

PROOF. By Theorem 1, $q \in V_a(p_1)$. Since:

(16)
$$c = \int_{x_1}^{x_2} f(x, y, y')\,dx = \int_{x_1}^{x_3} f(x, y, y')\,dx + \int_{x_3}^{x_2} f(x, y, y')\,dx = a + \int_{x_3}^{x_2} f(x, y, y')\,dx$$

it follows by theorem 1 that $p_2 \in V_{c-a}(p_3)$. Therefore $p_3 \in V_{c-a}^{-1}(p_2)$ by corollary (1) to theorem 1.

COROLLARY. *Under the hypotheses of the above theorem* $q \in V_a(p; Y) \cap \cap V_{c-a}^{-1}(p_2; Y)$ *where* Y *is the graph of* $y=y(x)$.

PROOF. The proof is as that of the main theorem but with corollary (2) of theorem 1 replacing theorem 1.

REMARK 2. The set $V_a(p_1) \cap V_{c-a}^{-1}(p_2)$ need not always reduce to a single point even when there is a unique minimizing arc joining p_1 to p_2. The example of the function $f \equiv \sqrt{1+y'^2}$ with $p_1 \equiv (-2, 0)$ and $p_2 \equiv (2, 0)$ illustrates this when the space H is the plane with the interior of the unit circle at the origin together with its semicircular boundary below the x-axis removed.

REMARK 3. The set $V_c(p)$ need not be connected. As an example let $p_1 \equiv (0, 0)$ and $p_2 \equiv (2\pi+6, 6)$. The space, figure 2, consists of the line joining p_1 to $(6, 6)$, the line joining $(2\pi, 0)$ to p_2 as well as the portion of the sinusoidal curves $y = 6 - \frac{1}{2^{n-2}} + \frac{1}{2^n} \sin\left[2^n\left(x-6+\frac{1}{2^{n-2}}\right)\right]$ $(n=0, 1, 2, \ldots)$, included between these two lines. It should be noted that the length of the sinusoidal segments is constant and equal to the length of one period of $\sin x$. Using the arc length functional $f \equiv \sqrt{1+y'^2}$, the set $V_{6\sqrt{2}+2\pi}(p_1)$ consists of the left line and a portion of each of the sinusoidal arcs of length $2\pi + \sqrt{2}/2^{n-2}$, $(n=0, 1, 2, \ldots)$ together with p_2.

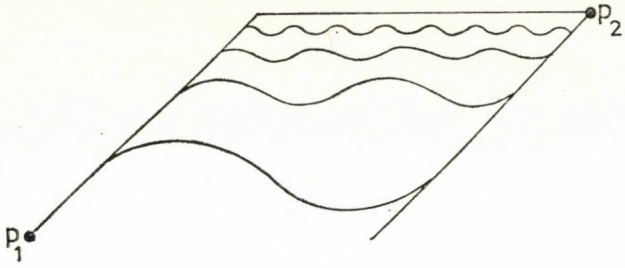

Figure 2

The purpose in introducing the previous definitions, in particular the c-vicinities of p, is to permit the definition of the σ-separation of two points relative to a subset X of H.

DEFINITION. If $p_1 \equiv (x_1, y_1)$, $p_2 \equiv (x_2, y_2)$ are points of a subset $X \subseteq H$, with $x_1 \leqq x_2$, the *σ-separation* of these points relative to X written $\sigma(p_1, p_2; X)$, is the quantity:

$$\sigma(p_1, p_2; X) = \inf \{c | p_2 \in V_c(p_1; X)\} \tag{17}$$

whenever this exists as a finite number.

It is especially to be noted that this function is defined only when $x_1 \leqq x_2$. Where $\sigma(p_1, p_2; X)$ is defined, it has some of the important properties of a metric as the following theorem shows.

THEOREM 3. *If* $p_1 \equiv (x_1, y_1)$, $p_2 \equiv (x_2, y_2)$ *and* $p_3 \equiv (x_3, y_3)$ *then*

(a) $\sigma(p_1, p_2; X) = 0$;

(b) $\sigma(p_1, p_2; X) > 0$ *if* $x_1 < x_2$ *and the quantity is defined;*

(c) $\sigma(p_1, p_3; X) \leqq \sigma(p_1, p_2; X) + \sigma(p_2, p_3; X)$ *if* $x_1 < x_2 < x_3$ *and the σ-separations are all defined.*

PROOF (a) Since $p_1 \in S_0(p_1; X) = T_0(p_1; X)$ it follows from the definition that $p_1 \in V_0(p_1; X)$ and thus $\sigma(p_1, p_1; X) = 0$.

(b) If $\sigma(p_1, p_2; X) = c$ is defined, then lemma 2 shows that $p_2 \in V_c(p_1; X)$. If $c = 0$, then $p_2 = p_1$ and $x_1 \not< x_2$.

(c) Let $\sigma(p_1, p_2; X) = a$, $\sigma(p_2, p_3; X) = b$ and $\sigma(p_1, p_3; X) = c$; then by lemma 2, $p_2 \in V_a(p_1; X)$, $p_3 \in V_b(p_2; X)$, $p_3 \in V_c(p_1; X)$. Let $\varepsilon > 0$ be arbitrary. By definition of these vicinities, there exist partitions $a_1 + a_2 + \ldots + a_j = a$, $b_1 + b_2 + \ldots + b_k = b$ of norm ε and finite sets of positive numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$; $\varepsilon'_1, \varepsilon'_2, \ldots, \varepsilon'_n$ such that $\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_m \leqq \varepsilon$ and $\varepsilon'_1 + \varepsilon'_2 + \ldots + \varepsilon'_n \leqq \varepsilon$ with the properties that:

$$T_{a_1, a_2, \ldots, a_j}(p_1; X) \cap T^{-1}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m}(p_2; X) \neq \varnothing, \tag{18}$$

$$T_{b_1, b_2, \ldots, b_k}(p_1; X) \cap T^{-1}_{\varepsilon'_1, \varepsilon'_2, \ldots, \varepsilon'_n}(p_3; X) \neq \varnothing.$$

It follows that $p_2 \in T_{a_1, \ldots, a_j, \varepsilon_m, \varepsilon_{m-1}, \ldots, \varepsilon_1}(p_1; X)$ and hence this set has at least one point, i.e. p_2, in common with $T_{b_1, b_2, \ldots, b_k}(p_2; X)$. Thus

$$T_{a_1, \ldots, a_j, \varepsilon_m, \ldots, \varepsilon_1, b_1, \ldots, b_k}(p_1; X) \cap T^{-1}_{\varepsilon'_1, \varepsilon'_2, \ldots, \varepsilon'_n}(p_3; X) \neq \varnothing \tag{19}$$

where $a_1+a_2+\ldots+a_j+\varepsilon_m+\varepsilon_{m-1}+\ldots+\varepsilon_1+b_1+\ldots+b_k \leqq a+b+\varepsilon$ and $\varepsilon_1'+\varepsilon_2'+\ldots+\varepsilon_n' \leqq \varepsilon$. Since the norm ε can be arbitrarily chosen, $p_3 \in V_{a+b+\varepsilon}(p_1; X)$ for any ε. Thus, $c \leqq a+b+\varepsilon$ and the result follows.

COROLLARY. *Under the hypotheses of the theoem, if* $\sigma(p, q; X)$ *is defined and* $\sigma(p, q; X)=0$ *then* $p=q$.

PROOF. This is inherent in the proofs of parts (a) and (b) above.

When $\sigma(p_1, p_2)$ is defined the definitions we have introduced may be applied to the original problem of constructing a path in H joining p_1 to p_2 to minimize $J[y]$. For any value of the parameter a, $0 \leqq a \leqq \sigma(p_1, p_2)$ let

$$C(a; p_1, p_2) = V_a(p_1) \cap V^{-1}_{\sigma(p_1, p_2)-a}(p_2). \tag{20}$$

It is to be noted that $C(0; p_1, p_2) \equiv p_1$, and $C(\sigma(p_1, p_2); p_1, p_2) \equiv p_2$. For any given value of a, $C(a; p_1, p_2)$ may be a null set, a singleton set, or a set consisting of more than one point. If $C(a; p_1, p_2)$ is a singleton set for each $a \in [0, \sigma(p_1 p_2)]$ then it is a single-valued function of a, and $C(a; p_1, p_2)$ is a point $p(a)$ of the original space H with well-defined coordinates $p \equiv (x(a), y(a))$. This does not immediately imply that $(x(a), y(a))$ can be expressed in terms of x as independent variable in in the form of a single-valued function $y=y(x)$. When this is possible we shall adopt the notation $y=R(x)$ to indicate the fact, but it must be remarked that there is not any immediate implication that $R(x)$ is continuous, differentiable, an arc, or an admissible arc.

THEOREM 4. *If* $r \in C(a; p_1, p_2)$ *then* $\sigma(p_1, r)$, $\sigma(r, p_2)$ *exist, and* $\sigma(p_1, r)+\sigma(r, p_2)=\sigma(p_1, p_2)$.

PROOF. There is a value a, $0 \leqq a \leqq \sigma(p_1, p_2)$ such that $r \in V_a(p_1) \cap V^{-1}_{\sigma(p_1, p_2)-a}(p_2)$. Since $r \in V_a(p_1)$, the definition of $\sigma(p_1, r)=\inf\{\alpha | r \in V_\alpha(p_1)\}$ implies that this quantity exists. Since $r \in V^{-1}_{\sigma(p_1, p_2)-a}(p_2)$, $p_2 \in V_{\sigma(p_1, p_2)-a}(r)$, and by the same argument as above this implies $\sigma(r, p_2)$ exists. Now, $\sigma(p_1, r) \leqq a$ and $\sigma(r, p_2) \leqq \sigma(p_1, p_2)-a$. Thus, $\sigma(p_1, r)+\sigma(r, p_2) \leqq a+\sigma(p_1, p_2)-a=\sigma(p_1, p_2)$. Theorem 3 (c) completes the proof.

LEMMA 3. (a) *If* $q_1 \in \bigcup_{a=0}^{\sigma(p_1, p_2)} C(a; p_1, p_2)$ *and if* $q_2 \in \bigcup_{b=0}^{\sigma(q_1, p_2)} C(b; q_1, p_2)$, *then* $\bigcup_{c=0}^{\sigma(q_1, q_2)} C(c; q_1, q_2) \subseteqq \bigcup_{a=0}^{\sigma(p_1, p_2)} C(a; p_1, p_2)$.

(b) *If* $q_2 \in \bigcup_{a=0}^{\sigma(p_1, p_2)} C(a; p_1, p_2)$ *and if* $q_1 \in \bigcup_{b=0}^{\sigma(p_1, q_2)} C(b; p_1, q_2)$ *then* $\bigcup_{c=0}^{\sigma(q_1, q_2)} C(c; q_1, q_2) \subseteqq \bigcup_{a=0}^{\sigma(p_1, p_2)} C(a; p_1, p_2)$.

(c) *If* q_1, q_2 *are as above,* $\sigma(p_1, q_1)$, $\sigma(q_1, q_2)$ *and* $\sigma(q_2, p_2)$ *exist.*

(d) *If* q_1, q_2 *are as above,* $\sigma(p_1, q_1)+\sigma(q_1, q_2)+\sigma(q_2, p_2)=\sigma(p_1, p_2)$.

PROOF of (c) and (d): By theorem 4 applied to p_1, q_1, p_2, $\sigma(p_1, q_1)$ and $\sigma(q_1, p_2)$ exist and $\sigma(p_1, q_1)+\sigma(q_1, p_2)=\sigma(p_1, p_2)$; applied to q_1, q_2, p_2, $\sigma(q_1, p_2)$, $\sigma(q_2, p_2)$ exist and $\sigma(q_1, q_2)+\sigma(q_2, p_2)=\sigma(q_1, p_2)$. Substituting the last expression in the first gives (d).

PROOF of (a): Let $r \in \bigcup_{c=0}^{\sigma(q_1, q_2)} C(c; q_1, q_2)$ and γ be the specific value of c such that $r \in V_\gamma(q_1) \cap V^{-1}_{\sigma(q_1, q_2)-\gamma}(q_2)$. Now $q_1 \in V_{\sigma(p_1, q_1)}(p_1)$ and $q_2 \in V^{-1}_{\sigma(q_2, p_2)}(p_2)$. By definition of the sets V and V^{-1} it follows that:

$$r \in V_{\sigma(p_1, q_1)}(p_1) \cup V_\gamma(q_1) \subseteqq V_{\sigma(p_1, q_1)+\gamma}(p_1)$$

and

$$r \in V^{-1}_{\sigma(q_2, p_2)}(p_2) \cup V^{-1}_{\sigma(q_1, q_2)-\gamma}(q_2) \subseteqq V^{-1}_{\sigma(q_2, p_2)+\sigma(q_1, q_2)-\gamma}(p_2).$$

Thus:

$$r \in V_{\sigma(p_1, q_1)+\gamma}(p_1) \cap V^{-1}_{\sigma(q_2, p_2)+\sigma(q_1, q_2)-\gamma}(p_2) =$$

(21)

$$= C(\sigma(p_1, q_1)+\gamma; p_1, p_2) \subseteqq \bigcup_{a=0}^{\sigma(p_1, p_2)} C(a; p_1, p_2)$$

as stated. The proof of (b) is similar to that of (a).

THEOREM 5. *If $R(x)$ is an admissible arc joining p_1 to p_2 then $J[R] = \sigma(p_1, p_2; R)$.*

PROOF. By theorem 1 and the definition of σ, $\sigma(p_1, p_2; R) \leqq J[R]$.

By the condition of admissibility, $R(x)$ is of class $C^{(1)}$ and since the interval $[x_1, x_2]$ is compact, for any $\eta > 0$ there exists a uniform δ such that $\left| R'(a) - \frac{R(b)-R(a)}{b-a} \right| < \eta$ when $|a-b| < \delta$ and $a, b \in [x_1, x_2]$.

By property (ii) of $f(\alpha, \beta, \gamma)$, for any $\varepsilon > 0$ there is an η such that $|f(a, R(a), R'(a)) - f(a, R(a), \gamma)| < \varepsilon$ when $|R(a) - \gamma| < \eta$. Combining this with the preceding paragraph, for any $\varepsilon > 0$ there is a δ such that

$$\left| f(a, R(a), R'(a)) - f\left(a, R(a), \frac{R(b)-R(a)}{b-a}\right) \right| < \varepsilon \tag{22}$$

when $|b-a| < \delta$ and $a, b \in [x_1, x_2]$.

By lemma 2, $p_2 \in V_{\sigma(p_1, p_2)}(p_1)$, and therefore by the definition of $V_{\sigma(p_1, p_2)}(p_1)$, for any $\Delta > 0$ there is a partition $a_0 + a_1 + \ldots + a_{n-1} = \sigma(p_1, p_2; R)$ of norm Δ and a set of positive numbers $e_0, e_1, \ldots, e_k$ with $e_0 + e_1 + \ldots + e_k \leqq \Delta$ such that

$$p_2 \in T_{a_0, a_1, \ldots, a_{n-1}, e_0, \ldots, e_k}(p_1; R). \tag{23}$$

Choose $\Delta < \delta$, then by definition of $R(x)$ there is a subdivision $x_1 = z_0 < z_1 < \ldots < z_n = x_2$ with $\max_i |z_i - z_{i+1}| < \Delta$ and a set of points $(z_0, R(z_0)), (z_1, R(z_1)), \ldots, (z_n, R(z_n))$ on the arc $y = R(x)$ such that:

$$f\left[z_i, R(z_i), \frac{R(z_{i+1})-R(z_i)}{(z_{i+1}-z_i)}\right](z_{i+1}-z_i) \leqq a_i \tag{24}$$

with $(z_{i+1}, R(z_{i+1})) \in T_{a_i}(z_i, R(z_i); R)$. Therefore, for these points

$$\sigma(p_1, p_2; R) \geqq \sum_{i=0}^{n-1} f\left(z_i, R(z_i), \frac{R(z_{i+1})-R(z_i)}{(z_{i+1}-z_i)}\right)(z_{i+1}-z_i) \tag{25}$$

$$\geqq \sum_{i=0}^{n-1} \left[f(z_i, R(z_i), R'(z_i))-\varepsilon\right](z_{i+1}-z_i)$$

$$= \sum_{i=0}^{n-1} f(z_i, R(z_i), R'(z_i))(z_{i+1}-z_i)-\varepsilon(x_2-x_1).$$

But $\lim\limits_{\delta\to 0} \sum\limits_{i=0}^{n-1} f(z_i, R(z_i), R'(z_i))(z_{i+1}-z_i) = \int\limits_{x_1}^{x_2} f(x, R(x), R'(x))\,dx$ exists. Letting $\delta\to 0$ in (25) leads to:

$$\sigma(p_1, p_2; R)+\varepsilon(x_2-x_1) \geqq \int_{x_1}^{x_2} f(x, R(x), R'(x))\,dx. \tag{26}$$

Since ε may be chosen arbitrarily small, it follows that $\sigma(p_1, p_2; R) \geqq J[R]$. Combining this with the first statement of the proof, the stated result follows.

THEOREM 6. *If:* (i) $y=y(x)$ *is an admissible arc joining* p_1 *to* p_2*;* (ii) $J[y]=\sigma(p_1, p_2)$; (iii) $C(a)$ *is singleton for* $0 \leqq a \leqq \sigma(p_1, p_2)$; *then,* $C(a)$ *is the same point set as* $y=y(x)$, *and* $R(x)$ *exists and* $R(x) \equiv y(x)$.

PROOF. The theorem is a consequence of theorem 2 and the definitions of $C(a)$ and $R(x)$.

When $C(a)$ is singleton for all a, the two preceding theorems settle the question of the existence of a unique minimizing, admissible arc, under the condition that $\inf J[y]=\sigma(p_1, p_2)$, where the infimum is taken over the class of admissible arcs. For, if $C(a)$ is singleton and $R(x)$ were undefined or inadmissible, there is not an actual path which is admissible that minimizes $J[y]$.

In our introductory paragraphs, it was required that admissible functions be of class $C^{(1)}$. It may be desirable to permit a broader admissible class, say, of arcs with a finite number of corners, and to generalize the integral accordingly to improper Riemann integration. To illustrate the point, suppose that $f \equiv \sqrt{1+y'^2}$ and the space H is the half plane $y \geqq 0$ with the points $x=0$, $0 \leqq y < 1$ removed. If $p_1 \equiv (-2, 0)$ and $p_2 \equiv (2, 0)$, there is no everywhere differentiable arc that minimizes $\int\limits_{-2}^{2} \sqrt{1+y'^2}\,dx$. But if corners are permitted and improper Riemann integration is allowed, then a minimizing function does exist and would correspond to $R(x)$. Again, let the space H be a nowhere differentiable function joining the endpoints $(0, 0)$ and $(1, 0)$; the functional $J[y]=\int\limits_0^1 \sqrt{1+y'^2}\,dx$ is undefined anywhere even in the Lebesgue sense, and yet the arc length concept is meaningful. As another example, suppose that $(0, 0)$ is joined to $(1, 0)$ by a smooth arc of length 3, and also by one of length 2, but with a corner. In this instance $J[y]=\int\limits_0^1 \sqrt{1+y'^2}\,dx$ is mini-

mized in the admissible class by the arc of length 3 when proper Riemann integration is considered, but by the function of length 2 when improper Riemann integration is permitted; also, $\sigma(p_1, p_2)=2$. This last example shows that unless some definite purpose exists for remaining within the confines of the original problem definition, it is desirable to broaden the admissible class. In so doing, it is essential that the definition of $J[y]$ be extended beyond the constraints implied by the use of derivatives and integrals.

An extension of $J[y]$ to functions $y=y(x)$ which are arcs but may lack differentiability properties is possible with the use of σ. Let $y=y(x)$ be a function which is an arc with endpoints $p_1\equiv(x_1, y_1)$ and $p_2\equiv(x_2, y_2)$, and with graph Y. Let H be restricted to Y. If $q\equiv(x, y)$ is any point on Y then $\sigma(p_1, q; Y)=\inf\{c|c\in V_c(p_1; Y)\}$. Lemmas 2 shows that $q\in V_{(p_1, q; Y)}(p_1; Y)$. If $y(x)$ is admissible in the original sense, the corollary to theorem 2 shows that $y=R(x)$ in the restricted space of Y, and that $R(x)$ is identical with $y=y(x)$; then theorem 5 shows that $J[y]=\sigma(p_1, p_2; Y)$. Thus $\sigma(p_1, p_2; Y)$ is a quantity that is equal in value to $J[y]$ whenever the latter is defined. If $q\equiv(x, y(x))$ is an ambient point on Y, it similarly follows that

$$\int_{x_1}^{x} f(x, y, y')dx=\sigma(p_1, q; Y).$$

To be truly accurate, it is required now to extend the definition of the $V_c(p; X)$ for any point $p\equiv(r, s)\in X\subseteq H$ to include $V_\infty(p; X)$. This set will consist of p together with all points $q\equiv(x, y)\in X$, $x>r$ which are arcwise connected to p by the graph of a function defined on the interval $[r, x]$ lying in X, whether or not the point q belongs to any $V_c(p; X)$ for a finite value of c in the sense of the earlier definition. It is clear that $V_c(p; X)\subset V_\infty(p; X)$ for any finite c. As earlier $\sigma(p, q; X)=$ $=\inf\{c|q\in V_c(p; X)\}$ where c can be infinite.

Whenever $J[y]$ is defined as a finite quantity, $\sigma(p_1, p_2; Y)$ has the same value, so that if $\sigma(p_1, p_2; Y)$ is infinite, $J[y]$ is undefined. For every arc, $\sigma(p_1, p_2; Y)$ is defined as $\inf\{c|p_2\in V_c(p_1; Y)\}$ so that this quantity exists either as a finite or infinite value. It is to be remarked that $\sigma(p_1, p_2; Y)$ can exist as a finite quantity for arcs, for which $J[y]$ remains undefined.

The generalization of the original minimizing problem is now that of minimizing $\sigma(p_1, p_2; Y)$ in the class of all arcs which are functions $\{y=y(x)\}$, the graphs of which lie entirely in H, and that have endpoints p_1, p_2. It should be noted that if $\sigma(p_1, p_2; Y)$ is infinite for every function of the class, no optimizing problem existed from the outset. Also, if there is no arc connecting p_1 to p_2, no optimizing problem existed. Thus, it may be assumed that at least one arc which is a function defined on $[x_1, x_2]$ exists connecting p_1 to p_2, for which $\sigma(p_1, p_2; Y)$ is finite.

EXAMPLE 1. Let $f\equiv 1+y'$. The condition (i) on f requires $\gamma\geqq 0$ so that when the derivative is replaced with the difference quotient in the definition (2) of the sets $S_a(p)$ this difference quotient must be non-negative. Generalized admissible curves are therefore limited to monotone non-decreasing continuous functions. If H is the whole plane, the sets $V_c(p)$ are seen to be the right triangular sets $(x-r)+$ $+(y-s)\leqq c$, $x>r$, $y>s$, together with $p\equiv(r, s)$. The value $\sigma(p, q; Y)$ for any $q\in Y$ is the same as the least value of c such that $q\in V_c(p; H)$. Then $\sigma(p, q; Y)=$ $=\inf\{c|q\in V_c(p; H)\}=(x-r)+(y-s)$ and $\sigma(p, q; Y)$ is seen to be the appropriate generalization of $J[y]=\int_{(r, s)}^{(x, y)}(1+y')dx=(x-r)+(y-s)$ from the original admis-

sible class of curves to curves which are continuous, monotonic non-decreasing, and which could have a set of corners of measure zero.

EXAMPLE 2. Let $f=\sqrt{1+y'^2}$. If H is the whole plane, the $S_a(p)$ are semicircular sets $(x-r)^2+(y-s)^2\leqq a$, $x>r$, together with the point p. Suppose that $y=y(x)$ is a continuous function with endpoints p_1 and p_2. Restricting the space to Y, then $T_a(p_1; Y)$ is the portion of Y which is arcwise connected to p_1 lying in the semicircular set $S_a(p_1)$ of H. The set $T_{a_1, a_2, \ldots, a_k}(p_1; Y)$ is the portion of the graph arcwise connected to p_1 which is covered by a chain of the semicircular sets of H of respective radii $a_1, a_2, \ldots, a_k$, adjacent links of which have at least one common point of Y. Since $y=y(x)$ arcwise connects p_2 to p_1, for any norm ε, it is possible to reach p_2 by a chain for which $\max[a_1, a_2, \ldots, a_k]<\varepsilon$. It is further possible to cover Y by a chain for which $a_1+a_2+\ldots+a_k<\sigma(p_1, p_2; Y)$. A polygon may be constructed from p_1 to p_2 with vertices on Y which are common points of adjacent links of the covering. It follows that the length of the arc: l, defined as the supremum of the lengths of all polygons inscribed to the arc, satisfies $\sigma(p_1, p_2; Y)\leqq l$. On the other hand, Y has [1] Caratheodory linear measure: L, satisfying $L\leqq\sigma(p_1, p_2; Y)$, since the above covering by semicircular sets is a subclass of Caratheodory coverings. Thus $L\leqq\sigma(p_1, p_2; Y)\leqq l$; but Caratheodory measure equals the length; i.e. $L=\sigma(p_1, p_2; Y)=l$, whenever l is defined.

LEMMA 4. *If* $x\subseteqq H$, *then* $T_{a_1, a_2}(p; X)\subseteqq T_{a_1+a_2}(p; X)$.

PROOF. $T_{a_1, a_2}(p; X)\subseteqq S_{a_1+a_2}(p; X)$ and $T_{a_1, a_2}(p; X)$ is an arcwise connected subset of X. Since $T_{a_1+a_2}(p; X)$ is the largest connected component of p in $S_{a_1+a_2}(p; X)$ it follows $T_{a_1, a_2}(p; X)\subseteqq T_{a_1+a_2}(p; X)$.

LEMMA 5. *If* $X\subseteqq H$ *and* $\alpha=\sigma(p, q; X)$ *then* $T_{\alpha+\varepsilon}(p; X)\supseteqq V_\alpha(p; X)$ *for every* $\varepsilon>0$.

PROOF. The cases for $\alpha=0$ and $\alpha=\infty$ are trivial.

Let $\varepsilon>0$ be given and r be an arbitrary point of $V_\alpha(p; X)$. By definition of $V_\alpha(p; X)$ there are partitions $a_1+a_2+\ldots+a_k=\alpha$ and $\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_n\leqq\varepsilon$ such that $T_{a_1, a_2, \ldots, a_k}(p; X)\cap T^{-1}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n}(r; X)\neq\emptyset$. This implies that $r\in T_{a_1, a_2, \ldots, a_k, \varepsilon_n, \ldots, \varepsilon_1}(p; X)$. By lemma 4, $r\in T_{a_1+a_2+\ldots+a_k+\varepsilon_n+\ldots+\varepsilon_1}(p; X)\subseteqq T_{\alpha+\varepsilon}(p; X)$.

LEMMA 6. *If for every* $q\equiv(x_q, y_q)$ *and* $r\equiv(x_r, y_r)\in R$ *with* $x_q<x_r$, *the set* $C(a; q, r)\neq\emptyset$ *for all* $0\leqq a\leqq\sigma(q, r)$, *then* $R(x; q, r)$ *exists,* $R(q, r)$ *is a subset of* R, *and* $\sigma(p_1, q)+\sigma(q, r)+\sigma(r, p_2)=\sigma(p_1, p_2)$.

PROOF. Let x be any value $x_q\leqq x\leqq x_r$ and let $s\equiv(x, R(x))$ with the coordinates of s now denoted as (x_s, y_s). We observe that $y_s(x)$ is a single-valued function defined on $[x_q; x_r]$, since $R(x)$ has this property. By theorem 4, $\sigma(p_1, s)$, $\sigma(s, p_2)$ exist and $\sigma(p_1, s)+\sigma(s, p_2)=\sigma(p_1, p_2)$.

Choose a such that $q=V_a(p_1)\cap V^{-1}_{\sigma-a}(p_2)$ and consider $C(a; p_1, s)$ which, by hypothesis, is non-empty. If $q'\in C(a; p_1, s)$ then $q'\in V_a(p_1)$ and $q'\in V^{-1}_{(p_1, s)-a}(s)$; and since $\sigma(p_1, s)-a+\sigma(s, p_2)=\sigma(p_1, p_2)-a$ and $s\in V^{-1}_{\sigma(s, p_2)}(p_2)$ it follows $q'\in V^{-1}_{\sigma(p_1, p_2)-a}(p_2)$. Thus $q'\in V_a(p_1)\cap V^{-1}_{\sigma(p_1, p_2)-a}(p_2)\equiv C(a; p_1, p_2)$ which is singleton for each value of a; therefore, $q'=q=C(a; p_1, s)\subseteqq \bigcup_{b=0}^{\sigma(p_1, s)} C(b; p_1, s)$. The

hypotheses of lemma 3 (b) are satisfied, so that $\bigcup_{c=0}^{\sigma(q,s)} C(c; q, s) \subseteq R$. Similarly, it may be shown that $\bigcup_{c=0}^{\sigma(s,r)} C(c; s, r) \subseteq R$. The hypotheses of lemma 3 (d) are satisfied when $s=r$, and therefore $\sigma(p_1 q)+\sigma(q, r)+\sigma(r, p_2)=\sigma(p_1, p_2)$.

In particular $s \in C(\sigma(q, s);\ q, s)=V_{\sigma(q,s)}(q) \cap V_0^{-1}(s)$ and $s \in C(0; s, r)= =V_0(s) \cap V_{\sigma(s,r)}^{-1}(r)$, so that $s \in V_{\sigma(q,s)}(q) \cap V_{\sigma(s,r)}^{-1}(r)$. By lemma 3 (d) applied to p_1, q, s, p_2 it follows: $\sigma(p_1, q)+\sigma(q, s)+\sigma(s, p_2)=\sigma(p_1, p_2)$ and applied to p_1, s, r, p_2, it follows: $\sigma(p_1, s)+\sigma(s, r)+\sigma(r, p_2)=\sigma(p_1, p_2)$. Adding these two equalities and reducing with those established in the preceding paragraphs gives: $\sigma(q, s)+ +\sigma(s, r)=\sigma(q, r)$. This result, with the fact that $s \in V_{\sigma(q,s)}(q) \cap V_{\sigma(s,r)}^{-1}(r)$, shows that $s \in C(\sigma(q, s);\ q, r)$. This set is singleton, as follows by an argument similar to that employed in the preceding paragraph. Thus, not only is $y_s(x)$ a single valued function of x defined everywhere on the interval $[x_r, x_s]$, but the point $s \equiv (x_s, y_s) \equiv \equiv C(\sigma(q, s);\ q, r)$; therefore $R(x; q, r)$ exists. Since $s \equiv (x, R(x))$ it follows that $R(q, r)$ is a subset of R.

THEOREM 7. *If:* (a) $R(x)$ *is an arc joining* p_1 *to* p_2*; and:* (b) *for every* $q \equiv (x_q, y_q)$ *and* $r \equiv (x_r, y_r) \in R$ *with* $x_q < x_r$ *the set* $C(a; q, r) \neq \varnothing$ *for all* $0 \leqq a \leqq \sigma(q, r)$; *then* $\sigma(p_1, p_2)=\sigma(p_1, p_2; R)$.

PROOF. Let $\eta>0$ be given, and assume as remarked earlier that $0<\sigma(p_1, p_2)<\infty$. Denote p_1 as q_0 and pick $q_1 \equiv (x_1, R(x_1))$ on R so that $\eta/2<\sigma(q_0, q_1)<\eta$. Then by theorem 4, $\sigma(q_1, p_2)$ exists and equals $\sigma(p_1, p_2)-\sigma(q_0, q_1)$. By lemma 6, $R(x, q_1, p_2)$ exists and is a subset of R. As before, select $q_2 \in R(q_1, p_2)$ so that $\eta/2< <\sigma(q_1, q_2)<\eta$. The existence of $\sigma(q_2, p_2)$ is guaranteed and $\sigma(q_2, p_2)=\sigma(q_1, p_2)- -\sigma(q_1, q_2)=\sigma(p_1, p_2)-\sigma(q_0, q_1)-\sigma(q_1, q_2)$. Proceeding in this way, in a finite number of steps, a set of points $\{q_0, q_1, \ldots, q_n\}$ on R can be found so that $\sigma(q_{i-1}, q_i)<\eta$ and $\sum_{i=1}^{n} \sigma(q_{i-1}, q_i)=\sigma(p_1, p_2)$.

Let $\sigma(q_{j-1}, q_j)=a_j$, $j=1, 2, \ldots, n$. By the definition of $\sigma(q_{j-1}, q_j)$ and by lemma 2, $q_j \in V_{a_j}(q_{j-1})$ and if q is an ambient point on the subarc of $R(x)$ from q_{j-1} to q_j, then $q \in V_{a_j}(q_{j-1})$. Let ζ and ε be numbers satisfying $\zeta>0$, $0<\varepsilon<\zeta/n$. By lemma 5, $q \in T_{a_j+\varepsilon}(q_{j-1})$. Thus, the entire subarc of $R(x)$ from q_{j-1} to q_j lies in the sphere $S_{a_j+\varepsilon}(q_{j-1})$ and since the subarc is arcwise connected to q_{j-1} it follows that is lies in $T_{a_j+\varepsilon}(q_{j-1}; R)$. Therefore $q_j \in T_{a_j+\varepsilon}(q_{j-1}; R)$ and $p_2 \in \in T_{a_1+\varepsilon, a_2+\varepsilon, \ldots, a_n+\varepsilon}(p_1; R)$. Since $\eta>0$ was arbitrary, this implies that $p_2 \in V_{\sigma(p_1, p_2)+\zeta}(p_1; R)$. Since $\zeta>0$ may be chosen arbitrarily small, by lemma 2 on the space R, it follows that $p_2 \in V_{\sigma(p_1, p_2)}(p_1; R)$ and therefore $\sigma(p_1, p_2; R) \leqq \sigma(p_1, p_2)$. The converse that $\sigma(p_1, p_2) \leqq \sigma(p_1, p_2; R)$ is immediate since R is a subset of H.

Without condition (b), examples exist for which $\sigma(p_1, p_2)<\sigma(p_1, p_2; R)$. We have omitted a description of such an example, due to its detailed nature.

THEOREM 8. *If* (a) $p_1 \equiv (x_1, y_1)$ *and* $p_2 \equiv (x_2, y_2)$, $x_1 < x_2$ *are points of* H*;* (b) *the graph* R *of* $R(x)$ *is a bounded and closed function from* $[x_1, x_2]$ *into* $E^{(2)}$*;* (c) *if* $\sigma(p_1, p_2; R)<\infty$*;* (d) *for every* $q \equiv (x_q, y_q)$ *and* $r \equiv (x_r, y_r) \in R$ *with* $x_q < x_r$*, the set* $C(a; q, r) \neq \varnothing$ *for all* $0 \leqq a \leqq \sigma(q, r)$*; then,* $R(x)$ *is a unique minimizing function in*

the sense that $\sigma(p_1, p_2; R) < \sigma(p_1, p_2; F)$ *where* F *is the graph of any other function* $F(x)$ *which is an arc joining* p_1 *to* p_2 *lying in* H.

PROOF. It will first be shown that $R(x)$ is continuous. Suppose that t, $x_1 \leqq t \leqq x_2$ were a point of discontinuity of $R(x)$. There exists a sequence $\{r_i : i=1, 2, \ldots\}$ with $r_i \to t$ for which $R(r_i)$ does not converge to $R(t)$. A subsequence $r_{ij}=t_j$, $j=1, 2, \ldots$ either monotonically increasing or decreasing properly to t can be extracted from $\{r_i\}$. Assume the former, as the proof in the latter case is analogous.

Consider the sequence of points $\{t_j, R(t_j)\}$ in $E^{(2)}$; it is bounded and must have an accumulation point with coordinates (t, ϱ) which is a point on R, since R is a closed point set. The single-valuedness of $R(x)$ guarantees that $\varrho = R(t)$. This contradicts the assumption of discontinuity at $x=t$.

Now $R(x_2) = V_{\sigma(p_1, p_2; R)}(p_1) \cap V_0^{-1}(p_2)$ is singleton, and since $V_0^{-1}(p_2) = p_2$ it follows $R(x_2) = p_2$. Similarly $R(x_1) = p_1$. Continuity and single-valuedness of $R(x)$ show that $R(x)$ describes an arc with endpoints p_1 and p_2.

Suppose $F(x)$, $x_1 \leqq x \leqq x_2$ were an arc lying in H with endpoints p_1, p_2 such that $d = \sigma(p_1, p_2; F) \leqq \sigma(p_1, p_2; R)$. If $d < \sigma(p_1, p_2; R)$ then $p_2 \in V_d(p_1)$ and $\sigma(p_1, p_2) \leqq d$. By theorem 7, $\sigma(p_1, p_2; R) = \sigma(p_1, p_2)$ which is a contradiction. If $d = \sigma(p_1, p_2; R)$, let $q \equiv (s, F(s))$, $x_1 \leqq s \leqq x_2$ be any point of F. Suppose that a is any value such that $q \in V_a(p_1; F) \cap V^{-1}_{\sigma(p_1, p_2; F)-a}(p_2; F)$; then $q \in V_a(p_1) \cap V^{-1}_{d-a}(p_2) \equiv$ $\equiv (s, R(s))$ since $d = \sigma(p_1, p_2)$ and by the definition of $R(x)$ as a singleton point set. Hence $F(x) = R(x)$, $x_1 \leqq x \leqq x_2$.

REMARK 4. It is apparent that $R(x)$ must be both a bounded and a closed function when the interval $[x_1, x_2]$ is finite.

LEMMA 7. *If for every pair of points* $q_1, q_2 \in H$ *for which* $\sigma(q_1, q_2)$ *is defined,* $C(a; q_1, q_2)$, $0 \leqq a \leqq \sigma(q_1, q_2)$ *is non-empty and if* $\bigcup_{a=0}^{\sigma(q_1, q_2)} C(a; q_1, q_2)$ *is compact, then for every* x, $x_1 \leqq x \leqq x_2$, *there is a point* $p \in \bigcup_{a=0}^{\sigma(q_1, q_2)} C(a; q_1, q_2)$ *for which* x *is the value of the* x*-coordinate of* p.

PROOF. Suppose the contrary, and let $p_1 \equiv (x_1, y_1)$, $p_2 \equiv (x_2, y_2)$, $x_1 < x_2$ be a pair of points of H, with $\sigma(p_1, p_2)$ defined, such that for some x^*, $x_1 \leqq x^* \leqq x_2$ there is no point of $\bigcup_{a=0}^{\sigma(p_1, p_2)} C(a; p_1, p_2)$ which has x^* for its x-coordinate.

Since $\bigcup_{a=0}^{\sigma(p_1, p_2)} C(a; p_1, p_2)$ is compact there is a point $q_1 \equiv (x_3, y_3)$ of this set with $x_3 < x^*$ such that no other point of $\bigcup_{a=0}^{\sigma(p_1, p_2)} C(a; p_1, p_2)$ exists with an x-coordinate on the half-open interval $(x_3, x^*]$. We note that $\sigma(q_1, p_2)$ exists by theorem 4 and that $\sigma(q_1, p_2) > 0$ by the corollary to theorem 3, since $x_3 < x^* < x_2$.

Consider $\bigcup_{b=0}^{\sigma(q_1, p_2)} C(b; q_1, p_2)$. This set is compact by hypothesis and contained in $\bigcup_{a=0}^{\sigma(p_1, p_2)} C(a; p_1, p_2)$ by lemma 3 (a). By compactness there is a point $q_2 \equiv (x_4, y_4)$ of this set with $x^* < x_4$ such that no other point exists in $\bigcup_{b=0}^{\sigma(q_1, p_2)} C(b; q_1, p_2)$ with x-coordinate on the half open interval $[x^*, x_4)$.

We note that $\sigma(q_1, q_2)$ exists by lemma 3 (c). Let $\sigma(q_1, q_2)=d$, then since $x_3<x^*<x_4$ it follows that $d>0$ by the corollary to theorem 3. By the definition of q_1, q_2 there is no point in $\bigcup_{b=0}^{\sigma(q_1,p_2)} C(b; q_1, p_2)$ with an x-coordinate in the interval $x_3<x<x_1$, and since $\bigcup_{c=0}^{\sigma(q_1,q_2)} C(c; q_1, q_2) \subseteq \bigcup_{b=0}^{\sigma(q_1,p_2)} C(b; q_1, p_2)$ it follows by lemma 3 (a), that there is no point in $\bigcup_{c=0}^{\sigma(q_1,q_2)} C(c; q_1, q_2)$ with x-coordinate in the interval $x_3<x<x_1$. Thus, by the remark following formula (20), $\bigcup_{c=0}^{\sigma(q_1,q_2)} C(c; q_1, q_2)$ reduces to the two points q_1 and q_2 only. Therefore $C(d/2; q_1, q_2)$ is vacuous contradicting the hypotheses of this theorem.

THEOREM 9. *If for every pair of points $q_1, q_2 \in H$ for which $\sigma(q_1, q_2)$ is defined, $C(a; q_1, q_2)$, $0 \leqq a \leqq \sigma(q_1, q_2)$ is non-empty, and if $\bigcup_{a=0}^{\sigma(q_1,q_2)} C(a; q_1, q_2)$ is compact, then every pair of points $p_1, p_2 \in H$ for which $\sigma(p_1, p_2)$ is defined are connected by a minimizing function $y=M(x)$ in H in the sense that if $F(x)$ is any function which is an arc joining p_1 to p_2, $\sigma(p_1, p_2; M) \leqq \sigma(p_1, p_2; F)$.*

PROOF. Let the unit interval $[0, 1]$ be represented by dyadic decimals, and $d_k \in [0, 1]$ denote any of those terminating dyadic decimals for which all digits after the kth are zero. The numbers $\xi_k = x_1 + d_k(x_2 - x_1)$, $(k=0, 1, 2, \ldots)$ constitute an everywhere dense subset of the interval $[x_1, x_2]$.

We construct a function $M(x)$ on this dense subset. Let $M(x_1)=y_1$, $M(x_2)=y_2$ where $p_1 \equiv (x_1, y_1)$, $p_2 \equiv (x_2, y_2)$. By lemma 7, let a_1 be any value of a such that there is a point in $C(a_1; p_1, p_2)$ with x-coordinate equal to $x_1 + 2^{-1}(x_2 - x_1)$. Let $M(x_1 + 2^{-1}(x_2 - x_1))$ be assigned the y-coordinate of one of these points, and let $(x_1 + 2^{-1}(x_2 - x_1), M(x_1 + 2^{-1}(x_2 - x_1)))$ be denoted as q_1. By theorem 4, $\sigma(p_1, q_1)$ and $\sigma(q_1, p_2)$ both exist. We now proceed with an inductive definition. Suppose that all the values of $M(x)$ have been chosen corresponding to all the possible numbers $\xi_k = x_1 + d_k(x_2 - x_1)$ as d_k ranges through all its kth dyadic values. Suppose that $d_k^{(1)}$ and $d_k^{(2)}$ are any two k^{th} dyadic numbers such that $d_k^{(2)} - d_k^{(1)} = 2^{-k}$ and that $\xi_k^{(1)} = x_1 + d_k^{(1)}(x_2 - x_1)$, $\xi_k^{(2)} = x_1 + d_k^{(2)}(x_2 - x_1)$; also, let $q_k^{(1)} \equiv (\xi_k^{(1)}, M(\xi_k^{(1)}))$, $q_k^{(2)} \equiv (\xi_k^{(2)}, M(\xi_k^{(2)}))$. By the inductive hypothesis it is assumed that $q_k^{(1)}$, $q_k^{(2)}$ have been chosen so as to satisfy the conditions of lemma 3 (c) and that therefore $\sigma(q_k^{(1)}, q_k^{(2)})$ exists. Let d_{k+1} represent that $(k+1)^{\text{st}}$ dyadic which is midway between $d_k^{(1)}$ and $d_k^{(2)}$, i.e. $d_{k+1} = (d_k^{(1)} + d_k^{(2)})/2$, and $\xi_{k+1} = x_1 + d_{k+1}(x_2 - x_1)$ so that $\xi_{k+1} = (\xi_k^{(1)} + \xi_k^{(2)})/2$. By the hypotheses of this theorem and lemma 7 there is a point $q_{k+1} \in \bigcup_{a=0}^{\sigma(q_k^{(1)}, q_k^{(2)})} C(a; q_k^{(1)}, q_k^{(2)})$ which has ξ_{k+1} as its x-coordinate; and by theorem 4 applied to $C(a; q_k^{(1)}, q_k^{(2)})$ with a equal to the value that corresponds to q_{k+1}, $\sigma(q_k^{(1)}, q_{k+1})$, $\sigma(q_{k+1}, q_k^{(2)})$ both exist. As k ranges through the integers, the definition of $M(x)$ on the everywhere dense set is completed.

We next prove that $M(x)$ is continuous on the everywhere dense subset. In accordance with the notations used above, let $\xi_k = x_1 + d_k(x_2 - x_1)$ be any point

of the everywhere dense subset, and suppose that $\{\eta(j): j=1, 2, \ldots\}$ are also points of this subset with $\eta(j)\to\xi_k$. Let $\alpha=\xi_k-2^{-(k+1)}$ and $\beta=\xi_k+2^{-(k+1)}$, $q_\alpha\equiv$ $\equiv(\alpha, M(\alpha))$, $q_\beta\equiv(\beta, M(\beta))$. By the method of construction of $M(x)$ we know that $\sigma(\alpha, \xi_k)$, $\sigma(\xi_k, \beta)$ both exist. There is an integer J such that for all $j\geqq J$, $\alpha\leqq\eta(j)\leqq\beta$; therefore, by the method of construction of $M(x)$, the points $(\eta(j), M(\eta(j)))$ lie in either $\bigcup_{a=0}^{\sigma(\alpha,\xi_k)} C(a; \alpha, \xi_k)$ or $\bigcup_{a=0}^{\sigma(\xi_k,\beta)} C(a; \xi_k, \beta)$ for all $j\geqq J$. By the compactness of these sets, the points lying in the former, if infinite in number have at least one limit point (ξ_k, y) in the set; and similarly for the latter set, if infinite in number have a limit point (ξ_k, z) in the set. But by the remark following formula (20), $C(\sigma(\alpha, \xi_k); \alpha, \xi_k)\equiv(\xi_k, M(\xi_k))$ and therefore this point must be (ξ_k, y); and similarly $C(0; \xi_k, \beta)\equiv(\xi_k, M(\xi_k))$ and must be (ξ_k, z). Thus, $M(x)$ is continuous at the arbitrary point ξ_k of the dense subset.

It is well known [2] that a function which is continuous on an everywhere dense subset of the interval $[x_1, x_2]$ has a unique continuous extension in $E^{(2)}$. Since $\bigcup_{a=0}^{\sigma(p_1,p_2)} C(a; p_1, p_2)$ is compact, this unique extension is contained in this union, and hence in H. The function so constructed is designated as $M(x)$, $x_1\leqq x\leqq x_2$.

We observe that if p, q, r are any three points on M in that order, then $\sigma(p, q)+$ $+\sigma(q, r)=\sigma(p, r)$ by lemma 3 and the hypotheses of this theorem.

To prove that $\sigma(p_1, p_2; M)\leqq\sigma(p_1, p_2; F)$ where $F(x)$ is any function defined in the statement of this proposition, it is first necessary to show that $\sigma(p_1, p_2; M)=$ $=\sigma(p_1, p_2)$. Since $\sigma(p_1, p_2)\leqq\sigma(p_1, p_2; F)$ whenever F is a subset of H, the conclusion that $\sigma(p_1, p_2; M)\leqq\sigma(p_1, p_2; F)$ will be an immediate conclusion.

We show first that for any $\eta>0$ there is always a large enough k so that $\sigma(q_k^{(1)}, q_k^{(2)})<\eta$ where $q_k^{(1)}$, $q_k^{(2)}$ are as defined in the second paragraph of the proof of this proposition with $d_k^{(1)}$ and $d_k^{(2)}$ any k^{th} dyadics which satisfy $d_k^{(2)}-d_k^{(1)}=2^{-k}$. Suppose the contrary, then there is an infinite sequence of values of k and at least one pair of dyadics $d_k^{(1)}$ and $d_k^{(2)}$ with $d_k^{(2)}-d_k^{(1)}=2^{-k}$ for which $\sigma(q_k^{(1)}, q_k^{(2)})\geqq\eta$. Since the interval $[x_1, x_2]$ is compact the x-coordinates $\zeta_k^{(1)}$ of the $q_k^{(1)}$ have at least one limit point; we denote any one of these as ϱ and the point $(\varrho, M(\varrho))$ as r. We observe that if r lies between $q_k^{(1)}$ and $q_k^{(2)}$ then since $\sigma(q_k^{(1)}, r)+\sigma(r, q_k^{(2)})=$ $=\sigma(q_k^{(1)}, q_k^{(2)})$ either $\sigma(q_k^{(1)}, r)\geqq\eta/2$ or $\sigma(r, q_k^{(2)})\geqq\eta/2$; and if r lies to the left of $q_k^{(1)}$ then $\sigma(r, q_k^{(2)})=\sigma(r, q_k^{(1)})+\sigma(q_k^{(1)}, q_k^{(2)})>\eta/2$, while if r lies to the right of $q_k^{(2)}$ then $\sigma(q_k^{(1)}, r)=\sigma(q_k^{(1)}, q_k^{(2)})+\sigma(q_k^{(2)}), r)>\eta/2$. Two cases are to be distinguished according as an infinite subsequence of the above set of values of k exist with the property, in case (a), that $\sigma(q_k^{(1)}, r)\geqq\eta/2$; or, in case (b), that $\sigma(r, q_k^{(2)})\geqq\eta/2$; we note both conditions may simultaneously occur. In case (a) a further subsequence may be extracted for which the x-coordinates converge in a monotone non-decreasing manner to the x-coordinate of r, i.e. $\zeta_l^{(1)}\uparrow\varrho$, $(l=1, 2, \ldots)$. Let $t\equiv(\tau, M(\tau))$ be any point on M with $\tau<\varrho$; then there is a value L such that for all $l\geqq L$, $\tau<\zeta_l^{(1)}\leqq\varrho$, and $\sigma(t, r)=\sigma(t, q_l^{(1)})+\sigma(q_l^{(1)}, r)\geqq\eta/2$. This implies that if $t\in V_\alpha^{-1}(r)$ then $\alpha\geqq\eta/2$, for every t with $\tau<\varrho$. But then $V_\alpha^{-1}(r)=\varnothing$ for $\alpha<\eta/2$, which contradicts the first hypothesis of this theorem. In case (b) a subsequence may be extracted for which $\zeta_l^{(2)}\downarrow\varrho$. Let $t\equiv(\tau, M(\tau))$ be any point on M with $\tau>\varrho$; then there is a value L such that for all $l\geqq L$, $\varrho\leqq\zeta_l^{(2)}<\tau$ and $\sigma(r, t)=\sigma(r, q_l^{(2)})+\sigma(q_l^{(2)}, t)\geqq\eta/2$. This

implies that if $t \in V_\alpha(r)$ then $\alpha \geqq \eta/2$ for every t with $\tau > \varrho$. But then $V_\alpha(r) = \varnothing$ for $\alpha < \eta/2$ which also contradicts the first hypothesis of this theorem.

The remainder of the proof is the same as the proof of theorem 7 after the first paragraph, with R replaced by M and the selected points simply replaced by a subset of the dense subset $(\zeta_k, M(\zeta_k))$ of M as defined above. We suppress the unnecessary repetition of the arguments.

The proof is now complete.

COROLLARY. *Under the hypotheses of the theorem, there exists a unique minimizing arc joining p_1 to p_2 if and only if $C(a; p_1, p_2)$ is singleton, i.e. reduces to $R(x)$.*

PROOF. We show first that for any $0 \leqq a \leqq \sigma(p_1, p_2)$ there is at most one point of M in the corresponding $C(a; p_1, p_2)$. Suppose the contrary, and that two distinct points q, r of M belong to $C(a; p_1, p_2)$ for some value of a. Since $M(x)$ is a continuous function on $[x_1, x_2]$, the points q, r have different x-coordinates; we assume that the x-coordinate of q is less than that of r. By lemma 3 (d), $\sigma(p_1, q) + \sigma(q, r) + \sigma(r, p_2) = \sigma(p_1, p_2)$ and by theorem 3 (b), $\sigma(q, r) > 0$. Since q, $r \in C(a; p_1, p_2) = V_a(p_1) \cap V^{-1}_{\sigma(p_1, p_2) - a}(p_2)$ then $\sigma(p_1, q) \leqq a$, $\sigma(p_1, r) \leqq a$ and $\sigma(q, p_2) \leqq \sigma(p_1, p_2) - a$, $\sigma(r, p_2) \leqq \sigma(p_1, p_2) - a$. Since $\sigma(p_1, q) + \sigma(q, p_2) = \sigma(p_1, p_2)$, if $\sigma(p_1, q) < a$ set $\sigma(p_1, q) = a - \alpha$ for $\alpha > 0$; then $\sigma(p_1, p_2) = a - \alpha + \sigma(q, p_2) \leqq a - \alpha + \sigma(p_1, p_2) - a$ which is impossible. Thus, $\sigma(p_1, q) = a$. Again, since $\sigma(p_1, r) + \sigma(r, p_2) = \sigma(p_1, p_2)$, if $\sigma(r, p_2) < \sigma(p_1, p_2) - a$, set $\sigma(r, p_2) = \sigma(p_1, p_2) - a - \beta$ for $\beta > 0$; then $\sigma(p_1, p_2) = \sigma(p_1, r) + \sigma(p_1, p_2) - a - \beta \leqq a + \sigma(p_1, p_2) - a - \beta$ which is impossible. Thus, $\sigma(r, p_2) = \sigma(p_1, p_2) - a$. Substituting into the relation $\sigma(p_1, q) + \sigma(q, r) + \sigma(r, p_2) = \sigma(p_1, p_2)$ we conclude $\sigma(q, r) = 0$. This contradicts $\sigma(q, r) > 0$. Thus, there is at most one point of M in $C(a; p_1, p_2)$ for any value of a, $0 \leqq a \leqq \sigma(p_1, p_2)$.

If $C(a; p_1, p_2)$ is not singleton for some value of a, there is at least one point in $\bigcup_{a=0}^{\sigma(p_1, p_2)} C(a; p_1, p_2)$ which is not on M. By a trivial modification of the construction of $M(x)$ in the main theorem, it is possible to construct a function $L(x)$ which includes this point on its graph and which has the same property as $M(x)$, namely that $\sigma(p_1, p_2; L) = \sigma(p_1, p_2)$. L differs from M at at least one point, and therefore there is not a unique minimizing arc whenever $C(a; p_1, p_2)$ is not singleton for any value of a, $0 \leqq a \leqq \sigma(p_1, p_2)$.

The significance of the above theorem and its corollary may be elaborated upon. The corollary is in some sense a generalization of Jacobi's condition in the classical theory of the calculus of variations, necessary for the existence of a minimizing arc joining two fixed endpoints; for if $C(a; p_1, p_2)$ is not singleton then p_1 and p_2 may be said to be conjugate points. The theorem provides a set of sufficient conditions for the entire space H to have the property that every pair of points can be connected by an arc which minimizes $J[y]$ in the generalized sense we have introduced.

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ON THE INTEGRAL REPRESENTATION OF THE ENTROPY RATE

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A representation theorem for the entropy rate is proved for stationary discrete information sources with a countably infinite alphabet the statistical properties of which are described by a finitely additive invariant probability.

1. Preliminary discussion

It follows from the results of Kryloff and Bogoliouboff in ergodic theory that any discrete stationary source μ with a countable (i.e. finite or countably infinite) alphabet may be represented by an integral in the form

$$\mu = \int_R \mu_z \mu(dz), \tag{1.1}$$

where R is the set of regular points in the space of (infinite) messages, and where μ_z runs over the class of ergodic sources if z runs over the set R (cf. the systematic survey in [6]). PARTHASARATHY [7] proved that the entropy rate $H(\mu)$ of a source μ with a finite alphabet may be represented by an integral in the form

$$H(\mu) = \int_R H(\mu_z) \mu(dz). \tag{1.2}$$

WINKELBAUER [11] showed that the relation (1.2) is valid, in general, for sources with a countably infinite alphabet. The proof was based on an approximation theorem for the entropy rate (cf. [9], [8], or [11], Lemma 8.4, and Theorem 8.2, p. 812). However, Winkelbauer used the original concept of the entropy rate, called sometimes Kolmogoroff's entropy (cf. the relation (4.1) of the present paper), and, on the other hand, Parthasarathy and Rokhlin used a modified concept due to SINAJ [10]. As well-known, these concepts are equivalent provided Kolmogoroff's entropy is finite. Especially, for finite alphabet sources the relation (1.2) is valid independently on the concept of the entropy rate accepted.

The aim of this paper is to generalize the representation formula (1.2) for such sources the statistical properties of which are described by a finitely additive probability. A motivation of this research will be given in Section 3.

2. Basic notations

Throughout this paper the symbol A^B will be used to denote the set of all mappings of the set B into the set A. The discrete time will be represented by the set I of all integers. The symbol X will be used to designate the alphabet; it will be either the set N of all natural numbers, or one of the finite sets $\{1, \ldots, k\}$ $(k=1, 2, \ldots)$. We shall addopt the following convention: if an object will be defined, in general, for an arbitrary alphabet X, it will be indicated by the subscript X. If $X=N$, the subscript will be omitted; if $X=\{1, \ldots, k\}$, the subscript k will be used. The symbol $\mathbf{A}_X^I$ will represent the field of all finite-dimensional cylinders in the space X^I; a finite-dimensional cylinder set is, by definition, any set of the form

$$\{z : z\in X^I, \{z_j\}_{j\in J}\in E\} \quad \text{for} \quad E\subset X^J; \tag{2.1}$$

J being a finite non-empty subset of the set I. The symbol $\mathbf{F}_X^I$ will denote the σ-field generated by the field $\mathbf{A}_X^I$; in symbols $\mathbf{F}_X^I=\sigma(\mathbf{A}_X^I)$. Clearly

$$\mathbf{A}_k^I = \mathbf{A}^I\cap\{1, \ldots, k\}^I, \quad \mathbf{F}_k^I = \mathbf{F}^I\cap\{1, \ldots, k\}^I. \tag{2.2}$$

The symbol X^n will be used as an abbreviation for the set

$$\{(x_1, \ldots, x_n) : x_i\in X,\ i = 1, \ldots, n\}.$$

We shall put

$$[x] = \{z : z\in X^I, \{z_j\}_{0\leqq j<n} = x\} \quad \text{for} \quad x\in X^n, \tag{2.3}$$

$$[E]_{i,n} = \{z : z\in X^I, \{z_j\}_{i\leqq j<i+n}\in E\} \tag{2.4}$$

for $i\in I$, $n\in N$, and $E\subset X^n$. The coordinate-shift transformation of the space X^I (onto itself) will be denoted by T (without any subscript); it is defined by the property that

$$(Tz)_i = z_{i+1} \quad \text{for} \quad z\in X^I, \quad i\in I. \tag{2.5}$$

For fixed $n\in N$, the family $\{T^{-i}[x]: x\in X^n, i\in I\}$ will be designated by the symbol $\mathbf{P}_X^n$. Elements of the class

$$\mathbf{P}_X = \cup\{\mathbf{P}_X^n : n\in N\}$$

are said to be elementary cylinders.

The following concepts will be used frequently throughout the paper. A point $z\in X^I$ is said to be quasiregular provided the limit

$$\lim_n \frac{1}{n}\sum_{j=0}^{n-1}\chi_A(T^j z)$$

exists for every $A\in\mathbf{P}_X$ (the symbol χ_A denotes the indicator function of the set A). Consequently, the set Q_X of all quasiregular points in the space X^I can be expressed in the form

$$Q_X = \bigcap_{A\in\mathbf{P}_X}\left\{z : z\in X^I, \overline{\lim_n}\frac{1}{n}\sum_{j=0}^{n-1}\chi_A(T^j z) = \underline{\lim_n}\frac{1}{n}\sum_{j=0}^{n-1}\chi_A(T^j z)\right\}.$$

For $z \in Q_X$, let us define a set function μ_z on the class $\mathbf{P}_X$ by the property that

$$\mu_z(A) = \lim_n \frac{1}{n} \sum_{j=0}^{n-1} \chi_A(T^j z), \quad A \in \mathbf{P}_X. \tag{2.6}$$

If the set function μ_z satisfies the conditions

$$\sum_{A \in \mathbf{P}_X^n} \mu_z(A) = 1 \quad \text{for} \quad n = 1, 2, \ldots, \tag{2.7}$$

then it determines uniquely a T-invariant probability measure on the σ-field $\mathbf{F}_X^I$ (cf. Kolmogoroff's theorem, e.g. in [3], § 49). Let us recall that a set function μ is said to be T-invariant provided that $\mu = \mu T^{-1}$ (as to the symbol μT^{-1}, cf. [3], § 39). Let us note that if X is finite, the conditions (2.7) are automatically satisfied.

3. Existence of finitely additive invariant probabilities

In the present Section we shall consider exclusively the case $X = N$. For any invariant probability μ on the field $\mathbf{A}^I$ we shall set

$$\mu_n(E) = \mu([E]_{i,n}), \quad E \subset N^n; \quad n = 1, 2, \ldots \tag{3.1}$$

(cf. (2.4)). Since $\mu = \mu T^{-1}$, the relation (3.1) determines uniquely a probability μ_n on the field $\mathscr{P}(N^n)$ of all subsets of the set N^n ($n = 1, 2, \ldots$). Now the σ-additivity of all μ_n's implies the σ-additivity of μ itself by Kolmogoroff's theorem. Consequently, we have reduced the problem of existence of a finitely additive invariant probability on the field $\mathbf{A}^I$ to the problem of existence of a finitely additive probability on the field $\mathscr{P}(N^n)$ of all subsets of a countably infinite set N^n. We may assume (without any loss of generality) that $n = 1$.

The argument used to justify the existence of such probabilities will be based on the axiom of choice, therefore it is not constructive. The author was not successful in finding a constructive method. In fact, it seems that a constructively defined probability can be given only on a class of sets smaller than the class of all recursively enumerable subsets of N. But we do not go in details concerning this problem in this paper.

The set N being an infinite set of isolated points is not compact. Thus there is at least one ultrafilter of subsets of N which does not converge to a point in N (cf. e.g. [1], Lemma 12, p. 30). For example, let us consider the ultrafilter U containing the filter of complements of all finite sets in N. The existence of this ultrafilter is justified by Zorn's lemma. Let us define a set function μ by the property

$$\mu(E) = \begin{cases} 1, & \text{if} \quad E \in U, \\ 0, & \text{if} \quad E \notin U. \end{cases} \tag{3.2}$$

Clearly μ is an additive non-negative and normalized set function. Since U is an ultrafilter, for each $E \subset N$, either $E \in U$ or $E \notin U$, consequently μ is defined on the field $\mathscr{P}(N)$ of all subsets of N. Finally, it cannot be σ-additive since the set $N \in U$ is a countable union of disjoint one-point sets, which are not members of U.

A formally different (but based on the same non-constructive idea) justification may be based on Hahn—Banach theorem for functionals in normed spaces. Consider the field **A** generated by the filter mentioned above. The let us define a set function μ on **A** by the properties

$$\mu(E) = \begin{cases} 1, & \text{if } E \in \mathbf{A}, E \text{ infinite,} \\ 0, & \text{if } E \text{ is finite.} \end{cases} \tag{3.3}$$

Let us consider the space $S(\mathbf{A})$ of all simple functions defined on $(N, \mathbf{A})$. If $f \in S(\mathbf{A})$ then there are real (but not necessarily different) numbers $f_0, f_1, \ldots, f_n$ having the property that

$$f = \sum_{i=1}^{n} f_i \chi_{\{i\}} + f_0 \chi_{N-\{1,\ldots,n\}}.$$

The space $S(\mathbf{A})$ in the supremal norm is a normed linear subspace of the linear space $l^\infty(N)$ of all bounded sequences of real numbers. The functional f_μ^* defined on $S(\mathbf{A})$ by the property

$$f_\mu^* f = \int f \, d\mu \quad (= f_0)$$

is linear and evidently $\| f_\mu^* \|_{\text{on } S(\mathbf{A})} = 1$. By Hahn—Banach theorem there is a linear functional $\tilde{f}$ on the space $l^\infty(N)$, possessing the following properties:

(a) $$\text{for } f \in S(\mathbf{A}), \quad \tilde{f} f = f_\mu^* f,$$

(b) $$\| \tilde{f} \|_{\text{on } l^\infty(N)} = \| f_\mu^* \|_{\text{on } S(\mathbf{A})} = 1.$$

If $E \subset N$, then its indicator function χ_E may be identified with the sequence $\{\chi_E(n)\}_{n=1}^\infty \in l^\infty(N)$. Consequently, the restriction of $\tilde{f}$ to the family of all indicator functions of subsets in N yields under this identification a finitely additive probability on the field $\mathcal{P}(N)$ of all subsets of the set N.

Finally, let us make the following remark to finish this Section. If $z \in N^I$ is a point with coordinates

$$z_i = 1 \quad \text{for} \quad i \leqq 1, \quad z_i = i \quad \text{for} \quad i > 1, \quad i \in I,$$

then $z \in Q$ and $\mu_{z,1}$ (cf. (3.1)) is just the set function μ defined by (3.3). It is possible to find even a recursively enumerable subset of N, for which the limit μ_z will not exist. Consequently, we have to use the extension method given above to obtain a probabilility on the field $\mathcal{P}(N)$. But we cannot speak of the resulting set function on the field $\mathbf{A}^I$ as of a set function generated by the point z because the extension is, generally speaking, not unique.

4. The notion of entropy rate

The entropy rate of a σ-additive invariant probability measure μ on the field $\mathbf{A}_X^I$ (or, equivalently, on the σ-field $\mathbf{F}_X^I$) is, by definition, the limit

$$H(\mu) = -\lim_n \frac{1}{n} \sum_{x \in X^n} \mu[x] \log \mu[x] \tag{4.1}$$

(cf. (2.3)). The symbol log means $\log_2$, and we addopt the usual convention $0 \cdot \log 0 = 0$. The existence of the latter limit for invariant probabilities is a well-known fact from information theory. If $X = \{1, \ldots, k\}$ then it follows immediately from (4.1) that the inequality

$$H(\mu) \leqq \log k \tag{4.2}$$

is valid. Before proceeding to the modified concept of the entropy rate making sense also for finitely additive probabilities, let us introduce some useful notations. Let A denote an abstract non-empty set. If ζ is a finite decomposition of the set A, the symbol ζ^n will be used to denote the finite decomposition of the set A^n consisting of all sets expressible in the form $C^1 \times \ldots \times C^n$ with $C^i \in \zeta$ $(i = 1, \ldots, n)$; $n \in N$. If ζ and ξ are finite decompositions of the set A, the symbol $\zeta \vee \xi$ will designate the finite decomposition of the same set A, defined by the formula

$$\zeta \vee \xi = \{c \cap D \colon C \in \zeta, D \in \xi\}. \tag{4.3}$$

By induction,

$$\zeta_1 \vee \ldots \vee \zeta_k = (\zeta_1 \vee \ldots \vee \zeta_{k-1}) \vee \zeta_k \quad (k = 1, 3, \ldots). \tag{4.4}$$

Using the operation $\vee$ the relation $\succ$ of partial ordering can be introduced in the set of all finite decompositions of the given set A in the following way:

$$\zeta \succ \xi \quad \text{iff} \quad \zeta \cup \xi = \zeta. \tag{4.5}$$

Moreover, if ζ, ξ are finite decompositions then

$$\zeta \cup \xi \succ \zeta, \quad \zeta \vee \xi \succ \xi,$$

thus the set of all finite decompositions of a given set A is a directed set by means of the relation $\succ$ of partial ordering. The set of all finite decompositions of the alphabet N will be denoted by the symbol Z.

Let μ be an invariant finitely additive probability on the field $\mathbf{A}^I$, let $\xi \in Z$, respectively. Then let us set

$$H(\mu, \xi) = -\lim_n \frac{1}{n} \sum_{E \in \xi^n} \mu([E]_{i,n}) \log \mu([E]_{i,n}) \tag{4.6}$$

(cf. (2.4)). Since $\mu = \mu T^{-1}$, the limit does not depend on $i \in I$. The entropy rate $H(\mu)$ is then defined as the supremum

$$H(\mu) = \sup_{\xi \in Z} H(\mu, \xi). \tag{4.7}$$

Let us note that the definition (4.7) yields a quantity that equals the Sinaj's entropy when using finite decompositions of the space N^I consisting of sets in $\mathbf{A}^I$ in his definition.

5. Integral representation of the entropy rate

First let us introduce some concepts concerning invariant finitely additive probabilities. Let $\mathbf{A}$ be a field of subsets of an abstract non-empty set X. Let T disignate a one-to-one bi-$\mathbf{A}$-measurable transformation of the set X (onto itself). An invariant finitely additive probability μ defined on the field $\mathbf{A}$ is said to be ergodic

if there is no sequence $\{A_n\}_{n=1}^{\infty}$ of sets in $\mathbf{A}$, and no $\delta>0$, respectively, such that the following relations hold:

(1) $\lim_{m,n} \mu(A_n \Delta A_m) = 0$,
(2) $\lim_{n} \mu(A_n \Delta T^{-1} A_n) = 0$,
(3) $\delta < \mu(A_n) < 1-\delta$ for all $n \in N$,

the symbol Δ denoting as usually the symmetric difference; cf. [5]. We state some important properties of this concept of ergodicity. It is shown in [5] that the concept of ergodicity coincides with that of indecomposability in the family of all invariant finitely additive probabilities on $\mathbf{A}$. For μ σ-additive, the concept of ergodicity coincides with the familiar one. Let us recall that a σ-additive invariant probability measure is said to be ergodic provided it assumes only the values 0 and 1 on the sub σ-field of all invariant sets.

The set of all ergodic probabilities will be designated by $E(\mathbf{A})$. Given $A \in \mathbf{A}$, let us define the function f_A following the relation

$$f_A(v) = v(A), \quad v \in E(\mathbf{A}).$$

The symbol $\mathscr{B}(E(\mathbf{A}))$ is used to denote the smallest σ-field of subsets of $E(\mathbf{A})$ such such that all functions f_A are measurable. It is the well-known Kolmogoroff's σ-field. OLSHEN [5] proved that given a T-invariant finitely additive probability μ on the field $\mathbf{A}$ there is a unique probability measure $\hat{\mu}$ on the σ-field $\mathscr{B}(E(\mathbf{A}))$ having the property that

$$\mu(A) = \int_{E(\mathbf{A})} v(A)\hat{\mu}(dv), \quad A \in \mathbf{A}. \tag{5.1}$$

This representation formula will replace (1.1) in the finitely additive case. Now we shall state and prove the representation theorem for the entropy rate.

THEOREM. *The entropy rate $H(\mu)$ of a finitely additive invariant probability μ defined on the field $\mathbf{A}^I$ can be represented by an integral in the form*

$$H(\mu) = \int_{E(\mathbf{A}^I)} H(v)\hat{\mu}(dv). \tag{5.2}$$

PROOF. 1. In order to be possible to introduce probability measures induced by finite decompositions of the alphabet N, a fixed one-to-one numeration of elements of the finite decompositions must be established. Let $\xi \in Z$. Let C_1 denote that $C \in \xi$ for which $1 \in C$ (clearly there is just one such set). Then let us define $a_1 = \min\{n: n \in N - C_1\}$. Let C_2 denote that $C \in \xi$ for which $a_1 \in C$. Then let us define $a_2 = \min\{n: n \in N - C_1 \cup C_2\}$, etc. Finally, let us define

$$C_{\pi(\xi)} = N - \cup\{C_k : k = 1, 2, \ldots, \pi(\xi)-1\};$$

the symbol $\pi(\xi)$ denotes the number of elements of the finite decomposition ξ. We assume this way of one-to-one numeration as fixed throughout the proof. The formula

$$(\tau_\xi z)_i = k, \quad \text{if} \quad z_i \in C_k \quad \text{for} \quad z \in N^I, \quad i \in I \tag{1}$$

unambiguously determines a mapping $\tau_\xi\colon N^I \to \{1, \ldots, \pi(\xi)\}^I$. Since τ_ξ^{-1} takes cylinders to cylinders, i.e.

$$\tau_\xi^{-1}\mathbf{A}^I_{\pi(\xi)} \subset \mathbf{A}^I,$$

we can, given $\xi \in Z$, define the induced set function $\mu\tau_\xi^{-1}$ on the field $\mathbf{A}^I_{\pi(\xi)}$. Due to the fact that

$$\mu\tau\xi^{-1}(\{1, \ldots, \pi(\xi)\}^I) = 1,$$

$\mu\tau_\xi^{-1}$ is a σ-additive probability measure, which is invariant, too. The symbol $\bar{\mu}_\xi$ will be used to denote its unique extension to the σ-field $\mathbf{F}^I_{\pi(\xi)}$. The entropy rate $H(\bar{\mu}_\xi)$ is defined as the limit

$$H(\bar{\mu}_\xi) = -\lim \frac{1}{n} \sum \bar{\mu}_\xi[x] \log \bar{\mu}_\xi[x] \tag{2}$$

(cf. (4.1)); the summation being taken over the finite set $\{1, \ldots, \pi(\xi)\}^n$. Let us note that, by (2), the entropy rate depends only on the values assuming by the measure $\bar{\mu}_\xi$ on a subfamily of $\mathbf{A}^I_{\pi(\xi)}$. Consequently, no difference will be made between the symbols $H(\bar{\mu}_\xi)$ and $H(\mu\tau_\xi^{-1})$, respectively.

2. Now we shall show that for every $\xi \in Z$ the equality

$$H(\bar{\mu}_\xi) = \int_{E(\mathbf{A}^I)} H(\bar{\nu}_\xi)\hat{\mu}(d\nu) \tag{3}$$

is valid. Using (2) and (4.2) for $k = \pi(\xi)$ it can be easily checked that the non-negative function $H(\bar{\nu}_\xi)$ of the variable ν is bounded and $\mathscr{B}(E(\mathbf{A}^I))$-measurable, thus $\hat{\mu}$-integrable. Therefore the integral (3) has a definite meaning. Now let us fix an arbitrary $\xi \in Z$. For the sake of simplicity, let us denote $k = \pi(\xi)$. The relation (5.1) implies, especially, that

$$\mu(\tau_\xi^{-1}A) = \int_{E(\mathbf{A}^I)} \nu(\tau_\xi^{-1}A)\hat{\mu}(d\nu), \quad A \in \mathbf{A}^I_k.$$

It follows the relation

$$\bar{\mu}_\xi(E) = \int_{E(\mathbf{A}^I)} \bar{\nu}_\xi(E)\hat{\mu}(d\nu), \quad E \in \mathbf{F}^I_k. \tag{4}$$

Actually, using the extension theorem for measures we see that the right-hand side of (4) is the unique σ-additive extension of the right-hand side of the preceeding relation. To prove the equality sign in (4) it suffices to use repeatedly the extension theorem. The space $\{1, \ldots, k\}^I$ is a metric space, e.g. with respect to the distance function

$$\varrho(z, z') = \begin{cases} \max\left\{\dfrac{1}{1+|i|} : z_i \neq z'_i,\ i \in I\right\}, & \text{if} \quad z \neq z', \\ 0, \quad \text{if} \quad z = z';\ z, z' \in \{1, \ldots, k\}^I. \end{cases}$$

The topology induced by the distance function ϱ is equivalent with the product topology of the space $\{1, \ldots, k\}^I$, each space $\{1, \ldots, k\}$ considered with the discrete topology. Thus the Tychonoff theorem implies that $(\{1, \ldots, k\}^I, \varrho)$ is a compact metric space. We conclude that $\bar{\mu}_\xi$ is an invariant probability measure on the Borel σ-field $\mathbf{F}^I_k$ in a compact metric space, consequently

$$\bar{\mu}_\xi(E) = \int_{R_k} \mu_z(E)\bar{\nu}_\xi(dz), \quad E \in \mathbf{F}^I_k \tag{5}$$

(cf. [6], (2.6), p. 120). Here, $R_k = R \cap \{1, \ldots, k\}^I$, where R is the set of regular points in the space N^I, in symbols

$$R = \{z : z \in Q,\ \mu_z \text{ is } \sigma\text{-additive and ergodic}\}.$$

Both (4) and (5) imply that

$$\int_{E(\mathbf{A}^I)} \bar{\nu}_\xi(E)\,\hat{\mu}(dv) = \int_{R_k} \mu_z(E)\,\bar{\mu}_\xi(dz),$$

i.e.

$$\int_{E(\mathbf{A}^I)} \left[\int \chi_E(z)\,\bar{\nu}_\xi(dz)\right]\hat{\mu}(dv) = \int_{R_k} \left[\int \chi_E(z)\,\mu_x(dz)\right]\bar{\mu}_\xi(dx). \tag{6}$$

Next we shall extend the domain of validity of the relation (6). Let $\{E_1, \ldots, E_n\}$ be a finite $\mathbf{F}_k^I$-measurable decomposition of the space $\{1, \ldots, k\}^I$. Let f be a simple function on the decomposition given, then clearly

$$\int_{E(\mathbf{A}^I)} \left[\int f(z)\,\bar{\nu}_\xi(dz)\right]\hat{\mu}(dv) = \int_{R_k} \left[\int f(z)\,\mu_x(dz)\right]\bar{\mu}_\xi(dx) \tag{7}$$

using the fact that f is of the form $\sum_{i=1}^{n} a_i \chi_{E_i}$, and the definition of the σ-field $\mathscr{B}(E(\mathbf{A}^I))$ respectively.

Let f be a nonnegative, $\mathbf{F}_k^I$-measurable function on $\{1, \ldots, k\}^I$, bounded, say, by the constant $\log k$. Then f is $\bar{\nu}_\xi$-integrable (and μ_x-integrable as well), and we shall denote

$$F(v) = \int f(z)\,\bar{\nu}_\xi(dz), \quad F(z) = \int f(z)\,\mu_x(dz).$$

Evidently

$$F(v) \leqq \log k \quad \text{for all} \quad v \in E(\mathbf{A}^I),$$

$$F(x) \leqq \log k \quad \text{for all} \quad x \in R_k.$$

On the other hand, there is an increasing sequence $\{f_n\}_{n=1}^{\infty}$ of non-negative simple functions which converges to f pointwise. The monotone convergence theorem (cf. [3], § 27, Theorem B) implies that

$$F(v) = \int f(z)\bar{\nu}_\xi(dz) = \lim_n \int f_n(z)\bar{\nu}_\xi(dz), \quad v \in E(\mathbf{A}^I)$$

and

$$F(x) = \int f(z)\mu_x(dz) = \lim_n \int f_n(z)\mu_x(dz), \quad x \in R_k,$$

respectively. Let us denote

$$F_n(v) = \int f_n(z)\bar{\nu}_\xi(dz), \quad F_n(x) = \int f_n(z)\mu_x(dz).$$

Then

$$\int F_n(v)\,\hat{\mu}(dv) = \int F_n(x)\,\bar{\mu}_\xi(dx), \quad n = 1, 2, \ldots \tag{8}$$

(cf. (7)). Now $\{F_n(v)\}_{n=1}^{\infty}$ is a sequence of $\hat{\mu}$-integrable functions, uniformly bounded by $\log k$ and $F(v)$ is $\mathscr{B}(E(\mathbf{A}^I))$-measurable being the pointwise limit of measurable

functions. Moreover $F(\nu) \leqq \log k$, $\nu \in E(\mathbf{A}^I)$. Using the bounded convergence theorem (cf. [3], § 26, Theorem D) we conclude that $F(\nu)$ is $\hat{\mu}$-integrable and

$$\int_{E(\mathbf{A}^I)} F(\nu)\hat{\mu}(d\nu) = \lim_n \int_{E(\mathbf{A}^I)} F_n(\nu)\hat{\mu}(d\nu). \tag{9}$$

Analogously

$$\int_{R_k} F(x)\bar{\mu}_\xi(dx) = \lim_n \int_{R_k} F_n(x)\bar{\mu}_\xi(dx). \tag{10}$$

Now (9), (10), and (8) together imply that

$$\int_{E(\mathbf{A}^I)} \left[\int f(z)\bar{\nu}_\xi(dz)\right]\hat{\mu}(d\nu) = \int_{R_k} \left[\int f(z)\mu_x(dz)\right]\bar{\mu}_\xi(dx)$$

holds for every non-negative $\mathbf{F}_k^I$-measurable function bounded by $\log k$. Especially (cf. (4.2)), we obtain

$$\int_{E(\mathbf{A}^I)} \left[\int H(\mu_z)\bar{\nu}_\xi(dz)\right]\hat{\mu}(d\nu) = \int_{R_k} \left[\int H(\mu_z)\mu_x(dz)\right]\bar{\mu}_\xi(dx). \tag{11}$$

WINKELBAUER ([12], (1.8), p. 132) introduced a "universal" function $h_\mu(z)$ such that

(a) $h_\mu(z)$ equals with probability one the entropy rate of the ergodic component μ_z of any given stationary source with finite entropy, in symbols, if $H(\mu) < \infty$ then

$$\mu\{z : z \in R,\ h_\mu(z) = H(\mu_z)\} = 1$$

(cf. [12], Lemma 5, p. 139);

(b) if $H(\mu) < \infty$, then

$$\int h_\mu(z)\mu(dz) = H(\mu),$$

(cf. [12], (1.9), p. 132). Moreover, $h_\mu(z)$ is an invariant function. Both (a) and (b) apply in our case since the entropy rates are bounded by $\log k$ for every fixed $\xi \in Z$ with $\pi(\xi) = k$. If $\nu \in E(\mathbf{A}^I)$, then $\bar{\nu}_\xi$ is an ergodic probability measure, therefore there is a constant K such that

$$\bar{\nu}_\xi\{z : z \in R_k,\ h_{\bar{\nu}_\xi}(z) = K\} = 1. \tag{12}$$

Using (a), (b), and (12) for $\mu = \bar{\nu}_\xi$, we conclude that

$$\bar{\nu}_\xi\{z : z \in R_k,\ H(\mu_z) = H(\bar{\nu}_\xi)\} = 1.$$

Hence the left-hand side of (11) can be written in the form

$$\int_{E(\mathbf{A}^I)} \left[\int_{\{z : H(\mu_z) = H(\bar{\nu}_\xi)\}} H(\mu_z)\bar{\nu}_\xi(dz)\right]\hat{\mu}(d\nu) = \int_{E(\mathbf{A}^I)} H(\bar{\nu}_\xi)\hat{\mu}(d\nu). \tag{13}$$

The same consideration applied to the right-hand side of (1) yields the equality

$$\int_{R_{\pi(\xi)}} \left[\int_{\{z : H(\mu_z) = H(\mu_x)\}} H(\mu_z)\mu_x(dz)\right]\bar{\mu}_\xi(dx) = \int_{R_{\pi(\xi)}} H(\mu_x)\bar{\mu}_\xi(dx). \tag{14}$$

Since $\bar{\mu}_\xi(\{1, \dots, k\}^I) = 1$ for $k = \pi(\xi)$, the representation theorem of PARTHASARATHY [7] gives the equality

$$\int_{R_{\pi(\xi)}} H(\mu_x)\bar{\mu}_\xi(dx) = H(\bar{\mu}_\xi). \tag{15}$$

Now (11), (13), (14), and (15) together imply (3).

3. Let us consider the net $\{H(\mu, \xi)\}_{\xi \in Z}$ of non-negative real numbers. To show that it is monotone, we express $H(\mu, \xi)$ in a more convenient form. If $\xi \in Z$, we shall put

$$[\xi^n]_{0,n} = \{[C]_{0,n} : C \in \xi^n\}$$

(cf. (2.4)). If ζ is a $\mathbf{A}^I$-measurable finite decomposition of the space N^I, then its entropy given a probability μ on $\mathbf{A}^I$, is expressed by the formula

$$H_\mu(\zeta) = -\sum_{E \in \zeta} \mu(E) \log \mu(E).$$

Now

$$H(\mu, \xi) = \lim \frac{1}{n} H_\mu([\xi^n]_{0,n}) \quad \text{for} \quad \xi \in Z \tag{16}$$

(cf. (4.6)). Let $\xi, \zeta \in Z$, let $\xi \succ \zeta$ (cf. (4.4)). Then $\xi^n \succ \zeta^n$ for $n = 1, 2, \dots$, hence $[\xi^n]_{0,n} \succ [\zeta^n]_{0,n}$. Using a well-known property of the entropy of a finite probability field (cf. e.g. [2], Lemma 3, p. 10) we conclude

$$H_\mu([\xi^n]_{0,n}) \geqq H_\mu([\zeta^n]_{0,n}), \quad n = 1, 2, \dots,$$

hence

$$H(\mu, \xi) \geqq H(\mu, \zeta).$$

It follows immediately that

$$\lim_Z H(\mu, \xi) = \sup_{\xi \in Z} H(\mu, \xi) = H(\mu) \tag{17}$$

(cf. (4.7)); the limit on the left-hand side is the limit of the monotone net of non-negative real numbers indexed by the directed set Z.

4. From the definition of the mapping τ_ξ (cf. (1)) it follows that

$$H(\mu, \xi) = H(\mu\tau_\xi^{-1}) \; (= (\bar{\mu}_\xi)). \tag{18}$$

Actually, by (4.6) and (2), respectively, the relation (18) is a consequence of the one-to-one correspondence between the sets $\{E: E \in \xi^n\}$ and $\{1, \dots, \pi(\xi)\}^N$, respectively, established by the mapping τ_ξ.

5. If f is a non-negative real function defined on the set X, the symbol $f(x) \wedge n$ $(x \in X)$ means

$$f(x) \wedge n = \begin{cases} f(x), & \text{if } f(x) \leqq n, \\ n, & \text{if } f(x) > n. \end{cases}$$

The sequence $\{H(\cdot, \xi) \wedge n\}_{n=1}^\infty$ of $\hat{\mu}$-integrable functions pointwise increases to the $\mathscr{B}(E(\mathbf{A}^I))$-measurable limit $H(\cdot, \xi)$. The sequence $\{H(\cdot) \wedge n\}_{n=1}^\infty$ of $\hat{\mu}$-integrable functions pointwise increases to the $\mathscr{B}(E(\mathbf{A}^I))$-measurable limit $H(\cdot)$. The net $\{H(\cdot, \xi) \wedge n\}_{\xi \in Z}$ of $\hat{\mu}$-integrable functions pointwise increases to the integrable limit

$H(\cdot)\wedge n$. Therefore, given $\varepsilon>0$, $\delta>0$ there is $\xi(\varepsilon,\delta)\in Z$ such that for $\xi\succ\xi(\varepsilon,\delta)$ we have

$$\hat{\mu}\{v : |H(v,\xi)\wedge n - H(v)\wedge n| > \varepsilon\} < \delta. \tag{19}$$

Now for every $n\in N$ we have the obvious inequality

$$\left|\int_{E(\mathbf{A}^I)} H(v,\xi)\hat{\mu}(dv) - \int_{E(\mathbf{A}^I)} H(v)\hat{\mu}(dv)\right| \leqq \left|\int H(v,\xi)\hat{\mu}(dv) - \int (H(v,\xi)\wedge n)\,d\hat{\mu}\right| + \tag{20}$$

$$+\left|\int (H(v,\xi)\wedge n)\,d\hat{\mu} - \int (H(v)\wedge n)\,d\hat{\mu}\right| + \left|\int (H(v)\wedge n)\,d\hat{\mu} - \int H(v)\,d\hat{\mu}\right|,$$

and for the middle term we have

$$\left|\int (H(v,\xi)\wedge n)\,d\hat{\mu} - \int (H(v)\wedge n)\,d\hat{\mu}\right| \leqq \tag{21}$$

$$\leqq \int \left|(H(v,\xi)\wedge n) - (H(v)\wedge n)\right| d\hat{\mu} \leqq$$

$$\leqq \varepsilon + \int_A (H(v,\xi)\wedge n)\,d\hat{\mu} + \int_A (H(v)\wedge n)\,d\hat{\mu},$$

where

$$A = \{v : |H(v,\xi)\wedge n - H(v)\wedge n| > \varepsilon\}.$$

Now choose and fix $\varepsilon>0$. To the third and to the first member on the right-hand side of (20) the monotone convergence theorem (cf. [3], § 27, Theorem D) applies, consequently, there is $n_0\in N$ such that for $n\geqq n_0$ the first and the third terms are bounded by ε, respectively. Let us take the middle term for $n=n_0$. Then using (19), and the fact that $H(v,\xi)\wedge n_0\leqq n_0$, $H(v)\wedge n_0\leqq n_0$ for all $v\in E(\mathbf{A}^I)$, we obtain as a consequence of (21) the inequality

$$\left|\int (H(v,\xi)\wedge n_0)\,d\hat{\mu} - \int (H(v)\wedge n_0)\,d\hat{\mu}\right| \leqq \varepsilon + 2n_0\delta.$$

Taking $\delta=\varepsilon/2n_0$, we can summarize: given $\varepsilon>0$ there is $\xi_\varepsilon\in Z$ such that for $\xi\succ\xi_\varepsilon$ we have the inequality

$$\left|\int_{E(\mathbf{A}^I)} H(v,\xi)\,d\hat{\mu}(v) - \int_{E(\mathbf{A}^I)} H(v)\hat{\mu}(dv)\right| < 4\varepsilon.$$

Consequently

$$\lim_Z \int_{E(\mathbf{A}^I)} H(v,\xi)\,d\hat{\mu} = \int_{E(\mathbf{A}^I)} \lim_Z H(v,\xi)\hat{\mu}(dv) = \int_{E(\mathbf{A}^I)} H(v)\hat{\mu}(dv). \tag{22}$$

Summarizing (3), (17), and (22) we obtain the conclusion of the theorem.

REMARK. The methods and concepts used in the paper allow a more general formulation of the result. Namely, it was possible to consider an abstract alphabet X with a field $\mathbf{A}$ of subsets of the set X. The author prefered the formulation addopted in the paper since this shows in a more illustrative way the differences between the σ-additive and the finitely additive cases, respectively.

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ON DISTANCES NEAR INTEGERS, I.

by

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1. Throughout this paper, the distance between some points P, Q in the plane will be denoted by $\varrho(P, Q)$. We shall write $\|x\|$ for the distance of x from the nearest integer (the "norm" of x), i.e. $\|x\| = \min\{x-[x], [x]+1-x\}$. δ will denote some fixed real number satisfying

$$0 < \delta < 1/2. \tag{1}$$

For $X(>0)$, δ fixed, let $P_1, P_2, \ldots, P_n$ be points in the circle of radius X, such that

$$\|\varrho(P_i, P_j)\| \geqq \delta \quad \text{for} \quad 1 \leqq i < j \leqq n \tag{2}$$

(i.e. each of the distances between the given points is further from any integer than δ). Let us denote the maximal number of points with these properties by $N(X, \delta)$.

It is easy to prove that for some absolute constant c_δ (depending on δ) and X large, $N(X, \delta) < c_\delta X$ (see Lemma 2). P. Erdős conjectured (oral communication) that

$$\lim_{X \to +\infty} \frac{N(X, \delta)}{X} = 0 \tag{3}$$

for any fixed δ satisfying (1) and, on the other hand,

$$\lim_{X \to +\infty} N(X, \delta_0) = +\infty \tag{4}$$

for some fixed $\delta_0(>0)$. (Note that the latter conjecture reflects a phenomenon different from the one dimensional case. Namely, if $P_1, P_2, \ldots, P_k$ are any points on a fixed line then by the matchbox-principle, there exist i, j such that $i \neq j$ and $\|\varrho(P_i, P_j)\| \leqq \frac{1}{k}$; thus in the one dimensional case,

$$\min_{1 \leqq i < j \leqq k} \|\varrho(P_i, P_j\| \to 0 \quad \text{as} \quad k \to +\infty.$$

Conjecture (3) has not been proved yet while (4) has been proved by R. L. Graham (Erdős's oral communication).

Part I of this paper is devoted to the proof of a slightly sharper version of conjecture (3). (While in Part II, we shall improve Graham's lower estimate for $N(X, \delta)$.) In fact, we shall prove the following theorem here:

THEOREM. *For any* δ *satisfying* (1), *we have*

$$N(X,\delta)<\frac{4\cdot 10^4}{\delta^3}\frac{X}{\log\log X}$$

if X *is large enough (depending on* δ*).*

In order to prove this theorem, we shall prove five lemmas at first. Section 2 contains three lemmas: Lemma 1 will formulate the simple principle on that the proof of our theorem is based while Lemmas 2 and 3 will be corollaries of Lemma 1. Section 3 will consist of two lemmas (Lemmas 4 and 5). Both Lemmas 2 and 4 will be needed only for the proof of Lemma 5 which is the most difficult part of the proof. The role of Lemma 5 is to force out the applicability of Lemma 3 in the case when we have "many" points (in the circle of radius X). In fact, it will be shown in Section 4 that Lemmas 3 and 5 will imply our theorem easily.

2. LEMMA 1. *Let* e *be some line,* $Q_1, Q_2, \ldots, Q_m$ *be points in the plane. Let* Q_i' *denote the perpendicular projection of the point* Q_i *on the line* e. *Let us assume that*

$$m>\frac{9}{\delta} \tag{5}$$

and for $i=1,2,\ldots,m$, *either*

$$\varrho(Q_i,Q_i')<\delta/4 \tag{6}$$

or for $j=1,2,\ldots,i-1,i+1,\ldots,m$, *both*

$$\varrho(Q_i',Q_j')>1 \tag{7}$$

and

$$\varrho^2(Q_i,Q_i')<\frac{\delta}{4}\varrho(Q_i',Q_j') \tag{8}$$

hold. Then there exist i, j $(i\neq j)$ *such that*

$$\|\varrho(Q_i,Q_j)\|<\delta. \tag{9}$$

PROOF. Throughout the proof of Lemma 1, $\theta_1, \theta_2, \ldots$ will denote real numbers such that $|\theta_i|\leqq 1$ (for $i=1,2,\ldots$).

Let us introduce Cartesian coordinates in the following way: the x-axis should be the line e while the y-axis is arbitrary. Let $Q_i=(x_i,y_i)$. Then (6), (7), (8) can be written in form

$$|y_i|<\delta/4, \tag{10}$$

$$|x_j-x_i|>1 \tag{11}$$

and

$$y_i^2<\frac{\delta}{4}|x_j-x_i|. \tag{12}$$

Furthermore, define the positive integer a by

$$a-1\leqq\frac{8}{\delta}<a. \tag{13}$$

By (1),

$$a \leqq \frac{8}{\delta} + 1 < \frac{8}{\delta} + \frac{1}{2\delta} < \frac{9}{\delta}. \tag{14}$$

Hence, with respect to (5), $m > a$. Thus the matchbox principle gives that there exist integers k, i, j such that both

$$\frac{k-1}{a} \leqq x_i - [x_i] < \frac{k}{a} \quad \text{and} \quad \frac{k-1}{a} \leqq x_j - [x_j] < \frac{k}{a}$$

hold. Then obviously,

$$\|x_i - x_j\| = \|(x_i - x_j) - ([x_i] - [x_j])\| = \|(x_i - [x_i]) - (x_j - [x_j])\| < \frac{1}{a},$$

i.e. $x_i - x_j$ can be written in form

$$x_i - x_j = q + \frac{\theta_1}{a}$$

where q is integer. Hence, by (13),

$$x_i - x_j = q + \frac{\theta_2 \delta}{8}. \tag{15}$$

For any real numbers u, v,

$$(u+v)^2 \leqq 2(u^2+v^2).$$

Thus

$$\varrho(Q_i, Q_j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} = \sqrt{(x_i - x_j)^2 + 2\theta_3(y_i^2 + y_j^2)}. \tag{16}$$

Let us discuss the case (6) (i.e. case (10)) at first. Then we get from (16) with respect to (15) that

$$\varrho(Q_i, Q_j) = \sqrt{\left(q + \frac{\theta_2 \delta}{8}\right)^2 + 2\theta_3\left(\theta_4 \frac{\delta^2}{16} + \theta_5 \frac{\delta^2}{16}\right)} = \tag{17}$$

$$= \sqrt{\left(q + \frac{\theta_2 \delta}{8}\right)^2 + 4\theta_6 \frac{\delta^2}{16}} = \sqrt{\left(q + \frac{\theta_2 \delta}{8}\right)^2 + \theta_6 \frac{\delta^2}{4}}.$$

If $q = 0$ then

$$\varrho(Q_i, Q_j) = \sqrt{\frac{\theta_2^2 \delta^2}{64} + \frac{\theta_6 \delta^2}{4}} \leqq \sqrt{\frac{\delta^2}{64} + \frac{\delta^2}{4}} = \delta \sqrt{\frac{17}{64}} < \delta$$

which implies (9).

If $q \neq 0$ then by (1) and (17)

$$\varrho(Q_i, Q_j) = \sqrt{q^2 + \frac{q\theta_2 \delta}{4} + \frac{\theta_2^2 \delta^2}{64} + \theta_6 \frac{\delta^2}{4}} = \sqrt{q^2 + \frac{q\theta_2 \delta}{4} + \frac{17\theta_7 \delta^2}{64}} = \tag{18}$$

$$= \sqrt{q^2 + \frac{q\theta_2 \delta}{4} + \frac{17}{128}\theta_8 \delta q} = \sqrt{q^2 + \frac{49}{128}\theta_9 \delta q} = \sqrt{q^2 + \frac{\theta_{10} \delta q}{2}} =$$

$$= |q| \sqrt{1 + \theta_{10} \frac{\delta}{2q}} = |q| \left(1 + \theta_{11} \frac{\delta}{2q}\right) = |q| + \theta_{12} \frac{\delta}{2}$$

since

$$|\sqrt{1+x}-1| = \frac{|x|}{\sqrt{1+x}+1} \leqq |x| \quad \text{for} \quad |x| < 1.$$

But from (18)

$$\|\varrho(Q_i, Q_j)\| = \left\| |q| + \theta_{12}\frac{\delta}{2} \right\| = \left\| \theta_{12}\frac{\delta}{2} \right\| \leqq \left| \theta_{12}\frac{\delta}{2} \right| \leqq \frac{\delta}{2} < \delta,$$

i.e. (9) holds.

Finally, in the case when (7) and (8), i.e. (11) and (12) hold, we get from (16) (with respect to (1) and (15)) that

$$\varrho(Q_i, Q_j) = \sqrt{(x_i-x_j)^2 + 2\theta_{13}\frac{\delta}{4}|x_j-x_i| + 2\theta_{14}\frac{\delta}{4}|x_j-x_i|} =$$

$$= \sqrt{(x_i-x_j)^2 + \theta_{15}\delta|x_i-x_j|} = |x_i-x_j|\sqrt{1+\theta_{15}\frac{\delta}{|x_i-x_j|}} =$$

$$= |x_i-x_j|\left(1+\frac{3}{5}\theta_{16}\frac{\delta}{|x_i-x_j|}\right) = |x_i-x_j| + \frac{3}{5}\theta_{16}\delta =$$

$$= \left|q+\frac{\theta_2\delta}{8}\right| + \frac{3}{5}\theta_{16}\delta = |q| + \frac{1}{8}\theta_{17}\delta + \frac{3}{5}\theta_{16}\delta = |q| + \frac{29}{40}\theta_{18}\delta$$

since

$$|\sqrt{1+x}-1| = \frac{|x|}{\sqrt{1+x}+1} < \frac{|x|}{\sqrt{1/2}+1} < \frac{3}{5}|x| \quad \text{for} \quad |x| < \frac{1}{2}.$$

Hence

$$\|\varrho(Q_i, Q_j)\| = \left\| |q| + \frac{29}{40}\theta_{18}\delta \right\| = \left\| \frac{29}{40}\theta_{18}\delta \right\| \leqq \left| \frac{29}{40}\theta_{18}\delta \right| \leqq \frac{29}{40}\delta < \delta$$

and the proof of Lemma 1 is completed.

LEMMA 2. *If Y is large enough (depending on δ) then*

$$N(Y, \delta) < \frac{90}{\delta^2}Y.$$

PROOF. We have to show that if

$$n \geqq \frac{90}{\delta^2}Y \tag{19}$$

and $R_1, R_2, \ldots, R_n$ are points in the circle of radius Y then (for large enough Y) there exist i, j such that

$$\varrho(R_i, R_j) < \delta$$

(and $i \neq j$).

Let us introduce Cartesian coordinates in the following way: the origin should be the centre of the circle, otherwise the x and y-axes are arbitrary. Let us form the vertical strips

$$k\frac{\delta}{4} \leqq x < (k+1)\frac{\delta}{4}$$

where

$$k = 0, \pm 1, \pm 2, \ldots, \pm\left(\left[\frac{4}{\delta}Y\right]+1\right).$$

These strips cover the given circle and their number is

$$2\left(\left[\frac{4}{\delta}Y\right]+1\right)+1 < \left(2\frac{4}{\delta}Y+\frac{Y}{\delta}\right)+\frac{Y}{\delta} = \frac{10}{\delta}Y$$

for large enough Y. Thus the matchbox principle gives that at least one of these strips contains more than

$$\frac{n}{\frac{10}{\delta}Y} \geqq \frac{\frac{90}{\delta^2}Y}{\frac{10}{\delta}Y} = \frac{9}{\delta}$$

of the points $R_1, R_2, \ldots, R_n$ (with respect to (19)). Let k_0 denote that value of k which determines this strip, i.e. this strip should be defined by the inequality

$$k_0\frac{\delta}{4} \leqq x < (k_0+1)\frac{\delta}{4}. \tag{20}$$

Furthermore, denote the points R_t lying in this strip by $R_{t_1}, R_{t_2}, \ldots, R_{t_m}$. As we said,

$$m > \frac{9}{\delta}. \tag{21}$$

We are going to apply Lemma 1, choosing $Q_r = R_{t_r}$ $(r=1, 2, \ldots, m)$ and the line e as the line $x = k_0\frac{\delta}{4}$ (bounding the strip (20)). Then (5) holds by (21) and also (6) holds trivially since the width of the strip (20) is $\delta/4$. Thus Lemma 1 gives the existence of i, j $(i \neq j)$ such that $\|\varrho(R_{t_i}, R_{t_j})\| < \delta$ and the proof of Lemma 2 is completed.

LEMMA 3. *Let* $R_1, R_2, \ldots, R_s$ $(s \geqq 2)$ *be any points in the plane such that*

$$\sum_{i=2}^{s}\frac{1}{\varrho(R_1, R_i)} > \frac{18\pi}{\delta^2}. \tag{22}$$

Then there exist i, j *such that* $1 \leqq i < j \leqq s$ *and*

$$\|\varrho(R_i, R_j)\| < \delta. \tag{23}$$

PROOF. If $\varrho(R_1, R_u) < \delta$ for some $2 \leqq u \leqq s$ then i, j in (23) can be chosen as $i=1, j=u$. Thus we may assume that

$$\varrho\{R_1, R_i) \geqq \delta \quad \text{for} \quad i = 2, 3, \ldots, s. \tag{24}$$

Let us fix some line e_0 passing through R_1. For any real number φ, let e_φ denote that line which can be obtained from e_0 by rotating it through the angle φ. For any real number φ and any point $R(\neq R_1)$ in the plane, let us define the function $f_{R_1}(R, \varphi)$ in the following way: let $f_{R_1}(R, \varphi)=1$ if the distance from the point R to the line e_φ is less than $\delta/4$ and $f_{R_1}(R, \varphi)=0$ otherwise.

For some point $R\,(\neq R_1)$, let us denote the directed angle from e_0 to the line passing through R_1 and R by $\psi(R)$ (i.e. the line passing through R_1 and R should be identical to $e_{\psi(R)}$). For $\varrho(R_1, R) > \delta/4$, an easy geometrical consideration gives the existence of a real number $\gamma(R)$ such that

$$0 < \gamma(R) < \pi/2 \tag{25}$$

and $f_{R_1}(R, \varphi)=1$ holds if and only if φ lies in the (open) interval $(\psi(R)-\gamma(R), \psi(R)+\gamma(R))$ modulo π. (This angle $\gamma(R)$ depends only on $\varrho(R_1, R)$ thus we might put also $\gamma(R)=\Gamma(\varrho(R_1, R))$). We may determine $\gamma(R)$ from the right triangle whose hypotenuse is R_1R, the angle at R is $\gamma(R)$ and the leg opposite this angle is $\delta/4$ (Fig. 1).

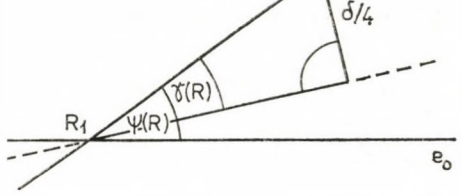

Fig. 1

Hence

$$\sin\gamma(R) = \frac{\delta/4}{\varrho(R_1, R)} = \frac{\delta}{4\varrho(R_1, R)} \quad (\text{for } \varrho(R_1, R) > \delta/4). \tag{26}$$

From (25) and (26),

$$\gamma(R) > \sin\gamma(R) = \frac{\delta}{4\varrho(R_1, R)} \quad (\text{for } \varrho(R_1, R) > \delta/4). \tag{27}$$

Let us investigate now the integral

$$I = \int_{-\pi/2}^{+\pi/2} \left(\sum_{i=2}^{s} f_{R_1}(R_i, \varphi) \right) d\varphi. \tag{28}$$

With respect to (22), (24) and (27):

$$I = \sum_{i=2}^{s} \int_{-\pi/2}^{+\pi/2} f_{R_1}(R_i, \varphi)\, d\varphi = \sum_{i=2}^{s} 2\gamma(R_i) > \sum_{i=2}^{s} \frac{\delta}{2\varrho(R_1, R_i)} > \frac{\delta}{2} \cdot \frac{18\pi}{\delta^2} = \frac{9\pi}{\delta}.$$

Since the value of the integral (28) is greater than $9\pi/\delta$ and the length of the interval on that we integrate is π, there must exist at least one real number $\varphi=\varphi_0$ for which the value of the integrand is greater than $\frac{9\pi/\delta}{\pi}=9/\delta$:

$$\sum_{i=2}^{s} f_{R_1}(R_i, \varphi_0) > 9/\delta.$$

The left hand side is equal to the number of the points R_i for that

$$f_{R_1}(R_i, \varphi_0) = 1. \tag{29}$$

Thus if $R_{i_1}, R_{i_2}, \ldots, R_{i_m}$ denote all the points R_i satisfying (29), i.e.

$$f_{R_1}(R_{i_j}, \varphi_0) = 1 \quad \text{for} \quad j = 1, 2, \ldots, m, \tag{30}$$

then

$$m > 9/\delta. \tag{31}$$

Let us choose $Q_1=R_{i_1}, Q_2=R_{i_2}, \ldots, Q_m=R_{i_m}$ and $e=e_{\varphi_0}$ in Lemma 1. Then (5) holds by (31) and (6) holds by (30) and the definition of the function $f_{R_1}(R, \varphi)$, thus Lemma 1 can be applied. We obtain the existence of u, v $(2 \leqq u < v \leqq s)$ such that

$$\|\varrho(Q_u, Q_v)\| = \|\varrho(R_{i_u}, R_{i_v})\| < \delta$$

and the proof of Lemma 3 is completed.

3. LEMMA 4. *Let a be any positive real number, P some point in the plane, e some line passing through P, $\mathbf{S}$ some strip of width a, bounded by lines parallel to e, and containing P. Let Q be an other point in the strip $\mathbf{S}$ such that*

$$\varrho(P, Q) > 2a. \tag{32}$$

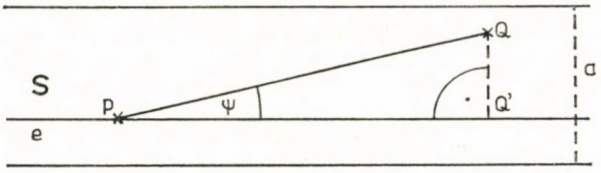

Fig. 2

Finally, let ψ denote that value of the angle between e and the line passing through P and Q, for which $0 \leqq \psi \leqq \pi/2$ holds. (Fig. 2.) Then

$$(0 \leqq)\ \psi < \frac{\pi}{3}\frac{a}{\varrho(P, Q)}. \tag{33}$$

PROOF. Let Q' denote the perpendicular projection of Q on the line e. Since the strip $\mathbf{S}$ contains both Q and Q', $\varrho(Q, Q')$ is less than or equal to the width of the strip:

$$\varrho(Q, Q') \leqq a. \tag{34}$$

ψ can be determined from the right triangle QPQ':

$$\sin\psi = \frac{\varrho(Q, Q')}{\varrho(P, Q)} \leqq \frac{a}{\varrho(P, Q)} \tag{35}$$

(by (34)). With respect to (32):

$$(0 \leqq) \sin\psi < \frac{a}{2a} = \frac{1}{2}. \tag{36}$$

The function $\frac{\psi}{\sin\psi}$ is increasing in the interval $[0, \pi/6]$ and $\frac{\pi/6}{\sin \pi/6} = \frac{\pi/6}{1/2} = \frac{\pi}{3}$. Thus (36) implies that

$$\psi < \frac{\pi}{3} \sin\psi. \tag{37}$$

(33) follows from (35) and (37), and thus Lemma 4 is proved.

LEMMA 5. *Let X be sufficiently large depending on δ, n be some positive integer satisfying*

$$n > \frac{10^4}{\delta^3} \frac{X}{\sqrt{\log X}}, \tag{38}$$

$P_1, P_2, \ldots, P_n$ be points in the circle **C** *of radius X, such that*

$$\| \varrho(P_i, P_j) \| \geqq \delta; \tag{39}$$

finally, let U be some real number satisfying

$$\log X \leqq U < X/3. \tag{40}$$

Then

$$\sum_{\substack{1 \leqq i, j \leqq n \\ \sqrt{U} \leqq \varrho(P_i, P_j) < 5U^2/\delta}} \frac{1}{\varrho(P_i, P_j)} > \frac{\delta}{216} \frac{n^2}{X} \tag{41}$$

(provided that X is large enough).

PROOF. By (40), we may assume that also U is sufficiently large (depending on δ).

For every $0 \leqq \varphi \leqq 2\pi$, let us introduce Cartesian coordinates x_φ, y_φ in the following way: the origin should be at the center of the circle **C**, the x_0, y_0-axes are arbitrary while the x_φ, y_φ-axes are obtained by rotating the x_0, y_0-axes through the (directed) angle φ.

For each

$$r = 0, \pm 1, \pm 2, \ldots, \pm([X\delta/4U^2]+1), \tag{42}$$

$$s = 0, \pm 1, \pm 2, \ldots, \pm([X/U]+1), \tag{43}$$

let us form the rectangle $\mathbf{T}_{r,s}(\varphi)$ determined by the points (x_φ, y_φ) satisfying

$$r\frac{4}{\delta}U^2 \leqq x_\varphi < (r+1)\frac{4}{\delta}U^2, \tag{44}$$

$$sU \leqq y_\varphi < (s+1)U. \tag{45}$$

These rectangles cover the circle $\mathbf{C}$ for every φ. For each of these rectangles $\mathbf{T}_{r,s}(\varphi)$, let $\tau_{r,s}(\varphi)$ denote the number of those P_i which are in $\mathbf{T}_{r,s}(\varphi)$:

$$\tau_{r,s}(\varphi) = \sum_{\substack{1\leqq i\leqq n \\ P_i\in\mathbf{T}_{r,s}(\varphi)}} 1.$$

Finally, let us define the function $\alpha_{i,j}(\varphi)$ for $1\leqq i,j\leqq n$, $0\leqq\varphi\leqq 2\pi$ in the following way: let $\alpha_{i,j}(\varphi)=1$ if the points P_i, P_j are in the same rectangle $\mathbf{T}_{r,s}(\varphi)$ (i.e. the coordinates of $P_i=(x_\varphi^{(i)}, y_\varphi^{(i)})$ and $P_j=(x_\varphi^{(j)}, y_\varphi^{(j)})$ satisfy (44) and (45) for the same pair (r, s)) and $\alpha_{i,j}(\varphi)=0$ otherwise.

We shall investigate the integral

$$I = \int_0^{2\pi}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{i,j}(\varphi)\right)d\varphi.$$

Deducing a lower and an upper estimate for this integral and comparing these estimates, we shall obtain the desired inequality (41).

We are going to give a lower estimate at first. For this purpose, we need a lower estimate for

$$\sum_{\substack{r \\ \tau_{r,s_0}(\varphi)>0}} 1$$

(i.e. for the number of those rectangles $\mathbf{T}_{r,s_0}(\varphi)$ which contain at least one of the points P_i) for each fixed s_0 and φ; this estimate will be based on Lemma 1.

For some s_0, φ fixed, let us denote those r for which $\tau_{r,s_0}(\varphi)>0$ holds, by $r_1, r_2, \ldots, r_t$ where

$$r_1 < r_2 < \ldots < r_t. \tag{46}$$

Let $2z-1$ denote the greatest odd integer not exceeding t, i.e.

$$z = t-[t/2]. \tag{47}$$

Let us pick up one of the points P_k from each of the rectangles $\mathbf{T}_{r_1,s_0}(\varphi)$, $\mathbf{T}_{r_3,s_0}(\varphi)$, $\mathbf{T}_{r_5,s_0}(\varphi)$, $\ldots$, $\mathbf{T}_{r_{2z-1},s_0}(\varphi)$; let, say, $P_{k_1}\in\mathbf{T}_{r_1,s_0}(\varphi)$, $P_{k_2}\in\mathbf{T}_{r_3,s_0}(\varphi)$, $\ldots$, $P_{k_z}\in\mathbf{T}_{r_{2z-1},s_0}(\varphi)$.

Let e denote the line $y_\varphi=s_0U$ which bounds the strip

$$s_0U \leqq y < (s_0+1)U \tag{48}$$

and let us denote the perpendicular projection of P_{k_i} $(1\leqq i\leqq z)$ on the line e by P'_{k_i}. Obviously, $|p-q|\geqq 2$, $A\in\mathbf{T}_{p,s}(\varphi)$, $B\in\mathbf{T}_{q,s}(\varphi)$ imply that

$$\varrho(A, B) > \frac{4}{\delta}U^2$$

since the length of the longer sides of the rectangles $\mathbf{T}_{r,s}(\varphi)$ is $\frac{4}{\delta}U^2$. Thus

$$\varrho(P'_{k_i}, P'_{k_j}) > \frac{4}{\delta}U^2 \quad \text{for} \quad 1 \leqq i < j \leqq z \tag{49}$$

(by (46)). This implies that if U is large enough (depending on δ) then

$$\varrho(P'_{k_i}, P'_{k_j}) > 1 \quad \text{for} \quad 1 \leqq i < j \leqq z. \tag{50}$$

Finally, $\varrho(P_{k_i}, P'_{k_i})$ is less than the width of the strip (48):

$$\varrho(P'_{k_i}, P'_{k_i}) < U \quad \text{for} \quad i = 1, 2, \ldots, z. \tag{51}$$

By (49), (50) and (51), both (7) and (8) in Lemma 1 hold for the points $Q_1 = P_{k_1}, Q_2 = P_{k_2}, \ldots, Q_z = P_{k_z}$ (and for the line e defined by $y_\varphi = s_0 U$). Thus

$$z \leqq 9/\delta, \tag{52}$$

since otherwise Lemma 1 would give the existence of points P_{k_i}, P_{k_j} $(i \neq j)$ such that

$$\varrho(P_{k_i}, P_{k_j}) < \delta$$

which would contradict to our assumption (39).

From (47) and (52):

$$9/\delta \geqq z = t - [t/2] \geqq t - t/2 = t/2,$$

hence

$$t = \sum_{\substack{r \\ \tau_{r,s_0}(\varphi) > 0}} 1 \leqq 18/\delta$$

(for each fixed s_0).

The number of the strips of form (45), i.e. the number of the integers s in (43) is

$$2([X/U]+1)+1 \leqq 2\frac{X}{U} + 3 < 3\frac{X}{U},$$

since $3 < X/U$ by (40). Thus

$$\sum_{\substack{r,s \\ \tau_{r,s}(\varphi) > 0}} 1 < \frac{18}{\delta} \cdot 3\frac{X}{U} = \frac{54X}{\delta U}. \tag{53}$$

The integrand in I can be rewritten in the following form:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{i,j}(\varphi) = \sum_{\substack{r,s \\ \tau_{r,s}(\varphi)>0}} \left(\sum_{\substack{1 \leqq i \leqq n \\ P_i \in \mathbf{T}_{r,s}(\varphi)}} \sum_{\substack{1 \leqq j \leqq n \\ P_j \in \mathbf{T}_{r,s}(\varphi)}} 1 \right) = \sum_{\substack{r,s \\ \tau_{r,s}(\varphi)>0}} \left(\sum_{\substack{1 \leqq i \leqq n \\ P_i \in \mathbf{T}_{r,s}(\varphi)}} 1 \right)^2 = \sum_{\substack{r,s \\ \tau_{r,s}(\varphi)>0}} (\tau_{r,s}(\varphi))^2.$$

By (53), this sum has less than $\frac{54X}{\delta U}$ terms. Thus by Cauchy's inequality:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{i,j}(\varphi) = \sum_{\substack{r,s \\ \tau_{r,s}(\varphi)>0}} (\tau_{r,s}(\varphi))^2 \geqq \frac{\left(\sum\limits_{\substack{r,s \\ \tau_{r,s}(\varphi)>0}} \tau_{r,s}(\varphi) \right)^2}{\frac{54X}{\delta U}} = \frac{n^2}{\frac{54X}{\delta U}} = \frac{n^2 \delta U}{54X}.$$

This inequality holds for each $0 \leqq \varphi \leqq 2\pi$ thus we get the following lower estimate for I:

$$I = \int_0^{2\pi} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{i,j}(\varphi)\, d\varphi \geqq \int_0^{2\pi} \frac{n^2 \delta U}{54\, X}\, d\varphi = \frac{n^2 \delta U \pi}{27X}. \tag{54}$$

To obtain an upper estimate for I, we change the order of integration and summation:

$$I = \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^{2\pi} \alpha_{i,j}(\varphi)\, d\varphi. \tag{55}$$

If $\alpha_{i,j}(\varphi) \neq 0$ (i.e. $\alpha_{i,j}(\varphi) = 1$) then P_i, P_j are in the same rectangle $\mathbf{T}_{r,s}(\varphi)$, the lengths of the sides of whose are U and $\frac{4}{\delta} U^2$. Thus in this case,

$$\varrho(P_i, P_j) \leqq \sqrt{U^2 + \left(\frac{4}{\delta} U^2\right)^2} = \sqrt{U^2 + \frac{16}{\delta^2} U^4} < \sqrt{\frac{U^2}{\delta^2} U^2 + \frac{16}{\delta^2} U^4} = \sqrt{\frac{17}{\delta^2} U^4} < \frac{5}{\delta} U^2$$

for large enough U. Consequently, we may drop all the terms with $\varrho(P_i, P_j) \geqq \frac{5}{\delta} U^2$ in (55):

$$I = \sum_{i=1}^{n} \left(\sum_{\substack{1 \leqq j \leqq n \\ \varrho(P_i, P_j) < 5U^2/\delta}} \int_0^{2\pi} \alpha_{i,j}(\varphi)\, d\varphi. \right). \tag{56}$$

To estimate the inner sum, we split it into three parts:

$$\sum_{\substack{1 \leqq j \leqq n \\ \varrho(P_i, P_j) < 5U^2/\delta}} \int_0^{2\pi} \alpha_{i,j}(\varphi)\, d\varphi = \sum_{\substack{1 \leqq j \leqq n \\ \varrho(P_i, P_j) < \sqrt{U}}} \int_0^{2\pi} \alpha_{i,j}(\varphi)\, d\varphi + \tag{57}$$

$$+ \sum_{\substack{1 \leqq j \leqq n \\ \sqrt{U} \leqq \varrho(P_i, P_j) \leqq 2U}} \int_0^{2\pi} \alpha_{i,j}(\varphi)\, d\varphi + \sum_{\substack{1 \leqq j \leqq n \\ 2U < \varrho(P_i, P_j) < 5U^2/\delta}} \int_0^{2\pi} \alpha_{i,j}(\varphi)\, d\varphi = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3.$$

The first term can be estimated by applying Lemma 2:

$$\sum\nolimits_1 \leqq \sum_{\substack{1 \leqq j \leqq n \\ \varrho(P_i, P_j) < \sqrt{U}}} \int_0^{2\pi} 1\, d\varphi = 2\pi \sum_{\substack{1 \leqq j \leqq n \\ \varrho(P_i, P_j) < \sqrt{U}}} 1 \leqq 2\pi N(\sqrt{U}, \delta) < 2\pi \frac{90}{\delta^2} \sqrt{U} \tag{58}$$

for large enough U.

The second term:

$$\sum\nolimits_2 \leqq \sum_{\substack{1 \leqq j \leqq n \\ \sqrt{U} \leqq \varrho(P_i, P_j) \leqq 2U}} \int_0^{2\pi} 1\, d\varphi = \tag{59}$$

$$= 2\pi \sum_{\substack{1 \leqq j \leqq n \\ \sqrt{U} \leqq \varrho(P_i, P_j) \leqq 2U}} 1 \leqq 2\pi \sum_{\substack{1 \leqq j \leqq n \\ \sqrt{U} \leqq \varrho(P_i, P_j) \leqq 2U}} \frac{2U}{\varrho(P_i, P_j)} = 4\pi U \sum_{\substack{1 \leqq j \leqq n \\ \sqrt{U} \leqq \varrho(P_i, P_j) \leqq 2U}} \frac{1}{\varrho(P_i, P_j)}.$$

The estimate of the third term is based on Lemma 4. Namely, if $\alpha_{i,j}(\varphi)\neq 0$ (i.e. $\alpha_{i,j}(\varphi)=1$) then P_i, P_j are in the same rectangle $\mathbf{T}_{r,s}(\varphi)$ (of width U). Since in $\sum_3$, $2U<\varrho(P_i, P_j)$ is assumed, (32) in Lemma 4 holds with $P=P_i$, $Q=P_j$, $a=U$ in case $\alpha_{i,j}(\varphi)\neq 0$. Thus denoting the directed angle from the x_0-axis to the line passing through P_i and P_j by $\varphi_{i,j}$, Lemma 4 gives that if P_i, P_j are in the same rectangle $\mathbf{T}_{r,s}(\varphi)$ of width U then

$$|\psi| = |\varphi_{i,j}-\varphi| < \frac{\pi}{3}\frac{U}{\varrho(P_i,P_j)}.$$

modulo π; i.e.

$$\left\|\frac{\psi}{\pi}\right\| = \left\|\frac{\varphi_{i,j}-\varphi}{\pi}\right\| < \frac{U}{3\varrho(P_i, P_j)}.$$

Hence, for any i, j satisfying $2U<\varrho(P_i, P_j)$,

$$\int_0^{2\pi} \alpha_{i,j}(\varphi)\,d\varphi = 2 \int_{|\varphi_{i,j}-\varphi|<\frac{\pi U}{3\varrho(P_i,P_j)}} \alpha_{i,j}(\varphi)\,d\varphi \leqq \tag{60}$$

$$\leqq 2 \int_{|\varphi-\varphi_{i,j}|<\frac{\pi U}{3\varrho(P_i,P_j)}} 1\,d\varphi = \frac{4\pi U}{3}\frac{1}{\varrho(P_i, P_j)}.$$

Summarizing, we obtain from (56), (57), (58), (59) and (60) that

$$I=\sum_{i=1}^{n}(\sum_1+\sum_2+\sum_3) < \sum_{i=1}^{n}\left(2\pi\frac{90}{\delta^2}\sqrt{U}+4\pi U \sum_{\substack{1\leqq j\leqq n\\ \sqrt{U}\leqq\varrho(P_i,P_j)\leqq 2U}}\frac{1}{\varrho(P_i, P_j)}+\right. \tag{61}$$

$$\left.+\sum_{\substack{1\leqq j\leqq n\\ 2U<\varrho(P_i,P_j)<5U^2/\delta}}\frac{4\pi U}{3}\cdot\frac{1}{\varrho(P_i,P_j)}\right) \leqq \frac{180\pi}{\delta^2}\sqrt{U}n+4\pi U\sum_{\substack{1\leqq i,j\leqq n\\ \sqrt{U}\leqq\varrho(P_i,Pj)<5U^2/\delta}}\frac{1}{\varrho(P_i,P_j).}$$

Comparing the lower estimate (54) with the upper estimate (61), we obtain that

$$\frac{\pi n^2\delta U}{27X} < \frac{180\pi\sqrt{U}n}{2}+4\pi U\sum_{\substack{1\leqq i,j\leqq n\\ \sqrt{U}\leqq\varrho(P_i,P_j)<5U^2/\delta}}\frac{1}{\varrho(P_i, P_j)}$$

or in equivalent form

$$\sum_{\substack{1\leqq i,j\leqq n\\ \sqrt{U}\leqq\varrho(P_i,P_j)<5U^2/\delta}}\frac{1}{\varrho(P_i, P_j)} > n\left(\frac{n\delta}{108X}-\frac{45}{\delta^2\sqrt{U}}\right).$$

Hence, by (38) and (40),

$$\sum_{\substack{1\leqq i,j\leqq n\\ \sqrt{U}\leqq\varrho(P_i,P_j)<5U^2/\delta}}\frac{1}{\varrho(P_i, P_j)} > n\left(\frac{n\delta}{108X}-\frac{45}{\delta^2\sqrt{\log X}}\right) > n\left(\frac{n\delta}{108X}-\frac{45}{\delta^2}\cdot\frac{\delta^3 n}{10^4 X}\right) =$$

$$=\frac{\delta n^2}{X}\left(\frac{1}{108}-\frac{45}{10^4}\right) > \frac{\delta n^2}{X}\left(\frac{1}{108}-\frac{1}{216}\right) = \frac{\delta n^2}{X}\cdot\frac{1}{216}$$

and the proof of Lemma 5 is completed.

4. Completion of the proof of the theorem

Let us assume indirectly that

$$n \geqq \frac{4 \cdot 10^4}{\delta^3} \frac{X}{\log\log X} \tag{62}$$

and $P_1, P_2, \ldots, P_n$ are points in the circle of radius X such that

$$\|\varrho(P_i, P_j)\| \geqq \delta \quad \text{whenever} \quad 1 \leqq i < j \leqq n. \tag{63}$$

We have to show that if is X large enough (depending on δ), these assumptions are contradictory.

Let

$$U_1 = \log X \quad \text{and} \quad U_{k+1} = 25U_k^4/\delta^2 \quad \text{for} \quad k = 1, 2, \ldots, \tag{64}$$

and let us define k_0 by

$$U_{k_0} < X/3 \leqq U_{k_0+1}. \tag{65}$$

It can be shown easily by straight induction that for large enough X,

$$\log X = U_1 < U_{k+1} = \frac{25}{\delta^2} U_k^4 < U_k^5 < U_1^{5^k} \quad \text{for} \quad k = 1, 2, \ldots. \tag{66}$$

Thus (65) implies (for large X) that

$$\log\log X/3 \leqq \log\log U_{k_0+1} < \log\log U_1^{5^{k_0}} =$$

$$= k_0 \log 5 + \log\log U_1 = k_0 \log 5 + \log\log\log X.$$

Hence for large enough X,

$$k_0 > \frac{\log\log X/3 - \log\log\log X}{\log 5} = \frac{\log\log X}{\log 5}(1+o(1)) < \frac{\log\log X}{2}. \tag{67}$$

Odviously,

$$\sum_{i=1}^{n}\left(\sum_{\substack{1\leqq j\leqq n\\ j\neq i}} \frac{1}{\varrho(P_i, P_j)}\right) \geqq \sum_{\substack{1\leqq i,\, j\leqq n\\ \sqrt{U_1}\leqq\varrho(P_i,\,P_j)<5U_{k_0}^2/\delta}} \frac{1}{\varrho(P_i, P_j)} = \tag{68}$$

$$= \sum_{k=1}^{k_0}\left(\sum_{\substack{1\leqq i,\, j\leqq n\\ \sqrt{U_k}\leqq\varrho(P_i,\,P_j)<5U_k^2/\delta}} \frac{1}{\varrho(P_i, P_j)}\right)$$

(since $5U_k^2/\delta = \sqrt{U_{k+1}}$ for $k = 1, 2, \ldots$). By (65) and (66),

$$\log X \leqq U_k < X/3 \quad \text{for} \quad k = 1, 2, \ldots, k_0. \tag{69}$$

By (62), (63) and (69), each of (38), (39) and (40) in Lemma 5 holds for the points $P_1, P_2, \ldots, P_n$ and for $U = U_1, U_2, \ldots, U_{k_0}$ (provided that X is large enough),

thus Lemma 5 can be applied to estimate each of the inner sums obtained in (68). We get that

$$\sum_{i=1}^{n}\left(\sum_{\substack{1\leqq j\leqq n\\ j\neq i}}\frac{1}{\varrho(P_i,P_j)}\right)>\sum_{k=1}^{k_0}\frac{\delta}{216}\cdot\frac{n^2}{X}=\frac{\delta}{216}\cdot\frac{n^2}{X}\cdot k_0>\frac{\delta}{216}\cdot\frac{n^2}{X}\cdot\frac{\log\log X}{2}=$$

$$=\frac{\delta}{512}\cdot\frac{n^2\log\log X}{X}$$

with respect to (67). Thus the greatest of the inner sums on the left hand side:

$$\max_{1\leqq i\leqq n}\sum_{\substack{1\leqq j\leqq n\\ j\neq i}}\frac{1}{\varrho(P_i,P_j)}>\frac{\frac{\delta}{512}\cdot\frac{n^2\log\log X}{X}}{n}=\frac{\delta}{512}\cdot\frac{n\log\log X}{X}.$$

In other words, for some $1\leqq i_0\leqq n$,

$$\sum_{\substack{1\leqq j\leqq n\\ j\neq i_0}}\frac{1}{\varrho(P_{i_0},P_j)}>\frac{\delta}{512}\,n\,\frac{\log\log X}{X}\geqq \tag{70}$$

$$\geqq\frac{\delta}{512}\,\frac{4\cdot 10^4}{\delta^3}\,\frac{X}{\log\log X}\,\frac{\log\log X}{X}=\frac{10^4}{128\delta^2}>\frac{18\pi}{\delta^2}$$

by (62).

Let us choose R_1 as P_{i_0} and $R_2,\ldots,R_s$ as $P_1,\ldots,P_{i_0-1},P_{i_0+1},\ldots,P_n$ in Lemma 3. Then (22) holds by (70), thus Lemma 3 is applicable and we obtain the existence of points $R_i=P_u$, $R_j=P_v$ such that

$$\|\varrho(R_i,R_j)\|=\|\varrho(P_u,P_v)\|<\delta\quad(\text{where } i\neq j,\ u\neq v).$$

This contradicts to our indirect assumption (63) which proves the theorem.

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APPLICATION OF THE AVERAGING METHOD TO SOLVING BOUNDARY PROBLEMS FOR SYSTEMS OF INTEGRO-DIFFERENTIAL EQUATIONS WITH A LINEAR BOUNDARY CONDITION

by

D. D. BAINOV and S. D. MILUSHEVA

The averaging method was first applied to solving the Cauchy problem for systems of integro-differential equations by A. N. FILATOV and his school [1].

The present paper gives the proofs of two theorems which jusitfy the averaging method when solving boundary problems for systems of integro-differential equations of a standard type.

Consider the system

$$\dot{x} = \varepsilon X\Big(t, x, \int_0^t \varphi(t, s, x(s))\, ds\Big) \tag{1}$$

with boundary conditions

$$\sum_{i=0}^{N} A_i X(t_i) = 0, \tag{2}$$

where $x, X \in R_n$, $\varphi \in R_p$, $A_i = (a_{jk}^{(i)})_1^n$, $t_i = \alpha_i T$, $i = \overline{0, N}$, $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_N = 1$, $T = L\varepsilon^{-1}$, $L = \text{const} > 0$, and $\varepsilon > 0$ is a small parameter.

For the system (1) several schemes of averaging are known [1]. Here we will consider two of them.

1. First scheme of averaging

Assume that the integral in (1) can be calculated if t and $x(s)$ are suitable constants (parameters),

$$\int_0^t \varphi(t, s, x)\, ds = \varphi_1(t, x).$$

If the limit

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T X(t, x, \varphi_1(t, x))\, dt = \overline{X}(x) \tag{3}$$

exists, then an averaged system for the first approximation of the solution $x(t)$ of the system (1) is defined to be the system

$$\dot{\xi} = \varepsilon \overline{X}(\xi) \tag{4}$$

with boundary conditions

$$\sum_{i=0}^{N} A_i \xi(t_i) = 0. \tag{5}$$

Note that if $x=(x^{(1)}, \ldots, x^{(n)})$ and $A=(a_{jk})_{lm}$, then by definition

$$\|x\| = \left[\sum_{i=1}^{n} (x^{(i)})^2\right]^{\frac{1}{2}}, \quad \|A\| = \left[\sum_{k=1}^{m}\sum_{j=1}^{l} a_{jk}^2\right]^{\frac{1}{2}}.$$

THEOREM 1. *Let the functions $X(t, x, y)$ and $\varphi(t, s, x)$ be defined and continuous in the domain $\Omega\{t\geqq 0, s\geqq 0, x\in D\subset R_n\}$ and let us make the following six assumptions:*

1. *These functions satisfy in Ω*

$$\|X(t, x, y)-X(t, x', y')\| \leqq \lambda\{\|x-x'\|+\|y-y'\|\}, \quad \lambda = \text{const},$$

$$\|\varphi(t, s, x)-\varphi(t, s, x')\| \leqq \mu(t, s)\|x-x'\|,$$

with

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t d\tau \int_0^\tau \mu(\tau, s)\, ds = 0.$$

2. *The boundary problem* (1), (2) *has a unique continuous solution $x(t)\in D$ for $0\leqq t\leqq T$.*

3. *For every $x\in D$ the limit* (3) *exists and the function $\overline{X}(x)$, defined and continuous in the domain D, satisfies the conditions*

$$\|\overline{X}(x)\| \leqq M, \quad \|\overline{X}(x)-\overline{X}(x')\| \leqq \nu\|x-x'\|,$$

where M and ν are positive constants.

4. *The boundary problem* (4), (5) *has a unique continuous solution $\xi(t)\in D$ for $0\leqq t\leqq T$.*

6. *The matrices A_i, $i=\overline{1, N}$, depend on ε and the function $b(\varepsilon)=\max_i \|A_i(\varepsilon)\|$ is continuous at sufficiently small values of ε, finally $\lim_{\varepsilon\to 0} b(\varepsilon)=0$.*

Then, for every $\eta>0$ there can be found a number $\varepsilon^0>0$, such that when $0<\varepsilon\leqq\varepsilon^0$, the inequality

$$\|x(t)-\xi(t)\| < \eta$$

holds for $0\leqq t\leqq T$.

PROOF. Under the conditions of the theorem for the solutions of boundary problems (1), (2) and (4), (5), the following equalities hold:

$$x(t) = x_0+\varepsilon\int_0^t X\Big(\tau, x(\tau), \int_0^\tau \varphi(\tau, s, x(s))\, ds\Big)\, d\tau, \tag{6}$$

$$\xi(t) = \xi_0+\varepsilon\int_0^t \overline{X}(\xi(\tau))\, d\tau, \tag{7}$$

where

$$x_0 = -\varepsilon A^{-1}(\varepsilon)\sum_{i=1}^{N} A_i(\varepsilon)\int_0^{t_i} X\Big(\tau, x(\tau), \int_0^\tau \varphi(\tau, s, x(s))\, ds\Big)\, d\tau,$$

$$\xi_0 = -\varepsilon A^{-1}(\varepsilon)\sum_{i=1}^{N} A_i(\varepsilon)\int_0^{t_i} \overline{X}(\xi(\tau))\, d\tau, \quad A(\varepsilon) = \sum_{i=0}^{N} A_i(\varepsilon).$$

From (6) and (7) and the conditions of the theorem, at sufficiently small values of ε $(0<\varepsilon\leqq\varepsilon_1)$ we obtain the inequality

$$\|x(t)-\xi(t)\| \leqq \varepsilon h(\varepsilon) \sum_{i=1}^{N} \left\| \int_0^{t_i} \left[X\left(\tau, x(\tau), \int_0^{\tau} \varphi(\tau, s, x(s))\, ds\right) - \overline{X}(\xi(\tau)) \right] d\tau \right\| +$$

$$+\varepsilon \left\| \int_0^{t} \left[X\left(\tau, x(\tau), \int_0^{\tau} \varphi(\tau, s, x(s))\, ds\right) - \overline{X}(\xi(\tau)) \right] d\tau \right\|, \tag{8}$$

where

$$h(\varepsilon) = b(\varepsilon)\|A_0^{-1}\|\|Q^{-1}(\varepsilon)\|, \quad Q(\varepsilon) = E + A_0^{-1} \sum_{i=1}^{N} A_i(\varepsilon), \ \det Q(\varepsilon) \neq 0$$

and E is the unit matrix.

We will estimate, following [1], the second summand on the right hand side of (8). For this purpose we use the representation

$$\varepsilon \int_0^{t} \left[X\left(\tau, x(\tau), \int_0^{\tau} \varphi(\tau, s, x(s))\, ds\right) - \overline{X}(\xi(\tau)) \right] d\tau = \tag{9}$$

$$= \varepsilon \int_0^{t} \left[X\left(\tau, x(\tau), \int_0^{\tau} \varphi(\tau, s, x(s))\, ds\right) - X\left(\tau, \xi(\tau), \int_0^{\tau} \varphi(\tau, s, \xi(s))\, ds\right) \right] d\tau +$$

$$+\varepsilon \int_0^{t} \left[X\left(\tau, \xi(\tau), \int_0^{\tau} \varphi(\tau, s, \xi(s))\, ds\right) - X\left(\tau, \xi(\tau), \int_0^{\tau} \varphi(\tau, s, \xi(\tau))\, ds\right) \right] d\tau +$$

$$+\varepsilon \int_0^{t} \left[X\left(\tau, \xi(\tau), \int_0^{\tau} \varphi(\tau, s, \xi(\tau))\, ds\right) - \overline{X}(\xi(\tau)) \right] d\tau.$$

We introduce the notations

$$X\left(\tau, \xi(\tau), \int_0^{\tau} \varphi(\tau, s, \xi(\tau))\, ds\right) \equiv X(\tau, \xi(\tau), \varphi_1(\tau, \xi(\tau))) \equiv \hat{X}(\tau, \xi(\tau)),$$

$$\hat{X}(\tau, \xi(\tau)) - \overline{X}(\xi(\tau)) = \psi(\tau, \xi(\tau)),$$

according to which

$$\varepsilon \int_0^{t} \left[X\left(\tau, \xi(\tau), \int_0^{\tau} \varphi(\tau, s, \xi(\tau))\, ds\right) - \overline{X}(\xi(\tau)) \right] d\tau = \varepsilon \int_0^{t} \psi(\tau, \xi(\tau))\, d\tau.$$

The function ψ satisfies a Lipschitz condition of the form

$$\|\psi(\tau, \xi) - \psi(\tau, \xi')\| \leqq \|\hat{X}(\tau, \xi) - \hat{X}(\tau, \xi')\| + \|\overline{X}(\xi) - \overline{X}(\xi')\| \leqq$$

$$\leqq \lambda\left(\|\xi - \xi'\| + \int_0^{\tau} \mu(\tau, s)\|\xi - \xi'\|\, ds\right) + \nu\|\xi - \xi'\| \leqq (\lambda + \nu + \lambda\mu_0(\tau))\|\xi - \xi'\|,$$

where

$$\mu_0(\tau) = \int_0^{\tau} \mu(\tau, s)\, ds.$$

According to the conditions of the theorem

$$\bar{\mu}_0(t) = \frac{1}{t}\int_0^t \mu_0(\tau)\,d\tau \to 0 \quad \text{when} \quad t \to \infty.$$

We will show that to every positive number ε there can be associated a function $a(\varepsilon, m)$, $a(\varepsilon, m) \to 0$ when $m \to \infty$, $\varepsilon \to 0$, for which the following inequality holds:

$$\varepsilon \left\| \int_0^t \psi(\tau, \xi(\tau))\,d\tau \right\| \leqq a(\varepsilon, m). \tag{10}$$

Indeed, let us divide the interval $[0, T]$ into m equal parts using the points $t_k = \dfrac{kL}{\varepsilon m}$, $k = \overline{0, m}$.

We assume that to the right of t the function ψ is identically equal to zero, and we extend in (10) the integration onto the whole interval $[0, T]$. Thus we obtain

$$\left\|\varepsilon \int_0^t \psi(\tau, \xi(\tau))\,d\tau\right\| = \left\|\varepsilon \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \psi(\tau, \xi(\tau))\,d\tau\right\| \leqq \tag{11}$$

$$\leqq \left\|\varepsilon \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} [\psi(\tau, \xi(\tau)) - \psi(\tau, \xi(t_k))]\,d\tau\right\| + \left\|\varepsilon \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \psi(\tau, \xi(t_k))\,d\tau\right\|.$$

For the first summand on the right hand side of (11) the following inequalities are fulfilled:

$$\left\|\varepsilon \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} [\psi(\tau, \xi(\tau)) - \psi(\tau, \xi_k)]\,d\tau\right\| \leqq \varepsilon \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (\lambda + \nu + \lambda\mu_0(\tau))\,\|\xi(\tau) - \xi_k\|\,d\tau \leqq$$

$$\leqq \varepsilon^2 M \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \{(\lambda+\nu)|\tau - t_k| + \lambda\mu_0(\tau)|\tau - t_k|\}\,d\tau \leqq \frac{ML^2}{2m}(\lambda+\nu) +$$

$$+ \frac{\varepsilon ML\lambda}{m} \int_0^{\frac{L}{\varepsilon}} \mu_0(\tau)\,d\tau \leqq \frac{ML^2}{2m}(\lambda+\nu) + \frac{\lambda ML}{m}\delta(\varepsilon) \equiv \beta(\varepsilon, m),$$

where

$$\xi_k = \xi(t_k),$$

$$\varepsilon \int_0^t \mu_0(\tau)\,d\tau = \varepsilon t \bar{\mu}_0(t) \leqq \sup_{0 \leqq \tau \leqq L} \bar{\mu}\tau_0\left(\frac{\tau}{\varepsilon}\right) \equiv \delta(\varepsilon),$$

$$\delta(\varepsilon) \to 0 \quad \text{when} \quad \varepsilon \to 0.$$

Therefore, $\beta(\varepsilon, m) \to 0$ when $m \to \infty$.

We introduce the notation

$$\Phi(t, \xi) = \left\|\frac{1}{t}\int_0^t [X(\tau, \xi, \varphi_1(\tau, \xi)) - \overline{X}(\xi)]\,d\tau\right\|.$$

At every fixed $\xi \in D$, $\Phi(t, \xi) \to 0$ when $t \to \infty$.

For the second summand from (11) we obtain

$$\varepsilon \left\| \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \psi(\tau, \xi_k)\, d\tau \right\| = \varepsilon \left\| \int_0^{t_1} \psi(\tau, \xi_0)\, d\tau + \sum_{k=1}^{m-1} \left\{ \int_0^{t_{k+1}} \psi(\tau, \xi_k)\, d\tau - \right.\right.$$

$$\left.\left. - \int_0^{t_k} \psi(\tau, \xi_k)\, d\tau \right\} \right\| \leqq \varepsilon \left\| \int_0^{t_1} \psi(\tau, \xi_0)\, d\tau \right\| + \varepsilon \sum_{k=1}^{m-1} \left\| \int_0^{t_{k+1}} \psi(\tau, \xi_k)\, d\tau \right\| +$$

$$+ \varepsilon \left\| \int_0^{t_1} \psi(\tau, \xi_1)\, d\tau \right\| + \varepsilon \sum_{k=2}^{m-1} \left\| \int_0^{t_k} \psi(\tau, \xi_k)\, d\tau \right\|.$$

Since

$$\varepsilon \left\| \int_0^{t} \psi(\sigma, \xi)\, d\sigma \right\| = \varepsilon t \Phi(t, \xi) = \tau \Phi\left(\frac{\tau}{\varepsilon}, \xi\right),$$

we have

$$\varepsilon \left\| \int_0^{t_1} \psi(\sigma, \xi_0)\, d\sigma \right\| \leqq \sup_{0 \leqq \tau \leqq \frac{L}{m}} \tau \Phi\left(\frac{\tau}{\varepsilon}, \xi_0\right) \to 0 \qquad \text{as} \quad \varepsilon \to 0,$$

$$\varepsilon \left\| \int_0^{t_1} \psi(\sigma, \xi_1)\, d\sigma \right\| \leqq \sup_{0 \leqq \tau \leqq \frac{L}{m}} \tau \Phi\left(\frac{\tau}{\varepsilon}, \xi_1\right) \to 0 \qquad \text{as} \quad \varepsilon \to 0,$$

$$\varepsilon \left\| \int_0^{t_{k+1}} \psi(\sigma, \xi_k)\, d\sigma \right\| \leqq L \Phi\left(\frac{(k+1)L}{\varepsilon m}, \xi_k\right) \to 0 \quad (k = \overline{1, m-1}) \quad \text{as} \quad \varepsilon \to 0,$$

$$\varepsilon \left\| \int_0^{t_k} \psi(\sigma, \xi_k)\, d\sigma \right\| \leqq L \Phi\left(\frac{kL}{\varepsilon m}, \xi_k\right) \to 0 \quad (k = \overline{1, m-2}) \qquad \text{as} \quad \varepsilon \to 0,$$

and at fixed values of K, m and ξ_k.

Therefore,

$$\varepsilon \left\| \sum_{k=0}^{m-1} \int_0^{t_{k+1}} \psi(\tau, \xi_k)\, d\tau \right\| \leqq \sup_{0 \leqq \tau \leqq \frac{L}{m}} \tau \Phi\left(\frac{\tau}{\varepsilon}, \xi_0\right) + \sup_{0 \leqq \tau \leqq \frac{L}{m}} \tau \Phi\left(\frac{\tau}{\varepsilon}, \xi_1\right) +$$

$$+ L \sum_{k=1}^{m-1} \Phi\left(\frac{(k+1)L}{\varepsilon m}, \xi_k\right) + L \sum_{k=2}^{m-1} \Phi\left(\frac{kL}{\varepsilon m}, \xi_k\right) \equiv \alpha(\varepsilon, m).$$

At fixed values of m the right hand side $\alpha(\varepsilon, m)$ of the last inequality approaches zero as $\varepsilon \to 0$.

In this way inequality (10) is proved.

We introduce the notations

$$c(\varepsilon, m) = a(\varepsilon, m) + \lambda M L \delta(\varepsilon),$$

$$a(\varepsilon, m) = \alpha(\varepsilon, m) + \beta(\varepsilon, m).$$

From (9), according to the conditions of the theorem, we obtain the inequality

$$\varepsilon \left\| \int_0^t \left[X\left(\tau, x(\tau), \int_0^\tau \varphi(\tau, s, x(s))\, ds\right) - \overline{X}(\xi(\tau))\right] d\tau \right\| \leqq \tag{12}$$

$$\leqq c(\varepsilon, m) + \varepsilon\lambda \int_0^t \left[\|x(\tau) - \xi(\tau)\| + \int_0^\tau \mu(\tau, s)\|x(s) - \xi(s)\|\, ds \right] d\tau$$

which is satisfied at every $t_0 \in [0, T]$.

We put $c(\varepsilon, m)u(t) = x(t) - \xi(t)$ and introduce the notation

$$\|u\|_T = \sup_{0 \leqq t \leqq T} \|u(t)\|.$$

Then, from (8), and having in mind (12) we obtain

$$\|x(t) - \xi(t)\| \leqq h(\varepsilon) \sum_{i=1}^{N} \left\{ c(\varepsilon, m) + \varepsilon\lambda \int_0^{t_i} \left[\|x(\tau) - \xi(\tau)\| + \right.\right. \tag{13}$$

$$\left.\left. + \int_0^\tau \mu(\tau, s)\|x(s) - \xi(s)\|\, ds \right] d\tau \right\} + c(\varepsilon, m) + \varepsilon\lambda \int_0^t \left[\|x(\tau) - \xi(\tau)\| + \right.$$

$$\left. + \int_0^\tau \mu(\tau, s)\|x(s) - \xi(s)\|\, ds \right] d\tau \leqq c(\varepsilon, m)\left\{1 + h(\varepsilon)\left[N + \lambda(\alpha L + N\delta(\varepsilon))\|u\|_T\right]\right\} +$$

$$+ \varepsilon\lambda \int_0^t \left[\|x(\tau) - \xi(\tau)\| + \int_0^\tau \mu(\tau, s)\|x(s) - \xi(s)\|\, ds \right] d\tau, \quad \alpha = \sum_{i=1}^{N} \alpha_i.$$

From (13) we obtain

$$\|u(t)\| \leqq \left\{1 + h(\varepsilon)\left[N + \lambda(\alpha L + N\delta(\varepsilon))\|u\|_T\right]\right\} +$$

$$+ \varepsilon\lambda \int_0^t \left[\|u(\tau)\| + \int_0^\tau \mu(\tau, s)\|u(s)\|\, ds \right] d\tau.$$

In order to estimate the function $\|u(t)\|$ we will use the following

LEMMA ([1]). *Let the following conditions be satisfied when* $t \geqq \tau \geqq t_0$:

1. *The functions* $u(t)$, $v(t)$ *and* $\mu(t, \tau)$ *are non-negative and continuous.*
2. *The function* $u(t)$ *satisfies the inequality*

$$u(t) \leqq \alpha + \beta \int_{t_0}^t \left[v(\tau)u(\tau) + \int_{t_0}^\tau \mu(\tau, s)u(s)\, ds \right] d\tau,$$

where α *and* β *are non-negative numbers.*

Then, the following inequality holds:

$$u(t) \leqq \alpha \exp\left\{\beta \int_{t_0}^t \left[v(\tau) + \int_{t_0}^\tau \mu(\tau, s)\, ds \right] d\tau \right\}.$$

We apply the lemma and obtain

$$\|u(t)\| \leqq \left\{1+h(\varepsilon)\left[N+\lambda(\alpha L+N\delta(\varepsilon))\|u\|_T\right]\right\} \exp\left\{\varepsilon\lambda \int_0^t \left[1+\int_0^\tau \mu(\tau, s)\, ds\right] d\tau\right\}.$$

Therefore,

$$\|u\|_T \leqq \left\{1+h(\varepsilon)\left[N+\lambda(\alpha L+N\delta(\varepsilon))\|u\|_T\right]\right\} \exp\left\{\lambda(L+\delta(\varepsilon))\right\}. \tag{14}$$

Since $\lim_{\varepsilon\to 0} h(\varepsilon)=0$, there exists a number $\varepsilon_2>0$, such that if $\varepsilon\in(0, \varepsilon_2]$, the inequality

$$\lambda(\alpha L+N\delta(\varepsilon))h(\varepsilon)\exp\left\{\lambda(L+\delta(\varepsilon))\right\} < 1$$

holds. Then from (14) we get

$$\|u\|_T \leqq \frac{(1+Nh(\varepsilon))\exp\left\{\lambda(L+\delta(\varepsilon))\right\}}{1-\lambda(\alpha L+N\delta(\varepsilon))h(\varepsilon)\exp\left\{\lambda(L+\delta(\varepsilon))\right\}} = \Delta$$

i.e.

$$\|x(t)-\xi(t)\| \leqq c(\varepsilon, m)\Delta.$$

We choose the numbers m and $\varepsilon_3>0$ in such a way that if $\varepsilon\in(0, \varepsilon_3]$ the inequality

$$c(\varepsilon, m)\Delta < \eta$$

be satisfied. Then, when $\varepsilon^0=\min(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ the conclusion of Theorem 1 holds.

2. Second scheme for averaging

Suppose that the integral

$$\int_0^\infty \varphi(t, s, x)\, ds$$

in which we consider t and x as parameters, can be calculated, and we put

$$\int_0^\infty \varphi(t, s, x)\, ds = \varphi_2(t, x).$$

So, if the limit

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T X(t, x, \varphi_2(t, x))\, dt = \overline{X}(x) \tag{15}$$

exists, then the system

$$\dot{\xi} = \varepsilon\overline{X}(\xi) \tag{16}$$

with a boundary condition

$$\sum_{i=0}^{N} A_i\xi(t_i) = 0 \tag{17}$$

will be called an averaged system for the first approximation of the solution $x(t)$ of system (1).

THEOREM 2. *We assume the following:*

1. *The conditions* 1, 2, 5, *and* 6 *of Theorem* 1 *are satisfied.*

2. *For every* $x \in D$ *the limit* (15) *exists, and the function* $\overline{X}(x)$, *defined and continuous in* D, *satisfies the conditions*

$$\|\overline{X}(x)\| \leqq M = \text{const}, \quad \|\overline{X}(x)-\overline{X}(x')\| \leqq v\|x-x'\|, \quad v = \text{const}.$$

3. *The boundary problem* (16), (17) *has a unique continuous solution* $\xi(t)$ *for* $0 \leqq t \leqq T$.

4. *Along the trajectory* $\xi(t)$ *of boundary problem* (16), (17)

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t d\tau \left\| \int_0^\infty \varphi(\tau, s, \xi(\tau))\, ds \right\| = 0.$$

5. *The function* $\varphi_2(t, x)$ *satisfies the Lipschitz condition*

$$\|\varphi_2(t, x)-\varphi_2(t, x')\| \leqq \beta\|x-x'\|, \quad \beta = \text{const}.$$

Then, to every $\eta > 0$ *we may associate a number* $\varepsilon^0 > 0$, *such that when* $0 < \varepsilon \leqq \varepsilon^0$, *the inequality*

$$\|x(t)-\xi(t)\| < \eta \quad \text{for} \quad 0 \leqq t \leqq T$$

will be fulfilled.

The proof of theorem 2 is analogous to that of theorem 1.

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ERDŐS—RÉNYI TYPE MODULUS OF CONTINUITY THEOREMS FOR BROWNIAN SHEETS

by

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1. Introduction

Let $W(\mathbf{x})$, $\mathbf{x}\in[0, \infty)^d$, be a d-parameter Brownian motion (a d-parameter Wiener measure) on some underlying probability space $(\Omega, \mathscr{A}, P)$. Before giving a formal definition for W, we first introduce some notations and definitions.

NOTATIONS. Let $\mathbf{x}=(x_1, \ldots, x_d)$ and $\mathbf{y}=(y_1, \ldots, y_d)$ be in $[0, \infty)^d$ and let $a\in[0, \infty)$. We define:

(1.1) $$|\mathbf{x}| = x_1 \cdot x_2 \cdot \ldots \cdot x_d;$$

(1.2) $$\mathbf{x} \leqq \mathbf{y} \quad (\mathbf{x} < \mathbf{y}) \quad \text{if} \quad x_i \leqq y_i \quad (x_i < y_i), \quad \text{for all} \quad 1 \leqq i \leqq d;$$

(1.3) $$\mathbf{x}+\mathbf{y} = (x_1+y_1, \ldots, x_d+y_d);$$

(1.4) $$\mathbf{x}-\mathbf{y} = (x_1-y_1, \ldots, x_d-y_d) \quad \text{if} \quad \mathbf{y} \leqq \mathbf{x};$$

(1.5) $$\mathbf{xy} = (x_1y_1, \ldots, x_dy_d);$$

(1.6) $$a\mathbf{x} = (ax_1, \ldots, ax_d).$$

Suppose that $\mathbf{x}\leqq\mathbf{y}$. Define

(1.7) $$\langle \mathbf{x}, \mathbf{y} \rangle = \{\mathbf{z}\in[0, \infty)^d : \mathbf{x} \leqq \mathbf{z} \leqq \mathbf{y}\}, \ \langle \mathbf{x} \rangle = \langle \mathbf{0}, \mathbf{x} \rangle \quad \text{and}$$

$$\operatorname{int}\langle \mathbf{x}, \mathbf{y} \rangle = \{\mathbf{z}\in[0, \infty)^d : \mathbf{x} < \mathbf{z} < \mathbf{y}\};$$

(1.8) For each $\mathbf{x}\in[0, \infty)^d$, define $F_{\mathbf{x}}^{(i)}:[0, \infty)^d \to [0, \infty)^d$ by

$$F_{\mathbf{x}}^{(i)}(\mathbf{y}) = (y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_d), \quad 1 \leqq i \leqq d,$$

Also, let us reserve the symbols $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ for the points in $[0, \infty)^d$ having integer coordinates.

Now, we proceed to define a d-parameter Brownian motion.

DEFINITION. A d-parameter Brownian motion $W(\mathbf{x})$, $\mathbf{x}\in[0, \infty)^d$, on an underlying probability space $(\Omega, \mathscr{A}, P)$ is a separable Gaussian process satisfying the following conditions:

(1.9) $W(\cdot)$ is continuous on $[0, \infty)^d$ almost surely,

(1.10) $W(\mathbf{x})=0$ a.s. if $|\mathbf{x}|=0$; and, if $|\mathbf{x}|>0$, then

$$P\{W(\mathbf{x}) < \lambda\} = \Phi(\lambda|\mathbf{x}|^{-\frac{1}{2}}), \quad -\infty < \lambda < \infty,$$

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where

$$\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-t^2/2}\, dt.$$

(1.11) $W(\mathbf{x})$ has independent increments in the following sense: Suppose that $\mathbf{x} < \mathbf{y}$. Define

$$\begin{aligned} W(\mathbf{x}, \mathbf{y}) = W(\mathbf{y}) &+ (-1) \sum_{i=1}^{d} W\big(F_{\mathbf{x}}^{(i)}(\mathbf{y})\big) + \\ &+ (-1)^2 \sum_{i<j} W\big(F_{\mathbf{x}}^{(i)} F_{\mathbf{x}}^{(j)}(\mathbf{y})\big) + \dots \\ &+ (-1)^{d-1} \sum_{i} W\big(F_{\mathbf{y}}^{(i)}(\mathbf{x})\big) + \\ &+ (-1)^d W(\mathbf{x}). \end{aligned}$$

Then $W(\mathbf{x}, \mathbf{y})$ and $W(\mathbf{x}', \mathbf{y}')$ are independent if $\operatorname{int}\langle \mathbf{x}, \mathbf{y}\rangle \cap \operatorname{int}\langle \mathbf{x}', \mathbf{y}'\rangle = \varnothing$. Furthermore,

$$P\{W(\mathbf{x}, \mathbf{y}) < \lambda\} = \Phi(\lambda\, |\mathbf{y}-\mathbf{x}|^{-\frac{1}{2}}), \quad -\infty < \lambda < \infty.$$

When $d=1$, the ERDŐS—RÉNYI [2] law of large number has the following special form in the Gaussian case.

THEOREM A. (Erdős and Rényi.) *Let $W(x)$, $x \geqq 0$, be a standard Brownian motion ($d=1$). Denote by $[y]$ the greatest integer less than y. Then, for each $c>0$, we have*

$$\lim_{N\to\infty} \max_{0\leqq j\leqq N-[c\ln N]} \frac{W(j+[c\ln N])-W(j)}{[c\ln N]} = \sqrt{\frac{2}{c}} \quad \text{a.s.,} \tag{1.12}$$

where j runs through all integers between 0 and $N-[c\ln N]$.

Also, in [2], it is mentioned that Theorem A yields

$$\lim_{h\to 0} P\left\{ \left| \sup_{0\leqq x\leqq 1-h} \frac{W(x+h)-W(x)}{\sqrt{2h\ln h^{-1}}} - 1 \right| > \varepsilon \right\} = 0 \tag{1.13}$$

for each $\varepsilon>0$. This, of course, is the "in probability" form of P. LÉVY's modulus of continuity for Brownian motion (cf. [3]):

THEOREM B. (P. Lèvy.) *Let $W(x)$, $0\leqq x\leqq 1$, be a Brownian motion on $[0, 1]$. Then*

$$\overline{\lim_{h\to 0}} \sup_{0\leqq x\leqq 1-h} \frac{W(x+h)-W(x)}{\sqrt{2h\ln h^{-1}}} = 1 \quad \text{a.s..} \tag{1.14}$$

REMARK. (1.14) actually is true with lim instead of $\overline{\lim}$ (cf. OREY and TAYLOR [5]). Also, OREY and PRUITT [4] have extended Theorem B to the case of the d-parameter Brownian motion (cf. Theorem C, section 4).

In this paper, we first extend Theorem A to the case when $d>1$. Then, we make an attempt to link these two types of modulus of continuity together. (Anticipating a strong connection between them, we call the result of Theorem A the Erdős—Rènyi type modulus of continuity).

2. Main results

Throughout this section and in the sequel $W(\mathbf{x})$ is a d-parameter Brownian motion on some underlying probability space $(\Omega, \mathscr{A}, P)$.

To extend Theorem A to the case when $d>1$, we must first give an appropriate definition for the Erdős—Rènyi type modulus of continuity for $d>1$ which, of course, should result in Theorem A when $d=1$. Observe that, according to our definition (1.11) of $W(\mathbf{x}, \mathbf{y})$, $W(j+k)-W(j)=W(j, j+k)$, when $d=1$. We can rewrite (1.12) as

$$\lim_{N\to\infty} \max_{0\leqq j\leqq N-[c\ln N]} \frac{W(j, j+[c\ln N])}{[c\ln N]} = \sqrt{\frac{2}{c}} \quad \text{a.s..}$$

Also, $E(W(j, j+[c\ln N])^2)=[c\ln N]$. It is quite natural then to hope for a result of the following form:

$$\lim_{N\to\infty} \max_{0\leqq j\leqq N-a_N} \frac{W(\mathbf{j}, \mathbf{j}+\mathbf{a}_N)}{|\mathbf{a}_N|} = \alpha \quad \text{a.s..}$$

for some $\mathbf{a}_N$'s and α, where $\mathbf{N}=(N, \ldots, N)$.

Indeed, we prove the following theorem which will also render the above conjecture precise.

THEOREM 1. *Let* $\mathbf{a}_N\in[0, \infty)^d$, $N=1, 2, \ldots$, *be such that it has positive integer coordinates and satisfies the following condition*

$$\lim_{N\to\infty} \frac{|\mathbf{a}_N|}{N^\delta} = 0, \quad \text{for each} \quad \delta > 0. \tag{2.1}$$

Then, for each $c>0$,

$$\lim_{N\to\infty} \max_{0\leqq j\leqq N-a_N} \frac{W(\mathbf{j}, \mathbf{j}+\mathbf{a}_N)}{\sqrt{|\mathbf{a}_N|[c\ln N]}} = \sqrt{\frac{2d}{c}} \quad \text{a.s..} \tag{2.2}$$

The following corollary is an immediate consequence of Theorem 1.

COROLLARY 1. *Suppose that* $\{\mathbf{a}_N\}$ *satisfies*

$$\lim_{N\to\infty} \frac{|\mathbf{a}_N|}{[\ln N]} = c > 0. \tag{2.3}$$

Then, we have

$$\lim_{N\to\infty} \max_{0\leqq \mathbf{j}\leqq \mathbf{N}-\mathbf{a}_N} \frac{W(\mathbf{j}, \mathbf{j}+\mathbf{a}_N)}{|\mathbf{a}_N|} = \sqrt{\frac{2d}{c}} \quad \text{a.s..} \tag{2.4}$$

In particular, when $d=1$ and $a_N=[c\ln N]$, we have Theorem A.

REMARK. Suppose that $c=0$ or $c=\infty$ in (2.3). Then (2.4) diverges if $c=0$ and converges to 0 if $c=\infty$. Therefore, $\ln N$ is the best rate in (2.4), resulting in the limit $\sqrt{\frac{2d}{c}}$, the Erdős—Rényi characteristic number of $W(\mathbf{x})$, $\mathbf{x}\in[0, \infty)^d$ (cf. [2]).

In the case of $d=1$, if we let $a_N=1$, then we have

$$\max_{0\leq j\leq N-1}\frac{W(j,j+1)}{\sqrt{[\ln N]}}\xrightarrow{\text{a.s.}}\sqrt{2},\quad\text{as}\quad N\to\infty.$$

Thus, we proved

COROLLARY 2. *Let* $X_1, X_2, \ldots, X_N, \ldots$ *be a sequence of i.i.d.r.v.'s with* $P\{X_1<\lambda\}=\Phi(\lambda)$, $-\infty<\lambda<\infty$, i.e. $X_1\sim N(0,1)$. *Then we have*

$$\lim_{N\to\infty}\max_{1\leq i\leq N}\frac{X_i}{\sqrt{2[\ln N]}}=1\quad\text{a.s.}. \tag{2.5}$$

Insofar, we were invoking the spirit of Theorem A when formulating our results. Let us give another approach to the problem which, we think, is also appropriate.

THEOREM 2. *Let* $\{\mathbf{a}_N\}\subseteq[0,\infty)^d$ *be such that the coordinates of each* $\mathbf{a}_N$ *are positive integers. Suppose that* $\{\mathbf{a}_N\}$ *satisfies* (2.1) *and, letting* $\mathbf{a}_N=(a_{1N},\ldots,a_{dN})$, *assume also*

$$\lim_{N\to\infty}a_{iN}=\infty,\quad 1\leq i\leq N. \tag{2.6}$$

Define, for each N *and* $\mathbf{k}\leq\mathbf{N}$,

$$S(N,\mathbf{k})=\sup_{0\leq\mathbf{y}\leq\mathbf{k}}\sup_{0\leq\mathbf{x}\leq\mathbf{N}-\mathbf{y}}\frac{W(\mathbf{x},\mathbf{x}+\mathbf{y})}{\sqrt{2d|\mathbf{k}|[\ln N]}}. \tag{2.7}$$

Then, for each $\varepsilon>0$, *there is a* $\gamma=\gamma(\varepsilon)>0$ *such that*

$$P\{S(N,\mathbf{a}_N)>1+\varepsilon\}\leq AN^{-\gamma}, \tag{2.8}$$

for some $A>0$. *Furthermore,*

$$\lim_{N\to\infty}S(N,\mathbf{a}_N)=1\quad\text{a.s.}. \tag{2.9}$$

The above theorem is a stronger version of the Erdős—Rényi type modulus of continuity of (2.2). Indeed, the following corollary manifests the importance of Theorem 2.

COROLLARY 3. *Let* $\{\mathbf{a}_N\}$ *be defined as in Theorem* 2. *Then*

$$\lim_{N\to\infty}\sup_{0\leq\mathbf{x}\leq\mathbf{N}-\mathbf{a}_N}\frac{W(\mathbf{x},\mathbf{x}+\mathbf{a}_N)}{\sqrt{2d|\mathbf{a}_N|[\ln N]}}=1\quad\text{a.s.}. \tag{2.10}$$

In particular, when $d=1$, *we have, for each* $c>0$

$$\lim_{N\to\infty}\sup_{0\leq x\leq N-[\ln N]}\frac{W(x+[c\ln N])-W(x)}{[c\ln N]}=\sqrt{2}\quad\text{a.s.}. \tag{2.11}$$

REMARK. When $d=1$, we can say much more than (2.11). We shall discusss this in our final section.

3. Some lemmas and proof of Theorems 1 and 2.

Let us first state, without proofs, some of the known results.

LEMMA A. *For each* $\lambda>0$, *we have*

$$\frac{1}{\sqrt{2\pi}}\frac{\lambda}{\lambda^2+1}e^{-\lambda^2/2} \leqq 1-\Phi(\lambda) \leqq \frac{1}{\sqrt{2\pi}}\frac{1}{\lambda}e^{-\lambda^2/2}. \tag{3.1}$$

LEMMA B. (BARNDORFF—NIELSEN.) *Let* $\{E_N\}$ *be a sequence of arbitrary events. If* $P(E_N)\to 0$ *and* $\sum P(E_N\bar{E}_{N+1})<\infty$, *then* $P(E_N \text{ i.o.})=0$.

The following lemma can be found in [4] (Lemma 1.2).

LEMMA C. (OREY and PRUITT) *Let* W *be a* d*-parameter Brownian motion* ($d\geqq 1$). *Suppose that* $I=\langle \mathbf{x}, \mathbf{y}\rangle$ *and* $\mathbf{x}<\mathbf{y}$. *Let* $W(I)=W(\mathbf{x}, \mathbf{y})$. *Then, there is a constant* $A_1=A_1(d)$, *such that*

$$P\{\sup_{J\subseteq I} W(J) > \lambda\} \leqq A_1 P\{W(I) > \lambda\} \tag{3.2}$$

where J *runs through all intervals in* I.

For the sake of simplicity, let us introduce some further notations:

NOTATIONS. Let $N\geqq 1$, $\mathbf{N}=(N, \ldots, N)$ and $\mathbf{k}\in[0, \infty)^d$ with integer coordinates.

$$\theta(N, \mathbf{k}) = \max_{0\leqq \mathbf{j}<\mathbf{N}-\mathbf{k}} \frac{W(\mathbf{j}, \mathbf{j}+\mathbf{k})}{\sqrt{2d|\mathbf{k}|[\ln N]}}; \tag{3.3}$$

$$\theta^*(N, \mathbf{k}) = \max_{0\leqq \mathbf{j}\leqq \mathbf{N}-\mathbf{k}} \sup_{J\subseteq\langle \mathbf{j}, \mathbf{j}+\mathbf{k}\rangle} \frac{W(J)}{\sqrt{2d|\mathbf{k}|[\ln N]}}, \tag{3.4}$$

as before, J denotes an interval.

First, we prove some Lemmas which are going to be relevant in the sequel.

LEMMA 1. *Let* $\varepsilon>0$ *be arbitrary and* $\delta>0$ *such that* $0<\delta<2\varepsilon+\varepsilon^2$. *Then, for large* N,

$$P\{\max_{1\leqq|\mathbf{k}|\leqq N^\delta} \theta^*(N, \mathbf{k}) \geqq 1+\varepsilon\} \leqq AN^{-\gamma} \tag{3.5}$$

for some $A>0$ *and* $\gamma=\gamma(\varepsilon)>0$. *Furthermore,*

$$\overline{\lim_{N\to\infty}} \max_{1\leqq|\mathbf{k}|\leqq N^\delta} \theta^*(N, \mathbf{k}) \leqq 1+\varepsilon \quad \text{a.s.}. \tag{3.6}$$

PROOF. Observe that the number of all $\mathbf{k}$'s such that $1\leqq|\mathbf{k}|\leqq N^\delta$ will not exceed $N^{\delta d}$. Since, by Lemma 2,

$$P\left\{\sup_{J\subseteq\langle \mathbf{j}, \mathbf{j}+\mathbf{k}\rangle} \frac{W(J)}{\sqrt{2d|\mathbf{k}|[\ln N]}} > 1+\varepsilon\right\} \leqq A_1 P\{W(\mathbf{j}, \mathbf{j}+\mathbf{k}) \geqq (1+\varepsilon)\sqrt{2d|\mathbf{k}|[\ln N]}\}$$

which, in turn, is bounded by $AN^{-(1+\varepsilon)^2 d}$, for some constant $A>0$, we have

$$P\{\max_{1\leqq|\mathbf{k}|\leqq N^{\delta}} \theta^*(N,\mathbf{k}) > 1+\varepsilon\} \leqq$$
$$\leqq \sum_{1\leqq|\mathbf{k}|\leqq N^{\delta}} \sum_{\mathbf{j}\leqq \mathbf{N}-\mathbf{k}} AN^{-(1+\varepsilon)^2 d} \leqq$$
$$\leqq AN^{-(1+\varepsilon)^2 d+(1+\delta)} =$$
$$= AN^{-\gamma}$$

where $\gamma=((1+\varepsilon)^2-(1+\delta))d>0$. This proves (3.5).

Let us denote the event in (3.5) by E_N. Suppose that $[\ln N]=[\ln N+1]$. Then $E_N\bar{E}_{N+1}=\varnothing$ and $P(E_N\bar{E}_{N+1})=0$. But $[\ln N]\neq[\ln N+1]$ if and only if $N=[e^k]$ for some k, uniquely determined by N. Let $Q=\{N:[\ln N]\neq[\ln N+1]\}$. Thus,

$$\sum_N P(E_N\bar{E}_{N+1}) = \sum_{N\in Q} P(E_N\bar{E}_{N+1}) \leqq$$
$$\leqq \sum_{N\in Q} P(E_N) \leqq$$
$$\leqq \sum_k A([e^k]-1)^{-\gamma} <$$
$$< \infty.$$

Since (3.5) implies that $P(E_N)\to 0$, it is evident that (3.6) follows from Lemma B. This completes our proof.

LEMMA 2. *Let $\varepsilon>0$ be arbitrary small, say $\varepsilon>\frac{1}{2}$. Suppose that $|a_N|/N^r\to 0$, for each $r>0$. Then for large N, we have*

$$P\{\theta(N,\mathbf{a}_N)\leqq 1-\varepsilon\} \leqq e^{-BN^{\alpha}} \tag{3.7}$$

for some $\alpha>0$ and $B>0$.

PROOF. Let $\mathbf{a}_N=(a_{1N},\ldots,a_{dN})$ and define $\mathbf{k}_N=\left(\left[\frac{N}{a_{1N}}\right],\ldots,\left[\frac{N}{a_{dN}}\right]\right)$. Then $|\mathbf{k}_N|\geqq N^d/|\mathbf{a}_N|$. Let $\varepsilon'>0$ be small enough such that $\alpha=(2\varepsilon-\varepsilon^2)d-\varepsilon'>0$. Then

$$P\{\theta(N,\mathbf{a}_N)\leqq 1-\varepsilon\} \leqq$$
$$\leqq P\left\{\frac{W(\mathbf{ja}_N,\mathbf{ja}_N+\mathbf{a}_N)}{\sqrt{2d|\mathbf{a}_N|[\ln N]}}\leqq 1-\varepsilon,\ \mathbf{0}\leqq\mathbf{j}\leqq\mathbf{k}_N\right\} \leqq$$
$$\leqq \prod_{\mathbf{j}} P\{W(\mathbf{ja}_N,\mathbf{ja}_N+\mathbf{a}_N)\leqq\sqrt{(1-\varepsilon)^2|\mathbf{a}_N|[\ln N]}\} \leqq$$
$$\leqq \left(1-\frac{N^{-(1-\varepsilon)^2 d-\varepsilon'}}{\sqrt{2\pi}}\right)^{N^d/|\mathbf{a}_N|} \leqq$$
$$\leqq e^{-BN^{\alpha}}$$

for some appropriate $B>0$ and for N large enough such that

$$\sqrt{(1-\varepsilon)^2 2d[\ln N]}/(1-\varepsilon)^2 2d[\ln N] > N^{-\varepsilon'}.$$

This completes our proof.

Now, we are able to prove our results.

PROOF OF THEOREM 1. Lemma 2 implies that $\underline{\lim}\, \theta(N, \mathbf{a}_N) \geqq 1$ a.s. . Observe that $\theta(N, \mathbf{a}_N) \leqq \max_{1<|\mathbf{k}|<N^\delta} \theta^*(N, \mathbf{k})$, whenever $|\mathbf{a}_N| < N^\delta$. By (3.6), we have $\overline{\lim}\, \theta(N, \mathbf{a}_N) \leqq 1$ a.s. . This forces that $\lim \theta(N, \mathbf{a}_N) = 1$ a.s. . This proves Theorem 1 for the case $c=1$. When $c \neq 1$, the result (2.2) follows from the observation that $[\ln N]/[c \ln N] \to \frac{1}{c}$.

REMARK. As we have seen, in the above proof, that it suffices to show only the case when $c=1$. So, it is easy to see that Theorem 2 can be more flexible in the sense that if we put $[c \ln N]$, $c>0$, to substitute $[\ln N]$ in $S(N, \mathbf{a}_N)$, then the limit goes to $\frac{c}{\sqrt{c}}$ almost surely.

PROOF OF THEOREM 2. For each $\mathbf{0}<\mathbf{y}\leqq\mathbf{a}_N$ and $\mathbf{0}\leqq\mathbf{x}\leqq\mathbf{N}-\mathbf{y}$ there are $\mathbf{j}\leqq\mathbf{N}$ and $\mathbf{k}\leqq\mathbf{a}_N+\mathbf{1}$ such that $\langle \mathbf{x}, \mathbf{x}+\mathbf{y}\rangle \subseteq \langle \mathbf{j}, \mathbf{j}+\mathbf{k}\rangle$. Let $\varepsilon>0$ and $\delta>0$ such that $\delta<2\varepsilon+\varepsilon^2$. For N large enough such that $|\mathbf{a}_N+1|<N\delta$, we have

$$S(N, \mathbf{a}_N+1) \leqq \max_{1\leqq|\mathbf{k}|\leqq N^\delta} \theta^*(N, \mathbf{k}).$$

(Note that (2.1) and (2.6) implies that $|\mathbf{a}_N|/N^\delta \to 0$ and $|\mathbf{a}_N+1|/|\mathbf{a}_N| \to 1$; thus $|\mathbf{a}_N+\mathbf{1}|/N^\delta \to 0$ also.) Combining lemma 1 and the above inequality, we have $P\{S(N, \mathbf{a}_N+1) \geqq 1+\varepsilon\} \leqq AN^{-\gamma}$ and $\overline{\lim}\, S(N, \mathbf{a}_N+\mathbf{1}) \leqq 1$ a.s. . Since $\frac{|\mathbf{a}_N+\mathbf{1}|}{|\mathbf{a}_N|} \to 1$ and

$$S(N, \mathbf{a}_N) \leqq \left(\frac{\mathbf{a}_N+\mathbf{1}}{|\mathbf{a}_N|}\right)^{\frac{1}{2}} S(N, \mathbf{a}_N+1),$$

we have (2.8) and $\overline{\lim}\, S(N, \mathbf{a}_N) \leqq 1$ a.s. But $\theta(N, \mathbf{a}_N) \leqq S(N, \mathbf{a}_N)$, for each N, and Theorem 1 implies that $\underline{\lim}\, S(N, \mathbf{a}_N) \geqq 1$ a.s. . Hence, we prove (2.9) also. This completes the proof of Theorem 2.

4. Some consequences

We have mentioned in section 1 that Theorem A implies a weaker form of Theorem B, but it does not seem to be obvious that Theorem A implies Theorem B. Indeed, if Theorem A is formulated as it is, then we were unable to prove this latter implication. However, the duality of Theorems A and B actually follows from Theorem 2 for the case $d=1$. We discuss this question now.

Taking $h_N=[\ln N]/N$ and using the fact that $\sup_{0\leqq x\leqq 1} W(x) \overset{\mathcal{D}}{=} \sup_{0\leqq x\leqq 1} N^{-\frac{1}{2}} W(Nx)$

for each $N>0$, and letting $\ln_2 N=\ln\ln N$, we get for each $\varepsilon>0$

$$P\left\{\sup_{0\leqq x\leqq 1-h_N}\frac{W(x+h_N)-W(x)}{\sqrt{2h_N\ln h_N^{-1}}}\leqq 1-\varepsilon\right\}\leqq$$

$$\leqq P\left\{\max_{0\leqq j\leqq N-[\ln N]}\frac{N^{-\frac{1}{2}}(W(j+[\ln N])-W(j))}{\sqrt{2[\ln N]N^{-1}[\ln N-\ln_2 N]}}\leqq 1-\varepsilon\right\}\leqq$$

$$\leqq P\{\theta(N,[\ln N])\leqq 1-\varepsilon\}\leqq$$

$$\leqq e^{-BN^{\alpha}},$$

for some $\alpha>0$ by Lemma 2. Thus

$$\text{(4.1)}\quad \overline{\lim_{h\to 0}}\sup_{0\leqq x\leqq 1+h}\frac{W(x+h)-W(x)}{\sqrt{2h\ln h^{-1}}}\geqq \underline{\lim_{N\to\infty}}\sup_{0\leqq x\leqq 1-h_N}\frac{W(x+h_N)-W(x)}{\sqrt{2h_N\ln h_N^{-1}}}\geqq 1$$

almost surely.

On the other hand, let $\varepsilon_0>0$ be arbitrary and choose $\varepsilon>0$ such that $(1+\varepsilon)^{3/2}\leqq 1+\varepsilon_0$. Let us fix ε and let $\beta=1+\varepsilon>1$. Define $h_N=N[\beta^N]^{-1}$. Using (2.8) of Theorem 2, we have

$$P\left\{\sup_{h_{N+1}\leqq h\leqq h_N}\sup_{0\leqq x\leqq 1-h}\frac{W(x+h)-W(x)}{\sqrt{2h_N[\ln[\beta^N]]}}\geqq 1+\varepsilon\right\}\leqq$$

$$\leqq P\{S([\beta^N],N)\geqq 1+\varepsilon\}\leqq$$

$$\leqq A[\beta^N]^{-\gamma}.$$

This means that

$$\text{(4.2)}\quad \overline{\lim_{N\to\infty}}\sup_{h_{n+1}\leqq h\leqq h_N}\sup_{0\leqq x\leqq 1-h}\frac{W(x+h)-W(x)}{\sqrt{2h_N\ln[\beta^N]}}\leqq 1+\varepsilon\quad\text{a.s.}.$$

Since $h_N[\ln[\beta^N]]/h_{N+1}\ln h_{N+1}^{-1}\to\frac{1}{\beta}=(1+\varepsilon)^{-1}$, we have

$$\text{(4.3)}\quad \overline{\lim_{N\to\infty}}\sup_{h_{N+1}\leqq h\leqq h_N}\sup_{0\leqq x\leqq 1-h}\frac{W(x+h)-W(x)}{\sqrt{2h_{N+1}\ln h_{N+1}^{-1}}}\leqq(1+\varepsilon)^{\frac{3}{2}}\leqq 1+\varepsilon_0\quad\text{a.s.}.$$

Since $\sqrt{2h_{N+1}\ln h_{N+1}^{-1}}\leqq\sqrt{2h\ln h^{-1}}$ whenever $h_{N+1}\leqq h\leqq h_N$, for N large we get

$$\text{(4.4)}\quad \overline{\lim_{h\to 0}}\sup_{0\leqq x\leqq 1-h}\frac{W(x+h)-W(x)}{\sqrt{2h\ln h^{-1}}}\leqq$$

$$\leqq \overline{\lim_{N\to\infty}}\sup_{h_{N+1}\leqq h\leqq h_N}\sup_{0\leqq x\leqq 1-h}\frac{W(x+h)-W(x)}{\sqrt{2h_{N+1}\ln h_{N+1}^{-1}}}\leqq$$

$$\leqq 1+\varepsilon_0.\quad\text{a.s.}$$

Since $\varepsilon_0>0$ is arbitrary, we can conclude that

$$\overline{\lim_{h\to 0}} \sup_{0\leqq x\leqq 1-h} \frac{W(x+h)-W(x)}{\sqrt{2h\ln h^{-1}}} \leqq 1 \quad \text{a.s.}.$$

Combining (4.1) with the latter statement we have (1.14) of Theorem B.

REMARK. We emphasise that we have not proved that the Erdős—Rényi type modulus of continuity implies that of the Lévy type, for it is not the conclusion (2.9) of Theorem 2 but the rate of convergence (2.8) that establishes the result. Nevertheless, we see that the two types of modulus of continuity came from the same source.

In [4], OREY and PRUITT give the following extension of Theorem B to the case when $d>1$.

THEOREM C. (Orey and Pruitt.) *Let J denote an interval and $I=\langle \mathbf{0}, \mathbf{1}\rangle$.* Then

$$\lim_{|J|\to 0} \sup_{J\subseteq I} \frac{W(J)}{\sqrt{2|J|\ln |J|^{-1}}} = 1 \quad \text{a.s.}, \tag{4.5}$$

where $|J|=|\mathbf{y}-\mathbf{x}|$ if $J=\langle \mathbf{x}, \mathbf{y}\rangle$.

It is natural to ask if one can prove (4.5) from Theorem 2. The answer is "not quite", because one lacks total ordering in $[0, \infty)^d$. However, using a similar argument as above, one can show that

$$\overline{\lim_{h\to 0}} \sup_{0\leqq \mathbf{x}\leqq \mathbf{1}-\mathbf{h}} \frac{W(\mathbf{x}, \mathbf{x}+\mathbf{h})}{\sqrt{2|\mathbf{h}|\ln |\mathbf{h}|^{-1}}} = 1 \quad \text{a.s.} \tag{4.6}$$

where $\mathbf{h}=(h, h, \ldots, h)>\mathbf{0}$, and this latter statement is weaker than (4.5). Judging from the case when $d=1$, it is inviting to conjecture that, when $d>1$, one should be able to prove Theorem C from a Theorem 2-type result.

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SAMPLING FROM A FINITE POPULATION. A REMAINDER TERM ESTIMATE

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1.

Let $x_1, \ldots, x_N$ be a sequence of real numbers. Fix $0<n<N$ and define $F(t)=A(t)\Big/\binom{N}{n}$, $t\in R$, where $A(t)$ stands for the number of sets $S\subset\{1, 2, \ldots, N\}$ of cardinality n satisfying $\sum_{k\in S} x_k \leqq t$. That is, F is the distribution function of the sum of n elements chosen at random from $x_1, \ldots, x_N$.

The object of the present paper is to show the following

THEOREM. *For all real* t

$$\left|F(t)-\varphi\left(\frac{t-n\bar{x}}{s\sqrt{pq}}\right)\right| \leqq \frac{C}{\sqrt{pq}}\,\frac{\sum_{k=1}^{N}|x_k-\bar{x}|^3}{s^3}. \tag{1}$$

Here C *is an absolute constant,*

$$\varphi(x) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x}\exp\left(-\frac{1}{2}y^2\right)dy,$$

$$\bar{x} = \frac{1}{N}\sum_{1}^{N} x_k, \quad s^2 = \sum_{k=1}^{N}(x_k-\bar{x})^2$$

and

$$p = n/N,\ q = (N-n)/N.$$

BIKELIS [1] gave the bound

$$\frac{C}{\sqrt{n/N}}\,\frac{\max_{1\leqq k\leqq N}|x_k-x|}{s}, \quad n \leqq N/2. \tag{2}$$

Our choice of bound is motivated by the approximation

$$F(t)-\varphi(t^*) \approx \frac{p-q}{6}\,\varphi^{(3)}(t^*)\,\frac{1}{\sqrt{pq}}\,\frac{\sum_{1}^{N}(x_k-\bar{x})^3}{s^3} \tag{3}$$

which I think can be made precise in the form of a limit theorem. Here and below p and q have the same meaning as in the theorem, and $t^*=\frac{t-n\bar{x}}{s\sqrt{pq}}$. (The right hand side of (3) has to be modified if $x_1, \ldots, x_N$ take their values in a coset of a proper subgroup of **R**.)

2.

Put $y_k=(x_k-\bar{x})/s$. Then

$$\sum_1^N y_k = 0, \quad \sum_1^N y_k^2 = 1. \tag{4}$$

Denote the Fourier—Stieltjes transform of the function $t \to F(n\bar{x}+ts)$ by f. Then (ERDŐS and RÉNYI [2])

$$f(\xi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \prod_{k=1}^{N} \beta(\eta+\xi y_k)\, d\eta \Big/ \binom{N}{n} p^n q^{N-n}, \tag{5}$$

where $\beta(\eta)=qe^{-ip\eta}+pe^{iq\eta}$. The left side of (1) is by Esseen's lemma dominated by

$$\frac{1}{\pi} \int_{-T}^{T} \left| \frac{f(\xi)-e^{-\frac{1}{2}\xi^2 pq}}{\xi} \right| d\xi + \frac{24}{\pi\sqrt{2\pi}} \frac{1}{T} \tag{6}$$

for all $T>0$ (See FELLER [3], page 538). If we note that $f(\xi)-\exp\left(-\frac{1}{2}\xi^2 pq\right) = f(\xi)-f(0)\exp\left(-\frac{1}{2}\xi^2 pq\right)$, and use (5) and the inequality to the left in Lemma 1 below, then we conclude

$$|f(\xi)-e^{-\frac{1}{2}\xi^2 pq}| \leqq \frac{\sqrt{2Npq}}{\pi} \int_{-\pi}^{\pi} \left| \prod_1^N \beta(\eta+\xi y_k) - e^{-\frac{1}{2}\xi^2 pq} \beta(\eta)^N \right| d\eta. \tag{7}$$

So the theorem will follow if we can find an absolute constant $c>0$ and a polynomial $P(\xi, \eta)$ whose coefficients are absolute constants such that the expression within the absolute value signs to the right in (7) is dominated by

$$e^{-\frac{1}{2}(N\eta^2+\xi^2)cpq} |\xi|\, P\left(|\xi\sqrt{pq}|,\ |\eta\sqrt{Npq}|\right) \tag{8}$$

for $|\eta| \leqq \pi$ and $|\xi| \leqq T = a \Big/ \sum_1^N |y_k|^3$. Here a is an absolute constant to be chosen later.

LEMMA 1.

$$\frac{1}{2}\sqrt{\pi} \leqq \binom{N}{n} (n/N)^n (1-n/N)^{N-n} \sqrt{2\pi N (n/N)(1-n/N)} < 1 \tag{9}$$

for $0<n<N$.

PROOF. Put for $x>0$, $f(x)=(1+1/x)^{x+\frac{1}{2}}$. Then

$$\frac{d}{dx}\log f(x) = -\log\left(1-\frac{1}{1+x}\right) - \frac{1}{1+x} - \frac{1}{2}\frac{1}{x(1+x)} = \tag{10}$$

$$= \sum_{k=2}^{\infty} \frac{1}{k}\left(\frac{1}{1+x}\right)^k - \frac{1}{2}\frac{1}{x(1+x)} < \sum_{k=2}^{\infty} \frac{1}{2}\left(\frac{1}{1+x}\right)^k - \frac{1}{2}\frac{1}{x(1+x)} = 0,$$

which shows that $f(x)$ *decreases* towards e. If we denote the middle term in (9) by $A(N, n)$, then $A(N+1, n)/A(N, n)=f(N-n)/f(N)>1$ for $N=n+1, n+2, \ldots$. Hence

$$A(n+1, n) \leqq A(N, n) < A(\infty, n) = \frac{\sqrt{2\pi n}\,(n/e)^n}{n!}. \tag{11}$$

The inequality to the left in (9) now follows from the fact that $A(n+1, n)=$ $=\sqrt{2\pi}/f(n) \geqq \sqrt{2\pi}/f(1)=\frac{1}{2}\sqrt{\pi}$. To obtain the remaining inequality we note that $A(\infty, n+1)/A(\infty, n)=f(n)/e>1$, which shows that $A(\infty, n)<\lim_{n\to\infty} A(\infty, n)=1$. (Here we used Stirling's formula.)

3.

Putting $\gamma(\eta)=\beta(\eta)\exp\left(\frac{1}{2}pq\eta^2\right)$ and recalling (4) we find that the integrand in (7) equals

$$e^{-\frac{1}{2}pq(N\eta^2+\xi^2)}\left|\prod_1^N \gamma(\eta+\xi y_k)-\gamma(\eta)^N\right|, \tag{12}$$

and the expression between the absolute value signs in (12) is dominated by

$$\sum_{k=1}^{N}\left(\prod_1^{k-1}|\gamma(\eta+\xi y_j)|\right)|\gamma(\eta+\xi y_k)-\gamma(\eta)||\gamma(\eta)|^{N-k}. \tag{13}$$

(Here we used the identity $\prod_1^N a_j-\prod_1^N b_j=\sum_{k=1}^{N}\left(\prod_1^{k-1} a_j\right)(a_k-b_k)\prod_{k+1}^{N} b_j$ with

$$a_k=\gamma(\eta+\xi y_k), \quad b_k=\gamma(\eta)).$$

LEMMA 2. *Put* $\theta(b)=\left(\frac{2}{\pi}\frac{\pi-b}{\pi+b}\right)^2$, *then*

$$|1-p+pe^{i\eta}| \leqq \exp\left(-\frac{1}{2}\eta^2\theta(b)p(1-p)\right) \tag{14}$$

for $|\eta|\leqq\pi+b$, $0\leqq b<\pi$ *and* $0\leqq p\leqq 1$.

PROOF. We first show that (14) holds when $b=0$. We have $|1-p+pe^{i\eta}|^2=$ $=1-2p(1-p)(1-\cos\eta)$, and $1-\cos\eta=2\sin^2\frac{1}{2}\eta\geqq 2(\eta/\pi)^2$ for $|\eta|<\pi$. The inequality $1-x\leqq e^{-x}$ valid for $x\geqq 0$ now gives (14) when $b=0$. If $\pi<|\eta|<\pi+b$ then either $\pi\leqq\eta\leqq\pi+b$ or $-\pi-b\leqq\eta\leqq-\pi$. In the first case $|\eta-2\pi|\leqq\pi$ and hence (14) holds with $b=0$ and $\eta-2\pi$ in place of η, but the left side of (11) has period 2π and $(2\pi-\eta)^2\geqq\left(\eta\frac{\pi-b}{\pi+b}\right)^2$ for $\pi\leqq\eta\leqq\pi+b$. If $-\pi-b\leqq\eta\leqq-\pi$ then $|\eta+2\pi|\leqq\pi$ etc. The lemma follows.

In order to use Lemma 2 to estimate $|\gamma(\eta+\xi y_k)|$ for $|\eta|\leqq\pi$, $|\xi|\leqq T$ we have to require that $|Ty_k|\leqq b$. So let us introduce the set of k for which this is not true:

$$K=\left\{k\in[1,N]\,|\,a|y_k|>b\sum_{j=1}^{N}|y_j|^3\right\}. \tag{15}$$

(The constant $b<\pi$ will be chosen later.)

We use the estimate $|\gamma(\eta+\xi y_k)|\leqq\exp\left[\frac{1}{2}pq(1-\theta(b))(\eta+\xi y_k)^2\right]$ when k is outside K and the trivial bound $|\gamma(\eta+\xi y_k)|\leqq\exp\left[\frac{1}{2}pq(\eta+\xi y_k)^2\right]$ when $k\in K$. It now follows that

$$\left(\prod_{j=1}^{k-1}|\gamma(\eta+\xi y_i)|\right)|\gamma(\eta)|^{N-k} \tag{16}$$

is dominated by $\exp\frac{1}{2}pqQ$ where

$$Q=(1-\theta(b))\sum_{\substack{j=1\\ j\in K}}^{k}(\eta+\xi y_j)^2+\sum_{\substack{j=1\\ j\in K}}^{k}(\eta+\xi y_j)^2+(1-\theta(b))(N-k)\eta^2= \tag{17}$$

$$=\eta^2\left[(1-\theta(b))N+\theta(b)\sum_{\substack{j=1\\ j\in K}}^{k}1\right]+\xi^2\left[(1-\theta(b))\sum_{j=1}^{k}y_j^2+\theta(b)\sum_{\substack{j=1\\ j\in K}}^{k}y_j^2\right]+$$

$$+2\xi\eta\left[(1-\theta(b))\sum_{j=1}^{k}y_j+\theta(b)\sum_{\substack{j=1\\ j\in K}}^{k}y_j+\sum_{\substack{j=k+1\\ j\in K}}^{N}y_j\right].$$

But

$$\sum_1^N|y_k|^3\geqq\sum_{k\in K}|y_k|^3\geqq\frac{b}{a}\sum_1^N|y_i|^3\sum_{k\in K}y_k^2. \tag{18}$$

Hence

$$\sum_{k\in K}y_k^2\leqq a/b. \tag{19}$$

Also

$$\sum_1^N|y_k|^3\geqq\sum_{k\in K}|y_k|^3\geqq\left(\frac{b}{a}\sum_1^N|y_j|^3\right)^3\sum_{k\in K}1 \tag{20}$$

and by Hölder's inequality

$$1=\sum_1^N y_j^2\leqq\left(\sum_1^N|y_j|^3\right)^{2/3}N^{1/3}. \tag{21}$$

Hence

$$\sum_{k\in K}1\leqq\left(\frac{a}{b}\right)^3N. \tag{22}$$

The inequalities (19) and (22) together with Schwartz' inequality now give

$$\sum_{k\in K}|y_k| \leqq \sqrt{\sum_{k\in K} y_k^2 \sum_{k\in K} 1} \leqq \left(\frac{a}{b}\right)^2 \sqrt{N}. \tag{23}$$

If we recall the inequality $2|\xi\eta\sqrt{N}| \leqq \xi^2 + N\eta^2$, and note that the maximum of $\left|\sum_1^k y_j\right|$ under the constraints $\sum_1^N y_j = 0$, $\sum_1^N y_j^2 = 1$ equals $\sqrt{Nk/N(1-k/N)} \leqq \frac{1}{2}\sqrt{N}$, we therefore see that (16) is dominated by

$$\exp\frac{1}{2}pq\left\{N\eta^2\left[\frac{3}{2}(1-\theta(b))+\left(\frac{a}{b}\right)^2+\theta(b)\left(\frac{a}{b}\right)^3\right]+\right. \tag{24}$$

$$\left.+\xi^2\left[\frac{3}{2}(1-\theta(b))+\theta(b)\frac{a}{b}+\left(\frac{a}{b}\right)^2\right]\right\}.$$

It remains to estimate

$$\sum_{k=1}^N |\gamma(\eta+\xi y_k)-\gamma(\eta)|. \tag{25}$$

We have

$$\gamma(\eta+\xi y_k)-\gamma(\eta) = \xi y_k \int_0^1 \gamma'(\eta+t\xi y_k)\,dt, \tag{26}$$

and by Taylor's formula

$$\gamma'(\zeta) = \gamma'(0)+\gamma''(0)\zeta+\int_0^1 (1-t)\gamma'''(t\zeta)\,dt\,\zeta^2. \tag{27}$$

Using the identities $\beta'(0)=0$, $\beta''(0)=-pq$, and the inequalities $|\beta^{(j)}(\eta)| \leqq pq \leqq 1/4$, $j=1, 2, \ldots$ we find that $\gamma'(0)=\gamma''(0)=0$, and

$$|\gamma'''(\zeta)| \leqq \text{Const}\, e^{\frac{1}{2}pq\zeta^2} pq\left(1+\left|\zeta\sqrt{pq}\right|^3\right). \tag{28}$$

Here and below Const denotes absolute constants.

We apply (28) and (27) with $\zeta=\eta+t\xi y_k$ and use the inequalities $|\zeta|^r \leqq 2^{r-1}(|\eta|^r+|t\xi y_k|^r)$, $r=2, 3$. It follows that (25) is dominated by

$$\text{Const}\, pq\sum_{k=1}^N |\xi y_k|\left(\eta^2+(\xi y_k)^2\right)\left(1+|\eta|^3+\left|\xi y_k\sqrt{pq}\right|^3\right)\cdot\exp\left[pq\left(\eta^2+(\xi y_k)^2\right)\right]. \tag{29}$$

The factor $1+|\eta|^3+\left|\xi y_k\sqrt{pq}\right|^3$ is dominated by $1+\pi^3+\left|\xi\sqrt{pq}\right|^3$ for $|\eta| \leqq \pi$ and the exponent $pq\eta^2+pq(\xi y_k)^2 \leqq \pi^2/4+pq(\xi y_k)^2$. But $|\xi y_k| \leqq b$ when $|\xi| \leqq T$ and $k \notin K$, and by (19) $y_k^2 \leqq a/b$ when $k \in K$. By Hölder's inequality

$$\sum_1^N |y_k| \leqq \left(\sum_1^N |y_k|^3\right)^{1/3} N^{2/3}. \tag{30}$$

So it follows from (21) that

$$\sum_1^N |y_k| \leqq N\sum_1^N |y_k|^3. \tag{31}$$

Using these estimates in (29) we see that (25) is dominated by

$$\text{(32)} \qquad \text{Const } pq|\xi|(\eta^2 N+\xi^2)\left(1+\left|\xi\sqrt{pq}\right|^3\right)\cdot\exp\left(pq\xi^2 a/b\right)\sum_1^N |y_k|^3$$

when $|\eta|\leqq\pi$.

The expression (13) is dominated by the product of (24) and (32). We first choose $b>0$ so small that $\frac{3}{2}(1-\theta(b))<1$ (this is possible because $\theta(0)=$ $=(2/\pi)^2>1/3$) and then $a>0$ so small that both

$$\text{(33)} \qquad \frac{3}{2}(1-\theta(b))+\left(\frac{a}{b}\right)^2+\theta(b)\left(\frac{a}{b}\right)^3<1$$

and

$$\text{(34)} \qquad \frac{3}{2}(1-\theta(b))+(2+\theta(b))\frac{a}{b}+\left(\frac{a}{b}\right)^2<1.$$

It follows that (12) is dominated by

$$\text{(35)} \qquad \text{Const } pq|\xi|(\eta^2 N+\xi^2)\left(1+\left|\xi\sqrt{pq}\right|^3\right)\cdot\exp\left[-\frac{1}{2}pqc(N\eta^2+\xi^2)\right]$$

where c is a positive absolute constant. The theorem follows.

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DER IRRTUMSBEREICH BEI ZUSAMMENGESETZTEN ALTERNATIVEN*

von

J. LEHN

Summary

It is well-known that in the case of two simple hypotheses the pairs of error probabilities of the first and second kind form a convex subset of the unit square when randomized decision functions are considered. This paper establishes a similar result in the case of composite hypotheses and gives bounds on the error probabilities of the second kind for α-level tests.

Bezeichnungen

Sei $(M, \mathscr{K})$ ein meßbarer Raum. Die Alternativen H_1 und H_2 sind zwei nichtleere, disjunkte Mengen von Wahrscheinlichkeitsmaßen auf $\mathscr{K}$. Eine Alternative heißt einfach, falls sie nur ein Element enthält, sonst zusammengesetzt. Die Alternativen H_1 und H_2 heißen dominiert, wenn ein σ-endliches Maß μ auf $\mathscr{K}$ existiert, sodaß jedes Wahrscheinlichkeitsmaß aus $H_1 \cup H_2$ eine Dichte bezüglich μ hat. Eine $\mathscr{K}$-meßbare Funktion $\delta: M \to [0, 1]$ heißt Test. Die Menge aller Tests wird mit Δ bezeichnet. Für $\delta \in \Delta$ definieren wir

$$\alpha_1(\delta) = \sup_{p \in H_1} \int_M \delta \, dp \quad \text{und} \quad \alpha_2(\delta) = \sup_{p \in H_2} \int_M (1-\delta) \, dp.$$

Dabei heißt $\alpha_i(\delta)$ *Irrtumswahrscheinlichkeit i-ter Art* $(i = 1, 2)$ des Tests δ. Für $0 \leqq \alpha \leqq 1$ ist

$$\Delta^\alpha(H_1|H_2) = \left\{\delta \in \Delta: \alpha_1(\delta) \leqq \alpha, \ \alpha_2(\delta) = \inf \{\alpha_2(\delta'): \delta' \in \Delta, \ \alpha_1(\delta') \leqq \alpha\}\right\}$$

die *Klasse der Niveau-α-Tests* für H_1 gegen H_2.

1. Der Irrtumsbereich

(1.1) DEFINITION. Die Teilmenge

$$A(H_1|H_2) = \left\{(\alpha_1(\delta), \ \alpha_2(\delta)) \in \mathbf{R}^2: \ \delta \in \Delta, \ \alpha_1(\delta) + \alpha_2(\delta) \leqq 1\right\}$$

des Einheitsquadrats heißt *Irrtumsbereich für H_1 gegen H_2*.

Über die geometrische Gestalt dieses Bereiches macht der folgende Satz eine erste Aussage.

* Herrn Professor Dr. H. Richter zum 60. Geburtstag gewidmet.

(1.2) SATZ. *Der Irrtumsbereich $A(H_1|H_2)$ ist konvex und enthält die Diagonale des Einheitsquadrats mit den Endpunkten* (0, 1) *und* (1, 0).

BEWEIS. Seien $\delta_1, \delta_2 \in \Delta$, $\alpha_1(\delta_i)+\alpha_2(\delta_i) \leqq 1$ $(i=1, 2)$, $0 \leqq t \leqq 1$ und $\delta' \in \Delta$ gegeben durch

$$\delta' = t\delta_1 + (1-t)\delta_2.$$

Dann gilt mit den Abkürzungen $a_i = t\alpha_i(\delta_1) + (1-t)\alpha_i(\delta_2)$ und $b_i = \alpha_i(\delta')$ stets $a_i \geqq b_i$ $(i=1, 2)$. Gilt dabei $a_1 = b_1$ und $a_2 = b_2$, so ist δ' ein Test mit $\alpha_i(\delta') = a_i$ $(i=1, 2)$. Gilt jedoch $a_1 + a_2 > b_1 + b_2$, so ist wegen $1 \geqq a_1 + a_2$ durch

$$\delta'' = (1-b_1-b_2)^{-1} \cdot [(1-a_1-a_2)\delta' + a_1(1-b_2) - b_1(1-a_2)]$$

ein Test erklärt, für den sich nach kurzer Rechnung $\alpha_i(\delta'') = a_i$ $(i=1, 2)$ ergibt. Da die Tests $\delta^0 \equiv 0$ und $\delta^1 \equiv 1$ zu Δ gehören, folgt daraus die Behauptung. ∎

(1.3) BEMERKUNG. Sind die Alternativen dominiert, so existieren Niveau-α-Tests für $0 \leqq \alpha \leqq 1$ (vgl. z. B. [4], Satz 2.21). Daraus läßt sich dann zusammen mit (1.2) die Abgeschlossenheit von $A(H_1|H_2)$ ableiten.

2. Schranken für die Irrtumswahrscheinlichkeiten zweiter Art von Niveau-α-Tests

Für Wahrscheinlichkeitsmaße p und q auf $\mathcal{K}$ definieren wir

$$d(p, q) = \sup_{K \in \mathcal{K}} |p(K) - q(K)|.$$

Die konvexe Hülle einer Alternative H_i bezeichnen wir mit $\operatorname{co} H_i$ $(i=1, 2)$ und benützen die Abkürzung

$$d(\operatorname{co} H_1, \operatorname{co} H_2) = \inf_{p_i \in \operatorname{co} H_i} d(p_1, p_2).$$

Für $0 \leqq c \leqq 1$ bezeichne $D(c)$ das Dreieck mit den Ecken (1, 0), (0, 1) und $\left(\frac{1-c}{2-c}, \frac{1-c}{2-c}\right)$ sowie $V(c)$ das konvexe Viereck mit den Ecken (1, 0), (0, 1), $(0, 1-c)$ und $(1-c, 0)$. Im Fall dominierter Alternativen erhalten wir damit als Folgerung aus einem Ergebnis in [2] eine weitere Aussage über die geometrische Gestalt des Irrtumsbereiches.

(2.1) SATZ. *Sind die Alternativen H_1 und H_2 dominiert, so gilt mit der Abkürzung* $c = d(\operatorname{co} H_1, \operatorname{co} H_2)$

$$D(c) \subset A(H_1|H_2) \subset V(c).$$

BEWEIS. Nach Theorem 5 in [2] gilt für dominierte Alternativen

$$(*) \qquad 1 - d(\operatorname{co} H_1, \operatorname{co} H_2) = \min_{\delta \in \Delta} [\alpha_1(\delta) + \alpha_2(\delta)].$$

Es ist daher $\alpha_1(\delta) + \alpha_2(\delta) \geqq 1-c$ für alle $\delta \in \Delta$, also gilt $A(H_1|H_2) \subset V(c)$. Bezeichnet $T(t)$ das Dreieck mit den Ecken (0, 1), (1, 0) und $(t, 1-c-t)$, so gilt $D(c) \subset T(t)$ für $0 \leqq t \leqq 1-c$. Nach (*) existiert ein t_0 mit $0 \leqq t_0 \leqq 1-c$ und $(t_0, 1-c-t_0) \in A(H_1|H_2)$. Wegen Satz (1.2) folgt daraus $D(c) \subset T(t_0) \subset A(H_1|H_2)$. ∎

Aus diesem Satz erhalten wir das

(2.2) KOROLLAR. *Seien* $0 \leqq \alpha \leqq 1$, $\delta \in \Delta^x(H_1 | H_2)$ *und* $c = d(\operatorname{co} H_1, \operatorname{co} H_2)$. *Sind die Alternativen dominiert, so gilt für* $0 < c < 1$

$$\max\{0, 1-c-\alpha\} \leqq \alpha_2(\delta) \leqq \max\left\{\frac{1-c-\alpha}{1-c}, (1-c)(1-\alpha)\right\}.$$

Das für einfache Alternativen formulierte Korollar 2.26 in WITTING's Buch (vgl. [4]) kann nun auf den Fall dominierter zusammengesetzter Alternativen verallgemeinert werden.

(2.3) KOROLLAR. *Für* $\delta \in \Delta^x(H_1 | H_2)$, $0 < \alpha < 1$, *gilt* $\alpha_2(\delta) \leqq 1-\alpha$; *und zwar ist bei dominierten Alternativen genau einer der beiden Fälle möglich: Es ist entweder* $\alpha_2(\delta) < 1-\alpha$ *für alle* α *mit* $0 < \alpha < 1$, *oder es ist* $d(\operatorname{co} H_1, \operatorname{co} H_2) = 0$.

Das folgende Beispiel zeigt, daß im Satz (2.1) auf die Voraussetzung, daß die Alternativen dominiert sind, nicht ohne weiteres verzichtet werden kann. Es widerlegt auch die Vermutung, der Irrtumsbereich für zusammengesetzte Alternativen sei der Durchschnitt aller Irrtumsbereiche für die entsprechenden Paare einfacher Alternativen.

BEISPIEL. Seien $M = [0, 1]$, $\mathscr{K}$ das System der Borelschen Teilmengen von $[0, 1]$. Mit p_x bezeichnen wir das Einpunktmaß mit $p_x(\{x\}) = 1$ für $0 \leqq x \leqq 1$. Weiter sei $C \subset [0, 1]$ mit $C \notin \mathscr{K}$. Wählen wir $H_1 = \{p_x : x \in C\}$ und $H_2 \{p_x : x \in M \setminus C\}$, dann ist,

$$\sup_{K \in \mathscr{K}} |p_1(K) - p_2(K)| = 1 \quad \text{für} \quad p_i \in \operatorname{co} H_i \quad (i = 1, 2)$$

und daher $d(\operatorname{co} H_1, \operatorname{co} H_2) = 1$. Der Punkt $(0, 0)$ gehört jedoch nicht zu $A(H_1 | H_2)$, denn aus $\alpha_1(\delta) = \alpha_2(\delta) = 0$ folgt $\delta(x) = 0$ für $x \in C$ und $\delta(x) = 1$ für $x \in M \setminus C$, d. h., δ ist nicht $\mathscr{K}$-meßbar, gehört also nicht zu Δ.

3. Anwendung

Wir betrachten nun das Entscheidungsproblem bei einfachen Alternativen und unabhängigen Versuchswiederholungen. Wird ein Niveau-α-Test verwendet, so liegt die Irrtumswahrscheinlichkeit erster Art fest. Es ist dann wünschenswert, daß auch die Irrtumswahrscheinlichkeit zweiter Art in gewissen Grenzen bleibt. Durch passende Wahl der Anzahl der Versuchswiederholungen läßt sich dies (wegen der Voraussetzung der Verschiedenheit der Alternativen) erreichen. Abschätzungen dafür, wie groß diese Anzahl zu wählen ist, um die Irrtumswahrscheinlichkeit zweiter Art in bestimmten Grenzen zu halten, gibt der folgende Satz.

Wir bezeichnen das n-fache Produktmaß eines Maßes p auf $\mathscr{K}$ mit p^n.

(3.1) SATZ. *Seien* $0 \leqq \alpha \leqq 1$, $\delta \in \Delta^x(\{p^n\} | \{q^n\})$ *und* $c = d(p, q)$. *Ist* $c < 1$, *so gilt*

$$\max\{0, (1-c)^n - \alpha\} \leqq \alpha_2(\delta) \leqq \max\{1 - \alpha(1-c^2)^{-\frac{n}{2}}, \quad (1-c^2)^{\frac{n}{2}}(1-\alpha)\}.$$

BEWEIS. Mit der Abkürzung $d_n = d(p^n, q^n)$ gilt

$$(1-c)^n \leqq 1 - d_n \leqq (1-c^2)^{\frac{n}{2}}.$$

Zur ersten Ungleichung vergleiche man etwa [3], Seite 709, zur zweiten [1]. Satz 2. Damit folgt die Behauptung aus (2.2). ∎

Dieser Satz zeigt insbesondere, daß bei Anwendung von Niveau-α-Tests mit fest vogegebenem α die Folge der Irrtumswahrscheinlichkeiten zweiter Art mit wachsender Anzahl der Beobachtungen exponentiell gegen Null konvergiert.

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SOME REMARKS CONCERNING IRREGULARITIES OF DISTRIBUTION OF SEQUENCES OF INTEGERS IN ARITHMETIC PROGRESSIONS, II

by

A. SÁRKÖZY

1. We shall write $f(N) \ll g(N)$ if there exists an absolute constant C such that $|f(N)| \leqq C|g(N)|$ for $N=1, 2, \ldots$. For α real, we shall write $e^{2\pi i\alpha}=e(\alpha)$.

Let N be any positive integer, $\mathscr{A}$ a subsequence of $\{1, 2, \ldots, N\}$. For some positive integer q and some integers n, h, let

$$A_{q,h}(n) = \sum_{\substack{1 \leqq a \leqq n \\ a \equiv h (\mathrm{mod}\, q) \\ a \in \mathscr{A}}} 1,$$

in particular,

$$A(n) = A_{1,0}(n) = \sum_{\substack{1 \leqq a \leqq n \\ a \in \mathscr{A}}} 1.$$

We write

$$\eta = \frac{A(N)}{N}.$$

Furthermore, for some positive integer q and some integers h, u, v, let

$$d_{q,h}(m, n) = \sum_{\substack{m < a \leqq n \\ a \equiv h (\mathrm{mod}\, q) \\ a \in \mathscr{A}}} 1 - \eta \sum_{\substack{m < a \leqq n \\ a \equiv h (\mathrm{mod}\, q)}} 1,$$

$$e_{q,h}(m, n) = \sum_{\substack{m < a \leqq n \\ a \equiv h (\mathrm{mod}\, q) \\ a \in \mathscr{A}}} 1 - q^{-1} \sum_{\substack{m < a \leqq n \\ a \in \mathscr{A}}} 1,$$

in particular,

$$D_{q,h}(m) = d_{q,h}(0, m) = \sum_{\substack{1 \leqq a \leqq m \\ a \equiv h (\mathrm{mod}\, q) \\ a \in \mathscr{A}}} 1 - \eta \sum_{\substack{1 \leqq a \leqq m \\ a \equiv h (\mathrm{mod}\, q)}} 1,$$

$$E_{q,h}(m) = e_{q,h}(0, m) = \sum_{\substack{1 \leqq a \leqq m \\ a \equiv h (\mathrm{mod}\, q) \\ a \in \mathscr{A}}} 1 - q^{-1} \sum_{\substack{1 \leqq a \leqq m \\ a \in \mathscr{A}}} 1 = A_{q,h}(m) - q^{-1} A(m),$$

and let

$$V_q(m) = \sum_{h=0}^{q-1} D_{q,h}^2(m),$$

$$U_q(m) = \sum_{h=0}^{q-1} E_{q,h}^2(m).$$

K. F. ROTH proved in [2] the following

THEOREM 1. *For all positive integers N, Q and any sequences $\mathscr{A}$,*

$$\sum_{q=1}^{Q} q^{-1} \sum_{m=1}^{N} V_q(m) + Q \sum_{q=1}^{Q} V_q(N) \gg \eta(1-\eta)Q^2N, \tag{1}$$

where the implicit constant is absolute.

This theorem implies the

COROLLARY 1 (K. F. ROTH).

Let N, Q be positive integers for which $Q \leqq \sqrt{N}$. For any sequence $\mathscr{A}$, there exist integers $q_1, m_1 (\leqq N)$, h_1 such that

$$1 \leqq q_1 \leqq Q$$

and

$$|D_{q_1,h_1}(m_1)| = \Big| \sum_{\substack{1 \leqq a \leqq m_1 \\ a \equiv h_1 (\mathrm{mod}\, q_1) \\ a \in \mathscr{A}}} 1 - \eta \sum_{\substack{1 \leqq a \leqq m_1 \\ a \equiv h_1 (\mathrm{mod}\, q_1)}} 1 \Big| \gg \sqrt{\eta(1-\eta)Q}. \tag{2}$$

(In fact, (1) says that (2) holds on average.)

Choosing $Q = [\sqrt{N}]$, we obtain

COROLLARY 1′ (K. F. ROTH). *For all positive integers N and sequences $\mathscr{A}$, there exist some positive integer q_1 and integers $m_1 (\leqq N)$, h_1 such that*

$$1 \leqq q_1 \leqq \sqrt{N}$$

and

$$|D_{q_1,h_1}(m_1)| \gg \sqrt{\eta(1-\eta)} N^{1/4}. \tag{3}$$

As Roth writes, (1) says that a sequence $\mathscr{A}$ can not be "well-distributed simultaneously *among* and *within* all congruence classes". In the first part of this series (see [4]), my aim was to prove results saying that if a sequence $\mathscr{A}$ satisfies some possibly simple assumption then it can not be well-distributed purely *among* all congruence classes. (There I showed that such a theorem did not exist without any restriction on $\mathscr{A}$.) To formulate a theorem of this type, I studied the quantities $U_q(m)$ instead of $V_q(m)$, and I gave a lower estimate for a mean of them in terms of the number of the blocks constituting $\mathscr{A}$, i.e. in terms of

$$B_N(\mathscr{A}) = \sum_{\substack{1 \leqq n \leqq N \\ n-1 \notin \mathscr{A}, n \in \mathscr{A}}} 1.$$

If N or both N and $\mathscr{A}$ are fixed then we shall write $B(\mathscr{A}) = B_N(\mathscr{A})$ and $B = B_N(\mathscr{A})$, respectively. I proved the following results:

THEOREM 2. (A. SÁRKÖZY, [4]). *Let N be any positive integer, Q any real number such that $Q \geqq 2$. Then for any sequence $\mathscr{A}$,*

$$\sum_{2 \leqq q \leqq Q} q^{-1} \sum_{m=1}^{N} U_q(m) + \sum_{2 \leqq q \leqq Q} q U_q(N) \gg Q^2 B. \tag{4}$$

COROLLARY 2. *Let N, Q be positive integers for which $2 \leqq Q \leqq \sqrt{N}$. For any sequence $\mathscr{A}$, there exist integers $q_1, m_1 (\leqq N), h_1$ such that*

$$2 \leqq q_1 \leqq Q$$

and

$$|E_{q_1,h_1}(m_1)| = \Big| \sum_{\substack{1 \leqq a \leqq m_1 \\ a \equiv h_1 (\mathrm{mod}\, q_1) \\ a \in \mathscr{A}}} 1 - q_1^{-1} \sum_{\substack{1 \leqq a \leqq m_1 \\ a \in \mathscr{A}}} 1 \Big| \gg \sqrt{QB/N}. \tag{5}$$

Choosing $Q = [\sqrt{N}]$, we obtain

COROLLARY 2′. *For all positive integers $N \geqq 4$ and sequences $\mathscr{A}$, there exist integers $q_1, m_1 (\leqq N), h_1$ such that*

$$2 \leqq q_1 \leqq \sqrt{N}$$

and

$$|E_{q_1,h_1}(m_1)| \gg \sqrt{B}/\sqrt[4]{N}. \tag{6}$$

The aim of this paper is to discuss inequality (4) and to show that Theorem 2 and also Corollaries 2 and 2′ are near best possible. This paper relates to the first part of this series in the same way as the paper of H. L. MONTGOMERY—A. SÁRKÖZY (see [1]) to paper [2] OF ROTH. Thus we shall often refer to [1] and some details of this discussion will be analogous to the corresponding details in [1]; we shall leave these details to the reader. However, there will be also some important differences and we shall give all the details in these cases. Namely, in [1], we discussed Theorem 1 and Corollaries 1 and 1′ in terms of only *two* parameters, Q and N. On the other hand, in [4], we had to introduce the new parameter B of basic importance. Thus we have to discuss Theorem 2 and Corollaries 2 and 2′ in terms of *three* parameters, Q, N and B, and we shall be interested mostly in the role of the parameter B. This extra parameter will cause some serious difficulties. E.g. in Section 7, we shall not be able to adapt the method used in [1] (which was based on the Rudin—Shapiro construction); instead, we shall have to elaborate a combinatorial-statistical method (whose fundamental principle was just mentioned in [1]).

2. By Steiner's identity:

$$\sum_{i=1}^{n} (a_i - B)^2 = \sum_{i=1}^{n} (a_i - A)^2 + n(B - A)^2 \tag{7}$$

where $a_1, a_2, \ldots, a_n, B$ are arbitrary real numbers and

$$A = \Big(\sum_{i=1}^{n} a_i \Big) / n.$$

Furthermore, for any real numbers a, b,

$$(a+b)^2 \leqq 2a^2 + 2b^2. \tag{8}$$

Applying (7) and (8):

$$(9)\qquad U_q(m)=\sum_{h=0}^{q-1}E_{q,h}^2(m)=\sum_{h=0}^{q-1}\left(A_{q,h}(m)-q^{-1}A(m)\right)^2=$$

$$=\sum_{h=0}^{q-1}\left(A_{q,h}(m)-\eta q^{-1}m\right)^2-q\left(\eta q^{-1}m-q^{-1}A(m)\right)^2=$$

$$=\sum_{h=0}^{q-1}\Big\{\Big(A_{q,h}(m)-\eta\sum_{\substack{1\leqq a\leqq m\\ a\equiv h(\mathrm{mod}\,q)}}1\Big)+\eta\Big(\sum_{\substack{1\leqq a\leqq m\\ a\equiv h(\mathrm{mod}\,q)}}1-q^{-1}m\Big)\Big\}^2-q^{-1}\left(\eta m-A(m)\right)^2\leqq$$

$$\leqq 2\sum_{h=0}^{q-1}\Big(A_{q,h}(m)-\eta\sum_{\substack{1\leqq a\leqq m\\ a\equiv h(\mathrm{mod}\,q)}}1\Big)^2+2\sum_{h=0}^{q-1}\eta^2\Big(\sum_{\substack{1\leqq a\leqq m\\ a\equiv h(\mathrm{mod}\,q)}}1-q^{-1}m\Big)^2-q^{-1}\left(\eta m-A(m)\right)^2\leqq$$

$$\leqq 2\sum_{h=0}^{q-1}D_{q,h}^2(m)+2\sum_{h=0}^{q-1}\eta^2-q^{-1}\left(\eta m-A(m)\right)^2=$$

$$=2V_q(m)+2q\eta^2-q^{-1}m^2\left(A(N)N^{-1}-A(m)m^{-1}\right)^2\quad\left(\leqq 2V_q(m)+2q\right)$$

(cf. Lemma 1 in [1]). This inequality shows that if we have a lower estimate for some mean of the quantities $U_q(m)$ then we have practically the same lower bound for the corresponding mean of the $V_q(m)$. In fact, e.g. Corollary 1 can be deduced easily also from Theorem 2.

Applying (7) and the inequality

$$(a+b)^2\geqq\frac{1}{2}a^2-b^2,$$

it can be deduced in similar way that

$$(10)\qquad U_q(m)\geqq\frac{1}{2}V_q(m)-q\eta^2-q^{-1}m^2\left(A(N)N^{-1}-A(m)m^{-1}\right)^2.$$

Inequalities (9) and (10) express the connection of the quantities $U_q(m)$ with the quantities $V_q(m)$.

3. As we said in Part I of this series, the large sieve gave an upper estimate for some mean of the quantities $U_q(N)$. Thus one might like to deduce an estimate of type (4) without the quantities $U_q(1)$, $U_q(2)$, ..., $U_q(N-1)$; in other words, e.g. one might like to show that the first term on the left hand side can be dropped.

This can not be done, as the following example shows: $a\in\mathscr{A}$ should hold if and only if $1\leqq a\leqq N/2$ and a is odd, or $N/2<a\leqq N$ and a is even. It can be shown easily that $|E_{q,h}(N)|\leqq 3$ for any q, h. Consequently, the second term on the left hand side of (4) is $O(Q^3)$. On the other hand, $B\gg N$ for this sequence $\mathscr{A}$, hence, the right hand side is $\gg NQ^2$. Thus the second term on the left hand side is little o of the right hand side for $Q=o(N)$.

4. On the other hand, it can be shown that in the most interesting case $B\gg Q^2$ (this holds e.g. for $B\gg N$, $Q\ll\sqrt{N}$), the second term on the left hand side of (4) can be always dropped.

THEOREM 3. *Let δ be any positive number, $Q \geqq 2$ and N be positive integers. Let us assume that*

$$B \geqq \delta Q^2. \tag{11}$$

Then

$$\sum_{q=2}^{Q} q^{-1} \sum_{m=1}^{N} U_q(m) \gg_\delta Q^2 B, \tag{12}$$

where $\gg_\delta$ denotes that the implicit constant depends on δ.

PROOF. Let us apply Theorem 2 for the sequences $\mathscr{A} \cap [0, M]$ where $N - B_N(\mathscr{A})/2 \leqq M \leqq N$, and let us add the inequalities obtained in this way:

$$\sum_{q=2}^{Q} q^{-1} \sum_{N-B_N(\mathscr{A})/2 \leqq M \leqq N} \sum_{m=1}^{M} U_q(m) + \tag{13}$$

$$+ \sum_{q=2}^{Q} q \sum_{N-B_N(\mathscr{A})/2 \leqq M \leqq N} U_q(M) \gg Q^2 \sum_{N-B_N(\mathscr{A})/2 \leqq M \leqq N} B_M(\mathscr{A}).$$

In view of (11), the left hand side can be estimated in the following way:

$$\sum_{q=2}^{Q} q^{-1} \sum_{N-B_N(\mathscr{A})/2 \leqq M \leqq N} \sum_{m=1}^{M} U_q(m) + \sum_{q=2}^{Q} q \sum_{N-B_N(\mathscr{A})/2 \leqq M \leqq N} U_q(M) \leqq \tag{14}$$

$$\leqq \sum_{q=2}^{Q} \left\{ \left(\frac{B}{2} + 1 \right) q^{-1} + Q \right\} \sum_{m=1}^{N} U_q(m) \leqq \sum_{q=2}^{Q} (2Bq^{-1} + B\delta^{-1}Q^{-1}) \sum_{m=1}^{N} U_q(m) \leqq$$

$$\leqq \sum_{q=2}^{Q} B(2q^{-1} + \delta^{-1}q^{-1}) \sum_{m=1}^{N} U_q(m) = B(2+\delta^{-1}) \sum_{q=2}^{Q} q^{-1} \sum_{m=1}^{N} U_q(m).$$

The right hand side of (13):

$$Q^2 \sum_{N-B_N(\mathscr{A})/2 \leqq M \leqq N} B_M(\mathscr{A}) = Q^2 \sum_{0 \leqq t \leqq B_N(\mathscr{A})/2} B_{N-t}(\mathscr{A}) \geqq \tag{15}$$

$$\geqq Q^2 \sum_{0 \leqq t \leqq B_N(\mathscr{A})/2} (B_N(\mathscr{A}) - t) \geqq Q^2 \sum_{0 \leqq t \leqq B_N(\mathscr{A})/2} (B_N(\mathscr{A}) - B_N(\mathscr{A})/2) =$$

$$= Q^2 \sum_{0 \leqq t \leqq B_N(\mathscr{A})/2} B_N(\mathscr{A})/2 \geqq Q^2 B_N^2(\mathscr{A})/4 = Q^2 B^2/4.$$

(13), (14) and (15) yield (12).

5. As the construction given in Section 3 and Theorem 3 show, for B, Q satisfying (11) (e.g. for $B \gg N$, $Q \ll \sqrt{N}$), the first term on the left hand side of (4) plays the dominant role, while the second one plays only secondary role. Thus in this case, it is reasonable to replace inequality (4) by (12).

On the other hand, for "small" B and "large" Q, neither of the terms on the left hand side of (4) can be dropped. This has been shown in Section 3 for the first term. As regards the second term, this fact can be shown by investigating the sequence $\mathscr{A}$ consisting of the single integer N:

$$\mathscr{A} = \{N\}. \tag{16}$$

It can be checked easily that for this sequence $\mathcal{A}$ and $2 \leqq q \leqq Q \leqq N$,

$$U_q(m) = 0 \quad \text{for} \quad m = 1, 2, \ldots, N-1 \tag{17}$$

and

$$U_q(N) < 2. \tag{18}$$

(17) and (18) imply that the first term on the left hand side of (4):

$$\sum_{2 \leqq q \leqq Q} q^{-1} \sum_{m=1}^{N} U_q(m) \ll \log Q.$$

(While the second term:

$$\sum_{2 \leqq q \leqq Q} q U_q(N) \ll Q^2.)$$

On the other hand, the right hand side of (4):

$$Q^2 B = Q^2.$$

Thus in fact, the first term on the left hand side of (4) is little o of the right hand side for the sequence (16).

Summarizing: for B, Q which do not satisfy (11), both terms on the left hand side of (4) are needed. In particular, the role of the second term is to cover the case when B is small, Q is relatively large, and $i-1 \notin \mathcal{A}$, $i \in \mathcal{A}$ holds mostly for i near N.

6. In this section, we shall show that (4) is best possible (except the value of the implicit constant) for practically all B, Q, N:

THEOREM 4. *Let Q, N be positive integers greater than 1 and let B_0 be some positive integer satisfying*

$$1 \leqq B_0 \leqq N/16. \tag{19}$$

Then there exists a sequence $\mathcal{A} (\subset \{1, 2, \ldots, N\})$ such that for this sequence $\mathcal{A}$,

$$B_N(\mathcal{A}) = B_0 \tag{20}$$

and

$$\sum_{2 \leqq q \leqq Q} q^{-1} \sum_{m=1}^{N} U_q(m) + \sum_{2 \leqq q \leqq Q} q U_q(N) \ll Q^2 B_0 \tag{21}$$

(where the implicit constant is independent of B_0, Q, N).

PROOF. We will construct a sequence $\mathcal{A}$ having the desired properties.

As the first step of the construction, we are going to define an integer N^* satisfying

$$N^* \leqq 16 B_0 \tag{22}$$

and to construct a sequence $\mathcal{A}^* \subset \{1, 2, \ldots, N^*\}$ such that

$$B_{N^*}(\mathcal{A}^*) = B_0 \tag{23}$$

and (21) (with N^* in place of N) hold:

$$\sum_{2\leqq q\leqq Q} q^{-1} \sum_{m=1}^{N^*} U_q^*(m) + \sum_{2\leqq q\leqq Q} q U_q^*(N^*) \ll Q^2 B_0 \tag{24}$$

where $U_q^*(m)$ denotes the value $U_q(m)$ corresponding to the sequence $\mathscr{A}^*$.

This sequence $\mathscr{A}^*$ can be constructed similarly to the sequence $\mathscr{N}$ in Section 5 of [1]; thus we omit some details. The construction is based on the Rudin—Shapiro example in harmonic analysis (see [3], [5]): Let

$$P_0(z) = Q_0(z) = 1$$

and

$$P_n(z) = P_{n-1}(z) + z^{2^{n-1}} Q_{n-1}(z),$$

$$Q_n(z) = P_{n-1}(z) - z^{2^{n-1}} Q_{n-1}(z) \quad \text{for} \quad n = 1, 2, \ldots .$$

The aim of this construction is to show the existence of polynomials having coefficients -1 or $+1$ and such that their absolut value is "uniformly small" on the unit circle. In fact, it can be shown easily that for this polynomials $P_n(z)$, $Q_n(z)$,

$$\max_{|z|=1} |P_n(z)| \leqq \sqrt{2} \cdot 2^{n/2} \quad \text{and} \quad \max_{|z|=1} |Q_n(z)| \leqq \sqrt{2} \cdot 2^{n/2}. \tag{25}$$

This construction uniquely determines an infinite sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, \ldots$ such that

$$\varepsilon_k = -1 \quad \text{or} \quad +1 \quad \text{for} \quad k = 1, 2, \ldots \tag{26}$$

and

$$P_n(z) = \sum_{j=1}^{2^n} \varepsilon_j z^{j-1}.$$

Let

$$\varphi_k(\alpha) = \sum_{j=1}^{k} \varepsilon_j e(j\alpha) \quad \text{for} \quad k = 1, 2, \ldots .$$

It can be deduced easily from (25) that

$$\max_{0\leqq\alpha\leqq 1} |\varphi_k(\alpha)| < 5\sqrt{k} \quad \text{for} \quad k = 1, 2, \ldots \tag{27}$$

(see Lemma 2 in [1]).

We are ready to construct the sequence $\mathscr{A}^*$ of desired properties. We have to distinguish two cases.

CASE 1. Let

$$Q > 16B_0. \tag{28}$$

This case is near trivial. Namely, let

$$N^* = 16B_0 \tag{29}$$

and let $\mathscr{A}^* \subset \{1, 2, \ldots, N^*\}$ be any sequence satisfying $B_{N^*}(\mathscr{A}^*) = B_0$. (By $B_0 < N^*/2$, such sequence $\mathscr{A}^*$ exists.) Then (22) and (23) hold trivially.

Furthermore, for $q \leqq m$,

$$U_q^*(m) = \sum_{h=0}^{q-1} \left(A_{q,h}^*(m) - q^{-1}A^*(m)\right)^2 \leqq \sum_{h=0}^{q-1} (mq^{-1}+1)^2 \leqq \tag{30}$$

$$\leqq \sum_{h=0}^{q-1} (2mq^{-1})^2 = 4m^2q^{-1} \quad (\text{for } q \leqq m)$$

while for $q \geqq m$ $(\geqq 1)$,

$$U_q^*(m) = \sum_{h=0}^{q-1} \left(A_{q,h}^*(m) - q^{-1}A^*(m)\right)^2 = \tag{31}$$

$$= \sum_{h=0}^{m} \left(A_{q,h}^*(m) - q^{-1}A^*(m)\right)^2 + \sum_{h=m+1}^{q} \left(A_{q,h}^*(m) - q^{-1}A^*(m)\right)^2 \leqq$$

$$\leqq \sum_{h=0}^{m} 1 + \sum_{h=m+1}^{q} \left(q^{-1}A^*(m)\right)^2 \leqq (m+1) + \sum_{h=m+1}^{q} (q^{-1}m)^2 \leqq$$

$$\leqq 2m + q \cdot q^{-2}m^2 = 2m + q^{-1}m^2 \leqq 3m \quad (\text{for } q \geqq m).$$

By (28) and (29), $Q > N^*$. Thus by (30) and (31), the left hand side of (24):

$$\sum_{2 \leqq q \leqq Q} q^{-1} \sum_{m=1}^{N^*} U_q^*(m) + \sum_{2 \leqq q \leqq Q} q U_q^*(N^*) = \tag{32}$$

$$= \sum_{2 \leqq q \leqq N^*} q^{-1} \sum_{1 \leqq m \leqq q} U_q^*(m) + \sum_{2 \leqq q \leqq N^*} q^{-1} \sum_{q < m \leqq N^*} U_q^*(m) +$$

$$+ \sum_{N^* < q \leqq Q} q^{-1} \sum_{1 \leqq m \leqq N^*} U_q^*(m) + \sum_{2 \leqq q \leqq N^*} q U_q^*(N^*) + \sum_{N^* < q \leqq Q} q U_q^*(N^*) \leqq$$

$$\leqq \sum_{2 \leqq q \leqq N^*} q^{-1} \sum_{1 \leqq m \leqq q} 3m + \sum_{2 \leqq q \leqq N^*} q^{-1} \sum_{q < m \leqq N^*} 4m^2q^{-1} +$$

$$+ \sum_{N^* < q \leqq Q} q^{-1} \sum_{1 \leqq m \leqq N^*} 3m + \sum_{2 \leqq q \leqq N^*} q \cdot 4(N^*)^2 q^{-1} + \sum_{N^* < q \leqq Q} q \cdot 3N^* \leqq$$

$$\leqq \sum_{2 \leqq q \leqq N^*} q^{-1} \cdot 3q^2 + \sum_{2 \leqq q \leqq N^*} \sum_{1 \leqq m \leqq N^*} 4(N^*)^2 q^{-2} +$$

$$+ \sum_{N^* < q \leqq Q} (N^*)^{-1} \cdot 3(N^*)^2 + 4(N^*)^3 + \sum_{1 \leqq q \leqq Q} Q \cdot 3N^* <$$

$$< \sum_{2 \leqq q \leqq N^*} 3N^* + 4(N^*)^3 \sum_{q=1}^{+\infty} q^{-2} + \sum_{1 \leqq q \leqq Q} 3N^* + 4(N^*)^3 + 3Q^2N^* <$$

$$< 3(N^*)^2 + 8(N^*)^3 + 3QN^* + 4(N^*)^3 + 3Q^2N^* \leqq 21Q^2N^*.$$

By (29), the right hand side of (24):

$$Q^2 B_{N^*}(\mathscr{A}^*) = Q^2 B_0 > \frac{1}{16} Q^2 N^*. \tag{33}$$

(32) and (33) yield (24).

CASE 2.

$$Q \leqq 16 B_0. \tag{34}$$

Let

$$R = \begin{cases} Q & \text{if } 2 \leqq Q \leqq \sqrt{16 B_0}, \\ 16 B_0 / Q & \text{if } \sqrt{16 B_0} < Q \leqq 16 B_0, \end{cases} \tag{35}$$

and let P denote the smallest prime number greater than $8R$. By Bertrand's postulate,

$$8R < P \leqq 16R. \tag{36}$$

Let us define an infinite sequence $\delta_0, \delta_1, \ldots, \delta_k, \ldots$ in the following way: Let $\delta_0 = -1$. For $i = 1, 2, \ldots$, let

$$i \equiv h_i \pmod{P} \quad \text{where} \quad 1 \leqq h_i \leqq P,$$

and let $\delta_i = \varepsilon_{h_i}$. Then obviously, $\delta_k = -1$ or $+1$ for $k = 0, 1, 2, \ldots$.

Furthermore, let us define the positive integer N^* by

$$\sum_{\substack{1 \leqq i \leqq N^* - 1 \\ \delta_{i-1} = -1,\ \delta_i = +1}} 1 < B_0 = \sum_{\substack{1 \leqq i \leqq N^* \\ \delta_{i-1} = -1,\ \delta_i = +1}} 1. \tag{37}$$

Finally, the sequence $\mathscr{A}^*$ of desired properties can be constructed in the following way: for some positive integer i, $i \in \mathscr{A}^*$ should hold if and only if $1 \leqq i \leqq N^*$ and $\delta_i = +1$ hold.

We have to show that N^* and $\mathscr{A}^*$ satisfy (22), (23) and (24).

(23) holds trivially (by (37)). The next step is to show that also (22) holds.

By the definition of the sequence $\varepsilon_1, \varepsilon_2, \ldots$ in (26), for any positive integer k, either $\varepsilon_{4k-1} = -1$ and $\varepsilon_{4k} = +1$ or $\varepsilon_{4k-1} = +1$ and $\varepsilon_{4k} = -1$ hold. Consequently, for any positive integer l, there exists a positive integer i such that $l < i \leqq l + 8$ and $\varepsilon_{i-1} = -1$, $\varepsilon_i = +1$ hold. This implies (by $P > 8R \geqq 8$) that for $t = 0, 1, 2, \ldots$, there exists a positive integer i such that $t < i \leqq t + 16$ and $\delta_{i-1} = -1$, $\delta_i = +1$ hold. Thus

$$\sum_{\substack{1 \leqq i \leqq 16 B_0 \\ \delta_{i-1} = -1,\ \delta_i = +1}} 1 = \sum_{k=1}^{B_0} \Bigg(\sum_{\substack{16k - 16 < i \leqq 16k \\ \delta_{i-1} = -1,\ \delta_i = +1}} 1 \Bigg) \geqq \sum_{k=1}^{B_0} 1 = B_0.$$

This yields in conjunction with the definition of B_0 that (22) holds.

Finally, to show that also (24) holds, let us introduce the following notation: for $m = 1, 2, \ldots, N^*$, let

$$S(\alpha, m) = \sum_{\substack{1 \leqq j \leqq m \\ j \in \mathscr{A}^*}} e(j\alpha),$$

in particular,

$$S(\alpha, N^*) = S(\alpha).$$

As we showed in [1],

$$qU_q^*(m) = q\sum_{h=0}^{q-1}\left(E_{q,h}^*(m)\right)^2 = \sum_{k=1}^{q-1}|S(k/q, m)|^2 \tag{38}$$

for $m=1, 2, \ldots, N^*$, $q=2, 3, \ldots$. Thus to estimate $U_q^*(m)$, we need an estimate for $|S(\alpha, m)|$.

For some $1\leqq m\leqq N^*$, let $m=uP+v$ where $u\geqq 0$, v are integers and $1\leqq v\leqq P$. Then by the construction of the sequence $\mathscr{A}^*$,

$$S(\alpha, m) = \sum_{j=1}^{m}\frac{\delta_j+1}{2}e(j\alpha) = \frac{1}{2}\sum_{j=1}^{m}\delta_j e(j\alpha)+\frac{1}{2}\sum_{j=1}^{m}e(j\alpha) = \tag{39}$$

$$= \frac{1}{2}\sum_{l=0}^{u-1}\left(\sum_{j=lP+1}^{(l+1)P}\delta_j e(j\alpha)\right)+\frac{1}{2}\sum_{j=uP+1}^{uP+v}\delta_j e(j\alpha)+\frac{1}{2}\cdot\frac{e(\alpha)\left(1-e(m\alpha)\right)}{1-e(\alpha)} =$$

$$= \frac{1}{2}\sum_{l=0}^{u-1}e(lP\alpha)\sum_{j=1}^{P}\delta_j e(j\alpha)+\frac{1}{2}e(uP\alpha)\sum_{j=1}^{v}\delta_j e(j\alpha)+\frac{1}{2}\cdot\frac{e(\alpha)\left(1-e(m\alpha)\right)}{1-e(\alpha)} =$$

$$= \frac{1}{2}\sum_{l=0}^{u-1}e(lP\alpha)\varphi_p(\alpha)+\frac{1}{2}\cdot e(uP\alpha)\varphi_v(\alpha)+\frac{1}{2}\cdot\frac{e(\alpha)\left(1-e(m\alpha)\right)}{1-e(\alpha)} =$$

$$= \frac{1}{2}\cdot\frac{1-e(uP\alpha)}{1-e(P\alpha)}\varphi_p(\alpha)+\frac{1}{2}\cdot e(uP\alpha)\varphi_v(\alpha)+\frac{1}{2}\cdot\frac{e(\alpha)\left(1-e(m\alpha)\right)}{1-e(\alpha)}$$

for $\|P\alpha\|\neq 0$.

For any real number β,

$$|1-e(\beta)|\geqq 4\|\beta\|. \tag{40}$$

In view of (27) and (40), we obtain from (39) that

(41)

$$|S(\alpha, m)|\leqq\frac{1}{2}\cdot\frac{|1-e(uP\alpha)|}{|1-e(P\alpha)|}\cdot|\varphi_p(\alpha)|+\frac{1}{2}\cdot|e(uP\alpha)||\varphi_v(\alpha)|+\frac{1}{2}\cdot\frac{|e(\alpha)||1-e(m\alpha)|}{|1-e(\alpha)|}\leqq$$

$$\leqq\frac{1}{2}\cdot\frac{2}{4\|P\alpha\|}\cdot 5\sqrt{P}+\frac{1}{2}\cdot 5\sqrt{v}+\frac{1}{2}\cdot\frac{2}{4\|\alpha\|}\leqq\frac{5\sqrt{P}}{4\|P\alpha\|}+\frac{5}{2}\cdot\sqrt{P}+\frac{1}{4\|\alpha\|}$$

for $\|P\alpha\|\neq 0$.

By Cauchy's inequality,

$$(a+b+c)^2\leqq 3(a^2+b^2+c^2)$$

for any real numbers a, b, c. Furthermore, P is a prime number thus

$$P\nmid q \tag{42}$$

implies that $\|kP/q\| \neq 0$ for $k=1, 2, \ldots, q-1$ and the numbers $\|P/q\|, \|2P/q\|, \ldots, \|(q-1)P/q\|$ are equal to the numbers $\|1/q\|, \|2/q\|, \ldots, \|(q-1)/q\|$. Thus with respect to (36), we obtain from (38) and (41) that for all q satisfying (42),

$$qU_q^*(m) = \sum_{k=1}^{q-1} |S(k/q, m)|^2 \leqq 3 \sum_{k=1}^{q-1} \left(\frac{25P}{16\|kP/q\|^2} + \frac{25}{4} P + \frac{1}{16\|k/q\|^2} \right) =$$

$$= 3 \sum_{k=1}^{q-1} \left(\frac{25}{16} P + \frac{1}{16} \right) \frac{1}{\|k/q\|^2} + 3 \sum_{k=1}^{q-1} \frac{25}{4} P \leqq 3 \cdot \frac{26}{16} P \cdot 2 \sum_{k=1}^{q/2} \frac{1}{\|k/q\|^2} + 3 \cdot \frac{25}{4} \cdot qP <$$

$$< 10P \sum_{k=1}^{q/2} \frac{q^2}{k^2} + 19qP < 10Pq^2 \sum_{k=1}^{+\infty} \frac{1}{k^2} + 19qP <$$

$$< 20q^2P + 19q^2P = 39q^2P \leqq 624q^2R.$$

Hence (by (22) and (35))

$$\sum_{\substack{2 \leqq q \leqq Q \\ P \nmid q}} q^{-1} \sum_{m=1}^{N^*} U_q^*(m) + \sum_{\substack{2 \leqq q \leqq Q \\ P \nmid q}} qU_q^*(N^*) < \tag{43}$$

$$< \sum_{2 \leqq q \leqq Q} q^{-2} \sum_{m=1}^{N^*} 624q^2R + \sum_{2 \leqq q \leqq Q} 624q^2R \leqq$$

$$\leqq \sum_{2 \leqq q \leqq Q} 624N^*R + \sum_{2 \leqq q \leqq Q} 624Q^2R \leqq 624N^*QR + 624Q^2 \cdot QR \leqq$$

$$\leqq 624 \cdot 16B_0 \cdot Q \cdot Q + 624Q^2 \cdot 16B_0 < 20000Q^2B_0.$$

If $Q \leqq \sqrt{16B_0}$, then $2 \leqq q \leqq Q$ implies (by (35) and (36)) that

$$q \leqq Q = R < P,$$

hence $P \nmid q$. Thus for $Q \leqq \sqrt{16B_0}$, the sum estimated in (43) is identical to the left hand side of (24), consequently, (24) holds in this case.

If

$$Q > \sqrt{16B_0}, \tag{44}$$

then to complete the proof of (24), we have to show that

$$\sum_{\substack{2 \leqq q \leqq Q \\ P/q}} q^{-1} \sum_{m=1}^{N^*} U_q^*(m) + \sum_{\substack{2 \leqq q \leqq Q \\ P/q}} qU_q^*(N^*) \ll Q^2B_0. \tag{45}$$

This can be shown applying the trivial estimates (30) and (31). In view of (22), (30), (31), (34), (35), (36) and (44):

$$\sum_{\substack{2\leqq q\leqq Q\\ P/q}} q^{-1}\sum_{m=1}^{N^*} U_q^*(m)+\sum_{\substack{2\leqq q\leqq Q\\ P/q}} qU_q^*(N^*)\leqq$$

$$\leqq \sum_{\substack{2\leqq q\leqq Q\\ P/q}}\Big\{q^{-1}\sum_{1\leqq m\leqq q} U_q^*(m)+q^{-1}\sum_{q<m\leqq N^*} U_q^*(m)\Big\}+\sum_{\substack{2\leqq q\leqq Q\\ P/q}} qU_q^*(N^*)\leqq$$

$$\leqq \sum_{\substack{2\leqq q\leqq Q\\ P/q}}\Big\{q^{-1}\sum_{1\leqq m\leqq q} 3m+q^{-1}\sum_{q<m\leqq N^*} 4m^2q^{-1}\Big\}+\sum_{\substack{2\leqq q\leqq Q\\ P/q}} q\{3N^*+4(N^*)^2q^{-1}\}\leqq$$

$$\leqq \sum_{\substack{2\leqq q\leqq Q\\ P/q}}\Big\{q^{-1}\sum_{m=1}^{q} 3q+q^{-1}\sum_{m=1}^{N^*} 4(N^*)^2q^{-1}\Big\}+\sum_{\substack{2\leqq q\leqq Q\\ P/q}} Q\cdot 3N^*+\sum_{\substack{2\leqq q\leqq Q\\ P/q}} 4(N^*)^2\leqq$$

$$\leqq \sum_{\substack{2\leqq q\leqq Q\\ P/q}} 3q+4(N^*)^3\sum_{\substack{2\leqq q\leqq Q\\ P/q}} q^{-2}+3QN^*\cdot\frac{Q}{P}+4(N^*)^2\cdot\frac{Q}{P}\leqq$$

$$< \sum_{\substack{2\leqq q\leqq Q\\ P/q}} 3Q+4(N^*)^3\sum_{i=1}^{+\infty}(iP)^{-2}+\frac{3Q^2N^*}{P}+\frac{4(N^*)^2Q}{P}\leqq$$

$$\leqq 3Q\cdot\frac{Q}{P}+\frac{4(N^*)^3}{P^2}\cdot 2+\frac{3Q^2N^*}{P}+\frac{4(N^*)^2Q}{P}=$$

$$=(3Q^2+3^2QN^*)\frac{1}{P}+4(N^*)^2Q\cdot\frac{1}{P}+8(N^*)^3\cdot\frac{1}{P^2}<$$

$$<(3Q^2N^*+3Q^2N^*)+4(N^*)^2Q\cdot\frac{1}{8R}+8(N^*)^3\cdot\frac{1}{(8R)^2}<$$

$$<6Q^2\cdot 16B_0+4(16B_0)^2Q\cdot\frac{Q}{8\cdot 16B_0}+8(16B_0)^3\cdot\frac{Q^2}{(8\cdot 16B_0)^2}=106Q^2B_0,$$

thus (45) holds.

Summarizing: in both Cases 1 and 2, there exist an integer N^* and a sequence $\mathscr{A}^*$ satisfying (22), (23) and (24).

If such integer N^* and sequence $\mathscr{A}^*$ are given then the proof of Theorem 4 can be completed, i.e. a sequence $\mathscr{A}$ satisfying (20) and (21) can be constructed easily.

Namely, let us define the sequence $\mathscr{A}$ in the following way: $a\in\mathscr{A}$ should hold if and only if $a-(N-N^*)\in\mathscr{A}^*$. (In other words, to obtain $\mathscr{A}$, let us add $N-N^*$ to each element of $\mathscr{A}^*$).

By (19) and (22), $N^*\leqq N$. Consequently, $\mathscr{A}\subset\{1, 2, \dots, N\}$. Obviously, $B_N(\mathscr{A})=B_{N^*}(\mathscr{A}^*)$, thus (23) implies (20). Finally,

$$U_q(m)=\begin{cases}0 & \text{for} \quad m\leqq N-N^*,\\ U_q^*(m-(N-N^*)) & \text{for} \quad N-N^*<m\leqq N.\end{cases}$$

Thus the left hand side of (21) is equal to the left hand side of (24), consequently, also (21) holds.

(We remark that the application of the Rudin—Shapiro construction can be avoided. Namely, the existence of a sequence $\mathscr{A}$ satisfying (21) and (22) can be shown also by a combinatorial-statistical argument similar to the one which will be used in the next section.)

7. Throughout this section, the number of the elements of a set Γ will be denoted by $|\Gamma|$.

So far we have studied the estimates for the *average* of the discrepancies $|E_{q,h}(m)|$. In Sections 7, 8 and 9, we shall discuss the estimates for the *maximum* of these discrepancies, i.e. Corollaries 2 and 2′. In this section, we shall investigate the case when there exist "large" irregularities, i.e. QB/N is "large". It will be shown that in this case, Corollaries 2 and 2′ are best possible except at most a logarithmic factor on the right hand side (provided that $Q \leqq \sqrt{N}$).

THEOREM 5. *Let Q, B_0, N be positive integers, satisfying*

$$2 \leqq Q \leqq N/2, \tag{46}$$

$$B_0 < \frac{N}{240}, \tag{47}$$

$$\frac{QB_0}{N} > \frac{1}{4}. \tag{48}$$

Then there exists a sequence $\mathscr{A} \subset \{1, 2, \ldots, N\}$ such that

$$B(\mathscr{A}) = B_0 \tag{49}$$

holds and

$$|E_{q,h}(m)| = \Big| \sum_{\substack{1 \leqq a \leqq m \\ a \equiv h (\mathrm{mod}\, q) \\ a \in \mathscr{A}}} 1 - q^{-1} \sum_{\substack{1 \leqq a \leqq m \\ a \in \mathscr{A}}} 1 \Big| < 112 \sqrt{QB/N}\, \sqrt{\log Q} \tag{50}$$

for $2 \leqq q \leqq Q$ and any integers h, m.

Choosing $Q = [\sqrt{N}]$, we obtain

COROLLARY 3. *Let B_0, N be positive integers satisfying $N \geqq 4$ and*

$$\frac{\sqrt{N}}{4} < B_0 < \frac{N}{240}.$$

Then there exists a sequence $\mathscr{A} \subset \{1, 2, \ldots, N\}$ such that (49) *holds and*

$$|E_{q,h}(m)| = \Big| \sum_{\substack{1 \leqq a \leqq m \\ a \equiv h (\mathrm{mod}\, q) \\ a \in \mathscr{A}}} 1 - q^{-1} \sum_{\substack{1 \leqq a \leqq m \\ a \in \mathscr{A}}} 1 \Big| < 80 \frac{\sqrt{B}}{\sqrt[4]{N}} \sqrt{\log N}$$

for $2 \leqq q \leqq \sqrt{N}$ and any integers h, m.

PROOF OF THEOREM 5. Let

$$t = \left[\frac{48QB_0}{N}\right]$$

and let P denote the smallest prime number greater than Q. By (47) and (48),

$$t = \left[12\frac{4QB_0}{N}\right] \geqq [12 \cdot 1] = 12 \tag{51}$$

and

$$t < \frac{48QB_0}{N} = 48Q\frac{B_0}{N} < 48Q \cdot \frac{1}{240} = \frac{Q}{5}. \tag{52}$$

Furthermore, by Bertrand's postulate,

$$Q < P < 2Q. \tag{53}$$

Let Γ denote the set of those sequences $\mathscr{A}^*(\subset \{1, 2, \ldots, N\})$ for which the following two conditions hold:

(i)

$$A^*(P)\left(= \sum_{\substack{a \in \mathscr{A}^* \\ 1 \leqq a \leqq P}} 1\right) = t. \tag{54}$$

(ii) $\mathscr{A}^*$ is periodical in $[1, N]$ with period length P, i.e. if $a = kP + l$ where k, l are integers and $1 \leqq l \leqq P$, then $a \in \mathscr{A}^*$ should hold if and only if $1 \leqq a \leqq N$ and $l \in \mathscr{A}^*$.

Furthermore, let Γ_1 denote the set of those sequences $\mathscr{A}^*$ for which $\mathscr{A}^* \in \Gamma$ and

$$B(\mathscr{A}^*) < B_0 \tag{55}$$

hold.

Finally, let Γ_2 denote the set of those sequences $\mathscr{A}^*$ for which $\mathscr{A}^* \in \Gamma$ holds and there exist positive integers q, h, m such that

$$1 \leqq h \leqq q, \tag{56}$$

$$1 \leqq q \leqq Q, \tag{57}$$

$$1 \leqq m \leqq N \tag{58}$$

and

$$\left|A_{q,h}(m) - \frac{1}{q} \cdot \frac{t}{P} m\right| > 8\sqrt{t \log Q}. \tag{59}$$

We are going to show at first that if there exists a sequence $\mathscr{A}^*$ satisfying

$$\mathscr{A}^* \in \Gamma, \quad \mathscr{A}^* \notin \Gamma_1, \quad \mathscr{A}^* \notin \Gamma_2 \tag{60}$$

then also a sequence $\mathscr{A}$ satisfying the requirements of Theorem 5 must exist.

Since $\mathscr{A}^* \notin \Gamma_1$ (by (60)),

$$B(\mathscr{A}^*) \geqq B_0.$$

Thus there exists a positive integer N^* satisfying

$$1 \leqq N^* \leqq N$$

and

$$B_{N^*}(\mathscr{A}^*) = \sum_{\substack{1 \leqq n \leqq N^* \\ n-1 \notin \mathscr{A}^*,\ n \in \mathscr{A}^*}} 1 = B_0.$$

Let

$$\mathscr{A} = \mathscr{A}^* \cap \{1, 2, \ldots, N^*\}.$$

Then $\mathscr{A} \subset \{1, 2, \ldots, N\}$ and (49) hold trivially by the definition of $\mathscr{A}^*$, N^* and $\mathscr{A}$. It remains to show that the sequence $\mathscr{A}$ satisfies also (50) for $2 \leqq q \leqq Q$ and any integers h, m.

Obviously,

$$E_{q,h}(m) = \begin{cases} 0 & \text{for} \quad m < 1, \\ E^*_{q,h}(m) & \text{for} \quad 1 \leqq m \leqq N^*, \\ E^*_{q,h}(N^*) & \text{for} \quad N^* < m \end{cases}$$

where $E^*_{q,h}(m)$ denotes the value $E_{q,h}(m)$ corresponding to the sequence $\mathscr{A}^*$:

$$E^*_{q,h}(m) = A^*_{q,h}(m) - q^{-1}A^*(m).$$

Thus it suffices to show that

$$|E^*_{q,h}(m)| < 112\sqrt{QB/N}\sqrt{\log Q} \quad \text{whenever} \quad 1 \leqq m \leqq N^*. \tag{61}$$

Let us define the integer h_0 by

$$h \equiv h_0 \pmod q, \quad 1 \leqq h_0 \leqq q.$$

Obviously,

$$E^*_{q,h}(m) = E^*_{q,h_0}(m). \tag{62}$$

By (60), $\mathscr{A}^* \notin \Gamma_2$ thus

$$\left| A^*_{q,h_0}(m) - \frac{1}{q} \cdot \frac{t}{P} m \right| \leqq 8\sqrt{t \log Q}$$

and

$$\left| A^*(m) - \frac{t}{P} m \right| = \left| A^*_{1,1}(m) - \frac{1}{1} \cdot \frac{t}{P} m \right| \leqq 8\sqrt{t \log Q}.$$

Hence

$$|E^*_{q,h_0}(m)| = |A^*_{q,h_0}(m) - q^{-1}A^*(m)| = \tag{63}$$

$$= \left| \left(A^*_{q,h_0}(m) - \frac{1}{q} \cdot \frac{t}{P} m \right) + \left(\frac{1}{q} \cdot \frac{t}{P} m - \frac{A^*(m)}{q} \right| \leqq \right.$$

$$\leqq \left| A^*_{q,h_0}(m) - \frac{1}{q} \cdot \frac{t}{P} m \right| + \frac{1}{q} \left| A^*(m) - \frac{t}{P} m \right| \leqq$$

$$\leqq 8\sqrt{t \log Q} + \frac{1}{q} \cdot 8\sqrt{t \log Q} \leqq 16\sqrt{t \log Q} =$$

$$= 16\sqrt{\left[\frac{48QB_0}{N}\right] \log Q} < 16\sqrt{\frac{49QB_0}{N} \log Q} = 112\sqrt{QB/N}\sqrt{\log Q}.$$

(61) follows from (62) and (63) and thus the sequence $\mathscr{A}$ satisfies all the requirements in Theorem 5.

Thus to complete the proof of Theorem 5, it remains to show that there exists a sequence $\mathscr{A}^*$ satisfying (60). Obviously, it suffices to show that

$$(|\Gamma - \Gamma_1 - \Gamma_2| \geqq) \quad |\Gamma| - |\Gamma_1| - |\Gamma_2| > 0. \tag{64}$$

In order to prove (64), we have to estimate $|\Gamma|$, $|\Gamma_1|$ and $|\Gamma_2|$.

Obviously,

$$|\Gamma| = \binom{P}{t} \tag{65}$$

(since

$$P < N \tag{66}$$

by (46) and (53)).

To estimate $|\Gamma_1|$, let us investigate $B(\mathscr{A}^*)$ for some set $\mathscr{A}^* \in \Gamma$:

$$B(\mathscr{A}^*) = B_N(\mathscr{A}^*) = \sum_{k=0}^{[N/P]} \left(B_{(k+1)P}(\mathscr{A}^*) - B_{kP}(\mathscr{A}^*)\right) \geqq \tag{67}$$

$$\geqq \sum_{k=0}^{[N/P]-1} \left(B_P(\mathscr{A}^*) - B_0(\mathscr{A}^*) - 1\right) = \left[\frac{N}{P}\right]\left(B_P(\mathscr{A}^*) - 1\right) \geqq \frac{N}{2P}\left(B(\mathscr{A}^*) - 1\right)$$

by (66) and since $\mathscr{A}^*$ has property (ii) (by $\mathscr{A}^* \in \Gamma$). If also $\mathscr{A}^* \in \Gamma_1$ holds then we get from (55) and (67) that

$$B_p(\mathscr{A}^*) < \frac{2PB_0}{N} + 1.$$

In view of (48) and (53):

$$B_p(\mathscr{A}^*) < \frac{4QB_0}{N} + \frac{4QB_0}{N} = \frac{8QB_0}{N}.$$

In other words, if we put

$$s = B_P(\mathscr{A}^*) = \sum_{\substack{1 \leqq n \leqq P \\ n-1 \notin \mathscr{A}^*,\ n \in \mathscr{A}^*}} 1$$

and

$$S = \left[\frac{8QB_0}{N}\right]$$

then

$$1 \leqq s \leqq S. \tag{68}$$

By (47) and (53),

$$S \leqq \frac{8QB_0}{N} = 8 \cdot \frac{B_0}{N} \cdot Q < 8 \cdot \frac{1}{240} \cdot Q = \frac{Q}{30} < \frac{P}{30}. \tag{69}$$

Since $\mathscr{A}^* \in \Gamma$ holds, $\mathscr{A}^*$ has property (ii). Thus $\mathscr{A}^*$ is uniquely determined by the set of the integers n, k satisfying

$$1 \leqq n \leqq P, \quad n-1 \notin \mathscr{A}^*, \quad n \in \mathscr{A}^* \tag{70}$$

and

$$1 \leqq k \leqq P, \quad k-1 \in \mathscr{A}^*, \quad k \notin \mathscr{A}^*. \tag{71}$$

The number of the integers n satisfying (70) was denoted by s. For s fixed, the set of these integers n can be chosen in at most $\binom{P}{s}$ ways. If s is fixed then obviously, the number of the integers k satisfying (71) is either $s-1$ or s. Thus the set of these integers k can be chosen in at most

$$\binom{P}{s-1}+\binom{P}{s} \leqq 2\binom{P}{s}$$

ways (with respect to (68) and (69), $s<P/2$). Summarizing, we obtain for fixed s that $\mathscr{A}^*$ $(\in \Gamma_1)$ can be chosen in at most

$$\binom{P}{s}\cdot 2\binom{P}{s} = 2\binom{P}{s}^2$$

ways. Thus by (68) and (69),

$$|\Gamma_1| \leqq \sum_{1 \leqq s \leqq S} 2\binom{P}{s}^2 \leqq \sum_{1 \leqq s \leqq S} 2\binom{P}{S}^2 = 2S\binom{P}{S}^2 <$$

$$< 2S \sum_{k=0}^{2S} \binom{P}{k}\binom{P}{2S-k} = 2S\binom{2P}{2S} < \frac{1}{15} P\binom{2P}{2S} = \frac{1}{15}\binom{P}{1}\binom{2P}{2S} <$$

$$< \frac{1}{15} \sum_{k=0}^{2S+1} \binom{P}{k}\binom{2P}{2S+1-k} = \frac{1}{15}\binom{3P}{2S+1} = \frac{1}{15}\cdot\frac{3P(3P-1)(3P-2)\ldots(3P-2S)}{(2S+1)!}.$$

It follows easily from (69) that

$$3P-i < 4(P-i) \quad \text{for} \quad 0 \leqq i \leqq 2S.$$

Thus

$$|\Gamma_1| < \frac{1}{15}\cdot\frac{4P\cdot 4(P-1)\cdot 4(P-2)\cdot\ldots\cdot 4(P-2S)}{(2S+1)!} = \frac{4^{2S+1}}{15}\binom{P}{2S+1}.$$

By the definitions of S and t,

$$t = \left[\frac{48QB_0}{N}\right] = \left[6\cdot\frac{8QB_0}{N}\right] \geqq 6\left[\frac{8QB_0}{N}\right] = 6S. \tag{72}$$

Hence

$$|\Gamma_1| < \frac{4^{2S+1}}{15}\binom{P}{2S+1} = \frac{4^{2S+1}}{15}\binom{P}{t}\prod_{i=1}^{t-2S-1}\frac{2S+1+i}{P-t+i} = \tag{73}$$

$$= \frac{4^{2S+1}}{15}\binom{P}{t}\prod_{j=1}^{t-2S-1}\frac{t+1-j}{P-2S-j}.$$

We are going to show that for $j=1, 2, \ldots, t-2S-1$,

$$\frac{t+1-j}{P-2S-j} < \frac{1}{4}. \tag{74}$$

It suffices to prove that

$$t+1 < \frac{P}{4}-\frac{S}{2}+\frac{3j}{4},$$

$$t+1 < \frac{P}{4}-\frac{S}{2}+\frac{3}{4},$$

$$4t+2S+1 < P,$$

$$4t+3S < P.$$

In view of (52), (53) and (72), this inequality holds which proves (74).

Thus we get from (73) that

$$|\Gamma_1| < \frac{4^{2S+1}}{15}\binom{P}{t}\prod_{i=1}^{t-2S-1}\frac{1}{4} = \frac{4^{2S+1}}{15}\binom{P}{t}\frac{1}{4^{t-2S-1}} = \frac{4^{4S-t+2}}{15}\binom{P}{t}.$$

With respect to (72):

$$|\Gamma_1| < \frac{4^{4S-6S+2}}{15}\binom{P}{t} = \frac{4^{2(s-S)}}{15}\binom{P}{t} \leqq \frac{1}{15}\binom{P}{t}. \tag{75}$$

Finally, we have to estimate $|\Gamma_2|$.

Let us assume that for some sequence $\mathscr{A}^* \in \Gamma_2$, there exist integers h, q, m satisfying (56), (57), (58) and (59). Let us define the non-negative integer s by

$$sPq < m \leqq (s+1)Pq,$$

and let $m_0 = m - sPq$. Obviously,

$$1 \leqq m_0 \leqq Pq. \tag{76}$$

Since P is a prime number greater than Q, $1 \leqq q \leqq Q$ implies that

$$(q, P) = 1. \tag{77}$$

(77) and properties (i), (ii) imply obviously that

$$A^*_{q,h}(u+vPq) - A^*_{q,h}(u) = vA^*(P) = vt$$

whenever u, v are non-negative integers satisfying

$$1 \leqq u \leqq u+vPq \leqq N.$$

Thus

$$\left|A^*_{q,h}(m) - \frac{1}{q}\cdot\frac{t}{P}m\right| = \left|A^*_{q,h}(m_0) + \left(A^*_{q,h}(m) - A^*_{q,h}(m_0)\right) - \frac{1}{q}\cdot\frac{t}{P}m\right| = \tag{78}$$

$$= \left|A^*_{q,h}(m_0) + st - \frac{1}{q}\cdot\frac{t}{P}m\right| = \left|A^*_{q,h}(m_0) - \frac{1}{q}\cdot\frac{t}{P}(m-sPq)\right| = \left|A^*_{q,h}(m_0) - \frac{1}{q}\cdot\frac{t}{P}m_0\right|.$$

We get from (59) and (78) that

$$\left|A^*_{q,h}(m_0) - \frac{1}{q}\cdot\frac{t}{P}m_0\right| > 8\sqrt{t\log Q}. \tag{79}$$

Thus the integers h, q, m_0 satisfy (56), (57), (76) and (79). h, q, m_0 satisfying (56), (57) and (76) can be chosen in at most q, Q and Pq ways, respectively.

Let us put

$$\sum_{\substack{1 \leqq n \leqq m_0 \\ n \equiv h \pmod q}} 1 = r \tag{80}$$

and

$$A^*_{q,h}(m_0) = u; \tag{81}$$

then obviously,

$$\left| r - \frac{m_0}{q} \right| < 1 \tag{82}$$

and (79) can be rewritten in form

$$\left| u - \frac{1}{q} \cdot \frac{t}{P} m_0 \right| > 8\sqrt{t \log Q}. \tag{83}$$

By (82) and (83) and with respect to (46), (51), (52) and (53),

$$\left| u - \frac{rt}{P} \right| \geqq \left| u - \frac{1}{q} \cdot \frac{t}{P} m_0 \right| - \left| \frac{1}{q} \cdot \frac{t}{P} m_0 - \frac{rt}{P} \right| = \tag{84}$$

$$= \left| u - \frac{1}{q} \cdot \frac{t}{P} m_0 \right| - \frac{t}{P} \left| \frac{m_0}{q} - r \right| > 8\sqrt{t \log Q} - 1 > 8\sqrt{t \log Q} - \sqrt{12 \log 2} \geqq$$

$$\geqq 8\sqrt{t \log Q} - \sqrt{t \log Q} = 7\sqrt{t \log Q}.$$

Since $\mathscr{A}^* \in \Gamma$ and (77) holds,

$$A^*_{q,h}(Pq) = \sum_{\substack{n \in \mathscr{A}^* \\ 1 \leqq n \leqq Pq \\ n \equiv h \pmod q}} 1 = t, \tag{85}$$

furthermore,

$$\mathscr{A}^* \cap \{h, h+q, h+2q, \ldots, h+(P-1)q\} \tag{86}$$

determines $\mathscr{A}^* \cap \{1, 2, \ldots, P\}$ and thus also $\mathscr{A}^*$ uniquely. Fixing r and u in (80) and (81), respectively, the elements of

$$\mathscr{A}^* \cap \{h, h+q, h+2q, \ldots, h+(P-1)q\} \cap \{1, 2, \ldots, m_0\} \tag{87}$$

can be chosen in at most $\binom{r}{u}$ ways. By (85), the number of the remaining elements of the sequence (86) is $t-u$, and these elements can be chosen from the $P-r$ integers in

$$\{h, h+q, h+2q, \ldots, h+(P-1)q\} \cap \{m_0+1, m_0+2, \ldots, qP\}.$$

Thus fixing the sequence (87), the remaining elements of the sequence (86) can be chosen in $\binom{P-r}{t-u}$ ways. Consequently, for fixed h, q, m_0, r, u the sequence (86) and thus also the sequence $\mathscr{A}^*$ can be chosen in at most $\binom{r}{u}\binom{P-r}{t-u}$ ways.

Summarizing the estimates above, we obtain with respect to (53), (57) and (84) that

$$|\Gamma_2| \leqq q \cdot Q \cdot Pq \sum_{r=0}^{P} \sum_{\left|u-\frac{rt}{P}\right|>7\sqrt{t\log Q}} \binom{r}{u}\binom{P-r}{t-u} < \tag{88}$$

$$< 2Q^4 \sum_{r=0}^{P} \sum_{\left|u-\frac{rt}{P}\right|>7\sqrt{t\log Q}} \binom{r}{u}\binom{P-r}{t-u} = 2Q^4\left(\sum_{r=0}^{P} \sum_{u-\frac{rt}{P}>-7\sqrt{t\log Q}} \binom{r}{u}\binom{P-r}{t-u} + \right.$$

$$\left. + \sum_{r=0}^{P} \sum_{u-\frac{rt}{P}>7\sqrt{t\log Q}} \binom{r}{u}\binom{P-r}{t-u}\right) = 2Q^4(\Sigma_1+\Sigma_2).$$

We are going to show that

$$\Sigma_1 = \Sigma_2. \tag{89}$$

In fact, let us put $r^*=P-r$, $u^*=t-u$, i.e. $r=P-r^*$, $u=t-u^*$. Then the condition $u-\frac{rt}{P}>7\sqrt{t\log Q}$ in Σ_2 can be rewritten in form

$$-7\sqrt{t\log Q} > -u+\frac{rt}{P} = -(t-u^*)+\frac{(P-r^*)t}{P} = u^*-\frac{r^*t}{P}$$

and if r runs over $0, 1, 2, \ldots, P$, then also $r^*=P-r$ runs over these numbers. Thus

$$\Sigma_2 = \sum_{r^*=0}^{P} \sum_{u^*-\frac{r^*t}{P}<-7\sqrt{t\log Q}} \binom{P-r^*}{t-u^*}\binom{r^*}{u^*}.$$

Replacing r^* and u^* by r and u, respectively, we obtain (89).

Thus from (88):

$$|\Gamma_2| < 2Q^4(\Sigma_1+\Sigma_2) = 4Q^4\Sigma_1 = \tag{90}$$

$$= 4Q^4 \sum_{r=0}^{P} \sum_{0\leqq u<\frac{t}{P}r-7\sqrt{t\log Q}} \binom{r}{u}\binom{P-r}{t-u} = 4Q^4 \sum_{r=0}^{P} \sum_{M(r)\leqq u<\frac{t}{P}r-7\sqrt{t\log Q}} \binom{r}{u}\binom{P-r}{t-u}$$

where $M(r)=\max\{0, t-P+r\}$.

To estimate the inner sum, let us put

$$F(u) = \binom{r}{u}\binom{P-r}{t-u},$$

adn for $u\leqq P\frac{t}{r}$, let

$$\frac{t}{P}r-u = d \quad (\geqq 0).$$

Then for $M(r)+1 \leqq u \leqq \frac{r}{P}$,

$$\frac{F(u-1)}{F(u)} = \tag{91}$$

$$= \frac{r!(P-r)!}{(u-1)!(r-u+1)!(t-u+1)!(P-r-t+u-1)!} \cdot \frac{u!(r-u)!(t-u)!(P-r-t+u)!}{r!(P-r)!} =$$

$$= \frac{u(P-r-t+u)}{(r-u+1)(t-u+1)} < \frac{u(P-r-t+u)}{(r-u)(t-u+1)} = 1 - \frac{(rt-uP)+(r-u)}{(r-u)(t-u+1)} =$$

$$= 1 - \frac{Pd+(r-u)}{(r-u)(t-u+1)} < 1 - \frac{Pd}{rt}$$

$\left(\text{since } r-u>0 \text{ and } t-u+1>0 \text{ follow from } u \leqq \frac{t}{P} r\right)$.

Let us put $u_0 = \left[\frac{t}{P} r\right]$ and $u_1 = \left[\frac{t}{P} r - 7\sqrt{t \log Q}\right]$. If $u_1 \geqq M(r)$ (i.e. $\frac{t}{P} r - 7\sqrt{t \log Q} \geqq M(r)$) then we get from (91) that

$$F(u_1) = F(u_0) \prod_{u=u_1+1}^{u_0} \frac{F(u-1)}{F(u)} < F(u_0) \prod_{u=u_1+1}^{u_0} \left(1 - \frac{P\left(\frac{t}{P} r - u\right)}{rt}\right).$$

Writing $u = u_0 - j$ (and $e^z = \exp\{z\}$):

$$F(u_1) = F(u_0) \prod_{j=0}^{u_0-u_1-1} \left(1 - \frac{P\left(\frac{t}{P} r - u_0\right) + Pj}{rt}\right) \leqq \tag{92}$$

$$\leqq F(u_0) \prod_{j=1}^{u_0-u_1-1} \left(1 - \frac{Pj}{rt}\right) < F(u_0) \exp\left\{-\sum_{j=1}^{u_0-u_1-1} \frac{Pj}{rt}\right\} =$$

$$= F(u_0) \exp\left\{-\frac{P}{rt} \cdot \frac{(u_0-u_1-1)(u_0-u_1)}{2}\right\} < F(u_0) \exp\left\{-\frac{P}{rt} \cdot \frac{(u_0-u_1-1)^2}{2}\right\}$$

since $1-x<e^{-x}$ for $x>0$.

By (46) and (51):

$$u_0 - u_1 - 1 = \left[\frac{t}{P} r\right] - \left[\frac{t}{P} r - 7\sqrt{t \log Q}\right] - 1 \geqq$$

$$\geqq 7\sqrt{t \log Q} - 1 - 1 = 7\sqrt{t \log Q} - 2 > 7\sqrt{t \log Q} - \sqrt{12 \log 2} \geqq$$

$$\geqq 7\sqrt{t \log Q} - \sqrt{t \log Q} - 6\sqrt{t \log Q}.$$

Thus from (92):

$$F(u_1) < F(u_0)\exp\left\{-\frac{P}{rt}\cdot\frac{36t\log Q}{2}\right\} \leqq \tag{93}$$

$$\leqq F(u_0)\exp\left\{-\frac{P}{P}18\log Q\right\} = F(u_0)Q^{-18}.$$

By (91), $F(u)$ is increasing for $M(r)\leqq u\leqq\frac{t}{P}r$. Thus in view of (46), (52), (53) and (93), we get from (90) that

$$|\Gamma_2| < 4Q^4\sum_{r=0}^{P}\sum_{M(r)\leqq u\leqq\frac{t}{P}r-7\sqrt{t\log Q}}\binom{r}{u}\binom{P-r}{t-u} \leqq \tag{94}$$

$$\leqq 4Q^4\sum_{r=0}^{P}\sum_{M(r)\leqq u\leqq\frac{t}{P}r-7\sqrt{t\log Q}}\binom{r}{u_1}\binom{P-r}{t-u_1} =$$

$$= 4Q^4\sum_{r=0}^{P}\binom{r}{u_1}\binom{P-r}{t-u_1}\sum_{M(r)\leqq u\leqq\frac{t}{P}r-7\sqrt{t\log Q}}1 <$$

$$< 4Q^4\sum_{r=0}^{P}\binom{r}{u_0}\binom{P-r}{t-u_0}Q^{-18}\sum_{0\leqq u\leqq P}1 = 4Q^{-14}(P+1)\sum_{r=0}^{P}\binom{r}{u_0}\binom{P-r}{t-u_0} \leqq$$

$$\leqq 4Q^{-14}(P+1)\sum_{r=0}^{P}\sum_{0\leqq u\leqq t}\binom{r}{u}\binom{P-r}{t-u} = 8Q^{-14}(P+1)\sum_{r=0}^{P}\binom{P}{t} = 8Q^{-14}(P+1)^2\binom{P}{t} <$$

$$< 8Q^{-14}(2P)^2\binom{P}{t} < 8Q^{-14}(4Q)^2\binom{P}{t} = 2^7Q^{-12}\binom{P}{t} \leqq 2^7 2^{-12}\binom{P}{t} = \frac{1}{32}\binom{P}{t}.$$

(64) follows from (65), (75) and (94) which completes the proof of Theorem 5.

8. One may raise the question that what can be asserted on

$$\max_{q,h,m}|E_{q,h}(m)|$$

(without the restriction $q\leqq\sqrt{N}$)?

By Corollary 2′

$$\max_{q,h,m}|E_{q,h}(m)| \gg \sqrt{B}/\sqrt[4]{N}. \tag{95}$$

On the other hand, the construction given in Section 7 yields

THEOREM 6. *Let B_0, N be positive integers satisfying $N>60^2$ and*

$$\frac{\sqrt{N}}{4} < B_0 < \frac{N}{240}. \tag{96}$$

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Then there exists a sequence $\mathscr{A} \subset \{1, 2, \ldots, N\}$ *such that* $B(\mathscr{A}) = B_0$ *holds and*

$$|E_{q,h}(m)| \ll B^{1/3} \log^{2/3} N \tag{97}$$

for any positive integers h, q, m.

PROOF. Let $\mathscr{A}$ be the sequence defined in the proof of Theorem 5 for $Q = \left[\frac{N}{B_0^{1/3} \log^{2/3} N}\right]$. Then for $q \leqq Q$, (97) follows from (50). For $Q < q$, $1 \leqq m \leqq N$:

$$|E_{q,h}(m)| = \Big| \sum_{\substack{1 \leqq a \leqq m \\ a \equiv h \pmod q \\ a \in \mathscr{A}}} 1 - q^{-1} \sum_{\substack{1 \leqq a \leqq m \\ a \in \mathscr{A}}} 1 \Big| \leqq$$

$$\leqq \sum_{\substack{1 \leqq a \leqq m \\ a \equiv h \pmod q \\ a \in \mathscr{A}}} 1 + q^{-1} \sum_{\substack{1 \leqq a \leqq m \\ a \in \mathscr{A}}} 1 \leqq \sum_{\substack{1 \leqq a \leqq m \\ a \equiv h \pmod q}} 1 + q^{-1} \sum_{1 \leqq a \leqq m} 1 \leqq$$

$$\leqq \frac{m}{q} + 1 + \frac{m}{q} = \frac{2m}{q} + 1 < \frac{2N}{Q} + 1 < \frac{3N}{Q} = \frac{3N}{[N/B_0^{1/3} \log^{2/3} N]} \ll B_0^{1/3} \log^{2/3} N$$

while if $m > N$ then $E_{q,h}(m) = E_{q,h}(N)$, thus (97) holds also in this case which completes the proof of Theorem 6.

(We remark that (97) seems to be far from the best possible.)

9. By (95),

$$\lim_{N \to +\infty} \frac{B_N(\mathscr{A})}{\sqrt{N}} = +\infty \tag{98}$$

implies that

$$\max_{q,h,m} |E_{q,h}(m)| \to +\infty. \tag{99}$$

One may raise the question that what happens for $B_N(\mathscr{A}) = o(\sqrt{N})$? The assumption (98) can be improved, probably. However, it is worth to note that $B_N(\mathscr{A}) \to +\infty$ is not sufficient to obtain (99) (neither for infinite sequences).

For $x > e$, let us define the positive integer $L(x)$ by

$$\log_{L(x)+1} x < 1 \leqq \log_{L(x)} x$$

(where $\log_1 x = \log x$ and $\log_{k+1} x = \log(\log_k x)$ for $k = 1, 2, \ldots$).

THEOREM 7. *There exists an infinite sequence* $\mathscr{A}$ *for which*

$$B_N(\mathscr{A}) \gg L(N) \tag{100}$$

(for $N > N_0$*) and*

$$|E_{q,h}(m)| \leqq 3$$

for any positive integers q, h, m.

PROOF. Let us define an infinite sequence $\{n_1, n_2, \ldots\}$ in the following way: let $n_1=10$ and $n_{k+1}=n_k+[1, 2, \ldots, n_k]$ for $k=1, 2, \ldots$. Furthermore, let

$$\mathscr{A}_k = \{n_{2k}+1, n_{2k}+2, \ldots, n_{2k+1}\}$$

and

$$\mathscr{A} = \bigcup_{k=1}^{+\infty} \mathscr{A}_k.$$

For this sequence $\mathscr{A}$, (100) follows from the fact that

$$[1, 2, \ldots, n] < e^{(1+\varepsilon)n} \quad \text{for} \quad n > n_0(\varepsilon).$$

Furthermore, if q, m, k_1, k_2 are positive integers such that

$$(n_1 \leqq) \quad n_{k_1} < q \leqq n_{k_1+1} \leqq n_{k_2} < m \leqq n_{k_2+1} \tag{101}$$

then

$$|E_{q,h}(m)| \leqq |E_{q,h}(n_{k_1})| + |E_{q,h}(n_{k_1+1}) - E_{q,h}(n_{k_1})| +$$

$$+ |E_{q,h}(n_{k_2}) - E_{q,h}(n_{k_1+1})| + |E_{q,h}(m) - E_{q,h}(n_{k_2})| \leqq 1+1+0+1 = 3.$$

We may discuss also those cases similarly when for some positive integers q and m, integers k_1, k_2 satisfying (101) do not exist.

(We remark that for the sequence $\mathscr{A}$ defined above,

$$\lim_{n\to+\infty} \inf \frac{A(n)}{n} = 0 \quad \text{and} \quad \lim_{n\to+\infty} \sup \frac{A(n)}{n} = 1.$$

Also this fact shows that a lower estimate for the quantities $E_{q,h}(m)$ should be given in terms of rather $B_N(\mathscr{A})$ than $\frac{A(N)}{N}$.)

Finally, for *infinite* sequences, one might like to seek a condition e.g. on $B_N(\mathscr{A})$ which implies the existence of integers q, h such that

$$\lim_{m\to+\infty} |E_{q,h}(m)| = +\infty.$$

A theorem of this type does not exist, as the following construction shows:

THEOREM 8. *There exists an infinite sequence $\mathscr{A}$ for which*

$$B_N(\mathscr{A}) > \frac{N}{4} \tag{102}$$

and for $\varepsilon>0$ and any integers q, h, m,

$$|E_{q,h}(m)| \ll e^{(1+\varepsilon)q}. \tag{103}$$

PROOF. Let $q_1, q_2, \ldots$ denote the prime-powers: $q_1=2$, $q_2=3$, $q_3=4$, $q_4=5$, $q_5=7$, $q_6=8$, $q_7=9$, $q_8=11, \ldots$, and let us put $n_0=0$ and $n_k=2^{k-1}q_1q_2\ldots q_k$ for $k=1, 2, \ldots$. Let $\mathscr{A}_1=\{1\}$ and if $\mathscr{A}_k\subset\{n_{k-1}+1, n_{k-1}+2, \ldots, n_k\}$ (where $k=1, 2, 3, \ldots$) has been defined then let us define $\mathscr{A}_{k+1}$ in the following way: for some $n_k<a\leqq n_{k+1}$, let $a=un_k+v$ where u, v are positive integers and $1\leqq v\leqq n_k$. Then $a\in\mathscr{A}_{k+1}$ should hold if and only if u is even and $v\in\bigcup_{i=1}^{k}\mathscr{A}_i$ or u is odd and $v\notin\bigcup_{i=1}^{k}\mathscr{A}_i$. Finally, let

$$\mathscr{A}=\bigcup_{k=1}^{+\infty}\mathscr{A}_k.$$

(102) can be proved easily.

In order to prove (103), note that for any positive integer q, there exists a positive integer k ($\geqq 2$) for which q/n_k and $n_k\ll e^{(1+\varepsilon)q}$ hold; furthermore, for this k and any positive integers h, t, $E_{q,h}(2tn_k)=0$. We leave the details to the reader.

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ON DISTANCES NEAR INTEGERS, II

by

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1. We shall use the same notations as in Part I* of this paper. In particular, we denote the distance between the points P, Q by $\varrho(P, Q)$, the distance of the real number x from the nearest integer by $\|x\|$, finally, for some $X>0$ and

$$0 < \delta < 1/2, \tag{1}$$

$N(X, \delta)$ denotes the maximum of those positive integers m for which there exist points $P_1, P_2, \ldots, P_m$ in the circle of radius X such that

$$\|\varrho(P_i, P_j)\| \geqq \delta \quad \text{for} \quad 1 \leqq i < j \leqq n.$$

In Part I, we showed (proving a conjecture of P. Erdős) that for any fixed δ (satisfying (1)), $N(X, \delta)=o(X)$, more exactly,

$$N(X, \delta) < \frac{4 \cdot 10^4}{\delta^3} \cdot \frac{X}{\log\log X}$$

for large enough X.

On the other hand, Erdős conjectured that for some $\delta_0>0$,

$$\lim_{X \to +\infty} N(X, \delta_0) = +\infty.$$

This conjecture has been proved by R. L. Graham (Erdős's oral communication). Graham's construction is the following: let

$$x_1 = 10 \quad \text{and} \quad x_n = 2x_{n-1} \quad \text{for} \quad n = 2, 3, \ldots,$$

and let us define the positive integer m by

$$\sqrt{x_m^2 + x_m^4} \leqq X < \sqrt{x_{m+1}^2 + x_m^4}.$$

For $i=1, 2, \ldots, m$, let P_i denote that point whose Cartesian coordinates are $x=x_i$, $y=x_i^2$: $P_i=(x_i, x_i^2)$. These points are in the circle of radius X (with center at the origin) and it is easy to show that for large enough X,

$$m > \frac{1}{10} \log X$$

* Studia Sci. Math. Hung. **10** (1975), 37—50.

and

$$\varrho(P_i, P_j) > \frac{1}{10} \quad \text{for} \quad 1 \leqq i < j \leqq m.$$

Hence,

$$N(X, 1/10) > \frac{1}{10} \log X \tag{2}$$

for large enough X.

In this section, our aim is to improve on this estimate.

Our construction is in a closed connection with Graham's construction. Namely, both constructions are based on the simple principle formulated in the following

LEMMA. *Let δ be some real number satisfying* (1), *a arbitrary positive integer, b some positive number satisfying*

$$3\delta < \frac{b^2}{a} < 2(1-\delta). \tag{3}$$

Then

$$\|\sqrt{a^2+b^2}\| > \delta. \tag{4}$$

PROOF. Since a is integer, (4) would follow from

$$a+\delta < \sqrt{a^2+b^2} < a+(1-\delta).$$

Since $a>0$ and $\delta>0$, this inequality can be rewritten in the following form:

$$a^2+2a\delta+\delta^2 < a^2+b^2 < a^2+2a(1-\delta)+(1-\delta)^2,$$

$$\delta(2a+\delta) < b^2 < 2a(1-\delta)+(1-\delta)^2.$$

By $0<\delta<1/2$ and $a\geqq 1$, it is sufficient to show that

$$\delta(2a+a) < b^2 < 2a(1-\delta).$$

But this inequality is equivalent to (3) and thus the lemma is proved.

We are going to prove the following sharpening of the estimate (2):

THEOREM 1. *Let*

$$0 < \delta \leqq \frac{1}{6 \cdot 8^4} \tag{5}$$

and X be sufficiently large depending on δ. Then

$$N(X, \delta) > X^{1/2-\delta^{1/7}}. \tag{6}$$

The theorem implies obviously the

COROLLARY. *For* $\varepsilon>0$ *arbitrary small, there exists* $\delta_0=\delta_0(\varepsilon)>0$ *such that for* $0<\delta<\delta_0(\varepsilon)$,

$$N(X,\delta) > X^{1/2-\varepsilon}$$

whenever X *is large enough (depending on* ε *and* δ*).*

PROOF OF THEOREM 1. Let us define the positive integer k by

$$\frac{1}{6k^4} \geqq \delta > \frac{1}{6(k+1)^4}. \tag{7}$$

(5) and (7) imply that

$$k \geqq 8. \tag{8}$$

From (7) and (8)

$$\delta > \frac{1}{6(k+1)^4} > \frac{1}{k\cdot(2k)^4} > \frac{1}{k\cdot(\sqrt{k}\cdot k)^4} = \frac{1}{k^7},$$

hence

$$k > \delta^{-1/7}. \tag{9}$$

Furthermore, define the positive integer t by

$$2k^{2t+3} \leqq X < 2k^{2(t+1)+3}. \tag{10}$$

Then

$$k^t > \sqrt{\frac{X}{2k^5}}. \tag{11}$$

Again from (10):

$$k^{2t} \leqq \frac{X}{2k^3} < X,$$

hence

$$t < \frac{\log X}{2\log k}. \tag{12}$$

Let us take all the points $P^{(u)}=(x^{(u)}, y^{(u)})$ in the Cartesian plane whose coordinates can be written in the form

$$x^{(u)} = \sum_{i=0}^{t} \varepsilon_i^{(u)} k^{2i+2}, \quad y^{(u)} = \sum_{i=0}^{t} \varepsilon_i^{(u)} k^i \tag{13}$$

where the integers $\varepsilon_i^{(u)}$ satisfy

$$0 \leqq \varepsilon_i^{(u)} \leqq k-2 \quad \text{for} \quad i = 0, 1, \ldots, t. \tag{14}$$

For fixed i, $\varepsilon_i^{(u)}$ in (14) can be chosen in $k-1$ ways, furthermore i can be chosen in $t+1$ ways. Thus for large enough X, the number of the points $P^{(1)}, \ldots, P^{(u)}, \ldots, P^{(m)}$

with coordinates of form (13) is

$$m = (k-1)^{t+1} > (k-1)^t = k^t\left(1-\frac{1}{k}\right)^t = \tag{15}$$

$$= k^t e^{t\log\left(1-\frac{1}{k}\right)} > \sqrt{\frac{X}{2k^5}}\, e^{\frac{\log X}{2\log k}\left(-\frac{2}{k}\right)} =$$

$$= \sqrt{\frac{X}{2k^5}}\, e^{-\frac{\log X}{k\log k}} = X^{1/2} e^{-\frac{\log X}{k\log k}-\log\sqrt{2k^5}} =$$

$$= X^{1/2} e^{-\log X\left(\frac{1}{k\log k}+\frac{\log\sqrt{2k^5}}{\log X}\right)} > X^{1/2} e^{-\frac{\log X}{k}} = X^{1/2-1/k} > X^{1/2-\delta^{1/7}}$$

by (8), (9), (11), (12) and since

$$\log(1-x) = -\sum_{n=1}^{+\infty}\frac{x^n}{n} > -\sum_{n=1}^{+\infty} x^n = -\frac{x}{1-x} > -\frac{x}{1-1/2} = -2x \quad \text{for} \quad 0<x<1/2.$$

Furthermore, for some point $P^{(u)}=(x^{(u)}, y^{(u)})$ satisfying (13),

$$\sqrt{(x^{(u)})^2+(y^{(u)})^2} = \sqrt{\left(\sum_{i=0}^{t}\varepsilon_i^{(u)}k^{2i+2}\right)^2+\left(\sum_{i=0}^{t}\varepsilon_i^{(u)}k^i\right)^2} <$$

$$< \sqrt{(k^{2t+3})^2+(k^{t+1})^2} < \sqrt{2(k^{2t+3})^2} < 2k^{2t+3} \leqq X$$

by (10), i.e. these points are inside the circle with radius X and center at the origin. Since their number is greater than $X^{1/2-\delta^{1/7}}$ by (15), (6) (and thus also our theorem) would follow from the fact that

$$\|\varrho(P^{(u)}, P^{(v)})\| \geqq \delta \quad \text{whenever} \quad 1 \leqq u < v \leqq m. \tag{16}$$

Here

$$\|\varrho(P^{(u)}, P^{(v)})\| = \left\|\sqrt{(x^{(u)}-x^{(v)})^2+(y^{(u)}-y^{(v)})^2}\right\|. \tag{17}$$

To obtain a lower estimate for $\|\varrho(P^{(u)}, P^{(v)})\|$, we shall apply the lemma above with

$$a = x^{(u)}-x^{(v)}, \quad b = y^{(u)}-y^{(v)}. \tag{18}$$

To show the applicability of the lemma, we have to estimate both a and b.

Throughout the remaining part of the proof, $\theta_1, \theta_2, \ldots$ will denote real numbers such that $|\theta_i| \leqq 1$ for $i=1, 2, \ldots$.

We may assume that $x^{(u)} > x^{(v)}$. Let us define the integer i_0 ($i_0 \geqq 0$) by

$$\varepsilon_t^{(u)} = \varepsilon_t^{(v)}, \ \varepsilon_{t-1}^{(u)} = \varepsilon_{t-1}^{(v)}, \ \ldots, \ \varepsilon_{i_0+1}^{(u)} = \varepsilon_{i_0+1}^{(v)}, \tag{19}$$

$$\varepsilon_{i_0}^{(u)} > \varepsilon_{i_0}^{(v)} \tag{20}$$

(since $x^{(u)} > x^{(v)}$, (19) and $\varepsilon_{i_0}^{(u)} \neq \varepsilon_{i_0}^{(v)}$ imply that (20) holds). Then

$$(21) \qquad x^{(u)} - x^{(v)} = (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) k^{2i_0+2} + \sum_{i=0}^{i_0-1} (\varepsilon_i^{(u)} - \varepsilon_i^{(v)}) k^{2i+2} =$$

$$= (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) k^{2i_0+2} + \theta_1 \sum_{i=0}^{i_0-1} (k-2) k^{2i+2} = (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) k^{2i_0+2} + \theta_1 k^2 (k-2) \frac{k^{2i_0}-1}{k^2-1} =$$

$$= (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) k^{2i_0+2} + \theta_2 k^2 (k-1) \frac{k^{2i_0}}{k^2-1} = (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) k^{2i_0+2} + \theta_2 \frac{k^{2i_0+2}}{k+1} =$$

$$= (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) k^{2i_0+2} + \theta_3 (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) \frac{k^{2i_0+2}}{k+1} = (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) k^{2i_0+2} \left(1 + \frac{\theta_3}{k+1}\right).$$

Similarly,

$$(22) \quad y^{(u)} - y^{(v)} = (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) k^{i_0} + \sum_{i=0}^{i_0-1} (\varepsilon_i^{(u)} - \varepsilon_i^{(v)}) k^i = (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) k^{i_0} + \theta_4 \sum_{i=0}^{i_0-1} (k-2) k^i =$$

$$= (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) k^{i_0} + \theta_4 (k-2) \frac{k^{i_0}-1}{k-1} = (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) k^{i_0} + \theta_5 (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) (k-2) \frac{k^{i_0}}{k-1} =$$

$$= (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) k^{i_0} \left(1 + \theta_5 \frac{k-2}{k-1}\right).$$

By (21), we get that the number a defined in (18) can be estimated in the following way:

$$(23) \qquad a = x^{(u)} - x^{(v)} \geqq (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) k^{2i_0+2} \left(1 - \frac{1}{k+1}\right) = (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) \frac{k^{2i_0+3}}{k+1}$$

and

$$(24) \qquad a = x^{(u)} - x^{(v)} \leqq (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) k^{2i_0+2} \left(1 + \frac{1}{k+1}\right) <$$

$$< (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) k^{2i_0+2} \frac{k+2}{k} = (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) k^{2i_0+1} (k+2).$$

Similarly, we get from (22) that

$$(25) \qquad b = y^{(u)} - y^{(v)} \geqq (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) k^{i_0} \left(1 - \frac{k-2}{k-1}\right) >$$

$$> (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) k^{i_0} \frac{1}{k} = (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) k^{i_0-1}$$

and

$$(26) \qquad b = y^{(u)} - y^{(v)} \leqq (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) k^{i_0} \left(1 + \frac{k-2}{k-1}\right) < (\varepsilon_{i_0}^{(u)} - \varepsilon_{i_0}^{(v)}) 2k^{i_0}.$$

With respect to (5), (7) and (8), we get from (23), (24), (25) and (26) that

$$\frac{b^2}{a} = \frac{(y^{(u)}-y^{(v)})^2}{x^{(u)}-x^{(v)}} < \frac{((\varepsilon_{i_0}^{(u)}-\varepsilon_{i_0}^{(v)})k^{i_0-1})^2}{(\varepsilon_{i_0}^{(u)}-\varepsilon_{i_0}^{(v)})k^{2i_0+1}(k+2)} = \tag{27}$$

$$= \frac{\varepsilon_{i_0}^{(u)}-\varepsilon_{i_0}^{(v)}}{k^3(k+2)} > \frac{1}{k^3\cdot 2k} = \frac{1}{2k^4} \leqq 3\delta$$

and

$$\frac{b^2}{a} = \frac{(y^{(u)}-y^{(v)})^2}{x^{(u)}-x^{(v)}} < \frac{((\varepsilon_{i_0}^{(u)}-\varepsilon_{i_0}^{(v)})2k^{i_0})^2}{(\varepsilon_{i_0}^{(u)}-\varepsilon_{i_0}^{(v)})\dfrac{k^{2i_0+3}}{k+1}} = \tag{28}$$

$$= \frac{4(\varepsilon_{i_0}^{(u)}-\varepsilon_{i_0}^{(v)})(k+1)}{k^3} \leqq \frac{4(k-2)(k+1)}{k^3} = \frac{4(k^2-k-2)}{k^3} < \frac{4k^2}{k^3} = \frac{4}{k} < 1 < 2(1-\delta).$$

By (27) and (28), (3) in the lemma holds for the numbers a, b defined by (18), and also (1) holds by (5). Thus the lemma can be applied to estimate the quantities in (17). We get that (16) holds which proves our theorem.

We remark that we have been interested here mostly in the case $\delta \to 0$. This is the reason of that that the upper bound given for δ in (5) is so small. It can be shown (by the same method) that this upper bound can be increased, replacing $1/2-\delta^{1/7}$ on the right hand side of (6) by an other exponent. In other words, e.g. it can be shown that the right hand side of (2) can be replaced by X^C (for some absolute constant C):

$$N(X, 1/10) > X^C.$$

2. In this section, we discuss an other related problem of P. Erdős. Namely, he raised the problem whether there exist infinitely many points $P_1, P_2, \ldots$ in the plane such that for all pairs P_i, P_j, $\|\varrho(P_i, P_j)\|$ is near $1/2$?

THEOREM 2. *For arbitrary $\delta > 0$, there exist infinitely many points $P_1, P_2, \ldots$ in the plane such that for $1 \leqq i < j$,*

$$\left| \|\varrho(P_i, P_j)\| - \frac{1}{2} \right| < \delta. \tag{29}$$

PROOF. We prove Theorem 2 by modifying Graham's construction.

Let N be a large integer (sufficiently large depending on δ) and let us define the integers $n_1, n_2, \ldots$ by

$$n_1 = N \quad \text{and} \quad n_{k+1} = n_k^2 \quad \text{for} \quad k = 1, 2, \ldots . \tag{30}$$

For $k = 1, 2, \ldots$, let P_k denote that point in the Cartesian plane whose coordinates

are $x_k=n_k$, $y_k=n_k^2$: $P_k=(n_k, n_k^2)$. Then for $N\to+\infty$ and $1\leqq i<j$, we have

$$\left|\varrho(P_i, P_j)-\left(y_j-y_i+\frac{1}{2}\right)\right| = \left|\sqrt{(x_j-x_i)^2+(y_j-y_i)^2}-\left(y_j-y_i+\frac{1}{2}\right)\right| =$$

$$= \frac{\left|((x_j-x_i)^2+(y_j-y_i)^2)-\left(y_j-y_i+\frac{1}{2}\right)^2\right|}{\left|\sqrt{(x_j-x_i)^2+(y_j-y_i)^2}+\left(y_j-y_i+\frac{1}{2}\right)^2\right|} =$$

$$= \frac{\left|(x_j-x_i)^2+(y_j-y_i)^2-(y_j-y_i)^2-(y_j-y_i)-\frac{1}{4}\right|}{\left|\sqrt{(x_j-x_i)^2+(y_j-y_i)^2}+\left(y_j-y_i+\frac{1}{2}\right)\right|} =$$

$$= \frac{|(1+o(1))x_j^2-(1+o(1))y_j+O(1)|}{(1+o(1))(y_j+y_j)} = \frac{|(1+o(1))n_j^2-(1+o(1))n_j^2+O(1)|}{(1+o(1))\cdot 2n_j^2} = o(1)$$

hence

$$\left|\|\varrho(P_i, P_j)\|-\frac{1}{2}\right| = o(1)$$

(uniformly for $1\leqq i<j$) which completes the proof of Theorem 2.

If we write $n_{k+1}=an_k$ in (30) in place of $n_{k+1}=n_k^2$ (where $a=a(\delta)$ is a positive integer, sufficiently large in terms of δ) then we may show in this way that for large X, there exist points $P_1, P_2, \ldots, P_m$ in the circle of radius X such that

$$m > c_1(\delta)\log X \tag{31}$$

(where the factor $c_1(\delta)>0$ depends on δ but it is independent of X) and (29) holds for $1\leqq i<j\leqq m$. Applying the method used in the proof of Theorem 1, this estimate can be sharpened; namely, it can be shown that (31) can be replaced by

$$m > X^{c_2(\delta)}$$

where $c_2(\delta)>0$ is independent of X. Furthermore,

$$\lim_{\delta\to 0^+} c_2(\delta) = 0$$

for this exponent. On the other hand, P. Erdős conjectured that if ε is arbitrary small positive number, $\delta=\delta(\varepsilon)>0$ is sufficiently small, X is large (in terms of ε) and $P_1, P_2, \ldots, P_m$ are points in the circle of radius X, such that (29) holds for $1\leqq i<j\leqq m$ then, perhaps,

$$m < X^{\varepsilon}$$

must hold. I have not been able to prove or disprove this conjecture.

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CHROMATIC NUMBER OF HYPERGRAPHS AND LINEAR ALGEBRA

by

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B. TOFT [5] raised the following problem. Assume H is an r-uniform hypergraph on n points which is 3-critical, i.e. not 2-colorable but the deletion of any edge results in a 2-colorable hypergraph. What is the maximum number of edges in such a hypergraph? He constructed such a hypergraph with $c_r \cdot n^{r-1}$ edges and quotes a result of Erdős saying that such a hypergraph has at most $c_r' \cdot n^{r-\varepsilon_r}$ edges, where $\varepsilon_r = 1/r - 1$. The aim of this note is to prove

THEOREM 1. *Let H be an r-uniform 3-critical hypergraph on n points. Then*

$$|E(H)| \leqq \binom{n}{r-1}.$$

Toft's construction shows that this theorem is sharp, at least in the order of magnitude.

The proof uses linear algebra. The idea of applying linear algebra in such combinatorial problems is not new; one can mention the paper by P. SEYMOUR [4], several linear algebra proofs of Fisher's inequality and its extensions [1], [3]; also paper [2], which applies linear algebra in chromatic grahp theory.

In fact we prove the following stronger version of Theorem 1:

THEOREM 1′. *Let $E_1, \ldots, E_m$ be the edges of an r-uniform hypergraph H and let $A_1, \ldots, A_N$ be all $(r-1)$-tuples of points of H. Define*

$$a_{ij} = \begin{cases} 1 & \text{if } A_i \subset E_j, \\ 0 & \text{otherwise.} \end{cases}$$

If H is 3-critical then the matrix (a_{ij}) has rank m.

Since the rank of the matrix (a_{ij}) is clearly at most $N \leqq \binom{n}{r-1}$, Theorem 1′ implies Theorem 1.

PROOF. Suppose indirectly the rank of (a_{ij}) were less than m, then we would have numbers $\alpha_1, \ldots, \alpha_m$, not all 0, satisfying

$$\sum_{E_i \supset A_j} \alpha_i = 0 \quad (j = 1, \ldots, N).$$

Let e.g. $\alpha_1 \neq 0$, and let V_1, V_2 be a 2-coloration of $H - \{E_1\}$. Clearly E_1 must be monochromatic, e.g. $E_1 \subseteq V_1$. Then E_1 is the only edge contained in either V_1 or V_2. On the other hand, we prove the following lemma, which clearly contradicts the assumption that $\alpha_1 \neq 0$:

LEMMA. *Let H be an r-uniform hypergraph with edges $E_1, \dots, E_m$. Let $\alpha_1, \dots, \alpha_m$ be real numbers such that for every $(r-1)$-element subset A of $V(H)$,*

$$\sum_{E_i \supset A} \alpha_i = 0. \tag{1}$$

Then for every partition $\{V_1, V_2\}$ of $V(H)$ the following holds:

$$\sum_{E_i \subseteq V_1} \alpha_i = (-1)^r \sum_{E_i \subseteq V_2} \alpha_i.$$

PROOF OF THE LEMMA. Let

$$I_v = \{i : E_i \cap V_1 = v\}, \quad \beta_v = \sum_{i \in I_v} \alpha_i.$$

Also let, as in Theorem 1′, $A_1, \dots, A_N$ denote all $(r-1)$-element subsets of $V(H)$ and define

$$J_v = \{j : A_j \cap V_1 = v\}.$$

Then sum (1) over all $A = A_j$, $j \in J_v$:

$$0 = \sum_{j \in J_v} \sum_{E_i \supset A_j} \alpha_i.$$

Here only edges E_i with $|E_i \cap V_1| = v$ or $v+1$ occur, and in fact, $r - v$ times and $v+1$ times, respectively. Thus

$$0 = (r-v) \sum_{i \in I_v} \alpha_i + (v+1) \sum_{i \in I_{v+1}} \alpha_i = (r-v)\beta_v + (v+1)\beta_{v+1}.$$

This can be considered as a simple recurrence relation on β_v, whose solution is

$$\beta_v = (-1)^v \binom{r}{v} \beta_0.$$

In particular,

$$\beta_r = (-1)^r \beta_0,$$

which proves the Lemma.

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ON SUBSPACES OF DYADIC COMPACTA

by
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1. Notations. Introduction

Denote by D the two-point discrete space. A Hausdorff continuous image of a product space D^τ will be said a dyadic compactum.

Recently JAMES HAGLER proved [7] that if R is a dyadic compactum with weight $\varkappa$ where $\varkappa > \omega_0$ is a regular cardinality, then R contains a topological copy of $D^\varkappa$.

In this paper we shall examine the validity of this theorem also for singular cardinalities. It will turn out e.g. that it is true if $\varkappa$ is not cofinal with ω_0. Our proof is more straightforward and perhaps simpler even for regular $\varkappa$'s.

We point out another result of the paper. A well-known theorem of ESENIN—VOLPIN [6] asserts that if the set $\{x \in R;\ \chi(x, R) \leqq \varkappa\}$ is dense in the dyadic compactum R then $w(R) \leqq \varkappa$. We extend this theorem for $\pi\chi$, the π-character of R.

We shall use the usual set-theoretic notations; cardinals are identified with initial ordinals. All undefined terms can be found in [4] and [8].

If x is a point of the topological space R, the system $\mathscr{U}$ is called a local π-base at x if the elements of $\mathscr{U}$ are non-empty open subsets in R and for each neighbourhood V of x in R there exists a $U \in \mathscr{U}$ with $U \subset V$.

The π-character of x in R, resp. the π-character of R is defined by

$$\pi\chi(x, R) = \min \{|\mathscr{U}|;\ \mathscr{U} \text{ is a local } \pi\text{-base of } x \text{ in } R\},$$

$$\pi\chi(R) = \sup \{\pi\chi(x, R);\ x \in R\}.$$

The density of the topological space R is denoted by $d(R)$; that is $d(R) = \min \{|H|;\ H \subset R, \overline{H} = R\}$.

By a mapping we mean a continuous function from a topological space to another; $f: X \to Y$ denotes a mapping from the topological space X into the space Y. If $A \subset X$ then $f|A$ denotes the restriction of f to the subspace A.

Let E be a topological space, τ be a set. We shall regard the points of the product space E^τ as functions mapping τ into E. If $T \subset \tau$, $p \in E^\tau$, denote by $\pi_T(p)$ the restriction of the function p to the set T; it is a point of E^T. If $T = \{t\}$, denote $\pi_{\{t\}}$ simply by π_t.

A set $U = \times \{U_t;\ t \in \tau\}$ is said to be a basic open set if the sets U_t are open in E and $\tau(U) = \{t \in \tau;\ U_t \neq E\}$ is finite; the family of basic open sets in E^τ forms a base for the topology of E^τ.

If $p, \vartheta \in E^\tau$, denote by $\tau(p, \vartheta) = \{t \in \tau;\ p(t) \neq \vartheta(t)\}$. If $\varkappa$ is a cardinality, $\vartheta \in E^\tau$ put

$$\sum(\varkappa, \vartheta) = \{p \in E^\tau;\ |\tau(p, \vartheta)| < \varkappa\}.$$

2. The subspace $S(\varkappa)$. The π-character in dyadic spaces

Let R be a dyadic compactum, $f: D^\tau \to R$ a mapping onto R. Put

$$S(\varkappa, \vartheta) = f\left(\sum(\varkappa, \vartheta)\right) \subset R,$$

$$S(\varkappa) = \bigcap \{S(\varkappa, \vartheta);\ \vartheta \in D^\tau\}.$$

LEMMA 1. *Let $\varkappa \geqq \omega_0$ be a regular cardinality, $\vartheta \in D^\tau$, $A \subset S(\varkappa, \vartheta)$, $|A| < \varkappa$, then $w(\bar{A}) < \varkappa$.*

PROOF. Select a set $B \subset \sum(\varkappa, \vartheta)$, $|B| = |A| < \varkappa$, $f(B) = A$. Put $T = \bigcup \{\tau(p, \vartheta);\ p \in B\}$. The regularity of $\varkappa$ implies that $|T| < \varkappa$.

Denoting by $C = \bigcap \left\{\pi_t^{-1}(\{\vartheta(t)\});\ t \in \tau - T\right\}$, $B \subset C$, C is a compact set, $w(C) \leqq$ $\leqq |T| < \varkappa$ so $A \subset f(C) = K$. By a well-known theorem of Arhangelskii (see e.g. [4] Corollary 2 to Th. 3.1.11) $w(K) \leqq w(C) < \varkappa$, hence $w(\bar{A}) < \varkappa$, too. ∎

THEOREM 1. *If $x \in R$, $\pi\chi(x, R) < \varkappa$ then $x \in S(\varkappa)$.*

PROOF. Let $\mathcal{U}$ be a local π-base at x, $|\mathcal{U}| < \varkappa$. For each $U \in \mathcal{U}$ select a B_U elementary open set in D^τ with $f(B_U) \subset U$, and put $T = \bigcup \{\tau(B_U);\ U \in \mathcal{U}\}$. Now $T \subset \tau$, $|T| < \varkappa$.

If $\vartheta \in D^\tau$ is any point, denote by $C = \bigcap \left\{\pi_t^{-1}(\{\vartheta(t)\});\ t \in \tau - T\right\}$, $G = D^\tau - C$. The set C is compact in D^τ and $C \cap B_U \neq \emptyset$ for each $U \in \mathcal{U}$ so the open set G does not contain any B_U. Hence G is not a neighbourhood of the set $f^{-1}(x)$ that is $C \cap f^{-1}(x) \neq \emptyset$. This means that there exists a $p \in D^\tau$, $f(p) = x$, $p|\tau - T = \vartheta|\tau - T$ so $x \in S(\varkappa, \vartheta)$. ∎

THEOREM 2. *If $\varkappa \geqq \omega_1$ is a regular cardinality then $w\left(\overline{S(\varkappa)}\right) < \varkappa$.*

PROOF. On account of the Lemma we need only to prove that $d\left(S(\varkappa)\right) < \varkappa$. Let us suppose that $d\left(S(\varkappa)\right) \geqq \varkappa$ then there exists an $\{x_\xi;\ \xi < \varkappa\} \subset S(\varkappa)$ left-separated sequence in R [8], that is such that for $\xi < \varkappa$ $x_\xi \notin \overline{\{x_\eta;\ \eta < \xi\}}$.

Denote by $C_\xi = f^{-1}(x_\xi)$ and select for $\xi < \varkappa$ a G_ξ clopen set in D^τ with $C_\xi \subset G_\xi$, $G_\xi \cap C_\eta = \emptyset$ if $\eta < \xi < \varkappa$.

We can suppose without loss of generality that for each $\xi < \varkappa$ $G_\xi = \bigcup \{B_\xi^i;\ i < n\}$, where $n < \omega_0$, B_ξ^i are elementary open sets in D^τ and $\tau(B_\xi^i) = T_\xi$ for each $i < n$.

Making use of the well-known Erdős—Rado theorem (see e.g. [8] A.2) we can also suppose that the family $\{T_\xi;\ \xi < \varkappa\}$ is a Δ-system; that is, for each $\xi, \eta < \varkappa$, $\xi \neq \eta$ $T_\xi \cap T_\eta = \bigcap \{T_\xi;\ \xi < \varkappa\} = T$. Put $S_\xi = T_\xi - T$ $(\xi < \varkappa)$, $S = \bigcup \{S_\xi;\ \xi < \varkappa\}$, then the sets S_ξ are pairwise disjoint.

The set D^T being finite we can also assume that $\pi_T(B_\xi^i) = \pi_T(B_\eta^i)$ for each $\xi, \eta < \varkappa$, $i < n$.

Let now $\varphi: \varkappa \to n$ a function for which $|\varphi^{-1}(i)| = \varkappa$ for each $i < n$. If $t \in S_\xi$ let $\vartheta_t \in \pi_t(B_\xi^{\varphi(\xi)})$; finally let $\vartheta \in D^\tau$ be such a point that $\vartheta(t) = \vartheta_t$ for $t \in S$.

Select a point $p_0 \in C_0 \cap \Sigma(\varkappa, \vartheta)$. Now $|\tau(p_0, \vartheta)| < \varkappa$; the disjointness of the family $\{S_\xi;\ \xi < \varkappa\}$ implies that there exists a $\xi_0 < \varkappa$ such that for each ξ, $\xi_0 \leqq \xi < \varkappa$, $S_\xi \cap \tau(p_0, \vartheta) = \emptyset$.

By $p_0 \in C_0 \subset \cup \{B_0^i; i<n\}$, there exists an $i_0<n$, $p_0 \in B_0^{i_0}$; choose a ξ, $\xi_0 \leqq \xi < \varkappa$, $\varphi(\xi)=i_0$. If $t \in T$ then $p_0(t) \in \pi_t(B_0^{i_0})=\pi_t(B_\xi^{i_0})$; if $t \in T_\xi - T = S_\xi$ then $p_0(t)=\vartheta(t)=\vartheta_t \in \pi_t(B_\xi^{i_0})$. This means that $p_0 \in C_0 \cap B_\xi^{i_0} \subset C_0 \cap G_\xi = \varnothing$, a contradiction. ∎

COROLLARY. *If $\varkappa \geqq \omega_1$ is a regular cardinality, R is a dyadic compactum and $w(R) \geqq \varkappa$ then the set $\{x \in R; \pi\chi(x, R) < \varkappa\}$ is not dense in R; especially for a dyadic compactum R $\pi\chi(R)=w(R)$.* ∎

We can also go further. For a topological space R denote by $d\pi\chi(R)$ the least cardinal $\varkappa$ such that the set $\{x \in R; \pi\chi(x, R) \leqq \varkappa\}$ is dense in R. Evidently $d\pi\chi(R) \leqq \leqq \pi\chi(R)$ for each topological space R and Theorem 1 and 2 shows that for a dyadic compactum $d\pi\chi(R)=\pi\chi(R)=w(R)$.

Call a topological space R dyadic iff it is a dense subspace of a dyadic compactum [5].

We need the following simple remark.

LEMMA. *Let Y be a dense subset of the regular space X. For each $x \in Y$ $\pi\chi(x, Y)=\pi\chi(x, X)$.*

PROOF. If $\mathcal{U}$ is a local π-base of x in X, then $\mathcal{U}|Y=\{U \cap Y; U \in \mathcal{U}\}$ is a local π-base of x in Y. On the other hand, if $\mathcal{U}$ is a local π-base of x in Y, select for each $U \in \mathcal{U}$ an open set $\tilde{U}$ in X, $\tilde{U} \cap Y = U$. The regularity of X then implies that $\tilde{\mathcal{U}}=\{\tilde{U}; U \in \mathcal{U}\}$ is a local π-base of x in X. ∎

COROLLARY. *If R is a dyadic space, $d\pi\chi(R)=\pi\chi(R)=w(R)$.*

PROOF. Let αR be a dyadic compactum, R dense in αR. If $T \subset R$ is dense in R, $\pi\chi(x, R) \leqq \varkappa$ for each $x \in T$ then T is dense in αR, $\pi\chi(x, \alpha R) \leqq \varkappa$ for each $x \in T$, hence $w(\alpha R)=d\pi\chi(\alpha R) \leqq \varkappa$. This shows that

$$w(\alpha R) \leqq d\pi\chi(R) \leqq \pi\chi(R) \leqq w(R) \leqq w(\alpha R). \quad \blacksquare$$

3. D^λ in dyadic compacta

DEFINITION. Let E, F be topological spaces $f: E^\tau \to F$ a mapping, $p \in E^\tau$. We say that the set $T \subset \tau$ determines f at p iff $f(p|T)=f(p)$; that is if $q \in E^\tau$, $q|T=p|T$ implies that $f(q)=f(p)$.

Let us use the following notation:

$$v(p, f) = \min \{|T|; T \subset \tau, T \text{ determines } f \text{ at } p\}.$$

LEMMA 2. *Let R be a dyadic compactum, $w(R) \geqq \lambda > \omega_0$, and $f: D^\tau \to R$ a mapping onto R. Then we have two possibilities.*

a) *There exists a point $p \in D^\tau$ with $v(p, f) \geqq \lambda$.*

b) *$\mathrm{cf}(\lambda)=\omega_0$, $w(R)=\lambda$ and $R=\cup\{C_n; n<\omega_0\}$ where the sets C_n are compact subspaces with $w(C_n)<\lambda$.*

PROOF. Suppose that $v(p, f)<\lambda$ for each $p \in D$. We immediately obtain that $S(\lambda)=R$. If λ is a regular cardinality, Theorem 2 shows that $w(R)= =w(\overline{S(\lambda)})<\lambda$, a contradiction.

Let now λ be a singular cardinality, $\alpha=\mathrm{cf}(\lambda)$, $\alpha<\lambda$, $\lambda=\sup\{\lambda_\xi;\ \xi<\alpha\}$ and we can suppose that for each $\xi<\alpha$, λ_ξ is regular, $\omega_0<\lambda_\xi<\lambda$.

Put $C_\xi=\overline{S(\lambda_\xi)}$; by $S(\lambda)=\bigcup\{S(\lambda_\xi);\ \xi<\alpha\}$ we get that $R=\bigcup\{C_\xi;\ \xi<\alpha\}$, $w(C_\xi)=w(\overline{S(\lambda_\xi)})<\lambda_\xi<\lambda$. If $\alpha=\mathrm{cf}(\lambda)>\omega_0$ then α is a non-countable regular cardinality hence it is a caliber for R [10]. Using the fact that $C_\xi\subset C_\eta$ if $\xi<\eta<\alpha$ we get in this case that for a suitable $\xi<\alpha$ $C_\xi=R$ hence $w(R)<\lambda$, a contradiction.

Finally, if $\alpha=\mathrm{cf}(\lambda)=\omega_0$, then $R=\bigcup\{C_n;\ n<\omega_0\}$, $w(C_n)=\lambda_n<\lambda$ and a well-known theorem of ARHANGELSKII [1] implies that $w(R)\leqq\sum w(C_n)=\sum\lambda_n=\lambda$. ∎

Note that a similar theorem was proved by EFIMOV [3].

We shall need the following

LEMMA. *Let $\{T_h;\ h\in H\}$ be a family of finite sets; $\{S_h;\ h\in H\}$ a family of pairwise disjoint sets, $S_h\subset T_h$ for each $h\in H$. If $|H|>\omega_0$ then there exists a set $K\subset H$, $|K|=|H|$ with $k,k'\in K$, $k\neq k'$ implies $T_k\cap S_{k'}=\emptyset$.*

PROOF. A theorem of S. PICCARD [9] asserts that if R is a relation on a non-countable set H such that the sets $R(h)=\{h'\in H;\ hRh'\}$ are finite for each $h\in H$ then there exists a set $K\subset H$, $|K|=|H|$ with $h,h'\in K$, hRh' implies $h=h'$.

Define the relation R on H by hRh' iff $T_h\cap S_{h'}=\emptyset$. The disjointness of the sets S_h implies that $|R(h)|\leqq|T_h|<\omega_0$ for $h\in H$. ∎

PROPOSITION. *Let $f:D^\tau\to R$ be a mapping onto the dyadic space R. If there exists a point $p\in D^\tau$ with $v(p,f)\geqq\lambda>\omega_0$ then R contains a topological copy of D^λ.*

PROOF. Select a point $p\in D^\tau$ with $v(p,f)\geqq\lambda$. We shall define by transfinite induction a family $\{U_\xi^\varepsilon;\ \xi<\lambda,\ \varepsilon<2\}$ of basic open sets in D^τ with

(i) $p\in U_\xi^0$, $\tau(U_\xi^0)=\tau(U_\xi^1)=T_\xi$ $(\xi<\lambda)$,

(ii) $f(U_\xi^0)\cap f(U_\xi^1)=\emptyset$ $(\xi<\lambda)$,

(iii) If $\xi<\zeta<\lambda$, $\varepsilon<2$ then $U_\xi^0\cap U_\zeta^\varepsilon\neq\emptyset$.

Let us suppose that $\eta<\lambda$ and for $\xi<\eta$, $\varepsilon<2$ the sets U_ξ^ε are defined. Put

$$T=\bigcup\{\tau(U_\xi^\varepsilon);\ \xi<\eta,\ \varepsilon<2\};$$

evidently $T\subset\tau$, $|T|<\lambda$.

The definition of $v(p,f)$ implies that there exists a point $q\in D^\tau$, $q|T=p|T$, $f(q)\neq f(p)$. Select two basic open sets U_η^0. U_η^1 with $\tau(U_\eta^0)=\tau(U_\eta^1)=T_\eta$, $p\in U_\eta^0$, $q\in U_\eta^1$, $f(U_\eta^0)\cap f(U_\eta^1)=\emptyset$. Now, if $\xi<\eta$, $\varepsilon<2$ and $t\in\tau(U_\xi^0)\cap\tau(U_\eta^\varepsilon)$ then $t\in T$ hence $p(t)\in\pi_t(U_\eta^\varepsilon)\cap\pi_t(U_\xi^0)$. This shows that $U_\xi^0\cap U_\eta^\varepsilon\neq\emptyset$.

Denote now by $S_\xi=\{t\in\tau;\ \pi_t(U_\xi^0)\cap\pi_t(U_\xi^1)=\emptyset\}$ $(\xi<\lambda)$. Then $\{S_\xi;\ \xi<\lambda\}$ is a disjoint family of non-empty finite subsets of τ. Indeed, $S_\xi\neq\emptyset$ by (ii); and if $t\in S_\xi\cap S_\eta$, $\xi\leqq\eta<\lambda$ then $\pi_t(U_\xi^0)=\pi_t(U_\eta^0)=\{p(t)\}$ and $\pi_t(U_\eta^0)\cap\pi_t(U_\eta^1)=\emptyset$; hence $U_\xi^0\cap U_\eta^1=\emptyset$ and so, by (iii), $\xi=\eta$.

Making use of the Lemma, we can suppose that

$$(*)\qquad T_\xi\cap S_\eta\neq\emptyset\quad\text{iff}\quad\xi=\eta.$$

Note that if $\xi<\lambda$, $t\in T_\xi-S_\xi$, then $\pi_t(U_\xi^0)=\pi_t(U_\xi^1)=\{p(t)\}$. We assert that if $\xi,\eta<\lambda$, $\xi\neq\eta$, $\varepsilon,\varepsilon'<2$ then $U_\xi^\varepsilon\cap U_\eta^{\varepsilon'}\neq\emptyset$. Indeed, if $t\in T_\xi\cap T_\eta$ then, by $(*)$, $t\notin S_\xi\cup S_\eta$ hence $\pi_t(U_\xi^\varepsilon)=\pi_t(U_\eta^{\varepsilon'})=\{p(t)\}$. This implies, of course, that for any

finite sets $\{\xi_i; i<n\}\subset\lambda$, $\xi_i\neq\xi_j$ if $i<j<n$, $\{\varepsilon_i; i<n\}$ $\varepsilon_i<2$ $(i<n)$ the intersection $\bigcap\{U_{\xi_i}^{\varepsilon_i}; i<n\}$ is non-empty. The space D^τ being compact and the sets U_ξ^ε closed, we get that for any $x\in D^\lambda$

$$\bigcap\{U_\xi^{x(\xi)}; \xi<\lambda\}\neq\emptyset.$$

Put

$$S=\bigcup\{S_\xi;\ \xi<\lambda\}\subset\tau,$$

$$F=\bigcap\{\pi_t^{-1}(\{p(t)\});\ t\in\tau-S\}\subset D^\tau,$$

$$C=F\cup\bigcap\{U_\xi^0\cup U_\xi^1;\ \xi<\lambda\}\subset D^\tau.$$

The mapping $\varphi: D^\lambda\to C$ defined by $\varphi(x)=F\cap\bigcap\{U_\xi^{x(\xi)};\ \xi<\lambda\}$ is a homeomorphism of D^λ onto C, and $h=f\circ\varphi$ is a continuous mapping of D^λ into R. If $x, y\in D^\lambda$, $x\neq y$ then there exists a $\xi<\lambda$ with $x(\xi)\neq y(\xi)$ hence $\varphi(x)\in U_\xi^0$ and $\varphi(y)\in U_\xi^1$ or $\varphi(x)\in U_\xi^1$ and $\varphi(y)\in U_\xi^0$. In each case, $h(x)=f(\varphi(x))\neq f(\varphi(y))=h(y)$, hence h is an embedding of D^λ into R. ∎

As a corollary we immediately obtain

THEOREM 3. *Let R be a dyadic compactum, $w(R)\geqq\lambda$, $\mathrm{cf}(\lambda)>\omega_0$. Then R contains a topological copy of D^λ.* ∎

This is not true in general if $\mathrm{cf}(\lambda)=\omega_0$. Indeed, if $\lambda=\sup\{\lambda_n; n<\omega_0\}$, $\lambda_n<\lambda$, denote by X the discrete topological sum of the spaces D^{λ_n} $(n<\omega_0)$; finally let R be the one-point compactification of X. R is a dyadic compactum (it is the continuous image of $(\omega_0+1)\times D^\lambda$) and $w(R)=\lambda$; but evidently R does not contain any topological copy of D^λ.

Our least theorem characterizes the "λ-great" dyadic compacta.

THEOREM 4. *Let R be a dyadic compactum, $\lambda>\omega_0$ a cardinality. The following conditions are equivalent.*

(a) R contains each zero-dimensional T_0-space with weight $\leqq\lambda$.

(b) R contains D^λ.

(c) R can be mapped onto I^λ.

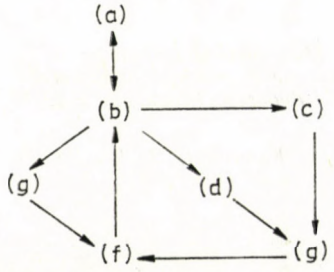
(d) R contains a compact subspace C with $\chi(x, C)\geqq\lambda$ for each $x\in C$.

(e) R contains a closed subset F which can be mapped onto a compactum C such that $\chi(x, C)\geqq\lambda$ for each $x\in C$.

(f) If R_n are closed subsets with $w(R_n)<\lambda$ $(n<\omega_0)$ then $R\neq\bigcup\{R_n; n<\omega_0\}$.

(g) There exists a point $x\in R$ and a set $A\subset R$ with $x\in\bar{A}$, $t(x, A)\geqq\lambda$.

PROOF. The graph of our reasoning will be as follows:



The implications (b)→(c), (b)→(d), (b)→(g), (c)→(e), (d)→(e) are evident.

(a)↔(b) is a well-known theorem of VEDENISOFF (see e.g. [4] Th. 6.2.11).

(e)→(f). Let $F \subset R$, $\varphi : F \xrightarrow{\text{onto}} C$, $\chi(x, C) \geqq \lambda$ for each $x \in C$ and let $\{R_n; n < \omega_0\}$ be a family of compact subspaces in R, $w(R_n) < \lambda$ $(n < \omega_0)$. Put $F_n = F \cap R_n$, $C_n = \varphi(F_n)$. $w(F_n) \leqq w(R_n) < \lambda$ hence $w(C_n) < \lambda$, too (see e.g. [4] Corollary 2 to Th. 3.1.11). Using the fact that a non-empty open set in C necessarily has weight $\geqq \lambda$ we obtain that the sets C_n are nowhere dense compact subsets of C. The Baire-category theorem implies that $C \neq \bigcup \{C_n; n < \omega_0\}$ hence $F - \bigcup \{F_n; n < \omega_0\} \neq \emptyset$, so $R \neq \bigcup \{R_n; n < \omega_0\}$.

(g)→(f). Let $R = \bigcup \{R_n; n < \omega_0\}$, $R_n = \overline{R}_n$, $w(R_n) < \lambda$ $(n < \omega_0)$, $x \in R$, $A \subset R$, $t(x, A) \geqq \lambda > \omega_0$.

We can suppose without loss of generality that A contains the closure of each of its countable subset. Put $A_n = A \cap R_n$; by $t(R_n) \leqq w(R_n) < \lambda$, $x \notin \overline{A}_n$ $(n < \omega_0)$.

Select a sequence $\{G_n; n < \omega_0\}$ of open neighbourhood of x, $G_n \cap A_n = \emptyset$, $\overline{G}_{n+1} \subset G_n$ $(n < \omega_0)$ and a point $p_n \in G_n \cap A$. If p is a cluster point of the sequence $\{p_n; n < \omega_0\}$ then evidently $p \in \bigcap \{\overline{G}_n; n < \omega_0\} = \bigcap \{G_n; n < \omega_0\}$ and $p \in A = \overline{\bigcup \{A_n; n < \omega_0\}}$, a contradiction.

(f)→(b) Compare the Proposition with Theorem 3. ∎

NOTE. If $\lambda = \omega_0$ then by classical results the conditions (a)—(f) are equivalent.

ADDED IN PROOF. A result similar to our Theorem 4 was proved by B. A. Efimov (Mat. Sbornik **103** (1977), pp. 52—68.

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JOINT DISTRIBUTIONS BASED ON RUNS AND ON THE NUMBER OF SOJOURNS AND INTERSECTIONS

by

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1. Introduction

NARAYANA [1] considered the partial order defined on the compositions of a positive integer n and gave some of its applications in probability theory. He presented a geometric interpretation of the r-compositions of n and gave a solution to the problem of the number of r-compositions of n dominated by the r-compositions of n. In the notations of the lattice path, this is the number of paths from $(0, 0)$ to (n, n) not crossing the line $y=x$, each path having exactly r horizontal and r vertical components. Using this approach of partial orders defined on compositions of an integer, MOHANTY and NARAYANA [2] gave two simple alternative solutions to 'ballot problems' [3, p. 69] in probability.

SUJAN [4] derived the joint distribution and the joint limiting distribution of the statistics based on the number of runs and that based on the number of intersections by using the random walk model and introducing the notion of composed path. A composed path is made up of runs where two consecutive runs are not necessarily of different kind.

In this paper, the joint distribution of the statistics based on the number of runs, number of sojourns and the number of intersections have been derived by using Sujan's method of composed paths. Concerning a possible statistical application of the results derived we refer to the paper of REIMANN and VINCZE [5] in which, for samples of slightly different sizes, advantageous statistical properties of sample functions connected with the process $nF_n(x)-mG_m(x)$ are treated. The random quantities occurring in the present paper can also be considered as two sample statistics in the same respect.

2. Notations

In the sequel, sequences $\Theta=(\Theta_1, \ldots, \Theta_{m+n})$ of m $(+1)$'s and n (-1)'s will be considered, each possible sequence having the same probability $1\Big/\binom{m+n}{n}$. The partial sums of Θ_i's will be denoted by S_i i.e.

$$S_i = \Theta_1+\Theta_2+\ldots+\Theta_i \quad (i = 1, 2, \ldots, m+n),$$

$$S_0 = 0 \quad \text{and} \quad S_{m+n} = m-n.$$

The array $S_0, S_1, \ldots, S_{m+n}$ will be called the path of the particle. Thus to each sequence $(\Theta_1, \Theta_2, \ldots, \Theta_{m+n})$ there corresponds a random path, starting at the origin

and reaching after $m+n$ steps the point $(m+n, m-n)$, $(0 \leqq n \leqq m)$. A figure illustrating the path of the particle can be got by representing the points (i, S_i) in a plane, and then connecting each one of them with the next one. Further, a path as defined above will be said to have R runs if the total number of changes from positive direction to negative direction and vice-versa is $R-1$.

In the sequel we shall consider the joint distributions of

$$(\lambda_{n+k,n-k},\ \alpha_{n+k,n-k},\ R_{n+k,n-k}),\quad (\lambda_{n+k,n-k},\ \alpha_{n+k,n-k},\ \alpha^{+}_{n+k,n-k},\ R_{n+k,n-k}),$$

$$(\alpha_{n+k,n-k},\ R_{n+k,n-k}) \quad \text{and} \quad (\alpha_{n+k,n-k},\ \alpha^{+}_{n+k,n-k},\ R_{n+k,n-k})$$

where we denote by

$\lambda_{m,n}$: the number of changes of sign in the sequences $S_1, S_2, \ldots, S_{m+n}$ $(S_{m+n}=m-n)$ for random samples of sizes m and n.

$\alpha_{m,n}$: the number of points (i, S_i) such that $S_i=0$ for random samples of sizes m and n.

$\alpha^{+}_{m,n}$: the number of points (j, S_j) such that $S_j=0$ and $S_{j-1}=+1$ for random samples of sizes m and n.

$R_{m,n}$: the number of runs in the sequence $\vartheta_1, \vartheta_2, \ldots, \vartheta_{m+n}$.

We shall employ the random walk model [6] and SUJAN's method of composed paths [4] and consider paths $E_{2n,2k}$ starting at the origin and reaching after $2n$ steps the point $(2n, 2k)$, $0 \leqq k \leqq n$. Then $\lambda_{n+k,n-k}$, $\alpha_{n+k,n-k}$, $\alpha^{+}_{n+k,n-k}$ and $R_{n+k,n-k}$ denote the number of intersections, sojourns, positive sojourns on the x-axis and runs respectively.

For ease in presentation we introduce the following symbols:

T-point: a point (i, S_i) of the path $E_{2n,2k}$ for which $S_i=0$, $S_{i-1}S_{i+1}=-1$ holds. This will be called the intersection point in the line $y=0$.

V-point: a point (i, S_i) where a path reaches the line $y=0$ i.e. for which $S_i=0$. We call it a 'sojourn' or 'return to origin'.

$V^{+}(V^{-})$: a V-point (i, S_i) such that $S_{i-1}=+1$ $(S_{i-1}=-1)$ and we call it a positive (negative) sojourn.

W=wave: The segment of a path included between two consecutive V-points.

$W^{+}(W^{-})$: a wave (W) with $S_i>0$ $(S_i<0)$ in between and is called a positive (negative) wave.

S=section: the segment of a path included between two consecutive T-points.

$S^{+}(S^{-})$: a section (S) with $S_i \geqq 0$ $(S_i \leqq 0)$ in between.

T_r^{+}=positive trail: The segment of a path starting on the x-axis, ending with a positive position and not crossing the x-axis in between. More precisely, for a positive trail comprised between the ith and the nth step, $S_i=0$, $S_j \geqq 0$, $j=i+1, \ldots, n-1$ and $S_n>0$.

T^{+}=positive tail: The segment of a path starting on the x-axis, ending with a positive position and not touching the x-axis in between.

$F^{R,l}_{2n,2k}$: an $E_{2n,2k}$ path having R runs and l T-points.

$F^{R+,l}_{2n,2k}(F^{R-,l}_{2n,2k})$: an $F^{R,l}_{2n,2k}$ path starting with a positive (negative) step.

$F^{R+,l,p(q)}_{2n,2k}(F^{R-,l,p(q)}_{2n,2k})$: an $F^{R+,l}_{2n,2k}(F^{R-,l}_{2n,2k})$ path having p V-points and q $(0 \leqq q \leqq p)$ V^{+}-points.

$F^{2r+,\ldots,j}_{2n,2k}(F^{(2r+1)+,\ldots,j}_{2n,2k})$: a composed path from $(0, 0)$ to $(2n, 2k)$ starting with a positive step, having r $(r+1)$ positive runs and $(j+r)$ negative runs where

the last j runs are negative and the remainder of the runs alternate, the $(j+1)$th run from the end being negative (positive).

$F_{2n,2k}^{2r+,l,p(q),j}(F_{2n,2k}^{(2r+1)+,l,p(q),j})$: an $F_{2n,2k}^{2r+,\ldots,j}(F_{2n,2k}^{(2r+1)+,\ldots,j})$ path having l T-points, p V-points and q $(0 \leqq q \leqq p)$ V^+-points.

$N(A)$: number of all possible paths of type A.

3. Some joint distributions

We note from (3.2) of [4] that

(1)
$$N(F_{m,n}^{2r+,0,.,j}) = \binom{\frac{1}{2}(m+n)-1}{r-1}\binom{\frac{1}{2}(m-n)-1}{r+j-1} - \binom{\frac{1}{2}(m+n)-1}{r-2}\binom{\frac{1}{2}(m-n)-1}{r+j},$$

$$n > 0.$$

THEOREM 1.

(2) $$N(F_{2n,2k}^{2r+,0,p(p),j} = \binom{n+k-p-2}{r-p-1}\binom{n-k-1}{r+j-1} - \binom{n+k-p-2}{r-p-2}\binom{n-k-1}{r+j}, \quad k > 0,$$

(3) $$N(F_{2n,0}^{2r+,0,p+1(p+1),j}) = \frac{j(n-p-1)+(n-r)(p+1)}{(n-r)(r+j)}\binom{n-p-2}{r-p-1}\binom{n-1}{r+j-1}.$$

PROOF. To obtain (2) let us consider an $F_{2n,2k}^{2r+,0,p(p),j}$ path having its p returns to origin at $O_1, O_2, \ldots, O_p$ (see Fig. 1). This path consists of p W^+ and a T^+. Let m out of p W^+ be of length two each i.e. each having two runs as well. Then the remaining $(p-m)$ W^+ and a T^+ have $(2r-2m)$ runs. Let us draw a line $y=1$. Then on joining the segments above the line $y=1$ end-to-end, in order, we get a path $OP_1Q_1P_2Q_2 \ldots P_{p-m}Q_{p-m}R$ from $(0, 0)$ to $(2n-2p-1, 2k-1)$ having $(2r-2m)$ runs and consisting of $(p-m)$ S^+ and one T_r^+ (see Fig. 2). $Q_1, Q_2, \ldots, Q_{p-m}$ are the terminal points of $(p-m)$ ordered S^+. $P_1, P_2, \ldots, P_{p-m}$ are the last turning points of the path before terminating at $Q_1, Q_2, \ldots, Q_{p-m}$ on the x-axis, where a turning point is a point where the path changes its direction from positive to negative or vice versa (see Fig. 2).

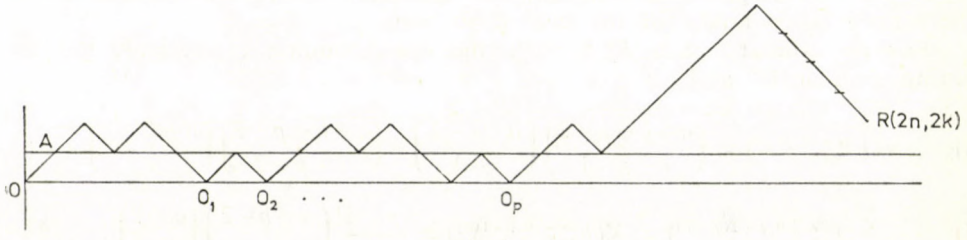

Fig. 1

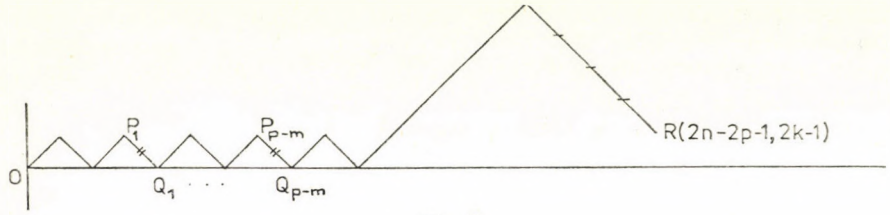

Fig. 2

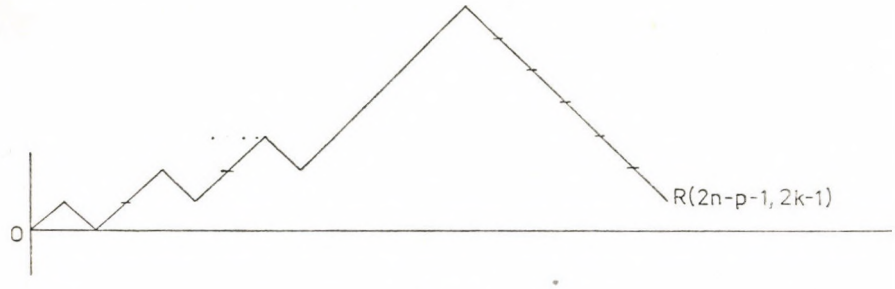

Fig. 3

Now, make the following transformation to the path in Fig. 2. To the end P_1, attach, in order, the segments $Q_1P_2, Q_2P_3, \ldots, Q_{p-m-1}P_{p-m}, Q_{p-m}R$. Then attach to this path, in order, the remaining segments

$$P_{p-m}Q_{p-m}, \quad P_{p-m-1}Q_{p-m-1}, \ldots, P_1Q_1.$$

Part of the transformed path between the origin and its last turning point is considered to be an ordinary path. In the remaining part the run end points are kept as they are, moreover, each section attached forms a separate run. The resulting is an $F^{(2r-2p)+,0,\ldots,j+p-m}_{2n-2p-1,2k-1}$ path (see Fig. 3). The reversing of the above procedure gives back the complete original path. Hence the transformation is 1:1. Thus

$$N(F^{2r+,0,p(p),j}_{2n,2k}) = \sum_{m=0}^{p} \binom{p}{m} N(F^{(2r-2p)+,0,\ldots,j+p-m}_{2n-2p-1,2k-1}).$$

The factor $\binom{p}{m}$ is due to the reason that there can be any m W^+ each of length two out of p W^+. This gives, on using (1), the required result (2) for the case $k>0$. The same relation holds for the case $k=0$ too.

For an ordinary path $F^{2r+,0,p(p)}_{2n,2k}$, the corresponding expressions, got by putting $j=0$ in (2) and (3) are:

$$(4) \qquad N(F^{2r+,0,p(p)}_{2n,2k}) = \binom{n+k-p-2}{r-p-1}\binom{n-k-1}{r-1} - \binom{n+k-p-2}{r-p-2}\binom{n-k-1}{r}$$

$$(5) \qquad N(F^{2r+,0,p+1(p+1)}_{2n,0}) = N(F^{2r-,0,p+1(0)}_{2n,0}) = \frac{p+1}{r}\binom{n-p-2}{r-p-1}\binom{n-1}{r-1}$$

DEDUCTIONS. (i) Putting $p=0$ in (5), we get

$$N(F_{2n,0}^{2r+,0,1(1)}) = N(F_{2n,0}^{2r-,0,1(0)}) = \frac{1}{r}\binom{n-2}{r-1}\binom{n-1}{r-1}.$$

(ii) Summing (5) over $0 \leqq p \leqq r-1$, we get

$$N(F_{2n,0}^{2r+,0}) = N(F_{2n,0}^{2r-,0}) = \frac{1}{n}\binom{n}{r}\binom{n}{r-1},$$

verifying a result in [1].

(iii) Summing (5) over $p+1 \leqq r \leqq n$, we get

$$N(F_{2n,0}^{+,0,p+1(p+1)}) = N(F_{2n,0}^{-,0,p+1(0)}) = \frac{p+1}{2n-p-1}\binom{2n-p-1}{n},$$

verifying a result in [3, p. 90].

THEOREM 2.

(6)
$$N(F_{2n,2k}^{2r+,2l,p(.),j}) =$$

$$= \binom{p}{2l}\left[\binom{n+k-p-2}{r+l-p-1}\binom{n-k-1}{r+l+j-1} - \binom{n+k-p-2}{r+l-p-2}\binom{n-k-1}{r+l+j}\right],$$

(7)
$$N(F_{2n,0}^{2r+,2l,p+1(.),j}) = \frac{j(n-p-1)+(p+1)(n-r-l)}{(n-r-l)(r+l+j)}\binom{p}{2l}\binom{n-p-2}{r+l-p-1}\binom{n-1}{r+l+j-1}.$$

PROOF. To obtain (6) we consider an $F_{2n,2k}^{2r+,2l,p(.),j}$ path. This path will consist of $2l$ S and one T_r^+ or p W and one T^+. On changing the signs of $\mathcal{Y}$'s for all such segments lying below the x-axis we get an $F_{2n,2k}^{(2r+2l)+,0,p(p),j}$ path. Thus

$$N(F_{2n,2k}^{2r+,2l,p(.),p}) = \binom{p}{2l} N(F_{2n,2k}^{(2r+2l)+,0,p(p),j}).$$

The factor $\binom{p}{2l}$ is due to the reason that any $2l$ V-points out of p can be made as intersections by taking reflections of the inbetween segments. The theorem now follows from (2).

For an ordinary path $F_{2n,2k}^{2r+,2l,p(.)}$, the corresponding expressions, got by putting $j=0$ in (6) and (7) are:

(8)
$$N(F_{2n,2k}^{2r+,2l,p(.)}) =$$

$$= \binom{p}{2l}\left[\binom{n+k-p-2}{r+l-p-1}\binom{n-k-1}{r+l-1} - \binom{n+k-p-2}{r+l-p-2}\binom{n-k-1}{r+l}\right],$$

(9) $$N(F_{2n,0}^{2r+,2l,p+1(.)}) = N(F_{2n,0}^{2r-,2l,p+1(.)}) = \frac{p+1}{r+l}\binom{p}{2l}\binom{n-p-2}{r+l-p-1}\binom{n-1}{r+l-1}.$$

DEDUCTIONS. (i) Putting $l=0$ in (9), we get

$$N(F_{2n,0}^{2r+,0,p+1(p+1)}) = N(F_{2n,0}^{2r-,0,p+1(0)}) = \frac{p+1}{r}\binom{n-p-2}{r-p-1}\binom{n-1}{r-1},$$

verifying result (5).

(ii) Summing (9) over $2l \leqq p \leqq r+l-1$, we get

$$N(F_{2n,0}^{2r+,2l}) = N(F_{2n,0}^{2r-,2l}) = \frac{2l+1}{n}\binom{n}{r-l-1}\binom{n}{r+l},$$

veryfying results (3.6) and (3.10) of [4].

(iii) Summing (9) over $p-l+1 \leqq r \leqq n-l$, we get

$$N(F_{2n,0}^{\cdot+,2l,p+1(\cdot)}) = N(F_{2n,0}^{\cdot-,2l,p+1(\cdot)}) = \frac{p+1}{2n-p-1}\binom{p}{2l}\binom{2n-p-1}{n},$$

verifying a result in [7].

(iv) Summing (9) over l, we get an important result, viz.,

$$N(F_{2n,0}^{2r+,\cdot,p+1(\cdot)}) = N(F_{2n,0}^{2r-,\cdot,p+1(\cdot)}) = \sum_l \frac{p+1}{r+l}\binom{p}{2l}\binom{n-p-2}{r+l-p-1}\binom{n-1}{r+l-1}. \tag{10}$$

THEOREM 3.

$$N(F_{2n,2k}^{(2r+1)-,2l+1,p(\cdot),j}) = \tag{11}$$

$$= \binom{p}{2l+1}\left[\binom{n+k-p-2}{r+l-p}\binom{n-k-1}{r+l+j} - \binom{n+k-p-2}{r+l-p-1}\binom{n-k-1}{r+l+j+1}\right],$$

$$N(F_{2n,0}^{(2r+1)-,2l+1,p+1(\cdot),j}) = \tag{12}$$

$$= \frac{j(n-p-1)+(p+1)(n-r-l-1)}{(n-r-l-1)(r+l+j+1)}\binom{p}{2l+1}\binom{n-p-2}{r+l-p}\binom{n-1}{r+l+j}.$$

PROOF. An $F_{2n,2k}^{(2r+1)-,2l+1,p(\cdot),j}$ path consists of l S^+, one T_r^+ and $(l+1)$ S^- (or p W and one T^+). On changing the signs of ϑ's for all such segments below the axis we get an $F_{2n,2k}^{2(r+l+1)+,0,p(p),j}$ path. Hence

$$N(F_{2n,2k}^{(2r+)-,2l+1,p(\cdot),j}) = \binom{p}{2l+1} N(F_{2n,2k}^{2(r+l+1)+,0,p(p),j}).$$

The theorem now follows from (2). For an ordinary path $F_{2n,2k}^{(2r+1)-,2l+1,p(\cdot)}$, the corresponding expressions, got by putting $j=0$ in (11) and (12) are:

$$N(F_{2n,2k}^{(2r+1)-,2l+1,p(\cdot)}) = \tag{13}$$

$$= \binom{p}{2l+1}\left[\binom{n+k-p-2}{r+l-p}\binom{n-k-1}{r+l} - \binom{n+k-p-2}{r+l-p-1}\binom{n-k-1}{r+l+1}\right],$$

$$N(F_{2n,0}^{(2r+1)-,2l+1,p+1(\cdot)}) = N(F_{2n,0}^{(2r+1)+,2l+1,p+1(\cdot)}) = \tag{14}$$

$$= \frac{p+1}{r+l+1}\binom{p}{2l+1}\binom{n-p-2}{r+l-p}\binom{n-1}{r+l}.$$

DEDUCTIONS. (i) Summing (14) over $2l+1 \leqq p \leqq r+l$, we get

$$N(F_{2n,0}^{(2r+1)-,2l+1}) = N(F_{2n,0}^{(2r+1)+,2l+1}) = \frac{2l+2}{n}\binom{n}{r-l-1}\binom{n}{r+l+1},$$

verifying results (3.8) and (3.12) of [4].

(ii) Summing (14) over $p-l \leqq r \leqq n-l-1$, we get

$$N(F_{2n,0}^{-,2l+1,p+1(\cdot)}) = N(F_{2n,0}^{+,2l+1,p+1(\cdot)}) = \frac{p+1}{2n-p-1}\binom{p}{2l+1}\binom{2n-p-1}{n},$$

verifying a result in [7].

(iii) Summing (14) over l, we get

$$N(F_{2n,0}^{(2r+1)-,\cdot,p+1(\cdot)}) = N(F_{2n,0}^{(2r+1)+,\cdot,p+1(\cdot)}) = \tag{15}$$

$$= \sum_l \frac{p+1}{r+l+1}\binom{p}{2l+1}\binom{n-p-2}{r+l-p}\binom{n-1}{r+l}.$$

THEOREM 4.

$$N(F_{2n,2k}^{2r+,2l,p(q),j}) = \tag{16}$$

$$= \binom{q}{l}\binom{p-q-1}{l-1}\left[\binom{n+k-p-2}{r+l-p-1}\binom{n-k-1}{r+l+j-1} - \binom{n+k-p-2}{r+l-p-2}\binom{n-k-1}{r+l+j}\right],$$

$$N(F_{2n,0}^{2r+,2l,p+1(q+1),j}) = \tag{17}$$

$$= \frac{j(n-p-1)+(p+1)(n-r-l)}{(n-r-l)(r+l+j)}\binom{q}{l}\binom{p-q-1}{l-1}\binom{n-p-2}{r+l-p-1}\binom{n-1}{r+l+j-1}.$$

PROOF. An $F_{2n,2k}^{2r+,2l,p(q),j}$ path consists of l S^+, one T_r^+ and l S^- (or q W^+, $(p-q)$ W^- and one T^+). On changing the signs of the ϑ's for all such segments below the axis we get an $F_{2n,2k}^{(2r+2l)+,0,p(p),j}$ path. Since l S^+, one T_r^+ and l S^- can be constructed out of q ordered W^+, one T^+ and $(p-q)$ ordered W^- in $\binom{q}{l}\binom{p-q-1}{l-1}$ ways. Hence

$$N(F_{2n,2k}^{2r+,2l,p(q),j}) = \binom{q}{l}\binom{p-q-1}{l-1} N(F_{2n,2k}^{(2r+2l)+,0,p(p),j}).$$

The theorem now follows from (2). For an ordinary path $F_{2n,2k}^{2r+,2l,p(q)}$, the corresponding expressions, got by putting $j=0$ in (16) and (17) are:

$$N(F_{2n,2k}^{2r+,2l,p(q)}) = \tag{18}$$

$$= \binom{q}{l}\binom{p-q-1}{l-1}\left[\binom{n+k-p-2}{r+l-p-1}\binom{n-k-1}{r+l-1} - \binom{n+k-p-2}{r+l-p-2}\binom{n-k-1}{r+l}\right],$$

$$N(F_{2n,0}^{2r+,2l,p+1(q+1)}) = \frac{p+1}{r+l}\binom{q}{l}\binom{p-q-1}{l-1}\binom{n-p-2}{r+l-p-1}\binom{n-1}{r+l-1}. \tag{19}$$

Similarly we can prove that

$$N(F_{2n,0}^{2r-,2l,p+1(q+1)}) = \frac{p+1}{r+l}\binom{q}{l-1}\binom{p-q-1}{l}\binom{n-p-2}{r+l-p-1}\binom{n-1}{r+l-1}. \tag{20}$$

DEDUCTIONS. (i) For $l=0$, (19) and (20) reduce to

$$N(F_{2n,0}^{2r+,0,p+1(p+1)}) = N(F_{2n,0}^{2r-,0,p+1(0)}) = \frac{p+1}{r}\binom{n-p-2}{r-p-1}\binom{n-1}{r-1},$$

verifying result (5).

(ii) Summing (19) over $l \leqq q \leqq p-l$ and (20) over $l-1 \leqq q \leqq p-l-1$ we get

$$N(F_{2n,0}^{2r+,2l,p+1(.)}) = N(F_{2n,0}^{2r-,2l,p+1(.)}) = \frac{p+1}{r+l}\binom{p}{2l}\binom{n-p-2}{r+l-p-1}\binom{n-1}{r+l-1},$$

verifying result (9).

(iii) Summing (19) and (20) over $p-l+1 \leqq r \leqq n-l$, we get

$$N(F_{2n,0}^{+,2l,p+1(q+1)}) = \frac{p+1}{2n-p-1}\binom{2n-p-1}{n}\binom{q}{l}\binom{p-q-1}{l-1},$$

$$N(F_{2n,0}^{-,2l,p+1(q+1)}) = \frac{p+1}{2n-p-1}\binom{2n-p-1}{n}\binom{q}{l-1}\binom{p-q-1}{l}.$$

(iv) Summing (19) and (20) over l, we get

$$N(F_{2n,0}^{2r+,\cdot,p+1(q+1)}) = \sum_l \frac{p+1}{r+l}\binom{q}{l}\binom{p-q-1}{l-1}\binom{n-p-2}{r+l-p-1}\binom{n-1}{r+l-1}, \tag{21}$$

$$N(F_{2n,0}^{2r-,\cdot,p+1(q+1)}) = \sum_l \frac{p+1}{r+l}\binom{q}{l-1}\binom{p-q-1}{l}\binom{n-p-2}{r+l-p-1}\binom{n-1}{r+l-1}. \tag{22}$$

THEOREM 5.

$$N(F_{2n,2k}^{(2r+1)-,2l+1,p(q),j}) = \tag{23}$$

$$= \binom{q}{l}\binom{p-q-1}{l}\left[\binom{n+k-p-2}{r+l-p}\binom{n-k-1}{r+l+j} - \binom{n+k-p-2}{r+l-p-1}\binom{n-k-1}{r+l+j+1}\right],$$

$$N(F_{2n,0}^{(2r+1)-,2l+1,p+1(q+1),j}) = \tag{24}$$

$$= \frac{j(n-p-1)+(p+1)(n-r-l-1)}{(n-r-l-1)(r+l+j+1)}\binom{q}{l}\binom{p-q-1}{l}\binom{n-p-2}{r+l-p}\binom{n-1}{r+l+j}.$$

PROOF. On changing the signs of the ϑ's for all such segments below the axis in an $F_{2n,2k}^{(2r+1)-,2l+1,p(q),j}$ path we get an $F_{2n,2k}^{2(r+l+1)+,0,p(p),j}$ path. Since l S^+, one T_r^+ and $(l+1)$ S^- can be constructed out of q ordered W^+, one T^+ and $(p-q)$ ordered W^- in $\binom{q}{l}\binom{p-q-1}{l}$ ways. Hence the theorem follows from (2). For an ordinary path $F_{2n,2k}^{(2r+1)-,2l+1,p(q)}$, the corresponding expressions are:

$$N(F_{2n,2k}^{(2r+1)-,2l+1,p(q)}) = \tag{25}$$

$$= \binom{q}{l}\binom{p-q-1}{l}\left[\binom{n+k-p-2}{r+l-p}\binom{n-k-1}{r+l} - \binom{n+k-p-2}{r+l-p-1}\binom{n-k-1}{r+l+1}\right],$$

$$N(F_{2n,0}^{(2r+1)-,2l+1,p+1(q+1)}) = N(F_{2n,0}^{(2r+1)+,2l+1,p+1(q+1)}) = \tag{26}$$

$$= \frac{p+1}{r+l+1}\binom{q}{l}\binom{p-q-1}{l}\binom{n-p-2}{r+l-p}\binom{n-1}{r+l}.$$

DEDUCTIONS. (i) Summing (26) over $l \leqq q \leqq p-l-1$, we get

$$N(F_{2n,0}^{(2r+1)-,2l+1,p+1(.)}) = N(F_{2n,0}^{(2r+1)+,2l+1,p+1(.)}) =$$

$$= \frac{p+1}{r+l+1}\binom{p}{2l+1}\binom{n-p-2}{r+l-p}\binom{n-1}{r+l},$$

verifying result (14).

(ii) Summing (26) over $p-l \leqq r \leqq n-l-1$, we get

$$N(F_{2n,0}^{\cdot-,2l+1,p+1(q+1)}) = N(F_{2n,0}^{\cdot+,2l+1,p+1(q+1)}) = \frac{p+1}{2n-p-1}\binom{2n-p-1}{n}\binom{q}{l}\binom{p-q-1}{l}.$$

(iii) Summing (26) over l, we get

$$N(F_{2n,0}^{(2r+1)-,.,p+1(q+1)}) = N(F_{2n,0}^{(2r+1)+,.,p+1(q+1)}) = \tag{27}$$

$$= \sum_l \frac{p+1}{r+l+1}\binom{q}{l}\binom{p-q-1}{l}\binom{n-p-2}{r+l-p}\binom{n-1}{r+l}.$$

The preceding give immediately the joint distributions of

$$(\lambda_{n+k,n-k}, \alpha_{n+k,n-k}, R_{n+k,n-k}),\ (\lambda_{n+k,n-k}, \alpha_{n+k,n-k}, \alpha^+_{n+k,n-k}, R_{n+k,n-k}),$$

$$(\alpha_{n+k,n-k}, R_{n+k,n-k}) \quad \text{and} \quad (\alpha_{n+k,n-k}, \alpha^+_{n+k,n-k}, R_{n+k,n-k}).$$

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SELF-AND SOCIAL OPTIMIZATION IN QUEUES

by

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Abstract. Following NAOR [5], YECHIALI [7], [8] and KUNDSEN [3], we consider a GI/M/c queue. After the service each customer gets a positive reward which is lessened by his waiting loss depending on his waiting time. The customers' strategy is defined by their common threshold-number, n: each customer joins the queue iff he observes less then n custormers in the queue at his arrival. The customers are selfish or collectivistic if they are maximizing private or collective net gains, alternatively. The corresponding threshold-numbers are n_s and n_0, respectively. If a toll collector levies tolls maximizing his own revenue, then the modified thresholdnumbers are $\bar{n}_s$ and $\bar{n}_0$, respectively. Under alternative assumptions we have proved that $\bar{n}_0 \leqq \bar{n}_s \leqq n_0 \leqq n_s$. The second and third inequalities were proved by the mentioned authors in different special cases through the determination of these values explicitly, contrary to our direct, mostly elementary and general method.

Introduction

Recently some authors investigated the alternatives of self- and social optimizations in a standard queueing model, namely, in GI/M/c and its special variants. Their main suggestion is that the self-optimization is seldom identical with the social optimization.

(i) Most of their results were proved for *special* cases. NAOR [5] dealt with M/M/1 (i.e. Poisson arrival, exponential service and one server), KNUDSEN [3] was concerned with its multi-server generalization: M/M/c. YECHIALI's first paper [7] analysed GI/M/1 and his second one [8] generalized this for GI/M/c, being the single paper which applied the same model as the present paper does. (Here *GI* stands for a general distribution of independent inter-arrival times.) At the same time [8] did not analyse the problem of $\bar{n}_s \leqq \bar{n}_0$. The first purpose of this paper is to generalize previous results, to "prove" Naor's following statement: "The results in qualitative form are independent of the specifics of the model". [5, p. 23.]

(ii) Beside these restrictions they proved the inequalities mentioned in the Abstract *indirectly,* with the aid of explicit formulae. Consequently, these proofs were often unnecessary restrictive and complicated. I am aware that my results are sometimes rather evident. But the results originally were proved so sophisticatedly that for example the triviality of the inequality $n_0 \leqq n_s$ was rather obscured.

(iii) The third purpose of this paper is to analyse further problems, e.g. to prove $\bar{n}_0 \leqq \bar{n}_s$, and its paradoxical character. Unfortunately, most of these new results are proved only for the special case of M/M/1.

1. The GI/M/c/n queue

We consider a multi-server queue with general interarrival time distribution and with exponential service time: GI/M/c.

We assume that the primary random variables (interarrival times and service times) are independent. Let $A(X)$ be the common distribution function of the interarrival times.

Let i_t be the number of the customers in the system just before the arrival of the t th customer, $t=1, 2\ldots$. An arriving customer can join or balk at his arrival but not later, depending only on the system-state i and the corresponding probabilities D_i and $1-D_i$, respectively. Assume that for some $n<\infty$, $D_n=0$.

Evidently, $\{i_t\}_{t=1}^{\infty}$ is an imbedded ergodic Markov chain. Let R be the customer's reward which he gets at the completion of his service and let $h(\omega)$ be the waiting loss — waiting time function, which is non-negative and non-decreasing. (This assumption was used by Knudsen, generalizing the assumption of $h(\omega)=\omega$, used by all the other authors.) Here ω denotes the waiting time (or more precisely: the time in system) of a customer. Let ω_i be the random waiting time of a customer who observes i (other) customers in the system at his arrival and joins; and let W_i be the corresponding waiting loss: $W_i=Eh(\omega_i)$. If $h(\omega)=\omega$, then W_i is equal to $w_i=b$ $(0\leqq i<c)$ and $w_i=\dfrac{i+1}{c}b$ $(c\leqq i)$ where $b=E\tau$ is the expected service time.

Evidently, $W_i\leqq W_{i+1}$ and $W_i=Eh(\tau)$ if $0\leqq i\leqq c-1$. We assume $W_0<R<W_\infty$, which holds for w_i's.

We need the following definition and notation: Let n be a positive integer, and let us assume that $D_i=1$ if $i<n$ and $D_n=0$ if $i\geqq n$. We shall denote this policy by S_n and we shall refer to n as the *threshold-number* or maximal queue length. Usually the order of the services is the order of the arrivals.

2. A lemma

We need the following notations: The admission rate (i.e. utilization) $p(n)$ is the probability that an arriving customer joins the queue when the threshold-number is n. The corresponding conditional expected waiting time $w(n)$ is the expected waiting time of the joining customer. We shall prove the following

LEMMA. *$p(n)$ and $w(n)$ are increasing, but* $w(n+1)<w(n)+\dfrac{b}{c}$.

PROOF. Consider the three strategies S_n, S_{n+1}, and S_{n+1}^*. S_{n+1}^* is the following modification of S_{n+1}: A customer who observes n customers in the queue at his arrival is "polite" unless another polite customer is waiting and he will be served only when no other customer is waiting for service, and will be preempted if necessary, resuming service.

Under these strategies let $i(x, S_n)$, $i(x, S_{n+1})$ and $i(x, S_{n+1}^*)$ be the number of customers in the system at time x. Then, for fixed arrival and service times, we have $i(x, S_n)\leqq i(x, S_{n+1}^*)$ for each x. Since the different service order has no effect on the process (exponential service times!), the processes $i(x, S_{n+1})$ and $i(x, S_{n+1}^*)$ have the same distribution.

This establishes a one-to-one correspondence between the outcomes under S_n and S_{n+1}, having the same probability and the property $i(x, S_n) \leqq i(x, S_{n+1})$ for all x. Now it can be seen that for a fixed realization, each customer that joins under S_n will also join under S_{n+1}^*, but not vice versa, since polite customers will join under S_{n+1}^* but not under S_n. Hence $p(n) < p(n+1)$.

Similarly, $w(n) < w(n+1)$, because for a fixed realization, arriving customers who join under both S_n and S_{n+1}^* experience the same waiting time under both rules, while those customers who do not join under S_n, but join under S_{n+1}^* as "polite" customers have an expected waiting time exceeding $(n+1)b/c$ which in turn is greater than $w(n)$.

To prove $w(n+1) < w(n) + \frac{b}{c}$ we need the following result due to TAKÁCS [6]: Let $p_i(n)$ be the limiting probability that just before an arrival i customers were in the system, $0 \leqq i \leqq n$. Then $p_n(n)$ is the probability that the arriving customer balks: $p(n) = 1 - p_n(n)$. For each c there exists a sequence $\{q_i\}_{i=0}^{\infty}$ of positive numbers and the conditional arrival probabilities of joining are given by

$$\tilde{p}_i(n) = \frac{1 - p(n)}{p(n)} q_{n-i}, \quad c \leqq i \leqq n-1.$$

(Since $\tilde{p}_i(n)$ $(0 \leqq i \leqq c-1)$ is not needed it is omitted.)

This result is a simple consequence of the much more complicated original one (8) [6, p. 75] and it directly follows from the intermediate formula (16) [6, p. 76]. This is the single point where we really need non-elementary devices!

By $p(n) < p(n+1)$ our last formula implies $\tilde{p}_i(n) > \tilde{p}_{i+1}(n+1)$, $c \leqq i \leqq n-1$. By definition the mean queuing-time $\tilde{w}(n) = w(n) - b$ is equal to

$$\tilde{w}(n) = \sum_{i+c}^{n-1} \tilde{p}_i(n) \frac{b}{c} (i - c + 1).$$

Similarly,

$$\tilde{w}(n+1) = \sum_{i=c}^{n} \tilde{p}_i(n+1) \frac{b}{c} (i - c + 1).$$

Thus

$$\tilde{w}(n+1) = \sum_{i=c}^{n} \tilde{p}_i(n+1) \frac{b}{c} + \sum_{i=1}^{n-1} \tilde{p}_{i+1}(n+1) \frac{b}{c} (i - c + 1)$$

where the first summand is smaller than $\frac{b}{c}$ because of $\sum_{i=1}^{n} \tilde{p}_i(n+1) < 1$ and the second one is smaller than $\tilde{w}(n)$, because of $\tilde{p}_i(n) > \tilde{p}_{i+1}(n+1)$. Hence $w(n+1) < w(n) + \frac{b}{c}$. Q.E.D.

3. Self-optimization

By definition, under *self-optimization* an arriving customer joins the queue iff $w_i \leqq R$. I.e. *he joins the queue iff the queue size* (= the number of customers in the system) *was smaller than* n_s *just before his arrival:* $n_s = \max\{i, w_{i-1} \leqq R\}$. Thus $D_i = 1$ if $i < n_s$ and $D_{n_s} = 0$.

As Naor remarks, n_s does not depend on the arrival rate, ... "throwing serious doubt on the social optimality of the strategy n_s". [5, p. 19].

4. Social optimization

In the case of *social optimization* the customers' expected benefit should be maximized. (We remark that here customers mean joining *and* balking ones.) Now it is not so evident that any optimal strategy exists at all and if it exists then it is as simple as the self-optimal one. (Cf. YECHIALI [7] and [8].) But it is evident that *a strategy is not socially optimal if it prescribes to wait for more than* $n_s - 1$ *customers with positive probability*. Namely, if the t th customers joins and $i_t \geqq n_s$, then the t th customer's benefit would be negative because of $W_{i_t} > R$. Thus his balking is advantageous for him (zero benefit) as well as for those customers who arrive later and should have waited for him.

By a common theorem of BLACKWELL [1] and DERMAN [2], for a finite state space with a finite number of actions, there exists a non-randomized optimal strategy. Let us denote it by $\{D_i\}_{i=0}^{n_s}$ and let j be the first (i.e. minimal) integer for which $D_j = 0$. Evidently, now $D_i = 0$ if $i \geqq j$, while $D_i = 1$ if $i < j$. Thus *the socially optimal strategy is determined by a thresholdnumber again,* say $n_0 = j$. Of course, $n_0 \leqq n_s$. We have

PROPOSITION I. *The (maximal) queue length and the admission rate at social optimization is generally smaller and never larger than at self optimization:* $n_0 \leqq n_s$.

REMARK. Proposition I is the generalization of the corresponding statements, see [5, p. 20 and p. 23], [7, Theorem 7, p. 361], [3, Theorem 1, p. 520] and [8], Theorem 3, p. 440]. Usually $n_0 < n_s$. (Cf. [5] and Section 6 of this paper.)

5. Toll charges

Until now the customer's reward was fixed. What happens if a toll collecting agency levies (extra) *toll* for each service? Then R is replaced by $R - \theta$, n_s by $n_s(\theta)$ and n_0 by $n_0(\theta)$. Evidently, $n_s(\theta) = \max\{i, W_{i-1} \leqq R - \theta\}$, i.e. $n_s(\theta) = n$ iff $\theta \in (\theta_{n+1}, \theta_n]$ where $\theta_n = R - W_{n-1}$, $(1 \leqq) c \leqq n \leqq n_s$.

If the toll collecting is controlled by the collectivistic customers, then levying toll $\theta^* \in (\theta_{n_0+1}, \theta_{n_0}]$ reduces the self-optimal threshold-number to the socially optimal one: $n_s(\theta^*) = n_0$. (Cf. [5, (26), p. 22.], [7, (30) p. 364], [8, p. 442] and [3, (5.1) p. 521].)

"Clearly, if the toll revenue may be used for redistribution of income among the population ... the proposed imposition of tolls is an optimal procedure." [5, p. 22.].

But... "the toll collecting agency may be completely divorced from the individual and the collectivistic interests of the customers. In that case the agency will seek to impose a toll"... ($\bar{\theta}_s$ or $\bar{\theta}_0$ respectively, — A. S.) ... "designed to maximize its own revenue rather than to optimize the whole system". [5, p. 22]. As a matter of fact Naor confined his attention to the exploitation of selfish customers by a toll-collector, and it was YECHIALI [7], who introduced the exploitation of collectivistic customers.

Let $g(\theta, n)$ be the agency's revenue per arriving customer when the toll is θ and the strategy is S_n. In the case of customers' self-optimization the agency maximizes $g_s(\theta)=g(\theta, n_s(\theta))$ and in the case of social optimization it maximizes $g_0(\theta)=g(\theta, n_0(\theta))$; the corresponding tolls are denoted by $\bar{\theta}_s$ and $\bar{\theta}_0$, respectively. Let $f(\theta, n), f_s(\theta)$ and $f_0(\theta)$ be the corresponding net gains per customer. To maximize its revenue the agency must choose a toll at a discontinuity point of $n_s(\theta)$ and $n_0(\theta)$, respectively.

First we remark that both $n_s(\theta)$ and $n_0(\theta)$ are non-increasing. Our first statement is trivial. To prove the second one let $\theta'>\theta$. By definition $f(\theta', n)=f(\theta, n)-p(n)(\theta'-\theta)$. Since $f(\theta, n)\leqq f(\theta, n_0(\theta))$ and $p(n)(\theta'-\theta)>p(n_0(\theta))(\theta'-\theta)$ if $n>n_0(\theta)$, $f(\theta', n)<f(\theta', n_0(\theta))$ if $n>n_0(\theta)$. Hence $n_0(\theta')\leqq n_0(\theta)$.

By the way, $\theta'>\theta$ implies $f(\theta', n)<f(\theta, n)$. Hence the same relation holds for their maxima: $f_0(\theta')<f_0(\theta)$.

1. Let us begin with the *selfish* customers. Studying the behavior of $f_s(\theta)$, we can prove that $f_s(\theta_n)<f_s(\theta_n+0)$ if $W_0<W_{n-1}$. Indeed, a very small increase of θ at θ_n (say by ε) excludes the service of only those customers, whose net gain was zero while it reduces the waiting times (and losses) of the others by $b/c-\varepsilon$ (or $W_i-W_{i-1}-v(\varepsilon)$, where $\lim_{\varepsilon\to 0} v(\varepsilon)/\varepsilon=0$).

This paradox (i.e. an increase in toll increases the customer's expected net gain) also points out the social disadvantage of the self-optimization.

We need the following notation; $f_s\{n\}=f_s(\theta_n)$ and $g_s\{n\}=g_s(\theta_n)$. These sequences substitute the whole functions $f_s(\theta)$ and $g_s(\theta)$ in this problem.

We shall prove that *under* $h(\omega)=\omega$, $f_s\{n\}$ *is increasing*, i.e. the paradox disappears at this point. *From now on* $h(\omega)=\omega$.

By $h(\omega)=\omega$, $\theta_n=R-\dfrac{nb}{c}$, hence $f_s\{n\}=p(n)\left[\dfrac{nb}{c}-w(n)\right]$. By our Lemma

$$0<p(n)<p(n+1),\quad 0<\frac{nb}{c}-w(n)<\frac{(n+1)b}{c}-w(n+1),\quad \text{thus } f_s\{n\}<f_s\{n+1\}.$$

PROPOSITION II. *If the toll collecting agency tends to maximize its own revenue, the tolls levied under self-optimization will generally be higher and never lower than that under social optimization and the socially optimal utilization will generally not be attained.*

In mathematical form: for GI/M/c $\bar{\theta}_s\geqq\theta_{n_0}$ and $\bar{n}_s\leqq n_0$.

REMARK. This proposition is the generalization of NAOR's statement [5. p. 23] and related to KNUDSEN's Theorem 2 [3, p. 523.] (The present result is neither more general nor less general then Knudsen's Theorem 2.)

PROOF. If the agency worked for the collective interest of the customers, then the net gain would be $f(\theta)=f_s(\theta)+g_s(\theta)$. As $(\theta_{n_0+1}, \theta_{n_0}]$ is the interval of the

optimal tolls for the collectivisitic customers, $\theta \leqq \theta_{n_0+1}$ implies $f(\theta) < f(\theta_{n_0})$, i.e. $n > n_0$ implies $f\{n\} < f\{n_0\}$. We have seen that $n > n_0$ implies $f_s\{n\} > f_s\{n_0\}$. Consequently, $n > n_0$ implies $g_s\{n\} < g_s\{n_0\}$. As $g_s\{n_0\} \leqq g_s\{\bar{n}_s\}$, $\bar{n}_s \leqq n_0$. Further, $\bar{\theta}_s = \theta_{\bar{n}_s} \geqq \theta_{n_0}$.

2. Now we turn to the *collectivistic* customers. According to Proposition I, $n_0(\theta) \leqq n_s(\theta)$, hence $p(n_0(\theta)) \leqq p(n_s(\theta))$ and $w(n_0(\theta)) \leqq w(n_s(\theta))$ by our Lemma. Thus $g_0(\theta) \leqq g_s(\theta)$ and similarly for their maxima: $g_0(\bar{\theta}_0) \leqq g_s(\bar{\theta}_s)$; i.e. *the customers' cooperation does not increase the agency's toll revenue.* Similar questions arise for the optimal tolls, for the corresponding thresholdnumbers and for the corresponding customers' expected benefits. This problem is much more difficult than the former ones. Actually I was able to prove only

PROPOSITION III. *For $M/M/1$ the "optimal" utilisation of the system by the collectivistic customers is generally smaller and never greater than that by the individualistic customers: $\bar{n}_0 \leqq \bar{n}_s$.*

PROOF. Naor's (19) and (22) give the condition of $f_0(n, \theta) \leqq f_0(n, \theta)$. From (22) it is easy to see that

$$n_0(\theta) = \max\left\{n; \sum_{i=0}^{n-1} (n-i)(\lambda b)^i \leqq \frac{R-\theta}{b}\right\}$$

and for each n; $1 \leqq n \leqq n_0$, there exists θ_n^*, for which $\theta \in (\theta_{n+1}^*, \theta_n^*]$ implies $n_0(\theta) = n$. Moreover, θ_n^* is determined by $f_0(n, \theta_n^*) = f_0(n-1, \theta_n^*)$, i.e. by

$$\theta_n^* = R - b \sum_{i=1}^{n-1} (n-i)(\lambda b)^i.$$

It is evident that $\theta_n - \theta_n^* > 0$ increases by an increase of n. Since $g_0\{n\} = g_s\{n\} - p(n)(\theta_n - \theta_n^*)$ where $g_0\{n\} = g_0(\theta_n^*)$, by our standard method $\bar{n}_0 \leqq \bar{n}_s$. Q.E.D.

It looks like evident that $\theta_n - \theta_n^*$ increases in general which would prove

CONJECTURE I. *Proposition III. holds in general; for $GI/M/c$: $\bar{n}_0 \leqq \bar{n}_s$.*

Our last result is of doubtful value, because it does not tell us whether the smaller utilization is the result of an advantageous collectivistic self-restraint, (as in $n_0 < n_s$), or a disadvantageous "exploitation" by the toll collector (as in $\bar{n}_s < n_0$). In the first case $\bar{f}_0 > \bar{f}_s$:, in the second one $\bar{f}_0 < \bar{f}_s$. (The neutral third case is not interesting.) In the case of $\bar{f}_0 < \bar{f}_s$ the customers' cooperation against the toll-collector is not advantageous, throwing serious doubt on the relevance of our assumption, according to which the customers choice firstly and the toll-collector secondly. We have the following

COUNTER-EXAMPLE ($\bar{f}_0 < \bar{f}_s$). Let $c = \lambda = g = b = 1$. Then $p(n) = n/(n+1)$ (see: (7) in [5]), implying

$$g_s\{n\} = n(R-n)/(n+1) \quad \text{and} \quad g_0\{n\} = n\big(R - n(n+1)/2\big)/(n+1).$$

Let $5 < R < 6$. Then $n_s = 5$ and $n_0 = 2$. By Proposition II $\bar{n}_s \leqq 2$ and anyway $\bar{n}_0 \leqq 2$. By simple calculation we get $g_s\{1\} < g_s\{2\}$ and $g_0\{1\} > g_0\{2\}$, i.e. $\bar{n}_s = 2$ and $\bar{n}_0 = 1$. Therefore $\bar{\theta}_0 = \theta_1$ and $\bar{f}_0 = 0$ while $\bar{\theta}_s = \theta_2$ and $\bar{f}_s > 0$.

6. The role of the arrival intensity

Until now the arrival intensity was fixed. Now we concentrate on the influence of λ on the main characteristics of the system to analyse some further problems. Especially the two extreme cases are interesting because of their simplicity:

1. $\lambda \sim 0$,
2. $\lambda \sim \infty$.

We shall assume that the analysed inter-arrival time distributions change linearly; i.e. $A_\lambda(x) = A(\lambda x)$ where the intensity of any random renewal process with distribution function $A(x)$ is 1. To avoid further complications we assume that $R > n_s b$. Under the assumptions of Proposition II we shall prove the following asymptotic results:

PROPOSITION IV. 1. *If λ is less than a certain λ^* depending on R, c and b, then* $n_0[\lambda] = n_s$.

2. *If λ is greater than a certain λ^{**} depending on R, c and b, then* $n_0[\lambda] = c$.

In both cases $\bar{n}_0[\lambda] = \bar{n}_s[\lambda] = c$, $\bar{\theta}_0[\lambda] = \bar{\theta}_s[\lambda]$ *and* $\bar{f}_0[\lambda] = \bar{f}_s[\lambda] = 0$.

REMARK. By Proposition IV, Conjecture I is proved for the extreme arrival intensities, when the toll collector totally exploits the customers, regardless of their organization.

PROOF. 1. If λ is very small, then the collective has no reason to divert a newly arrived customer with positive expected benefit (i.e. if $i_t < n_s$), because the probability is almost zero that the next customer arrives before the completion of this customer's service. Hence $n_0[\lambda] = n_s$ if $\lambda < \lambda_1$.

Let $p[n, c, \lambda]$ be the probability of the event that at an arrival at most $c-1$ customers were in the system, i.e. nobody will wait for service. Then $p[n, \lambda] \sim \sim p[n, c, \lambda]$ asymptotic equality holds if $\lambda \sim 0$. Now $g_s[n, \lambda] \sim p[n, c, \lambda] \cdot (R - w_n)$. To maximize $g_s[n, \lambda]$, $\bar{n}_s[\lambda] = c$ must prevail if $\lambda < \lambda_2$. Let $\lambda^* = \min(\lambda_1, \lambda_2)$ and $\lambda < \lambda^*$. Evidently, for $\theta > 0$ $n_0[\theta, \lambda] = n_s[\theta, \lambda]$ remains valid. Hence $\bar{n}_0[\lambda] = \bar{n}_s$, implying $\bar{\theta}_0[\lambda] = \bar{\theta}_s[\lambda] = R - b$, etc.

2. Since the expected service time is b, during an interval of length T at most $c\,T/b$ customers can be served in avarage.. Thus the maximal expected gross gain of the customers during this period is at most cRT/b. During each service the expected waiting time is at least b, during the period at least T.

This minimum net gain is arbitrarily approximated if λ is very large and $n = c$. Thus $n_0[\lambda] = c$ if $\lambda > \lambda^{**}$. Since $c \leqq \bar{n}_0[\lambda] \leqq n_0[\lambda]$, $n_0[\lambda] = c$. Similarly, $c \leqq \bar{n}_s[\lambda] \leqq \leqq n_0[\lambda]$ (Proposition II) implies $\bar{n}_s[\lambda] = c$.

REMARK. We have already noticed that for the M/M/1 queue $n_0[\lambda] = = \max\left\{n; \sum_{i=0}^{n-1} (n-i)(\lambda b)^i \leqq \frac{R}{b}\right\}$. This formula implies that $n_0[\lambda]$ is a non-increasing function of λ.

In general it seems to be evident that the increase of the arrival intensity never increases the frequency of joining compared to that of balking. This suggests

CONJECTURE II. *The threshold-number $n_0[\lambda]$ is a non-increasing function of the arrival intensity.*

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COMPLETE SYSTEMS OF EIGENFUNCTIONS OF THE WAVE EQUATION IN SOME SPECIAL CASES

by

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To the ninetieth birthday of Professor G. Pólya.

1. Let us consider the differential equation

$$u_{xx}+u_{yy}+\lambda u = 0 \tag{1.1}$$

with the boundary condition

$$u = 0 \quad \text{on} \quad B, \tag{1.2}$$

where B is a closed curve.

In the case when B is the boundary of the square S: $0 \leqq x, y \leqq \pi$, complete systems of eigenfunctions and eigenvalues are

$$u_{mn} = \sin mx \sin ny \tag{1.3}$$

and

$$\lambda_{mn} = m^2+n^2, \tag{1.4}$$

respectively, where $m, n = 1, 2, 3, \ldots$

If B is the boundary of the isosceles rectangular triangle T: $0 \leqq y \leqq x \leqq \pi$, it is known that the functions

$$u^a_{mn} = \sin mx \sin ny - \sin nx \sin my \quad (m > n > 0) \tag{1.5}$$

are eigenfunctions of the eigenvalue problem (1.1), (1.2) with eigenvalues $\lambda_{mn} = m^2+n^2$ $(m>n>0)$.

In this section we shall give a verification of this fact and prove the orthogonality and completeness of the system (1.5). Our considerations are based on a symmetry property of the functions (1.5).

We begin with the observation that the eigenvalues (1.4) of the boundary problem of the square are of multiplicity at least 2, if $m \neq n$. Indeed $\lambda_{mn} = \lambda_{nm}$ and from this follows that the functions $\alpha u_{mn} + \beta u_{nm}$ (α, β constants), too, are eigenfunctions of the square S. In particular

$$u^s_{mn} = u_{mn} + u_{nm} \tag{1.6}$$

and

$$u^a_{mn} = u_{mn} - u_{nm} \tag{1.7}$$

are eigenfunctions.

We shall make use of the following

DEFINITION. If $f(x, y)$ is defined on S and $f(x, y)=f(y, x)$, then it is a *symmetric function* of x and y. If, however, $f(x, y)=-f(y, x)$, then $f(x, y)$ is *antisymmetric.*

Along the straight line $x=y$ the value of an antisymmetric function is 0. Similarly the normal derivative of a symmetric function with respect to the line $x=y$, if it exists, vanishes on that line. The functions (1.6) and (1.7) are symmetric and antisymmetric, respectively. Thus the elements of the set of functions (1.7) vanish not only on the boundary of the square S, but also on the hypotenuse of T, so they are eigenfunctions of the first boundary problem connected with T. Since $u^a_{mn}=-u^a_{mn}$ (in particular $u^a_{mm}=0$) we can restrict ourselves to the set (1.5) of the eigenfunctions of the triangle T.

If f^a, g^a, h^a are antisymmetric functions integrable in S, then trivially

$$\iint_S f^a\,dx\,dy = 0, \quad \iint_S g^a h^a\,dx\,dy = 2\iint_T g^a h^a\,dx\,dy. \tag{1.8}$$

Since the spectrum of the boundary problem of the isosceles rectangular triangle may contain multiple eigenvalues, it is not obvious that the system (1.5) is an orthogonal one. Yet we have

$$I = \iint_T u^a_{mn} u^a_{m'n'}\,dx\,dy = 0, \quad \text{if} \quad |m-m'|+|n-n'| \neq 0. \tag{1.9}$$

Indeed by (1.7) and (1.8)

$$2I = \iint_S (u_{mn}-u_{nm})(u_{m'n'}-u_{n'm'})\,dx\,dy = \tag{1.10}$$

$$= \iint_S (u_{mn}u_{m'n'}+u_{nm}u_{n'm'})\,dx\,dy - \iint_S (u_{mn}u_{n'm'}+u_{m'n'}u_{nm})\,dx\,dy.$$

Since $m>n$ and $m'>n'$, by the known orthogonality properties of the functions (1.3) the last integral vanishes and so does the last but one, too, save if $m=m'$, $n=n'$.

The system (1.5) is complete in T, i.e. if $\varphi(x, y)$ is an L^2 integrable function in T and it is orthogonal in T to every function of the set (1.5), then

$$\varphi(x, y) = 0 \quad \text{a.e.} \tag{1.11}$$

For seeing this let us extend the definition of $\varphi(x, y)$ from T to S so that the extended function $\varphi^a=\varphi^a(x, y)$ should be antisymmetric. φ^a is L^2 integrable in S, thus by (1.8) and by the assumption on φ we have

$$\iint_S \varphi^a u^s_{mn}\,dx\,dy = 0, \tag{1.12}$$

$$\iint_S \varphi^a u^a_{mn}\,dx\,dy = 2\iint_T \varphi u^a_{mn}\,dx\,dy = 0. \tag{1.13}$$

The last two formulas, when added and subtracted show that φ^a is orthogonal in the square to every function of the complete system (1.3), hence the statement (1.11) follows.

2. Let us now consider the second eigenvalue problem connected with the two dimensional wave equation, i.e.

$$v_{xx}+v_{yy}+\lambda v = 0, \tag{2.1}$$

$$\partial v/\partial_{\mathbf{n}} = 0 \quad \text{on} \quad B, \tag{2.2}$$

where $\partial v/\partial \mathbf{n}$ means the normal derivative.

If B is the boundary of the square S, a complete orthogonal system of eigenfunctions in S is

$$v_{mn} = \cos mx \cos ny \tag{2.3}$$

with eigenvalues (1.4), where $m, n = 0, 1, 2, \ldots$

If, on the other hand, B is the boundary of T, then by the same considerations as in Section 1 we get that the symmetric functions

$$v^s_{mn} = v_{mn}+v_{nm} \quad (m \geqq n \geqq 0) \tag{2.4}$$

constitute a complete orthogonal system of eigenfunctions on T with corresponding eigenvalues $\lambda_{mn}=m^2+n^2$ $(m \geqq n \geqq 0)$. Note that the extension of the function analogous to φ in Section 1 should now be a symmetric function in order to have the same symmetry property as the eigenfunctions (2.4).

3. B. Sz.-Nagy raised the question whether the foregoing reasoning could be generalized to m dimensions. In the following we shall give an account of one of the possible generalizations. We note that the proof in this section is longer than that in Section 1. In the two dimensional case the analysis was short, because one could in a natural way find classes of functions determined by their symmetry properties so that (a) each function of two variables could be represented by a sum of functions belonging to different classes, (b) two functions belonging to different classes were orthogonal in S. This, however, seems no longer to be possible in higher dimensions.

Let us first recall the definition of the determinant and the permanent of a quadratic matrix $[a_{ij}]_{i,j=1}^m$:

$$\det [a_{ij}]_{i,j=1}^m = \sum (-1)^\varepsilon a_{1i_1} \ldots a_{mi_m} \quad \text{and} \quad \operatorname{per} [a_{ij}]_{i,j=1}^m = \sum a_{1i_1} \ldots a_{mi_m},$$

where $i_1, i_2, \ldots, i_m$ is a permutation of the numbers $1, 2, \ldots, m$, ε is the parity of this permutation and the summation is extended to all possible permutations.

Let further be

$$\Delta = \sum_{i=1}^m \frac{\partial^2}{\partial x_i^2},$$

S_m and $B(S_m)$ the interior and the boundary, respectively, of the cube $0 \leqq x_i \leqq \pi$ $(i=1, 2, \ldots, m)$, T_m and $B(T_m)$ the interior and the boundary of the simplex $0 \leqq x_1 \leqq$ $\leqq x_2 \leqq \ldots \leqq x_m \leqq \pi$, finally $\partial/\partial \mathbf{n}$ differentiation in the direction of the normal vector

in a point of the boundary. It is known that complete solutions of the eigenvalue problems

(3.1) $$\Delta w+\lambda w=0 \quad \text{in} \quad S_m, \quad w=0 \quad \text{on} \quad B(S_m)$$

and

(3.2) $$\Delta w+\lambda w=0 \quad \text{in} \quad S_m, \quad \partial w/\partial \mathbf{n}=0 \quad \text{on} \quad B(S_m)$$

are

(3.3a) $$w^1=\prod_{i=1}^{m} \sin n_i x_i \quad (n_i=1, 2, 3, \ldots)$$

and

(3.3b) $$w^2=\prod_{i=1}^{m} \cos n_i x_i \quad (n_i=0, 1, 2, \ldots),$$

respectively, with eigenvalues Σn_i^2.

We state that *complete solutions of the first and second boundary problem with respect to the simplex* T_m *are*

(3.4a) $$\det [\sin n_i x_j]_{i,j=1}^{m} \quad (0<n_1<\ldots<n_m)$$

and

(3.4b) $$\operatorname{per} [\cos n_i x_j]_{i,j=1}^{m} \quad (0 \leqq n_1 \leqq \ldots \leqq n_m),$$

respectively, with eigenvalues Σn_i^2, where the n_i's are integers.

In finding the functions (3.4a) and (3.4b) a lemma of COURANT [1], [2, vol. I. page 395] is useful, an obvious extension of a special case of which is the following:

Let D be a domain in R^m, having a plane face F. If w is an entire function of its variables $x_1, x_2, \ldots, x_m$, and it is a solution of $\Delta w+\lambda w=0$, P' is the reflexion of the arbitrary point P across the hyperplane of F, finally either $w\equiv 0$ on F or $\partial w/\partial \mathbf{n}\equiv 0$ on F, then $w(P)=-w(P')$ and $w(P)=w(P')$, respectively.[1]

A consequence of the lemma is that if Π is a polygonal domain in R^m, Π' the reflexion of Π across one of its faces, w is an entire function which is the solution of the first or second boundary problem in Π, then w is a solution of the same boundary problem in Π' and $\Pi\cup\Pi'$.

The planes $x_i=x_j$ $(i\neq j)$ dissect the cube S_m into $m!$ simplexes one of which is T_m. Two adjacent simplexes are reflexions of each other across their common faces. If either the first or the second boundary problem with respect to T_m admits a solution which is an entire function of all of its variables, then this function is

[1] To render this paper independent of existing literature we give a proof of this lemma. In the special case when F is the hyperplane $x_1=0$ the lemma states, that if $w|_{x_1=0}\equiv 0$ $[\partial w/\partial x_1|_{x_1=0}\equiv 0]$ then w is an odd [even] function of x_1. Indeed, if $w=\Sigma w_k x_1^k$, where $w_k=w_k(x_2, \ldots, x_m)$ are entire functions of their variables, we have inserting the power series in (3.1) and equating coefficients, that

$$(k+2)(k+1)w_{k+2}+\Delta w_k+\lambda w_k=0 \quad (k=0, 1, 2, \ldots)$$

Hence, if $w|_{x_1=0}=w_0\equiv 0$, then $w_{2\nu}\equiv 0$ [if $\partial w/\partial x_1|_{x_1=0}=w_1\equiv 0$, then $w_{2\nu+1}\equiv 0$], $\nu=1, 2, 3, \ldots$.

If F is an arbitrary hyperplane, one can always translate and rotate the system of coordinates in such a way that F becomes the hyperplane $x_1'=0$ of a new system of coordinates $x_1', x_2', \ldots, x_m'$.

an eigenfunction of the same boundary problem with respect to any of the remaining $m!-1$ simplexes and to the cube S_m, as well. Moreover, it is necessarily an antisymmetric [symmetric] eigenfunction of S_m in any two of the variables x_i.

Now, for any permutation of the numbers $n_1, n_2, \ldots, n_m$ the functions (3.3a) and (3.3b) remain eigenfunctions of S_m with unchanged eigenvalues. Any linear combination of these $m!$ functions is an eigenfunction of S_m and plainly the determinants (3.4a) [the permanents (3.4b)] are antisymmetric [symmetric] eigenfunctions of S_m and, meanwhile, they are eigenfunctions of T_m, too.

Now we proceed to the investigation of the orthogonality and completeness of the systems (3.4a) and (3.4b) .We restrict ourselves to the three dimensional case and to the first boundary problem.

The planes $x_1=x_2$, $x_1=x_3$, $x_2=x_3$ dissect the cube S_3 into six tetrahedrons, one of which is T_3. Out of these six tetrahedrons let us select two adjacent ones, say T_3' and T_3''. Since T_3' and T_3'' are reflections of each other across their common faces, in view of the antisymmetry property of the functions

$$w^a_{n_1 n_2 n_3} = w^a_{n_1 n_2 n_3}(x_1, x_2, x_3) = \begin{vmatrix} \sin n_1 x_1 & \ldots & \sin n_1 x_3 \\ \sin n_2 x_1 & \ldots & \sin n_2 x_3 \\ \sin n_3 x_1 & \ldots & \sin n_3 x_3 \end{vmatrix} \tag{3.5}$$

$$(0 < n_1 < n_2 < n_3)$$

we have that

$$\iiint_{T_3'} w^a_{n_1 n_2 n_3} w^a_{m_1 m_2 m_3}\, dx_1\, dx_2\, dx_3 = \iiint_{T_3''} w^a_{n_1 n_2 n_3} w^a_{m_1 m_2 m_3}\, dx_1\, dx_2\, dx_3. \tag{3.6}$$

Hence if δ_{ij} is Kronecker's delta, we have

$$\iiint_{T_3} w^a_{n_1 n_2 n_3} w^a_{m_1 m_2 m_3}\, dx_1\, dx_2\, dx_3 = \tag{3.7}$$

$$= \frac{1}{3!} \iiint_{S_3} w^a_{n_1 n_2 n_3} w^a_{m_1 m_2 m_3}\, dx_1\, dx_2\, dx_3 = A \prod_{i=1}^{3} \delta_{n_i m_i}$$

by the orthogonality properties of the functions (3.3). The positive number A is independent of the choice of n_1, n_2, n_3.

This proves the orthogonality of the system of functions $w^a_{n_1 n_2 n_3}$ in T_3.

For investigating the completeness of the same system we introduce again a function $\varphi \in L_2$ in T_3 and assume that in T_3 φ is orthogonal to any function $w^a_{n_1 n_2 n_3}$. Next we extend the definition of φ to S_3 so that the extended function φ^a should be antisymmetric in any two of its variables: $\varphi^a(x_1, x_2, x_3) = -\varphi^a(x_2, x_1, x_3) = -\varphi^a(x_1, x_3, x_2)$. Let us expand the function φ^a with respect to the orthogonal system (3.3a). Then we have

$$\varphi^a = \sum_{\nu_1, \nu_2, \nu_3 = 1}^{\infty} c_{\nu_1 \nu_2 \nu_3} w^1_{\nu_1 \nu_2 \nu_3} \quad \text{a.e.} \tag{3.8}$$

where

$$c_{\nu_1 \nu_2 \nu_3} = \left(\frac{2}{\pi}\right)^3 \iiint_{S_3} \varphi^a(x_1, x_2, x_3) w^1_{\nu_1 \nu_2 \nu_3}\, dx_1\, dx_2\, dx_3$$

and

$$c_{\nu_2\nu_1\nu_3} = \left(\frac{2}{\pi}\right)^3 \iiint_{S_3} \varphi^a(x_1, x_2, x_3) w^i_{\nu_2\nu_1\nu_3}(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 =$$

$$= \left(\frac{2}{\pi}\right)^3 \iiint_{S_3} -\varphi^a(x_2, x_1, x_3) w^1_{\nu_1\nu_2\nu_3}(x_2, x_1, x_3)\, dx_2\, dx_1\, dx_3 = -c_{\nu_1\nu_2\nu_3}.$$

Hence if ν_1, ν_2, ν_3 is any permutation of the nondecreasing numbers n_1, n_2, n_3 and ε is the parity of this permutation,

$$c_{\nu_1\nu_2\nu_3} = (-1)^\varepsilon c_{n_1 n_2 n_3} \tag{3.9}$$

and after a rearrangement of the terms in (3.8) we get by the definition of the determinant

$$\varphi^a = \sum_{0<n_1<n_2<n_3} c_{n_1 n_2 n_3} w^a_{n_1 n_2 n_3} \quad \text{a.e.},$$

since by (3.9) $c_{n_1 n_2 n_3}=0$, if any two of the numbers n_1, n_2, n_3 coincide.

Finally by our assumption on φ and by the orthogonality property (3.7) we get

$$\iiint_{T_3} \varphi^a w^a_{n_1 n_2 n_3}\, dx_1\, dx_2\, dx_3 = A c_{n_1 n_2 n_3} = 0$$

and thus $\varphi^a=0$ a.e. in S_3 $\varphi=0$ a.e. in T_3.

4. In a recent paper V. A. MIHAĬLEC has shown [3], that if $N_1(\lambda)$ and $N_2(\lambda)$ denote the number of eigenvalues less than λ of the first and second eigenvalue problems, respectively, then for certain cylindrical three-dimensional domains D of volume V and surface S one has

$$N_j(\lambda) = \frac{V}{6\pi^2}\lambda^{3/2} + (-1)^j \frac{S}{16\pi}\lambda + o(\lambda), \quad j = 1, 2. \tag{4.1}$$

By counting the number of lattice points in the inside and on the surface of the spherical sector

$$x^2+y^2+z^2 < \lambda, \quad x > y > z,$$

one gets that formula (4.1) holds also in the case when D is the tetrahedron $\pi \geqq x \geqq y \geqq z \geqq 0$.

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REMARKS ON DISCRETE CONVERGENCE

by

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This is a complementary remark to the recent paper "Discrete and equal convergence" by Á. CSÁSZÁR and M. LACZKOVICH [2]. You have to see for the necessary definitions and notations there.

Our aim is to extend Theorem 15 to a larger class of spaces.

In this paper by a space we shall mean always a Tychonoff space.

1. Call a space X *fair* if given any real valued function f on X such that for each closed set $F \subset X$ the points of discontinuity of $f|F$ constitute a rare subset of F, then necessarily $f \in C^d(X)$.

We shall begin with a necessary condition.

THEOREM 1. *Let X be a fair space. If $H \subset X$ is an open or closed subset of X then $H = \bigcup_{n=1}^{\infty} Z_n$ where the Z_n's are zero-sets in X.*

PROOF. Let $F \subset X$ be closed in X, put $G = X - F$. If $f = k_F$, the characteristic function of the set F, we immediately obtain that, for each set $A \subset X$, the points of discontinuity of the function $f|A$ constitute a rare set in the subspace A. X being a fair space, $f \in C^d(X)$ so k_F is the discrete limit of a sequence $(f_n) \subset C(X)$.

Put

$$Z_n = \bigcap_{k=n}^{\infty} \{x \in X; f_k(x) = 1\},$$

$$Z'_n = \bigcap_{k=n}^{\infty} \{x \in X; f_k(x) = 0\}.$$

The sets Z_n, Z'_n are zero-sets in X,

$$F = \bigcup_{n=1}^{\infty} Z_n, \quad G = \bigcup_{1}^{\infty} Z'_n. \qquad \blacksquare$$

The condition that each closed and each open set be the union of countably many zero-sets, is evidently equivalent with the following one:

(*) X is perfect and each closed set is the union of countably many zero-sets.

(A space is called perfect if each closed set is G_δ or equivalently if each open set is F_σ.)

A perfectly normal space clearly fulfils the condition (*); indeed then each closed set is a zero-set. On the other hand it is easily seen that the well-known

bubble-space ([3] Example 1.2.2) is a perfect space not fulfilling $(*)$. I do not know any example of a non-normal space with the property $(*)$.

We shall prove a partial converse of Theorem 1 but we need a definition and two Lemmas.

DEFINITION [1]. A topological space X is called *subparacompact* if each open cover of X has a σ-discrete refinement.

Note that if X is a regular space (and we deal only with Tychonoff-spaces) then X is subparacompact iff each open cover of X has a closed σ-discrete refinement. Indeed, if $\mathfrak{U}$ is an open cover of X, choose an open cover $\mathfrak{V}$ of X such that $\overline{\mathfrak{V}} = \{\overline{V};\ V \in \mathfrak{V}\}$ refines $\mathfrak{U}$; this is possible by the regularity of the space X. Let now $\mathfrak{A}$ be a σ-discrete refinement of the family $\mathfrak{V}$. The system $\overline{\mathfrak{A}} = \{\overline{A};\ A \in \mathfrak{A}\}$ is σ-discrete ([3] Th. 4.4.2), closed and evidently refines $\mathfrak{U}$.

Call a family $\mathfrak{S}$ of subsets of the set X a refinement of the family $\mathfrak{I}$ if $\cup\mathfrak{S} = \cup\mathfrak{I}$ and for each set $S \in \mathfrak{S}$ there exists a set $T \in \mathfrak{I}$ with $S \subset T$.

LEMMA 2. *A space X is perfect and subparacompact iff for each open system $\mathfrak{U}$ in X there exists a closed σ-discrete (in X!) refinement $\mathfrak{B}$ of $\mathfrak{U}$.*

PROOF. a) Let X be perfect and subparacompact, $\mathfrak{U}$ an open system in X and put $G = \cup\mathfrak{U}$, $G = \bigcup_{n=1}^{\infty} F_n$, F_n closed in X. For each natural number n, F_n is a subparacompact space hence there exists in F_n a closed σ-discrete refinement $\mathfrak{B}_n$ of $\mathfrak{U}(\cap)F_n$. The set F_n is closed hence $\mathfrak{B}_n$ is a closed σ-discrete system also in X. Put now $\mathfrak{B} = \bigcup_{1}^{\infty} \mathfrak{B}_n$; $\mathfrak{B}$ is the desired family.

b) Suppose that the condition is true for the space X. Certainly X is subparascompact. Let now $G \subset X$ be an open set and select a closed σ-discrete family $\mathfrak{B}$ which is a refinement of the family $\{G\}$. If $\mathfrak{B} = \bigcup_{1}^{\infty} \mathfrak{B}_n$ and the families $\mathfrak{B}_n$ are discrete and closed, $F_n = \cup\mathfrak{B}_n$ is closed and $G = \bigcup_{1}^{\infty} F_n$. ∎

LEMMA 3. *Let X be a normal space, f be a real-valued function on X. Then $f \in C^d(X)$ iff there exists a sequence $\{F_n;\ n = 1, 2, \ldots\}$ of closed subsets of X with $X = \bigcup_{1}^{\infty} F_n$ and $f|F_n \in C(F_n)$.*

PROOF. a) If $f \in C^d(X)$, f is the discrete limit of a sequence $(f_n) \subset C(X)$. Put

$$F_n = \{x \in X;\ f_k(x) = f_n(x) \text{ for } k \geqq n\} \quad (n = 1, 2, \ldots).$$

F_n is closed, $f|F_n = f_n|F_n \in C(F_n)$ and $X = \bigcup_{1}^{\infty} F_n$.

b) If $X = \bigcup_{1}^{\infty} F_n$, $F_n = \overline{F}_n$, and $f|F_n \in C(F_n)$, we can suppose without loss of generality that $F_n \subset F_{n+1}$ $(n = 1, 2, \ldots)$. Using now the Tietze—Urysohn extension theorem, we get a sequence $(f_n) \subset C(X)$ with $f_n|F_n = f|F_n$; evidently f is the discrete limit of the sequence (f_n). ∎

THEOREM 2. *A subparacompact and perfectly normal space X is fair.*

PROOF. Our proof is a refinement of the argument of Theorem 15 in [2]. Define $X_0=X$ and if the closed sets X_β are defined for $\beta<\alpha$ and $X_{\beta_1}\supset X_{\beta_2}$ for $\beta_1<\beta_2<\alpha$ define X_α to be the closure of the set of points of discontinuity of the restriction

$$f\,|\bigcap_{\beta<\alpha} X_\beta .$$

Then clearly $X_\alpha\subset X_\beta$ for $\beta<\alpha$ and X_α is closed. Moreover, if $\beta<\alpha$ and $X_\alpha\neq\emptyset$ then $X_\beta\neq X_\alpha$ since, by hypothesis, X_α is a rare subset of the (non-empty) closed set $\bigcap_{\beta<\alpha} X_\beta$.

Therefore the sets X_α must be empty from a certain ordinal on; let $\gamma=\gamma(f, X)$ be the smallest ordinal with $X_\gamma=\emptyset$.

We show $f\in C^d(X)$ by means of transfinite induction with respect to $\gamma(f, X)$. This is true if $\gamma(f, X)\leqq 1$; let $\gamma(f, X)=\alpha$ and suppose the assertion is true if $\gamma(f, X)<\alpha$.

Put $F=\bigcap_{\beta<\alpha} X_\beta$. Now $F=\emptyset$ or $f|F$ is continuous.

Denote $\mathscr{G}=\{X-X_\beta;\ \beta<\alpha\}$ and select a closed refinement $\mathscr{B}=\bigcup_{n=1}^{\infty}\mathscr{B}_n$ of $\mathscr{G}$ with $\mathscr{B}_n$ discrete. Now

$$\bigcup\mathscr{B}=\bigcup\mathscr{G}=X-F.$$

Note that if $B\in\mathscr{B}$ then there exists a $\beta<\alpha$ with $B\cap X_\beta=\emptyset$. By the evident relation $B_\beta\subset X_\beta$ we get that $\gamma(f|B, B)\leqq\beta<\alpha$ and so $f|B\in C^d(B)$. (We can speak about $\gamma(f|B, B)$ because subparacompactness and perfect normality are hereditary with respect to closed subpaces.) Using now Lemma 3, we can choose for each set $B\in\mathscr{B}$ a sequence $\{F_n(B);\ n=1, 2, \ldots\}$ of closed sets with

$$F_n(B)\subset B,\quad \bigcup_1^{\infty} F_n(B)=B,\quad f|F_n(B)\in C\big(F_n(B)\big).$$

Let now m, n be two fixed natural numbers. The family $\mathscr{B}_{m,n}=\{F_m(B);\ B\in\mathscr{B}_n\}$ is a discrete family of closed sets and f is continuous on each of its members. Put $\Phi_{m,n}=\bigcup\mathscr{B}_{m,n}$; then $\Phi_{m,n}$ is closed in X, $f|\Phi_{m,n}\in C(\Phi_{m,n})$; $f|F\in C(F)$ and $X=F\cup\bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\Phi_{m,n}$.

Making use now of Lemma 3 we get the assertion. ∎

2. The most interesting case of Theorem 2 is that of metrizable spaces. A metrizable space is subparacompact (indeed, it is paracompact) and perfectly normal. Unfortunately, our Theorem 2 is not an exact converse to Theorem 13 in [2] because a closed subset of a metrizable Baire-space need not be a Baire-space.

Another difficulty is that the property "the set of the points of discontinuity of $f|C$ is a rare set in C for each compact set $C\subset X$" does not imply that $f\in C^d(X)$ even if X is a separable metric Baire-space.

EXAMPLE. Denote by $\mathbf{R}^2$ the two-dimensional euclidean space and put

$$Y=\{(x, y)\in\mathbf{R}^2;\ y>0\}.$$

Choose a set Z on the real axis such that Z has the cardinality of the continuum and any compact set contained in Z is countable (see e.g. [4] p. 422). Put $X=Y\cup Z$ with the subspace topology inherited from $\mathbb{R}^2$; X is certainly a separable metric Baire-space.

Let now φ be the set of the real valued functions on X which are identically zero on Y. Φ has cardinality $\exp\exp\aleph_0$ hence $\Phi\notin C^d(X)$.

We assert that if $f\in\Phi$, $C\subset X$ is a compact set then the points of discontinuity of $f|C$ constitute a rare set in C.

Suppose this is false; there exists a non-empty set G open in the compact space $C\subset X$ such that $f|G$ is discontinuous on a dense subset of G. Now evidently $G\subset Z\cap C$ and so $\overline{G}$ is a countable compact subset of the real line without isolated points. This is an obvious contradiction. ∎

For spaces metrizable in a complete manner, however, the situation is more fortunate. First we need a lemma.

LEMMA 4. *Let (X,ϱ) be a complete metric space, f a real valued function on X. If f is discontinuous on a dense set in X, there exists a compact subset $C\subset X$ such that $f|C$ is discontinuous on a dense subset of C.*

PROOF. If $x\in X$, $H\subset X$, $\varepsilon>0$, put

$$S(x,\varepsilon)=\{y\in X;\ \varrho(x,y)<\varepsilon\},$$

$$S(H,\varepsilon)=\bigcup\{S(x,\varepsilon);\ x\in H\}.$$

Denote by A the set of the points of discontinuity of f in X; A is dense in X. For each $x\in A$ there exists an $\varepsilon(x)>0$ and a sequence $(x_n)\subset X$ with $x_n\to x$, $|f(x_n)-f(x)|\geqq\varepsilon(x)$.

If $x\in A$, n is a natural number, let $\varphi_n(x)=y$ be a point in X with $\varrho(x,y)<2^{-n}$, $|f(x)-f(y)|\geqq\varepsilon(x)$.

If $x\in X-A$, let $\varphi_n(x)=y$ be a point in A with $\varrho(x,y)<2^{-n}$; such a point always can be selected by the density of the set A.

Let now $p_0\in X$ be any point, $F_0=\{p_0\}$, and put

$$F_{n+1}=F_n\cup\{\varphi_{n+1}(x);\ x\in F_n\},$$

$$G_n=S(F_n,2^{-n})\quad(n=0,1,2\ \ldots).$$

The sets F_n are finite sets and $F_n\subset F_{n+1}$ $(n=0,1,2,\ldots)$. We assert that $\overline{G}_{n+1}\subset G_n$ $(n=0,1,2,\ldots)$. Indeed, if $x\in\overline{G}_{n+1}$ then $\varrho(x,F_{n+1})\leqq 2^{-n-1}$ so there exists a point $y\in F_{n+1}$ with $\varrho(x,y)\leqq 2^{-n-1}$. By the definition of F_{n+1} there exists a point $z\in F_n$ with $\varrho(y,z)<2^{-n-1}$, hence $\varrho(x,z)<2^{-n}$ so $x\in S(F_n,2^{-n})=G_n$.

Note that $F_n\subset G_m$ for each n,m. Indeed, if $n\leqq m$ then $F_n\subset F_m\subset G_m$; if $n>m$ then $F_n\subset G_n\subset G_m$.

Denote now

$$C=\bigcap_{n=0}^{\infty}G_n=\bigcap_{n=0}^{\infty}\overline{G}_n.$$

C is closed in X and totally bounded hence ([3] Th. 4.3.14) it is compact.

Evidently $H=\bigcup^{\infty} F_n$ is dense in C and $H\cap A$ is dense in H. If $x\in H\cap A$ then $f|C$ is discontinuous at x; this shows that $f|C$ is discontinuous on a dense subset of C. ∎

THEOREM 3. *Let X be a space metrizable in a complete manner, f a real valued function on X. The following conditions are equivalent:*

a) $f\in C^d(X)$.

b) *For each subspace $A\subset X$, the points of discontinuity of $f|A$ constitute a rare subset in A.*

c) *For each closed subset $F\subset X$, the points of discontinuity of $f|F$ constitute a rare subset in F.*

d) *For each compact subset $C\subset X$, the points of discontinuity of $f|C$ constitute a rare subset in C.*

PROOF. a→d: Theorem 13 in [2].

d→c: Suppose that $F\subset X$ is closed and denote by $A\subset F$ the set of the points of discontinuity of $f|F$. If A is not a rare set in F, there exists a non-empty set $G\subset\bar{A}$ open in F. The subspace G is metrizable in a complete manner ([3] Theorem 4.3.9) and $f|G$ is discontinuous on a dense subset of G. By Lemma 4 there exists a compact set $C\subset G\subset X$ with $f|C$ discontinuous on a dense subset of C. This shows that condition d) is not satisfied.

c→b: Suppose $A\subset X$ and that the points of discontinuity of $f|A$ do not constitute a rare subset in A. There exists a non-empty set $G\subset A$ open in A such that $f|G$ is discontinuous on a dense subset of G. Put $F=\bar{G}$; F is closed in X and $f|F$ is discontinuous on a dense subset of the closed set F.

b→c: Trivial.

c→a: This is a consequence of Theorem 2. ∎

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UNTERSUCHUNGEN ÜBER RICHTUNGSSTRUKTUREN, II. ÜBER $\mathfrak{R}$-INTERVALLSYSTEME VON ALLGEMEINER LAGE

von

E. DEÁK

In diesem Teil unserer Serie versuchen wir, den im Teil I eingeführten Begriff des $\mathfrak{R}$-Kleinheitsgrades, der dort eine fundamentale Rolle als Hilfsmittel gespielt hat, in den Vordergrund zu stellen. Unser Ziel ist zu zeigen, daß unter gewissen Bedingungen das System $\mathscr{I}(\mathfrak{R})$ der $\mathfrak{R}$-offenen Intervalle ($\mathfrak{R}$ eine KRS eines Raumes X) ein Teilsystem $\mathscr{I}^*$ enthält, so daß auch noch $\mathscr{I}^*$ eine Basis der Topologie von X ist, und daß der Durchschnitt der Begrenzungen (ja sogar der Rahmen) von k verschiedenen Mengen aus $\mathscr{I}^*$ als Vereinigung von endlich vielen, abgeschlossenen, $(k, \mathfrak{R})$-kleinen Mengen dargestellt werden kann.

Unser Ergebnis (Satz (2.2) in Verbindung mit Satz (1.5)) steht also in einer gewissen Beziehung zu [1] (vgl. u. a. [1], (1.8)), aber auch zu den Begriffen der hübschen Überdeckungen bzw. Basen, die in den beiden ersten Teilen ([3], [4]) unserer anderen Serie entwickelt und untersucht werden.

Wir benutzen die Begriffe und Bezeichnungen aus [1] (und verweisen wieder auch auf die Übersicht [2] der Theorie der RS-en). (Das Zeichen $\subset$ bedeutet hier, in Einklang mit [1], $\subsetneqq$.) Einige weitere Begriffe, Bezeichnungen und Tatsachen bezüglich RS-en werden im § 0 angeführt. Im § 1 bringen wir die nötigen neuen Begriffe und Hilfsmittel für § 2.

§ 0.

(0.1) Eine Richtung $\mathscr{R}$ auf einer Menge $X \neq \varnothing$ ist durch die Relation

$$(G_1, F_1) \prec (G_2, F_2) \quad \text{genau dann, wenn}$$

$$F_1 \subseteqq G_2, \quad G_1 \subset F_2 \quad ((G_1, F_1), (G_2, F_2) \in \mathscr{R})$$

linear geordnet. ($\prec$ bedeutet hier $\precneqq$.)

(0.2) Für eine Richtung $\mathscr{R}$ auf einer Menge $X \neq \varnothing$ gilt

$$G_1 \subset G_2 \Rightarrow F(\mathscr{R}; G_1) \subseteqq G_2 \quad (G_1, G_2 \in \mathscr{G}(\mathscr{R})).$$

(0.3) DEFINITION. Eine Richtung $\mathscr{R}$ auf einer Menge $X \neq \varnothing$ wird *saturiert* genannt, wenn

(0.3.1) $$M \in \mathscr{G}(\mathscr{R}) \cap \mathscr{F}(\mathscr{R}) \Rightarrow (M, M) \in \mathscr{R}$$

(das ist eine Verschärfung des Richtungsaxioms (0.1),(I) in [1]).

(0.4) BEMERKUNG. Eine nichtsaturierte Richtung $\mathscr{R}$ kann durch Hinzunahme der Paare unter (0.3.1) ohne weiteres zu einer saturierten Richtung erweitert werden.

(0.5) DEFINITIONEN. Es sei X eine nichtleere Menge.

(a) Eine *ordentliche Richtung* auf X ist eine Richtung $\mathscr{R}$ auf X mit der Eigenschaft

$$\cup\{F\setminus G\colon (G,F)\in\mathscr{R}\} = X.$$

(b) Das in diesem Fall zu jedem $x\in X$ eindeutig gehörende Paar $(G, F)\in\mathscr{R}$ mit $x\in F\setminus G$ (vgl. (0.4),(b) in [1]) wird mit $(G_x(\mathscr{R}), F_x(\mathscr{R}))$ bezeichnet.

(c) Eine *ordentliche RS (ORS)* auf X ist eine RS $\mathscr{R}$ auf X, so daß jedes $\mathscr{R}\in\mathfrak{R}$ eine ordentliche Richtung ist.

(d) Wir benutzen die Abkürzung KORS für „kompatible ORS" (bezüglich einer Topologie auf X).

(0.6) DEFINITION. Ein Raum wird *schwach ordentlich* genannt, wenn er eine KORS zuläßt.

(0.7) SATZ. *Ein Raum ist schwach ordentlich genau dann, wenn er vollständig regulär ist.*

(0.8) Sind $\mathscr{R}$ eine ordentliche Richtung auf einer Menge $X\neq\varnothing$, $G\in\mathscr{G}(\mathscr{R})$, $F\in\mathscr{F}(\mathscr{R})$ und $F\subset G$, so gilt

$$F\subseteqq G_x(\mathscr{R}),\quad F_x(\mathscr{R})\subseteqq G\quad (x\in G\setminus F).$$

§ 1.

(1.1) Bezeichnungen. Es seien $X\neq\varnothing$ eine Menge, $\mathfrak{R}$ eine RS auf X und $\varnothing\neq I\in\mathscr{I}(\mathfrak{R})$, also

$$I = \cap\{G(\mathscr{R})\setminus F(\mathscr{R})\colon \mathscr{R}\in\mathfrak{R}^*\} \tag{1.1.1}$$

für ein endliches $\mathfrak{R}^*$ mit $\varnothing\neq\mathfrak{R}^*\subseteqq\mathfrak{R}$ und für $\mathfrak{R}$-Halbräume

$$G(\mathscr{R})\in\mathscr{G}(\mathscr{R}),\quad F(\mathscr{R})\in\mathscr{F}(\mathscr{R})\quad (\mathscr{R}\in\mathfrak{R}^*)$$

mit notwendigerweise

$$F(\mathscr{R})\subset G(\mathscr{R})\quad (\mathscr{R}\in\mathfrak{R}^*).$$

Auf diese Darstellung der Menge I als $\mathfrak{R}$-Intervall (die nicht unbedingt die einzig mögliche ist) bezogen seien

$$S_1(\mathscr{R}) = \underline{F}(\mathscr{R};\ G(\mathscr{R}))\setminus G(\mathscr{R}), \tag{1.1.2}$$

$$S_2(\mathscr{R}) = F(\mathscr{R})\setminus\overline{G}(\mathscr{R};\ F(\mathscr{R})), \tag{1.1.3}$$

$$T(\mathscr{R}) = \underline{F}(\mathscr{R};\ G(\mathscr{R}))\setminus\overline{G}(\mathscr{R};\ F(\mathscr{R})), \tag{1.1.4}$$

$$D_i(\mathscr{R}) = S_i(\mathscr{R})\cap\cap\{T(\mathscr{R}')\colon \mathscr{R}'\in\mathfrak{R}^*\}\quad (i=1,2), \tag{1.1.5}$$

$$D(\mathscr{R}) = D_1(\mathscr{R})\cup D_2(\mathscr{R})\quad (\mathscr{R}\in\mathfrak{R}^*), \tag{1.1.6}$$

und der $\mathfrak{R}$-*Rahmen* von I mit der Darstellung (1.1.1) soll die Menge

$$\mathscr{B}(\mathfrak{R}^*; I) = \bigcup\{D(\mathscr{R}): \mathscr{R}\in\mathfrak{R}^*\} \tag{1.1.7}$$

bedeuten.

(1.1a) BEMERKUNG. *Für eine beliebige (also nicht unbedingt kompatible) RS* $\mathfrak{R}$ *eines Raumes* X *und für jede Darstellung nach* (1.1.1) *eines beliebigen* $I\in\mathscr{I}(\mathfrak{R})$ *gilt*

$$\operatorname{Gr} I \subseteq \mathscr{B}(\mathfrak{R}^*; I)$$

(da I eine offene Menge, $I\cap\mathscr{B}(\mathfrak{R}^*; I)=\varnothing$ und $I\cup\mathscr{B}(\mathfrak{R}^*; I)$ gleich der abgeschlossenen Menge

$$\cap\{\underline{F}(\mathscr{R}; G(\mathscr{R}))\setminus\overline{G}(\mathscr{R}; F(\mathscr{R})): \mathscr{R}\in\mathfrak{R}^*\}$$

sind).

(Anschaulich erfaßt man das, indem man das $\mathfrak{R}$-offene Intervall als eine Art „Quader" auffaßt, dessen „Seiten" eben die Mengen $D_i(\mathscr{R})$ ($i=1, 2$, $\mathscr{R}\in\mathfrak{R}^*$) sind. Dies ist tatsächlich so im Spezialfall, daß $X=\mathbf{R}^n$ ($n=1, 2, \ldots$) und $\mathfrak{R}$ die „gewöhnliche" RS (d. i. die Descartessche Koordinatenstruktur) von X sind, und das ist ja der anschauliche Hintergrund des RS-Begriffs überhaupt.)

(1.2) DEFINITION. Es seien $X\neq\varnothing$ eine Menge und $\mathfrak{R}$ eine RS auf X.

Zwei nichtleere $\mathfrak{R}$-offene Intervalle $I^{(1)}$ und $I^{(2)}$ sind *von allgemeiner Lage* (genauer ausgedrückt: *von* $\mathfrak{R}$*-allgemeiner Lage**) zueinander, wenn es einen endlichen Teil $\mathfrak{R}^*\neq\varnothing$ von $\mathfrak{R}$ und Darstellungen
Teil $\mathfrak{R}^*\neq\varnothing$ von $\mathfrak{R}$ und Darstellungen

$$I^{(j)} = \bigcap\{G^{(j)}(\mathscr{R})\setminus F^{(j)}(\mathscr{R}): \mathscr{R}\in\mathfrak{R}^*\} \quad (j=1, 2) \tag{1.1.8}$$

(mit

$$G^{(j)}(\mathscr{R})\in\mathscr{G}(\mathscr{R}),\ F^{(j)}(\mathscr{R})\in\mathscr{F}(\mathscr{R}) \quad (j=1, 2;\ \mathscr{R}\in\mathfrak{R}^*))$$

gibt, so daß mit den Bezeichnungen

$$S_1^{(j)}(\mathscr{R}),\ S_2^{(j)}(\mathscr{R}),\ T^{(j)}(\mathscr{R}),\ D_i^{(j)}(\mathscr{R}),\ D^{(j)}(\mathscr{R}) \quad (j=1, 2,\ i=1, 2), \tag{1.1.9}$$

die jenen unter (1.1) sinngemäß entsprechen (der obere Index weist auf $I^{(1)}$ bzw. $I^{(2)}$ hin),

$$\bigcap_{j=1,2} D^{(j)}(\mathscr{R}) = \varnothing \quad (\mathscr{R}\in\mathfrak{R}^*) \tag{1.1.10}$$

besteht.

(1.3) BEMERKUNGEN. (a) In dem unter (1.2) beschriebenen Fall zieht (1.1.10) die Gleichung

$$\mathscr{B}(\mathfrak{R}^*; I^{(1)})\cap\mathscr{B}(\mathfrak{R}^*; I^{(2)}) =$$

$$= \bigcup\{D^{(1)}(\mathscr{R}')\cap D^{(2)}(\mathscr{R}''): \mathscr{R}'\in\mathfrak{R}^*,\ \mathscr{R}''\in\mathfrak{R}^*\setminus\{\mathscr{R}'\}\}$$

nach sich.

* Die Beziehung auf $\mathfrak{R}$ hervorzuheben hat nur dann Bedeutung, wenn auf derselben Menge X mindestens noch eine, von $\mathfrak{R}$ verschiedene RS $\mathfrak{R}'$ mit im Spiel ist; dann kann es nämlich vorkommen, daß zwei Mengen $I^{(1)}, I^{(2)}\subseteq X$ sowohl als $\mathfrak{R}$-offene wie als $\mathfrak{R}'$-offene Intervalle dargestellt werden können, aber, obwohl von etwa $\mathfrak{R}$-allgemeiner Lage, nicht von $\mathfrak{R}'$-allgemeiner Lage sind.

(b) Eine Menge $\emptyset \neq I \subseteq X$ kann sogar für ein und dieselbe RS $\mathfrak{R}$ auf X evtl. mehrere verschiedene Darstellungen als $\mathfrak{R}$-offenes Intervall zulassen. Unsere Definition fordert die Existenz solcher Darstellungen der Mengen $I^{(1)}$ und $I^{(2)}$ mit der besagten Eigenschaft, unabhängig davon, wie diese Mengen, als $\mathfrak{R}$-offene Intervalle, ursprünglich vorgegeben sind.

(c) Es bedeutet keine echte Einschränkung, daß wir Darstellungen von $I^{(1)}$ und $I^{(2)}$ mit gemeinsamen $\mathfrak{R}^*$ gefordert haben. Gibt es nämlich Darstellungen

$$I^{(j)} = \bigcap \{G^{(j)}(\mathscr{R}) \setminus F^{(j)}(\mathscr{R}) \colon \mathscr{R} \in \mathfrak{R}^{(j)}\} \quad (j = 1, 2)$$

mit irgendwelchen verschiedenen $\mathfrak{R}^{(1)}, \mathfrak{R}^{(2)} \subseteq \mathfrak{R}$, so daß mit den Bezeichnungen unter (1.1) (mit $\mathscr{R} \in \mathfrak{R}^{(j)}$ anstelle von $\mathscr{R} \in \mathfrak{R}^*$) die Beziehung unter (1.1.10) (mit $\mathscr{R} \in \mathfrak{R}^{(1)} \cap \mathfrak{R}^{(2)}$ anstelle von $\mathscr{R} \in \mathfrak{R}^*$) besteht (was der eben definierten Beziehung sinngemäß entspricht), so können wir durch die ergänzende Bestimmungen

$$G^{(1)}(\mathscr{R}) = X, \quad F^{(1)}(\mathscr{R}) = \emptyset \quad (\mathscr{R} \in \mathfrak{R}^{(2)} \setminus \mathfrak{R}^{(1)}),$$

$$G^{(2)}(\mathscr{R}) = X, \quad F^{(2)}(\mathscr{R}) = \emptyset \quad (\mathscr{R} \in \mathfrak{R}^{(1)} \setminus \mathfrak{R}^{(2)})$$

und mit der Bezeichnung

$$\mathfrak{R}^* = \mathfrak{R}^{(1)} \cup \mathfrak{R}^{(2)}$$

Darstellungen (1.1.8) mit der unter (1.2) beschriebenen Eigenschaft herstellen.

(d) Anschaulich kann man den Inhalt des Begriffs unter (1.2) etwa so erfassen: faßt man die $\mathfrak{R}$-offenen Intervalle wieder als „Quader" auf, so sollen auf ein und derselben „Hyperebene" liegende „Seiten" verschiedener Quader keine gemeinsamen Punkte haben.

(1.4) DEFINITION. Ein System $\mathscr{I} \subseteq \mathscr{I}(\mathfrak{R})$ von $\mathfrak{R}$-offenen Intervallen ($\mathfrak{R}$ eine RS auf einer Menge $X \neq \emptyset$) ist *von allgemeiner Lage (von $\mathfrak{R}$-allgemeiner Lage)*, wenn das für je zwei, als Teilmengen von X verschiedene Elemente von $\mathscr{I}$ zutrifft.

Die Beziehung dieses Begriffs zum Begriff des $\mathfrak{R}$-Kleinheitsgrades erhellt aus dem folgenden Satz, wo wir uns allerdings auf ORS-en beschränken müssen.

(1.5) SATZ. *Es seien $X \neq \emptyset$ eine Menge (bzw. ein Raum), $\mathfrak{R}$ eine ORS (bzw. KORS) auf X, $\emptyset \neq \mathscr{I} \subseteq \mathscr{I}(\mathfrak{R})$ ein System von allgemeiner Lage und $k \geqq 1$ eine natürliche Zahl.*

Für beliebige, paarweise als Mengen verschiedene, Elemente $I^{(j)}$ $(j = 1, 2, \ldots, k)$ von $\mathscr{I}$ gibt es dann Darstellungen wie unter (1.1.8) *(aber jetzt mit $j = 1, 2, \ldots, k$), so daß mit den, auf $j = 1, 2, \ldots, k$ angewandten, Bezeichnungen* (1.1.9) *die Menge*

$$K = \bigcap_{j=1}^{k} \mathscr{B}(\mathfrak{R}^*; I^{(j)})$$

darstellbar ist als die Vereinigung von endlich vielen (im topologischen Fall abgeschlossenen) $(k, \mathfrak{R})$-kleinen Mengen. (Vgl. für den Fall $k = 1$ (1.1a) und [1], (0.18).)

BEWEIS. Es seien

$$\mathfrak{R}^* = \{\mathscr{R}_1, \mathscr{R}_2, \ldots, \mathscr{R}_n\}, \quad \mathscr{R}_{\nu_1} \neq \mathscr{R}_{\nu_2} \quad (\nu_1, \nu_2 \in \{1, 2, \ldots, n\}, \ \nu_1 \neq \nu_2)$$

($n=|\mathfrak{R}^*|$). Es gilt

$$K=\bigcap_{j=1}^{k}\bigcup_{\nu=1}^{m}D^{(j)}(\mathscr{R}_\nu)=\bigcup\left\{\bigcap_{j=1}^{k}D^{(j)}(\mathscr{R}_\nu)_j\colon(\nu_1,\nu_2,\ldots,\nu_k)\in V\right\},$$

wo V die Menge aller Variationen zur k-ten Klasse, u. zw. — wegen (1.1.10) — ohne Wiederholung, bedeutet (das ist auch für $k>n$ sinnvoll, da dann $V=\varnothing$ aber auch $K=\varnothing$ ist), und daher — nach (1.1.5) und (1.1.6) —

$$K\subseteqq\bigcup\left\{\bigcap_{j=1}^{k}S_{i_j}^{(j)}(\mathscr{R}_{\nu_j})\colon i_j\in\{1,2\}\quad(j=1,2,\ldots,k),\ (\nu_1,\nu_2,\ldots,\nu_k)\in V\right\}.$$

Jede der Mengen $S_{i_j}^{(j)}(\mathscr{R}_{\nu_j})$ ist $(k,\mathfrak{R})$-klein (und, im topologischen Fall, abgeschlossen), und es kommen nur endlich viele verschiedene $(k+1)$-Tupel

$$(\nu_1,\nu_2,\ldots,\nu_k,i_j)$$

vor, womit der Satz bewiesen ist.

(1.6) BEMERKUNG. Es fragt sich, ob die Voraussetzung der Ordentlichkeit von $\mathfrak{R}$ unter (1.5) wesentlich ist, oder etwa durch irgendwelche Voraussetzungen über die Topologie von X (Trennungseigenschaften) ersetzt werden könnte. Auf diese Frage gehen wir diesmal nicht ein.

§ 2.

(2.1) HILFSSATZ. *Es seien $\mathfrak{R}$ eine ordentliche Richtung auf einer Menge $X\neq\varnothing$,*

$$(G_1,F_1),\ (G_2,F_2)\in\mathscr{R},\quad F_1\subset G_2,$$

und nehmen wir an, daß es keine Menge M mit

$$(M,M)\in\mathscr{R},\quad F_1\subseteqq M\subseteqq G_2$$

gibt. Für die Menge

$$\mathscr{S}=\{(G,F)\in\mathscr{R}\colon F_1\subseteqq G,\ F\subseteqq G_2\}$$

gilt dann

$$|\mathscr{S}|\geqq 2^{\aleph_0}.$$

BEWEIS. 1° Man sieht auf Grund von (0.5),(a), (0.8) und [1], (0.1),(a), (III), (IV) (die sehr leichten Einzelheiten seien dem Leser überlassen), daß $\mathscr{S}$ mit der Ordnung $\prec$ unter (0.1) eine (im Dedekindschen Sinne) stetig geordnete Menge mit $|\mathscr{S}|>1$ ist.

2° Eine solche geordnete Menge enthält nun aber bekanntlich eine Teilmenge, die zur Menge etwa der reellen Zahlen zwischen 0 und 1 ordnungsisomorph ist.

(2.2) SATZ. *Es seien X ein (notwendigerweise schwach ordentlicher) Raum mit $w(X)\leqq 2^{\aleph_0}$ und $\mathfrak{R}$ eine KORS dieses Raumes, deren sämtliche Elemente saturierte Richtungen sind.*

Dann gibt es eine Basis der Topologie von X bestehend aus $\mathfrak{R}$-offenen Intervallen, die ein System von allgemeiner Lage bilden.

BEWEIS. 1° Wir gehen aus vom System

$$\mathfrak{I} = \Big\{\Big(\bigcap_{l=1}^{n} (\Gamma_l \setminus \Phi_l),\ \bigcap_{l=1}^{n} (G_l \setminus F_l)\Big): \quad n = 1, 2, \ldots; \tag{2.2.1}$$

$$\Gamma_l, G_l \in \mathcal{G}(\mathcal{R}_l), \quad \Phi_l, F_l \in \mathcal{F}(\mathcal{R}_l), \quad \mathcal{R}_l \in \mathfrak{R},$$

$$\underline{F}(\mathcal{R}_l; \Gamma_l) \subseteq G_l, \quad F_l \subseteq \overline{G}(\mathcal{R}_l; \Phi_l) \quad (k = 1, 2, \ldots, n)\Big\}$$

(dessen Elemente also Paare (J, I) von $\mathfrak{R}$-offenen Intervallen sind, die je eine $\mathfrak{R}^*(\subseteq \mathfrak{R})$-Darstellung mit $J \cup \mathcal{B}(\mathfrak{R}^*; J) \subseteq I$ zulassen.

2° Zu jedem $J \in \mathcal{I}(\mathfrak{R})$ (mit $J \neq \varnothing$) gibt es ein $I \in \mathcal{I}(\mathfrak{R})$ mit $(J, I) \in \mathfrak{I}$.

(Das ist eine Trivialität, man nimmt etwa einfach $I = X$, und diese Menge kann, was auch die Darstellung

$$\bigcap_{l=1}^{n} (\Gamma_l \setminus \Phi_l) \quad (\Gamma_l \in \mathcal{G}(\mathcal{R}_l), \Phi_l \in \mathcal{F}(\mathcal{R}_l), \mathcal{R}_l \in \mathfrak{R} \quad (l = 1, 2, \ldots, n))$$

von J sein mag, als

$$X = \bigcap_{l=1}^{n} (X \setminus \varnothing)$$

dargestellt werden, was wegen $X \in \mathcal{G}(\mathcal{R})$, $\varnothing \in \mathcal{F}(\mathcal{R})$ $(\mathcal{R} \in \mathfrak{R})$ der Behauptung entspricht.)

3° Zu jedem $I \in \mathcal{I}(\mathfrak{R})$ mit $I \neq \varnothing$ gibt es ein $J \in \mathcal{I}(\mathfrak{R})$ mit $J \neq \varnothing$ und $(J, I) \in \mathfrak{I}$.

Es seien nämlich $x \in I$ und

$$I = \bigcap_{l=1}^{n} (G_l \setminus F_l)$$

eine Darstellung von I mit

$$G_l \in \mathcal{G}(\mathcal{R}_l), \quad F_l \in \mathcal{F}(\mathcal{R}_l) \quad (l = 1, 2, \ldots, n),$$

wobei daher

$$F_l \subseteq G_x(\mathcal{R}_l), \quad F_x(\mathcal{R}_l) \subseteq G_l \quad (l = 1, 2, \ldots, n).$$

Mit der Wahl

$$\Phi_l = \begin{cases} F_y(\mathcal{R}_l) & (G_x(\mathcal{R}_l) \setminus F_l \neq \varnothing,\ y \in G_x(\mathcal{R}_l) \setminus F_l \text{ beliebig gewählt}) \\ \text{bzw.} & \\ F_l & (G_x(\mathcal{R}_l) \setminus F_l = \varnothing) \end{cases} \tag{2.2.2}$$

$$\Gamma_l = \begin{cases} G_y(\mathcal{R}_l) & (G_l \setminus F_x(\mathcal{R}_l) \neq \varnothing,\ y \in G_l \setminus F_x(\mathcal{R}_l) \text{ beliebig gewählt}) \\ \text{bzw.} & \\ G_l & (G_l \setminus F_x(\mathcal{R}_l) = \varnothing) \end{cases} \tag{2.2.3}$$

$$(l = 1, 2, \ldots, n)$$

genügt dann

$$J = \bigcap_{l=1}^{n} (\Gamma_l \setminus \Phi_l)$$

den Erfordernissen unter 1°. Tatsächlich haben wir

$$\bar{G}(\mathscr{R}_l;\, \Phi_l) = \left\{\begin{matrix} G_y(\mathscr{R}_l) \\ \text{bzw.} \\ \bar{G}(\mathscr{R}_l;\, F_l) \end{matrix}\right\} \supseteq F_l \quad (l = 1, 2, \ldots, n)$$

(im ersten Fall nach (0.8), im zweiten aber, weil, wegen der Saturiertheit von $\mathscr{R}_l$, $(F_l, F_l) \in \mathscr{R}_l$ und daher $\bar{G}(\mathscr{R}_l;\, F_l) = F_l$ sind) und

$$\underline{F}(\mathscr{R}_l;\, \Gamma_l) = \left\{\begin{matrix} F_y(\mathscr{R}_l) \\ \text{bzw.} \\ \underline{F}(\mathscr{R}_l; G_l) \end{matrix}\right\} \subseteq G_l \quad (l = 1, 2, \ldots, n)$$

(im ersten Fall wieder nach (0.8), und im zweiten, weil, wegen der Saturiertheit von $\mathscr{R}_l$, $(G_l, G_l) \in \mathscr{R}_l$ und daher $F(\mathscr{R}_l;\, G_l) = G_l$ sind).

(Man sieht obendrein, daß für ein beliebiges $x \in I$ noch $x \in I$ gefordert werden kann.)

4° Die Familie sowohl der ersten wie diejenige der zweiten Komponenten der Elemente von $\mathfrak{I}$ ist also, nach 2° und 3°, eben $\mathscr{I}(\mathfrak{R}) \setminus \{\varnothing\}$ (wir brauchen aber im weiteren eigentlich nur die Tatsache auszunutzen, daß jede dieser Mengenfamilien eine Basis der Topologie von X abgibt, was aus der Kompatibilität der RS $\mathfrak{R}$ folgt).

Es gibt dann eine Basis $\mathscr{I}^* \subseteq \mathscr{I}(\mathfrak{R})$ mit $|\mathscr{I}^*| = w(X)$, und wir setzen

$$\mathfrak{I}^* = \mathfrak{I} \cap (\mathscr{I}^* \times \mathscr{I}^*). \tag{2.2.4}$$

Dann gilt

$$|\mathfrak{I}^*| = w(X),$$

und jede der Mengenfamilien

$$\{J \in \mathscr{I}(\mathfrak{R}):\ \exists I \in \mathscr{I}(\mathfrak{R}),\ (J, I) \in \mathfrak{I}^*)\},$$

$$\{I \in \mathscr{I}(\mathscr{R}):\ \exists J \in \mathscr{I}(\mathfrak{R}),\ (J, I) \in \mathfrak{I}^*)\}$$

ist eine Basis für den Raum X.

5° Mit der Bezeichnung $\varkappa$ für die zur Mächtigkeit $w(X)$ gehörige Anfangszahl sei

$$(J^{(0)}, I^{(0)}), (J^{(1)}, I^{(1)}), \ldots, (J^{(\alpha)}, I^{(\alpha)}), \ldots \quad (\alpha < \varkappa)$$

eine Wohlordnung von $\mathfrak{I}^*$ vom Ordnungstypus $\varkappa$. Weiter seien

$$\mathfrak{R}^{(\alpha)} \subseteq \mathfrak{R},\ |\mathfrak{R}^{(\alpha)}| < \aleph_0 \quad (\alpha < \varkappa)$$

derart, daß Darstellungen

$$I^{(\alpha)} = \bigcap \{G^{(\alpha)}(\mathscr{R}) \setminus F^{(\alpha)}(\mathscr{R}):\ \mathscr{R} \in \mathfrak{R}^{(\alpha)}\},$$

$$J^{(\alpha)} = \bigcap \{\Gamma^{(\alpha)}(\mathscr{R}) \setminus \Phi^{(\alpha)}(\mathscr{R}):\ \mathscr{R} \in \mathfrak{R}^{(\alpha)}\}$$

$(\alpha < \varkappa)$ mit

$$G^{(\alpha)}(\mathscr{R}),\quad \Gamma^{(\alpha)}(\mathscr{R}) \in \mathscr{G}(\mathscr{R}),\quad F^{(\alpha)}(\mathscr{R}),\quad \Phi^{(\alpha)}(\mathscr{R}) \in \mathscr{F}(\mathscr{R})$$

$$(\mathscr{R} \in \mathfrak{R}^{(\alpha)},\ \alpha < \varkappa)$$

existieren.

6° Jetzt werden durch transfinite Induktion $\mathfrak{R}$-Halbräume

$$\tilde{G}^{(\alpha)}(\mathscr{R}) \in \mathscr{G}(\mathscr{R}), \quad \tilde{F}^{(\alpha)}(\mathscr{R}) \in \mathscr{F}(\mathscr{R}) \tag{2.2.5}$$

$$(\mathscr{R} \in \mathfrak{R}^{(\alpha)}, \ \alpha < \varkappa)$$

mit

$$\Gamma^{(\alpha)}(\mathscr{R}) \subseteq \tilde{G}^{(\alpha)}(\mathscr{R}) \subseteq G^{(\alpha)}(\mathscr{R}),$$

$$F^{(\alpha)}(\mathscr{R}) \subseteq \tilde{F}^{(\alpha)}(\mathscr{R}) \subseteq \Phi^{(\alpha)}(\mathscr{R})$$

$$(\mathscr{R} \in \mathfrak{R}^{(\alpha)}, \ \alpha < \varkappa)$$

und mit einer nachträglich zu formulierenden zusätzlichen Eigenschaft definiert.

a) Es seien

$$\tilde{G}^{(0)}(\mathscr{R}) = G^{(0)}(\mathscr{R}), \quad \tilde{F}^{(0)}(\mathscr{R}) = F^{(0)}(\mathscr{R}) \quad (\mathscr{R} \in \mathfrak{R}^{(0)}).$$

b) Ist $0 < \alpha < \varkappa$, und sind für jedes $\beta < \alpha$ und jedes $\mathscr{R} \in \mathfrak{R}^{(\beta)}$ die $\mathscr{R}$-Halbräume $\tilde{G}^{(\beta)}(\mathscr{R})$, $\tilde{F}^{(\beta)}(\mathscr{R})$ schon definiert, so unterscheiden wir zwei Fälle. Es sei ein beliebiges $\mathscr{R} \in \mathfrak{R}^{(\alpha)}$ festgesetzt.

b_1) Gibt es ein Element $(G, G) \in \mathscr{R}$ mit

$$\underline{F}\big(\mathscr{R}; \ \Gamma^{(\alpha)}(\mathscr{R})\big) \subseteq G \subseteq G^{(\alpha)}(\mathscr{R}),$$

was — wegen der Saturiertheit von $\mathscr{R}$ — insbesondere (aber evtl. nicht ausschließlich) dann zutrifft, wenn $F\big(\mathscr{R}; \Gamma^{(\alpha)}(\mathscr{R})\big) = G^{(\alpha)}(\mathscr{R})$ ist, so sei

$$\tilde{G}^{(\alpha)}(\mathscr{R}) = G.$$

b_2) Im entgegengesetzten Fall haben wir gewiß $F\big(\mathscr{R}; \Gamma^{(\alpha)}(\mathscr{R})\big) \subset G^{(\alpha)}(\mathscr{R})$ (vgl. (2.2.1)). Es sei dann $z \in G^{(\alpha)}(\mathscr{R}) \setminus \underline{F}\big(\mathscr{R}; \Gamma^{(\alpha)}(\mathscr{R})\big)$ u. zw. so gewählt, daß

$$\big(G_z(\mathscr{R}), F_z(\mathscr{R})\big) \neq \big(\tilde{G}^{(\beta)}(\mathscr{R}), F\big(\mathscr{R}; \tilde{G}^{(\beta)}(\mathscr{R})\big)\big)$$

$$(\beta < \alpha)$$

ausfällt. (Die Möglichkeit hierzu ist durch den Hilfssatz (2.1) und — nach der Definition von $\varkappa$ unter 5° — wegen

$$\big|\big\{\big(\tilde{G}^{(\beta)}(\mathscr{R}), \tilde{F}^{(\beta)}(\mathscr{R})\big) : \beta < \alpha\big\}\big| < w(X) \quad (\leqq 2^{\aleph_0})$$

gesichert.) Dann sei

$$\tilde{G}^{(\alpha)} = G_z(\mathscr{R}).$$

b^1) Gibt es ein $(F, F) \in \mathscr{R}$ mit

$$F^{(\alpha)}(\mathscr{R}) \subseteq F \subseteq \bar{G}\big(\mathscr{R}; \ \Phi^{(\alpha)}(\mathscr{R})\big)$$

was — wegen der Saturiertheit von $\mathscr{R}$ — insbesondere (aber evtl. nicht ausschließlich) dann zutrifft, wenn $F^{(\alpha)}(\mathscr{R}) = \bar{G}\big(\mathscr{R}; \Phi^{(\alpha)}(\mathscr{R})\big)$ ist, so sei

$$\tilde{F}^{(\alpha)}(\mathscr{R}) = F.$$

b^2) Im entgegengesetzten Fall haben wir gewiß $F^{(\alpha)}(\mathscr{R}) \subset \bar{G}\big(\mathscr{R}; \Phi^{(\alpha)}(\mathscr{R})\big)$ (vgl. (2.2.)). Es sei also dann

$$z \in \bar{G}\big(\mathscr{R}; \ \Phi^{(\alpha)}(\mathscr{R})\big) \setminus F^{(\alpha)}(\mathscr{R})$$

und zwar wieder so gewählt, daß

$$(G_z(\mathscr{R}),\ F_z(\mathscr{R})) \neq (G(\mathscr{R};\ \tilde{F}^{(\beta)}(\mathscr{R})),\ \tilde{F}^{(\beta)}(\mathscr{R}))$$
$$(\beta < \alpha)$$

ausfällt (vgl. den analogen Fall unter b_2). Dann sei

$$\tilde{F}^{(\alpha)}(\mathscr{R}) = F_z(\mathscr{R}).$$

7° Die angedeutete zusätzliche Eigenschaft der eben definierten $\mathfrak{R}$-Halbräume unter (2.2.5) kann nun, gedrängt, so formuliert werden:

Für beliebige $\alpha_1 < \alpha_2 < \varkappa$ sind

$$\text{(2.2.6)} \qquad \bigcap_{i=1,2} \left[(\tilde{F}^{(\alpha_i)}(\mathscr{R}) \setminus \bar{G}(\mathscr{R};\ \tilde{F}^{(\alpha_i)}(\mathscr{R})) \cup (\underline{F}(\mathscr{R};\ \tilde{G}^{(\alpha_i)}(\mathscr{R})) \setminus \tilde{G}^{(\alpha_i)}(\mathscr{R}))\right] = \varnothing$$
$$(\mathscr{R} \in \mathfrak{R}^{(\alpha_1)} \cap \mathfrak{R}^{(\alpha_2)}),$$

was aus der Konstruktion unter 6° hervorgeht. (Man bedenke bezüglich der Fälle b_1) und b^1), daß dort

$$\underline{F}(\mathscr{R};\ \tilde{G}^{(\alpha)}(\mathscr{R})) = \tilde{G}^{(\alpha)}(\mathscr{R}) \quad \text{bzw.} \quad \bar{G}(\mathscr{R};\ \tilde{F}^{(\alpha)}(\mathscr{R})) = \tilde{F}^{(\alpha)}(\mathscr{R})$$

gilt, und bezüglich der Fälle b_2) und b^2) die Bemerkung unter (0.4),(b) in [1].)

8° Nach 6° und wegen $J^{(\alpha)} \neq \varnothing$ $(\alpha < \varkappa)$ haben wir

$$F^{(\alpha)}(\mathscr{R}) \subseteqq \tilde{F}^{(\alpha)}(\mathscr{R}) \subseteqq \bar{G}(\mathscr{R};\ \Phi^{(\alpha)}(\mathscr{R})) \subseteqq \Phi^{(\alpha)}(\mathscr{R}) \subset$$
$$\subset \Gamma^{(\alpha)}(\mathscr{R}) \subseteqq \underline{F}(\mathscr{R};\ \Gamma^{(\alpha)}(\mathscr{R})) \subseteqq \tilde{G}^{(\alpha)}(\mathscr{R}) \subseteqq G^{(\alpha)}(\mathscr{R})$$
$$(\mathscr{R} \in \mathfrak{R}^{(\alpha)},\ \alpha < \varkappa)$$

und somit ist für jedes $\alpha < \varkappa$

$$\tilde{I}^{(\alpha)} = \bigcap \{\tilde{G}^{(\alpha)}(\mathscr{R}) \setminus \tilde{F}^{(\alpha)}(\mathscr{R}):\ \mathscr{R} \in \alpha^{(\alpha)}\}$$

ein Element von $\mathscr{I}(\mathfrak{R})$ mit

$$\text{(2.2.7)} \qquad J^{(\alpha)} \subseteqq \tilde{I}^{(\alpha)} \subseteqq I^{(\alpha)} \quad (\alpha < \varkappa).$$

9° Aus (2.2.6) ist zu ersehen, daß das System $\mathfrak{R}$-offener Intervalle

$$\text{(2.2.8)} \qquad \tilde{\mathscr{I}}^* = \{\tilde{I}^{(\alpha)}:\ \alpha < \varkappa\}$$

von allgemeiner Lage in X bezüglich $\mathfrak{R}$ ist.*

10° Wir zeigen endlich, daß das System $\tilde{\mathscr{I}}^*$ eine Basis der Topologie von X ist.

(a) Für einen beliebigen Punkt $x \in X$ und eine beliebige Umgebung U von x in X gibt es, nach 4°, einen endlichen Teil $\mathfrak{R}^*$ von $\mathfrak{R}$ und $\mathscr{R}$-Halbräume

$$G(\mathscr{R}) \in \mathscr{G}(\mathscr{R}),\quad F(\mathscr{R}) \in \mathscr{F}(\mathscr{R}) \quad (\mathscr{R} \in \mathfrak{R}^*),$$

so daß mit der Bezeichnung

$$I = \bigcap \{G(\mathscr{R}) \setminus F(\mathscr{R}):\ \mathscr{R} \in \mathfrak{R}^*\}$$

sowohl

$$I \in \mathscr{I}^*$$

* Unter (2.2.6) wurde eigentlich viel mehr ausgesagt, als dies Eigenschaft; es scheint aber nicht besonders wichtig, dieses Mehr etwa auch in der Formulierung unseres Satzes auszudrücken.

wie auch

(2.2.9) $$x \in I \subseteq U$$

gilt. Wir haben dann, nach (0.8),

$$F(\mathscr{R}) \subseteq G_x(\mathscr{R}) \subset F_x(\mathscr{R}) \subseteq G(\mathscr{R}) \quad (\mathscr{R} \in \mathfrak{R}^*).$$

(b) Jetzt soll zu diesem I ein $J \in \mathscr{I}^*$ mit $(J, I) \in \mathfrak{I}^*$ gesucht werden. Mit den Bezeichnungen

(2.2.10$_1$) $$\mathfrak{R}_1^* = \{\mathscr{R} \in \mathfrak{R}^* : F(\mathscr{R}) \in \mathscr{G}(\mathscr{R})\}$$

und

(2.2.10$_2$) $$\mathfrak{R}_2^* = \{\mathscr{R} \in \mathfrak{R}^* : G(\mathscr{R}) \in \mathscr{F}(\mathscr{R})\}$$

(was sich insbesondere, aber evtl. nicht ausschließlich, auf die $\mathscr{R} \in \mathfrak{R}^*$ mit $F(\mathscr{R}) = G_x(\mathscr{R})$ bzw. $F_x(\mathscr{R}) = G(\mathscr{R})$ bezieht), gibt es nach (0.8) Paare

(2.2.11)
$$(G_1(\mathscr{R}), F_1(\mathscr{R})) \in \mathscr{R} \quad (\mathscr{R} \in \mathfrak{R}^* \setminus \mathfrak{R}_1^*),$$
$$(G_2(\mathscr{R}), F_2(\mathscr{R})) \in \mathscr{R} \quad (\mathscr{R} \in \mathfrak{R}^* \setminus \mathfrak{R}_2^*)$$

mit

(2.2.12)
$$F(\mathscr{R}) \subseteq G_1(\mathscr{R}) \subset F_1(\mathscr{R}) \subseteq G_x(\mathscr{R}) \quad (\mathscr{R} \in \mathfrak{R}^* \setminus \mathfrak{R}_1^*),$$
$$F_x(\mathscr{R}) \subseteq G_2(\mathscr{R}) \subset F_2(\mathscr{R}) \subseteq G(\mathscr{R}) \quad (\mathscr{R} \in \mathfrak{R}^* \setminus \mathfrak{R}_2^*).$$

Aus Bezeichnungsgründen seien

(2.2.13)
$$G'(\mathscr{R}) = \begin{cases} G(\mathscr{R}) & (\mathscr{R} \in \mathfrak{R}_2^*), \\ G_2(\mathscr{R}) & (\mathscr{R} \in \mathfrak{R}^* \setminus \mathfrak{R}_2^*), \end{cases}$$
$$F'(\mathscr{R}) = \begin{cases} F(\mathscr{R}) & (\mathscr{R} \in \mathfrak{R}_1^*), \\ F_1(\mathscr{R}) & (\mathscr{R} \in \mathfrak{R}^* \setminus \mathfrak{R}_2^*), \end{cases}$$

(2.2.13a) $$I' = \bigcap \{G'(\mathscr{R}) \setminus F'(\mathscr{R}) : \mathscr{R} \in \mathfrak{R}^*\}.$$

Es gibt nun, im Sinne von 4°, ein $J \in \mathscr{I}^*$ mit

(2.2.14) $$x \in J \subseteq I',$$

und wir können, wieder aus Bezeichnungsgründen, o. B. d. A. annehmen, daß J eine Darstellung

(2.2.15) $$J = \bigcap \{\Gamma(\mathscr{R}) \setminus \Phi(\mathscr{R}) : \mathscr{R} \in \mathfrak{R}^*\}$$
$$(\Gamma(\mathscr{R}) \in \mathscr{G}(\mathscr{R}),\ \Phi(\mathscr{R}) \in \mathscr{F}(\mathscr{R})\ (\mathscr{R} \in \mathfrak{R}^*))$$

zuläßt.

Ist nämlich

$$J = \bigcap \{\Gamma(\mathscr{R}) \setminus \Phi(\mathscr{R}) : \mathscr{R} \in \mathfrak{R}^{**}\}$$

mit $\mathfrak{R}^{**} \neq \mathfrak{R}^*$, so sei

$$\mathfrak{R}^{***} = \mathfrak{R}^* \cup \mathfrak{R}^{**};$$

wenn wir nun

(2.2.16)
$$G(\mathscr{R}) = X,\ F(\mathscr{R}) = \varnothing \quad (\mathscr{R} \in \mathfrak{R}^{**} \setminus \mathfrak{R}^*),$$
$$\Gamma(\mathscr{R}) = X,\ \Phi(\mathscr{R}) = \varnothing \quad (\mathscr{R} \in \mathfrak{R}^* \setminus \mathfrak{R}^{**})$$

setzen, weiter (2.2.13) sinngemäß durch

$$(2.2.17) \qquad \begin{aligned} G'(\mathscr{R}) &= G(\mathscr{R}) \\ F'(\mathscr{R}) &= F(\mathscr{R}) \end{aligned} \qquad (\mathscr{R}\in\mathfrak{R}^{**}\setminus\mathfrak{R}^{*})$$

erweitern und dementsprechend (2.2.10) durch

$$(2.2.18_1) \qquad \mathfrak{R}_1^{***} = \{\mathscr{R}\in\mathfrak{R}^{***}\colon\ F(\mathscr{R})\in\mathscr{G}(\mathscr{R})\},$$

$$(2.2.18_2) \qquad \mathfrak{R}_2^{***} = \{\mathscr{R}\in\mathfrak{R}^{***}\colon\ G(\mathscr{R})\in\mathscr{F}(\mathscr{R})\}$$

ersetzen (wobei dann $\mathfrak{R}^{**}\setminus\mathfrak{R}^{*}\subseteq\mathfrak{R}_1^{***}\cap\mathfrak{R}_2^{***}$ wird), so erhalten wir die Darstellungen

$$I = \bigcap\{G(\mathscr{R})\setminus F(\mathscr{R})\colon\ \ \mathscr{R}\in\mathfrak{R}^{***}\},$$

$$I' = \bigcap\{G'(\mathscr{R})\setminus F'(\mathscr{R})\colon\ \mathscr{R}\in\mathfrak{R}^{***}\},$$

$$J = \bigcap\{\Gamma(\mathscr{R})\setminus\Phi(\mathscr{R})\colon\ \ \mathscr{R}\in\mathfrak{R}^{***}\},$$

wo — nach (2.2.10), (2.2.16), (2.2.17) und (2.2.18) — auch noch (2.2.11) und (2.2.12) sinngemäß (d. h. mit $\mathfrak{R}^{***}$, $\mathfrak{R}_1^{***}$ bzw. $\mathfrak{R}_2^{***}$ an Stelle von $\mathfrak{R}^{*}$, $\mathfrak{R}_1^{*}$ bzw. $\mathfrak{R}_2^{*}$) erhalten bleibt.

(c) Um einzusehen, daß für dieses J, mit der Darstellung unter (2.2.15), tatsächlich $(J, I)\in\mathfrak{I}^{*}$ ist, muß außer

$$(2.2.19) \qquad J\subseteq I,$$

was aus (2.2.13), (2.2.13a), (2.2.12), und (2.2.14) folgt, nach (2.2.4) noch

$$(J, I)\in\mathfrak{I}$$

gezeigt werden (vgl. (2.2.1)). Aber auch das wird selbstverständlich, wenn man bedenkt, daß J, nach (2.2.13a), (2.2.14) und (2.2.15), die Darstellung

$$J = \bigcap\{\Gamma'(\mathscr{R})\setminus\Phi'(\mathscr{R})\colon\ \mathscr{R}\in\mathfrak{R}^{*}\}$$

mit

$$\Gamma'(\mathscr{R}) = \Gamma(\mathscr{R})\cap G'(\mathscr{R}), \quad \Phi'(\mathscr{R}) = \Phi(\mathscr{R})\cup F'(\mathscr{R}) \quad (\mathscr{R}\in\mathfrak{R}^{*})$$

zuläßt, wobei nach den Richtungsaxiomen ([1], (0.1),(a), (III), (IV))

$$\Gamma'(\mathscr{R})\in\mathscr{G}(\mathscr{R}), \quad \Phi'(\mathscr{R})\in\mathscr{F}(\mathscr{R}) \quad (\mathscr{R}\in\mathfrak{R}^{*})$$

sind und nach (2.2.13), (2.2.10) und (2.2.12) tatsächlich

$$\underline{F}(\mathscr{R};\ \Gamma'(\mathscr{R}))\subseteq G(\mathscr{R}), \quad F(\mathscr{R})\subseteq\overline{G}(\mathscr{R};\ \Phi'(\mathscr{R})) \quad (\mathscr{R}\in\mathfrak{R}^{*})$$

gilt.

(d) Endlich haben wir, für das unter 5°, 6° und 8° eingeführte, zu unserem $(J, I)\in\mathfrak{I}^{*}$ gehörende $\mathfrak{R}$-offene Intervall $\tilde{I}$, nach (2.2.9), (2.2.14), (2.2.19) und (2.2.7),

$$x\in\tilde{I}\subseteq U,$$

also ist das $\tilde{\mathscr{I}}^{*}$ unter (2.2.8) tatsächlich eine Basis des Raumes X.

Damit ist, nach dem Ergebnis unter 9°, der Satz bewiesen. □

(2.3) BEMERKUNG. (a) Daß für den Satz (2.2) die Voraussetzung der Ordentlichkeit von $\mathfrak{R}$ wesentlich ist, leuchtet einem unmittelbar ein. Man fragt sich aber, ob die Bedingung $w(X)\leqq 2^{\aleph_0}$ nicht weggelassen oder wenigstens abgeschwächt

werden könnte, zumal diese Bedingung — vgl. die Fußnote zum Teil 9° des Beweises — eigentlich ein schärferes Ergebnis zeitigt, als die Behauptung des Satzes aussagt.

(b) Ein Beispiel von J. DEÁK (mündliche Mitteilung) zeigt, daß dem doch nicht so ist.

Es seien η die Anfangszahl der kleinsten Mächtigkeit größer als $2^{\aleph_0}$, $X=\eta\times[0,1]$ mit der lexikographischen Ordnung $\prec$ versehen und $\mathfrak{R}=\{\mathscr{R}\}$ mit

$$\mathscr{R} = \{(\varnothing, \varnothing)\}\cup(G_x, F_x)\colon x\in X\}\cup\{(X, X)\},$$

wobei

$$\begin{aligned} G_x &= \{x'\in X: x'\prec x\} \\ F_x &= \{x'\in X: x'\leqq x\} \end{aligned} \qquad (x\in X).$$

Dieses $\mathscr{R}$ ist eine saturierte Richtung des ordnungstopologischen Raumes X, und $\mathfrak{R}$ ist im wesentlichen die einzige MKRS von X; $\mathfrak{R}$ ist obendrein eine ORS (der Raum X ist nicht nur schwach ordentlich, sondern sogar ordentlich, vgl. etwa [2]).

Ist nun $\mathscr{B}\subseteqq\mathscr{I}(\mathfrak{R})$ eine Basis der Topologie von X, so sei für jedes $\alpha<\eta$ ein $I_\alpha\in\mathscr{B}$ mit

$$\alpha\in I_\alpha, \quad \alpha+1\notin I_\alpha$$

(wobei dann $I_{\alpha_1}\neq I_{\alpha_2}$ für $\alpha_1\neq\alpha_2$) gewählt, und wir setzen

$$f(0)=0,$$

$$f(\alpha)=\min\{\beta\colon \beta+1\in I_\alpha\} \quad (0<\alpha<\eta).$$

Damit ist eine regressive Funktion $f\colon \eta\to\eta$ definiert (d. h. $f(\alpha)<\alpha$ für $\alpha<\eta$), und nach einer Verallgemeinerung des klassischen Satzes von ALEXANDROFF—URYSOHN über regressive Funktionen ([5], 80) gibt es ein $\xi<\eta$ mit $|f^{-1}(\xi)|>2^{\aleph_0}$. Das bedeutet aber, daß im Ordnungsintervall $((\xi,0),(\xi+1,0))$, das von der Mächtigkeit des Kontinuums ist, die linken Endpunkte der I_α ($\alpha\in A$) für eine Teilmenge A von η von größerer Mächtigkeit als die des Kontinuums liegen, also verschiedene I_α mit gemeinsamen linken Endpunkt existieren müssen.

Das System $\mathscr{B}$ kann also nicht von allgemeiner Lage sein.

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ON TRANSFORMATIONS OF THE DISCRETE OPERATOR FIELD

by

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Introduction

In this paper we shall give an operational treatment of linear transformations of the discrete Mikusinski-operator field M, being a simple analogue of the discussions of GESZTELYI [1] and FÉNYES [2], [3].

This paper consists of four parts. The brief summary of the elements of the discrete operational calculus is given in the first chapter. Some elementary properties of the linear and continuous operator transformations are proved in the second chapter. The third chapter contains the differentiation and series expansion of the linear operator transformations. Finally, the fourth chapter contains the detailed operational discussion of the integration of operator transformations and of differential equations related to operator transformations.

The discrete operational notations and symbols of BUTZER—SCHULTE [4] will be generally used.

1. The elements of the discrete operational calculus

The symbol $a=\{a(n)\}$ denotes an arbitrary finite and complex valued function defined in the discrete points $n=0, 1, 2, \ldots$, of the positive half line. The symbol $a(n)$ denotes the values of the function $\{a(n)\}$ for $n=0, 1, \ldots$. The set of the above functions is denoted by E. Two operations are defined in E:

Addition: $a+b = \{a(n)\}+\{b(n)\} = \{a(n)+b(n)\}$,

Multiplication: $ab = \{a(n)\}\{b(n)\} = \left\{\sum_{k=0}^{n} a(k)b(n-k)\right\}$.

The set E is a commutative ring with respect to the addition and multiplication defined above. It has no divisors of zero and can be extended to a quotient field M. The elements of M are called discrete convolution quotients or operators. They are of the form

$$\frac{\{a(n)\}}{\{b(n)\}} \quad (\{a(n)\}, \{b(n)\} \in E, \ b \neq 0).$$

The field of the complex numbers is denoted by K.

E and K can be embedded isomorphic in the field M. The unit element of E is $\{\delta_0(n)\}$ where $\delta_m(n)$ denotes the Kronecker symbol. The unit element of E, M

and K can be identified algebraically. It is denoted by 1. Similarly the zero element of E, M, K will be denoted by 0.

Every number is a function in the discrete operational calculus, the value of the null-th component of which equals the value of the given number, the other components are zero.

Special operators

The summing operator. The function $h=\{1\}$ having the value 1 for every n, defines the summing operator, since $h\{a(n)\}=\{1\}\{a(n)\}=\left\{\sum_{n=0}^{n} a(k)\right\}$ for $\{a(n)\}\in E$.

The difference operator. The operator

$$q=\frac{1}{h-1} \tag{1.1}$$

defines the difference operator. Obviously $q\notin E$. Its fundamental property is

$$q\{a(n)\}=\{\Delta a(n)\}+(1+q)a(0) \quad \text{for} \quad \{a(n)\}\in E$$

where

$$\{\Delta a(n)\}=\{a(n+1)-a(n)\}.$$

More generally for the i-th difference the formula

$$q^i\{a(n)\}=\{\Delta^i a(n)\}+(1+q)\sum_{\nu=0}^{i-1} q^{i-1-\nu}\Delta^\nu a(0), \quad i=1,2,\ldots, \tag{1.2}$$

holds (see BUTZER—SCHULTE [4]).

The translation operator. The translation operator is defined by $v=\frac{1}{1+q}$.

It can be easily seen that

$$\frac{1}{(1+q)^m}=\{\delta_m(n)\}, \quad m=0,1,\ldots,$$

and

$$v^m\{a(n)\}=\{b(n)\}, \quad b(n)=\begin{cases} a(n-m), & \text{if} \quad n\geqq m \\ 0, & \text{if} \quad 0\leqq n<m \end{cases} \quad \text{for} \quad a\in E \tag{1.3}$$

hold. Moreover $v^i v^j=v^{i+j}$ for every integers i, j.

In the following the symbol $a_k=\{a_k(n)\}$ denotes a sequence of discrete functions. BERG [5] defines the convergence in M as follows.

A. A sequence $a_k=\{a_k(n)\}\in E$ converges pointwise to $\{a(n)\}\in E$ if for every fixed n

$$a_k(n)\to a(n) \quad \text{for} \quad k\to\infty \tag{1.4}$$

B. More generally, a sequence

$$x_k=\frac{\{a_k(n)\}}{\{b_k(n)\}}\in M$$

converges to $x=\frac{\{a_k(n)\}}{\{b_k(n)\}}\in M$, for $k\to\infty$, if a_k, b_k converge in the sense of **A** and $b\neq 0$.

For every fixed $x \in M$, there exists a sequence $\{x_k(n)\} \in E$ such that x_k converges to x in the sense of B. (see BERG [5]).

The above definition can be extended to the case of series:

A series $\sum_{\nu=0}^{\infty} \{\alpha_\nu(n)\}$ is said to be convergent if the sequence

$$\{a_k(n)\} = \sum_{\nu=0}^{k} \{\alpha_\nu(n)\} \quad \alpha_\nu \in E \tag{1.5}$$

converges in the sense of **A**. More generally, a series

$$\sum_{\nu=0}^{\infty} y_\nu \quad y_\nu \in M \tag{1.6}$$

is said to be convergent, if the sequence

$$x_k = \sum_{\nu=0}^{k} y_\nu$$

converges in the sense of **B**.

Obviously, the convergence of $\{x_k(n)\}$ in the sense of **A** implies its convergence in the sense of **B**. In the sequel, we shall say that a sequence (or a series) of operators is convergent in the sense of the operational calculus, if it converges in the sense of **B**.

Operationally convergent sequences have the following properties:

$$\lim_{k\to\infty} (a_k + b_k) = \lim_{k\to\infty} a_k + \lim_{k\to\infty} b_k,$$

$$\lim_{k\to\infty} (a_k b_k) = (\lim_{k\to\infty} a_k)(\lim_{k\to\infty} b_k) \quad (a_k, b_k \in M),$$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \frac{\lim\limits_{k\to\infty} a_k}{\lim\limits_{k\to\infty} b_k} \quad (\lim_{k\to\infty} b_k \neq 0).$$

Moreover for every $x \in M$

$$x = \sum_{\nu=\varrho}^{\infty} \frac{x(\nu)}{(1+q)^\nu} \quad x(\nu) \in K \quad (\varrho \text{ is some integer}) \tag{1.7}$$

holds. The number $x(\nu)$ will be called the ν-th component of the operator x. If $x(\nu)=0$, for $\nu<0$, we have that for every $\{x(n)\} \in E$

$$\{x(n)\} = \sum_{\nu=0}^{\infty} \frac{x(\nu)}{(1+q)^\nu} \tag{1.8}$$

holds (see also BUTZER—SCHULTE [4]).

2. Linear operator transformations

A single valued map F of M into M will be called a linear operator transformation if for every $x, y \in M$ and $\alpha, \beta \in K$

$$F(\alpha x+\beta y) = \alpha F(x)+\beta F(y) \tag{2.1}$$

holds.

An operator transformation F is said to be continuous at $x_0 \in M$, if for every sequence $x_k \in M$ converging operationally to $x_0 \in M$, the sequence $F(x_k)$ converges to $F(x_0)$ in the sense of **B**. F is said to be continuous, if it is continuous at every $x \in M$.

LEMMA 2.1. *Let F be an arbitrary linear and continuous operator transformation. Moreover let $x \in M$ be an arbitrary discrete operator, such that*

$$x = \sum_{\nu=\varrho}^{\infty} \frac{x(\nu)}{(1+q)^{\nu}}$$

holds. Then

$$F(x) = \sum_{\nu=\varrho}^{\infty} x(\nu) F\left[\frac{1}{(1+q)^{\nu}}\right] \tag{2.2}$$

holds.

The proof is obvious.

An operator transformation F is multiplicative, if for every $x, y \in M$

$$F(xy) = F(x)\,F(y) \tag{2.3}$$

holds.

LEMMA 2.2. *Let F be an arbitrary linear, continuous and multiplicative operator transformation. Moreover let $x \in M$ be an arbitrary discrete operator, such that $x = \sum_{\nu=\varrho}^{\infty} \frac{x(\nu)}{(1+q)^{\nu}}$ holds. Then*

$$F(x) = \sum_{\nu=\varrho}^{\infty} x(\nu) \frac{1}{[1+F(q)]^{\nu}} \tag{2.4}$$

holds.

PROOF. (2.4) is a simple consequence of the fact that for every $c \in K$, $F(c) = c$ holds.

Analogously to the continuous case (see GESZTELYI [1], page 172), we denote the set of the linear operator transformations by τ. The addition and multiplication is defined in τ by the definition

$$\begin{aligned} (F+G)(x) &= F(x)+G(x) \\ (FG)(x) &= F[G(x)] \end{aligned} \qquad (F, G \in \tau,\ x \in M).$$

Obviously, the set τ is a ring, moreover the sum and product of continuous operator transformations are also continuous.

EXAMPLES

I. Let us consider the so-called algebraic derivative defined by

$$D(a) = \{-na(n)\} \quad \text{for} \quad a \in E,$$

$$D(x) = \frac{bD(a) - aD(b)}{b^2} \quad \text{for} \quad x = \frac{a}{b} \in M \quad (a, b \in E)$$

(see FÉNYES—KOSIK [6]). Obviously D is linear, and it can be easily shown that it is also continuous.

Operator transformations of the form

$$F = cD, \quad c \in M$$

are also derivations of M. In the sequel we shall make use also of the algebraic derivative defined by

$$D^* = \frac{1}{1+q} D. \tag{2.5}$$

For $a \in E$ we have

$$D^*(a) = b = \{b(n)\}$$

where

$$b(n) = \begin{cases} 0, & \text{if} \quad n = 0 \\ -(n-1)f(n-1) & \text{if} \quad n > 0. \end{cases}$$

It is easy to see, that D^* is continuous.

Since $D[(1+q)^m] = m(1+q)^m$ for every integer m (see [6]) by (2.2) we have that for $x = \sum_{\nu=\varrho}^{\infty} \frac{x(\nu)}{(1+q)^\nu}$

$$D(x) = \sum_{\nu=\varrho}^{\infty} \frac{-\nu x(\nu)}{(1+q)^\nu} \tag{2.6}$$

holds.

II. Let us consider the multiplicative operator transformation $F = T^\alpha$ defined for every $\alpha \in K$ $(\alpha \neq 0)$ by

$$T^\alpha[\{a(n)\}] = \{\alpha^n a(n)\} \quad \text{for} \quad a \in E, \alpha \in K,$$

$$T^\alpha(x) = \frac{T^\alpha(a)}{T^\alpha(b)} \quad \text{for} \quad x = \frac{a}{b} \in M, \quad a, b \in E. \tag{2.7}$$

The linearity of T^α is obvious and it can be easily seen that it is also continuous. The multiplicativity of T^α follows simply from the identity

$$\alpha^n \sum_{k=0}^{n} f(k)g(n-k) = \sum_{k=0}^{n} \alpha^k f(k) \alpha^{n-k} g(n-k), \quad f, g \in E, \ \alpha \in K.$$

Since $T^\alpha\left[\frac{1}{(1+q)^\nu}\right]=\frac{\alpha^\nu}{(1+q)^\nu}$ holds for every integer ν, by (2.4) we have for $x=\sum_{\nu=\varrho}^{\infty}\frac{x(\nu)}{(1+q)^\nu}$

$$T^\alpha(x)=\sum_{\nu=\varrho}^{\infty}\frac{x(\nu)}{[1+T(q^\alpha)]^\nu}=\sum_{\nu=\varrho}^{\infty}\frac{\alpha^\nu x(\nu)}{(1+q)^\nu}.$$

REMARK. The multiplicative transformation U_k is well known in Mikusinski's operational calculus based on the continuous variable t:

$$U_k(f)=\{kf(kt)\} \quad \text{for} \quad f\in C, \quad k>0,$$

$$U_k(x)=\frac{U_k(f)}{U_k(g)} \quad \text{for} \quad x=\frac{f}{g}, \quad f, g\in C, \quad g\neq 0.$$

A discrete analogue of U_k may be defined in the ring E as follows:

$$U_\mu[\{a(n)\}]=\{a(\mu n)\}, \quad \mu=1, 2, \ldots .$$

The multiplicativity of U_μ follows from the formula

$$\sum_{k=0}^{\mu n}a(k)b(\mu n-k)=\sum_{k=0}^{n}a(\mu k)b[\mu(n-k)], \quad a, b\in E. \tag{2.8}$$

However the above definition can not be extended to the discrete operator field M since $U_\mu[\{a(n)\}]=0$ does not imply $\{a(n)\}=0$. For fixed $\mu>1$, the set of the functions $\{a(n)\}$ for which $U_\mu[\{a(n)\}]=0$ holds, is the kernel of the homomorphism of U_μ.

In the sequel we shall prove a theorem of GESZTELYI [1] in the discrete case.

THEOREM 2.1. *Let $F\neq 0$ be an arbitrary linear, continuous and multiplicative operator transformation such that*

$$FD=DF$$

holds. Then $F=T^\alpha$.

PROOF. For $x=q$ we have

$$D[F(q)]=(DF)(q)=(FD)(q)=F(1+q)=F(1)+F(q). \tag{2.9}$$

By introducing the denotation $F(q)=w$ we obtain the algebraic differential equation

$$Dw-w=1. \tag{2.10}$$

The general solution of (2.11) is of the form

$$w=F(q)=\cdot\cdot 1+\frac{1+q}{\alpha}, \quad \alpha\in K$$

(see FÉNYES—KOSIK [6]). We have that

$$1+F(q)=\frac{1+q}{\alpha} \tag{2.11}$$

holds. By lemma 2.2 we obtain

$$F(x) = \sum_{i=e}^{\infty} \frac{x(i)}{[1+F(q)]^i} = \sum_{i=e}^{\infty} \frac{\alpha^i x(i)}{(1+q)^i} = T^\alpha(x)$$

for every $x \in M$ and

$$F = T^\alpha. \tag{2.12}$$

3. The differentiation and series expansion of operator transformations

For every $F \in \tau$ we define the derivative of F by the definition

$$F' = aF - Fa \tag{3.1}$$

where $a \in M$ is arbitrary but fixed, see GESZTELYI [1]. Higher order derivatives of F are defined as usual. Obviously the above definition is correct since $F' \in \tau$ and

$$(F+G)' = F'+G', \quad (FG)' = F'G + FG'$$

hold (see also BERG [7]). It is easy to show that the continuity of F implies the continuity of F'.

EXAMPLES 1. $F = cI$, $c \in M$, $I(x) = x$ for every $x \in M$. We have

$$F'(x) = acx - cax = 0$$

and

$$F' = 0. \tag{3.2}$$

2. F is multiplicative. We have

$$F'(x) = aF(x) - F(ax) = aF(x) - F(a)F(x) = [a - F(a)]F(x).$$

In the sequel we assume that $a = q$.

For $F = T^\alpha$ we have

$$(T^\alpha)' = [q - T^\alpha(q)]T^\alpha = \left(1 - \frac{1}{\alpha}\right)(1+q)T^\alpha \quad (\alpha \neq 0). \tag{3.3}$$

3. $F = D^*$. By the application of $Dq = 1+q$ we have

$$(D^*)'(x) = \left(\frac{1}{1+q} D\right)'(x) = \frac{1}{1+q}(D')(x) = \frac{1}{1+q}(qD - Dq)(x) =$$

$$= \frac{q}{1+q} D(x) - \frac{1}{1+q} D(qx) = \frac{q}{1+q} D(x) - \frac{(1+q)x + qD(x)}{1+q} = -x$$

and

$$(D^*)' = -I. \tag{3.4}$$

(3.4) can be extended to the case of

$$F = (D^*)^n \tag{3.5}$$

where n is an arbitrary positive integer and $(D^*)^0=I$.

$$[(D^*)^n]' = -n(D^*)^{n-1} \quad (n \geqq 1) \tag{3.6}$$

holds. We prove (3.6) by total induction. For $n=1$ we obtain (3.4). If (3.6) holds for some n, then

$$[(D^*)^{n+1}]'(x) = q(D^*)^{n+1}(x) - (D^*)^{n+1}(qx) =$$

$$= q(D^*)^{n+1}(x) - \sum_{\mu=0}^{n+1} \binom{n+1}{\mu} (D^*)^{\mu}(q)(D^*)^{n+1-\mu}(x) =$$

$$= q(D^*)^{n+1}(x) - q(D^*)^{n+1}(x) = (n+1)(D^*)^n(x) -$$

$$-(n+1)(D^*)^n(x), \quad x \in M$$

also holds. We have used here that $(D^*)^{\mu}(q)=0$ for $\mu>1$, which is true since

$$(D^*)^2(q) = D^*[D^*(q)] = D^*\left[\frac{1}{1+q}D(q)\right] = D^*\left[\frac{1}{1+q}(1+q)\right] = \frac{1}{1+q}D(1) = 0$$

holds.

REMARK. In the original operational calculus based on the variable t GESZTELYI [1] defines the derivative of F as

$$F' = sF - Fs$$

where s is the differentiation operator. GESZTELYI [1] proves that the commutability of F and D implies the commutability of F' and D. This is not true in the discrete case. For example $(T^{\alpha})'$ and D do not commute since

$$(T^{\alpha})'[D(x)] = \left(1-\frac{1}{\alpha}\right)(1+q)T^{\alpha}D(x)$$

and

$$[D(T^{\alpha})'](x) = D\left[\left(1-\frac{1}{\alpha}\right)(1+q)T^{\alpha}(x)\right] = \left(1-\frac{1}{\alpha}\right)D[(1+q)T^{\alpha}(x)] =$$

$$= \left(1-\frac{1}{\alpha}\right)(1+q)T^{\alpha}(x) + \left(1-\frac{1}{\alpha}\right)(1+q)D[T^{\alpha}(x)] = (T^{\alpha})'(x) + (T^{\alpha})'[D(x)].$$

In the following we shall deal with the Taylor-series expansion of linear, continuous, multiplicative operator transformations (see FÉNYES [3]). If for a given linear, continuous and multiplicative operator transformation F there exists a subdomain $M_F^* \subset M$ such that for every $x \in M_F^*$

$$F(x) = \sum_{k=0}^{\infty} \frac{(-1)^k F^{(k)}(1)}{k!} (D^*)^k(x) \tag{3.7}$$

holds in the sense of the operational convergence, we say that F is analytic in M_F^*. In the case of $M_F^* = M$ we say that F is analytic. Obviously $F^{(k)}$ denotes the k-th derivative of F, $F^{(0)}=I$.

It can be seen that M_F^* is a linear space over the field K of the complex numbers. Moreover if (3.7) holds for a fixed $x \in M$, we say that F is analytic at x.

THEOREM 3.1. *If F is analytic at $x\in M$, then it is so at qx.*

PROOF. Since $F^{(k)}=[q-F(q)]^k F$ we have that

$$\sum_{k=0}^{\infty}\frac{(-1)^k F^{(k)}(1)}{k!}(D^*)^k(qx)=\sum_{k=0}^{\infty}\frac{[F(q)-q]^k}{k!}(D^*)^k(qx)= \tag{3.8}$$

$$=qx+\sum_{k=1}^{\infty}\frac{[F(q)-q]^k}{k!}\sum_{\mu=0}^{k}\binom{k}{\mu}(D^*)^{\mu}(q)(D^*)^{k-\mu}(x)=$$

$$=qx+\sum_{k=1}^{\infty}\frac{[F(q)-q]^k}{k!}[q(D^*)^k(x)+k(D^*)^{k-1}(x)]=$$

$$=q\sum_{k=0}^{\infty}\frac{[F(q)-q]^k}{k!}(D^*)^k(x)+[F(q)-q]\sum_{k=1}^{\infty}\frac{[F(q)-q]^{k-1}}{(k-1)!}(D^*)^{k-1}(x)=$$

$$=qF(x)+[F(q)-q]F(x)=F(q)F(x)=F(qx).$$

COROLLARY 1. F is analytic at $x=\sum_{v=0}^{n}a(v)q^v$, $a_v\in K$, n is arbitrary. This property of F holds trivially for $n=0$ and $a(0)=1$. It follows from the above theorem, that it also holds for $x=q, x=q^2, \ldots, x=q^n$. Since M_F^* is a linear space over K, we obtain that the Corollary holds.

COROLLARY 2. Let x be an arbitrary discrete operator such that

$$x=\sum_{v=\varrho}^{\infty}\frac{a(v)}{(1+q)^v},\quad a(v)\in K,$$

holds. If $a(v)=0$, for $v>0$, then F is analytic at x. Corollary 2 is a simple consequence of Corollary 1.

COROLLARY 3. If F is analytic at a function $\{a(n)\}$, then it is so at $\{a(n+N)\}$ where N is an arbitrary positive integer. The fundamental property of the difference operator q is

$$q\{a(n)\}=\{\Delta a(n)\}+(1+q)a(0)=\{a(n+1)-a(n)\}+(1+q)a(0);$$

we have

$$\{a(n+1)\}=q\{a(n)\}+\{a(n)\}-(1+q)a(0). \tag{3.9}$$

From Corollary 1 and 2 we see that Corollary 3 holds for $N=1$. Consequently it must hold for arbitrary N.

COROLLARY 4. If F is not analytic at $x\in M$, then it is also not analytic at $\frac{x}{1+q}$.

Let us suppose that F is analytic at $\frac{x}{1+q}$. By Theorem 3.1 one has that F is analytic at $\frac{qx}{1+q}+\frac{x}{1+q}=\frac{(q+1)x}{q+1}=x$. Contradiction.

LEMMA 3.1. *If F is not analytic at the function $x=\{\delta_1(n)\}=\frac{1}{1+q}$, then F is also not analytic at $\{x(n)\}=\sum_{\nu=0}^{N}\frac{x(\nu)}{(1+q)^\nu}$ for arbitrary $N>1$, and $x(\nu)\in K$.*

PROOF. Let us fix N and $x(\nu)$ arbitrarily. Since F is not analytic at $\frac{1}{1+q}$, one has by the Corollary 1 of the above Theorem that F is also not analytic at

$$a(1)+\frac{a(0)}{1+q}, \quad a(0), a(1)\in K, \quad a(0)\neq 0.$$

But then, applying Corollary 4 we obtain that F is also not analytic at $a(2)+\frac{a(1)}{1+q}+$ $+\frac{a(0)}{(1+q)^2}$, $a(2)\in K$.

Repeating this procedure we obtain that F is not analytic at

$$a(N)+\frac{a(N-1)}{1+q}+\ldots+\frac{a(0)}{(1+q)^N}, \tag{3.10}$$

and by choosing $a(N)=x(0)$, $a(N-1)=x(1)$, ..., $a(0)=x(N)$ the lemma is proved.

REMARK. It would be interesting to prove that Lemma 3.1 also holds in the case of $N=\infty$. We shall prove this only in the special case of $F=T^\alpha$.

THEOREM 3. 2. *Let us consider $F=T^\alpha$. If $\operatorname{Re}[\alpha]>\frac{1}{2}$, then T^α is analytic. Moreover for $\operatorname{Re}[\alpha]\leqq\frac{1}{2}$, T^α is analytic at a function $\{x(n)\}=\sum_{\nu=0}^{\infty}\frac{x(\nu)}{(1+q)^\nu}$ if and only if $x(\nu)=0$, for $\nu>0$.*

PROOF. I. *The case of* $\operatorname{Re}[\alpha]>\frac{1}{2}$. From the Corollary 2 of the Theorem 3.1 we have that T^α is analytic for every α ($\alpha\neq 0$) at operators of the form

$$x=\sum_{\mu=0}^{N}a(\nu)(1+q)^\mu.$$

So we have only to show that T^α is analytic at an arbitrary function of the form

$$\{f(n)\}=\sum_{\nu=0}^{\infty}\frac{f(\nu)}{(1+q)^\nu}.$$

From the definition of D^* follows that

$$(D^*)^k(f)=\{u_k(n)\} \tag{3.11}$$

where

$$u_k(n)=\begin{cases}0, & \text{for} \quad n<k\\ (-1)^k(n-1)(n-2)\ldots(n-k)f(n-k) & \text{for} \quad n\geqq k.\end{cases} \tag{3.12}$$

(3.3) gives that

$$(T^{\alpha})^{(k)} = \left(1+\frac{1}{\alpha}\right)^{k}(1+q)^{k}T^{\alpha}$$

and

$$(T^{\alpha})^{(k)}(1) = \left(1-\frac{1}{\alpha}\right)^{k}(1+q)^{k}.$$

Applying (1.3), (3.12) and (3.7) for $F=T^{\alpha}$, $x=f=\{f(n)\}$ we obtain that

$$(3.13)\quad T^{\alpha}(f) = \sum_{k=0}^{\infty}\frac{(-1)^{k}(T^{\alpha})^{(k)}(1)}{k!}(D^{*})^{k}(f) = \sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!}\left(1-\frac{1}{\alpha}\right)^{k}(1+q)^{k}(D^{*})^{k}(f) =$$

$$= f+\sum_{k=1}^{\infty}\frac{\left(\frac{1}{\alpha}-1\right)^{k}}{k!}(1+q)^{k}\{u_{k}(n)\} = f+\sum_{k=1}^{\infty}\frac{\left(\frac{1}{\alpha}-1\right)^{k}}{k!}(-1)^{k}\{n(n+1)\dots$$

$$\dots(n+k-1)f(n)\} = \left\{f(n)\left[1+\sum_{k=1}^{\infty}\frac{\left(1-\frac{1}{\alpha}\right)^{k}}{k!}n(n+1)\dots(n+k-1)\right]\right\} =$$

$$= \left\{f(n)\left[1-\left(1-\frac{1}{\alpha}\right)\right]^{-n}\right\} = \{\alpha^{n}f(n)\}$$

holds in the sense of **A**, since $\left|1-\frac{1}{\alpha}\right|<1$ only if $\operatorname{Re}[\alpha]>\frac{1}{2}$.

II. *The case of* $\operatorname{Re}[\alpha]\leqq\frac{1}{2}$. Obviously without loss of generality we have only to show that a sequence of the form

$$(3.14)\qquad \{a_{j}(n)\} = \left\{f(n)\left[1+\sum_{k=1}^{j}\frac{\left(1-\frac{1}{\alpha}\right)^{k}}{k!}n(n+1)\dots(n+k-1)\right]\right\}$$

converges operationally for $j\to\infty$, if and only if $f(n)=0$ for $n>0$. (Since T^{α} is analytic for $\{f(n)\}=\gamma\in K$ we may assume that $f(0)=0$.)

Suppose the converse. Then there exists a sequence of functions $\{c_{j}(n)\}$ converging pointwise to $\{c(n)\}\neq 0$ such that the product

$$\{a_{j}(n)\}\{c_{j}(n)\} = \{b_{j}(n)\}$$

converges pointwise for every fixed n. We have

$$(3.15)\qquad b_{j}(n) = \sum_{i=0}^{n}a_{j}(i)c_{j}(n-i).$$

Let $c(L_{1})\neq 0$, and $c(n)=0$, for $n<L_{1}$. Such a number L_{1} exists since $\{c(n)\}\neq 0$.

Moreover let $f(L_2)\neq 0$, and $f(n)=0$, for $n<L_2$. We obtain from (3.14), (3.15) that

$$b_j(L_1+L_2)=\sum_{i=0}^{L_1+L_2} a_j(i)c_j(L_1+L_2-i)=a_j(L_2)c_j(L_1)= \tag{3.16}$$

$$=c_j(L_1)\left[f(L_2)\left(1+\sum_{k=1}^{j}\frac{\left(1-\frac{1}{\alpha}\right)^k}{k!}L_2(L_2+1)\dots(L_2+k-1)\right)\right]$$

holds. However the sequence (3.16) can converge for $j\to\infty$, only if $c_j(L_1)$ is a null-sequence, consequently $c(L_1)=0$. Contradiction. The Theorem is proved.

4. On the integration of operator transformations. Differential equations

DEFINITION. If for a given $G\in\tau$ there exists an $F\in\tau$ such that

$$G=qF-Fq=F' \tag{4.1}$$

holds, we say that G is integrable and F is an integral of G. This is denoted by

$$G^{(-1)}=F$$

(see also FÉNYES [2]). Obviously the operator transformation $G^{(-1)}+cI$ is also an integral of G for every $c\in M$.

The repeated integration of G leads to the concept of the m-th integral of G.

DEFINITION. If for a given $G\in\tau$ there exists an F such that

$$G=qF^{(m-1)}-F^{(m-1)}q,\quad m\geqq 1,\quad F^{(0)}=I, \tag{4.2}$$

we say that F is an m-th integral of G denoted by

$$G^{(-m)}=F.$$

It is obvious that the formula

$$(c_1G+c_2H)^{(-m)}=c_1G^{(-m)}+c_2H^{(-m)},\quad c_1,c_2\in M,\quad G,H\in\tau$$

holds.

LEMMA 4.1. *$(D^*)^m$ is integrable and the formula*

$$[(D^*)^m]^{(-1)}=-\frac{(D^*)^{m+1}}{m+1}\quad (m\geqq 0) \tag{4.3}$$

holds (see also FÉNYES [2]).

PROOF. (4.3) is a simple consequence of (3.6).

COROLLARY 1. $(D^*)^l$ has integrals of arbitrary high order and

$$[(D^*)^l]^{(-m)}=(-1)^m\frac{(D^*)^{l+m}}{(l+1)(l+2)\dots(l+m)}+\sum_{k=1}^{m+1}c_k(D^*)^k \tag{4.4}$$

holds, where $l \geqq 0, m \geqq 1$ and the operators c_k are arbitrary. For $l=0$ we have

$$I^{(-m)} = (-1)^m \frac{(D^*)^m}{m!}. \tag{4.5}$$

COROLLARY 2. If $F \in \tau$ is an m-th order integral of G, then

$$F + \sum_{k=0}^{m-1} c_k (D^*)^k, \quad c_k \in M,$$

are also m-th order integrals of G.

Now we prove a theorem of GESZTELYI [1] in the discrete case.

THEOREM 4.1. *If $F \in \tau$ is continuous such that*

$$F' = 0$$

then $F = cI$, $c \in M$.

PROOF. First we show that the formula

$$F\left[\frac{1}{(1+q)^k}\right] = \frac{F(1)}{(1+q)^k}, \quad k = 0, 1, 2, \ldots, \tag{4.6}$$

holds.

(4.6) is trivial for $k=0$. For $k>0$ we prove by total induction. $F'=0$ means that $F'(x)=(qF-Fq)(x)=0$ holds for every $x \in M$. For $k=1$ and $x=\frac{1}{1+q}$ we have

$$(qF)\left(\frac{1}{1+q}\right) = F\left(\frac{q}{1+q}\right) = F\left(\frac{q+1-1}{1+q}\right) = F(1) - F\left(\frac{1}{1+q}\right), \tag{4.7}$$

so

$$(1+q)F\left(\frac{1}{1+q}\right) = F(1),$$

and (4.6) holds for $k=1$. Suppose that (4.6) holds for a fixed k, we prove that it also holds for $k+1$.

$$qF\left[\frac{1}{(1+q)k+1}\right] = F\left[\frac{q}{(1+q)^{k+1}}\right] = F\left[\frac{1+q-1}{(1+q)^{k+1}}\right] = \tag{4.8}$$

$$= F\left[\frac{1}{(1+q)^k}\right] - F\left[\frac{1}{(1+q)^{k+1}}\right] = \frac{F(1)}{(1+q)^k} - F\left[\frac{1}{(1+q)^{k+1}}\right].$$

Consequently $(1+q)F\left[\frac{1}{(1+q)^{k+1}}\right] = \frac{F(1)}{(1+q)^k}$, and (4.6) will be obtained.

Now let $x \in M$ be an arbitrary discrete operator. Then there exists a sequence of functions $\{x_k(n)\} \in E$, such that

$$\lim_{k \to \infty} \{x_k(n)\} = x$$

holds in the sense of the operational convergence. From the continuity of F and (4.6) follows that

$$F(\{x_k(n)\}) = F\left[\sum_{i=0}^{\infty} \frac{x_k(i)}{(1+q)^i}\right] = \sum_{i=0}^{\infty} F\left[\frac{x_k(i)}{(1+q)^i}\right] = \tag{4.9}$$

$$= \sum_{i=0}^{\infty} \frac{x_k(i)F(1)}{(1+q)^i} = F(1)\{x_k(n)\}.$$

Again from the continuity of F follows that $F(x)=F(1)x$. Let $F(1)=c$, so we have that $F(x)=cx$ holds for every $x\in M$. Consequently

$$F = cI.$$

THEOREM 4.2. *If $F\in\tau$ is continuous such that*

$$F' = (D^*)^m \quad m = 0, 1, \dots,$$

then

$$F = -\frac{(D^*)^{m+1}}{m+1} + cI, \quad c\in M.$$

The proof goes quite analogously to that of the Theorem 4.1, so we omit it here.

THEOREM 4.3. *If $F\in\tau$ is continuous such that*

$$F^{(m)} = 0, \quad m = 1, 2, \dots,$$

then

$$F = \sum_{k=0}^{m-1} c_k (D^*)^k, \quad c_k\in M. \tag{4.10}$$

PROOF. (4.10) is a simple consequence of the above two Theorems since the derivative of a continuous F is also continuous.

COROLLARY. *If a continuous $F_0\in\tau$ is a solution of*

$$F^{(m)} = G, \quad m = 1, 2, \dots,$$

for a given $G\in\tau$, then

$$F = F_0 + \sum_{k=0}^{m-1} c_k (D^*)^k, \quad c_k\in M, \tag{4.11}$$

provided that $F\in\tau$ is also continuous.

REMARK. We can also write D instead of D^* in the formulae (4.10), (4.11) since it is not difficult to show that every linear combination of the powers of D^* is also a linear combination of powers of D

$$\sum_{k=0}^{m} c_k (D^*)^k = \sum_{k=0}^{m} \bar{c}_k D^k, \quad c_k, \bar{c}_k\in M. \tag{4.12}$$

THEOREM 4.4. *Every linear, continuous and multiplicative operator transformation is infinitely many times integrable.*

PROOF. Let F be such an operator transformation for which

$$F(q) \neq q.$$

It can be seen that

$$F^{(-m)} = \frac{1}{[q-F(q)]^m} F, \quad m = 1, 2, \ldots, \tag{4.13}$$

holds since $F' = [q-F(q)]F$. In the case of $F(q)=q$ we choose an arbitrary $x \in M$. We have by Lemma 2.2

$$F(x) = F\left(\sum_i \frac{x(i)}{(i+q)^i}\right) = \sum_i \frac{x(i)}{[1+F(q)]^i} = \sum_i \frac{x(i)}{(1+q)^i} = x. \tag{4.14}$$

So $F(x)=x$ holds for every $x \in M$, consequently $F=I$ holds. However (see (4.5)):

$$I^{(-m)} = \frac{(-1)^m}{m!}(D^*)^m$$

In the sequel we shall deal with the linear m-th order, homogeneous differential equation of the form

$$\sum_{k=0}^{m} a_k F^{(k)} = 0, \quad F \in \tau, \tag{4.15}$$

where the operators $a_k \in M$ are given. The continuity of F will be assumed.

DEFINITION. Let

$$A_j = (1+q)^j F\left[\frac{1}{(1+q)^j}\right]$$

for every integer j,

$$\nabla(A_j) = A_j - A_{j-1},$$
$$\nabla^k(A_j) = \nabla[\nabla^{k-1}(A_j)] \quad (k \geqq 1),$$
$$\nabla^0(A_j) = A_j.$$

LEMMA 4.2.

$$\nabla^k(A_j) = (1+q)^{j-k} F^{(k)}\left[\frac{1}{(1+q)^j}\right], \quad -\infty < j < \infty, \; k \geqq 0. \tag{4.16}$$

PROOF. The proof goes by total induction. For $k=0$ we obtain the definition of A_j. For $k=1$ we have

$$\nabla(A_j) = A_j - A_{j-1} = (1+q)^j F\left[\frac{1}{(1+q)^j}\right] - (1+q)^{j-1} F\left[\frac{1}{(1+q)^{j-1}}\right] =$$

$$= (1+q)^{j-1}\left((1+q) F\left[\frac{1}{(1+q)^j}\right] - F\left[\frac{1}{(1+q)^{j-1}}\right]\right) =$$

$$= (1+q)^{j-1}\left(qF\left[\frac{1}{(1+q)^j}\right] - F\left[\frac{1}{(1+q)^{j-1}} - \frac{1}{(1+q)^j}\right]\right) =$$

$$= (1+q)^{j-1}\left(qF\left[\frac{1}{(1+q)^j}\right] - F\left[\frac{q}{(1+q)^j}\right]\right) = (1+q)^{j-1} F'\left[\frac{1}{(1+q)^j}\right].$$

So (4.16) holds for $k=1$. Suppose that (4.16) holds for an arbitrary fixed k. We have

$$\nabla^{k+1}(A_j) = (1+q)^{j-k}F^{(k)}\left[\frac{1}{(1+q)^j}\right] - (1+q)^{j-1-k}F^{(k)}\left[\frac{1}{(1+q)^{j-1}}\right] =$$

$$= (1+q)^{j-k-1}\left((1+q)F^{(k)}\left[\frac{1}{(1+q)^j}\right] - F^{(k)}\left[\frac{1}{(1+q)^{j-1}}\right]\right) =$$

$$= (1+q)^{j-k-1}\left(qF^{(k)}\left[\frac{1}{(1+q)^j}\right] - F^{(k)}\left[\frac{1}{(1+q)^{j-1}} - \frac{1}{(1+q)^j}\right]\right) =$$

$$= (1+q)^{j-k-1}\left(qF^{(k)}\left[\frac{1}{(1+q)^j}\right] - F^{(k)}\left[\frac{q}{(1+q)^j}\right]\right) =$$

$$= (1+q)^{j-k-1}F^{(k+1)}\left[\frac{1}{(1+q)^j}\right],$$

and the Lemma is proved.

From the differential equation (4.15) follows that

$$\sum_{k=0}^{m} a_k F^{(k)}[x] = 0$$

must hold for every $x \in M$. By choosing $x = \frac{1}{(1+q)^j}$ $(-\infty < j < \infty)$ we get

$$\sum_{k=0}^{m} a_k F^{(k)}\left[\frac{1}{(1+q)^j}\right] = 0,$$

and with (4.16) the equation

$$\sum_{k=0}^{m} a_k (1+q)^{k-j}\nabla^k(A_j) = 0 \quad (-\infty < j < \infty) \tag{4.17}$$

will be obtained. Multiplying both sides of (4.17) with $(1+q)^j$ we have

$$\sum_{k=0}^{m} a_k (1+q)^k \nabla^k(A_j) = 0. \tag{4.18}$$

(4.18) is a so-called operator-vector difference equation. This concept has been introduced in BUTZER—SCHULTE [4]. In this manner the solution of the differential equation (4.15) has been reduced to the solution of the operator vector difference-equation (4.18).

We solve (4.15) only in the case of $a_k = \frac{c_k}{(1+q)^k}$, where c_k are arbitrary numbers. Then (4.18) is of the form

$$\sum_{k=0}^{m} c_k \nabla^k(A_j) = 0 \tag{4.19}$$

being an ordinary (not an operator-vector) difference equation, the solutions of which are numbers for every value of j. It is well known that every particular solution of (4.19) reads as

$$A_j = A_k(j) = j^{\mu_k}\beta_k^j \tag{4.20}$$

$\mu_k =$ non negative integer, $\beta_k \in K$ with $\beta_k = \frac{1}{1-u_k}$, where $u_k \neq 1$ is a root of the characteristic equation

$$\sum_{k=0}^{m} c_k u^k = 0. \tag{4.21}$$

From the definition of A_j and (4.20) we have that for every particular solution F_k of (4.15)

$$F_k\left[\frac{1}{(1+q)^j}\right] = \frac{j^{\mu_k}\beta_k^j}{(1+q)^j} = (-1)^{\mu_k} D^{\mu_k} T^{\beta_k}\left[\frac{1}{(1+q)^j}\right] \tag{4.22}$$

holds. Let $X = \sum_j \frac{x(j)}{(1+q)^j}$ be an arbitrary operator. From the continuity of D, T^{β_k}, F_k we have that

$$F_k(x) = F_k\left[\sum_j \frac{x(j)}{(1+q)^j}\right] = \sum_j F_k\left[\frac{x(j)}{(1+q)^j}\right] = \tag{4.23}$$

$$= \sum_j (-1)^{\mu_k} D^{\mu_k} T^{\beta_k}\left[\frac{x(j)}{(1+q)^j}\right] = (-1)^{\mu_k} D^{\mu_k} T^{\beta_k}\left[\sum_j \frac{x(j)}{(1+q)^j}\right] =$$

$$= (-1)^{\mu_k} D^{\mu_k} T^{\beta_k}[x]$$

holds. Consequently

$$F_k = (-1)^{\mu_k} D^{\mu_k} T^{\beta_k} \tag{4.24}$$

is a particular solution of (4.15). If $u=1$ is not a root of the characteristic equation (4.21), then (4.15) has m linearly independent solutions and the general solution is of the form

$$F = \sum_{k=1}^{m} b_k F_k \tag{4.25}$$

where $b_k \in M$ are arbitrary. If $u=1$ is a σ-tuple root of (4.21) we obtain that (4.15) has less than m linearly independent solutions and the general solution is of the form

$$F = \sum_{k=1}^{m-\sigma} b_k F_k.$$

So we have proved the following

THEOREM 4.5. *Let us consider the differential equation*

$$\sum_{k=0}^{m} \frac{c_k}{(1+q)^k} F^{(k)} = 0, \quad c_k \in K, \tag{4.26}$$

and the difference equation

$$\sum_{k=0}^{m} c_k \nabla^k(A_j) = 0 \tag{4.27}$$

having the characteristic equation

$$\sum_{k=0}^{m} c_k u^k = 0. \tag{4.28}$$

To every particular solution

$$A_j = A_k(j) = j^{\mu_k}\beta_k^j$$

$\mu_k =$ *non negative integer,* $\beta_k \in K$, *of the difference equation* (4.27) *there corresponds a particular solution*

$$F_k = (-1)^{\mu_k} D^{\mu_k} T^{\beta_k} \tag{4.29}$$

of the differential equation (4.26). *If* $u=1$ *is not a root of the characteristic equation* (4.28), *then* (4.26) *has exactly* m *linearly independent solutions. The general solution reads as*

$$F = \sum_{k=1}^{m} b_k F_k \tag{4.30}$$

where $b_k \in M$ *are arbitrary.*

If $u=1$ *is a* σ*-tuple root of* (4.28), *then* (4.26) *has* $m-\sigma$ *linearly independent solutions, the general solution is of the form*

$$F = \sum_{k=1}^{m-\sigma} b_k F_k. \tag{4.31}$$

In the case of $\sigma = m$, *the differential equation* (4.26) *has the only solution* $F=0$.

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ÜBER DISKRETE MIKUSIŃSKISCHE OPERATOREN, DIE AUF GRUND DER DIRICHLETSCHEN PRODUKTENFORMEL ERZEUGT WERDEN

von

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Im folgenden bezeichnen wir mit $a(n)$ den Wert einer Funktion a der nichtnegativen ganzen (später nur der natürlichen) Zahl n an der Stelle n. Falls wir die Funktion selbst als *einen* Gegenstand betrachten, so drücken wir sie nach dem Vorschlag Mikusińskis mit dem Zeichen $\{a(n)\}$ aus. Mit den verschiedenen algebraischen Begründungen der Operatorenrechnung, fußend auf für diskrete Werte definierte Funktionen, beziehungsweise mit ihren praktischen Anwendungen haben sich mehrere Verfasser beschäftigt; siehe die sehr umfangreiche Literatur, zitiert in [1], [2], [3] und [4]. (Die Zahlen in den eckigen Klammern beziehen sich auf das Literaturverzeichnis am Ende des Artikels.) Insofern wir uns auf den eindimensionalen Fall beschränken, haben es alle diese Arbeiten gemeinsam, daß das Produkt zweier Funktionen $\{a(n)\}$ und $\{b(n)\}$ der nichtnegativen ganzen variablen Zahl n durch die auf Cauchy zurückgehende Formel

$$\{a(n)\} \cdot \{b(n)\} = \left\{ \sum_{k=0}^{n} a(k) b(n-k) \right\} \tag{1}$$

oder durch eine ähnliche Formel definiert wird. (Die rechte Seite von (1) wird als das „Cauchysche Produkt" der beiden Funktionen $\{a(n)\}$ und $\{b(n)\}$ bezeichnet — s. [8], S. 119.) E. GESZTELYI hat sich in seiner ausgezeichneten Arbeit [5] als erster mit den Anwendungen der Operatorenrechnung in der Zahlentheorie sowie mit der Theorie der eigens zu diesem Zweck eingeführten D-Operatoren beschäftigt. Die Grundlage seiner Untersuchungen bildet die Konstruktion eines M_l-Operatorenkörpers, der mit dem bekannten Mikusińskischen Operatorenkörper M isomorph ist, wobei ein Teilring von M_l wiederum mit dem Ring aller für die natürlichen Zahlen n definierten Funktionen isomorph ist. Im letzteren wird die Addition in der üblichen Weise, die Multiplikation aber durch die Formel

$$\{a(n)\}\{b(n)\} = \left\{ \sum_{\nu \mid n} a(\nu) b\left(\frac{n}{\nu}\right) \right\} \tag{2}$$

definiert. Hierbei durchläuft die Zahl ν alle Teiler von n (die uneigentlichen Teiler 1 und n miteinbegriffen). Die rechte Seite von (2) wird als das Dirichletsche Produkt der für die natürlichen Zahlen n definierten Funktionen $\{a(n)\}$ und $\{b(n)\}$ bezeichnet. (S. [8], S. 119).

In dieser Arbeit führen wir, ausgehend von dem Ring E der für die natürlichen Zahlen definierten Funktionen $\{a(n)\}$ und aus der Definition (2) des Produktes zweier Funktionen, die Begriffe der Operatorenrechnung direkt ein. (Die Addition wird hierbei wie gewöhnlich definiert.) In dem erhaltenen Operatorenkörper M definieren wir den Ring E^*, der die Menge derjenigen Operatoren x ist, die in der

Form

$$x = \frac{\{a(n)\}}{\{\delta(n, N)\}}$$

darstellbar sind, wobei $\{a(n)\}$ ein beliebiges Element von E und $\{\delta(n, N)\}$ diejenige Funktion in E ist, die für $n=N$ den Wert eins, für alle übrigen n den Wert null hat (die sogenannte „diskrete Dirac-Funktion"). $\delta(n, N)$ ist das Kroneckersche Symbol, N kann jede festgesetzte natürliche Zahl sein.

Weiterhin definieren wir den Begriff des sogenannten algebraischen Integrals, welches wir durch die inverse Operation der von GESZTELYI [5] eingeführten (nicht der klassischen Mikusińskischen) Derivation erhalten und finden die notwendige und hinreichende Bedingung dafür, daß ein Operator $x \in E^*$ in dem Ring E^* integrierbar sei. Die Frage der Integrierbarkeit eines beliebigen Operators $x \in M$ bleibt dabei unentschieden. Daraufhin diskutieren wir detailliert die in dem Ring E^* definierte algebraische Differentialgleichung

$$Dx - fx = h, \tag{3}$$

wobei $f=\{f(n)\}$ und $h=\{h(n)\}$ gegebene Funktionen der Variable n sind und $x \in E^*$ der gesuchte Operator ist. D ist das Symbol des Derivierens. Wir geben die notwendigen und hinreichenden Bedingungen dafür an, daß für die Gleichung (3) Lösungen in E^* existieren und stellen die Lösungen in einer expliziten Form dar. Wir beweisen fernerhin den sogenannten Alternativsatz für die Gleichung (3) in bezug auf Lösungen, die Funktionen aus E sind und führen einige Anwendungen der Gleichung (3) auf gewisse zahlentheoretische Funktionen an.

Unsere Untersuchungen können zum Teil als das Analogon der von FÉNYES und KOSIK durchgeführten Untersuchungen in [3] betrachtet werden, wo diese Verfasser, Annahme einer Cauchyschen Multiplikationsregel (1), eine Gleichung vom Typus (3) behandelten.

In dieser Arbeit kommen die in dem Artikel von BUTZER und SCHULTE [4] benutzten Zeichen und Symbole für Operatoren und Funktionen zur Anwendung.

§ 1. Diskrete Mikusińskische Operatoren, gegründet auf der Dirichletschen Produktenformel

Es sei mit $a=\{a(n)\}$ eine beliebige, auf der Menge der natürlichen Zahlen definierte Funktion, welche endliche und reelle Werte annimmt, bezeichnet. Das Symbol $a(n)$ bezeichne den Wert dieser Funktion für eine beliebige Stelle n. E sei das Zeichen für die Menge aller dieser Funktionen.

Wir definieren zwei Operationen in E:
die Addition

$$a+b = \{a(n)\}+\{b(n)\} = \{a(n)+b(n)\}$$

und die Multiplikation

$$ab = \{a(n)\}\{b(n)\} = \left\{\sum_{\nu|n} a(\nu)\, b\left(\frac{n}{\nu}\right)\right\}.$$

SATZ 1.1. *Die mit den beiden obigen Operationen versehene Funktionenmenge E ist ein kommutativer und nullteilerfreier Ring.*

BEWEIS. Vor allem überzeugen wir uns, daß für beliebige $a, b, c \in E$ die folgenden Beziehungen gelten.

(I) $a+b=b+a$,
(II) $a+(b+c)=(a+b)+c$,
(III) $ab=ba$,
(IV) $(ab)c=a(bc)$,
(V) $a(b+c)=ab+ac$.

Die Beziehungen (I), (II), (III) und (V) folgen einfach aus der Tatsache, daß die entsprechenden Beziehungen für Zahlenringe gültig sind. Die Gültigkeit der Beziehung (IV) wird wie folgt nachgewiesen.

Gegeben eine natürliche Zahl n, konstruieren wir Paare (ν, μ) von natürlichen Zahlen ν und μ derart, daß ν ein Teiler von n und μ ein Teiler von $\frac{n}{\nu}$ sei. Betrachten wir für eine festgesetzte Zahl n alle diese (offenbar endlich viele) Paare, deren Menge wir durch Q_n bezeichnen. Es ist nicht schwer einzusehen, daß $(\mu, \nu) \in Q_n$, wenn $(\nu, \mu) \in Q_n$. Ist nämlich μ ein Teiler von $\frac{n}{\nu}$, so auch ein Teiler von n und wir haben

$$\frac{n}{\nu} = \lambda\mu,$$

wo λ eine natürliche Zahl ist. Ferner gilt $n=\lambda\mu\nu$, woraus $\frac{n}{\mu}=\lambda\nu$ folgt, d. h. ν ist ein Teiler von $\frac{n}{\mu}$.

Hierauf haben wir:

$$(ab)c = \left\{\sum_{\nu|n} a(\nu)\, b\left(\frac{n}{\nu}\right)\right\}\{c(n)\} = \left\{\sum_{\mu|n} c(\mu) \sum_{\nu\left|\frac{n}{\mu}\right.} a(\nu)\, b\left(\frac{n}{\nu\mu}\right)\right\} =$$

$$= \left\{\sum_{\mu|n} \sum_{\nu\left|\frac{n}{\mu}\right.} a(\nu)\, b\left(\frac{n}{\mu\nu}\right) c(\mu)\right\},$$

$$a(bc) = \{a(n)\}\left\{\sum_{\nu|n} b\left(\frac{n}{\nu}\right) c(\nu)\right\} = \left\{\sum_{\mu|n} a(\mu) \sum_{\nu\left|\frac{n}{\mu}\right.} b\left(\frac{n}{\mu\nu}\right) c(\nu)\right\} =$$

$$= \left\{\sum_{\mu|n} \sum_{\nu\left|\frac{n}{\mu}\right.} a(\mu)\, b\left(\frac{n}{\mu\nu}\right) c(\nu)\right\}.$$

Aus der Richtigkeit der obigen Behauptung betreffs der Menge Q_n folgt sofort $(ab)c=a(bc)$, d. h. die Beziehung (IV). Hiermit haben wir bewiesen, daß E ein kommutativer Ring ist. Es bleibt zu zeigen, daß E nullteilerfrei ist.

Es seien $\{a(n)\} \neq 0$, $\{b(n)\} \neq 0$ und

$$a(n) = 0 \quad \text{wenn} \quad n < m; \quad m \geqq 1; \quad a(m) \neq 0,$$
$$b(n) = 0 \quad \text{wenn} \quad n < l; \quad l \geqq 1; \quad b(l) \neq 0.$$

Bestimmen wir den Wert der Funktion $\{a(n)\}\{b(n)\}$ an der Stelle $n=ml$. Wir erhalten:

$$\sum_{v|ml} a(v)b\left(\frac{ml}{v}\right) = a(m)b(l) \neq 0,$$

d. h. $\{a(n)\}\{b(n)\} \neq 0$, oder, der Ring E besitzt keinen Nullteiler, womit der Satz 1.1 bewiesen ist.

Führen wir nun einige bemerkenswerte Funktionen des Ringes E an.

I. Die „diskrete Dirac-Funktion" $\{\delta(n, N)\}$ (N eine festgesetzte natürliche Zahl)

$$\delta(n, N) = \begin{cases} 0 & \text{wenn} \quad n \neq N, \\ 1 & \text{wenn} \quad n = N. \end{cases}$$

Eine Grundeigenschaft dieser Funktion ist, daß für eine beliebige Funktion $\{a(n)$

$$\{\delta(n, N)\{a(n)\} = \left\{\sum_{v|n} \delta(v, N)a\left(\frac{n}{v}\right)\right\} = \{b(n)\}$$

gilt, wobei

$$b(n) = \begin{cases} a\left(\dfrac{n}{N}\right) & \text{wenn} \quad N|n, \\ 0 & \text{sonst}, \end{cases}$$

oder $b(kn)=a(k)$ $(k=1, 2, \ldots)$ und $b(n)=0$ sonst ist. Wählen wir $N=1$, so sehen wir, daß $\{\delta(n, 1)\}$ das Einheitselement des Ringes E ist. Fernerhin ergibt sich, daß für beliebige gegebene natürliche Zahlen N_1 und N_2

$$\{\delta(n, N_1)\}\{\delta(n, N_2)\} = \{\delta(n, N_1 N_2)\}$$

gilt.

II. Die Funktion $\{1\}$.

Für diese Funktion gilt, daß für eine beliebige Funktion $a=\{a(n)\} \in E$ die Gleichheit

$$\{1\}\{a(n)\} = \left\{\sum_{v|n} a(v)\right\}$$

besteht.

Die Funktion $\{1\}$ ist dafür geeignet, um einige bekannte zahlentheoretische Funktionen und Zusammenhänge zwischen ihnen operatorenmäßig einfach darzustellen. So haben wir z. B.

$\{1\}\{1\} = \left\{\sum_{v|n} 1\right\} = \{\tau(n)\}$, die Anzahl der Teiler von n,

$\{1\}\{n\} = \left\{\sum_{v|n} v\right\} = \{\sigma(n)\}$, die Summe der Teiler von n,

$\{1\}\{n^\alpha\} = \left\{\sum_{v|n} v^\alpha\right\} = \{\sigma_\alpha(n)\}$, die Summe der α-ten Potenzen der Teiler von n,

$\{1\}\{\varphi(n)\} = \left\{\sum_{v|n} \varphi(v)\right\} = \{n\}$, wo $\varphi(n)$ die „Eulersche Funktion",

die Anzahl der natürlichen Zahlen $\leqq n$ und teilerfremd zu n ist.

III. Die Möbiussche Funktion $\{\mu(n)\}$.

Ihre DEFINITION: $\mu(1)=1$ und $\mu(n)=0$, wenn n durch ein Primzahlquadrat teilbar ist; $\mu(n)=(-1)^{\alpha}$ wenn $n=p_1, p_2, \ldots, p_\alpha$, wo die Primfaktoren p_k verschieden sind.

Es ist bekannt, daß

$$\{1\}\{\mu(n)\} = \left\{\sum_{\nu|n} \mu(\nu)\right\} = \{\delta(n, 1)\}; \tag{1.4}$$

siehe PÓLYA—SZEGŐ [8], Teil VIII, Aufgabe No. 52.

IV. Die Mangoldtsche Funktion $\{\Lambda(n)\}$.

DEFINITION:

$\Lambda(n)=\log p$, wenn $n=p^x$, $x\geqq 1$, p Primzahl, x ganz,

$\Lambda(n)=0$ sonst.

Aus der Tatsache, daß der Ring E kommutativ und nullteilerfrei ist, folgt, daß er zu einem kommutativen Körper erweiterbar ist. Wir werden diesen Körper mit M bezeichnen und seine Elemente „diskrete Mikusińskische Operatoren" nennen.

Betrachtet man das Verhalten der Funktionen $\{c\delta(n, 1)\}\in E\subset M$ für alle c $(-\infty<c<\infty)$ in bezug auf Addition, Multiplikation und Division, so konstatiert man, daß sie einen Körper bilden, der mit dem Körper K der reellen Zahlen isomorph ist so, daß wir sie mit den reellen Zahlen identifizieren und als solche in den Körper M eingebettet denken können. In diesem Sinne sagen wir, daß der Körper M den Körper K der reellen Zahlen und den Ring E enthält. Jede relle Zahl c ist gleichzeitig eine Funktion, deren Wert für $n=1$ gleich c ist; die übrigen ihrer Werte sind gleich null. Das gemeinsame Einheitselement von K, E und M ist die Funktion $\{\delta(n, 1)\}$ für die wir das Zeichen 1 beibehalten:

$$\{\delta(n\,1)\} = 1.$$

Auch setzen wir $\{c\delta(n, 1)\}=c$ für jedes $c\in K$. Es ist $c\{a(n)\}=\{ca(n)\}$.

Alle Operatoren $x=\dfrac{\{a(n)\}}{\{b(n)\}}$ sind Funktionen, wenn $b(1)\neq 0$. Das sieht man ein, wenn man

$$\{a(n)\} = \{b(n)\}\{c(n)\}$$

setzt und die Werte $c(n)$ der Reihe nach berechnet.

Die Rechnung gelingt, wenn $b(1)\neq 0$:

$$\{b(n)\}\{c(n)\} = \left\{\sum_{\nu|n} b(\nu)c\left(\frac{n}{\nu}\right)\right\},$$

$$a(1) = b(1)c(1), \quad \text{woraus} \quad c(1) = \frac{a(1)}{b(1)};$$

$$a(2) = b(1)c(2)+b(2)c(1), \quad c(2) = \frac{a(2)-b(2)c(1)}{b(1)};$$

und so weiter (der Nenner ist immer $b(1)\neq 0$). Wir können somit die Gleichung (1.4) so schreiben:

$$\{1\}\{\mu(n)\} = 1 \quad \text{und} \quad \{\mu(n)\} = \frac{1}{\{1\}}.$$

Definition der Konvergenz im Ring. Es sei $\{a_k(n)\}\in E$ $(k=1, 2, \ldots)$ eine unendliche Folge von Funktionen. Wir führen im Ring E den Begriff der Konvergenz einer solchen Folge ein (siehe BUTZER—SCHULTE [4]), indem, wir definieren:

$$\lim_{k\to\infty}\{a_k(n)\} = \{a(n)\}, \quad \text{wenn für jede festgesetzte}$$

$$\text{Stelle } n, \lim_{k\to\infty} a_k(n) = a(n) \quad \text{im gewöhnlichen Sinne}$$

Wir übertragen diesen Konvergenzbegriff, im üblichen Sinne, auch auf unendliche Reihen.

Laut dieser Definition gilt offenbar der

SATZ 1.2. *Für jedes* $\{a(n)\}\in E$ *ist*

$$\{a(n)\} = \sum_{k=1}^{\infty} a(k)\{\delta(n, k)\}. \tag{1.5}$$

Definition des Ringes E^*. Es sei $E^*\subset M$ diejenige Teilmenge des Operatorenkörpers M, deren Elemente x in der Form

$$x = \frac{\{a(n)\}}{\{\delta(n, N)\}} \tag{1.6}$$

darstellbar sind ($\{a(n)\}\in E$, N eine natürliche Zahl).

LEMMA 1.1. *Es gilt* $E\subset E^*\subset M$, *und* E^* *ist ein Ring.*

BEWEIS. Setzen wir $N=1$, so haben wir:

$$x = \frac{\{a(n)\}}{\{\delta(n,1)\}} = \{a(n)\}\in E,$$

somit ist $E\subset E^*$ und laut Definition $E^*\subset M$. Daß E^* ein Ring ist, zeigt man durch Bildung der Summe und des Produktes zweier Operatoren von der Form (1.6), q. e. d.

Bildet man das Produkt

$$\{b(n)\}\{\delta(n, N)\} = \{a(n)\},$$

so konstatiert man: Für diejenigen n-Werte, die keine Vielfachen von N sind, ist $a(n)=0$, und für $n=\mu N$ haben wir:

$$\{a(\mu N)\} = \{b(\mu)\}.$$

Dies folgt aus der Produktenformel $\{a(n)\}=\left\{\sum_{\nu|n} a(\nu)\delta\left(\frac{n}{\nu}, N\right)\right\}$, und somit ist

$$x = \frac{\{a(n)\}}{\{\delta(n, N)\}}\in E \tag{1.7}$$

dann und nur dann, wenn $a(n)=0$ für alle n, die nicht durch N teilbar sind. Insbesondere gilt

$$\frac{\{\delta(n, N_1)\}}{\{\delta(n, N_2)\}} \in E \tag{1.8}$$

dann, wenn N_1 durch N_2 teilbar ist. Ist dies nicht der Fall, so stellt dieser Bruch einen Operator dar.

Definition der Operatorenfunktion $\delta(\alpha)$. Es sei $\alpha \geqq 1$ eine beliebige rationale Zahl, also $\alpha=\frac{N_1}{N_2}$ (N_1 und N_2 natürliche Zahlen); dann ist

$$\delta(\alpha) = \frac{\{\delta(n, N_1)\}}{\{\delta(n, N_2)\}}.$$

Es ist leicht einzusehen, daß diese Definition von der Darstellung der rationalen Zahlen α unabhängig ist; insbesondere für $\alpha=N$ haben wir:

$$\delta(\alpha) = \{\delta(n, N)\}.$$

LEMMA 1.2. *Für beliebige rationale Zahlen* α *und* β *gilt:*

$$\delta(\alpha)\delta(\beta) = \delta(\alpha\beta). \tag{1.9}$$

BEWEIS. Es seien $\alpha=\frac{N_1}{N_2}$ und $\beta=\frac{N_3}{N_4}$; dann sind $\alpha\beta=\frac{N_1N_3}{N_2N_4}$ und

$$\delta(\alpha)\delta(\beta) = \delta\left(\frac{N_1}{N_2}\right)\delta\left(\frac{N_3}{N_4}\right) = \frac{\{\delta(n, N_1)\}}{\{\delta(n, N_2)\}}\,\frac{\{\delta(n, N_3)\}}{\{\delta(n, N_4)\}} = \tag{1.10}$$

$$= \frac{\{\delta(n, N_1N_3)\}}{\{\delta(n, N_2N_4)\}} = \delta\left(\frac{N_1N_3}{N_2N_4}\right) = \delta(\alpha\beta),$$

q. e. d.

§ 2. Analytische Funktionen des Ringes E. Algebraische Derivation und Integration

Ist $f(z)=\sum_{k=0}^{\infty}\beta_k z^k$ eine beliebige ganze Funktion der komplexen Veränderlichen z, so definieren wir für ein beliebiges $a=\{a(n)\}\in E$ die „analytische Funktion" von $\{a(n)\}$:

$$f(\{a(n)\}) = \sum_{k=0}^{\infty}\beta_k\{a(n)\}^k, \quad \{a(n)\}^0 = 1 \tag{2.1}$$

wobei die Potenzen von a im Sinne der Dirichletschen Multiplikationsregel (2) zu verstehen sind. Diese Definition ist sinnvoll kraft des folgenden Satzes.

SATZ 2.1. *Es sei* $f(z)=\sum_{k=0}^{\infty}\beta_k z^k$ *eine beliebige ganze Funktion von* z. *Dann ist die Reihe*

$$\sum_{k=0}^{\infty}\beta_k\{a(n)\}^k$$

für eine beliebige Funktion $a\in E$ *konvergent im Sinne des im vorigen Paragraphen eingeführten Konvergenzbegriffes.*

Der Beweis ist Wort für Wort derselbe wie der des entsprechenden Satzes in der Arbeit [3] von Fényes und Kosik (wobei dort die Cauchysche Produktenregel gültig war); aus diesem Grunde übergehen wir ihn hier.

In der klassischen Theorie der linearen Differentialgleichungen spielt die Funktion $y=e^x$ eine bedeutende Rolle, so werden auch hier in der weiterhin zu entwickelnden Theorie der „algebraischen Differentialgleichungen erster Ordnung für den Ring E" die Funktionen

$$e^{\{a(n)\}}=\sum_{k=0}^{\infty}\frac{\{a(n)\}^k}{k!}=\exp\{a(n)\} \tag{2.2}$$

eine bedeutende Rolle spielen. Es ist nicht schwer einzusehen, daß für beliebige $a, b\in E$

$$e^a\cdot e^b=e^{a+b} \tag{2.3}$$

gilt, und daß, wenn man der Kürze halber

$$e^{\{a(n)\}}=\{e(n)\}$$

setzt,

$$e_a(1)=e^{a(1)}$$

ist.

Die bekannte Definition der algebraischen Derivation D (siehe [5]).

$$D\{a(n)\}=\{-\log na(n)\}\quad\text{wenn}\quad\{a(n)\}\in E. \tag{2.4}$$

Aus dieser Definition folgt für beliebige $a, b\in E$

$$D(a+b)=D(a)+D(b),$$

$$D(ab)=aD(b)+bD(a).$$

Die Definition (2.4) der Derivation kann durch die Formel

$$D\left(\frac{a}{b}\right)=\frac{bD(a)-aD(b)}{b^2}$$

auf den Quotientenkörper M erweitert werden.

LEMMA 2.1. *Der Ring* E^* *ist in bezug auf die Operation des algebraischen Derivierens abgeschlossen, das heißt, wenn* $x\in E^*$, *so auch* $D(x)\in E^*$.

BEWEIS. Da

$$D(\{\delta(n, N)\})=\{-\log n\delta(n, N)\}=-\log N\{\delta(n, N)\},$$

ist, haben wir, für beliebige

$$x = \frac{\{a(n)\}}{\{\delta(n, N)\}} \in E^*,$$

$$D(x) = \frac{D(a)\{\delta(n, N)\} - aD\{\delta(n, N)\}}{\{\delta(n, N)\}^2} = \tag{2.5}$$

$$= \frac{\{-\log na(n)\}\{\delta(n, N)\} + \log N\{a(n)\}\{\delta(n, N)\}}{\{\delta(n, N)\}^2} =$$

$$= -\frac{\left\{-\log \frac{n}{N} a(n)\right\}}{\{\delta(n, N)\}} \in E^*,$$

q. e. d.

BEMERKUNG. Es ist leicht zu sehen, daß, für eine beliebige rationale Zahl $\alpha = \frac{N_1}{N_2}$, $x = \frac{\{a(n)\}}{\delta(\alpha)} \in E^*$ gilt.

LEMMA 2.2. *Es gilt die Formel:*

$$D\frac{\{a(n)\}}{\delta(\alpha)} = \frac{\left\{-\log \frac{n}{\alpha} a(n)\right\}}{\delta(\alpha)}, \quad \alpha = \frac{N_1}{N_2}. \tag{2.6}$$

BEWEIS. Auf Grund des vorigen Lemmas sind

$$D(\delta(\alpha)) = D\frac{\{\delta(n, N_1)\}}{\{\delta(n, N_2)\}} = \frac{\left\{-\log \frac{n}{N_2} \delta(n, N_1)\right\}}{\{\delta(n, N_2)\}} = -\log \frac{N_1}{N_2} \frac{\{\delta(n, N_1)\}}{\{\delta(n, N_2)\}} =$$

$$= -\log \alpha\delta(\alpha),$$

und so bekommen wir für die Derivierte von $x = \frac{\{a(n)\}}{\delta(\alpha)}$,

$$D(x) = \frac{\delta(\alpha)D(a) - aD[\delta(\alpha)]}{[\delta(\alpha)]^2} = \frac{\delta(\alpha)\{-\log na(n)\} + \log \alpha\delta(\alpha)\{a(n)\}}{[\delta(\alpha)]^2} = \tag{2.8}$$

$$= \frac{\left\{-\log \frac{n}{\alpha} a(n)\right\}}{\delta(\alpha)},$$

q. e. d.

LEMMA 2.3. *Für eine beliebige ganze Funktion $f(z)$ und beliebige $a = \{a(n)\} \in E$ besteht die Gleichheit:*

$$D[f(\{a(n)\})] = D(\{a(n)\}) \cdot f'(\{a(n)\}). \tag{2.9}$$

BEWEIS.

$$D[f(\{a(n)\})] = D\left(\sum_{k=0}^{\infty} \beta_k a^k\right) = \sum_{k=0}^{\infty} \beta_k D(a^k) = \sum_{k=1}^{\infty} \beta_k k a^{k-1} D(a) = D(a) f'(a)$$

q. e. d.

Hier haben wir von der Formel $D(a^k) = ka^{k-1} D(a)$, die eine einfache Folgerung aus der Regel (2.4) der Derivation eines Produktes ist, Gebrauch gemacht. Die gliedweise Derivation der unendlichen Reihe ist erlaubt, da für festgesetzte n-Werte die Derivation nur eine Multiplikation mit der Konstanten $-\log n$ bedeutet.

Aus (2.9) ergibt sich sofort die wichtige Formel

$$D(e^{\{a(n)\}}) = e^{\{a(n)\}} \cdot D(\{a(n)\}). \tag{2.10}$$

Wir führen nun den Begriff des algebraischen Integrals ein.

DEFINITION. Wenn zu einem gegebenen Operator $x \in M$ ein Operator $y \in M$ existiert, so, daß $Dy = x$, so sagen wir, daß x algebraisch integrierbar ist und nennen y ein algebraisches Integral von x. In Zeichen:

$$y = \int x. \tag{2.11}$$

LEMMA 2.4. *Sind die Operatoren $p \in M$ und $q \in M$ algebraisch integrierbar, so gilt für beliebige Zahlen c_1 und c_2, daß auch*

$$c_1 p + c_2 q$$

integrierbar ist, und wir haben dann

$$\int (c_1 p + c_2 q) = c_1 \int p + c_2 \int q. \tag{2.12}$$

Weiterhin: sind $\int p = \varrho$ und c eine beliebige Zahl, so gilt auch

$$\int p = \varrho + c;$$

das heißt, wenn ein Operator integrierbar ist, so besitzt er unendlich viele Integrale.

BEWEIS. Die obigen Behauptungen ergeben sich einfach aus den Regeln (2.4) und aus der Tatsache, daß die Derivierte einer beliebigen Zahl gleich null ist.

In dem nun folgenden Teil dieser Arbeit befassen wir uns mit den Operatoren des Ringes E^*. Wir geben eine notwendige und hinreichende Bedingung dafür an, daß ein Operator des Ringes E^* algebraisch integrierbar sei. Hierfür haben wir die folgenden Überlegungen nötig.

LEMMA 2.5. *Sind $x \in E^*$ und $D(x) = 0$, so ist x eine Zahl $(x = c \in K)$.*

BEWEIS. Es sei $x = \dfrac{\{a(n)\}}{\{\delta(n, N)\}} = \dfrac{a}{\delta(N)}$.

Aus der Voraussetzung und aus (2.5) erhalten wir:

$$D\left(\frac{a}{\delta(N)}\right) = \frac{\left\{-\log \frac{n}{N} a(n)\right\}}{\delta(N)} = 0,$$

das heißt

$$\log\frac{n}{N}\cdot a(n)=0 \quad \text{für alle } n.$$

Dies ist nur dann möglich, wenn $a(n)=0$ für $n\neq N$ und $a(n)=c$ für $n=N$ $(c\in K)$, oder $\{a(n)\}=c\delta(N)$ und daher

$$x=\frac{a}{\delta(N)}=\frac{c\delta(N)}{\delta(N)}=c,$$

q. e. d.

BEMERKUNG. Wir haben nicht etwa bewiesen, daß aus $D(x)=0$ und $x\in M$ folgt, daß x eine Zahl sei. Für die folgenden Überlegungen genügt uns das eben bewiesene Lemma.

DEFINITION. Existiert für $x\in E^*$ ein $y\in E^*$ mit $D(y)=x$, so sagen wir, daß x integrierbar in dem Ring E^* und y sein Integral in E^* sind.

Aus dem Lemma 2.5. ist zu ersehen, daß, wenn $x\in E^*$ in E^* integrierbar ist, zwei seiner Integrale nur bis auf eine beliebige additive konstante Zahl verschieden sein können.

SATZ 2.2. *Ein Operator* $x=\dfrac{\{a(n)\}}{\delta(n)}\in E^*$ *ist dann und nur dann integrierbar in* E^*, *wenn* $a(N)=0$. *Ist dies der Fall, so wird ein beliebiges seiner Integrale durch die Formel*

$$\int x=\frac{\left\{-\dfrac{a(n)}{\log\dfrac{n}{N}}\right\}}{\delta(N)} \tag{2.13}$$

ausgedrückt, wobei vereinbarungsgemäß unter $\dfrac{a(N)}{\log\dfrac{N}{N}}$ *eine beliebige konstante Zahl* c *verstanden wird:*

$$\frac{a(N)}{\log\dfrac{N}{N}}=c\in K.$$

BEWEIS. Die Bedingung ist hinreichend, da im Falle $a(N)=0$ durch die algebraische Derivation von (2.13) auf Grund der Formel (2.5) wieder $x=\dfrac{a(n)}{\{\delta(N)\}}$ entsteht. Weiterhin ist es klar, daß durch Änderung der Konstanten c, die entstehenden rechten Seiten von (2.13) sich nur um eine konstante Zahl voneinander unterscheiden können und somit wir durch die Formel (2.13) alle Integrale von x bekommen können.

Die Bedingung ist notwendig. Nehmen wir an, daß $a(N)\neq 0$. Dann können wir zeigen, daß in diesem Falle x nicht integrierbar in E^* ist. Es sei vorerst

$\{a(n)\}=\{c\delta(n, N)\}=c\delta(N)$, wo $c\neq 0$; dann ist

$$x=\frac{\{a(n)\}}{\delta(N)}=\frac{c\delta(N)}{\delta(N)}=c,$$

und $c\neq 0$ ist wirklich unintegrierbar in E^*. Nehmen wir nämlich das Gegenteil an, so existiert ein Operator $\frac{\{b(n)\}}{\delta(N_1)}$ so, daß

$$D\frac{\{b(n)\}}{\delta(N_1)}=\frac{\left\{-\log\frac{n}{N_1}b(n)\right\}}{\delta(N_1)}=c \quad \text{und} \quad \left\{-\log\frac{n}{N_1}b(n)\right\}=c\delta(N_1).$$

Die zuletzt bekommene Gleichung ist aber unmöglich, da die Funktion auf der linken Seite für $n=N_1$ den Wert null hat, die rechte Seite aber für $n=N_1$ ungleich null ist. Wenn nun $\{a(n)\}$ eine beliebige Funktion und $a(N)\neq 0$ sind, so haben wir

$$x=\frac{\{a(n)\}}{\delta(N)}=\frac{\{a(n)-\delta(n, N)a(N)\}}{\delta(N)}+\frac{a(N)\delta(N)}{\delta(N)}= \tag{2.14}$$

$$=\frac{\{a(n)-\delta(n, N)a(N)\}}{\delta(N)}+a(N).$$

x ist sicher unintegrierbar in E^*, da im gegenteiligen Falle auch der Operator

$$x-\frac{\{a(n)-\delta(n, N)a(N)\}}{\delta(N)}=a(N)\neq 0$$

integrierbar wäre, was, wie wir es eben sahen, nicht der Fall ist. Damit ist der Satz 2.2. bewiesen.

BEMERKUNG. Es ist insbesondere zu sehen, daß die Zahlen in E^* nicht integrierbar sind (die Zahl 0 ausgenommen).

Wählen wir $N=1$, so folgt aus der Formel (2.13), daß $a\in E$ nur dann integrierbar ist, wenn $a(1)=0$. In diesem Falle ist

$$\int\{a(n)\}=\left\{-\frac{a(n)}{\log n}\right\},$$

wobei vereinbarungsgemäß $\frac{a(1)}{\log 1}=c$ gesetzt wird ($c\in K$ beliebig). Wir bemerken ferner, daß aus dem Lemma 2.2 in Übereinstimmung mit dem Satz 2.2 sofort folgt, daß im Falle einer beliebigen nicht-ganzen rationalen Zahl $\alpha>1$

$$\int\frac{\{a(n)\}}{\delta(\alpha)}=\frac{\left\{-\frac{a(n)}{\log\frac{n}{\alpha}}\right\}}{\delta(\alpha)}. \tag{2.15}$$

Des weiteren gilt

$$\int \delta(\alpha) = \frac{\delta(\alpha)}{-\log \alpha} \tag{2.16}$$

für eine beliebige rationale Zahl $\alpha > 1$.

BEMERKUNG. Die Frage nach der Integrierbarkeit eines beliebigen Operators $x \in M$ bleibt hier unbeantwortet.

§ 3. Lineare inhomogene algebraische Differentialgleichungen erster Ordnung

Eine Funktionalgleichung zwischen einem unbekannten Operator $x \in E^*$ und seiner Derivierten $D(x)$ werden wir im folgenden eine „Differentialgleichung" nennen.

Hier befassen wir uns mit der Lösung der linearen, inhomogenen algebraischen Differentialgleichung

$$D(x) - fx = h, \tag{3.1}$$

wobei f und $h \in E$ gegebene Funktionen sind. Wir suchen die Operatoren $x \in E^*$, die die Gleichung (3.1) befriedigen. Wir geben die notwendigen und hinreichenden Bedingungen für die Existenz einer Lösung x an, und untersuchen, wann diese Lösung eine Funktion $x \in E$ darstellt. Zuerst beschäftigen wir uns mit der homogenen Gleichung

$$D(x) - fx = 0, \quad f \in E. \tag{3.2}$$

Die Lösung suchen wir in der Form

$$x = q \exp\left[\int [\{f\} - f(1)]\right], \quad q \in E^*, \tag{3.3}$$

wo der Wert des Integrals für $n = 1$ irgendwie fixiert, z. B. gleich null gesetzt wird. (Von nun an benutzen wir anstatt des Zeichens $e^{\{a(n)\}}$ das Zeichen $\exp\{a(n)\}$.)

Die Lösung in dieser Form zu suchen ist sinnvoll, da im Falle $x \in E^*$ auch

$$q = x \exp\left[-[\{f\} - f(1)]\right]$$

zum Ring E^* gehört. In (3.2) eingesetzt ergibt der Ausdruck (3.3), mit Rücksicht auf (2.10),

$$Dq \cdot \exp[\ldots] + q(\{f\} - f(1)) \exp[\ldots] - fq \exp[\ldots] = 0$$

oder

$$Dq - f(1)q = 0. \tag{3.4}$$

Wir suchen die nichttrivialen Lösungen $q \in E^*$ dieser einfachen Differentialgleichung. Wir machen folgenden Ansatz:

$$q = \frac{\{a(n)\}}{\{\delta(n, N)\}}, \quad \text{wobei} \quad \{a(n)\} \in E.$$

Eingesetzt in (3.4) bekommen wir:

$$\frac{\left\{-\log\frac{n}{N}a(n)\right\}}{\{\delta(n,N)\}}-f(1)\frac{\{a(n)\})}{\{\delta(n,N\}}=0 \tag{3.5}$$

oder

$$-\log\frac{n}{N}a(n)=f(1)a(n) \tag{3.6}$$

für alle n.

Aus (3.6) ist zu ersehen, daß $a(n)$ höchstens für einen Wert n von null verschieden sein kann; es sei dieser Wert $n=N_1$. Dann ist

$$\{a(n)\}=c\{\delta(n,N_1)\}, \tag{3.7}$$

und aus (3.6) bekommt man, daß

$$f(1)=-\log\frac{N_1}{N} \tag{3.8}$$

sein muß. Somit muß $f(1)$ der Logarithmus einer rationalen Zahl sein. Dann sind

$$q=\frac{\{a(n)\}}{\{\delta(n,N)\}}=c\frac{\{\delta(n,N_1)\}}{\{\delta(n,N)\}}=c\delta(\alpha),\quad \text{wo}\quad \alpha=\frac{N_1}{N}=\exp\left(-f(1)\right). \tag{3.9}$$

Die Gleichung (3.4) kann keine andere linear unabhängige Lösung haben. Nehmen wir nämlich an, daß $q_1=b\delta(\beta)$ eine Lösung von (3.4) wäre, wobei q und $q_1(\neq 0)$ linear unabhängig sind. Durch Einsetzen von q_1 in die Gleichung (3.4) erhält man, daß $\beta=\exp[-f(1)]=\alpha$ und $q_1=b\delta(\alpha)$, d. h. $\frac{q}{q_1}=\frac{c}{b}=\gamma$ eine Zahl ist, und somit besteht ein linearer Zusammenhang zwischen q und q_1. Auf diese Weise ergibt (3.9) in (3.3) eingesetzt die allgemeine Lösung von (3.2) in der Gestalt

$$x=c\delta(\alpha)\exp\left[\int(\{f\}-f(1))\right], \tag{3.10}$$

wobei $\alpha=\exp[-f(1)]$ eine rationale Zahl ist.

(3.10) stellt nur dann eine Funktion dar, wenn α eine natürliche Zahl ist. Unter der Annahme daß die rationale Zahl α keine natürliche Zahl und $x\in E$ sind, würden wir nämlich

$$x=c\frac{\delta(N_1)}{\delta(N)}\exp[\ldots]$$

erhalten, wobei $\frac{N_1}{N}=\alpha$, $N\neq 1$ und $(N,N_1)=1$ wären.

Hieraus würde die Gleichung

$$\{\delta(n,N)\}\{x(n)\}=c\{\delta(n,N_1)\}\exp[\ldots] \tag{3.11}$$

folgen, was aber unmöglich ist. Der Wert der Funktion auf der linken Seite ist nämlich für $n=N_1$ (wegen $(N,N_1)=1$) bestimmt gleich null, wogegen auf der rechten Seite eine Funktion steht, die für $n=N_1$ einen von null verschiedenen

Wert hat (da der Wert der Exponentialfunktion für $n=1$ nicht gleich null sein kann).

Somit haben wir den folgenden Satz bewiesen:

SATZ 3.1. *Betrachten wir die homogene lineare Differentialgleichung*

$$D(x)-fx=0, \tag{3.12}$$

mit $f\in E$. *Ist* $\alpha=\exp[-f(1)]$ *irrational, so hat die Gleichung* (3.12) *nur die triviale Lösung* $x=0$ *in* E^*. *Ist* $\exp[-f(1)]$ *rational, so hat* (3.12) *nichttriviale Lösungen und die allgemeine Lösung ist dann von der Form:*

$$x=c\delta(\alpha)\exp\left[\int(\{f\}-f(1))\right], \quad c\in K \quad \textit{beliebig}. \tag{3.13}$$

Diese Formel stellt dann und nur dann eine Funktion dar, wenn α *eine natürliche Zahl ist.*

EINE ANWENDUNG AUF ZAHLENTHEORETISCHE FUNKTIONEN. Betrachten wir die Differentialgleichung

$$D(x)+\{\Lambda(n)\}x=0,$$

wobei $\Lambda(n)$ die Mangoldtsche Funktion ist. Nach (3.13) ist

$$x=\exp\left\{\frac{\Lambda(n)}{\log n}\right\} \tag{3.14}$$

eine der Lösungen. Hier vereinbaren wir, die willkürliche Konstante $\frac{\Lambda(n)}{\log 1}$ gleich null zu setzen.

Andererseits genügt auch die Funktion $\{1\}$ der obigen Gleichung. Um dies einzusehen, brauchen wir nur zu zeigen, daß

$$\{1\}\{\Lambda(n)\}=-D\{1\}=\{\log n\}.$$

Es sei $n=p_1^{\alpha_1}\cdot p_2^{\alpha_2}\dots p_k^{\alpha_k}$, wo die Zahlen p_μ die verschiedenen Primfaktoren von n sind. Bei Bildung des Produktes

$$\{1\}\{\Lambda(n)\}=\left\{\sum_{v|n}\Lambda(v)\right\}$$

brauchen wir nur diejenigen v-Werte zu berücksichtigen, für welche $\Lambda(v)\neq 0$ ist. Diese sind die Zahlen $p_1, p_1^2, \dots p_1^{\alpha_1}, p_2, p_2^2\dots p_2^{\alpha_2}\dots, p_k, p_k^2, \dots p_k^{\alpha_k}$. So bekommen wir, daß

$$\{1\}\{\Lambda(n)\}=\{\alpha_1\log p_1+\alpha_2\log p_2+\dots+\alpha_k\log p_k\}=\{\log n\}$$

(siehe DOETSCH [7], S. 187). Berücksichtigt man, daß die Werte der beiden Lösungen für $n=1$ übereinstimmen, so erhellt, daß die Gleichung

$$\exp\left\{\frac{\Lambda(n)}{\log n}\right\}=\{1\} \tag{3.15}$$

richtig ist. Da weiterhin, für die Möbiussche Funktion $\mu(n)$, $\{\mu(n)\}=\frac{1}{\{1\}}$ gilt, haben wir für diese Funktion die Darstellung:

$$\{\mu(n)\} = \exp\left\{-\frac{\Lambda(n)}{\log n}\right\}. \tag{3.16}$$

Beschäftigen wir uns nunmehr mit der Untersuchung der Lösungen der inhomogenen Gleichung (3.1). Zuerst betrachten wir den Fall, daß die entsprechende homogene Gleichung nichttriviale Lösungen in E^* besitzt, das heißt, daß α eine rationale Zahl ist. Es seien x_0 eine nichttriviale Lösung der homogenen und x_p eine partikuläre Lösung der inhomogenen Gleichung (3.1).

SATZ 3.2. (Siehe SCHATTE [9]). *Im Falle des Vorhandenseins einer nichttrivialen Lösung x_0 der homogenen Gleichung, existiert eine partikuläre Lösung $x_p\in E^*$ der inhomogenen Gleichung* (3.1) *dann und nur dann, wenn das Integral $\int\frac{h}{x_0}$ existiert.*

BEWEIS. *Die Bedingung ist hinreichend.* Vor allem bemerken wir, daß wegen $h\in E$ auch $\frac{h}{x_0}\in E^*$. Wie wir früher gesehen haben (siehe Satz 2.2), gehört das Integral $\int\frac{h}{x_0}$, wenn es überhaupt existiert, dem Ring E^* an, und mit Hilfe der Methode der Variation der Konstanten (die auf den gegenwärtigen Fall ohne wesentliche Änderung übertragbar ist) finden wir die partikuläre Lösung

$$x_p = x_0\int\frac{h}{x_0}$$

in E^*.

Die Bedingung ist notwendig. Es existiere die partikuläre Lösung $x_p\in E^*$. Dann bekommen wir

$$D\left[\frac{x_p}{x_0}\right] = \frac{x_0D(x_p)-x_pD(x_0)}{x_0^2} = \frac{x_0(h+fx_p)-x_pD(x_0)}{x_0^2} =$$

$$= \frac{hx_0+x_p(fx_0-D(x_0))}{x_0^2} = \frac{hx_0}{x_0^2} = \frac{h}{x_0}\in E^*,$$

das heißt: $\int\frac{h}{x_0}$ existiert.

Damit ist der Satz 3.2. bewiesen.

Mit Hilfe der Formel (3.10) können wir das Integral $\int\frac{h}{x_0}$, das bei der Methode der Variation der Konstanten zur Anwendung kommt, in ausführlicher Weise in der Form

$$\int\frac{h}{x_0} = \int\left(\frac{h}{\delta(\alpha)}\exp\left[-\int(\{f\}-f(1))\right]\right) \tag{3.17}$$

schreiben. Diese Schreibweise vereinfachen wir, indem wir die Bezeichnung

$$\{\mathscr{H}(n)\} = \{h(n)\} \exp\left[-\int(\{f\}-f(1))\right]$$

einführen. Dann haben wir laut (2.15)

$$\int \frac{\{h(n)\}}{x_0} = \int \frac{\{(\mathscr{H}n)\}}{\delta(\alpha)} = \frac{\left\{\begin{matrix}\mathscr{H}(n)\\ -\log\frac{n}{\alpha}\end{matrix}\right\}}{\delta(\alpha)}.$$

I. *Es sei α keine ganze Zahl*; dann ist (3.17) gültig und aus (3.10) bekommen wir die partikuläre Lösung

$$x_p = \left\{-\frac{\mathscr{H}(n)}{\log\frac{n}{\alpha}}\right\} \exp\left[\int(\{f\}-f(1))\right], \tag{3.18}$$

und diese Lösung stellt eine Funktion des Ringes E dar. Die allgemeine Lösung von (3.1) wird daher

$$x = \left[c\delta(\alpha)+\left\{\frac{-\mathscr{H}(n)}{\log\frac{n}{\alpha}}\right\}\right] \exp\left[\int(\{f\}-f(1))\right] \tag{3.19}$$

sein. Diese Formel stellt dann und nur dann eine Funktion E dar, wenn $c=0$.

II. *Es sei α ganz.* Auf Grund des Satzes 2.2 existiert das Integral (3.17) dann und nur dann, wenn $\mathscr{H}(\alpha)=0$. Wenn also $\mathscr{H}(\alpha)\neq 0$, so ist die inhomogene Gleichung (3.1) in E^* unlösbar. Ist $\mathscr{H}(\alpha)=0$, so stellt (3.19) auch die allgemeine Lösung von (3.1) dar, wobei an der Stelle $n=\alpha$ für den Wert der Funktion

$$\left\{\frac{-\mathscr{H}(n)}{\log\frac{n}{\alpha}}\right\}$$

eine beliebige festgesetzte Konstante zu nehmen ist. Die Formel (3.19) ergibt in diesem Falle für alle Werte von c eine Funktion.

Betrachten wir schließlich den Fall, daß α irrational ist. In diesem Falle hat die homogene Gleichung keine nichttriviale Lösung in E^*. Die Methode der Variation der Konstanten verliert hier ihren Sinn.

III. *Es sei α irrational.* Trotz der Tatsache, da die vorige Methode nicht anwendbar ist, hat der Operator (3.18) einen Sinn, er stellt eine Funktion dar. Es ergibt sich leicht, da diese Funktion, in die Gleichung (3.1) eingesetzt, diese befriedigt. Daher ist α irrational, und so hat (3.1) eine Lösung in E^*, die gleichzeitig eine zu E gehörige Funktion ist.

Wir haben somit den folgenden Satz bewiesen:

SATZ 3.3. *Betrachten wir die Differentialgleichung*

$$D(x)-fx = h, \quad f, h\in E. \tag{3.20}$$

Es seien $\alpha = e^{-f(1)}$ *und*

$$\{\mathscr{H}(n)\} = \{h(n)\} \exp\left[-\int (\{f\} - f(1))\right].$$

I. *Ist* α *irrational, so hat die Gleichung* (3.20) *eine einzige Lösung in* E^*, *die eine Funktion und in der Form*

$$x = \left\{-\frac{\mathscr{H}(n)}{\log \frac{n}{\alpha}}\right\} \exp\left[\int (\{f\} - f(1))\right] \tag{3.21}$$

darstellbar ist.

II. *Ist* α *rational, aber keine ganze Zahl, so ist*

$$x = \left[c\delta(\alpha) + \left\{\frac{-\mathscr{H}(n)}{\log \frac{n}{\alpha}}\right\}\right] \exp\left[\int (\{f\} - f(1))\right] \tag{3.22}$$

die allgemeine Lösung von (3.20) *in* E^*, *die dann und nur dann eine Funktion ist, wenn die, sonst beliebige reelle Zahl c gleich null ist.*

III. *Wenn* α *eine ganze Zahl ist, so hat die Gleichung* (3.20) *dann und nur dann eine Lösung in* E^*, *wenn* $\mathscr{H}(\alpha) = 0$ *ist. Ist diese Bedingung erfüllt, so stellt die Formel* (3.22) *die allgemeine Lösung dar, die für jeden Wert von c eine Funktion ist.*

In diesem Falle ist der Wert der Funktion $\left\{\frac{-\mathscr{H}(n)}{\log \frac{n}{\alpha}}\right\}$ *an der Stelle* $n = \alpha$ *einer beliebigen festgesetzten Konstante gleich zu setzen.*

BEMERKUNG. Auf Grund der Definition der Funktion $\{\mathscr{H}(n)\}$ kann man einfach einsehen, daß im Falle der nicht ganzen α die in den Formeln (3.21) und (3.22) vorkommende Funktion

$$\left\{\frac{-\mathscr{H}(n)}{\log \frac{n}{\alpha}}\right\} \exp\left[\int (\{f\} - f(1))\right]$$

unabhängig von der Wahl der beim Integrieren vorkommenden Konstanten ist.

In bezug auf Lösungen, die Funktionen sind, können wir das vorherige Ergebnis auch so zusammenfassen:

SATZ 3.4 (Alternativsatz). *Hat die zur Gleichung* (3.20) *gehörige homogene Gleichung keine nichtriviale Lösung* $x \in E$, *so hat die inhomogene Gleichung genau eine Lösung in E. Hat die homogene Gleichung nichttriviale Lösungen* $x \in E$, *so hat die inhomogene Gleichung entweder keine Lösung in E, oder sie hat dort unendlich viele Lösungen je nachdem, wie wir die an der rechten Seite vorkommende Funktion* $\{h(n)\}$ *gewählt haben.*

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A GENERALIZATION OF UNIFORM BASES

by

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In 1960, P. S. ALEXANDROV introduced the notion of a uniform basis and proved that a topological space X is metrizable iff X is collectionwise normal and has a uniform basis, see [1]. In this paper we give a natural generalization of uniform bases, and we show that in several cases this generalization reduces to the notion of point-$\varkappa$ (point-countable) basis. On the other hand, we also show that the new notion can differ from the earlier one.

We shall use the usual set-theoretic notation; cardinals are identified with initial ordinals. All undefined terms can be found in [5] or [9].

If X is a space, $p\in X$ and $\mathscr{A}$ is collection of subsets of X, recall that ord $(p, \mathscr{A})$ is defined as the cardinality of $\{A\in\mathscr{A}; p\in A\}$. A uniform basis $\mathscr{B}$ for X is an open basis such that for every open set U the family $\{B\in\mathscr{B}; B-U\neq\emptyset\}$ is point-finite in U.

DEFINITION 1. A collection of sets $\mathscr{A}$ in a topological space is point-$\varkappa$ if for each point p of the space we have ord $(p, \mathscr{A})<\varkappa$.

DEFINITION 2. If $\mathscr{B}$ is an open basis of a topological space X we shall say that $\mathscr{B}$ satisfies property $A(\varkappa)$, if for every point p of X and every subfamily $\mathscr{B}'$ of $\mathscr{B}$ with $p\in\bigcap\mathscr{B}'$ and $|\mathscr{B}'|\geqq\varkappa$ the family $\mathscr{B}'$ forms a neighbourhood basis of p in X, We shall say that X has property $A(\varkappa)$ iff there exists an open basis $\mathscr{B}$ of X which satisfies property $A(\varkappa)$.

LEMMA 3. *The basis $\mathscr{B}$ has property $A(\varkappa)$ iff it has the following property:*
$(*)$ *for every non-empty open set U the following subfamily $\mathscr{B}'$ of $\mathscr{B}$* is point-$\varkappa$ in U: $\mathscr{B}'=\{B\in\mathscr{B}; B-U\neq\emptyset\}$, i.e. the elements of $\mathscr{B}$ which hang out from U.

PROOF. (i) If $\mathscr{B}$ satisfies $A(\varkappa)$ but $(*)$ is not fulfilled then there are an open set U and a point $p\in U$ such that at least $\varkappa$-many members of $\mathscr{B}$ contain p and hang out U. But then this subfamily of $\mathscr{B}$ does not form a neighbourhood basis at p. This contradicts the definition of $A(\varkappa)$.

(ii) If $(*)$ is satisfied but $A(\varkappa)$ is not, then there is a point p and a subfamily $\mathscr{B}'\subset\mathscr{B}$ with $|\mathscr{B}'|\geqq\varkappa$ such that $p\in B$ for every $B\in\mathscr{B}'$ and $\mathscr{B}'$ is not a neighbourhood basis at p. But then there is an open set U, $p\in U$ such that every element of $\mathscr{B}'$ hangs out from U, and thus $\mathscr{B}'$ is not point-$\varkappa$ in U.

DEFINITION 4. A basis $\mathscr{B}$ of a topological space X satisfies property $\mathscr{B}(\varkappa)$ if $\mathscr{B}$ is point-$\varkappa$.

X has property $B(\varkappa)$ iff there exists an open basis $\mathscr{B}$ which satisfies $B(\varkappa)$.

REMARK. From Lemma 3 it is clear that the existence of a uniform basis in the sense of P. S. Aleksandrov is equivalent to the property $A(\omega)$, property $B(\omega_1)$ is that of having point-countable basis. It is obvious that $B(\varkappa)$ implies $A(\varkappa)$.

Next we are going to investigate the question: when does $A(\varkappa)$ imply $B(\varkappa)$? The following lemmas and theorems are proved for a fixed regular cardinal $\varkappa > \omega$.

LEMMA 5. *If X has property $A(\varkappa)$ then $\chi(X) \leqq \varkappa$. Further if $\varkappa > \omega$ is a regular cardinal number, $\psi(p, X) < \varkappa$ and $\mathscr{B}$ satisfies property $A(\varkappa)$ then* $\operatorname{ord}(p, \mathscr{B}) < \varkappa$.

PROOF. The first statement is trivial.

Let $\{p\} = \bigcap \{G_\alpha; \alpha \in \tau\}$, $\psi(p, X) = \tau < \varkappa$ and suppose that $\operatorname{ord}(p, \mathscr{B}) \geqq \varkappa$. Then for every $B \in \mathscr{B}$ with $B \neq \{p\}$ and $p \in B$ there is an $\alpha \in \tau$ such that B hangs out from G_α. Since $\tau < \varkappa$, and $\varkappa$ is regular there is $\alpha_0 \in \tau$ such that $\varkappa$ many elements of $\mathscr{B}$ hang out from G_{α_0}. But this contradicts the fact that $\mathscr{B}$ satisfies $A(\varkappa)$.

DEFINITION 6. A point p of a topological space X is a $P_\varkappa$-point iff the intersection of less than $\varkappa$-many neighbourhoods of p is also a neighbourhood of p.

LEMMA 7. *If $\psi(p, X) = \varkappa$ and X has property $A(\varkappa)$ then p is a $P_\varkappa$-point. Consequently, p has a neighbourhood basis which is well-ordered by the inclusion.*

PROOF. Let $\mathscr{B}$ be an open basis which satisfies $A(\varkappa)$. Let $\{G_\alpha; \alpha \in \tau, \tau < \varkappa\}$ be a system of open neighbourhoods of p. Since $\psi(p, X) = \varkappa$ there are $\varkappa$-many elements of $\mathscr{B}$ which contain p. If $\bigcap \{G_\alpha; \alpha \in \tau\}$ is not a neighbourhood of p then for every element $B \in \mathscr{B}$ with $p \in B$ there is an index $\alpha \in \tau$ such that B hangs out from G_α. Since $\tau < \varkappa$ and $\varkappa$ is regular there is an index $\alpha_0 \in \tau$ such that $\varkappa$-many elements of $\mathscr{B}$ hang out from G_{α_0} and contain p. This contradicts Lemma 3.

EXAMPLES. (i) A Lindelöf space X which has property $A(\omega_1)$ but is not $B(\omega_1)$, i.e. does not have a point-countable basis. Let D be a set, $|D| = \omega_1$ and p be a point, $p \notin D$. Put $X = D \cup \{p\}$. Define a topology on X in the following way: the points of D are isolated, the neighbourhoods of p are the complements of countable subsets of D and the point p. Of course, this space X is regular and Lindelöf, and has no point-countable basis, since $\chi(p, X) = \omega_1$. But X has property $A(\omega_1)$. In fact, the following open basis $\mathscr{B}$ satisfies $A(\omega_1)$; first let $\{x_\alpha; \alpha \in \omega_1\}$ be a well-ordering of the set D and put $\mathscr{B} = \{\{x_\alpha\}; \alpha \in \omega_1\} \cup \{B_\alpha \cup \{p\}; \alpha \in \omega_1\}$ where $B_\alpha = \{x_\beta; \alpha < \beta < \omega_2\}$.

(ii) From the above example it is not difficult to get a Lindelöf regular space which has property $A(\omega_1)$, and $\psi(p, X) = \omega_1$ for all $p \in X$.

(iii) An ω_1-metrizable space without isolated points is also an example like that in (ii), see [7].

THEOREM 8. *Let $\varkappa$ be a regular cardinal number.*

1) If X is a T_1-space with property $A(\varkappa)$ and $d(X) < \varkappa$ then X has property $B(\varkappa)$.

2) If X is a Hausdorf space and has property $A(\varkappa)$ and $c(X) < \varkappa$ then X has property $B(\varkappa)$.

PROOF. Let $\mathscr{B}$ be an open basis which satisfies property $A(\varkappa)$. By Lemma 5 if $\psi(p, X) < \varkappa$ then $\operatorname{ord}(p, \mathscr{B}) < \varkappa$.

Since $\mathscr{B}$ satisfies property $A(\varkappa)$ by Lemma 5 we can suppose that there is a $p \in X$ with $\psi(p, X) = \varkappa$. By Lemma 6 p is a $P_\varkappa$-point.

(1) Let X be a T_1-space and $S \subset X$ be dense in X, $|S| = \lambda < \varkappa$. For every $q \in S$ we choose a neighbourhood V_q of p such that $q \notin V_q$. By Lemma 7, since p is a $P_\varkappa$-point $U = \bigcap \{V_q; q \in S\}$ is again a neighbourhood of p and $U \cap S = \emptyset$; this contradicts $p \in \overline{S}$.

(2) Let X be a Hausdorff space and $c(X) < \varkappa$. By transfinite recursion we define $\{U_\alpha, V_\alpha; \alpha \in \varkappa\}$ open subsets of X such that $\{U_\alpha; \alpha \in \varkappa\}$ is a disjoint family, $U_\alpha \cap V_\alpha = \emptyset$ and $p \in V_\alpha$ for every $\alpha < \varkappa$. Let $\alpha < \beta < \varkappa$ and suppose that U_α, V_α have been defined. Put $W = \bigcap \{V_\alpha; \alpha < \beta\}$. Since p is a $P_\varkappa$-point, W is a neighbourhood of p. Choose U_β, V_β open subsets of W, $V_\beta \cap U_\beta = \emptyset$, $p \in V_\beta$. Since $U_\beta \subset W$, $U_\beta \cap U_\alpha = \emptyset$ for every $\alpha < \beta$. Thus $\{U_\beta; \beta < \varkappa\}$ is a disjoint family of open subsets, which contradicts $c(X) < \varkappa$. Q.e.d.

EXAMPLE. Let X be a space which has property $B(\omega_1)$, $c(X) = \omega$ and X is the increasing union of ω_1 closed subsets: $X = \bigcup \{F_\alpha; \alpha \in \omega_1\}$ (Such a space can be constructed from a Suslin-line or the PIXLEY—ROY space, see [10], [8].) Let p be a point $p \notin X$. Put $Y = X \cup \{p\}$. Let $\{(X \setminus F_\alpha) \cup \{p\}; \alpha \in \omega_1\}$ be the neighbourhood basis of the point p. Thus we get a T_1-space. This has property $A(\omega_1)$ and $c(X) = \omega$ but not $B(\omega_1)$, because $\chi(p, X) = \omega_1$.

DEFINITION 9. A topological space X is $[\omega, \varkappa]$-compact if every subset S of X with $|S| < \varkappa$ has a complete accumulation point, i.e. a point p such that for every neighbourhood U of p, $|U \cap S| = |S|$.

LEMMA 10. *Let $\varkappa > \omega$ be a regular cardinal number. If a T_1 space X is $[\omega, \varkappa]$-compact and X has property $B(\varkappa)$ then $w(X) < \varkappa$.*

PROOF. Let $\mathscr{B}$ be an open point-$\varkappa$ basis of X. If X is also compact then this is a theorem of MIŠČENKO, see [3].

If X is not compact but $[\omega, \varkappa]$-compact then $L(X) = \tau \geqq \varkappa$. Then there is a family $\{G_\alpha; \alpha \in \tau \geqq \varkappa\}$ of open subsets of X such that $G_\alpha \subset G_\beta$ iff $\alpha < \beta < \tau$, $\bigcup \{G_\alpha; \alpha \in \tau\} = X$ but if $\eta < \tau$ then $\bigcup \{G_\alpha; \alpha \in \eta\} \neq X$. Since X is $[\omega, \varkappa]$-compact, we can suppose that τ is regular.

For every point p of X we can choose a smallest index $\alpha_p \in \tau$ and a $B_p \in \mathscr{B}$ such that $p \in B_p \subset G_{\alpha_p}$. Let $\mathscr{F} = \{X \setminus G_\alpha; \alpha \in \tau\}$. Let f be the function defined as follows; for every $p \in X$ $f(p) = B_p$. Then f is an $\mathscr{F}$-regressive function in the sense given in [6]. Then by [6] there is a point p and a subset C of X such that C is $\mathscr{F}$-cofinal and for every $y \in C$ the set $B_y \in \mathscr{B}$ contains p. Since C is $\mathscr{F}$-cofinal we can choose a subset C' of C such that C' is $\mathscr{F}$-cofinal and for every $y, z \in C'$, $y \neq z$, $B_y \neq B_z$. Since $\operatorname{cf}(\tau) = \tau \geqq \varkappa$ thus $|C'| = \lambda \geqq \varkappa$. Now we got at least τ-many elements of $\mathscr{B}$ containing p, which contradicts $B(\varkappa)$.

THEOREM 11. *Let $\varkappa > \omega$ be a regular cardinal. If X is a compact T_2 space, and has property $A(\varkappa)$ then it also has property $B(\varkappa)$ and thus $w(X) < \varkappa$.*

PROOF. Let $\mathscr{B}$ denote an open basis which satisfies property $A(\varkappa)$. By Lemma 5 if $\psi(p, X) < \varkappa$ then $\operatorname{ord}(p, \mathscr{B}) < \varkappa$. Thus we can suppose that $\psi(p, X) = \varkappa$ for some $p \in X$. Then by Lemma 7 there is a neighbourhood basis $\{G_\alpha; \alpha \in \varkappa\}$ of p which is well-ordered by the inclusion. X is a regular space, so we can also suppose $G_\alpha \subset \overline{G}_\alpha \subset G_\beta$ if $\beta < \alpha < \varkappa$. Then for every α we choose a point p_α, $p_\alpha \in G_\alpha \setminus \overline{G}_{\alpha+1}$. $\{p_\alpha; \alpha \in \varkappa\}$ is a discrete subset of X.

For every $\alpha\in\varkappa$ let $F_\alpha=\{\overline{p_\beta};\ \beta<\alpha\}$ and $Y=\cup\{F_\alpha;\ \alpha\in\varkappa\}$. If $y\in Y$ then $\psi(y,Y)<\varkappa$ since $y\in\{\overline{p_\beta};\ \beta<\alpha\}$ for an $\alpha<\varkappa$, and $A(\varkappa)$ is a hereditary property. Thus Y has property $A(\varkappa)$ and so by Lemma 5 has property $B(\varkappa)$. On the other hand, Y is clearly $[\omega,\varkappa]$-compact, hence by Lemma 10 we get $w(Y)<\varkappa$. But this contradicts the fact that Y has a discrete subspace of cardinality $\varkappa$.

Let $\mathrm{pl}\,(X)$ denote the plunning degree of the space X as introduced e.g. in [4].

COROLLARY 1. *If X has property $A(\varkappa)$ but not $B(\varkappa)$ then* $\mathrm{pl}\,(X)\geqq\varkappa$.

PROOF. If we have $\mathrm{pl}\,(X)<\varkappa$ then every $P_\varkappa$-point p in X has a compact neighbourhood. This contradicts Theorem 11.

COROLLARY 2. *If X is a p-space and has property $A(\omega_1)$ then X has a point-countable basis.*

EXAMPLE (iv). Now we construct a space H such that H has property $A(\omega_1)$ but H^2 does not. Let I denote the closed unit interval and X be the space constructed in example (i) above. Let H be the disjoint topological union of I and X. Of course, then H has property $A(\omega_1)$. Since $A(\omega_1)$ is a hereditary property it is enough to show that $X\times I$ does not have this property. If p denotes the non-isolated point of X then $\psi((p,t),X\times I)=\omega_1$ for every $t\in I$ but (p,t) is a not a P_{ω_1}-point. Therefore $X\times I$ does not have property $A(\omega_1)$ by Lemma 7.

PROBLEM. What can be said about the class of topological spaces having property $A(\varkappa)$, where $\varkappa$ is a singular cardinal number? Is Lemma 7 true in this case?

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DIE ADDITIVE STRUKTUR DER IM ENGEREN SINNE LINEAR KOMPAKTEN RINGE

von

PHAM NGOC ANH

§ 1. Einleitung

Die additive Struktur eines Moduls, insbesondere eines Ringes hat im allgemeinen großen Einfluss auf die modul-, insbesondere ringtheoretischen Eigenschaften des Moduls bzw. Ringes. Von diesem Standpunkt aus betrachtet ist es eine wichtige Aufgabe, auch die additive Gruppe der im engeren Sinne linear kompakten Moduln, insbesondere der im engeren Sinne linear kompakten Ringe zu bestimmen. Dadurch wird die Beantwortung einer Reihe interessanter Fragen bezüglich dieser Ringe ermöglicht. Zum Beispiel lassen sich auf diese Weise die transfinit nilpotenten Ringe [7] charakterisieren. In dieser Arbeit wollen wir die additive Gruppe der im engeren Sinne linear kompakten Moduln, also auch der im engeren Sinne linear kompakten Ringe beschreiben und mit der additiven Struktur zusammenhängende Fragen diskutieren.

§ 2. Vorbereitungen

Unter einem Ring verstehen wir stets einen assoziativen Ring. Bezüglich der Grundbegriffe verweisen wir auf die Lehrbücher der Algebra.

Unter einem R-Modul M verstehen wir immer eine additive Gruppe mit Linksoperatorbereich R, ferner wird M^+ bzw. R^+ die additive Gruppe des Moduls M bzw. des Ringes R bezeichnen. Wir sagen daß ein R-Modul M ein artinscher Modul ist oder er die Minimalbedingung für Untermoduln erfüllt, falls jede streng absteigende Kette von Untermoduln nach endlich vielen Schritten abbricht.

Zu unseren Untersuchungen brauchen wir einige Vorbereitungen. Die folgende Behauptung stammt von A. G. KUROSCH [4]:

(I) *Eine abelsche Gruppe G mit Minimalbedingung für Untergruppen ist genau dann eine direkte Summe endlich vieler Gruppen $C(p_i^{n_i})$:*

$$G = C(p_1^{n_1}) \oplus \ldots \oplus C(p_k^{n_k}) \quad (1 \leqq n_1, \ldots, n_k \leqq \infty),$$

wabei $C(p^n)$ die zyklische Gruppe der Ordnung p^n ist, falls n endlich ist, und die quasizyklische Gruppe vom Typ p^∞, falls n unendlich ist.

Für den Beweis weisen wir auf KUROSCH [4] oder FUCHS [1] Seite 65 hin.

Das Radikal eines Ringes wird immer das Jacobsonsche Radikal bedeuten.

Jetzt werden wir einige topologische Vorbereitungen ausführen. Es wird stets vorausgesetzt, daß die topologischen Räume hausdorffsch sind. Ein Ring R ist ein topologischer Ring, wenn R ein topologischer Raum ist und die Subtraktion und die Multiplikation stetige Funktionen bezüglich dieser Topologie sind.

Unter einem topologischen R-Modul M verstehen wir einen R-Modul im al-

gebraischen Sinne, wobei R ein topologischer Ring und M eine topologische Gruppe sind, so daß die Modulmultiplikation eine stetige Funktion von $R\times M$ in M ist.

Grundlegend ist für unsere Untersuchungen der Begriff des inversen Limes. Bezüglich der Definition des inversen Limes verweisen wir auf das Buch von GRÄTZER [3].

Das inverse System $\Omega=[M_\alpha, \pi^\alpha_\beta]$ ist die direkte Summe der inversen Systeme $\Omega_\lambda=[M_{\alpha\lambda}, \varrho^\alpha_{\beta\lambda}]$, falls $M_\alpha=\sum_\lambda M_{\alpha\lambda}$ gilt, und $\varrho^\alpha_{\beta\lambda}$ die Beschränkung von π^α_β auf $M_{\alpha\lambda}$ ist.

(II) ([8], Theorem 1.) *Es gilt* $\varprojlim \sum_\lambda \Omega_\lambda=\sum_\lambda \varprojlim \Omega_\lambda$ *im algebraischen und topologischen Sinne.*

Wir nennen einen topologischen R-Modul M linear topologisch, wenn M einen Basisfilter aus Untermoduln besitzt. Ein linear topologischer R-Modul M wird linear kompakt (kurz: l. k.) genannt, falls jeder Filter aus Restklassen nach abgeschlossenen Untermoduln in M Berührungspunkte hat. Ein topologischer Ring R heißt linear kompakt, wenn er als topologischer R-Modul l. k. ist. Es gilt dann

(III) ([9], Prop. 4.) *Der inverse Limes linear kompakter R-Moduln ist l. k.*

Unter den linear topologischen Moduln spielen jene Moduln eine sehr wichtige Rolle, die inverse Limites von diskreten artinschen Moduln sind. Leptin hat diese Moduln im engeren Sinne linear kompakt (kurz: i. e. S. l. k.) genannt. Da ein diskreter artinscher Modul stets l. k. ist (vgl. z. B. Zelinsky [9] Prop. 5), so sind infolge (III) die i. e. S. l. k. Moduln immer l. k. Ein topologischer Ring R heißt i. e. S. l. k., wenn er als topologischer R-Modul i. e. S. l. k. ist.

(IV) ([6], (1.2)) *Ein unitärer l. k. Modul über einem halbeinfachen l. k. Ring ist die komplette direkte Summe minimaler Untermoduln.*

§ 3. Die additive Gruppe der i. e. S. l. k. Moduln und Ringe

SATZ 1. *Für die additive Gruppe M^+ eines i. e. S. l. k. Moduls M existiert eine direkte Zerlegung der Form*

$$M^+ = \sum{}^{\oplus} K_0 \oplus \sum_i{}^{\oplus} \sum{}^{\oplus} C(p_i^\infty) \oplus B,$$

wobei K_0 die additive Gruppe der rationalen Zahlen und B ein inverser Limes beschränkter Gruppen sind.

BEWEIS. Sei M ein i. e. S. l. k. Modul, also $M=\varprojlim [M_\alpha, \pi^\alpha_\beta]$, wobei die M_α artinsch sind. Für die Untermoduln nM_α ergeben sich zwei Fälle:

a) Es ist $nM_\alpha=0$ für eine geeignete natürliche Zahl n, also ist M_α eine beschränkte Gruppe.

b) Es ist $nM_\alpha\neq 0$ für alle natürlichen Zahlen n. Unter den Untermoduln nM_α gibt es also einen minimalen, etwa $mM_\alpha=D_\alpha$. Dann gilt für jede Primzahl p

$$pD_\alpha = D_\alpha.$$

Daher ist D_α eine teilbare abelsche Gruppe und folglich ein direkter Summand von M_α^+:

$$M_\alpha^+ = D_\alpha \oplus B_\alpha.$$

Wegen $D_\alpha=mM_\alpha=mD_\alpha\oplus mB_\alpha=D_\alpha\oplus mB_\alpha$ folgt $mB_\alpha=0$, also ist B_α beschränkt.

Bezeichnet ϱ_β^α bzw. σ_β^α die Beschränkung von π_β^α auf D_α bzw. B_α, so ist leicht zu sehen, daß $[D_\alpha, \varrho_\beta^\alpha]$ und $[B_\alpha, \sigma_\beta^\alpha]$ inverse Systeme bilden und $[M_\alpha, \pi_\beta^\alpha]$ ihre direkte Summe ist.

Wir zeigen jetzt, daß $\varprojlim [D_\alpha, \varrho_\beta^\alpha]$ teilbar ist. Es sei nämlich k eine beliebige natürliche Zahl. Die Gesamtheit der Elemente $x=(\ldots, x_\alpha, \ldots)$ von $\varprojlim [D_\alpha, \varrho_\beta^\alpha]$, für die $kx_\alpha=0$ gilt, bildet einen abgeschlossenen Untermodul K_α von M. Es sei jetzt $y=(\ldots, y_\alpha, \ldots)\in\varprojlim [D_\alpha, \varrho_\beta^\alpha]$ ein beliebiges Element. Es sei $N_\alpha=\{(\ldots, a_\alpha, \ldots)\in \in\varprojlim D_\alpha : ka_\alpha=y_\alpha\}$. Es sit offensichtlich, daß N_α eine Restklasse nach dem abgeschlossenen Untermodul K_α ist und $\{N_\alpha\}$ einen Filter bildet. Wegen der linearen Kompaktheit von M gibt es ein Element y_0 in $\bigcap N_\alpha$. Für y_0 gilt $ky_0=y$, also ist $\varprojlim D_\alpha$ teilbar.

Nach (II) besitzt M die folgende direkte Zerlegung:

$$M = \varprojlim D_\alpha \oplus \varprojlim B_\alpha$$

Damit haben wir den Beweis des Satzes 1 erbracht.

Daraus ergibt sich unmittelbar die

FOLGERUNG 1. *Für die additive Gruppe R^+ eines i. e. S. l. k. Ringes R existiert eine direkte Zerlegung der Form*

$$R^+ = \sum{}^{\oplus} K_0 \oplus \sum_i{}^{\oplus} \sum{}^{\oplus} C(p_i^\infty) \oplus B,$$

wobei B ein inverser Limes beschränkter Gruppen ist.

Mit Hilfe von diesem Ergebnis beweisen wir die

FOLGERUNG 2. *Jede Untergruppe vom* Typ $C(p^\infty)$ *eines i. e. S. l. k. Ringes R liegt im Linksannullator von R.*

BEWEIS. Es sei A eine Untergruppe vom Typ $C(p^\infty)$ in R. Es sei $R=\varprojlim [R_\alpha, \pi_\beta^\alpha \beta]$, wobei die R_α artinsche R-Moduln sind. Also genügt es zu zeigen, daß $AR_\alpha=0$ ist. In anderen Worten: wegen $D_\alpha\oplus B_\alpha=R_\alpha$ beweisen wir $AD_\alpha=AB_\alpha=0$, wobei D_α die maximale teilbare Untergruppe und B_α eine beschränkte Untergruppe von R_α sind.

Es seien nämlich $x\in A$ beliebig und $t\in D_\alpha$. Da A eine Torsionsgruppe ist, hat x die endliche Ordnung p^k. Da D_α teilbar ist, gibt es ein $t'\in D_\alpha$ mit $p^k t'=t$. Folglich gilt $xt=x(p^k t')=(p^k x)t'=0$. Ist jetzt $b\in B_\alpha$ ein anderes beliebiges Element, so gibt es eine geeignete natürliche Zahl n mit $nb=0$. Da A teilbar ist, gibt es ein $x'\in A$ mit $nx'=x$. Folglich gilt $xb=nx'b=x'(nb)=0$, d. h. $AR_\alpha=0$ für jedes R_α, also ist $AR=0$.

§ 4. Zusammenhang mit noetherschen Moduln

FUCHS und SZELE [2] zeigt, daß ein artinscher Ring genau dann noethersch ist, wenn er keine Untergruppe vom Typ $C(p^\infty)$ besitzt. Es sei jetzt $R=\varprojlim [R_\alpha, \pi_\beta^\alpha]$ ein i. e. S. l. k. Ring, und wir fragen nach einer notwendigen und hinreichenden Bedingung dafür, daß die artinschen Moduln R_α noethersch sind; d. h. wann ist ein i. e. S. l. k. Ring inverser Limes von noetherschen Moduln. Es gilt der

SATZ 2. *Sei* $R=\varprojlim [R_\alpha, \pi_\beta^\alpha]$ *ein i. e. S. l. k. Ring mit Radikal* J, *wobei die* R_α *artinsche R-Moduln sind. Alle* R_α *sind genau dann noethersch, wenn* $\cap \overline{J^i}=0$ *gilt und* R *keine Untergruppe vom Typ* $C(p^\infty)$ *besitzt, wobei der Querstrich zur Bezeichnung des topologischen Abschlusses einer Menge eines topologischer Raumes dient.*

BEWEIS. Sei $\{U_\alpha\}$ ein Basisfilter von Linksidealen in R, so daß $R/U_\alpha=R_\alpha$ gilt. Zunächst nehmen wir an, daß alle R_α noethersch sind. Dann hat jedes R_α eine Kompositionsreihe

$$R_\alpha = H_0 \supset H_1 \supset \ldots \supset H_{m_\alpha} = 0.$$

Daraus folgt, daß die H_i/H_{i+1} $(i=0, \ldots, m_\alpha-1)$ einfache R-Moduln sind. Für ein beliebiges von 0 verschiedenes Element $g\in H_i/H_{i+1}$ gilt $H_i/H_{i+1}=Rg$ und somit

$$H_i/H_{i+1} \cong R/(0\colon g) \quad ((0; g) = \{r\in R: rg = 0\}),$$

wobei das Linksideal $(0:g)$ modular und wegen der Einfachheit des Moduls H_i/H_{i+1} maximal ist. Dies zeigt, daß $J\subseteq(0:g)$, d. h. $JH_i\subseteq H_{i+1}$. Es folgt also $J^{m_\alpha}R_\alpha=0$ und somit $J^{m_\alpha}\subseteq U_\alpha$. Es gilt dann $\overline{J^{m_\alpha}}\subseteq\overline{U}_\alpha=U_\alpha$, da U_α ein offenes und so auch abgeschlossenes Linksideal ist. Wir haben also $\cap\overline{J^i}\subseteq\cap U_\alpha=0$. Andererseits folgt aus dem Beweis von Satz 1 $R_\alpha^+=D_\alpha\oplus B_\alpha$, wobei D_α die maximale teilbare Untergruppe und B_α eine beschränkte Untergruppe von R_α sind. Es bezeichne T_α die maximale teilbare Torsionsuntergruppe von R_α. Es gilt dann

$$T_\alpha^+ = \sum_p^{\oplus} T_\alpha(p),$$

wobei die $T_\alpha(p)$ p-Komponenten von T_α sind.

Wir zeigen jetzt, daß $T_\alpha(p)$ für jede Primzahl p ein Untermodul von R_α ist, d. h. wir beweisen $RT_\alpha(p)\subseteq T_\alpha(p)$. Weil $T_\alpha(p)$ eine p-Torsionsgruppe ist, so auch $RT_\alpha(p)$. Wegen der Teilbarkeit von $T_\alpha(p)$ ist $RT_\alpha(p)$ teilbar, d. h. $RT_\alpha(p)\subseteq T_\alpha(p)$.

Bezeichne A_n für jede natürliche Zahl n die Untergruppe aller Elemente x von $T_\alpha(p)$, für die $p^n x=0$ gilt. Dann haben wir offensichtlich

$$RA_n \subseteq A_n,$$

d. h. A_n ist ein Untermodul von R_α. Andererseits gelten $A_n\subseteq A_{n+1}$ für alle natürlichen Zahlen n. Also haben wir eine aufsteigende Kette von Untermoduln in R_α, die nach endlich vielen Gliedern nicht abbricht. Folglich ist jedes R_α noethersch, und enthält daher kein Untergruppe vom Typ $C(p^\infty)$, d. h. es gibt keine Untergruppe vom Typ $C(p^\infty)$ in R. Damit haben wir gezeigt, daß die Voraussetzung notwendig ist.

Jetzt nehmen wir an, daß $\cap\overline{J^i}=0$ gilt und R keine Untergruppe vom Typ $C(p^\infty)$ besitzt.

Da die Moduln R_α artinsch sind, gibt es für alle α eine kleinste nichtnegative ganze Zahl n_α mit $J^k+U_\alpha=J^{n_\alpha}+U_\alpha$ für alle $k\geqq n_\alpha$. Nach Satz 1 [5] gilt $U_\alpha=$ $=\bigcap_{i=1}^{\infty}\overline{J^i}+U_\alpha=\bigcap_{i=1}^{\infty}(J^i+U_\alpha)=J^{n_\alpha}+U_\alpha$, d. h. J^{n_α} liegt in U_α. Wir haben also $J^{n_\alpha}R_\alpha=0$.

Nach Satz 1 hat die additive Gruppe von $J_{n_\alpha-1}=(J^{n_\alpha-1}+U_\alpha)/U_\alpha$ die Form $D\oplus B$, wobei D die maximale teilbare Untergruppe und B inverser Limes be-

schränkter Gruppen sind. Da R_α keine quasizyklische Untergruppe enthält, ist D torsionsfrei.

Wir haben also

$$R_\alpha/I_{n_\alpha-1} \cong R_\alpha/D/I_{n_\alpha-1}/D.$$

Als teilbare torsionsfreie Untergruppe von R_α ist D ein direkter Summand von R_α; d.h. R_α/D ist die direkte Summe einer teilbaren torsionsfreien Gruppe, denn R_α ist ein artinscher R-Modul, der keine Untergruppen vom Typ $C(p^\infty)$ enthält. Wegen $I_{n_\alpha-1}/D \cong D$ ist $R_\alpha/I_{n_\alpha-1} \cong R_\alpha/D/I_{n_\alpha-1}/D$ die direkte Summe einer teilbaren torsionsfreien Untergruppe und einer beschränkten Gruppe; d.h. $R_\alpha/I_{n_\alpha-1}$ enthält keine Untergruppen vom Typ $C(p^\infty)$. Mit R_α enthalten also auch $R_\alpha/I_{n_\alpha-1}$ und $I_{n_\alpha-1}$ keine Untergruppen vom Typ $C(p^\infty)$.

Der Modul R_α ist genau dann noethersch, wenn es $R_\alpha/J_{n_\alpha-1}$ und $J_{n_\alpha-1}$ sind.

Wir betrachten jetzt den Modul $J_{n_\alpha-1}$. Er wird zu einem Modul über dem halbeinfachen l. k. Ring R/J, indem man für ein beliebiges Element $\bar{r}=r+J$ $(r\in R)$ des Faktorringes R/J die Verknüpfung $\bar{r}s$ durch $\bar{r}s=rs$ $(s\in J_{n_\alpha-1})$ definiert. Diese Definition ist zulässig, da aus $\bar{r}_1=\bar{r}_2\in R/J$, d. h. aus $r_1-r_2\in J$ für jedes $s\in J_{n_\alpha-1}$ $(r_1-r_2)s\in J$. $J_{n_\alpha-1}=0$, d. h. $r_1s=\bar{r}_2s$ folgt für alle $s\in J_{n_\alpha-1}$. So stimmen die R-Untermoduln und die R/J-Untermoduln von $J_{n_\alpha-1}$ überein. Insbesondere ist $J_{n_\alpha-1}$ als R/J-Modul artinsch. Für $J_{n_\alpha-1}$ gibt es eine direkte Zerlegung

$$J_{n_\alpha-1} = J^*_{n_\alpha-1} \oplus H,$$

wobei H ein trivialer und $J^*_{n_\alpha-1}$ ein unitärer R/J-Modul sind. Nach (IV) gibt es eine direkte Zerlegung

$$J^*_{n_\alpha-1} = \sum_{j\in\Gamma} E_j \tag{3}$$

wobei E_j für jedes $j\in\Gamma$ ein einfacher R/J-Modul ist. Da $J^*_{n_\alpha-1}$ artinsch ist, so ist Γ endlich. Andererseits ist H artinsch, und genügt daher der Minimalbedingung für Untergruppen. Also besitzt H nach (I) die Struktur

$$H = C(p_1^{n_1}) \oplus \ldots \oplus C(p_i^{n_i}) \quad (n_1, \ldots, n_i \leqq \infty).$$

H muß sogar endlich sein, da $J_{n_\alpha-1}$ keine quasizyklische Untergruppe enthält. Der R-Modul $J_{n_\alpha-1}$ wird also durch endlich viele noethersche R-Moduln erzeugt und ist demzufolge selbst noethersch.

Für $R_\alpha/J_{n_\alpha-1}=R_{n_\alpha-1}$ gilt $J^{n_\alpha-1}R_{n_\alpha-1}=0$. Man kann daher die obige Methode in $n_\alpha-1$ Schritten wiederholen, und damit haben wir bewiesen, daß R_α noethersch ist, q. e. d.

SATZ 3. *Gilt* $\bigcap_{i=1}^{\infty} J^i=0$ *für einen i. e. S. l. k. Ring* R *mit (Links-) Einselement, wobei* J *das Radikal von* R *ist, so ist er inverser Limes noetherscher Moduln.*

Wenn nämlich R ein (Links-) Einselement hat, so sind alle im Beweis von Satz 2 betrachteten Moduln unitär. Satz 3 läßt sich daher mit obiger Methode beweisen,

Es soll hier bemerkt sein, daß der inverse Limes noetherscher Moduln nich notwendig noethersch ist. Z. B. ist die komplette direkte Summe unendlich vieler noetherscher Moduln inverser Limes noetherscher Moduln, aber offensichtlich nicht noethersch.

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ON THE GLOBAL TRIANGULARIZABILITY OF PLANAR DIFFERENTIABLE DYNAMICAL SYSTEMS

by

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1. The main result

The subject of this paper is a special type of transformations of differential equation systems. Let the differential system have the form

$$\dot{x}_1 = v(x_1, x_2), \tag{1.1}$$

$$\dot{x}_2 = w(x_1, x_2)$$

where the real functions v, w are defined on the whole Euclidean plane $\mathbf{R}^2$ and the following conditions are assumed

A) $v, w \in C^n(\mathbf{R}^2)$ for some natural number $n \in \mathbf{N}$, i.e. they belong to the class of n-times continuously differentiable functions on $\mathbf{R}^2$,

B) $v^2+w^2>0$ everywhere on $\mathbf{R}^2$.

Then the differential system (1.1) induces a planar, unstable C^n-dynamical system φ. This means that the mapping

$$\varphi = (\varphi_1, \varphi_2)\colon \mathbf{R}^2 \times \mathbf{R} \to \mathbf{R}^2$$

has the following properties

D_1. $\varphi \in C^n(\mathbf{R}^3)$,

D_2. For every $p \in \mathbf{R}^2$ and $t, s \in \mathbf{R}$

$$\varphi(p, 0) = p$$

$$\varphi\big(\varphi(p, t), s\big) = \varphi(p, t+s)$$

$$\dot{\varphi}_1(p, t) = v\big(\varphi_1(p, t), \varphi_2(p, t)\big)$$

$$\dot{\varphi}_2(p, t) = w\big(\varphi_1(p, t), \varphi_2(p, t)\big),$$

D_3. For every point $p \in \mathbf{R}^2$ the trajectory $\overline{\varphi(p; \mathbf{R})}$ is not compact.

Concerning the further notions and theorems used in our paper we refer to the book of NIEMITZKIĬ and STEPANOV [2].

First we need the definition of triangularizability:

DEFINITION 1. The differential system (1.1) (or the induced dynamical system φ) is called a *triangularizable* one if it can be transformed to a triangularized form

$$\dot{u}_1 = \lambda(u_1), \tag{1.2}$$

$$\dot{u}_2 = \mu(u_1, u_2)$$

i.e. there exist a C^n-diffeomorphism

$$U\colon \mathbf{R}^2 \to \mathbf{R}^2$$

and a reparametrization $R:\mathbf{R}^2\times\mathbf{R}\to\mathbf{R}\in C^n(\mathbf{R}^3)$ such that $U(\varphi_t(p))=\psi_{R(p,t)}(U(p))$ for every $p\in\mathbf{R}^2$ and $t\in\mathbf{R}$, where ψ is the induced dynamical system of (1.2). The word 'transformation' means the common mapping of trajectories U and the parameter transformations R on different orbits. Obviously

$$R(p,t)+R(\varphi_t(p),s)=R(p,t+s) \quad \text{for every} \quad p\in\mathbf{R}^2 \quad \text{and} \quad t,s\in\mathbf{R}.$$

We introduce the notion of multiplicity of improper saddle point (See [3]).

DEFINITION 2. The dynamical system φ *has an improper saddle point of multiplicity at least* $k(\in\mathbf{N})$ *on the invariant set* $M\subset\mathbf{R}^2$ if there exist sequences

$$(q_n^0)_{n\in\mathbf{N}}\subset M \quad \text{and} \quad (t_n^i)_{n\in\mathbf{N}},\ (\tau_n^1)_{n\in\mathbf{N}},\ \left(\tau_n^j-\sum_{s=1}^{j-1}t_n^s\right)_{n\in\mathbf{N}}\to\infty$$

$$i=1,\ldots,k, \quad j=2,\ldots,k$$

such that

$$\lim_{n\to\infty}q_n^0=q^0\in M,\ \lim_{n\to\infty}q_n^i=\lim_{n\to\infty}\varphi\left(q_n^0,\sum_{s=1}^{i}t_n^s\right)=q^i\in M \quad i=1,2,\ldots,k$$

$$\sum_{s=1}^{i-1}t_n^s<\tau_n^i<\sum_{s=1}^{i}t_n^s, \quad i>1,\ 0<\tau_n^1<t_n^1$$

and for every $i=1,2,\ldots,k$ the set

$$T_i=\{\varphi(q_n^0,\tau_n^i):n\in\mathbf{N}\}\subset M$$

has no accumulation point (See Figure 1).

The trajectories $\varphi(q^j;\mathbf{R})$ $(j=0,\ldots,k)$ are said to be the limit orbits of the saddle point.

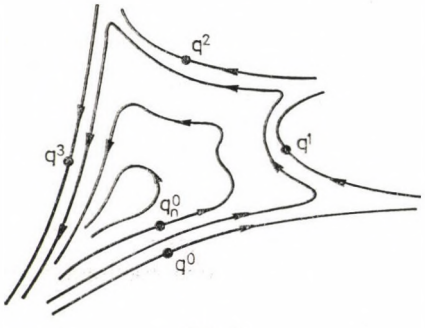

Fig. 1

The dynamical system φ has an improper saddle point of multiplicity (exactly) k on the set M if the previous case holds and it has no improper saddle point of multiplicity at least $k+1$ there.

V. S. SAMOVOL proved that a necessary condition for the triangularizability of planar, unstable dynamical systems is that they have no improper saddle point of multiplicity at least 3.

Our present aim is to give a necessary and sufficient condition for the triangularizability problem.

In the whole paper we shall denote by χ the stereographical projection of $\mathbf{R}^2$ into the Riemann sphere S^2 and let N be the north pole of S^2. Let φ be a C^n-dynamical system on the plane. If we apply the mapping χ, we can extend the dynamical system

$$\chi(\varphi)=\varphi^*=\{\varphi_t^*=\chi\circ\varphi_t\circ\chi^{-1}:\ t\in\mathbf{R}\}$$

by setting $\varphi_t^*(N)=N$ for every $t\in\mathbf{R}$.

On the other hand, every C^n-dynamical system on the sphere has at least one critical point. By choosing one of them (if there are many) as N the inverse of the corresponding stereographical projection defines a planar dynamical system. Thus the planar and spherical dynamical systems can be considered as equivalent ones in this sense.

Let φ^* be a dynamical system on S^2, having only one critical point N and denote the family of its trajectories by

$$\Phi^* = \{C(p) = \varphi^*(p; \mathbf{R}):\ p \in S^2\}$$

The trajectory $C(p) \in \Phi^*$ is called a loop, provided that $p \neq N$. If $C(p)$ is a loop, then $\overline{C(p)} = C(p) \cup \{N\}$ is a closed Jordan curve and so — by the Jordan curve theorem — $\overline{C(p)}$ yields a partition of the sphere

$$S^2 = \theta_1(\overline{C(p)}) \cup \theta_2(\overline{C(p)}) \cup \overline{C(p)}$$

DEFINITION 3. The set $\mathcal{M} \subset \Phi^*$ is called a *loop series* if the following conditions are valid

LS_1. $\mathcal{M}$ consists of loops,

LS_2. There exist loops $L_1, L_2 \in \Phi^*$ such that $\cup\{L_1, L_2\} \subset \overline{\cup \mathcal{M}}$ and

$$\cup \mathcal{M} = \theta_i(L_1) - \overline{\theta_k(L_2)} \quad \text{for some} \quad i, k \in \{1, 2\}.$$

We denote sometimes $\cup \mathcal{M}$ by $T(L_1, L_2)$ and $\cup \mathcal{M}$ is called a cell, L_1, L_2 are its consoles.

DEFINITION 4. The loop series $\mathcal{M}$ is called a *first class* one, if the dynamical system $\chi^{-1}(\varphi^*)|\overline{\chi^{-1}(\cup \mathcal{M})}$ is parallelizable (in homeomorphic sense). (See Figure 2.)

By the fundamental theorem of topological dynamics the parallelizability is equivalent to the condition that the dynamical system in question has no improper saddle point of multiplicity at least 1.

DEFINITION 5. The loop series $\mathcal{M}$ is called a *second class* one, if the dynamical system $\chi^{-1}(\varphi^*)|\overline{\chi^{-1}(\cup \mathcal{M})}$ has an improper saddle point of multiplicity 1 with limit trajectories $\chi^{-1}(L_1)$ and $\chi^{-1}(L_2)$, so that the dynamical system $\chi^{-1}(\varphi^*)|\chi^{-1}(\cup \mathcal{M})$ is parallelizable (See Figure 3.).

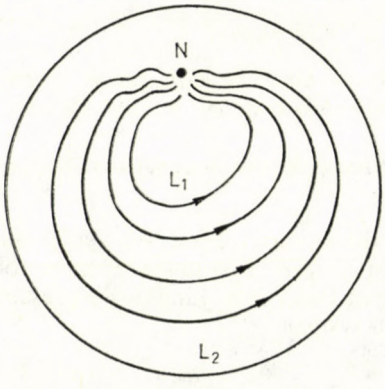

Fig. 2

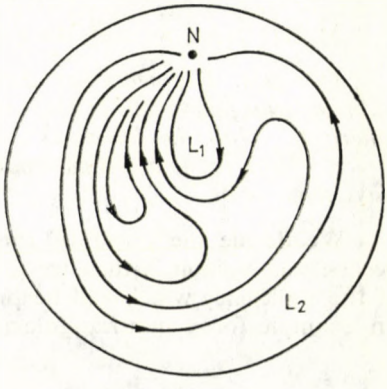

Fig. 3

DEFINITION 6. The loop series $\mathscr{M}$ is called a third class one, if there exists a loop $L \in \mathscr{M}$ such that

$$\theta_k(\bar{L}) \subset \cup \mathscr{M} \quad \text{for some index} \quad k \in \{1, 2\}$$

and the dynamical system

$$\chi^{-1}(\varphi^*)|\chi^{-1}(\cup \mathscr{M})$$

is parallelizable so that the system

$$\chi^{-1}(\varphi^*)|\overline{\chi^{-1}(\cup \mathscr{M} - \theta_k(\bar{L}))}$$

has an improper saddle point of multiplicity either 1 or 2 with limit trajectories $\chi^{-1}(L)$ and $\chi^{-1}(L_i)$ for some $i \in \{1, 2\}$ or $i = 1, 2$ resp. The set $\theta_k(\bar{L})$ is called a leaf (See Figures 4. and 5.).

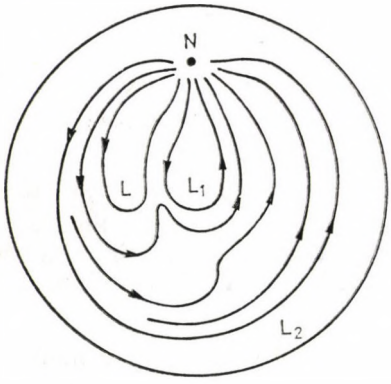

Fig. 4

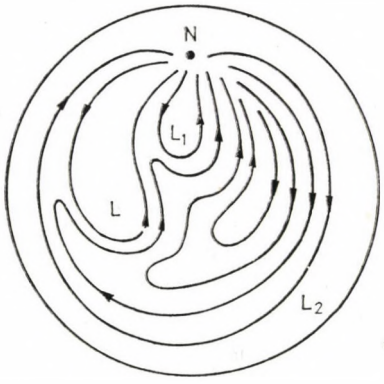

Fig. 5

At this point we can state the main result of this paper:

THEOREM. *The necessary and sufficient condition that the differential system* (1.1) *be triangularizable is that the induced dynamical system* φ^* *on the Riemann sphere* S^2 *fulfils the following property:*

Δ. There exists a cell partition of S^2 *such that the cells are first, second or third class loop series.*

We devide the proof of this theorem into three parts. First we verify the necessity in Section 2, then we give two lemmas in Section 3 to prepare the proof of the sufficiency which will be proved completely in Section 4. Finally we present an example for a nontriangularizable differential system.

2. Proof of the necessity

Since the type of each cell defined above is invariant under diffeomorphisms, it is sufficient to consider differential systems which are already triangularized. Therefore we shall investigate the system (1.2).

Let Λ be the set of all real roots of the function λ. We shall discuss three cases.

(a) $\Lambda=\emptyset$.

In this case we may suppose that $\lambda>0$ on $\mathbf{R}$. Let $p\in\mathbf{R}^2$ be an arbitrary point and let us consider the trajectory $\psi(p;\mathbf{R})$ starting from the point p. For the first component

$$\psi_1(p,.)\colon \mathbf{R}\to\mathbf{R}$$

is strictly increasing, we have

$$\int_{p_1}^{\psi_1(p,t)}\frac{1}{\lambda}=t \quad \text{for every} \quad t\in\mathbf{R}.$$

Therefore $\lim_{t\to\infty}\psi_1(p,t)=\infty$ and $\lim_{t\to-\infty}\psi_1(p,t)=-\infty$, i.e. the dynamical system ψ is a countable union of first class loop series.

(b) $\alpha,\beta\in\Lambda$ are consecutive zeros of the function λ, i.e. $\lambda(\alpha)=\lambda(\beta)=0$ and $(\alpha,\beta)\cap\Lambda=\emptyset$.

Without loss of generality we may suppose that $\lambda>0$ on (α,β). Then for every fixed point p of the open strip

$$S=(\alpha,\beta)\times\mathbf{R}$$

the function

$$\psi_1(p,.)\colon \mathbf{R}\to\mathbf{R}$$

is strictly increasing.

The lines $\{\alpha\}\times\mathbf{R}$ and $\{\beta\}\times\mathbf{R}$ are minimal invariant sets and so

$$\psi(p;\mathbf{R})\subset S \quad \text{for every} \quad p\in S.$$

From the relation

$$\int_{p_1}^{\psi_1(p,t)}\frac{1}{\lambda}=t, \quad t\in\mathbf{R},$$

it follows that

$$\lim_{t\to-\infty}\psi_1(p,t)=\alpha \quad \text{and} \quad \lim_{t\to\infty}\psi_1(p,t)=\beta. \tag{2.1}$$

Since the dynamical system ψ has no critical point $\operatorname{sgn}_{\{\alpha\}\times\mathbf{R}}\mu$ and $\operatorname{sgn}_{\{\beta\}\times\mathbf{R}}\mu$ are constants. From the unstability property D_3 it follows that

$$\lim_{t\to\infty}\psi_2(p,t)=\pm\infty \quad \text{and} \quad \lim_{t\to-\infty}\psi_2(p,t)=\pm\infty.$$

According to the cited result of Samovol we have the possibilities 0, 1, 2 for the multiplicity of any improper saddle point.

0*. $\psi|\bar{S}$ has no improper saddle point.

Then it is a first class loop series.

1*. $\psi|\bar{S}$ has an improper saddle point of multiplicity 1. Then there exist sequences

$$(p_n)_{n\in\mathbf{N}} \subset S \quad \text{and} \quad (t_n)_{n\in\mathbf{N}} \subset \mathbf{R}$$

such that $\lim_{n\to\infty} p_n = p$, $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty} \psi(p_n, t_n) = q$. Because of the properties (2.1) there is no improper saddle point on the open strip S and therefore a limit trajectory of the improper saddle point in $\bar{S}$ should lie on the boundary of S, i.e.

$$p\in\partial S \quad \text{or} \quad q\in\partial S.$$

Without loss of generality we may suppose that the first case holds. Moreover let

$$p\in\{\alpha\}\times\mathbf{R}$$

and let us consider the trajectory $\psi(q;\mathbf{R})$. We shall distinguish two cases.

(i) $q\in\{\beta\}\times\mathbf{R}$,

(ii) $q\in\operatorname{int} S$.

In the case (i) we get a second class loop series on $\bar{S}$.

In the case (ii) we consider the trajectory $\psi(q;\mathbf{R}) = C(q)$. If

$$\lim_{t\to-\infty} \psi(q, t) = -\lim_{t\to\infty} \psi(q, t),$$

i.e. $\psi(q;\mathbf{R})$ separates the lines $\{\alpha\}\times\mathbf{R}$ and $\{\beta\}\times\mathbf{R}$ then we get one second class loop series on the domain $T(C(p), C(q))$. The further classification of S depends on the fact whether there be an improper saddle point on the domain $T(C(q), \{\beta\}\times\mathbf{R})$ or not. If there is not then S is an union of one first and one second class loop series.

If there is an improper saddle point, then by using the previous argumentations we get for some $r\in\overline{T(C(q), \{\beta\}\times\mathbf{R})}$ that on the closure of the domain $T(C(r), \{\beta\}\times\mathbf{R})$ there is a second class loop series and if $C(q)\neq C(r)$ then on the domain $T(C(q), C(r))$ there is a first class loop series. If

$$\lim_{t\to\infty} \psi(q, t) = \lim_{t\to-\infty} \psi(q, t)$$

then on the set $\chi^{-1}(\theta_k(\chi(C(q))))\subset S$ the dynamical system is parallelizable and this case corresponds to the third class loop series.

2*. $\psi|\bar{S}$ has an improper saddle point of multiplicity 2. Then there exist seqences $(p_k)_{n\in\mathbf{N}}\subset S$ and $(t_n)_{n\in\mathbf{N}}, (s_n)_{n\in\mathbf{N}}, (\tau_n^i)_{n\in\mathbf{N}}\subset\mathbf{R}$ $(i=1,2)$ such that $\lim_{n\to\infty} \tau_n^1 = \lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} (\tau_n^2 - t_n) = \infty$ and

$$0 < \tau_n^1 < t_n < \tau_n^2 < t_n + s_n \quad \text{for every} \quad n\in\mathbf{N}$$

and

$$\lim_{n\to\infty} p_n = p, \ \lim_{n\to\infty} \psi(p_n, t_n) = q, \ \lim_{n\to\infty} \psi(p_n, t_n + s_n) = r$$

so that the sets

$$T_i = \{\psi(p_n, \tau_n^i) : n\in\mathbf{N}\}, \quad i = 1, 2,$$

have no accumulation point. Since on the open strip S the dynamical system is parallelizable it follows that

$$p\in\{\alpha\}\times\mathbf{R}, \quad q\in S, \quad r\in\{\beta\}\times\mathbf{R}.$$

It is easy to see that $\theta_k(C(q)) \subset S$ for some $k \in \{1, 2\}$ and so there is a third class loop series on the set S.

(c) $\gamma \in \Lambda$ is the last root of the function λ, i.e. $(\gamma, \infty) \cap \Lambda = \emptyset$. Then the line $\{\gamma\} \times \mathbf{R}$ is minimal invariant set and the following relations are valid:

$$\lim_{t \to -\infty} \psi_1(p, t) = \gamma, \quad \lim_{t \to \infty} \psi_1(p, t) = \infty \quad (p \in (\gamma, \infty) \times \mathbf{R})$$

provided that $\lambda > 0$ on (γ, ∞).
Without loss of generality we may suppose that $\mu > 0$ on $\{\gamma\} \times \mathbf{R}$. Let

$$E_- = \{p \in (\gamma, \infty) \times \mathbf{R} : \lim_{t \to -\infty} \psi_2(p, t) = -\infty\} \quad \text{and} \quad E_+ = (\gamma, \infty) \times \mathbf{R} - E_-.$$

Obviously the sets E_+ E_- are simply connected and

$$E_+ \cup E_- = (\gamma, \infty) \times \mathbf{R}, \quad E_+ \cap E_- = \emptyset, \quad E_- \supset \{\gamma\} \times \mathbf{R}.$$

If $E_+ = \emptyset$ then $\psi | [\gamma, \infty) \times \mathbf{R}$ consists of countable many first class loop series.
If $E_+ \neq \emptyset$ then $\psi | \bar{E}_-$ is a second class loop series and on the set E_+ the dynamical system is a countable union of first class loop series.
Thus the proof of the necessity is completed.

3. Two lemmas

Before plunging into the detailed proof of the sufficiency, we state two propositions.

LEMMA 1. *Let $D \subset \mathbf{R}^2$ be a closed domain such that the dynamical system $\varphi |$ int D-induced by the differential system* (1.1) *— compose a first class loop series. Then there exists a C^n-diffeomorphism*

$$\omega : D \to [0, 1] \times \mathbf{R}$$

such that for every $p \in D$ the set

$$\omega(\varphi(p; \mathbf{R}))$$

is a line, parallel with the second axis.

PROOF. Let us define a relation on the set $D \times D$ as follows. Two points p, q are said to be equivalent, if there exists a Jordan curve $\pi : [0, 1] \to D$, connecting the points p and q, i.e. $\pi(0) = p$ and $\pi(1) = q$ such that π consists of finitely many trajectory segments, belonging to the dynamical systems φ and φ^0, where φ^0 denotes the orthogonal system of φ (see Figure 6.).

This relation is clearly an equivalence and so it defines a partition of the domain D. The equivalence classes are open in the

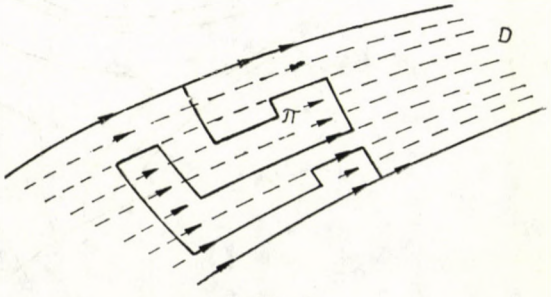

Fig. 6

relative topology of the subspace $D \subset \mathbf{R}^2$ and D is connected. Therefore there is only one equivalence class.

Let $p_0, q_0 \in \partial D$ be two points such that $\varphi(p_0; \mathbf{R}) \neq \varphi(q_0; \mathbf{R})$ and let us select a Jordan curve $\pi_0 : [0, 1] \to D$, connecting the points p_0 and q_0 such that it consists of trajectory segments belonging to the systems φ and φ^0.

Without loss of generality we may suppose that the number of trajectory segments, occuring in the curve π_0, is minimal.

We state that there is no whole trajectory $\varphi(\hat{p}; \mathbf{R})(\hat{p} \in D)$ such that

$$2 \leqq \operatorname{card}\left(\varphi(\hat{p}; \mathbf{R}) \cap \operatorname{graph} \pi_0\right) < \aleph_0.$$

Let us suppose indirectly, that there exist a point $p \in D$ and a real number $T \in \mathbf{R}$ such that $\varphi(p; [0, T]) \subset \operatorname{graph} \pi_0$ and

$$\varphi^0(p; [0, T_1]),\ \varphi^0(\varphi(p, T); [0, T_2]) \subset \operatorname{graph} \pi_0 \quad \text{for some} \quad T_1, T_2 \in \mathbf{R}_+$$

so that we can select points $p^* \in \varphi^0(p; (0, T_1))$, $q^* \in \varphi^0(q; (0, T_2))$ which are on the same trajectory, i.e. $\varphi(p^*; \mathbf{R}) = \varphi(q^*; \mathbf{R})$ $(\varphi(p, T) = q)$ (see Figure 7). Because of the minimality of π_0

$$\varphi(\varphi^0(p, T_1); \mathbf{R}) \cap \varphi^0(q; [0, T_2]) = \emptyset$$

and

$$\varphi(\varphi^0(q, T_2); \mathbf{R}) \cap \varphi^0(p; [0, T_1]) = \emptyset.$$

Let us consider the open intervals

$$E_1 = (0, t_1) = \left\{t \in (0, T_1]:\ \varphi(\varphi^0(p, t); \mathbf{R}) \cap \varphi^0(q; [0, T_2]) \neq \emptyset\right\}$$

and

$$E_2 = (0, t_2) = \left\{t \in (0, T_2]:\ \varphi(\varphi^0(q, t); \mathbf{R}) \cap \varphi^0(p; [0, T_1]) \neq \emptyset\right\}.$$

Obviously the trajectory family

$$\mathfrak{A} = \left\{\varphi\left(\varphi^0\left(p, t_1 - \frac{1}{n}\right); \mathbf{R}\right) : n \in \mathbf{N}\right\} \cup \left\{\varphi\left(\varphi^0\left(q, t_2 - \frac{1}{n}\right); \mathbf{R}\right) : n \in \mathbf{N}\right\}$$

creates an improper saddle point, which is contradiction.

Now let $\varphi(\tilde{p}, [0, \tau]) \subset \operatorname{graph} \pi_0$ be a trajectory segment and let us suppose that there exist real numbers $\varepsilon_1 < 0 < \varepsilon_2$ such that

$$\varphi^0(\tilde{p}, [\varepsilon_1, 0]),\ \varphi^0(\tilde{q}, [0, \varepsilon_2]) \subset \operatorname{graph} \pi_0, \quad \text{where} \quad \tilde{q} = \varphi(\tilde{p}, \tau).$$

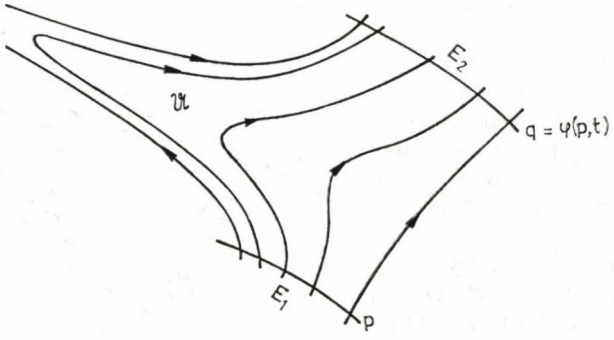

Fig. 7

Let us choose a positive number $\varepsilon<\min(-\varepsilon_1, \varepsilon_2)$ so that every trajectory, starting from the points of the segments $\varphi^0(\tilde{p}; (-\varepsilon, \varepsilon))$ and $\varphi^0(\tilde{q}; (-\varepsilon, \varepsilon))$ cuts the trajectories $\varphi^0(\tilde{q}; \mathbf{R})$ and $\varphi^0(\tilde{p}; \mathbf{R})$, resp. Then a real function $\mu:(-\varepsilon, \varepsilon)\to\mathbf{R}$ can be defined by the following relation

$$\varphi(\varphi^0(\tilde{p}, \varrho), \mu(\varrho))\in\varphi^0(\tilde{q}; \mathbf{R}), \quad \varrho\in(-\varepsilon, \varepsilon).$$

Let $D(\varepsilon)$ be defined by

$$D(\varepsilon) = \bigcup\{\varphi(\varphi^0(\tilde{p}, \varrho); [0, \mu(\varrho)]): \varrho\in(-\varepsilon, \varepsilon)\}.$$

It is clear that $\bigcap_{0<\varepsilon<\min(-\varepsilon_1, \varepsilon_2)} D(\varepsilon)=\varphi(\tilde{p}; [0, \tau])$. By the theorem of local parallelizability (See [1]) there exist a number $\varepsilon_0\in\mathbf{R}_+$ and a C^n-diffeomorphism

$$\delta\colon D(\varepsilon_0)\to[0, 1]\times[0, 1]$$

such that for every $\varrho\in(-\varepsilon_0, \varepsilon_0)$ the set

$$\delta(\varphi(\varphi^0(\tilde{p}, \varrho); [0, \mu(\varrho)]))$$

is a line segment, parallel with the second axis. Let us define a function $\hat{\psi}:(0, 1)\to$ $\to[0, 1]$ by

$$\hat{\psi}(x) = \sin^{n+1}\left(\frac{\pi}{2}x\right), \quad x\in[0, 1].$$

Then $\varphi^0(\tilde{p}; [\varepsilon_1, -\varepsilon_0])\cup\delta^{-1}(\text{graph}\,\hat{\psi})\cup\varphi^0(\tilde{q}; [\varepsilon_0, \varepsilon_2])$ is the underlying set of a C^n-Jordan curve.

When we perform the method above for every trajectory segment of the Jordan curve π_0 we get a C^n-curve γ which is a cross-section of the dynamical system $\varphi|D$. We may also suppose that $D(\gamma)=[0, 1]$. Now we construct the desired C^n-diffeomorphism

$$\omega\colon D\to[0, 1]\times\mathbf{R}$$

by the aid of

$$\omega(p) = (s(p), t(p))$$

where $p\in D$ and $\varphi(p, t(p))=\gamma(s(p))$. It is clear that $\omega\in C^n(D)$ and ω satisfies the desired properties of the Lemma 1 which accomplishes the proof.

LEMMA 2. *Let $f, g\in C^n[0, T)$ $(T>0)$ be functions such that the following conditions are valid:*

F_1. $f(0)=g(0)=0$ *and* $f=g$ *on* $[0, \varepsilon_0]$ $(0<\varepsilon_0<1)$,

F_2. f *and* g *are positive on* $(0, T)$,

F_3. $\lim_{\tau\to T} f(\tau)=\lim_{\tau\to T} g(\tau)=\infty$.

Let $m\in C^n[0, T)$ an arbitrary real function such that

$$\text{graph}\, m\subset\left\{\{t\}\times\left[\min\left(\frac{f}{4}, \frac{g}{4}\right), \min\left(\frac{f}{3}, \frac{g}{3}\right)\right]: t\in[0, T)\right\}$$

and let M be a domain defined by

$$M = \cup\{\{t\}\times[m(t), f(t)]:\ t\in[0, T)\} \subset \mathbf{R}^2.$$

Then there exists a positive $(n-1)$-times continuously differentiable function $h: M\to\mathbf{R}$ such that $h|\partial M\equiv 1$, $\partial_2^{(i)} h|\partial M\equiv 0$, $i=1, 2, \ldots, n-1$, and

$$\int_{m(t)}^{f(t)} h(t, s)\, ds = g(t)-m(t).$$

PROOF. Let us define the function $\varphi:\mathbf{R}\to\mathbf{R}$ by

$$\varphi(x) = \begin{cases} \exp\left(\dfrac{1}{x^2-1}\right), & \text{if} \quad |x| < 1, \\ 0 & \text{else,} \end{cases}$$

and consider the one-parameter function family $\psi_\varepsilon:\mathbf{R}\to\mathbf{R}$

$$\psi_0 \equiv 0 \quad \text{and} \quad \psi_\varepsilon(x) = \frac{1}{\int_{-\varepsilon}^{\varepsilon} \varphi\left(\dfrac{\tau}{\varepsilon}\right) d\tau} \int_{-\varepsilon}^{x} \varphi\left(\frac{\tau}{\varepsilon}\right) d\tau \quad \text{where} \quad \varepsilon > 0.$$

This family has the following properties

1° $\partial^{(i)}\psi_\varepsilon(-\varepsilon)=0$ for every $i\in\mathbf{N}\cup\{0\}$,

2° $\psi_\varepsilon(\varepsilon)=1$,

3° $\partial^{(k)}\psi_\varepsilon(\varepsilon)=0$ for every $k\in\mathbf{N}$.

Let us define the function $\xi: M\to\mathbf{R}$ by

$$\xi(t, x) = 1-\psi_{\delta(t)}(x-m(t)-\delta(t))+\psi_{\delta(t)}(x-f(t)+\delta(t)), \quad t\in(0, T),\ x\in[m(t), f(t)]$$

and $\xi(0, 0)=1$ where $\delta\in C^n[0, T)$ is choosen so that $\delta(0)=0, 2\delta=f-m$ on $\left[0, \frac{\varepsilon_0}{2}\right]$, $0<2\delta<f-m$ on $\left(\frac{\varepsilon_0}{2}, T\right)$ and for $t\in\left(\frac{\varepsilon_0}{2}, T\right)$, the inequality $\int_{m(t)}^{f(t)} \xi(t, x)\, dx < g(t)-m(t)$ is valid. Finally let $\lambda\in C^n[0, T)$

$$\lambda(t) = \frac{g(t)-m(t)-\int_{m(t)}^{f(t)} \xi(t, x)\, dx}{\int_{m(t)}^{f(t)} \varphi\left(\dfrac{2x-[m(t)+f(t)]}{f(t)-m(t)}\right) dx} \qquad t\in(0, T)\ (\lambda(0)=0).$$

and consider the function

$$h(t, x) = \lambda(t)\varphi\left(\frac{2x-[m(t)+f(t)]}{f(t)-m(t)}\right)+\xi(t, x), \quad (t, x)\in M.$$

Clearly $h \in C^{n-1}(M)$ and a simple calculation shows that the desired integral equality is valid. Because of the relations

$$\xi|\partial M \equiv 1, \eta|\partial M \equiv 0 \quad \text{and} \quad \partial_2^{(i)}\xi|\partial M \equiv \partial_2^{(i)}\eta|\partial M \equiv 0 \quad \text{for every} \quad i = 1, 2, \ldots, n-1$$

the boundary conditions are fulfilled too.

4. Proof of the sufficiency

Let φ be a dynamical system, induced by the differential system (1.1) which has the property Δ and denote the cell family and the set of the cell consoles by

$$\mathfrak{I} = \{D_\xi : \xi \in \mathbf{Z}\}$$

and

$$\mathfrak{G} = \{\Gamma_\xi : \xi \in \mathbf{Z}\}$$

resp. such that

$$\bar{D}_\xi \cap \bar{D}_{\xi+1} = \Gamma_\xi$$

for every $\xi \in \mathbf{Z}$ where $\mathbf{Z}$ is the set of integers.

We shall construct a C^n-diffeomorphism which transforms the dynamical system φ onto a triangularized one. Let us select points $\sigma_\xi \in \Gamma_\xi$ for every $\xi \in \mathbf{Z}$.

We investigate three different cases.

(1) D_ξ is a first class cell.

If we take $\sigma_{\xi-1} = p_0$ and $\sigma_\xi = q_0$ in the proof of the Lemma 1 then the closed domain $\bar{D}_\xi$ can be mapped onto the strip $[\xi-1, \xi] \times \mathbf{R}$ with some C^n-diffeomorphism such that $h(\sigma_{\xi-1}) = (\xi-1, 0)$, $h(\sigma_\xi) = (\xi, 0)$ and the trajectory images are parallel with the second axis.

(2) D_ξ is a second class cell.

Then $\varphi | \operatorname{int} D_\xi$ is parallelizable dynamical system and so using the method, described in the proof of the Lemma 1 we can prove that there exist C^n Jordan curves α, β and γ such that the following conditions are valid:

G_1. $D(\alpha) = D(\gamma) = [0, T]$, $D(\beta) = \mathbf{R}$, where T is a fixed positive number,

G_2. graph β, $\alpha([0, T))$, $\gamma([0, T)) \subset \operatorname{int} D_\xi$,

G_3. $\alpha(0) = \beta(0) = \gamma(0) = \sigma$ and $\alpha(T) = \sigma_{\xi-1}$, $\gamma(T) = \sigma_\xi$, where $\sigma \in \operatorname{int} D_\xi$ is a fixed point,

G_4. The sets $\alpha((0, T])$, $\beta(\mathbf{R} - \{0\})$, $\gamma((0, T])$ are mutually disjoint,

G_5. The sets graph α, graph β and graph γ are cross-sections and every trajectory $\varphi(p; \mathbf{R})$ starting from the point $p \in D_\xi$ cuts graph β,

G_6. $\dot{\alpha}(0)$, $\dot{\beta}(0)$, $\dot{\gamma}(0)$ are parallel vectors and $\langle \dot{\alpha}(T), \dot{\Gamma}_{\xi-1} \rangle = \langle \dot{\gamma}(T), \dot{\Gamma}_\xi \rangle = 0$. (See Figure 8.)

Before constructing the required mapping we perform a reparametrization of the dynamical system $\varphi | \bar{D}_\xi$.

Let $f : [0, T) \to \mathbf{R}$ be the function defined by the relation

$$\varphi(\alpha(t), f(t)) \in \operatorname{graph} \beta \quad t \in [0, T).$$

Obviously the function f has the properties F_1, F_2 and F_3 described in the Lemma 2. Maybe, we can assure the nonnegativity of f by the opposite reparametrization of the dynamical system $\varphi | \bar{D}_\xi$ if it is needed.

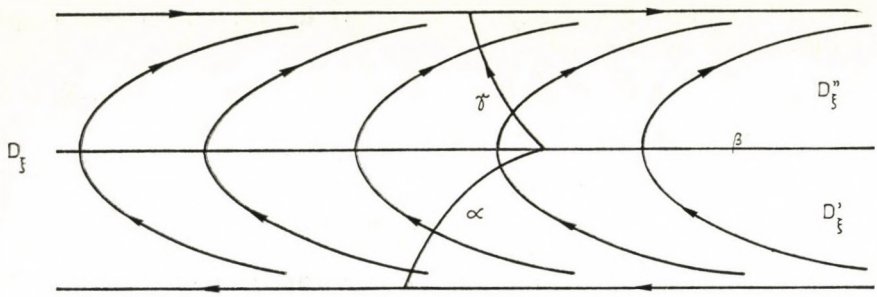

Fig. 8

Now let us construct a triangularized dynamical system φ^* on the strip $[0, 1]\times\mathbf{R}$, inducing the trajectories by the following one-parameter function system

$$v(t, \lambda) = \frac{1}{\sin(\pi t)} + \lambda, \quad \lambda \in \mathbf{R},\ t \in (0, 1).$$

Let $\beta^* = \left\{\left(\frac{1}{2}, 0\right)\right\}\times\mathbf{R}$ and let α^* be a circle arc joining the points $(0, 0)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$ with the center $\left(0, \frac{1}{2}\right)$. The Jordan curves β and β^* decompose the domains D_ξ and $(0, 1)\times\mathbf{R}$ resp., thus

$$D_\xi = D'_\xi \cup \operatorname{graph}\beta \cup D''_\xi.$$

Because of the symmetry it is sufficient to construct a mapping between the domains $\bar{D}'_\xi (\supset \operatorname{graph}\alpha)$ and $\left[0, \frac{1}{2}\right]\times\mathbf{R}$. Parametrizing the curves α^* and β^* such that

$$D(\alpha^*) = [0, T], \quad D(\beta^*) = \mathbf{R}, \quad \alpha^*(0) = \beta^*(0)$$

we are able to define a function

$$g\colon [0, T) \to \mathbf{R}$$

by the relation

$$\varphi^*\big(\alpha^*(t), g(t)\big) \in \operatorname{graph}\beta^*, \quad t \in [0, T).$$

α can be chosen such that g fulfils the properties F_1, F_2 and F_3 too.

Applying the Lemma 2 we get a function h to the functions f, g. We extend the function h to the domain

$$N = \cup\big\{\{t\}\times[0, m(t)]\colon\ t \in [0, T)\big\}$$

by taking $h|N \equiv 1$.

Let the function p be defined by

$$p(t, x) = \int_0^x h(t, s)\, ds, \quad t \in [0, T),\ x \in [0, f(t)].$$

The function p has the following properties

H. For every $t\in[0, T)$:

1. $p(t, 0)=0$,

2. $p(t,f(t))=\int_0^{f(t)} h(t, s)\,ds=m(t)+\int_{m(t)}^{f(t)} h(t, s)\,ds = g(t)$,

3. $\partial_2 p(t, 0)=h(t, 0)=1$,

4. $\partial_2 p(t,f(t))=h(t,f(t))=1$,

I. $p(t, x)=x$ for every $t\in[0, T)$ and $x\in[0, m(t)]$,

J. $p\in C^n(\cup\{\{t\}\times[0,f(t)]\colon t\in[0, T)\})$.

Then we give a reparametrization of the dynamical system $\varphi|\bar{D}'_\xi$, in the following manner:

Let the reparametrization be the identity outside the set

$$S=\{\varphi(\alpha(t), s)\colon t\in[0, T),\ 0<s<f(t)\}$$

and let the new parameter of the point

$$\varphi(\alpha(t), s)\in\bar{S}$$

be $p(t, s)$, where $t\in[0, T)$ and $s\in[0, f(t)]$. Then we can give the required C^n-mapping $\tilde{\omega}_\xi$ between the domains $\bar{D}'_\xi$ and $\left[0, \frac{1}{2}\right]\times\mathbf{R}$ as it follows.

If $p\in D'_\xi$ and $\beta(s)=\varphi(p, \tau(p))$ for some $s\in\mathbf{R}$ then let

$$\tilde{\omega}_\xi(p)=\varphi^*(\beta^*(s), -\tau(p)).$$

It follows from H that if $p\in D'_\xi$ and $\varphi(p;\mathbf{R})\cap\operatorname{graph}\alpha\neq\emptyset$ then

$$\tilde{\omega}_\xi(p)=\varphi^*(\alpha^*(u), -\zeta(p))$$

where $\alpha(u)=\varphi(p, \zeta(p))$ for some $u\in[0, T)$.
So if $p\in\Gamma_{\xi-1}$ the previous formula can serve as a definition of the mapping $\tilde{\omega}_\xi|\Gamma_{\xi-1}$.
It is obvious, that

$$\tilde{\omega}_\xi\colon \bar{D}'_\xi\to\left[0, \frac{1}{2}\right]\times\mathbf{R}$$

is an n-times continuously differentiable mapping and so the dynamical systems $\varphi|\bar{D}_\xi$ and φ^* are C^n-equivalent.

(3) D_ξ is a third class cell.

Let us define the Jordan curves α, β and γ similarly to the previous case with the additionally supplements that

$$(\operatorname{graph}\alpha\cup\operatorname{graph}\gamma)\cap L=\emptyset \quad\text{and}\quad \operatorname{graph}\beta\cap L\neq\emptyset$$

where L is the leaf of $\varphi|\bar{D}_\xi$ (See Figure 9.).

The Jordan curve β decomposes the domain D_ξ:

$$D_\xi=D'_\xi\cup\operatorname{graph}\beta\cup D''_\xi.$$

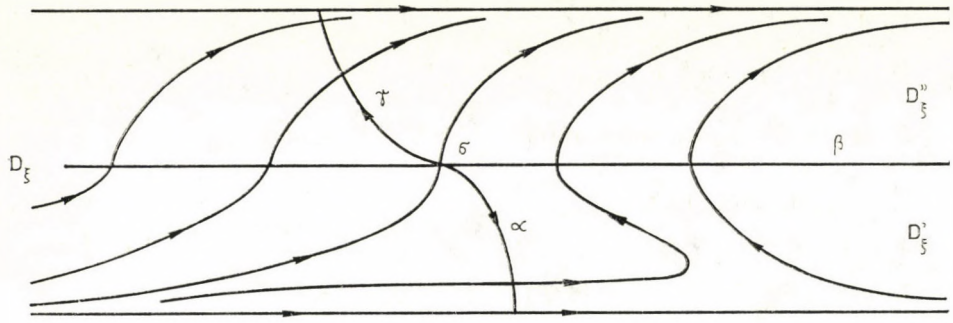

Fig. 9

Let us suppose that the improper saddle point of the dynamical system $\varphi|\bar{D}_\xi$ is on the domain $\bar{D}'_\xi$. It follows from the structure of third class loop series, that it is enough to give the required diffeomorphism on the domain $\bar{D}'_\xi$.

First we give a differential system on the strip $\left[0, \frac{\pi}{2}\right]\times\mathbf{R}$. Let Z be a real function, defined by

$$Z(x) = \begin{cases} \sqrt{2-x^2}, & \text{if} \quad |x| \leqq 1, \\ \dfrac{1}{x^2}, & \text{else,} \end{cases}$$

and consider the differential system

$$\dot{x}_1 = \cos^2 x_1 \int_{\operatorname{tg} x_1}^{1+\operatorname{tg} x_1} Z$$

$$\dot{x}_2 = \int_{\operatorname{tg} x_1}^{x_2} Z$$

on the strip $\left[0, \frac{\pi}{2}\right]\times\mathbf{R}$. In the points of the line $\left\{\frac{\pi}{2}\right\}\times\mathbf{R}$ we take the corresponding limits. Then the induced (semi) dynamical system has a unique improper saddle point. One limit trajectory of this improper saddle point is on the minimal invariant set graph $(\operatorname{tg} x+1)$. Let $L^* = \left\{(x_1, x_2) \in \left[0, \frac{\pi}{2}\right]\times\mathbf{R} : \operatorname{tg} x_1 + 1 < x_2\right\}$, and let $\gamma^*: [0, T] \to \left[0, \frac{\pi}{2}\right]\times\mathbf{R} - \bar{L}^*$ be a Jordan curve such that graph γ^* is a cross-section of φ^* and

$$\operatorname{graph}\beta^* \cap \operatorname{graph}\gamma^* = \{\gamma^*(0)\} = \{\sigma^*\},\ \gamma^*(T) = \left(\frac{\pi}{2}, 0\right),\ \dot{\gamma}^*(T) = (0, 1),$$

where graph $\beta^* = \{0\}\times\mathbf{R}$.

Defining the functions f and g similarly to the previous case we can apply the Lemma 2 to the functions

$$m = \min\left(\frac{f}{3}, \frac{g}{3}\right) \quad \text{and} \quad M = \min\left(\frac{2f}{3}, \frac{2g}{3}\right).$$

If we extend the obtained function h to the domain

$$Q = \cup\{\{t\}\times([0, m(t)]\cup[M(t), f(t)]):\ t\in[0, T)\}$$

by setting $h|Q\equiv 1$, then the C^n-diffeomorphism $\tilde{\omega}_\xi$ between the domains $\bar{D}'_\xi$ and $\left[0, \frac{\pi}{2}\right]\times\mathbf{R}$ can be constructed with repeating the proof of the previous case word for word.

It is clear from our construction that

$$\omega_\xi|\Gamma_{\xi+1} = \omega_{\xi+1}|\Gamma_{\xi+1}, \quad \xi\in\mathbf{Z},$$

and the joined mapping

$$\omega\colon \mathbf{R}^2 \to \mathbf{R}^2,$$

$$\omega|D_\xi = \omega_\xi \quad \text{for every} \quad \xi\in\mathbf{Z},$$

is a C^n-diffeomorphism, transforming the dynamical system φ onto a triangularized one. Thus the theorem is proved.

5. An example for nontriangularizable dynamical system.

Let f, s, v be functions defined by

$$f\colon \mathbf{R} \to \mathbf{R},$$

$$f(t) = \begin{cases} \sqrt{2}, & \text{if} \quad t \leqq 0, \\ \sqrt{2-t^2}, & \text{if} \quad 0 < t \leqq 1, \\ \dfrac{1}{t^2} & \text{else}, \end{cases}$$

$$s\colon \left[-1, \frac{\pi}{2}\right] \to \mathbf{R}$$

$$s(t) = \begin{cases} t+1, & \text{if} \quad -1 \leqq t < 0, \\ \cos t, & \text{if} \quad 0 \leqq t \leqq \dfrac{\pi}{2}, \end{cases}$$

and

$$v\colon \mathbf{R} \to \mathbf{R}$$

such that, if $x\leqq 1$, then

$$v(x) = \begin{cases} 0, & \text{if} \quad x \leqq 0, \\ \sqrt{2}x, & \text{if} \quad 0 < x \leqq 1 \end{cases}$$

and if $\frac{\pi}{2} > x > 0$ then

$$v(1+\operatorname{tg} x) = \frac{1}{\cos^3 x} f\left(1+\operatorname{tg} x - \frac{\pi}{2}\frac{\frac{\pi}{2}-x}{x}\right).$$

Let λ, ω be functions such that

$$\lambda, \omega\colon \mathbf{R}^2 \to \mathbf{R}$$

$$\lambda(x_1, x_2) = \begin{cases} f\left(x_2 + \dfrac{\pi|x_1|}{2|x_1| - \pi}\right), & \text{if} \quad -\dfrac{\pi}{2} \leqq x_1 \leqq \dfrac{\pi}{2}, \\ \sqrt{2} & \text{if} \quad \dfrac{\pi}{2} < |x_1| \leqq \dfrac{\pi}{2} + 1, \\ \sqrt{2}\cos^2\left(\dfrac{\pi}{2} + 1 - |x_1|\right) & \text{else,} \end{cases}$$

and

$$\omega(x_1, x_2) = \begin{cases} -s\left(\dfrac{\pi}{2} - |x_1|\right) v(x_2), & \text{if} \quad |x_1| \leqq \dfrac{\pi}{2} + 1 \\ \sin\left(\dfrac{\pi}{2} + 1 - |x_1|\right) & \text{else,} \end{cases}$$

where in the points of the lines $\left\{-\dfrac{\pi}{2}\right\} \times \mathbf{R}$, $\left\{\dfrac{\pi}{2}\right\} \times \mathbf{R}$ we should take the corresponding limits. Then the dynamical system induced by the differential system

$$\dot{x}_1 = \lambda(x_1, x_2)$$
$$\dot{x}_2 = \omega(x_1, x_2)$$

has the following properties:

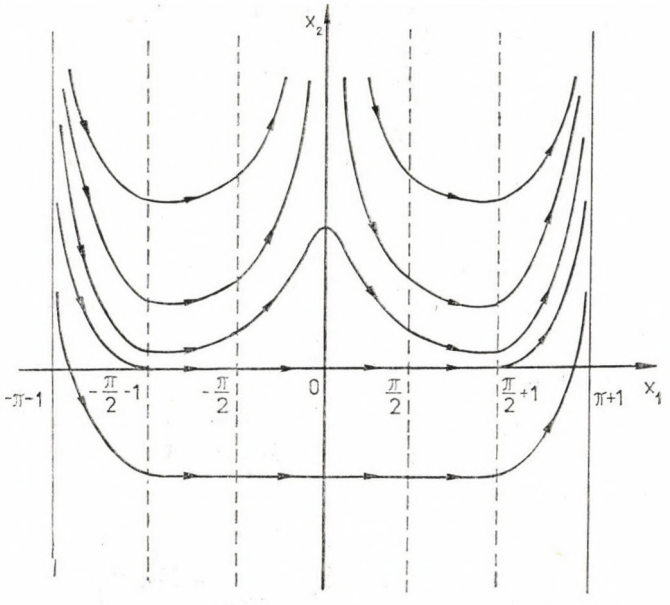

Fig. 10

K_1. The curves

$$x_2 = 1 - \operatorname{ctg} x_1 \quad -\frac{\pi}{2} \leqq x_1 < 0,$$

$$x_2 = 1 + \operatorname{ctg} x_1 \qquad 0 < x_1 \leqq \frac{\pi}{2}$$

are half-trajectories;

K_2. The trajectories which cross the line $\{0\}\times\mathbf{R}$, does it perpendicularly. (See Figure 10.)

The obtained dynamical system is not tirangularizable since it has a 'cell' on the strip $[-1-\pi, \pi+1]\times\mathbf{R}$ containing two 'leaves'.

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ÜBER GEWISSE VERSCHÄRFUNGEN DER BEIDEN KLASSISCHEN INDUKTIVEN TOPOLOGISCHEN DIMENSIONSBEGRIFFE I. RÄUME MIT „HÜBSCHEN" BASEN

von

E. DEÁK

§ 0. Einleitung*

Der vorliegende Artikel ist das erste Glied einer kleinen Serie, die von der folgenden Begriffsbildung ausgeht:

(0.1) DEFINITION. Ein System $\mathscr{U}$ offener Mengen eines Raumes soll ind-*n*-*hübsch* bzw. Ind-*n*-*hübsch* heißen, wenn für ein beliebiges $k \in \{1, 2, \ldots, n+1\}$ und für beliebige $U_1, U_2, \ldots, U_k \in \mathscr{U}$ mit $U_i \neq U_j$ $(i, j \in \{1, 2, \ldots, k\}, i \neq j)$

$$\text{ind} \bigcap_{i=1}^{k} \operatorname{Gr} U_i \leqq n-k \tag{0.1.1}$$

bzw.

$$\text{Ind} \bigcap_{i=1}^{k} \operatorname{Gr} U_i \leqq n-k \tag{0.1.2}$$

gilt.

Diese Eigenschaft gewinnt an Bedeutung, wenn $\mathscr{U}$ eine Überdeckung des Raumes X ist, und insbesondere lohnt es eine eigene Definition für den Fall zu geben, daß es sich sogar um eine Basis des Raumes handelt.

(0.2) DEFINITION. Eine *ind-hübsche* bzw. *Ind-hübsche* Basis eines Raumes der kleinen bzw. großen induktiven Dimension n bedeutet eine Basis, die im Sinne von (0.1) ind-*n*-hübsch bzw. Ind-*n*-hübsch ist.

* Bezeichnungen und Wortgebrauch. Das Wort *Raum* bedeutet einen topologischen Raum; als *Basis* wird eine offene Basis eines solchen bezeichnet. Das Symbol für die abgeschlossene Hülle bzw. Begrenzung einer Menge E in einem Raum X ist $E^{-(X)}$ bzw. $\operatorname{Gr}_X E$ (evtl. einfach E^- bzw. $\operatorname{Gr} E$). n bedeutet durchwegs — auch wenn dies nicht angedeutet wird — eine nichtnegative ganze Zahl. $\mathbf{R}^+$ bzw. $\mathbf{Q}^+$ ist das Zeichen für die Menge der positiven reellen bzw. positiven rationalen Zahlen. Für eine Menge A bedeutet $|A|$ ihre Mächtigkeit. Das Zeichen $\subset$ der Inklusion wird im Sinne $\subsetneqq$ gebraucht.

Wir legen für unsere Arbeit die folgende klassische Definition der beiden induktiven Dimensionsbegriffe zugrunde:

$$\text{ind } \emptyset = -1 \quad \text{bzw.} \quad \text{Ind } \emptyset = -1;$$

$$\text{ind } X \leqq n \qquad \text{bzw.} \quad \text{Ind } X \leqq n,$$

wenn für eine beliebige einelementige bzw. abgeschlossene nichtleere Menge H und eine beliebige offene Menge U im Raum X mit $H \subseteq U$ eine offene Menge G mit

$$H \subseteq G \subseteq U$$

und

$$\text{ind Gr } G \leqq n-1 \quad \text{bzw.} \quad \text{Ind Gr } G \leqq n-1$$

$(n=0, 1, 2, \ldots)$ existiert. ind $X=n$ bzw. Ind $X=n$ bedeutet, daß ind $X \leqq n$ bzw. Ind $X \leqq n$, aber nicht ind $X \leqq n-1$ bzw. Ind $X \leqq n-1$ gilt.

(0.3) BEMERKUNG. (a) Sind X ein Raum, $X^* \subset X$ und $\mathscr{U}$ ein ind-n-hübsches bzw. Ind-n-hübsches System in X, so ist, mit der Bezeichnung $M^* = X^* \cap M$ $(M \subseteq X)$, auch die Spur $\mathscr{U}^*$ von $\mathscr{U}$ auf X^* ind-n-hübsch bzw. Ind-n-hübsch in X^* (wegen $\mathrm{Gr}_{X^*} M^* \subseteq (\mathrm{Gr}\, M)^*$ für offene Mengen M in X).

(b) Insbesondere ist im Falle $\mathrm{ind}\, X^* = \mathrm{ind}\, X < \infty$ bzw. $\mathrm{Ind}\, X^* = \mathrm{Ind}\, X < \infty$ die Spur auf X^* einer ind-hübschen bzw. Ind-hübschen Basis in X eine ind-hübsche bzw. Ind-hübsche Basis in X^*.

(0.4) BEMERKUNGEN. (a) Läßt ein Raum X eine ind-n-hübsche Basis zu, so gilt $\mathrm{ind}\, X \leqq n$. (Man wende (0.1.1) auf den Fall $k = 1$ an). Es handelt sich also tatsächlich um eine Verschärfung des Begriffs der kleinen induktiven Dimension. (Die Existenz einer ind-0-hübschen Basis eines Raumes X ist sogar äquivalent mit $\mathrm{ind}\, X = 0$.)

(b) Natürlich fragt man sich, ob bzw. unter was für Bedingungen ind-hübsche Basen überhaupt existieren. Daß diese Fragestellung nicht ganz vage ist, kann durch ein höchst einfaches Beispiel für Räume X von beliebiger Dimension $\mathrm{ind}\, X < \infty$, deren *jede* Basis ind-hübsch ist, nahegelegt werden.

Es seien

$$X_{-1} = \varnothing,$$

$$X_n = \{1, 2, \ldots, n+1\} \quad (n = 0, 1, 2, \ldots),$$

und wir definieren eine Topologie auf jeder der Mengen X_n mit $n \geqq 0$: das System sämtlicher nichtleerer offenen Mengen sei

$$\{\{1, 2, \ldots, m\}\colon m = 1, 2, \ldots, n+1\}. \tag{0.4.1}$$

Für jeden dieser Räume gilt $\mathrm{ind}\, X_n = n$, der Raum läßt eine einzige Basis zu (eben nur das System aller nichtleeren offenen Mengen), und diese Basis ist ind-hübsch. (Für jedes $k \in \{1, 2, \ldots, n+1\}$ und für beliebige paarweise verschiedene offene Mengen $G_1, G_2, \ldots, G_k$ ist nämlich $\bigcap_{i=1}^{k} \mathrm{Gr}\, G_i$ einem Teilraum des X_{n-k} homöomorph.)

(c) Die Räume unter (b) sind allerdings nur T_0-Räume, schon die T_1-Eigenschaft besitzen sie nicht. Es kann aber für jeden endlichdimensionalen separablen metrisierbaren Raum X die Existenz von ind-hübschen Basen (etwa mit Hilfe des Menger-Nöbelingschen Einbettungssatzes) erwiesen werden. Es erübrigt sich, dies ausführlich darzulegen, da das Ergebnis sowohl im unter (0.5), (c) angeführten Satz wie im unter (1.6) zu beweisenden Satz des I. Teils enthalten ist. (Man beachte die Äquivalenz der drei klassischen Dimensionen ind, Ind und dim im Bereich der separablen metrisierbaren Räume.)

(0.5) BEMERKUNGEN. Was die große induktive Dimension im allgemeinen betrifft:

(a) Hier hängt die Existenz einer Ind-n-hübschen Basis nicht so einfach mit der Dimensionalität des betreffenden Raumes zusammen.

Es gibt z. B. — wie P. ROY in [10] gezeigt hat — sogar metrisierbare Räume X mit $0 = \mathrm{ind}\, X < \mathrm{Ind}\, X$. Jede Basis $\mathscr{B}$ mit $\mathrm{ind}\, \mathrm{Gr}\, B = -1$ $(B \in \mathscr{B})$ eines solchen Raumes ist selbstverständlich auch Ind-0-hübsch, und dennoch ist $\mathrm{Ind}\, X > 0$.

(b) Läßt aber ein Raum X eine Ind-n-hübsche Basis $\mathscr{B}$ zu, die zugleich σ-lokal-endlich ist, so ist er metrisierbar und es gilt $\operatorname{Ind} X \leqq n$, weil ja dazu schon $\operatorname{Ind} \operatorname{Gr} B \leqq n-1$ $(B \in \mathscr{B})$ genügend wäre (K. MORITA [7]; [8] 80).

(c) Es fragt sich andererseits, inwieweit die Existenz einer σ-lokal-endlichen und Ind-n-hübschen Basis eines metrisierbaren Raumes X eine strenge Verschärfung der Eigenschaft „$\operatorname{Ind} X \leqq n$" ist. Nun hat M. KATĚTOV 1956 in [5] bewiesen (ohne eigene Ausdrücke für die Eigenschaften unter (0.1) und (0.2) einzuführen), daß jeder endlich-dimensionale metrisierbare Raum eine σ-lokal-endliche und Ind-hübsche Basis zuläßt. Im Teil I, § 1, beweisen wir das (mit ganz anderen Methoden) aufs neue, aber mit sogar „σ-diskret" anstelle von „σ-lokal-endlich".

(d) Die Idee der „Hübschheit" ist also eigentlich nicht neu; einige Spezialfälle des unter (c) angeführten Satzes von Katětov wurden sogar schon viel früher als dieser bewiesen. Auch als Beweismittel taucht dieser Gedanke hie und da auf.

Wir sind aber auf anderem Wege zu diesem Begriff gekommen (s. Teil I, § 3), und möchten ihn einigermaßen systematisch, wenn auch noch in den Anfangsgründen verleibend, untersuchen.

§ 1. Metrisierbare Räume

(1.1) BEZEICHNUNGEN. Es seien (a, b) bzw. $[a, b]$ $(a, b \in \mathbf{R},\ a<b)$ das entsprechende offene bzw. abgeschlossene Zahlenintervall, A eine nichtleere Indexmenge,

$$\mathbf{0} = (0, 0, \ldots),$$

$$S_i = \{\mathbf{0}\} \cup \{(i, \alpha, x):\ x \in \mathbf{R}^+,\ \alpha \in A\} \quad (i \in \mathbf{N}),$$

$$P = \bigtimes_{i=1}^{\infty} S_i,$$

$$S_i^z(a, b) = \{(i, \alpha, x) \in S_i:\ a < x < b\}$$

$$(0 < a < b,\ \alpha \in A,\ i \in \mathbf{N}),$$

$$S_i(c) = \{(i, \alpha, x) \in S_i:\ x < c,\ \alpha \in A\}$$

$$(c \in \mathbf{R}^+,\ i \in \mathbf{N})$$

und für $p \in P \setminus \{\mathbf{0}\}$

$$p(i) = (i, \alpha_i(p), x_i(p))$$

die i-te Koordinate von p. (Für $p=\mathbf{0}$ setzen wir $x_i(p)=0$; im allgemeinen bezeichne $\pi_i[E]$ $(E \subseteq P)$ die i-te Projektion der Menge E.) Weiter sei für jedes $n \in \{0, 1, 2, \ldots\}$

$$P^{(n)} \subset P$$

die Menge jener Elemente p von P, unter deren von $\mathbf{0}$ verschiedenen Koordinaten $p(i)$ sich höchstens n mit rationalem $x_i(p)$ finden.

(1.2) DEFINITION. Es sei A eine nichtleere Menge. *Der Kowalskysche Sternraum vom Index A (oder $|A|$)* ist die Menge

$$S(A) = \{\mathbf{0}\} \cup \bigcup \{\mathbf{R}_\alpha^+:\ \alpha \in A\}$$

(man schreibt mitunter einfach S), wobei alle $\mathbf{R}_\alpha^+$ ($\alpha\in A$) Exemplare von $\mathbb{R}^+$ sind, versehen mit der Metrik

$$\varrho(x,y)=\begin{cases}|x-y| & (x,y\in\mathbf{R}_\alpha^+,\ \alpha\in A)\\ |x|+|y| & (x\in\mathbf{R}_\alpha^+,\ y\in\mathbf{R}_\beta^+,\ \alpha,\beta\in A,\ \alpha\neq\beta)\end{cases}$$

(H. J. KOWALSKY [6]).

(1.3) BEMERKUNGEN. (a) Mit unseren Bezeichnungen unter (1.1) kann die Topologie eines Kowalskyschen Sternraumes $S(A)$ (den Parameter i, der nur zur Unterscheidung verschiedener Exemplare von $S(A)$ diente, weglassend) dadurch beschrieben werden, daß das System

$$\{S^\alpha(a,b):\ 0<a<b,\ \alpha\in A\}\cup\{S(c):\ c\in\mathbf{R}^+\}$$

eine Basis dieser Topologie bildet.

(b) In der Definition unter (1.2) wird manchmal $E_\alpha=[0,1]$ anstelle von $\mathbf{R}_\alpha^+$ ($\alpha\in A$) genommen. In bezug auf die Anwendungen dieses Begriffs — s. etwa (1.5) — macht das keinen wesentlichen Unterschied.

(1.4) SATZ. (J. NAGATA [9].) *Für einen metrisierbaren Raum X gilt* $\dim X\leqq n$ *($n\in\{0,1,2,\ldots\}$) genau dann, wenn X in den Teilraum $P^{(n)}$ des entsprechenden (d. h. mit einer Indexmenge A, deren Mächtigkeit gleich dem topologischen Gewicht des Raumes X ist, gebildeten) topologischen Produktes P mit der Grundmenge unter* (1.1) *topologisch einbettbar ist.* □

(1.5) BEMERKUNG. Dieser Satz von Nagata — auf den wir den Beweis unseres folgenden Satzes (1.6) gründen — ist eigentlich eine dimensionstheoretische Erweiterung des bekannten Einbettungssatzes von KOWALSKY [6].

(1.6) SATZ. *Für jeden metrisierbaren Raum X mit* $\operatorname{Ind} X=n$, *n eine nichtnegative ganze Zahl, existiert eine σ-diskrete und* Ind-*n-hübsche Basis.*

BEWEIS. 1° Ist das topologische Gewicht des Raumes X durch die Mächtigkeit einer (festgesetzten) Menge A repräsentiert, so soll im folgenden die Menge A — um die Äste des Kowalskyschen Sternraumes $S(A)$ mit den Elementen α von A zu indizieren — zwar benutzt, das Zeichen A jedoch bei sämtlichen Symbolen, die sich auf den Kowalsky-Nagataschen Satz (1.4) beziehen, weggelassen werden.

2° Nach dem Satz (1.4) und der Bemerkung unter (0.3),(b) genügt es zu zeigen, daß der Raum $P^{(n)}$ eine σ-diskrete und Ind-n-hübsche Basis zuläßt.

3° Es sei $\mathcal{N}$ die Menge sämtlicher Paare $(N_0,N)\neq(\varnothing,\varnothing)$, wo N_0 und N jeweils disjunkte, endliche (evtl. leere) Teilmengen von $\mathbf{N}$ sind, und wir denken uns die abzählbar unendliche Menge $\mathcal{N}$ irgendwie als Folge

$$\mathcal{N}=\{(N_0^{(\nu)},N^{(\nu)}):\ \nu\in\mathbf{N}\}$$

mit

$$(N_0^{(\nu_1)},N^{(\nu_1)})\neq(N_0^{(\nu_2)},N^{(\nu_2)})\quad(\nu_1,\nu_2\in\mathbf{N},\ \nu_1\neq\nu_2)$$

geordnet.

4° Wir werden im weiteren eine Folge

$$\lambda_t=\left(N_0(t),N(t),\{q_i^{(1)}(t),q_i^{(2)}(t)\}_{i\in N(t)}\right)\quad(t\in\mathbf{N}) \tag{1.6.1}$$

von 3-Tupeln mit folgenden Eigenschaften als Hilfsmittel benutzen:

(I) $\{(N_0(t), N(t))\colon t\in\mathbf{N}\} = \mathcal{N}$

(es kann und muß auch für $t_1\neq t_2$ vorkommen, daß $(N_0(t_1), (N(t_1))=(N_0(t_2), N(t_2))$ ausfällt);

(II) $q_i^{(l)}(t)\in\mathbf{Q}^+$

$(l=1, 2,\ i\in N(t),\ t\in\mathbf{N})$;

(III) $q_{i_1}^{(l_1)}(t_1)\neq q_{i_2}^{(l_2)}(t_2)$

$(l_1, l_2\in\{1, 2\},\ i_1\in N(t_1),\ i_2\in N(t_2),\ t_1, t_2\in\mathbf{N},\ t_1\neq t_2)$;

(IV) Jedes der Systeme

$$\Big\{\bigtimes_{i\in N}(q_i^{(1)}(t), q_i^{(2)}(t))\colon (N_0(t), N(t)) = (N_0, N),\ t\in\mathbf{N}\Big\}$$
$$(i\in N,\ (N_0, N)\in\mathcal{N},\ N\neq\varnothing)$$

von offenen Würfeln bildet eine Basis für den entsprechenden Raum $\bigtimes_{i\in N}\mathbf{R}_i^+$ (mit $\mathbf{R}_i^+=\mathbf{R}^+$ $(i\in N)$).

5° Es gibt viele Möglichkeiten zur Konstruktion solcher Folgen $\{\lambda_t\}_{t\in\mathbf{N}}$. Einem (mündlichen) Vorschlag von J. DEÁK (Budapest) folgend geht man etwa so vor:

Es seien

$$\mathcal{M} = \{(r^{(0)}, r^{(1)}, r^{(2)}, r^{(3)})\in(\mathbf{Q}^+)^4\colon r^{(0)} < r^{(1)} < r^{(2)} < r^{(3)}\},$$

$\mathcal{M}_u=\mathcal{M}$ $(u\in\mathbf{N})$, und wir ordnen jene Teilmenge von $\bigtimes_{u\in\mathbf{N}}\mathcal{M}_u$, deren jedes Element höchstens endlich viele paarweise verschiedene Koordinaten hat, in eine Folge

$$\{\{r_i^{(0)}[\mu], r_i^{(1)}[\mu], r_i^{(2)}[\mu], r_i^{(3)}[\mu]\}_{i\in\mathbf{N}}\colon \mu\in\mathbf{N}\}.$$

Es sei weiter

$$\mathbf{Q}^+ = Q_0\cup\bigcup\{Q_{i,\nu}^{(l)}[\mu]\colon i\in N^{(\nu)},\ \nu, \mu\in\mathbf{N},\ N^{(\nu)}\neq\varnothing,\ l=1, 2\}$$

eine Zerlegung der Menge $\mathbf{Q}^+$ in (abzählbar unendlich viele) disjunkte Mengen Q_0 und $Q_{i,\nu}^{(l)}[\mu]$, d. h.

$$Q_0\cap Q_{i,\nu}^{(l)}[\mu]=\varnothing$$
$$(l\in\{1, 2\},\ i\in N^{(\nu)},\ \nu, \mu\in\mathbf{N},\ N^{(\nu)}\neq\varnothing),$$
$$Q_{i_1,\nu_1}^{(l_1)}[\mu_1]\cap Q_{i_2,\nu_2}^{(l_2)}[\mu_2]=\varnothing$$
$$(l_1, l_2\in\{1, 2\},\ i_1\in N^{(\nu_1)},\ i_2\in N^{(\nu_2)},$$
$$\nu_1, \nu_2, \mu_1, \mu_2\in\mathbf{N},\quad N^{(\nu_1)}\neq\emptyset,\quad N^{(\nu_2)}\neq\varnothing,$$
$$(l_1, i_1, \nu_1, \nu_2)\neq(l_2, i_2, \nu_2, \mu_2)).$$

Wählen wir nun für jedes Tripel

$$(i, \nu, \mu)\quad (\nu, \mu\in\mathbf{N},\ N^{(\nu)}\neq\varnothing,\ i\in N^{(\nu)})$$

Zahlen

$$q_{i,\nu}^{(l)}[\mu] \in Q_{i,\nu}^{(l)}[\mu] \quad (l = 1, 2)$$

mit

$$r_i^{(0)}[\mu] < q_{i,\nu}^{(1)}[\mu] < r_i^{(1)}[\mu],$$

$$r_i^{(2)}[\mu] < q_{i,\nu}^{(2)}[\mu] < r_i^{(3)}[\mu],$$

und ordnen wir die Menge

$$\left\{\{(q_{i,\nu}^{(1)}[\mu],\ q_{i,\nu}^{(2)}[\mu])\}_{i\in N^{(\nu)}}\colon\ \nu,\ \mu\in\mathbf{N},\ N^{(\nu)} \neq \varnothing\right\}$$

irgendwie als eine Folge

$$\left\{\{(q_i^{(1)}(t),\ q_i^{(2)}(t))_{i\in N(t)}\}\colon\ t\in\mathbf{N}\right\}$$

ohne Wiederholungen (wobei — da mit der Bezeichnung ν_t für das im t-ten Element dieser Folge auftretende ν, ν_t durch t eindeutig bestimmt ist — ν als unterer Index weggelassen und

$$N(t) = N^{(\nu_t)} \quad (t\in\mathbf{N})$$

gesetzt werden konnte), so wird die daraus gebildete Folge (1.6.1), mit

$$N_0(t) = N_0^{(\nu_t)} \quad (t\in\mathbf{N}),$$

allen Forderungen unter 4° entsprechen.

Die Menge Q_0 wurde bisher freigehalten.

6° Jetzt werde noch eine Folge

$$\{q(t)\}_{t\in\mathbf{N}} \subset Q_0$$

mit etwa

$$q(t) < \frac{1}{t} \quad (t\in\mathbf{N}) \tag{1.6.2}$$

(es kommt nur darauf an, daß $q(t)\to 0$ $(t\to\infty)$ gelte) und mit

$$q(t_1) \neq q(t_2) \quad (t_1, t_2\in\mathbf{N},\ t_1 \neq t_2) \tag{1.6.3}$$

ausgewählt.

7° Hiernach definieren wir

$$B_i(t;\alpha) = S_i^z(q_i^{(1)}(t),\ q_i^{(2)}(t)) \tag{1.6.4$_1$}$$

$$(t\in\mathbf{N},\ i\in N(t),\ \alpha\in A),$$

$$B_i(t) = \begin{cases} S_i(q(t)) & (i\in N_0(t)), \\ S_i & (i\in\mathbf{N}\setminus(N_0(t)\cup N(t)) \end{cases} \tag{1.6.4$_2$}$$

$$(t\in\mathbf{N}),$$

weiter

$$B(t;\{\alpha_i\}_{i\in N(t)}) = \Big(\bigtimes_{i\in N(t)} B_i(t;\alpha)\Big)\times\Big(\bigtimes_{i\in\mathbf{N}\setminus N(t)} B_i(t)\Big),$$

dann

$$\mathscr{B}(t) = \left\{B(t;\ \{\alpha_i\}_{i\in N(t)}):\ \alpha_i\in A\ \left(i\in N(t)\right)\right\}\quad (t\in\mathbf{N})$$

und endlich

$$\mathscr{B} = \bigcup_{t\in\mathbf{N}} \mathscr{B}(t).$$

8° Man sieht sofort, daß jeder der Teilräume

$$\mathrm{Gr}_{S_i} B_i(t;\alpha)\quad (t\in\mathbf{N},\ i\in N(t),\ \alpha\in A),$$

$$\mathrm{Gr}_{S_i} B_i(t)\qquad (t\in\mathbf{N},\ i\in\mathbf{N}\setminus N(t))$$

des entsptechenden Raumes S_i diskret (wenn nicht leer) ist. (Tatsächlich: für $i\in N(t)$ besteht dieser Teilraum aus genau zwei verschiedenen Punkten, für $i\in N_0(t)$ aus Punkten mit paarweise verschiedenen Indizes α, und bei $i\in\mathbf{N}\setminus\left(N_0(t)\cup N(t)\right)$ handelt es sich eben um die leere Menge; die Auswahl der $\alpha_i\in A$ spielt dabei überhaupt keine Rolle.)

9° Wir zeigen nun zunächst, daß das Mengensystem $\mathscr{B}$, dessen Elemente evident offene Mengen in P sind, sogar eine Basis dieses Raumes ist.

Sind $U\subseteq P$ und

$$p\in\operatorname{Int} U, \tag{1.6.5}$$

sonst beliebig, so gibt es ein $(N_0, N)\in\mathscr{N}$, Zahlen

$$0 < a_i < b_i\quad (i\in N),\quad c_i > 0\quad (i\in N_0)$$

und Indizes

$$\alpha_i\in A\quad (i\in N)$$

mit

$$a_i < x_i(p) < b_i,\quad \alpha_i = \alpha_i(p)\quad (i\in N)$$

und

$$x_i(p) < c_i\quad (i\in N_0),$$

so daß für die Menge

$$G = \bigtimes_{i\in\mathbf{N}} G_i$$

mit

$$G_i = \begin{cases} S_i(c_i) & (i\in N_0),\\ S_i^{\alpha}(a_i, b_i) & (i\in N),\\ S_i & \left(i\in\mathbf{N}\setminus(N_0\cup N)\right)\end{cases}$$

die Beziehung

$$p\in G\subseteq U \tag{1.6.6}$$

besteht.

Jetzt werde ein $t\in\mathbf{N}$ gewählt, das den folgenden Beschränkungen (1.6.7), (1.6.8) und (1.6.9) unterliegt.

Erstens sollen

$$N_0(t) = N_0,\quad N(t) = N \tag{1.6.7}$$

gelten. Solche $t\in\mathbf{N}$ gibt es unendlich viele.

Von diesen kommen aber, zweitens, nur diejenigen in Betracht, für welche

$$\{x_i(p)\}_{i\in N}\in\bigtimes_{i\in N}\left(q_i^{(1)}(t),\ q_i^{(2)}(t)\right)\subseteq\bigtimes_{i\in N}(a_i, b_i). \tag{1.6.8}$$

Wegen (IV) unter 4° ist das nicht nur erfüllbar, vielmehr verfügen wir noch immer über unendlich viele $t\in\mathbf{N}$.

Daher können wir t, nach (1.6.2), drittens noch der Beschränkung

$$g(t)\leqq\min[c_i:\ i\in N_0] \tag{1.6.9}$$

unterwerfen.

(Daß — offensichtlich — sogar jetzt noch unendlich viele geeignete t bleiben, tut nichts mehr zur Sache.)

Wir haben dann

$$p(i)\in B_i(t;\alpha_i)\subseteqq G_i\quad(i\in N),$$

$$p(i)\in B_i(t)\qquad(i\in\mathbf{N}\setminus N)$$

und daher

$$p\in B(t;\ \{\alpha_i\}_{i\in N})\subseteqq G.$$

Das bedeutet aber nach (1.6.5) und (1.6.6), daß $\mathscr{B}$ wahrhaftig eine Basis des Raumes P ist.

10° Jetzt soll gezeigt werden, daß jedes der Systeme $\mathscr{B}(t)$ $(t\in\mathbf{N})$ diskret im Raume P ist. Es seien zu diesem Zweck

$$t\in\mathbf{N},\quad p\in P$$

beliebig festgesetzt.

Im Falle $N(t)=\varnothing$ handelt es sich um eine Trivialität, da dann das System $\mathscr{B}(t)$ aus genau einer Menge besteht. Nehmen wir also an, daß

$$N(t)\neq\varnothing.$$

Gibt es nun ein $i\in N(t)$ mit

$$x_i(p)\notin[q_i^{(1)}(t),\ q_i^{(2)}(t)]$$

(was insbesondere dann der Fall ist, wenn $p(i)=0$ für irgendein $i\in N(t)$ gilt), so hat p im Raume P offensichtlich Umgebungen, die mit keiner der Mengen aus $\mathscr{B}(t)$ gemeinsame Punkte haben.

Sind aber, widrigenfalls,

$$x_i(p)\in[q_i^{(1)}(t),\ q_i^{(2)}(t)]\quad(i\in N(t))$$

(also, insbesondere, $x_i(p)\neq0$ $(i\in N(t))$, so ist die Menge

$$B(t;\ \{\alpha_i(t)\}_{i\in N(t)})$$

die einzige unter den Elementen von $\mathscr{B}(t)$, die mit den Umgebungen V von p, für welche

$$\pi_i[V]\subseteqq\{(i,\alpha_i(p),x):\ x\in\mathbf{R}^+\}\quad(i\in N(t)),$$

Punkte gemeinsam hat.

11° Da nach dem Bisherigen $\mathscr{B}$ eine σ-diskrete Basis von P ist, bildet auch die Spur $\mathscr{B}|P^{(n)}$ eine σ-diskrete Basis für den Teilraum $P^{(n)}$ von P. Nach der Bemerkung unter 2° bleibt also noch zu beweisen, daß das System $\mathscr{B}|P^{(n)}$ Ind-n-hübsch im Raume $P^{(n)}$ ist, was wegen

$$\mathrm{Gr}_{P^{(n)}}(B\cap P^{(n)})\subseteqq P^{(n)}\cup\mathrm{Gr}_P B\quad(B\in\mathscr{B})$$

auch so formuliert werden kann:

Für ein beliebiges

$$k \in \{1, 2, \ldots, n+1\}$$

und für beliebige

$$B^{(h)} \in \mathscr{B} \quad (h = 1, 2, \ldots, k)$$

mit

$$B^{(h_1)} \neq B^{(h_2)} \quad (h_1, h_2 \in \{1, 2, \ldots, k\},\ h_1 \neq h_2) \tag{1.6.10}$$

gilt

$$\operatorname{Ind} P^{(n)} \cap \bigcap_{h=1}^{k} \operatorname{Gr}_P B^{(h)} \leqq n-k. \tag{1.6.11}$$

12° Auf Grund von

$$\operatorname{Gr}_P B = \bigcup_{m=1}^{\infty} \prod_{i=1}^{\infty} B_i(m) \quad (B \in \mathscr{B}), \tag{1.6.12}$$

wobei

$$B_i(m) = \begin{cases} \operatorname{Gr}_{S_i} B_i & (i = m) \\ B_i^{-(S_i)} & (i \in \mathbf{N} \setminus \{m\}) \end{cases} \quad (i, m \in \mathbf{N}) \tag{1.6.13}$$

und

$$B_i = \pi_i[B] \quad (i \in \mathbf{N},\ B \in \mathscr{B})$$

sind (es kommt bei (1.6.12) lediglich auf den Begriff des topologischen Produkts und auf die spezielle Beschaffenheit der Mengen $B \in \mathscr{B}$ an), haben wir

$$\bigcap_{h=1}^{k} \operatorname{Gr}_P B^{(h)} = \cup \left\{ \bigcap_{h=1}^{k} \bigtimes_{i=1}^{\infty} B_i^{(h)}(m(h)) \colon \{m(h) \colon h = 1, 2, \ldots, k\} \subset \mathbf{N} \right\}.$$

Da es sich aber hier, den verschiedenen Möglichkeiten der Auswahl der endlichen Folge

$$\{m(h) \colon h = 1, 2, \ldots, k\} \subset \mathbf{N}$$

entsprechend, um die Vereinigung abzählbar vieler abgeschlossener Mengen handelt, genügt es zu beweisen (s. etwa [8] 53), daß für eine beliebige, im weiteren festgesetzte, solche endliche Folge (deren Elemente mit verschiedenen Indizes nicht unbedingt verschieden sind) und mit der Bezeichnung

$$D = \bigcap_{h=1}^{k} \bigtimes_{i=1}^{\infty} B_i^{(h)}(m(h)) \tag{1.6.14}$$

die Relation

$$\operatorname{Ind} P^{(n)} \cap D \leqq n-k \tag{1.6.15}$$

gültig ist; hieraus folgt dann auch (1.6.11).

13° Zu diesem Zweck sei zunächst bemerkt, daß jeder der Teilräume

$$D_{m(h)} = \pi_{m(h)}[D] \tag{1.6.16}$$

des entsprechenden Raumes $S_{m(h)}$ diskret ist; das folgt auf Grund der Bemerkung unter 8° daraus, daß nach (1.6.13) und (1.6.14)

$$D_{m(h)} \subseteq \operatorname{Gr}_{S_{m(h)}} B_{m(h)}^{(h)}(m(h)) \quad (h = 1, 2, \ldots, k). \tag{1.6.17}$$

Jetzt sollen zwei Fälle unterschieden werden.

14° Haben wir zunächst

$$m(h_1) \neq m(h_2) \quad (h_1, h_2 \in \{1, 2, \ldots, k\},\ h_1 \neq h_2), \tag{1.6.18}$$

so ist folgendes zu beachten:

$$P^{(n)} \cap D \subseteq D' \times D'',$$

wobei erstens

$$D' = \bigtimes_{h=1}^{k} D_{m(h)},$$

was — als topologisches Produkt endlich vieler, nach 13° diskreter Räume — ein diskreter Raum ist, der nach (1.6.17), (1.6.2) und nach (II) unter 4° nur solche Punkte enthält, deren sämtliche Koordinaten (k der Anzahl nach) „positiv-rational" (d. h. vom Typ (i, α, x) mit $x>0$, x rational) sind, und zweitens D'' die Menge jener Punkte des topologischen Produktes

$$P' = \bigtimes \{S_i\colon\ i \in \mathbf{N} \setminus \{m_1, m_2, \ldots, m_k\}\}$$

ist, unter deren von $\mathbf{0}$ verschiedenen Koordinaten sich höchstens $n-k$ rationale finden, also — da der Raum P' zu P homöomorph ist —

$$D'' \cong P^{(n-k)}$$

gilt.

Alles in allem haben wir daher

$$P^{(n)} \cap D \subseteq \bigoplus_{d \in D'} P_d^{(n-k)}$$

(genau genommen handelt es sich um topologische Einbettbarkeit) mit

$$P_d^{(n-k)} = P^{(n-k)} \quad (d \in D'),$$

womit in Falle (1.6.18) die Relation (1.6.15) — wegen der Monotonie der großen induktiven Dimension (in metrisierbaren Räumen), und da sich dieselbe bei der Bildung von topologischen Summen nicht vergrößert — bewiesen ist.

15° Gibt es aber, im Gegensatz zu (1.6.18), zwei Zahlen $h_1, h_2 \in \{1, 2, \ldots, k\}$ mit

$$h_1 \neq h_2, \quad m(h_1) = m(h_2), \tag{1.6.19}$$

so ist, wie wir jetzt zeigen werden,

$$D = \varnothing, \tag{1.6.20}$$

womit dann (1.6.15) auch im Falle (1.6.19) bewiesen sein wird.

Zum Beweis von (1.6.20) setzen wir

$$B^{(h_1)} = B(t_1;\ \{\alpha_i^{(1)}\}_{i \in N(t_1)}), \tag{1.6.21}$$

$$B^{(h_2)} = B(t_2;\ \{\alpha_i^{(2)}\}_{i \in N(t_2)})$$

(wo t_1 bzw. t_2 durch h_1 bzw. h_2 eindeutig bestimmt ist), und es folgt wieder eine Fallunterscheidung.

16° Im Falle

$$t_1 = t_2,$$

wo selbstverständlich

$$N_0(t_1) = N_0(t_2), \quad N(t_1) = N(t_2)$$

sind, haben wir

$$N(t_1) \neq \varnothing$$

(weil ja sonst, im Gegensatz zu (1.6.19) und (1.6.10), nach 7°, $B^{(h_1)} = B^{(h_2)}$ wäre), und darüber hinaus muß es sogar — vgl. (1.6.21) — ein $i_0 \in N(t_1)$ mit

$$\alpha_{i_0}^{(1)} \neq \alpha_{i_0}^{(2)} \tag{1.6.22}$$

geben (da widrigenfalls eben wieder $B^{(h_1)} = B^{(h_2)}$ sein müßte). Aus (1.6.22) folgt aber

$$(B_{i_0}^{(h_1)})^{-(S_{i_0})} \cap (B_{i_0}^{(h_2)})^{-(S_{i_0})} = \varnothing,$$

und daher, nach (1.6.12) und (1.6.13), sogar

$$(B^{(h_1)})^{-(P)} \cap (B^{(h_1)})^{-(P)} = \varnothing,$$

woran man, nach (1.6.14), die Gültigkeit von (1.6.20) ersieht.

17° Ist aber, andernfalls,

$$t_1 \neq t_2, \tag{1.6.23}$$

so gehen wir von der Inklusion

$$D_{m(h_1)}(=D_{m(h_2)}) \subseteqq \mathrm{Gr}_{S_{m(h_1)}} B_{m(h_1)}^{(h_1)} \cap \mathrm{Gr}_{S_{m(h_1)}} B_{m(h_1)}^{(h_2)} \tag{1.6.24}$$

aus, die sich als Folge von (1.6.14), (1.6.16) und (1.6.13) ergibt.

Haben wir hier, erstens,

$$m(h_1) \in N_0(t_1) \cup N_0(t_2),$$

so ist — da nach (1.6.3)

$$q(t_1) \neq q(t_2)$$

und nach der Konstruktion unter 5° auch

$$\begin{aligned} q(t_1) &\neq q_{m(h_1)}^{(l)}(t_2) \\ q(t_2) &\neq q_{m(h_1)}^{(l)}(t_1) \end{aligned} \qquad (l = 1, 2)$$

gelten —

$$\mathrm{Gr}_{S_{m(h_1)}} B_{m(h_1)}^{(h_1)} \cap \mathrm{Gr}_{S_{m(h_1)}} B_{m(h_1)}^{(h_2)} = \varnothing. \tag{1.6.25}$$

Dasselbe gilt, zweitens, auch wenn

$$m(h_1) \in N(t_1) \cap N(t_2),$$

u. zw. diesmal wegen (III) unter 4° und (1.6.23).

Zum dritten und letzten beachte man, daß wir, je nachdem

$$m(h_1) \in \mathbf{N} \setminus \big(N_0(t_1) \cup N(t_1)\big)$$

oder

$$m(h_1) \in \mathbf{N} \setminus \big(N_0(t_2) \cup N(t_2)\big)$$

ist, nach (1.6.4)

$$\left.\begin{array}{l}\mathrm{Gr}_{S_{m(h_1)}} B^{(h_1)}_{m(h_1)} \\ \text{bzw.} \\ \mathrm{Gr}_{S_{m(h_1)}} B^{(h_2)}_{m(h_1)}\end{array}\right\} = \mathrm{Gr}_{S_{m(h_1)}} S_{m(h_1)} = \varnothing$$

haben.

Es gilt also (1.6.25) in jedem der Fälle mit (1.6.19) und (1.6.23), woraus nach (1.6.13), (1.6.14), (1.6.16) und (1.6.24) wieder (1.6.20) folgt.

Damit ist der Satz vollständig bewiesen. □

§ 2. Über total-parakompakte Räume

(2.1) DEFINITION. Ein Raum heißt *total-parakompakt,* wenn jede seiner Basen ein lokal-endliches Überdeckungssystem desselben enthält. (R. M. FORD [3].)

(2.2) BEMERKUNGEN. (a) Es handelt sich hierbei um eine Verschärfung des Begriffs der Parakompaktheit und zugleich eine Abschwächung des Begriffs der Kompaktheit. (Für die Motive der Einführung dieses „Zwischenbegriffs" s. § 3.)

(b) Man sieht sofort, daß ein Raum X dann und nur dann total-parakompakt ist, wenn jedes offene Überdeckungssystem $\mathscr{U}$ desselben zu einem lokal-endlichen offenen Überdeckungssystem $\mathscr{B}^*$ verfeinert werden kann, dessen Elemente aus einer beliebig vorgeschriebenen Basis $\mathscr{B}$ von X gewählt sind.

(c) Ein total-parakompakter Raum braucht nicht einmal die T_0-Eigenschaft zu besitzen; ist er aber Hausdorffsch, so weist er schon sehr gute Trennungseigenschaften auf (nach (a) zumindest diejenigen, welche jedem parakompakten T_2-Raum zukommen, also — wie bekannt — die Normalität und überhaupt jede Abschwächung der Vollnormalität).

Für unsere hiesigen Zwecke genügt es vor Augen zu halten, daß jeder total-parakompakte T_2-Raum regulär ist.

(d) Jeder abgeschlossene Teilraum eines total-parakompakten Raumes ist total-parakompakt [3].

(2.3) SATZ. *Für einen total-parakompakten Raum X mit* $\operatorname{ind} X=0$ *gilt auch* $\operatorname{Ind} X=0$.

BEWEIS. Sind

$$\varnothing \neq F \subseteqq U \subseteqq X,$$

F abgeschlossen, U offen, so kann aus einer beliebigen Basis $\mathscr{B}$ des Raumes X mit

$$\mathrm{Gr}\, B = \varnothing \quad (B \in \mathscr{B})$$

das Teilsystem

$$\mathscr{B}^* = \mathscr{B}^*_1 \cup \mathscr{B}^*_2 \subseteqq \mathscr{B}$$

mit

$$\mathscr{B}^*_1 = \{B \in \mathscr{B}\colon B \cap F \neq \varnothing,\ B \subseteqq U\},$$

$$\mathscr{B}^*_2 = \{B \in \mathscr{B}\colon B \cap F = \varnothing\}$$

ausgewählt werden, das ebenfalls eine Basis von X ist. Für eine lokal-endliche Überdeckung $\mathscr{B}^{**} \subseteqq \mathscr{B}^*$ von X und für die Menge

$$G = \bigcup \{B\colon B \in \mathscr{B}^{**} \cap \mathscr{B}_1^*\}$$

gelten dann

$$F \subseteqq G \subseteqq U$$

und

$$\operatorname{Gr} G \subseteqq \bigcup \{\operatorname{Gr} B\colon B \in \mathscr{B}^{**} \cap \mathscr{B}_1^*\} = \varnothing,$$

womit der Satz bewiesen ist. □

(2.4) BEMERKUNGEN. (a) Die neuere Entwicklung der Untersuchungen über das Verhältnis der beiden klassischen induktiven Dimensionen zueinander ist vornehmlich durch den Begriff der Total-Parakompaktheit (bzw. dessen verschiedene Abschwächungen und Varianten) geprägt. Dieses Forschungsgebiet wurde 1963 von R. M. FORD eröffnet, u. zw. mit dem folgenden Satz in [3]:

Für jeden total-parakompakten metrisierbaren Raum X mit $\operatorname{ind} X < \infty$ *gilt* $\operatorname{Ind} X = \operatorname{ind} X$.

Seither ist schon eine ganze Reihe von Ergebnissen dieser Art erreicht worden (zu Orientierung s. etwa [4]).

(b) Wir möchten nun — durch Einbeziehung des Begriffs der ind-hübschen Basis — eine neue Richtung in diesem Gebiet einschlagen. Den heuristischen Ausgangspunkt bildet der Satz (2.3); man fragt nämlich nach zusätzlichen Bedingungen, unter denen die Äquivalenz der beiden induktiven Dimensionen total-parakompakter Räume X auch bei $\operatorname{ind} X > 0$ gesichert ist.

Nach dem Satz von Ford ist die Metrisierbarkeit von X eine solche Bedingung. Unser Ergebnis unter (2.7) wird eine andersgeartete Antwort auf die Frage geben. Wegweisend ist die folgende — nach (0.4),(a) äquivalente — Fassung des Satzes (2.3):

Läßt ein total parakompakter Raum X eine ind-0-*hübsche Basis, zu so gilt* $\operatorname{Ind} X = 0$.

Der besseren Übersichtlichkeit wegen spalten wir vom Beweis des Satzes (2.7) einen Hilfssatz ab.

(2.5) HILFSSATZ. *Es seien n eine nichtnegative ganze Zahl, $\mathscr{B}$ eine* ind-n-*hübsche Basis eines*

(a) *regulären*

bzw.

(b) T_1 —

Raumes X (mit notwendigerweise $\operatorname{ind} X \leqq n$*), $\mathscr{B}^* \subseteqq \mathscr{B}$ so, daß das System $\{B^-\colon B \in \mathscr{B}^*\}$ punktal-endlich in X ist, $k \in \{1, 2, \ldots, n+1\}$, $\mathscr{H}_k$ ein nichtleeres System von paarweise verschiedenen Mengen, deren jede als Durchschnitt der Begrenzungen von paarweise verschiedenen Mengen aus $\mathscr{B}^*$ darstellbar ist; endlich sei*

$$H_k = \bigcup \mathscr{H}_k.$$

Dann gilt

$$\operatorname{ind} H_k \leqq n - k.$$

BEWEIS. 1° Für $k = n+1$, n beliebig, ist die Aussage des Satzes eine Trivialität, da dann wegen der Hübschheit von $\mathscr{B}$ nur $\mathscr{H}_k = \{\varnothing\}$ sein kann; hier sind beide Voraussetzungen (a) und (b) belanglos.

Damit ist aber auch der Fall $n=0$ erledigt, und wir können im weiteren

$$n \geqq 1 \tag{2.5.1}$$

voraussetzen.

Der Beweis erfolgt nun mittels eines endlichen Induktionsverfahrens „von oben nach unten", d. h. als Ausgangspunkt nehmen wir den soeben erledigten Fall $k=n+1$.

2° Nehmen wir an, daß die Aussage des Satzes für irgendein k mit

$$1 < k \leqq n+1 \tag{2.5.2}$$

gültig ist; wir werden zeigen, daß sie dann auch für $k-1$ zutrifft.

Es seien also $\mathscr{B}^*$, $\mathscr{H}_{k-1}$ und H_{k-1} im Sinne der Voraussetzung des Satzes, sonst beliebig, vorgeschrieben. Selbstverständlich ist dann auch $\mathscr{H}_{k-1}$ ein punktalendliches System.

Es genügt zu zeigen, daß für ein beliebiges $x\in X$ und eine beliebige offene Menge $V\subseteqq X$ mit

$$x\in H_{k-1}\cap V \tag{2.5.3}$$

eine offene Menge $W\subseteqq X$ mit

$$x\in W \subseteqq V \tag{2.5.4}$$

(was mehr ist als wir brauchen, $H_{k-1}\cap W\subseteqq H_{k-1}\cap V$ anstelle von $W\subseteqq V$ würde ausreichen) und

$$\operatorname{ind} H_{k-1}\cap \operatorname{Gr} W \leqq n-k \tag{2.5.5}$$

existiert (woraus sich nämlich $\operatorname{ind} H_{k-1}\leqq n-k+1=n-(k-1)$ ergibt, was wir eben zu beweisen haben).

3° Es seien also x und V, wie unter 2°, festgesetzt. Weiter sei

$$\mathscr{B}^*(x) = \{B\in\mathscr{B}^*:\ x\in B^-\}.$$

Dann ist, da das System $\mathscr{B}^*(x)$ wegen der Voraussetzung über $\mathscr{B}^*$ nur endlich sein kann, die Menge

$$B^*(x) = \cap\mathscr{B}^*(x) \tag{2.5.6}$$

eine offene Umgebung von x.

4° Gibt es ein $W\in\mathscr{B}$ mit der Eigenschaft (2.5.4) und mit

$$W\in\mathscr{B}\setminus\mathscr{B}^*(x) \tag{2.5.7$_1$}$$

oder

$$\operatorname{Gr} W = \varnothing, \tag{2.5.7$_2$}$$

so ist mit diesem W wegen

$$H_{k-1}\cap\operatorname{Gr} W = \cup\left\{\left(\bigcap_{i=1}^{k-1}\operatorname{Gr} B_i\right)\cap\operatorname{Gr} W:\right. \tag{2.5.8}$$

$$B_i\in\mathscr{B}^*(x)\quad (i\in\{1,2,\ldots,k-1\}),\quad B_i\neq B_j$$

$$\left.(i,j\in\{1,2,\ldots,k-1\},\ i\neq j)\right\}$$

(man beachte (2.5.3), (2.5.1) und die erste Ungleichung unter (2.5.2)) auch (2.5.5) erfüllt.

Steht nämlich auf der rechten Seite dieser Inklusion die leere Menge, so gilt (2.5.5) wegen der zweiten Ungleichung unter (2.5.2).

Ist aber die rechtsstehende Menge unter (2.5.8) nicht die leere Menge, so kann angenommen werden, daß es sich um die Vereinigung von lauter nichtleeren Mengen handelt und demnach auch nur endlich viele Summanden vorhanden sind. Dann ist die Menge $H_{k-1} \cap \mathrm{Gr}\, W$ darstellbar als die endliche Vereinigung von Mengen, deren jede der Durchschnitt der Begrenzungen von k paarweise verschiedenen Mengen aus $\mathscr{B}$ ist. Auf diesen Fall angewendet ergibt die Induktionsvoraussetzung unter 2° die Gültigkeit von (2.5.5), und die Aussage des Satzes gilt demnach auch für $k-1$.

5° Es bleibt also die Existenz eines W mit den Eigenschaften (2.5.4) und mit $(2.5.7_1)$ oder $(2.5.7_2)$ nachzuweisen.

Nimmt man nun aber ein $W \in \mathscr{B}$ mit

$$x \in W \subseteqq W^- \subseteqq B^*(x) \cap V \tag{2.5.9a}$$

im Falle (a) bzw.

$$W = \{x\} \qquad (B^*(x) \cap V = \{x\}), \tag{2.5.9b$_1$}$$

$$x \in W \subset B^*(x) \cap V \quad (B^*(x) \cap V \neq \{x\}) \tag{2.5.9b$_2$}$$

im Falle (b) (für (2.5.9b$_2$) ist etwa $W = (B^*(x) \cap V) \setminus \{y\}$, mit $y \in (B^*(x) \cap V) \setminus \{x\}$, sonst beliebig, geeignet), so ist damit (2.5.4) schon gesichert; gilt weiter für dieses W (2.5.7) nicht, so kann es sich nur um die Variante unter (2.5.9a) oder jene unter (2.5.9b$_2$) handeln, und in beiden Fällen gilt dann $(2.5.7_1)$ wegen $W \subset W^-$ und (2.5.6).

Der Hilfssatz ist somit bewiesen. □

(2.6) SATZ. *Existiert für einen total-parakompakten T_2-Raum X eine* ind-n-*hübsche Basis, so ist*

$$\operatorname{Ind} X \leqq n \tag{2.6.1}$$

$(n = 0, 1, 2, \ldots)$.

BEWEIS. 1° Für $n=0$ gilt der Satz nach (2.4),(b).

2° Nehmen wir an, daß die Aussage des Satzes für ein $n-1$, $n \in \{1, 2, \ldots\}$, gültig ist, und es sei $\mathscr{B}$ eine ind-n-hübsche Basis eines Raumes X der besagten Art.

3° Sind F eine abgeschlossene und G eine offene Menge in X mit

$$\emptyset \neq F \subseteqq G,$$

sonst beliebig, so nehmen wir je eine offene Umgebung $U(x)$ von $x \in X$ mit

$$U(x) \subseteqq \begin{cases} G & (x \in F), \\ X \setminus F & (x \in X \setminus F). \end{cases}$$

Die offene Überdeckung $\mathscr{U} = \{U(x) : x \in X\}$ des Raumes X kann dann zu einem lokal-endlichen und ind-n-hübschen offenen Überdeckungssystem $\mathscr{B}^* (\subseteqq \mathscr{B})$ verfeinert werden (man wendet (2.2),(b) auf die Überdeckung $\mathscr{U}$ und die Basis $\mathscr{B}$ des Raumes an), und es seien

$$\mathscr{B}_1^* = \{B \in \mathscr{B}^* : B \cap F \neq \emptyset\},$$

$$B_1^* = \bigcup \mathscr{B}_1^*.$$

Dann haben wir

$$F \subseteq B_1^* \subseteq G,$$

und für den Teilraum

$$X^* = \operatorname{Gr} B_1^*$$

von X gilt

(2.6.2) $$X^* \subseteq \cup \{\operatorname{Gr} B\colon\ B \in \mathscr{B}_1^*\}.$$

4° Wir zeigen jetzt, daß X^* der Induktionsannahme unter 2° entspricht.

a) Dieser (T_2-) Raum ist nach (2.2),(d) total-parakompakt.

b) Die Spur $\mathscr{B}_2^* | X^*$ auf X^* des Systems

$$\mathscr{B}_2^* = \mathscr{B} \setminus \mathscr{B}_1^*$$

ist eine Basis der Topologie von X^*.

Tatsächlich: sind $x \in X^*$ und U eine Umgebung von x in X, so gibt es ein $B \in \mathscr{B}$ mit $x \in B \subseteq U$, also $x \in B \cap X^* \subseteq U \cap X^*$, und es ist bestimmt $B \in \mathscr{B}_2^*$, weil ja im Falle $B \in \mathscr{B}_1^*$ wegen $x \in B_1^*$ nicht $x \in X^*$ sein könnte.

c) Diese Basis von X^* ist ind-$(n-1)$-hübsch.

Es seien nämlich $k \in \{1, 2, \ldots, n\}$ und $B_i \in \mathscr{B}_2^*$ $(i = 1, 2, \ldots, k)$, so daß die Mengen

$$X^* \cap B_i \quad (i = 1, 2, \ldots, k)$$

paarweise verschieden sind, was dann natürlich auch für die B_i $(i = 1, 2, \ldots, k)$ zutrifft. Für die Menge

$$D = \bigcap_{i=1}^{k} (X^* \cap \operatorname{Gr} B_i) = X^* \cap \bigcap_{i=1}^{k} \operatorname{Gr} B_i$$

gilt nach (2.6.2)

$$D \subseteq \cup \left\{(\operatorname{Gr} B) \cap \bigcap_{i=1}^{k} \operatorname{Gr} B_i\colon\ B \in \mathscr{B}_1^*\right\}.$$

D ist also Teilmenge der Vereinigung von Mengen, deren jede als Durchschnitt der Begrenzungen (in X) von $k+1$ paarweise verschiedenen Mengen aus dem lokal-endlichen Teil $\mathscr{B}^*$ der ind-n-hübschen Basis $\mathscr{B}$ von X darstellbar ist. Nach dem Hilfssatz (2.5) haben wir also

$$\operatorname{ind} D \leqq n-(k+1) = (n-1)-k,$$

was eben die ind-$(n-1)$-Hübschheit des Systems $\mathscr{B}_2^* | X^*$ bedeutet.

5° Durch Anwendung auf X^* der Induktionsannahme ergibt sich nun

$$\operatorname{Ind} X^* \leqq n-1,$$

und somit ist $\operatorname{Ind} X \leqq n$, w. z. b. w. □

Nach diesen Vorbereitungen erhalten wir nun das unter (2.4) angedeutete Ergebnis:

(2.7) SATZ. *Existiert für einen total-parakompakten T_2-Raum X mit* $\operatorname{ind} X < \infty$ *eine* ind-*hübsche Basis, so ist*

$$\operatorname{Ind} X = \operatorname{ind} X.$$

Das ist der Spezialfall $n = \operatorname{ind} X$ des vorigen Satzes. □

(2.8) BEMERKUNG. Die Grundidee des Satzes (2.7) entsprang der Beweismethode eines unserer Sätze über das Verhältnis der beiden klassischen induktiven Dimensionen zu einem neuen Dimensionsbegriff, der sog. Richtungsdimension $\operatorname{Dim} X$ eines topologischen Raumes X (s. etwa den Bericht [1] über deren Begriff und Theorie):

Für jeden bzw. jeden total-parakompakten Raum X endlicher Richtungsdimension gilt $\operatorname{ind} X \leqq \operatorname{Dim} X$ *bzw. obendrein* $\operatorname{Ind} X \leqq \operatorname{Dim} X$. ([2] 444.)

Bei dem letztgenannten Satz (wo X nicht einmal die T_0-Eigenschaft zu besitzen braucht) ist der Beweis schwieriger als bei dem hiesigen Satz (2.4), obwohl in beiden Fällen eigentlich nur solche Eigenschaften der in Frage stehenden Dimensionsbegriffe ausgenutzt werden, die im großen und ganzen gemeinsam sind. (Beispiele: 1° Die Richtungsdimension ist ohne jede Beschränkung monoton, gleichwie die kleine induktive Dimension; wir brauchen aber die Monotonie nur auf abgeschlossene Teilräume bezogen, und diese Eigenschaft kommt auch der großen induktiven Dimension zu. 2° Jeder Punkt eines beliebigen Raumes der Richtungsdimension n, n eine natürliche Zahl, hat eine Umgebungsbasis bestehend aus offenen Mengen, deren Begrenzungen als Vereinigung von je endlich vielen abgeschlossenen Mengen darstellbar sind, so daß die Richtungsdimension dieser letzteren Mengen höchstens $n-1$ beträgt.)

§ 3. Ausblick

Anhand der Sätze unter (1.6), (2.4) und (2.5),(a) erhebt sich eine natürliche Frage: welche Räume endlicher kleiner induktiver Dimension haben eine ind-hübsche Basis?

Da diese Eigenschaft, nach (1.6) und (2.5),(a), jedem total-parakompakten metrisierbaren Raum X mit $\operatorname{ind} X < \infty$ zukommt, konkretisiert man die Fragestellung am besten so:

(3.1) PROBLEM. *Können im Bereich der Räume endlicher kleiner induktiver Dimension Klassen von metrisierbaren Räumen angegeben werden, die die Klasse der total-parakompakten metrisierbaren Räume als echten Teil enthalten, so daß jeder Raum aus einer solchen Klasse eine* ind-*hübsche Basis besitzt*?

Ergebnisse in dieser Richtung stehen noch aus. Solche Sätze würden aber, etwa mit (2.4) kombiniert, auch den angeführten Satz von Ford ergeben und überhaupt neuartige Einblicke in den Sachverhalt gewähren.

Zum Abschluß möchte ich meinem Kollegen Herrn J. Deák (Budapest) für seine wertvolle Kritik und für die Mitteilung der unter (1.6),5° beschriebenen Konstruktion meinen Dank aussprechen.

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ON A TWO-DIMENSIONAL RANDOM SPACE FILLING PROBLEM

by

ILONA PALÁSTI

In our paper [1] we have discussed the following problem: let us place, at random (with uniform distribution), disjoint domains which are congruent and parallel with a given domain D of unit area into a rectangle T_{xy} having sides x and y. The question is, how large a part of the rectangle will be filled in, on an average, with domains placed in such a way.

Let us denote by $M(x, y)$ the mean value of the number of domains placed in the manner as described in the sequel. We have proved that the limit

$$\lim_{\substack{x\to\infty \\ y\to\infty}} \frac{M(x, y)}{xy} = \alpha(D) \tag{1}$$

does exist. (Here α is a constant depending on D).

This was proved by using a two-dimensional generalization of a theorem of D. HYERS related to some functional inequalities (cf. his paper [2]) and by using the following

HYPOTHESIS. *There exists a constant* $A>0$ *for which the following inequalities hold:*

$$|M(x_1+x_2, y)-M(x_1, y)-M(x_2, y)| \leqq Ay \tag{2_1}$$

where $x_1>0, x_2>0, y>0$, *and*

$$|M(x, y_1+y_2)-M(x, y_1)-M(x, y_2)| \leqq Ax \tag{2_2}$$

for $y_1>0, y_2>0, x>0$.

Let us consider firstly the illustrative meaning of the hypothesis (2). We can immagine that the rectangle T_{xy} has already been filled by unit squares placed randomly in the above mentioned way. Then the hypothesis (2) means that if the rectangle T_{xy} is divided into two parts by a straight line (or by a band having unit width parallel to the axis X), then at most x unit squares would be intersected by this line (or at most $2x$ unit squares will be partly covered by the unit wide band). Naturally the same is true for the straight lines parallel to the axis, Y, that is at most y unit squares can be touched by these lines.

We have, moreover, conjectured that, if D is a unit square S, the sides of which are parallel to the axes X and Y resp., then $\alpha(S)=c^2$, where the constant c is equal to the constant given by A. RÉNYI in paper [3] as the solution of the analogous one-dimensional problem. This one-dimensional case of the problem is the following: let us place at random with uniform distribution disjoint unit intervals in the interval $(0, x)$ until this is possible. The constant c (meaning the

average filled portion of the interval $(0, x)$) is given by the expression

$$c = 2\int_0^\infty (1-e^{-t}) \exp\left(-2\int_0^t \frac{1-e^{-u}}{u}\,du\right) dt \sim 0.748. \tag{3}$$

Thus the numerical value of c^2 is approximately equal to 0.56.

This conjecture seems to be confirmed by the computer simulations carried out by H. SOLOMON and reported in his paper [4]. Later B. E. BLAISDELL and H. SOLOMON investigated the tenability of our conjecture in [5] by using numerical methods. Recently YOSHIAKI AKEDA and MOTOO HORI devised a computer program by which a Monte Carlo simulation of this problem could be made for relatively large T_{xy} in [6]. As the size of T_{xy} increases, then the mean packing density approaches $c^2 \sim 0.5589$, and our conjecture has thus been confirmed numerically.

In his work [7], P. A. P. MORAN gives a comprehensive study of a wide scope of this subject.

One-dimensional generalizations of Rényi's original problem are treated in the papers [8], [9], [10], [11], [12], [13] and also in the others of our References.

Now we shall prove a special case of our conjecture.

THEOREM. *Let us denote by $M(x, y)$ the mean value of the number of nonoverlapping unit squares, the sides of which are parallel to the axes X and Y resp., placed at random into a rectangle T_{xy} given by the coordinates* $(0, 0)$, $(x, 0)$, $(0, y)$, (x, y). *Let us prolong the sides of the unit square placed at first with uniform distribution, and let us fill the unit wide bands thus obtained with disjoint unit squares at random, with uniform distribution. The process continues until the remaining space is inadequate for a unit square. Then, supposing that the hypothesis* (2) *is true,*

$$M(x, y) \sim M(x)\,M(y), \tag{4}$$

where $M(x)$ is the mean value of the number of disjoint unit intervals placed at random with uniform distribution in the interval $(0, x)$.

REMARK. This procedure of random packing may also be viewed a simplified mathematical realization of BERNAL's concept [14] of the liquid state as "homogeneous, coherent and essentially irregular assemblages of molecules containing no crystalline regions or holes large enough to admit another molecule." In this latter paper he has again expressed the derivability for a mathematically treatable and physically acceptable model for the theory of liquids.

At this random space-filling problems the question is the restriction of random placing and the "rate of disorderity".

The centres of unit squares are the points of a lattice in a plane at regular or orderly placing, thus their centres are collinear in four directions (parallel to the axes X and Y and in two directions of diagonals), group by group.

The measurement of disorderity may be the discrepancy from regularity in a random placing procedure. E.g. irregular packing constructed by shaking together equal steal balls. These arrays are generally fixed by means of wax and the sphere center coordinates measured by special machines (see J. D. BERNAL and J. MASON [17] and G. D. SCOTT [18]). In this procedure a large number of collineation (or "chains of molecules") centres arranged in straight lines are observed (J. L. FINNEY [19]).

Consider now the order-disorder transitions.

A random arrangement of unit squares may be formed by placing unit intervals on both sides, x and y, of the rectangle T_{xy} at random, according to Rényi's procedure. We draw perpendiculars at both endpoints of each interval and place unit squares only in the common parts of unit wide bands obtained. This arrangement is irregular, differing from the orderly placing, because the unit squares are in random distance from each other, but their centres are collinear to bands in two directions (horizontaly and vertically). Clearly the number of unit squares is assimptotically equal to c^2xy with this placing method.

Another random space-filling procedure may be as follows: we draw parallel straight lines to the axis X only at the endpoints of unit intervals placed randomly on the y side of the rectangle T_{xy}. We fill these stripes with unit squares randomly in Rènyi's way. Thus the centers of all unit squares lie in straight lines parallel to the axis X, namely they are collinear, band by band, in one direction. The distances of equal squares arranged in lines and that of the unit bands are different, so we have disordered squares. It is easy to see that the number of unit squares which may be thus placed also tends to c^2xy. Naturally the same is true for randomly placed unit squares in the parallel stripes to the axis Y.

A third placing procedure is detailed in the theorem. Namely we place a unit square (basis unit square) randomly with uniform distribution into the inner side of T_{xy} and we prolong the sides of this unit square. We continue the placing of unit squares in a random way in the bands obtained. Then the centers of unit squares placed in the first basis step are collinear in both directions. The new bands obtained by prolongation of the sides of the secondly placed basis unit square meet the stripes filled before. The common part of bands are saturated, but there is no center of unit square placed before in the midle point of crossing almost surely. Accordingly the collinearity is broken in both directions with probability 1 at the second step of this filling procedure. The probability that any number of the basis pair chosen from uniformly distributed random numbers be the same as the number of pairs chosen before (in the order of choosing too, this probability) is nearly zero. Moreover, the possiblity of overlaping unit squares has been excluded. Similarly two-two breaks arise in collinarity in both directions almost surely at the third basis step. The numbers of breaks is $2(k-1)$ a.s. at the k-th basis step (thus the total number of them is $\leqq k(k-1)$), etc. Thus we have shorter and shorter "chains". Near to the end of this placing procedure there is a large number of unit squares through the center of which one may draw parallel lines to the sides of T_{xy} being no centre of another unit square so ("free molecules"). It is not trivial that the number of unit squares so placed is asymptotically equal to c^2xy. We will detail this case.

THE PROOF OF THE THEOREM. It is evident that $M(x, y)$ is a symmetric function of x and y, namely $M(x, y)=M(y, x)$.

Firstly we shall show that assuming the hypothesis (2), the function $M(x, y)$ satisfies the following integral question

$$M(x+1, y+1) = \frac{4}{xy}\int_0^x \int_0^y M(t, u)\, dt\, du + \tag{5}$$

$$+\frac{2}{x}\int_0^x M(t, 1)\, dt + \frac{2}{y}\int_0^y M(1, u)\, du + 1, \quad x > 0, y > 0.$$

Indeed if a unit square has already been placed randomly in the rectangle T_{xy} (by this we mean that the centre of unit square is placed according to the uniform law in the rectangle given by $\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, y-\frac{1}{2}\right), \left(x-\frac{1}{2}, \frac{1}{2}\right), \left(x-\frac{1}{2}, y-\frac{1}{2}\right)$) and let this unit square be given by its coordinates (t, u), $(t+1, u)$, $(t+1, u+1)$, $(t, u+1)$ (where $0 \leqq t \leqq x$; $0 \leqq u \leqq y$). Then, by making use of the additive property of the mean value, on an average $M(t, u)+M(x-t, u)+M(x-t, y-u)+M(t, y-u)$ further unit squares should be placed into the rectangle, and to this we also add the expression $M(t, 1)+M(x-t, 1)+M(t, u)+M(1, y-u)$ i.e. the average number of unit squares placed in the unit wide bands. Since t and u are uniformly distributed in the intervals $(0, x)$ and $(0, y)$ resp., we obtain

(6)

$$M(x+1, y+1)=\frac{1}{xy}\int_y^x \int_0^y [M(t, u)+M(t, y-u)+M(x-t, y-u)+M(x-t, u)]\, dt\, du+$$
$$+\frac{1}{xy}\int_0^x \int_0^y [M(t, 1)+M(x-t, 1)]\, dt\, du+$$
$$+\frac{1}{xy}\int_0^x \int_0^y [M(1, u)+M(1, y-u)]\, dt\, du+1.$$

Substituting the values $x-t=v$ and $y-u=z$ into the expression (6), then putting again $v=x$ and $z=y$, then integrating the second term with respect to y and the third one of (6) with respect to x we obtain the integral equation (5).

For the sake of making the solution of the integral equation (5) easier, let us transform it now into a differential equation. Multiplying both sides of (5) by xy and differentiating it with respect to x and then with respect to y, we obtain the following second order linear difference differential equation:

(7)
$$xy\frac{\partial^2 M(x+y, y+1)}{\partial x\, \partial y}+x\frac{\partial M(x+1, y+1)}{\partial x}+y\frac{\partial M(x+1, y+1)}{\partial y}+$$
$$+M(x+1, y+1)=4M(x, y)+2M(x, 1)+2M(1, y)+1.$$

Now we shall deal with the solution of differential equation (7). It follows from the result detailed in the introduction that a unique solution exists. Moreover, by the limit (1), we can also get the form of the solution. It follows not only from (1), but it is in the nature of the problem, that the function $M(x, y)=xy$ is a particular solution of (7). That is to say there exists an arrangement, namely the densest (systematical) packing of the unit squares in which case the whole rectangle can be filled. As it may be seen by a simple substitution, the function $M(x, y)=xy$ is indeed a particular solution of the differential equation (7). But then there exist a constant such that xy multiplied by it is also a solution. The question is, what kind of a constant is it, and under what conditions would it satisfy the equation (7). (It follows also from the nature of our problem, and from the placing method, that this constant must be less than one.)

By our conjecture and in the light of [4], [5], [6], let us try to find the general solution of (7) in product form, under those boundary conditions which can be

obtained from the hypothesis (2). That is let the value of $M(x, y)$ depend only on x in the band having width 1 and length x and which is parallel to the axis X. It depends on y similarly in the band being parallel with the axis Y and having unit width adn length y. Let $M(x, y)=1$ be the initial condition for $1 \leqq x \leqq 2$ and $1 \leqq y \leqq 2$.

Remembering that $M(x, y)$ is a symmetric function of x and y, let

$$M(x, y) = f(x)f(y). \tag{8}$$

Hence it follows from the initial condition that $M(1, 1)=[f(1)]^2$, that is $f(1)=1$, because $f(x)$ is evidently positive.

Now we can formulate the partial derivates of the function (8) as follows:

$$\frac{\partial M(x+1, y+1)}{\partial x} = f'(x+1)f(y+1),$$

$$\frac{\partial M(x+1, y+1)}{\partial y} = f(x+1)f'(y+1)$$

and

$$\frac{\partial^2 M(x+1, y+1)}{\partial x \partial y} = f'(x+1)f'(y+1).$$

Putting these into the differential equation (7) we obtain

$$xyf'(x+1)f'(y+1)+xf'(x+1)f(y+1)+yf(x+1)f'(y+1)+f(x+1)f(y+1) = \tag{9}$$
$$= 4f(x)f(y)+2f(x)f(1)+2f(1)f(y)+1.$$

Factoring out the expression $xf'(x+1)$ from the first and the second terms, and the function $f(x+1)$ from the third and fourth of the left hand side of (9), and taking into account that $f(1)=1$, we obtain

$$xf'(x+1)[yf'(x+1)+f(y+1)]+f(x+1)[yf'(y+1)+f(y+1)] = \tag{10}$$
$$= 4f(x)f(y)+2f(x)+2f(y)+1,$$

and thus

$$[xf'(x+1)+f(x+1)][yf'(y+1)+f(y+1)] = \tag{11}$$
$$= [2f(x)+1][2f(y)+1].$$

The latter may be written in the form

$$\frac{xf'(x+1)+f(x+1)}{2f(x)+1} = \frac{yf'(y+1)+f(y+1)}{2f(y)+1} = 1. \tag{12}$$

Now the differential equation of the problem solved by A. RÉNYI in [3] has the form

$$xM'(x+1)+M(x+1) = 2M(x)+1, \quad x > 0, \tag{13}$$

the initial condition of which is $M(x)=1$ for $1<x \leqq 2$, and its solution is

$$M(x) = cx-(1-c)+O\left(\frac{1}{x^2}\right), \tag{14}$$

where the value of the constant c is given by (3). Therefore the solution of our differential equation (7) may be given in the form

$$M(x, y) = c^2xy + o(xy). \tag{15}$$

For the sake of completeness we remark that

$$M(x, y) = c^2xy + \frac{c^2-1}{7} \tag{16}$$

indeed satisfies the differential equation (7).

So we have proved our statement.

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A NEW CHARACTERIZATION OF THE CLASS OF SUBSPACES OF A EUCLIDEAN SPACE

by

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Let M be a separable metric space. It is a consequence of the MENGER—NÖBELING embedding theorem that M is homeomorphic to a subspace of *some* Euclidean space iff $\dim M < \infty$. But how can the subsets of a *fixed* Euclidean space be characterized? E. DEÁK [2] has answered this question:

A separable metric space M is homeomorphic to a subspace of the Euclidean n-space $\mathbf{R}^n$ $(n=0, 1, \ldots)$ *iff* $\operatorname{Dim} M \leqq n$. (See Definition (0.3) below.)

In this note we offer another solution to this problem.

§ 0. Preliminaries

Here we list some definitions and theorems due to E. DEÁK. The reader interested in directional structures and directional dimension is referred to E. DEÁK's paper [3].

TERMINOLOGY AND NOTATIONS. "Space" means a topological space. An "order" $\prec$ is a total (linear) order in the sense $\lneqq$. The union of an empty collection is the empty set, the intersection of an empty collection of subsets of a fundamental set is the whole set. If x is a point in a Euclidean space, $x^{(i)}$ denotes its ith co-ordinate in a rectangular co-ordinate system.

(0.1) DEFINITION. A direction on a space X is an ordered family $\mathscr{R}=(\mathscr{R}, \prec)$ of pairs (G, F) with G open and F closed, satisfying the following conditions:

(i) $G \subset F$ for each $(G, F) \in \mathscr{R}$;

(ii) $(G_1, F_1) \prec (G_2, F_2)$ implies $F_1 \subset G_2$;

(iii) the family $\mathscr{F}(\mathscr{R}) = \{F: \exists G, (G, F) \in \mathscr{R}\}$ contains the intersection of an arbitrary subfamily of it;

(iv) the family $\mathscr{G}(\mathscr{R}) = \{G: \exists F, (G, F) \in \mathscr{R}\}$ contains the union of an arbitrary subfamily of it.

The elements of $\mathscr{G}(\mathscr{R})$ are called lower open $\mathscr{R}$-halfspaces. The complements of the elements of $\mathscr{F}(\mathscr{R})$ are the upper open $\mathscr{R}$-halfspaces. The lower and the upper open $\mathscr{R}$-halfspaces together are called open $\mathscr{R}$-halfspaces.*

* The notations $\mathscr{F}(\mathscr{R})$ and $\mathscr{G}(\mathscr{R})$ will be used even if $\mathscr{R}$ is only a so-called pseudo-direction satisfying only (i) and (ii).

(0.2) DEFINITION. A *directional structure* on a space X is a system $\mathfrak{R}$ of directions on X. $\mathfrak{R}$ is *compatible* if the family of all open $\mathfrak{R}$-halfspaces (i.e. open $\mathscr{R}$-halfspaces for $\mathscr{R}\in\mathfrak{R}$) is a subbase of X.

(0.3) DEFINITION. The *directional dimension* of a space X, denoted by $\operatorname{Dim} X$, is the minimum of the cardinalities of the compatible directional structures on X.

Note that $\operatorname{Dim} X=0$ iff X is indiscrete, since the empty set is a subbase of an indiscrete space (see our introductory remark on the terminology).

(0.4) DEFINITION. A direction $\mathscr{R}$ on a space X is *orderly* if $\cup\{F-G:(G, F)\in\mathscr{R}\}=X$. In this case, the sets $G_x(\mathscr{R})$ and $F_x(\mathscr{R})$ for a point $x\in X$ are defined by $(G_x(\mathscr{R}), F_x(\mathscr{R}))\in\mathscr{R}$, $x\in F_x(\mathscr{R})-G_x(\mathscr{R})$. A directional structure is orderly if each of its directions is orderly.

(0.5) REMARK. E. DEÁK's original definition of the direction includes the condition

(v) $(\emptyset, \emptyset)\in\mathscr{R}$, $(X, X)\in\mathscr{R}$,

which is omitted here for the sake of convenience. This, however, does not change Dim and has no influence on the validity of the theorems below.

(0.6) THEOREM. *Each direction on a perfectly normal space X is a subfamily of an orderly direction on X.*

(0.7) THEOREM. *A T_1-space X is sub-orderable* (i.e. embeddable into an order topological space) *iff* $\operatorname{Dim} X\leqq 1$.

E. DEÁK proves his embedding theorem stated in the introduction by proving:

(0.8) THEOREM. *Let M be a separable metric space with an orderly compatible directional structure $\mathfrak{R}=\{\mathscr{R}_i: 1\leqq i\leqq n\}$, $n=1, 2, \ldots$. Then there exist order-preserving functions $\varphi_i:\mathscr{R}_i\to\mathbf{R}$ $(1\leqq i\leqq n)$ such that the mapping $f: M\to\mathbf{R}^n$ defined by*

$$(f(x))^{(i)}=\varphi_i(G_x(\mathscr{R}_i),\ F_x(\mathscr{R}_i)) \quad (1\leqq i\leqq n,\ x\in M)$$

is a topological embedding.

§ 1. Halfdirections

In this paragraph we define the notion of halfdirections, introduce a new dimension function and prove a simple lemma.

(1.1) DEFINITION. An ordered family $N=(N, <)$ of open subsets of a space X is called a *nest* or a *regular nest* if $N_1<N_2$ implies $N_1\subset N_2$ or $\overline{N}_1\subset N_2$, respectively.

We shall later need the following solution of the sub-orderability problem (cf. Theorem (0.7)):

(1.2) THEOREM (VAN DALEN and WATTEL [1]). *A T_1-space is sub-orderable iff it has a subbase, which is the union of two nests.*

(1.3) DEFINITION. A halfdirection $\mathscr{H}$ on a space X is a regular nest on X containing the union of an arbitrary subfamily of it.

Note that the set of the lower (upper) open halfspaces of a direction is a halfdirection.

(1.4) DEFINITION. A *halfdirectional structure* on a space X is a non-void system $\mathfrak{H}$ of halfdirections on X. $\mathfrak{H}$ is *compatible* if $\cup\mathfrak{H}$ is a subbase of X.

(1.5) DEFINITION. The *halfdirectional dimension* of a space X, denoted by DIM X, is defined by DIM $X+1=$ the minimum of the cardinalities of the compatible halfdirectional structures on X.

Note that a) each space has a DIM; b) DIM is monotone; c) for an S_1-space X, DIM $X=0$ iff X is indiscrete, i.e. iff Dim $X=0$.

(1.6) LEMMA. *If $\mathscr{H}$ is a halfdirection on a space X, there exists a direction $\mathscr{R}$ on X with $\mathscr{H}=\mathscr{G}(\mathscr{R})$.*

PROOF. For each $H\in\mathscr{H}$ we define a closed set $F(H)$:

$$A(H)=\bigcap\{H^*\colon H^*>H\},$$

$$F(H)=\begin{cases}A(H) & \text{if it is a closed set,}\\ \overline{H} & \text{otherwise.}\end{cases}$$

Now $\mathscr{R}=\{(H, F(H))\colon H\in\mathscr{H}\}$ is a direction. Indeed, $\mathscr{R}$ evidently satisfies axioms (i), (ii) and (iv) of Definition (0.1). To prove (iii), let $\mathscr{F}^0$ be a subfamily of $\mathscr{F}(\mathscr{R})$ and $\mathscr{H}^0=\{H\colon F(H)\in\mathscr{F}^0\}$. We have to show that $\bigcap\mathscr{F}^0\in\mathscr{F}(\mathscr{R})$. If F^0 has a smallest element F_1, then $\bigcap\mathscr{F}^0=F_1\in\mathscr{F}(\mathscr{R})$. Otherwise, $\bigcap\mathscr{F}^0=\bigcap\mathscr{H}^0$. Now we take $H_1=\bigcup\{H\colon H\in\mathscr{H}, H\subset\bigcap\mathscr{H}^0\}$. Clearly, $H_1\in\mathscr{H}$ and $\bigcap\mathscr{H}^0=\bigcap\{H\colon H>H_1\}$, thus $A(H_1)$ is closed, so $\bigcap\mathscr{F}^0=\bigcap\mathscr{H}^0=A(H_1)=F(H_1)\in\mathscr{F}(\mathscr{R})$ and $\mathscr{R}$ is a direction. $\mathscr{H}=\mathscr{G}(\mathscr{R})$ is evident.

§ 2. An embedding theorem

(2.1) THEOREM. *A separable metric space M is homeomorphic to a subspace of $\mathbf{R}^n$ iff* DIM $M\leqq n$ $(n=0, 1, \ldots)$.

PROOF. The theorem is evidently true for $n=0$. Suppose $n>0$.

a) The halfdirections

$$\{H_i(\alpha)=\{x\colon x\in\mathbf{R}^n, x^{(i)}<\alpha\}\colon \alpha\in\mathbf{R}\}\quad (1\leqq i\leqq n)$$

and

$$\left\{H_{n+1}(\alpha)=\left\{x\colon x\in\mathbf{R}^n, \sum_{i=1}^{n}x^{(i)}>\alpha\right\}\colon \alpha\in\mathbf{R}\right\}$$

form a compatible halfdirectional structure on $\mathbf{R}^n$, since for $x\in\mathbf{R}^n$, $\varepsilon>0$:

$$\bigcap_{i=1}^{n}H_i(x^{(i)}+\varepsilon)\cap H_{n+1}\left(\sum_{i=1}^{n}x^{(i)}-\varepsilon\right)\subset\bigtimes_{i=1}^{n}(x^{(i)}-n\varepsilon, x^{(i)}+\varepsilon),$$

so DIM $\mathbf{R}^n\leqq n$, thus for an arbitrary subspace S of $\mathbf{R}^n$, DIM $S\leqq n$.

b) Suppose DIM $M \leqq n$. Let $\{\mathscr{H}_i : 1 \leqq i \leqq n+1\}$ be a compatible halfdirectional structure on M. By Lemma (1.6), there are directions $(\mathscr{R}_i, <^i)$ on M with $\mathscr{H}_i \subset \subset \mathscr{G}(\mathscr{R}_i)$ $(1 \leqq i \leqq n+1)$. Making use of Theorem (0.6), we may suppose that these directions are orderly. $\mathfrak{R} = \{\mathscr{R}_i : 1 \leqq i \leqq n+1\}$ is clearly a compatible directional structure on M. Moreover, the *lower* open $\mathfrak{R}$-halfspaces form a subbase of M.

Now we define an embedding $f: M \to \mathbf{R}^{n+1}$, $f(M) = N$ as given in Theorem (0.8). $f\left(\bigcup_{i=1}^{n+1} \mathscr{G}(\mathscr{R}_i)\right) | N$ is a subbase of N. For an arbitrary $(G, F) \in \mathscr{R}_i$, set $\alpha = \varphi_i((G, F))$. Now for a point x of M, $x \in G$ iff $(G_x(\mathscr{R}_i), F_x(\mathscr{R}_i)) <^i (G, F)$, so — since φ_i is order-preserving — $f(G) = \{x : x \in N, x^{(i)} < \alpha\}$. Thus $\{\{x : x \in N, x^{(i)} < \alpha\} : 1 \leqq i \leqq n+1, \alpha \in \mathbf{R}\}$ is a subbase of N. Therefore, N satisfies the condition

(*) *N has no two points x and y with $x^{(i)} < y^{(i)}$ for each $i = 1, \ldots, n+1$.*

c) To complete the proof, it is enough to show that the set N can be embedded into the set

$$P = \left\{x : x \in \mathbf{R}^{n+1}, \sum_{i=1}^{n+1} x^{(i)} = 0\right\},$$

which is homeomorphic to $\mathbf{R}^n$.

Let $\pi(x)$ $(x \in N)$ be defined by $(\pi(x))^{(i)} = x^{(i)} - \mu(x)$ $(1 \leqq i \leqq n+1)$, where $\mu(x) = \frac{1}{n+1} \sum_{i=1}^{n+1} x^{(i)}$. Clearly, $\pi(x) \in P$. It is to be proved that π is a topological embedding.

π is evidently continuous, moreovet for $x, y \in N$, $\xi = \pi(x)$, $\eta = \pi(y)$, we have

$$\varrho^2(\xi, \eta) = \sum_{i=1}^{n+1} \left[(x^{(i)} - y^{(i)}) - (\mu(x) - \mu(y))\right]^2 = \varrho^2(x, y) - (n+1)[\mu(x) - \mu(y)]^2,$$

where ϱ is the natural metric on $\mathbf{R}^{n+1}$.

If $\xi = \eta$ were true with $x \neq y$, we had $x^{(i)} - y^{(i)} = \mu(x) - \mu(y)$, $i = 1, \ldots, n+1$, i.e. $x^{(i)} - y^{(i)}$ were independent of i, in contradiction to (*). Thus π is one-to-one.

The continuity of π^{-1} must still be proved. According to (*), the set $A = \{1, \ldots, n+1\}$ can be written in the form $A = B \cup C$ with $B \cap C = \emptyset$, $|B| \leqq n$, $|C| \leqq n$, $x^{(i)} \geqq y^{(i)}$ $(i \in B)$, $x^{(i)} \leqq y^{(i)}$ $(i \in C)$. Then

$$(n+1)^2[\mu(x) - \mu(y)]^2 = \left[\sum_{i \in B} (x^{(i)} - y^{(i)}) - \sum_{i \in C} (y^{(i)} - x^{(i)})\right]^2 \leqq$$

$$\leqq \left[\sum_{i \in B} (x^{(i)} - y^{(i)})\right]^2 + \left[\sum_{i \in C} (x^{(i)} - y^{(i)})\right]^2 \leqq$$

$$\leqq |B| \sum_{i \in B} (x^{(i)} - y^{(i)})^2 + |C| \sum_{i \in C} (x^{(i)} - y^{(i)})^2 \leqq n\varrho^2(x, y),$$

thus

$$\varrho^2(x, y) = \varrho^2(\xi, \eta) + (n+1)[\mu(x) - \mu(y)]^2 \leqq \varrho^2(\xi, \eta) + \frac{n}{n+1} \varrho^2(x, y),$$

i.e. $\varrho(x, y) \leqq \sqrt{n+1}\, \varrho(\xi, \eta)$, so π^{-1} is continuous. The proof of the theorem is complete.

REMARK. The inequalities between $\varrho(x, y)$ and $\varrho(\xi, \eta)$ shown in part c) of the proof are quite obvious if one tries to visualize how the set N and the mapping π look like bearing condition $(*)$ in mind and observing that π is a perpendicular projection. We have chosen to give a proof with detailed calculations so as to avoid heuristic reasoning in higher dimensions.

§ 3. Remarks and problems

In this paragraph we consider the relationship of DIM to Dim and raise some questions on generalizations of Theorem (2.1).

(3.1) PROPOSITION. *For a not indiscrete space* X,

$$\operatorname{Dim} X - 1 \leqq \operatorname{DIM} X \leqq 2 \operatorname{Dim} X - 1.$$

In particular: if Dim X *or* DIM X *is infinite, then* DIM $X = \operatorname{Dim} X$.

PROOF. Lemma (1.6).

Theorem (2.1) can be stated in a somewhat different form:

(3.2) THEOREM. *For a separable metric space* M, DIM $M = \operatorname{Dim} M$.

PROOF. If Dim M is finite: Theorem (2.1) and E. Deák's embedding theorem; if Dim M is infinite: Proposition (3.1).

(3.3) PROBLEM. For what other classes of spaces does Theorem (3.2) hold?

(3.4) PROPOSITION. *For a* T_1*-space* X, DIM $X = 1$ *iff* $\operatorname{Dim} X = 1$.

PROOF. Theorems (0.7) and (1.2).

(3.5) EXAMPLE. *There is a compact* T_2*-space* C_n $(n = 2, 3, \ldots)$ *with*

$$\operatorname{Dim} C_n = n, \quad \operatorname{DIM} C_n = 2n - 1.$$

PROOF. $W(\omega_i)$ — ω_i the ith infinite initial number — denotes the order space of the ordinal numbers less than ω_i. Let C_n be the one-point compactification of the topological sum of the spaces $W(\omega_i)$ $(1 \leqq i \leqq 2n)$. GERLITS [4, p. 412] proves that $\operatorname{Dim} C_n = n$. In fact, his argument contains the proof of the statement $\operatorname{DIM} C_n = 2n - 1$.

(3.6) EXAMPLE. *There is a space* Y_n $(n = 1, 2, \ldots)$ *with*

$$\operatorname{Dim} Y_n = n, \quad \operatorname{DIM} Y_n = n - 1.$$

PROOF. Let the space Y_n be the set $\{0, 1, \ldots, n\}$ with the base $\{Y_k : 0 \leqq k \leqq n\}$.

(3.7) PROBLEM. Is $\operatorname{Dim} X \leqq \operatorname{DIM} X$ true for all S_1-spaces?

(3.8) PROBLEM. Is it true that a separable metric space is homeomorphic to a subspace of $\mathbf{R}^n$ $(n = 0, 1, \ldots)$ iff it has a subbase which is the union of $n + 1$ (regular) nests? A positive answer would generalize Theorem (2.1).

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COMPARISON THEOREM FOR FIRST ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DELAY

by

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In his book [1] A. D. MYŠKIS establishes a comparison theorem for linear differential equations with delay which have the form

$$x'(t) = -\int_0^{\sigma(t)} x(t-\tau)\, dr(t,\tau) \quad a \leqq t < b \quad (\leqq \infty), \tag{1}$$

where the function $r(t,\tau)$ is nondecreasing with respect to τ and the integral is taken in Stieltjes sense. Concerning the function $r(t,\tau)$ on $a \leqq t < b$ there are the following restrictions

$$r(t,\sigma(t)) \quad \text{is continuous,}$$

$$r(t,0) \equiv 0, \quad r(t,\tau) = r(t,\sigma(t)) \quad \text{for} \quad \tau \geqq \sigma(t) > 0, \tag{2}$$

$$\lim_{t' \to t} \int_0^{\min\{\sigma(t),\, \sigma(t')\}} |r(t,\tau) - r(t',\tau)|\, d\tau = 0.$$

The restrictions (2) guarantee that $x'(t)$ in (1) is continuous if $x(t-\tau)$ is also continuous. In order to get a solution of (1) we should give a continuous function $\varphi(t)$ on $[a', a]$, which is called initial function, such that

$$x(t-\tau) = \varphi(t-\tau) \quad \text{for} \quad t-\tau \leqq a$$

and $a' = \inf\{t-\sigma(t);\, a \leqq t < b\}$.

The comparison theorem mentioned above plays an essential role in studying the properties of solutions of (1). But the proof given there is lengthly which is caused by the function $\chi(t)$ introduced because an elaborated proof is necessary to show its required properties. We attempted to find another proof for Myškis' theorem avoiding the use of this function but failed. Instead of this we have succeeded in establishing another comparison theorem by which we can always compare the solutions, if the comparison is possible according to the Myškis' theorem, at the worst on a shorter interval, and our theorem is valid for some nonlinear differential equations, too. First we state our theorem and then we shall point out the differences of both theorems.

THEOREM. Let $x(t)$ be a solution of the differential equation

$$x'(t) = -\int_0^{\sigma(t)} f(x(t-\tau))\, dr(t,\tau) \quad \text{for} \quad -\infty < a \leqq t < b \quad (\leqq \infty) \tag{3}$$

with the continuous initial function $\varphi(t)$ on the initial set $E_a=[a', a]$, where the function $r(t, \tau)$ is the same as above.

Similarly let $\bar{x}(t)$ be a solution of the equation

$$\bar{x}'(t) = -\int_0^{\bar{\sigma}(t)} f(\bar{x}(t-\tau))\, d\bar{r}(t, \tau) \quad \text{for} \quad a \leqq t < b \tag{4}$$

with the continuous initial function $\bar{\varphi}(t)$ on $\bar{E}_a=[\bar{a}', a]$, where the function $\bar{r}(t, \tau)$ behaves as $r(t, \tau)$.

Let the following conditions be valid:

(i) The functions $r(t, \tau)$ and $\bar{r}(t, \tau)$ satisfy the relations

$$0 \leqq \bar{\sigma}(t) \leqq \sigma(t),$$

$$\bar{r}(t, \bar{\sigma}(t)) - \bar{r}(t, \tau) \leqq r(t, \sigma(t)) - r(t, \tau)$$

for $0 \leqq \tau \leqq \bar{\sigma}(t)$, $a \leqq t < b$.

From this condition it follows that $a' \leqq \bar{a}'$, $\bar{E}_a \subseteq E_a$.

(ii) The initial functions $\varphi(t)$ and $\bar{\varphi}(t)$ fulfill the relations

$$\varphi(a) = \bar{\varphi}(a) \underset{(<)}{\geqq} 0,$$

$$\max_{(\min)} \{\bar{\varphi}(t'): t \leqq t' \leqq a\} \underset{(\geqq)}{\leqq} \min_{(\max)} \{\varphi(t'): a' \leqq t' \leqq t\} \quad \text{for every} \quad t \in \bar{E}_a.$$

(iii) The function $f(u)$ is continuous and nondecreasing on $(-\infty, \infty)$ and further if $\varphi(a)>0$ then on $(0, \infty)$ $f(u)>0$ and the function $f(u)/u$ is non-increasing else $f(u)<0$ and $f(u)/u$ is nondecreasing on $(-\infty, 0)$.

(iv) The solution $x(t)$ has no zeros on $[a, b)$, i.e. $x(t) \underset{(<)}{\geqq} 0$ for $a \leqq t < b$.

Then the following comparison is true:

$$x(t) \underset{(\geqq)}{\leqq} \bar{x}(t) \quad \text{for} \quad a \leqq t < b.$$

If, in addition to (i)—(iv), the following relations are also valid:

(v) $$\int_0^{\sigma(a)} f(\varphi(a-\tau))\, dr(a, \tau) \underset{(<)}{\geqq} \int_0^{\bar{\sigma}(a)} f(\bar{\varphi}(a-\tau))\, d\bar{r}(a, \tau).$$

(vi) $r(t, \sigma(t)) > 0$ for $a \leqq t < b$.

(vii) Either for every $t \in [a, b)$ the sharp inequality $r(t, \sigma(t)) > \bar{r}(t, \bar{\sigma}(t))$ holds or the function $f(u)/u$ is strictly decreasing for $u>0$ (strictly increasing for $u<0$), or the function $f(u)$ is strictly increasing on $(0, \varphi(a)]$ $([\varphi(a), 0))$ and $r(t, \tau) \not\equiv \not\equiv r(t, \sigma(t))e(\tau)$ where

$$e(\tau) = \begin{cases} 0 & \text{for} \quad \tau \leqq 0, \\ 1 & \text{for} \quad \tau > 0. \end{cases}$$

Then the inequality in the comparison is sharp:

$$x(t) \underset{(>)}{\leqq} \bar{x}(t) \quad \text{for} \quad a < t < b.$$

REMARK 1. If we consider the linear case, i.e. when $f(u)=u$ then we can compare our theorem with Myškis' one. In the latter the condition (iv) is different,

namely it is sufficient to require the validity of the inequality $x(t-\tau)\underset{(\leqq)}{\geqq}0$ only for those $\tau\in[0, \sigma(t)]$ which are points of growth of $r(t, \tau)$ $a\leqq t\leqq b$. Hence the interval $[a, b]$ there may be larger than in our theorem. But in Myškis' theorem a restriction is placed on the solution $\bar{x}(t)$ which says: the differential equation (4) is valid on some $[a, \bar{b}]$, where $x(b)=\bar{x}(b)$ and $\bar{x}(t)\leqq\bar{x}(a)$ for $a\leqq t\leqq\bar{b}$. In our theorem there is no restriction on $\bar{x}(t)$.

The condition (i) for the functions $r(t, \tau)$ and $\bar{r}(t, \tau)$ is also weaker than in Myškis' theorem, where the corresponding relation reads as

$$\bar{r}(\bar{t}, \bar{\sigma}(t))-\bar{r}(\bar{t}, \tau)\leqq r(t, \sigma(t))-p(t, \tau) \quad \text{for} \quad 0\leqq\tau\leqq\bar{\sigma}(t)$$

and for $a\leqq t\leqq b$, $a\leqq\bar{t}\leqq\bar{b}$, $t\leqq\bar{t}$. We require the validity of this relation only for $\bar{t}=t$.

REMARK 2. We shall present an example to show that in general the range of the validity of the inequality $x(t)\underset{(\leqq)}{\leqq}\bar{x}(t)$ can not be extended to some interval $[a, b']$ where $b'>b$ and b' depends only on the equation (3), i.e. our theorem sharp in this sense. At the same time this example will be such one to which the Myškis' theorem can not be applied.

REMARK 3. In the special case $f(u)\equiv\varphi(a)$ the inequality $x(t)\underset{(\leqq)}{\leqq}\bar{x}(t)$ follows from the inequliaty

$$\int_a^t r(s, \sigma(s))\,ds\geqq\int_a^t\bar{r}(s, \bar{\sigma}(s))\,ds,$$

which is true since by (2) and by (i) for $\tau=0$ $r(t, \sigma(t))\geqq\bar{r}(t, \bar{\sigma}(t))$.

REMARK 4. Condition (iii) is equivalent to the existence of a nonincreasing (nondecreasing) continuous function $\tilde{\varphi}(t)$ on E_a satisfying $\tilde{\varphi}(t)\geqq\varphi(t)$ $(\tilde{\varphi}(t)\geqq\varphi(t))$ for $t\in E_a$ and $\bar{\varphi}(t)\geqq\tilde{\varphi}(t)$ $(\bar{\varphi}(t)\geqq\tilde{\varphi}(t))$ for $t\in\bar{E}_a$.

PROOF OF THE THEOREM. We shall restrict ourselves to the case $\varphi(a)>0$ since the proof in the other case goes in similar manner. Moreover it is sufficient to prove the second half of the Theorem where the sharp inequality $x(t)<\bar{x}(t)$ is stated. Then the first half of Theorem will follow from this by suitable limiting procedure.

The function $\bar{x}(t)$ is continuous and $\bar{x}(a)=\varphi(a)$ hence there is the value $\bar{b}$ defined by

$$\bar{b}=\sup\{t;\ \bar{x}(t)>0,\ a\leqq t<b\}.$$

It is clear that $a<\bar{b}\leqq b$. Let us introduce the functions $z(t)$ and $\bar{z}(t)$ by

$$z(t)=-\frac{x'(t)}{x(t)} \quad \text{for} \quad a\leqq t<b, \tag{5}$$

$$\bar{z}(t)=-\frac{\bar{x}'(t)}{\bar{x}(t)} \quad \text{for} \quad a\leqq t<\bar{b}. \tag{6}$$

This definition implies that

$$x(t) = \varphi(a)e^{-\int_a^{z(s)\,ds}}, \quad a \leqq t < b, \tag{7}$$

$$\bar{x}(t) = \varphi(a)e^{-\int_a^{t}\bar{z}(s)\,ds}, \quad a \leqq t < \bar{b}. \tag{8}$$

First we are going to show that

$$z(t) > \bar{z}(t) \quad \text{for} \quad a \leqq t < \bar{b}. \tag{9}$$

From the differential equations (3) and (4) we have for $t=a$

$$z(a) = \frac{\int_0^{\sigma(a)} f(\varphi(a-\tau))\,dr(a,\tau)}{\varphi(a)},$$

$$\bar{z}(a) = \frac{\int_0^{\bar{\sigma}(a)} f(\bar{\varphi}(a-\tau))\,d\bar{r}(a,\tau)}{\varphi(a)},$$

hence by (v) $z(a)>\bar{z}(a)$ thus (9) is valid at $t=a$. Due to the definitions (5), (6) the functions $z(t)$ and $\bar{z}(t)$ are continuous on $[a, \bar{b})$, therefore there is a value b^* with the property

$$b^* = \sup\{t;\ z(t') > \bar{z}(t'),\ a < t' < t < \bar{b}\}. \tag{10}$$

In other words the inequality $z(t)>\bar{z}(t)$ holds on $[a, b^*)$, thus by (7), (8)

$$x(t) < \bar{x}(t) \quad \text{for} \quad a < t < b^*. \tag{11}$$

Now it is clear that if $b^*=b$ then the Theorem is true, therefore it is sufficient to prove that the inequality $a<b^*<b$ contradicts to any assumption of the Theorem. We shall proceed along this way.

By (11) we have $\bar{x}(b^*)>0$ and therefore $\bar{b}>b^*$. According to (6) the function $\bar{z}(t)$ is continuous together with $z(t)$ on $[a, b^*]$ hence by (10)

$$z(b^*) = \bar{z}(b^*), \quad a < b^* < b, \tag{12}$$

and by (7), (8)

$$x(b^*) < \bar{x}(b^*). \tag{13}$$

By the condition (ii) $\varphi(t)\geqq\bar{\varphi}(a)=\varphi(a)>0$ for $t\leqq a$ and by (iv) $x(t)>0$ on $a\leqq t<b$ therefore (3) implies that $x'(t)\leqq 0$. Thus from (vii) it follows that

$$x'(t) = -\int_0^{\sigma(t)} f(x(t-\tau))\,dr(t,\tau) \leqq -f(x(t))r(t,\sigma(t)) < 0, \quad a \leqq t < b, \tag{14}$$

hence $x(t)$ is strictly decreasing on $[a, b)$.

Let us introduce the functions $\tilde{\varphi}(t)$ and $\psi(t)$ by

$$\tilde{\varphi}(t) = \begin{cases} \max\limits_{t \leqq t' \leqq a} \bar{\varphi}(t') & \text{for} \quad a' \leqq t \leqq a, \\ \tilde{\varphi}(a') & \text{for} \quad t \leqq a', \end{cases} \tag{15}$$

$$\psi(t) = \begin{cases} f(x(t)) & \text{for} \quad a \leqq t < b, \\ f(\tilde{\varphi}(t)) & \text{for} \quad t \leqq a. \end{cases} \tag{16}$$

Both functions are nonincreasing, continuous on their domain of definition and by (iii) $\varphi(t) \geqq \tilde{\varphi}(t) \geqq \bar{\varphi}(t)$ for $t \leqq a$ (see Remark 4.).

With the notations

$$\tau_1 = \min \{\sigma(b^*),\ b^* - a\}$$

$$\bar{\tau}_1 = \min \{\bar{\sigma}(b^*),\ b^* - a\}$$

we have from (3) at $t = b^*$ and from (15), (16)

$$-x'(b^*) = \int_0^{\tau_1} f(x(b^* - \tau))\, dr(b^*, \tau) + \int_{\tau_1}^{\sigma(b^*)} f(\varphi(b^* - \tau))\, dr(b^*, \tau) =$$

$$= \int_{\tau_1}^{\sigma(b^*)} [f(\varphi(b^* - \tau)) - f(\tilde{\varphi}(b^* - \tau))]\, dr(b^*, \tau) + \int_0^{\sigma(b^*)} \psi(b^* - \tau)\, dr(b^*, \tau) =$$

$$= I_1 + \int_0^{\sigma(b^*)} \psi(b^* - \tau)\, dr(b^*, \tau).$$

We shall make use of the following fact from the theory of Stieltjes integrations: if the function $h(t)$ is continuous on $[\alpha, \beta]$ and $g(t)$, $h(t)$ are functions of bounded variation there and $g(\alpha) = 0$ then

$$\int_\alpha^\beta h(\tau)\, dg(\tau) = h(\alpha) g(\beta) + \int_\alpha^\beta [g(\beta) - g(\tau)]\, dh(\tau). \tag{17}$$

Applying this fact to $h(t) = \psi(b^* - t)$, $g(\tau) = r(b^*, \tau)$, $\alpha = 0$, $\beta = \sigma(b^*)$ we obtain

$$\int_0^{\sigma(b^*)} \psi(b^* - \tau)\, dr(b^*, \tau) = \psi(b^*) r(b^*, \sigma(b^*)) + \int_0^{\sigma(b^*)} [r(b^*, \sigma(b^*)) -$$

$$- r(b^*, \tau)]\, d\psi(b^* - \tau) = [r(b^*, \sigma(b^*)) - \bar{r}(b^*, \bar{\sigma}(b^*))] \psi(b^*) + \bar{r}(b^*, \bar{\sigma}(b^*)) \psi(b^*) +$$

$$+ \int_0^{\sigma(b^*)} \{[r(b^*, \sigma(b^*)) - r(b^*, \tau)] - [\bar{r}(b, \bar{\sigma}(b^*)) - \bar{r}(b^*, \tau)]\}\, d\psi(b^* - \tau) +$$

$$+ \int_0^{\sigma(b^*)} [\bar{r}(b^*, \bar{\sigma}(b^*)) - \bar{r}(b^*, \tau)]\, d\psi(b^* - \tau) = I_2 +$$

$$+ \bar{r}(b^*, \bar{\sigma}(b^*)) \psi(b^*) + I_3 + \int_0^{\sigma(b^*)} [\bar{r}(b^*, \bar{\sigma}(b^*)) - \bar{r}(b^*, \tau)]\, d\psi(b^* - \tau).$$

If we apply again (17) to the same h, α, β as above but now let $g(t)=\bar{r}(b^*, \tau)$ then

$$-x'(b^*) = I_1+I_2+I_3+\int_0^{\sigma(b^*)} \psi(b^*-\tau)\, d\bar{r} = I_1+I_2+I_3+\int_0^{\bar{\sigma}(b^*)} \psi(b^*-\tau)\, d\bar{r},$$

where for the sake of simplicity $d\bar{r}$ stands for $d\bar{r}(b^*, \tau)$, hence by (16)

$$z(b^*) = \frac{-x'(b^*)}{x(b^*)} = \frac{I_1+I_2+I_3}{x(b^*)} + \int_0^{\bar{\tau}_1} \frac{f(x(b^*-\tau))}{x(b^*)}\, d\bar{r} + \int_{\bar{\tau}_1}^{\bar{\sigma}(b^*)} \frac{f(\tilde{\varphi}(b^*-\tau))}{x(b^*)}\, d\bar{r}.$$

The relations (7), (8) imply that

$$\frac{x(b^*-\tau)}{x(b^*)} = e^{\int_{b^*-\tau}^{b^*} z(s)\, ds};$$

$$\frac{\bar{x}(b^*-\tau)}{\bar{x}(b^*)} = e^{\int_{b^*-\tau}^{b^*} \bar{z}(s)\, ds},$$

hence

$$\int_0^{\bar{\tau}_1} \frac{f(x(b^*-\tau))}{x(b^*)}\, d\bar{r} = \int_0^{\bar{\tau}_1} \left[\frac{f(x(b^*-\tau))}{x(b^*-\tau)} - \frac{f(\bar{x}(b^*-\tau))}{\bar{x}(b^*-\tau)}\right] \frac{x(b^*-\tau)}{x(b^*)}\, d\bar{r} +$$

$$+ \int_0^{\bar{\tau}_1} \frac{f(\bar{x}(b^*-\tau))}{\bar{x}(b^*-\tau)} \left[e^{\int_{b^*-\tau}^{b^*} z(s)\, ds} - e^{\int_{b^*-\tau}^{b^*} \bar{z}(s)\, ds}\right] d\bar{r} +$$

$$+ \int_0^{\bar{\tau}_1} \frac{f(\bar{x}(b^*-\tau))}{\bar{x}(b^*)}\, d\bar{r} = I_4+I_5+\int_0^{\bar{\tau}_1} \frac{f(\bar{x}(b^*-\tau))}{\bar{x}(b^*)}\, d\bar{r}.$$

On the other hand

$$\int_{\bar{\tau}_1}^{\bar{\sigma}(b^*)} \frac{f(\tilde{\varphi}(b^*-\tau))}{x(b^*)}\, d\bar{r} = \int_{\bar{\tau}_1}^{\bar{\sigma}(b^*)} \left[\frac{1}{x(b^*)} - \frac{1}{\bar{x}(b^*)}\right] f(\tilde{\varphi}(b^*-\tau))\, d\bar{r} +$$

$$+ \int_{\bar{\tau}_1}^{\bar{\sigma}(b^*)} \frac{f(\tilde{\varphi}(b^*-\tau)) - f(\bar{\varphi}(b^*-\tau))}{\bar{x}(b^*)}\, d\bar{r} + \int_{\bar{\tau}_1}^{\bar{\sigma}(b^*)} \frac{f(\bar{\varphi}(b^*-\tau))}{\bar{x}(b^*)}\, d\bar{r} =$$

$$= I_6+I_7+\int_{\bar{\tau}_1}^{\bar{\sigma}(b^*)} \frac{f(\bar{\varphi}(b^*-\tau))}{\bar{x}(b^*)}\, d\bar{r},$$

and finally we have

$$z(b^*) = \frac{I_1+I_2+I_3}{x(b^*)} + I_4 + I_5 + I_6 + I_7 + \bar{z}(b^*).$$

It makes no difficulty to show that each of the quantities $I_1, I_2, \ldots, I_7$ is non-negative under our conditions. Taking into account (12) it follows from this that

$$I_1 = I_2 = \ldots = I_7 = 0. \tag{18}$$

Since

$$I_2 = \left[r\left(b^*, \sigma(b^*)\right) - \bar{r}\left(b^*, \bar{\sigma}(b^*)\right)\right]\psi(b^*)$$

and by (14), (iii) $\psi(b^*) = f\left(x(b^*)\right) > 0$, therefore

$$r\left(b^*, \sigma(b^*)\right) = \bar{r}\left(b^*, \bar{\sigma}(b^*)\right). \tag{19}$$

From $I_5 = I_6 = 0$ and from the definition of b^* by (10) we conclude that

$$\bar{r}(b^*, \tau) = r\left(b^*, \sigma(b^*)\right) e(\tau). \tag{20}$$

In the case when the function $f(u)/u$ is strictly decreasing for $u > 0$ then by (vi), (13), (20)

$$I_4 = \int_0^{\bar{r}_1} \left[\frac{f\left(x(b^*-\tau)\right)}{x(b^*-\tau)} - \frac{f\left(\bar{x}(b^*-\tau)\right)}{\bar{x}(b^*-\tau)}\right] \frac{x(b^*-\tau)}{x(b^*)}\, d\bar{r} =$$

$$= \left[\frac{f\left(x(b^*)\right)}{x(b^*)} - \frac{f\left(\bar{x}(b^*)\right)}{\bar{x}(b^*)}\right] r\left(b^*, \sigma(b^*)\right) > 0,$$

which contradicts (18).

When the function $f(u)$ is strictly increasing for $0 < u \leqq \varphi(a)$ then by (19), (20) the integral I_3 can be rewritten in the form

$$I_3 = \int_0^{\sigma(b^*)} \left[r\left(b^*, \sigma(b^*)\right) - r(b^*, \tau)\right] d\psi(b^*-\tau).$$

By (14) the solution $x(t)$ is strictly decreasing for $a \leqq t \leqq b^*$ therefore by (16) the function $\psi(b^*-\tau) = f\left(x(b^*-\tau)\right)$ is strictly increasing thus from $I_3 = 0$ we conclude that

$$r(b^*, \tau) = r\left(b^*, \sigma(b^*)\right) \quad \text{for} \quad 0 < \tau < \sigma(b^*),$$

hence by (2)

$$r(b^*, \tau) = r\left(b^*, \sigma(b^*)\right) e(\tau),$$

which contradicts (viii). Thus the proof is complete.

REMARK 5. Let us observe that from the conditions (i)—(iv) the relation

$$\int_0^{\sigma(a)} f\left(\varphi(a-\tau)\right) dr(a, \tau) \geqq \int_0^{\bar{\sigma}(a)} f\left(\bar{\varphi}(a-\tau)\right) d\bar{r}(a, \tau) \tag{21}$$

follows if $\varphi(a)>0$. Applying the method used above we have

$$\int_0^{\sigma(a)} f(\varphi(a-\tau))\,dr(a,\tau)=\int_0^{\sigma(a)}\left[f(\varphi(a-\tau))-f(\tilde{\varphi}(a-\tau))\right]dr(a,\tau)+$$

$$+f(\tilde{\varphi}(a))\left[r(a,\sigma(a))-\bar{r}(a,\bar{\sigma}(a))\right]+$$

$$+\int_0^{\sigma(a)}\left\{\left[r(a,\sigma(a))-r(a,\tau)\right]-\left[\bar{r}(a,\bar{\sigma}(a))-\bar{r}(a,\tau)\right]\right\}df(\tilde{\varphi}(a-\tau))+$$

$$+\int_0^{\bar{\sigma}(a)}\left[f(\tilde{\varphi}(a-\tau))-f(\bar{\varphi}(a-\tau))\right]d\bar{r}(a,\tau)+\int_0^{\bar{\sigma}(a)} f(\bar{\varphi}(a-\tau))\,d\bar{r}(a,\tau),$$

and the terms on the right side except the last one are nonnegative. According to (v) the second half of our theorem dealts with the case $>$ in (21). But the statement and the proof of our Theorem still remains true if instead of (v) the inequality $z(t)>\bar{z}(t)$ is assumed a priori in a small right neighbourhood of a. This assumption is satisfied for example for the differential equations

$$x'(t)=-x(qt), \quad t\geqq 0,$$

$$\bar{x}'(t)=-x(t), \quad t\geqq 0$$

where $0<q<1$, since by simple calculations we have $\bar{z}(t)\equiv 1$ and $z(t)=1+(1-q)t+O(t^2)$ for small t's.

EXAMPLE. Let $a=0$, $d>1$ and consider the differential equation with delay

$$x'(t)=-x(t-d), \quad t\geqq 0,$$

with the initial function $\varphi(t)=1$ for $t\leqq 0$. Then $x(t)=1-t$ for $0\leqq t\leqq d$. Since $x(1)=0$, therefore $b=1$.

Let ε, δ be small positive numbers, $\delta<\varepsilon$ and $k>1$ be large enough, and let the second differential equation be

$$x_\delta'(t)=-x_\delta(t-\Delta_\delta(t)),$$

where

$$\Delta_\delta(t)=\begin{cases}\varepsilon & \text{for} \quad 0\leqq t\leqq \varepsilon-\delta,\\ \dfrac{\varepsilon(\varepsilon-t)}{\delta} & \text{for} \quad \varepsilon-\delta\leqq t\leqq\varepsilon,\\ t-\varepsilon & \text{for} \quad \varepsilon\leqq t\leqq d,\end{cases}$$

and let the initial function $\varphi_\delta(t)$ be

$$\varphi_\delta(t)=\begin{cases}-k & \text{for} \quad t\leqq -\delta,\\ 1+\dfrac{k+1}{\delta}t & \text{for} \quad -\delta\leqq t\leqq 0.\end{cases}$$

It is easy to see that

$$x_\delta(t) = 1+kt \quad \text{for} \quad 0 \leqq t \leqq \varepsilon-\delta, \tag{22}$$

$$\lim_{\delta\to 0} x_\delta(t) = \begin{cases} 1+kt & \text{for} \quad 0 \leqq t \leqq \varepsilon, \\ (1+k\varepsilon)(1+\varepsilon-t) & \text{for} \quad \varepsilon \leqq t \leqq d. \end{cases}$$

On the other hand all conditions of the Theorem hold for these differential equations, hence $x_\delta(t) > x(t)$ on (0, 1). But we can not apply Myškis' theorem since by (22) $x_\delta(\varepsilon-\delta) > x_\delta(0) = 1$. Let $t_0 = 1+\varepsilon+1/k$ then we have for $t_\delta \in (0, d)$ defined by the relation $x(t_\delta) = x_\delta(t_\delta)$,

$$\lim_{\delta\to 0} t_\delta = t_0,$$

$$x_\delta(t) < x(t) \quad \text{for} \quad t_\delta < t \leqq d.$$

For the value of t_0 can be arbitrary close to 1, the interval (0, 1] is largest one on which the comparison $x_\delta(t) > x(t)$ is always true.

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AN INVARIANCE PRINCIPLE FOR THINNED RANDOM MEASURES

by

T. LINDVALL

1. Introduction, preliminaries

What really matters for the thinning of a wide class of point processes, including the class of renewal processes, is the structure of the thinning mechanism. This was the basic idea of the paper [4] by P. JAGERS and the author, in which rather general results about asymptotic poissonness of thinned point processes were established. The purpose of this paper is to partially extend Theorems 1 & 2 of [4] to an even wider class of rare events, such as visits to unusual states of certain ergodic Markov chains.

FORMALITIES. Let $(R, \mathscr{R})$ be the real line with its Borel sets. A *random measure* on $(R, \mathscr{R})$ is a measurable mapping from some probability space into $(\mathscr{M}, \mathscr{B}(\mathscr{M}))$, where $\mathscr{M}$ is the space of measures on $(R, \mathscr{R})$ giving finite mass to bounded intervals and $\mathscr{B}(\mathscr{M})$ the Borel algebra generated by the vague topology on $\mathscr{M}$: $\mu_n \to \mu$ vaguely if

$$\mu_n f = \int f \, d\mu_n \to \int f \, d\mu = \mu f$$

holds fore each $f \in C_{00}$, the space of continuous functions on R with compact support. A necessary and sufficient condition for ξ to be a random measure is that ξA is a random variable for each A in some class of Borel sets containing a base for the usual topology on R.

The *point processes* on $(R, \mathscr{R})$, denoted by $\mathscr{N}$, are the random measures that give mass $0, 1, 2, \ldots$ to the sets $\in \mathscr{R}$.

The convergence in distrubution of random measures is the usual one for random elements in a topological space: $\xi_n \xrightarrow{\mathscr{D}} \xi$ iff $E[\varphi(\xi_n)] \to E[\varphi(\xi)]$ for each bounded continuous function $\varphi: \mathscr{M} \to R$. Actually, this holds iff

$$\int f \, d\xi_n \xrightarrow{\mathscr{D}} \int f \, d\xi$$

for each $f \in C_{00}$. Another necessary and sufficient condition for $\xi_n \xrightarrow{\mathscr{D}} \xi$ is that

$$(\xi_n A_1, \ldots, \xi_n A_k) \xrightarrow{\mathscr{D}} (\xi A_1, \ldots, \xi A_k)$$

for bounded $A_i \in \mathscr{R}$ such that $P(\xi(\partial A_i) = 0) = 1$ for each A_i, ∂ denoting 'boundary of'.

For a point process ξ with infinitely many points of occurrence, i.e. points x such that $\xi\{x\} \geqq 1$, in both intervals $(-\infty, 0)$, $[0, \infty)$, let

$$X_j = \inf \{x \geqq 0; \ \xi[0, x] \geqq j+1\}, \quad j \geqq 0,$$

$$X_j = \sup \{x < 0; \ \xi[x, 0) \geqq -j\}, \quad j < 0,$$

ROMÁNYOS AZA... KÖNYVTÁRA

so that $\langle X_j\rangle_{j\in Z}$ is an enumeration of the occurrence points in the order they appear. For a point process on $[0, \infty)$, an enumeration $\langle X_j\rangle_{j=0}^{\infty}$ is natural.

For all these facts about random measures and point processes, see JAGERS [3].

2. The theorems

Let $(\Omega, \mathscr{B}, \mathbf{P})$ be the probability space on which all random elements discussed below are supposed to be defined. Let ξ be a point process on $(R, \mathscr{R})$ with occurrence points $\langle X_j\rangle_{j\in Z}$, spread out evenly along R in the sense that $\lim_{j\to\pm\infty} X_j/j = m$ exists a.s., where m is a constant, $0<m<\infty$. This is the same as

$$\lim_{x\to\infty} \xi[0, x]/x = -\lim_{x\to-\infty} \xi[-x, 0]/x = m^{-1}$$

a.s. Further, let ζ_n, $n=1, 2, \ldots$, be a sequence of random measures on $(R, \mathscr{R})$ and let η_n be the random measure giving mass $\zeta_n(X_{i-1}, X_i]$ to the integer i. In order to prove that η_n is indeed a random measure, it is sufficient to prove that $\zeta_n(X_{i-1}, X_i]$ is a random variable for each i. But let $\zeta_n(X_{i-1}, X_i] = \varphi((X_{i-1}, X_i), \zeta_n)$, where φ: $B\times\mathscr{M}\to R_+$ is defined by $((y_1, y_2), \mu)\to\mu(y_1, y_2]$. Here, $B=\{(y_1, y_2)\in R^2;\ y_1\leqq y_2\}$, B and $B\times\mathscr{M}$ are endowed with σ-algebras in the usual way. Then, since a mapping into $B\times\mathscr{M}$ is measurable if each component is, and the mapping φ from $B\times\mathscr{M}$ into R_+ is measurable if $\varphi((y_1, y_2), \cdot)$, $\varphi(\cdot, \mu)$ are measurable $\mathscr{M}\to R_+$, $B\to R_+$ respectively, some consideration renders $\zeta_n(X_{i-1}, X_i]$: $\Omega\to R_+$ a random variable. Justifications of this type will not be repeated.

For $a>0$, let C_a operate on $\mu\in\mathscr{M}$ so that

$$C_a\mu(A) = \mu(aA) = \mu\{ax;\ x\in A\}.$$

THEOREM 1. *Suppose* $a_n\to\infty$ *as* $n\to\infty$. *Then*

$$C_{a_n}\zeta_n \xrightarrow{\mathscr{D}} C_{m^{-1}}\eta$$

if and only if

$$C_{a_n}\eta_n \xrightarrow{\mathscr{D}} \eta.$$

PROOF. We prove only the interesting "if" part of the theorem, but the converse follows by the same kind of arguments.

Suppose $f\in C_{00}$. Let w be the modulus of continuity for f: with $\delta\geqq 0$,

$$w(\delta) = \sup_{|t_1-t_2|\leqq\delta} |f(t_1)-f(t_2)|.$$

Abbreviate $\zeta_n(X_{i-1}, X_i] = \tau_{ni}$. By assumption, $\sum_{|i|\leqq ca_n} \tau_{ni}$ is tight for each $c>0$. We shall prove that

$$\int f d[C_{a_n}\zeta_n] \xrightarrow{\mathscr{D}} \int f d[C_{m^{-1}}\eta].$$

Let ζ_n' be the random measure giving mass $\zeta_n(X_{i-1}, X_i]$ to the point X_i. We have

$$\int f d[C_{a_n}\zeta_n] = \int f d[C_{a_n}\zeta_n] - \int f d[C_{a_n}\zeta_n'] +$$
$$+\int f d[C_{a_n}\zeta_n'] - \int f d[C_{m^{-1}}C_{a_n}\eta_n] + \int f d[C_{m^{-1}}C_{a_n}\eta_n].$$

But the last integral tends to $\int f d[C_{m^{-1}}\eta]$ in distribution by assumption.

Let $b_n \uparrow \infty$ so slowly that $b_n/a_n \to 0$, and let $A_n = \{|X_i/i - m| < m/2 \text{ for } |i| \leqq b_n\}$. Throughout the rest of the proof, c denotes a generic constant. We have

$$\left|\int f d[C_{a_n}\zeta_n] - \int f d[C_{a_n}\zeta'_n]\right| =$$

$$= \left|\sum_i \left\{ \int_{(X_{i-1}/a_n, X_i/a_n]} f d[C_{a_n}\zeta_n] - \int_{(X_{i-1}/a_n, X_i/a_n]} f d[C_{a_n}\zeta'_n] \right\}\right| \leqq$$

$$\leqq \sum_{|i| \leqq b_n} \left| \int_{(X_{i-1}/a_n, X_i/a_n]} f d[C_{a_n}\zeta_n] - \int_{(X_{i-1}/a_n, X_i/a_n]} f d[C_{a_n}\zeta'_n] \right| +$$

$$+ 1_{A_n} \cdot \sum_{|i| > b_n} \left| \int_{(X_{i-1}/a_n, X_i/a_n]} f d[C_{a_n}\zeta_n] - \int_{(X_{i-1}/a_n, X_i/a_n]} f d[C_{a_n}\zeta'_n] \right| +$$

$$+ (1 - 1_{A_n}) \sum_{|i| > b_n} \left| \int_{(X_{i-1}/a_n, X_i/a_n]} f d[C_{a_n}\zeta_n] - \int_{(X_{i-1}/a_n, X_i/a_n]} f d[C_{a_n}\zeta'_n] \right|.$$

The last sum tends to 0 a.s., because $1_{A_n} \uparrow 1$ a.s. We also have

$$\sum_{|i| \leqq b_n} \left| \int_{(X_{i-1}/a_n, X_i/a_n]} f d[C_{a_n}\zeta_n] - \int_{(X_{i-1}/a_n, X_i/a_n]} f d[C_{a_n}\zeta'_n] \right| \leqq$$

$$\leqq \left\{ \max_{|i| \leqq b_n} w\left((X_i - X_{i-1})/a_n\right) \cdot \sum_{|i| \leqq b_n} \tau_{ni} \right.$$

But $\sum_{|i| \leqq b_n} \tau_{ni}$ is tight and

$$\left\{ \max_{|i| \leqq b_n} w\left((X_i - X_{i-1})/a_n\right) \right\} \leqq w\left(\max_{|i| \leqq b_n} \left| \frac{X_i - X_{i-1}}{1 + |i|} \right| \cdot \frac{b_n + 1}{a_n} \right)$$

which tends to 0 a.s. as $n \to \infty$, because $w(\delta) \to 0$ as $\delta \to 0$. Further,

$$1_{A_n} \cdot \sum_{|i| > b_n} \left| \int_{(X_{i-1}/a_n, X_i/a_n]} f d[C_{a_n}\zeta_n] - \int_{(X_{i-1}/a_n, X_i/a_n]} f d[C_{a_n}\zeta'_n] \right| \leqq$$

$$\leqq \left\{ \max_{b_n < |i| \leqq c a_n} w\left(\frac{X_i - X_{i-1}}{i} \cdot \frac{i}{a_n} \right) \right\} \cdot \sum_{b_n < |i| \leqq c a_n} \tau_{ni}$$

which tends to 0 a.s., because $\sum_{|i| \leqq c a_n} \tau_{ni}$ is tight and

$$\sup_{|i| > b_n} \frac{X_i - X_{i-1}}{i} \to 0 \quad \text{a.s.}$$

as $n \to \infty$.

Further,

$$\left|\int f d[C_{a_n}\zeta'_n] - \int f d[C_{m \cdot 1} C_{a_n}\eta_n]\right| =$$

$$= \left|\sum_i \left(f(X_i/a_n) - f(i \cdot m/a_n)\right)\tau_{ni}\right| \leqq \sum_{|i| \leqq b_n} |f(X_i/a_n) - f(i \cdot m/a_n)|\tau_{ni} +$$

$$+ 1_{A_n} \cdot \sum_{|i| > b_n} |f(X_i/a_n) - f(i \cdot m/a_n)|\tau_{ni} + (1 - 1_{A_n}) \cdot \sum_{|i| > b_n} |f(X_i/a_n) - f(i \cdot m/a_n)|\tau_{ni}.$$

Again, the last term tends to 0 a.s., and

$$\sum_{|i|\leqq b_n} |f(X_i/a_n)-f(i\cdot m/a_n)|\tau_{ni} \leqq w\left(\max_{|i|\leqq b_n}\left|\frac{X_i-im}{1+|i|}\cdot\frac{1+b_n}{a_n}\right|\right)\cdot\sum_{|i|\leqq b_n}\tau_{ni}$$

which tends to 0 a.s., and

$$1_{A_n}\cdot\sum_{|i|>b_n}|f(X_i/a_n)-f(i\cdot m/a_n)|\tau_{ni} \leqq w\left(\max_{b_n<|i|\leqq ca_n}\left|\frac{X_i-im}{i}\right|\cdot\frac{i}{a_n}\right)\cdot\sum_{b_n<|i|\leqq ca_n}\tau_{ni},$$

again ultimately small, since $\sum_{|i|\leqq ca_n}\tau_{ni}$ is tight and $\sup_{|i|\geqq b_n}\left|\frac{X_i-im}{i}\right|$ tends to 0 a.s. as $n\to\infty$.

REMARK. The theorem may of course be formulated for random measures on $[0, \infty)$ with points of occurrence $\langle X_j\rangle_{j=0}^{\infty}$ satisfying $X_j/j\to m$ as $j\to\infty$.

It is Theorem 2 below that will be used in our applications. Let $\Pi(\alpha)$ be a homogeneous Poisson process with intensity α on $(R, \mathscr{R})$ (or $(R_+, \mathscr{R}_+)$, if that is more appropriate), i.e., a point process where $\Pi(\alpha)A_1, \ldots, \Pi(\alpha)A_k$ are independent whenever the Borel sets A_i are disjoint and

$$\mathbf{P}(\Pi(\alpha)A = k) = e^{-\alpha t}\cdot(\alpha t)^k/k!$$

if A has Lebesque measure t. A compound Poisson process $\Pi(\alpha, \mu)$, where μ is a probability measure on $[0, \infty)$, is a random measure where a random mass with distribution μ is assigned to each occurrence point of a Poisson process $\Pi(\alpha)$, all assigned masses being independent of each other and of the underlying Poisson process.

Now, let ξ be a point process satisfying the assumption of Theorem 1, and let ζ_n be a sequence of random measures.

THEOREM 2. *Suppose the random variables $\zeta_n(X_{i-1}, X_i]$ are independent and identically distributed for each n. Let $0<p_n=\mathbf{P}(\zeta_n(X_{i-1}, X_i]>0)$, and suppose there exists a sequence $\beta_n, \beta_n>0$, and a probability measure μ on $(R_+, \mathscr{R}_+)$ such that*

$$\beta_n\cdot\zeta_n(X_{i-1}, X_i]\,|\,\zeta_n(X_{i-1}, X] > 0 \xrightarrow{\mathscr{D}} \mu$$

as $n\to\infty$. Then, if $p_n\to 0$ as $n\to\infty$,

$$\beta_n\cdot C_{p_n^{-1}}\zeta_n \xrightarrow{\mathscr{D}} \Pi(m^{-1}, \mu)$$

as $n\to\infty$.

PROOF. Let η_n give mass $\beta_n\cdot\zeta_n(X_{i-1}, X_i]$ to the integer i. All we need to prove is

$$C_{p_n^{-1}}\eta_n \xrightarrow{\mathscr{D}} \Pi(1, \mu).$$

A sufficient condition for this is

$$C_{p_n^{-1}}\eta_n(A) \xrightarrow{\mathscr{D}} \Pi(1, \mu)(A)$$

for every bounded interval A, cf. [3], Theorem 3, because of the independence of the variables $\zeta_n(X_{i-1}, X_i]$. Let μ_n be the distribution of $\beta_n\cdot\zeta_n[X_{i-1}, X_i]\,|\,\zeta_n(X_{i-1}, X_i]>0$,

$\hat{\mu}_n$ its Laplace transform. Then $C_{p_n^{-1}}\eta_n(A)$ has Laplace transform

$$(1+p_n(\hat{\mu}_n-1))^{\#\{i;\, i\in p^{-1}A\}}$$

which is easily seen to tend to

$$\exp[|A|(\hat{\mu}-1)]$$

because $p_n\cdot\#\{i; i\in p_n^{-1}A\}\to|A|$, the Lebesgue measure of A. But $\exp[|A|\cdot(\hat{\mu}-1)]$ is the Laplace transform of $\Pi(1,\mu)(A)$.

3. Applications

A. *Long success runs in Bernoulli trials*

Theorem 2 has perhaps its most natural applications to regenerative structures where $\langle X_i\rangle_{i=1}^{\infty}$ denotes the visits to a state where some process "starts from scratch", cf. KEILSON [5]. Let us see a first example of this in an investigation of long sucess runs in Bernoulli trials. Let τ_i, $i=1, 2, \ldots$, be i.i.d. with $\tau_i=1$ with probability p for success, $0<p<1$, $\tau_i=0$ for failure. We have a run of length n at time j if $j\geqq n$ and $\tau_i=1$ for $j-n+1\leqq i\leqq j$. Let $\langle X_i\rangle_{i=0}^{\infty}$ be the failure epochs, $X_0=0$ for convenience, and let ζ_n be the point process with mass 1 at each time j a run of length n is observed. We have, with $q=1-p$, that $m=q^{-1}$, $p_n=p^n$ and

$$\mathbf{P}(\zeta_n(X_0, X_1]=i\,|\,\zeta_n(X_0, X_1]\geqq 1)=qp^{i-1}.$$

Theorem 2 yields: $C_{p^{-n}}\zeta_n$ is asymptotically $\Pi(q, \langle qp^{i-1}\rangle_{i=0}^{\infty})$. If runs are not allowed to overlap, cf. FELLER [2], p. 305, $C_{p^{-n}}\zeta_n$ is easily seen to be asymptotically $\Pi(q)$, because then

$$\mathbf{P}(\zeta_n(X_0, X_1]\geqq 2\,|\,\zeta_n(X_0, X_1]\geqq 1)=p^n$$

since a run of length $2n$ is required.

This was a new look at an old problem; cf. [2], p. 341, problems 24—25.

B. *The hits of unusual states of a reflected random walk*

Let $Z=\langle Z_i\rangle_{i=0}^{\infty}$ be a Markov chain with the non-negative integers as state space, with transition probabilities $p_{n,n+1}=p$, $p_{n,n-1}=q=1-p$ when $n\geqq 1$, $p_{01}=1$. We assume $0<p<1/2$. Let $\langle u_i\rangle_{i=0}^{\infty}$ be the easily computed stationary distribution and let $\langle X_i\rangle_{i=0}^{\infty}$ be the epochs of visits to state 0. This is a renewal process, possibly delayed, with $m=u_0^{-1}$. For convenience, assume $Z_0=0$, so that $X_0=0$. We are interested in the point process ζ_n, where

$$\zeta_n(i)=\begin{cases}1, & \text{if } Z_i\geqq n,\\ 0, & \text{otherwise.}\end{cases}$$

As is common in the study of Markov chains, we shall be inconsistent about where Z is at time 0, but we return eventually to the case $Z_0=0$. We have

$$p_n=\mathbf{P}(\zeta_n(X_0, X_1]\geqq 1)=P(Z \text{ hits } n \text{ before } 0|Z_0=1).$$

But this probability is well known from classical ruin theory, cf. [2], p. 344 f. With $\varrho=p/q$, we have

$$p_n = 1-\frac{\varrho^{-n}-\varrho^{-1}}{\varrho^{-n}-1} \sim \varrho^{n-1}(1-\varrho).$$

Let f_n be the probability generating function of the random variable

$$\#\{i \geqq 0;\ i < \text{the } 1^{\text{st}} \text{ hit of } 0,\ Z_i \geqq n\}$$

if $Z_0=n$. Conditioning on the outcome of Z_1 and making use of the result [2], p. 272, formula (3.6) on the "waiting time for a gain" yield

$$f_n(s) = s\cdot\left[p\left(\frac{1-\sqrt{1-4pqs^2}}{2ps}\right)f_n(s)+q\left(\frac{\varrho^{-n}-\varrho^{-n+1}}{\varrho^{-n}-1}+\left(1-\frac{\varrho^{-n}-\varrho^{-n+1}}{\varrho^{-n}-1}\right)f_n(s)\right)\right]$$

and when $n\to\infty$, it is easily deduced that

$$f_n(s) \to f(s) = \frac{2qs(1-\varrho)}{1+\sqrt{1-4pqs^2}-2q\varrho s}.$$

We have

$$C_{p_n^{-1}}\zeta_n \xrightarrow{\mathcal{D}} \Pi(u_0, \mu)$$

where μ is the probability measure on $\{1, 2, \ldots\}$ with pgf f. But since $u_0=(1-\varrho)/2$ and $p_n\sim\varrho^{n-1}(1-\varrho)$, we conclude

$$C_{\varrho^{-n}}\zeta_n \xrightarrow{\mathcal{D}} \Pi\left(\frac{(1-\varrho)^2}{2\varrho},\ \mu\right).$$

C. *The high levels of certain birth and death processes, in particular of a $M/M/\infty$ queue*

Let customers arrive to a service system according to a Poisson process with intensity λ. The service times are exponentially distributed with parameter μ, there are infinitely many servers. Let $X(t)$ be the number of customers in the system at time t, $t\geqq 0$, $X(0)=0$. This is the wellknown $M/M/\infty$ queue, a birth and death process with upward and downward intensities λ, $n\mu$ from the state n.

Now, let ζ_{n1} be the random measure on $(R_+, \mathscr{R}_+)$ with density 1 if $X(t)\geqq n$, 0 otherwise, so that $\zeta_{n1}[0, t]$ is the amount of time with n or more customers in the system before t. In the search for the probability p_n that X hits n in a busy period, let

$$a_n = \mathbf{P}\big(X(t) \leqq n \text{ for all } t < \text{the first visit to } 0 \,|\, X(0) = n\big).$$

Clearly,

$$a_n = [n\mu/(\lambda+n\mu)]\cdot[a_{n-1}+(1-a_{n-1})a_n]$$

which, after elaboration, renders

$$a_n = \frac{n!}{\sum_0^n i!\,\varrho^{n-i}}$$

where $\varrho=\lambda/\mu$, and

$$p_n=\prod_{i=1}^{n-1}(1-a_i)=\frac{\varrho^{n-1}}{\sum_{i=1}^{n-1} i!\,\varrho^{n-1-i}}\sim\frac{\varrho^{n-1}}{(n-1)!}.$$

Since $a_n\to 1$ as $n\to\infty$, the probability that X leaves the set of states $\{n, n+1, \ldots\}$ and does not return during a busy period tends to 1 as $n\to\infty$, conditioned that the state n is hit. Further, the time spent at state n is exponentially distributed with parameter $\lambda+n\mu$, and since the expectation of the time from 0 to the end of the first busy period is e^{ϱ}/λ, cf. BREIMAN [1], p. 344 f., we obtain by Theorem 2

$$n\cdot C_{p_n^{-1}}\zeta_{n1}\xrightarrow{\mathcal{D}}\Pi(\lambda e^{-\varrho}, \nu_1)$$

where ν_1 is the exponential distribution with parameter μ.

In general, let $Y(t)$, $t\geqq 0$, be an ergodic birth and death process with stationary distribution $\langle u_i\rangle_{i=0}^{\infty}$ and upward transition intensities $\lambda_n>0$, $n\geqq 0$, and downward $\mu_n>0$, $n\geqq 1$, such that $\lambda_n/\mu_n\to 0$ as $n\to\infty$. If ζ_{n2} is the random measure with density 1 if $Y(t)\equiv n$, 0 otherwise, then

$$\mu_n\cdot C_{\alpha_n}\zeta_{n2}\xrightarrow{\mathcal{D}}\Pi(1, \nu_2)$$

where $\alpha_n=(\mu_n u_n)^{-1}$, and ν_2 is the exponential distribution with parameter 1. The proof of this fact is however lengthy, and it goes beyond the scope of this paper to give it here.

For results about the visits to unusual states of a Markov process, see also KEILSON [5], [6].

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ON A PROPERTY OF GENERALIZED RANDOM FIELDS

by

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Some properties of generalized random fields are given. These are extensions of results obtained by RAO (*Sankhya*, Ser. A. 1971) for real valued random variables.

1. Introduction

RAO [4] obtained some characterizations of probability laws on the real line through linear functions of real valued random variables. We obtain generalization of his results for generalized random fields. Section 2 contains preliminaries. Main theorems are stated and proved in section 3. Some remarks are made in that section.

2. Preliminaries

Let $\varkappa$ be the space of all functions $\varphi(x)=\varphi(x_1, x_2, \ldots, x_n)$ of n variables which are infinitely differentiable and have bounded supports. A sequence $\{\varphi_m\}$ of functions in $\varkappa$ is said to converge to zero if there exists a constant a such that φ_m vanishes for $|x| \geqq a$ and if for every q, the sequence $\{\varphi_m^{(q)}\}$ converges uniformly to zero, where $|x|=(x_1^2+\ldots+x_n^2)^{1/2}$ and $\varphi^{(q)}$ denotes the q-th derivative of φ. Any continuous linear functional on $\varkappa$ is called a generalized function.

A random functional Φ is defined on $\varkappa$ if for every $\varphi \in \varkappa$ there is associated a real valued random variable $\Phi(\varphi)$. In other words, for every m elements φ_i, $1 \leqq i \leqq m$ in $\varkappa$ one can specify the probability that

$$a_i \leqq \Phi(\varphi_i) \leqq b_i, \quad 1 \leqq i \leqq m$$

where $-\infty < a_i < b_i < +\infty$, $1 \leqq i \leqq m$ and these probability distributions are consistent. The random functional Φ is called linear if for any elements $\varphi, \psi \in \varkappa$ and any real numbers α and β

$$\Phi(\alpha\varphi+\beta\psi) = \alpha\Phi(\varphi)+\beta\Phi(\psi)$$

almost surely. The random functional Φ is said to be continuous if the convergence in $\varkappa$ of the functions φ_{kj} to φ_j, $1 \leqq j \leqq m$ implies that for every bounded continuous function $f(x_1, x_2, \ldots, x_m)$

$$\lim_{k \to \infty} \int f(x_1, \ldots, x_m)\, dP_k(x) = \int f(x_1, \ldots, x_m)\, dP(x)$$

where P is the probability measure corresponding to $(\Phi(\varphi_1), \ldots, \Phi(\varphi_m))$ and P_k is the probability measures corresponding to $(\Phi(\varphi_{k1}), \ldots, \Phi(\varphi_{km}))$.

Any continuous linear random functional on $\varkappa$ is called a generalized random function. If $\varkappa$ consists of functions of one variable the corresponding random function is called a generalized random process. If $\varkappa$ consists of functions of several variables, Φ is called a generalized random field.

Let Φ and Ψ be two generalized random fields on $\varkappa$. Φ and Ψ are said to be independent if the set of random variables $\{\Phi(\varphi): \varphi\in\varkappa\}$ is stochastically independent of the set $\{\Psi(\varphi): \varphi\in\varkappa\}$. Clearly this notion can be extended to any finite number of generalized random fields.

Let Φ be a generalized random field. Define

$$L(\varphi) = E\{e^{i\Phi(\varphi)}\}, \quad \varphi\in\varkappa. \tag{2.1}$$

$L(\cdot)$ is called the characteristic functional of the generalized random field Φ. It can be shown that $L(0)=1$, $L(-\varphi)=\overline{L(\varphi)}$, $L(\cdot)$ is continuous and positive-definite. Conversly if $L(\cdot)$ is a positive-definite continuous functional on $\varkappa$ such that $L(0)=1$, it can be shown that there exists a generalized random field Φ on $\varkappa$ whose characteristic functional is $L(\cdot)$. In fact, the correspondence between the characteristic functionals L and generalized random fields Φ on $\varkappa$ is one-one.

Let Φ and Ψ be two generalized random fields on $\varkappa$. The joint characteristic functional of the two dimensional generalized random field (Φ, Ψ) is defined by

$$L(\varphi, \psi) = E[e^{i\Phi(\varphi)+i\Psi(\psi)}], \quad \varphi\in\varkappa, \ \psi\in\varkappa. \tag{2.2}$$

A generalized random field Φ on $\varkappa$ is said to be Gaussian if its characteristic functional is of the form

$$e^{im(\varphi)-\frac{1}{2}B(\varphi,\varphi)}, \quad \varphi\in\varkappa \tag{2.3}$$

where $m(\cdot)$ is a generalized function and $B(\varphi, \psi)=E[\Phi(\varphi)\Phi(\psi)]$. It is said to be degenerate if the characteristic functional is of the form

$$e^{im(\varphi)}, \ \varphi\in\varkappa \tag{2.4}$$

where $m(\cdot)$ is a generalized function.

Let Φ_1 and Φ_2 be two generalized random fields on $\varkappa$. Let f and g be any two infinitely differentiable functions. We define the process $f\Phi_1+g\Phi_2$ to be the process for which to every $\varphi\in\varkappa$ corresponds the random variable $\Phi_1(f\varphi)+\Phi_2(g\varphi)$.

We refer the reader to GELFAND and WILENKIN [1] for more details on generalized random processes and generalized random fields.

Two generalized random fields Φ and Ψ are said to be determined "up to Gaussian generalized random field" if there exists a Gaussian generalized random field Λ such that $\Phi=\Psi+\Lambda$. They are said to be determined "up to translation" if there exists a generalized function m such that $\Phi=\Psi+m$.

3. Main result

We shall now obtain a theorem which gives a characterization of generalized random fields up to Gaussian factors.

THEOREM 3.1. *Let* Φ_i, $0 \leqq i \leqq 3$ *be four independent generalized random fields on* $\varkappa$ *and let*

$$\Psi_1 = \Phi_0 + \Phi_1 + \Phi_2 + \Phi_3 \tag{3.1}$$

$$\Psi_2 = \beta_0 \Phi_0 + \beta_1 \Phi_1 + \beta_2 \Phi_2 + \beta_3 \Phi_3$$

where β_i, $0 \leqq i \leqq 3$ *are infinitely differentiable functions such that* $\beta_i(x) \neq \beta_j(x)$ *for* $i \neq j$ *and for all* x. *Suppose that the joint characteristic functional* $H(\varphi, \psi)$ *of* (Ψ_1, Ψ_2) *is specified and does not vanish. If* $L_i(\varphi)$ *and* $M_i(\varphi)$ *are two alternate possible characteristic functionals of* Φ_i, *then*

$$L_i(\varphi) = M_i(\varphi) \exp\left\{m_i(\varphi) - \frac{1}{2} B_i(\varphi, \varphi)\right\}$$

where $m_i(\varphi)$, $0 \leqq i \leqq 3$ *are generalized functions and* $B_i(\varphi, \psi)$, $0 \leqq i \leqq 3$ *are continuous bilinear Hermitian functionals.*

PROOF. Let Γ_i, $0 \leqq i \leqq 3$ be independent generalized random fields on $\varkappa$ such that the two-dimensional generalized random field $(\sum_1, \sum_2)$ where

$$\Sigma_1 = \Gamma_0 + \Gamma_1 + \Gamma_2 + \Gamma_3$$

$$\Sigma_2 = \beta_0 \Gamma_0 + \beta_1 \Gamma_1 + \beta_2 \Gamma_2 + \beta_3 \Gamma_3$$

has the same joint characteristic functional $H(\varphi, \psi)$ as (Ψ_1, Ψ_2). Let $L_i(\cdot)$ and $M_i(\cdot)$, $0 \leqq i \leqq 3$ be the characteristic functionals of Φ_i and Γ_i, $0 \leqq i \leqq 3$ respectively. It can be easily seen that

$$H(\varphi, \psi) = \prod_{i=0}^{3} M_i(\varphi + \beta_i \psi) = \prod_{i=0}^{3} L_i(\varphi + \beta_i \psi) \tag{3.2}$$

for all φ, ψ in $\varkappa$. Since $H(\varphi, \psi) \neq 0$ for all φ, ψ in $\varkappa$, $J_i(\varphi) = \log \dfrac{L_i(\varphi)}{M_i(\varphi)}$ is well-defined where the logarithm is taken to be the continuous branch with $J_i(0) = 0$. (3.2) gives

$$\sum_{i=0}^{3} J_i(\varphi + \beta_i \psi) = 0 \tag{3.3}$$

for all φ, ψ in $\varkappa$. Let φ, ψ and λ be fixed in $\varkappa$ and let $\varphi' = \varphi - \beta_3 \lambda$ and $\psi' = \psi + \lambda$. Clearly φ' and ψ' belong to $\varkappa$. Hence (3.3) implies that

$$\sum_{i=0}^{3} J_i(\varphi' + \beta_i \psi') = 0. \tag{3.4}$$

Subtracting (3.3) from (3.4), one obtains the relation

$$\sum_{i=0}^{2} [J_i(\varphi' + \beta_i \psi') - J_i(\varphi + \beta_i \psi)] = 0 \tag{3.5}$$

for all φ, ψ, λ in $\varkappa$ since $\varphi' + \beta_3\psi' = \varphi + \beta_3\psi$. Let

$$W_i(\varphi) = J_i(\varphi + \lambda(\beta_i - \beta_3)) - J_i(\varphi), \quad 0 \leqq i \leqq 2 \tag{3.6}$$

for any fixed λ in $\varkappa$. (3.5) implies that

$$\sum_{i=0}^{2} W_i(\varphi + \beta_i\psi) = 0 \tag{3.7}$$

for any φ, ψ in $\varkappa$. Let φ, ψ and ν be fixed in $\varkappa$ and let $\varphi' = \varphi - \beta_2\nu$ and $\psi' = \psi + \nu$. Arguing as before, one obtains the relation

$$\sum_{i=0}^{1} Y_i(\varphi + \beta_i\psi) = 0 \tag{3.8}$$

where

$$Y_i(\varphi) = W_i(\varphi + \nu(\beta_i - \beta_2)) - W_i(\varphi), \quad 0 \leqq i \leqq 1 \tag{3.9}$$

for any fixed ν in $\varkappa$. Again fix φ, ψ and σ in $\varkappa$ and let $\varphi' = \varphi - \beta_1\sigma$ and $\psi' = \psi + \sigma$. Similar arguments show that

$$Z_0(\varphi + \beta_0\psi) = 0 \tag{3.10}$$

for any φ, ψ in $\varkappa$ where

$$Z_0(\varphi) = Y_0(\varphi + \sigma(\beta_0 - \beta_1)) - Y_0(\varphi). \tag{3.11}$$

Combining (3.6), (3.9), (3.10) and (3.11), it can be seen that

$$\begin{aligned} &J_0(\varphi + \beta_0\psi + \sigma(\beta_0 - \beta_1) + \nu(\beta_0 - \beta_2) + \lambda(\beta_0 - \beta_3)) - \\ &- J_0(\varphi + \beta_0\psi + \sigma(\beta_0 - \beta_1) + \nu(\beta_0 - \beta_2)) - \\ &- J_0(\varphi + \beta_0\psi + \sigma(\beta_0 - \beta_1) + \lambda(\beta_0 - \beta_3)) - \\ &- J_0(\varphi + \beta_0\psi + \nu(\beta_0 - \beta_2) + \lambda(\beta_0 - \beta_3)) + \\ &+ J_0(\varphi + \beta_0\psi + \sigma(\beta_0 - \beta_1)) + \\ &+ J_0(\varphi + \beta_0\psi + \nu(\beta_0 - \beta_2)) + \\ &+ J_0(\varphi + \beta_0\psi + \lambda(\beta_0 - \beta_3)) - \\ &- J_0(\varphi + \beta_0\psi) = \\ &= 0 \end{aligned} \tag{3.12}$$

for all $\varphi, \psi, \sigma, \nu$ and λ in $\varkappa$. Since $\beta_i(x) \neq \beta_j(x)$ for all $i \neq j$ and for all x, (3.12) implies that

$$\begin{aligned} &J_0(\psi + \sigma + \nu + \lambda) + J_0(\psi + \sigma) + J_0(\psi + \lambda) + J_0(\psi + \nu) = \\ &= J_0(\psi + \sigma + \nu) + J_0(\psi + \nu + \lambda) + J_0(\psi + \sigma + \lambda) + J_0(\psi) \end{aligned} \tag{3.13}$$

for all ψ, σ, ν and λ in $\varkappa$. Let $\nu=-\sigma$ and $\lambda=-\psi$. Since $J_0(0)=0$, it follows that

$$J_0(\psi+\sigma)+J_0(\psi-\sigma)= \tag{3.14}$$

$$= J_0(-\sigma)+J_0(\sigma)+2J_0(\psi)$$

for all ψ and σ in $\varkappa$. Since $J_0(\cdot)$ is a complex valued functional on $\varkappa$, it can be written in the form

$$J_0(\cdot) = R_0(\cdot)+iI_0(\cdot) \tag{3.15}$$

where $R_0(\cdot)$ and $I_0(\cdot)$ are real valued functionals on $\varkappa$. From the properties of characteristic functionals, it follows that $R_0(\cdot)$ and $I_0(\cdot)$ are continuous functionals on $\varkappa$ such that

$$R_0(\varphi) = R_0(-\varphi) \tag{3.16}$$

and

$$I_0(\varphi) = -I_0(-\varphi) \tag{3.17}$$

for all $\varphi\in\varkappa$. (3.14) implies that

$$R_0(\psi+\sigma)+R_0(\psi-\sigma) = 2R_0(\sigma)+2R_0(\psi) \tag{3.18}$$

and

$$I_0(\psi+\sigma)+I_0(\psi-\sigma) = 2I_0(\psi) \tag{3.19}$$

for all ψ, σ in $\varkappa$. (3.19) gives the relation

$$I_0(\sigma+\psi)+I_0(\sigma+\psi) = 2I_0(\sigma) \tag{3.20}$$

by interchanging ψ and σ. Since $I_0(\psi-\sigma)+I_0(\sigma-\psi)=0$ by (3.17), adding (3.19) and (3.20), one obtains the equation

$$2I_0(\psi+\sigma) = 2I_0(\psi)+2I_0(\sigma) \tag{3.21}$$

for all ψ, σ in $\varkappa$. Hence $I_0(\cdot)$ is a continuous linear functional on $\varkappa$. In other words, $I_0(\cdot)$ is a generalized function (say) $m_0(\cdot)$. Therefore, for every φ in $\varkappa$,

$$L_0(\varphi) = M_0(\varphi)e^{im_0(\varphi)+R_0(\varphi)} \tag{3.22}$$

where $m_0(\varphi)$ is a generalized function and $R_0(\cdot)$ satisfies the functional equation (3.18). Define

$$B_0(\psi,\sigma) = R_0(\psi)+R_0(\sigma)-R_0(\psi+\sigma) \tag{3.23}$$

for all ψ and σ in $\varkappa$.

Note that $B_0(\psi,\sigma)=B_0(\sigma,\psi)$, $B_0(\psi+\varphi,\sigma)=B_0(\psi,\sigma)+B_0(\varphi,\sigma)$. In fact $B_0(\psi,\sigma)$ is a continuous bilinear Hermitian functional on $\varkappa\times\varkappa$ such that $B_0(\varphi,\varphi)=$ $=-2R_0(\varphi)$. Hence

$$L_0(\varphi) = M_0(\varphi)e^{im_0(\varphi)-\frac{1}{2}B_0(\varphi,\varphi)} \tag{3.24}$$

for all φ in $\varkappa$ where $B_0(\varphi,\psi)$ satisfies the properties stated above.

REMARK 1. If $B_0(\varphi, \psi)$ is positive definite in the above theorem, then there exists a generalized Gaussian random field for which $B_0(\varphi, \psi)$ is the correlation functional (See [1], p. 252). But

$$e^{im_0(\varphi)-\frac{1}{2}B_0(\varphi,\varphi)}$$

is the characteristic functional of a generalized Gaussian random field with mean function $m_0(\varphi)$ and correlation functional $B_0(\varphi, \psi)$. Hence, by the one to one correspondence between characteristic functionals and measures (See [1], p. 346—47), it follows that Φ_0 and Γ_0 are determined up to generalized Gaussian random field.

REMARK 2. It is clear that if β_i, $0 \leqq i \leqq 3$ are constants all different from zero and different from each other, then the theorem holds.

REMARK 3. These results can be extended to multi-dimensional generalized random fields easily.

REMARK 4. Result obtained here is similar to the results for real-valued random variables by RAO [4] and for random elements on locally compact Abelian groups by PRAKASA RAO [3]. However, the theorem proved here is not a trivial consequence of the result of RAO [4] since the functions $\beta(x)$ are not constants and hence for a fixed $\varphi \in \varkappa$, $\Psi_2(\varphi)$ is not a linear combination of $\Phi_i(\varphi)$, $0 \leqq i \leqq 3$.

REMARK 5. If the two-dimensional generalized random field in theorem 3.1 is infinitely divisible, then its characteristic functional does not vanish and hence the conclusions of the theorem hold.

One can prove the following theorem by methods analogous to those given in the proof of Theorem 3.1. We shall state the theorem and omit its proof.

THEOREM 3.2. *Let* Φ_i, $0 \leqq i \leqq 2$ *be three independent generalized random fields on* $\varkappa$ *and let*

$$\Psi_1 = \Phi_0 + \Phi_1 + \Phi_2$$

$$\Psi_2 = \beta_0 \Phi_0 + \beta_1 \Phi_1 + \beta_2 \Phi_2$$

where β_i, $0 \leqq i \leqq 2$ *are infinitely differentiable functions such that* $\beta_i(x) \neq \beta_j(x)$ *for* $i \neq j$ *and all* x. *Suppose the joint characteristic functional of* (Ψ_1, Ψ_2) *does not vanish. Then the two-dimensional generalized random field* (Ψ_1, Ψ_2) *completely determines the generalized random fields* Φ_i, $0 \leqq i \leqq 2$ *up to translation.*

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RATE OF STRONG UNIFORM CONSISTENCY OF NONPARAMETRIC DENSITY ESTIMATORS

by

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Abstract

$(X_k)_1^\infty$ is an *iid* sequence of rv's with common density f. A density estimator f_n is said to be of the *averaging* type if, for each n, there are measurable functions $\varphi_{n1}, \ldots, \varphi_{nn}$ such that $f_n(x) = n^{-1} \sum_{k=1}^{n} \varphi_{nk}(x, X_k)$. Two general theorems are proved, giving the rate of strong uniform consistency of estimators of the averaging type. When applied to some estimators of the *δ-sequence* type, these theorems imply that if f satisfies a certain Lipschitz-type condition on a set A then $\tau_n \sup_A |f_n(x) - f(x)| \to 0$ *a.s.* even if τ_n is almost $n^{1/3}$. When applied to estimators of the *kernel* type, these theorems imply that if f has a bounded second derivative on an interval from a to b and $J(\varepsilon) = [a+\varepsilon, b-\varepsilon]$ then $\tau_n \sup_{J(\varepsilon)} |f_n(x) - f(x)| = 0$ *a.s.* even if τ_n is almost $n^{2/5}$.

1. Introduction and summary

Let $(X_k)_1^\infty$ be an *iid* sequence of random vectors, defined on $(\Omega, \mathcal{F}, \mathbf{P})$ and taking values in $\mathbf{R}^p$. Suppose that they are absolutely continuous (with respect to Lebesgue measure) and let f be their common density. When convenient, write X instead of X_1; use $\| \ \|$ to denote an arbitrary, but henceforth fixed, norm on $\mathbf{R}^p$. For real sequences $(a_n)_1^\infty$ and $(b_n)_1^\infty$, write $a_n \sim b_n$ to indicate that a_n is *asymptotically proportional* to b_n, i.e. that a_n/b_n converges to a finite nonzero limit as $n \to \infty$. The indicator (or characteristic function) of a set will be denoted by I. $\mathbf{E}$ is expectation and $\mathbf{V}$ is variance.

For each positive n, let $(\varphi_{nk})_{k=1}^n$ be a finite sequence of Borel measurable functions from $\mathbf{R}^p \times \mathbf{R}^p$ to $\mathbf{R}$ and define

$$f_n(x) = \frac{1}{n} \sum_{k=1}^{n} \varphi_{nk}(x, X_k),$$

a sequence of *random functions* which can be used to estimate f. We will say that such an estimator is of the *averaging* type. Estimators of the *kernel* type and estimators of the *δ-sequence* type are special cases of averaging-type estimators.

Our main results (Theorems 1 and 2) are very general. Because of their generality, they involve some conditions which are not very intuitive. More natural conditions will be encountered when these general theorems are applied (in § 2 and § 3) to some more specific types of estimators.

For a statement of the conditions, let A and $(C_n)_1^\infty$ be given Borel sets of $\mathbf{R}^p$, δ a given positive real, and $(\varrho_n)_1^\infty$ an arbitrary (but fixed) sequence of positive reals

such that $\varrho_n \to \infty$. Define

$$s_n = \left(\max_{1 \leqq k \leqq n} \sup \{\mathrm{V}\varphi_{nk}(x, X): x \in A \cap C_n\}\right)^{1/2} / \sqrt{n},$$

$$\chi_n = 1/(\varrho_n s_n \sqrt{\log n}),$$

$$b_n = \max_{1 \leqq k \leqq n} \sup \{|\varphi_{nk}(x, y)|: (x, y) \in \mathbf{R}^p \times \mathbf{R}^p\},$$

$$g_n = \sup \{|f(x) - \mathbf{E}f_n(x)|: x \in A\}.$$

If $\mathscr{P} = (A_1, A_2, \ldots, A_m)$ is a partition of some subset of $\mathbf{R}^p$, let $\# \mathscr{P}$ be the number of sets in the partition (i.e. m) and let $\Phi_n(\mathscr{P})$ be the supremum of the differences $|\varphi_{nk}(x_1, y) - \varphi_{nk}(x_2, y)|$ when x_1 and x_2 belong to the same subset of the partition $\mathscr{P}$; i.e.,

$$\Phi_n(\mathscr{P}) = \max_{\substack{1 \leqq k \leqq n \\ 1 \leqq j \leqq m}} \sup \{|\varphi_{nk}(x_1, y) - \varphi_{nk}(x_2, y)|: (x_1, x_2) \in A_j \times A_j, y \in \mathbf{R}^p\}.$$

Relative to the given sets (A and $C_1, C_2, \ldots$) and the given numbers (δ and $\varrho_1, \varrho_2, \ldots$), consider the following conditions:

(1.1) $$b_n / s_n = O\left(n / \sqrt{\log n}\right);$$

(1.2) there exist a positive L and a sequence $(\mathscr{P}_n)_1^\infty$ of partitions such that $\mathscr{P}_n$ is a partition of $A \cap C_n$,

$$\# \mathscr{P}_n \leqq n^L, \quad \text{and} \quad \lim_{n \to \infty} \chi_n \Phi_n(\mathscr{P}_n) = 0;$$

(1.3) there exists $c \in (0, 1)$ such that

$$\lim_{n \to \infty} \chi_n \max_{1 \leqq k \leqq n} \sup \{|\varphi_{nk}(x, y)|: \|x - y\| > c n^{1/\delta}\} = 0.$$

THEOREM 1. *If conditions* (1.1) *and* (1.2) *are satisfied then*

$$\lim_{n \to \infty} \left(\frac{1}{\varrho_n} \cdot \min \left\{ \frac{1}{s_n \sqrt{\log n}}, \frac{1}{g_n} \right\} \cdot \sup_{A \cap C_n} |f_n(x) - f(x)| \right) = 0 \quad a.s.$$

THEOREM 2. *Suppose that, for some* $\delta > 0$, $E\|X\|^\delta < \infty$. *Suppose also that, with this* δ *and* $C_n = \{x \in \mathbf{R}^p: \|x\| \leqq n^{1/\delta}\}$, *conditions* (1.1), (1.2), *and* (1.3) *are satisfied. Then*

$$\lim_{n \to \infty} \left(\frac{1}{\varrho_n} \cdot \min \left\{ \frac{1}{s_n \sqrt{\log n}}, \frac{1}{g_n} \right\} \cdot \sup_{A} |f_n(x) - f(x)| \right) = 0 \quad a.s.$$

The proof of these theorems is given in **§ 4**. In **§ 2**, Theorems 1 and 2 are applied to some estimators of the δ-sequence type, showing that if f satisfies a certain Lipschitz-type condition on a set A then $\tau_n \sup_A |f_n(x) - f(x)| \to 0$ *a.s.* even if τ_n is almost $n^{1/3}$. In **§ 3**, Theorems 1 and 2 are applied to some estimators of the kernel type, showing that if f has a bounded second derivative on an interval from a to b and $J(\varepsilon) = [a + \varepsilon, b - \varepsilon]$ then $\tau_n \sup_{J(\varepsilon)} |f_n(x) - f(x)| \to 0$ *a.s.*, even if τ_n is almost $n^{2/5}$.

Theorems 1 and 2 are generalizations of some results in [1], [5], and [8]. If $p = 1$, i.e. the X_k's are real rv's, then Theorems 1 and 2 are very similar to some results stated (without proof) in Theorems 2 of [1]. If our Theorem 2 is applied to a multi-

variate kernel estimator then one obtains a result very similar to that stated in Theorem 1 of [5]. If our Theorem 2 is applied to the estimators introduced in [8] then one finds that the rate of convergence stated in [8] for pointwise strong consistency is in fact valid for uniform strong consistency.

2. Application to δ-sequence estimators

Suppose that $(X_k)_1^\infty$ is an *iid* sequence of *real* rv's whose common density f vanishes on the complement of some known bounded interval; without loss of generality, let that interval be $[-\pi, \pi]$. In this case one can construct a density estimator by using the *Fejér kernel* K_ν which, for a positive integer ν, is given by

$$K_\nu(x) = \frac{1}{2\pi} \sum_{j=-\nu}^{\nu} \left(1 - \frac{|j|}{\nu+1}\right) e^{ijx}, \quad x \in \mathbf{R}.$$

Let $(\nu_n)_1^\infty$ be a sequence of positive integers such that $\nu_n \to \infty$ and put

$$\varphi_{nk}(x, y) = K_{\nu_n}(x-y), \quad 1 \leqq k \leqq n.$$

Then

$$\tilde{f}_n(x) = \frac{1}{n} \sum_{k=1}^{n} \varphi_{nk}(x, X_k) = \frac{1}{n} \sum_{k=1}^{n} K_{\nu_n}(x - X_k), \quad -\pi \leqq x < \pi,$$

is a nonparametric estimator of f; it is discussed extensively in [8]. If f satisfies a certain Lipschitz-type condition at x and $r_n \to \infty$ then, with probability one, $\lim_{n\to\infty} (n^{1/3}/(r_n \log n))|\tilde{f}_n(x) - f(x)| = 0$; see (3.2) in [8]. We shall extend this to a *uniform* convergence result.

DEFINITION. If $g: \mathbf{R} \to \mathbf{R}$ then g is *π-Lipschitz on A* iff there exists a finite M such that, for any $x \in A$ and any $y \in \mathbf{R}$

$$|x-y| < \pi \Rightarrow |g(x) - g(y)| \leqq M|x-y|.$$

NOTATION. If g is a function whose domain includes $[-\pi, \pi)$ then g^e denotes the 2π-periodic extension (to all of $\mathbf{R}$) of the restriction $g|[-\pi, \pi]$.

THEOREM 3. *Suppose that f vanishes off $[-\pi, \pi)$, f^e is π-Lipschitz on $A \subset [-\pi, \pi)$, and f is not identically zero on A. If $\nu_n \sim n^{1/3}$ and $r_n \to \infty$ then*

$$\lim_{n\to\infty} \left(\frac{n^{1/3}}{r_n \log n} \sup_A |\tilde{f}_n(x) - f(x)| \right) = 0 \quad a.s.$$

PROOF. For $g \in L_1[-\pi, \pi)$, let $\sigma_\nu(g, x)$ be the ν-th *Fejér sum* of the Fourier series of g. It is the arithmetic mean of the partial sums $S_0(g, x)$, $S_1(g, x)$, ..., $S_\nu(g, x)$ of the Fourier series of g; see (1.1) in [8]. Then $\mathbf{E}\tilde{f}_n(x) = \sigma_{\nu_n}(f, x)$, see (2.2) in [8].

Let $C_n = A$ for each n and let s_n, χ_n, b_n, g_n be as in § 1. The rate of asymptotic bias can be established from a corresponding result for Fourier series: if $g \in L_1[-\pi, \pi)$ and g^e is π-Lipschitz on A then

$$\sup_{x \in A} |\sigma_\nu(g, x) - g(x)| = O(\nu^{-1} \log \nu) \quad \text{as} \quad \nu \to \infty;$$

see p. 21 in [3] or p. 442 in [7]. Therefore, writing v instead of v_n,

(2.1) $\exists D > 0$ such that

$$g_n = \sup_{x \in A} |f(x) - \mathbf{E}\tilde{f}_n(x)| \leqq Dv^{-1} \log v.$$

Using the 2π-periodicity of K_v and f^e,

$$ns_n^2 = \sup_{x \in A} \mathbf{V}K_v(x - X) \leqq \sup_{x \in A} \int_{-\pi}^{\pi} K_v^2(x-y) f(y)\, dy = \sup_{x \in A} \int_{-\pi}^{\pi} K_v^2(y) f^e(x-y)\, dy \leqq$$

$$\leqq \|f^e\|_\infty \cdot \int_{-\pi}^{\pi} K_v^2(y)\, dy = (2\pi)^{-1} \|f^e\|_\infty \cdot \sum_{j=-v}^{v} \left(1 - \frac{|j|}{v+1}\right)^2.$$

Since f^e is π-Lipschitz on A, $\|f^e\|_\infty < \infty$; therefore the last bound is $O(v)$. — Now let $\xi \in A$ be such that $f(\xi) \neq 0$. Then (due to the Lipschitz condition) f is continuous at ξ and it follows that $\mathbf{V}K_v(\xi - X) = v_n t_n(\xi)$ where $t_n(\xi)$ converges to a finite nonzero constant; see [8], § 2. Therefore

(2.2) there exist m, M, and n_0 such that

$$n \geqq n_0 \Rightarrow 0 < mv_n \leqq ns_n^2 \leqq Mv_n.$$

The rate-factor in Theorem 1 involves $\min\{1/(s_n\sqrt{\log n}),\ 1/g_n\}$. By (2.1) and (2.2), if $n \geqq n_0$ then this is *greater than or equal to*

$$\beta_n = \min\left\{\frac{1}{(n^{-1}Mv_n)^{1/2}\sqrt{\log n}},\ \frac{v_n}{D \log v_n}\right\}.$$

If $v_n \sim n^\gamma$ for some $\gamma > 0$ then the *first* term in braces is $\sim n^{(1-\gamma)/2}/\sqrt{\log n}$ while the *second* term is $\sim n^\gamma / \log n$. Hence β_n is maximized by taking $\gamma = 1/3$ and, with that choice of γ, β_n becomes $c_n n^{1/3}/\log n$ for some sequence $(c_n)_1^\infty$ which converges to a finite nonzero limit. Put $\varrho_n = c_n r_n$; since $r_n \to \infty$, it follows that

(2.3) there exists a sequence $(\varrho_n)_1^\infty$ such that $\varrho_n \to \infty$ and

$$\frac{1}{\varrho_n} \cdot \min\left\{\frac{1}{s_n\sqrt{\log n}},\ \frac{1}{g_n}\right\} \geqq \frac{n^{1/3}}{r_n \log n}.$$

This argument shows why the theorem is stated with $v_n \sim n^{1/3}$ and why the rate-factor is as given in the theorem.

It is known that $|K_v(x)| \leqq (2\pi)^{-1}(v+1)$; see, e.g., § 18.27 in [2]. Therefore $n \geqq n_0 \Rightarrow b_n/s_n \leqq (v_n+1)/(2\pi)(n^{-1}mv_n)^{1/2}$. Since $v_n \sim n^{1/3}$, this bound is eventually smaller than $n/\sqrt{\log n}$; thus (1.1) is satisfied.

Since

$$\left|\frac{d}{dx} K_v(x)\right| \leqq (2\pi)^{-1} \sum_{j=-v}^{v} \left(1 + \frac{|j|}{v+1}\right)|j| = O(v^2),$$

it follows that

$$|\varphi_{nk}(x_1, y) - \varphi_{nk}(x_2, y)| = |K_{v_n}(x_1 - y) - K_{v_n}(x_2 - y)| \leqq |x_2 - x_1| \cdot O(v_n^2).$$

Divide $[-\pi, \pi)$ into n equal subintervals and intersect A with each subinterval. That yields a sequence $(\mathscr{P}_n)_1^\infty$ of partitions of $A=A\cap C_n$ such that

$$n \geqq n_0 \Rightarrow \chi_n \Phi_n(\mathscr{P}_n) = \left(1/(\varrho_n s_n \sqrt{\log n})\right) \Phi_n(\mathscr{P}_n) \leqq$$

$$\leqq \frac{1}{\varrho_n (n^{-1} m v_n)^{1/2} \sqrt{\log n}} \cdot \frac{2\pi}{n} \cdot O(v_n^2) \leqq \left(t_n/(\varrho_n \sqrt{\log n})\right) \cdot O(v_n^2)$$

where $t_n \sim 1/v_n^2$. Therefore $\chi_n \Phi_n(\mathscr{P}_n) \to 0$, i.e. condition (1.2) is satisfied.

In view of (2.3) and Theorem 1, the proof is complete. ∎

The assumption that f vanishes off $[-\pi, \pi)$ allows one to put $\tilde{f}_n$ into a form which offers some distinct computational advantages; see equation (1.3) in [8] and the last paragraph in § 4 of [8]. That cannot be done in the case of a completely arbitrary f but nonetheless one can devise on estimator which is closely related to $\tilde{f}_n$ and which attains the same rate of strong uniform consistency. That estimator, briefly described below, is discussed in § 4 of [8] where a *pointwise* strong consistency theorem is stated; in Theorem 4, below, that is extended to a *uniform* result.

Let $K_v^\#$ be the "basic pattern" of the 2π-periodic Fejér kernel K_v, namely

$$K^\#(x) = K_v(x) I_{[-\pi,\pi)}(x), \quad x \in \mathbf{R}.$$

Let $(v_n)_1^\infty$ be a sequence of positive integers such that $v_n \to \infty$ and put

$$f_n^\#(x) = \frac{1}{n} \sum_{k=1}^{n} K_{v_n}^\#(x - X_k), \quad x \in \mathbf{R}.$$

This is an estimator which is appropriate without any assumptions about the support of f.

THEOREM 4. *Suppose that f is π-Lipschitz on $A \subset \mathbf{R}$, that f is not identically zero on A, and that $(v_n)_1^\infty$ is a sequence of odd integers such that $v_n \sim n^{1/3}$ and that $r_n \to \infty$. If A is bounded then*

$$\lim_{n\to\infty} \left(\frac{n^{1/3}}{r_n \log n} \sup_A |f_n^\#(x) - f(x)| \right) = 0 \quad a.s.;$$

for unbounded A, the same conclusion is valid if $r_n/\sqrt{\log n} \to \infty$ and there exists $\delta > 0$ such that $\mathbf{E}|X|^\delta < \infty$.

PROOF. If A is unbounded and $\mathbf{E}|X|^\delta < \infty$, let $C_n = [-n^{1/\delta}, n^{1/\delta}]$; if A is bounded, let $C_n = [-n, n]$.

For $g \in L_1(\mathbf{R})$, let $\sigma_v^\#(g, x) = \int_{\mathbf{R}} g(t) K_v^\#(x-t)\, dt$. Then $\mathbf{E} f_n^\#(x) = \sigma_v^\#(f, x)$ and the rate-of-approximation result used in Theorem 3 extends to $\sigma_v^\#$: if $g \in \mathbf{L}_1(\mathbf{R})$ and g is π-Lipschitz on $A \subset \mathbf{R}$ then $\sup_{x\in A} |\sigma_v^\#(g, x) - g(x)| = O(v^{-1} \log v)$. It then follows that (2.1) is valid if $\tilde{f}_n$ is replaced by $f_n^\#$. Let $\|f\|_A = \sup\{f(x-y): x \in A, -\pi \leqq y < \pi\}$; this is finite because f is a probability density and is π-Lipschitz. If $x \in A$ then

$$\int_{\mathbf{R}} (K_v^\#(x-y))^2 f(y)\, dy = \int_{-\pi}^{\pi} K_v^2(y) f(x-y)\, dy \leqq \|f\|_A \int_{-\pi}^{\pi} K_v^2(y)\, dy,$$

from which one can show that the asymptotic behavior of $VK_v^{\#}(x-X)$ is the same as that of $VK_v(x-X)$; it follows that (2.2) is again valid. Now (arguing as in the proof of Theorem 3) we conclude that, under the conditions of Theorem 4, (2.3) is valid in relation to $f_n^{\#}$.

Conditions (1.1) and (1.2) are satisfied. For (1.1), argue as in Theorem 3. For (1.2), argue as follows. It is known (see e.g. [2], § 18.27) that

$$K_v(t) = (2\pi)^{-1}\{\sin((v+1)t/2)/\sin(t/2)\}^2;$$

it follows that $K_v(\pm\pi)=0=K_v^{\#}(\pm\pi)$ when v is *odd*. As noted previously,

$$|K_v(x_1)-K_v(x_2)| \leqq |x_2-x_1|\cdot O(v^2).$$

The same is true for $K_v^{\#}$ if $-\pi\leqq x_1, x_2<\pi$; if $-\pi\leqq x_1<\pi\leqq x_2$ then, with v odd,

$$|K_v^{\#}(x_1)-K_v^{\#}(x_2)| \leqq |K_v^{\#}(x_1)-K_v^{\#}(\pi)|+|K_v^{\#}(\pi)-K_v^{\#}(x_2)| =$$
$$= |K_v(x_1)-K_v(\pi)| \leqq |x_2-x_1|\cdot O(v^2).$$

It is likewise in the remaining possible cases, hence $|K_v^{\#}(x_1)-K_v^{\#}(x_2)|\leqq$ $\leqq|x_2-x_1|\cdot O(v^2)$. — If A is unbounded and $\mathbf{E}|X|^{\delta}<\infty$, let L be an integer such that $L\geqq 1+(1/\delta)$; if A is bounded, put $L=2$. Let n_0 be as in (2.2); for $n\geqq n_0$, divide C_n into n^L equal subintervals and intersect A with each of them. The result is a sequence $(\mathscr{P}_n)_1^{\infty}$ of partitions of $A\cap C_n$. Arguing as in Theorem 3 one sees that $\chi_n\Phi_n(\mathscr{P}_n)\to 0$, i.e. condition (1.2) is satisfied.

If A is bounded then $A\cap C_n=A$ for all sufficiently large n; in view of (2.3), the first assertion of Theorem 4 follows from Theorem 1.

Now suppose that $\mathbf{E}|X|^{\delta}<\infty$ and $r_n/\sqrt{\log n}\to\infty$. Then (1.3) is satisfied because $K_n^{\#}(x-y)=0$ as soon as $|x-y|>\pi$.

In view of (2.3), the second assertion of Theorem 4 now follows from Theorem 2. ∎

3. Application to kernel estimators

If f has a bounded second derivative then Theorems 1 and 2 can be applied to obtain results about the rate of convergence of some estimators of the kernel type. In this section we suppose that the X_k's are *real* rv's and that

$$f_n(x) = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_n}h\left(\frac{x-X_k}{a_n}\right), \tag{3.1}$$

where h is a given probability density function and $a_n\to 0$. Two results for this case are stated below. The *second* result is a slight strengthening of results contained in Theorem 7 of [6] or Theorem 1 of [5]. The result is given in [6] with rate-factor $n^{2/5}/n^{\gamma}$, with arbitrary positive γ, and in [5] with the better rate-factor $n^{2/5}/\log n$; the rate factor given here is yet another slight improvement. The *first* result, where A is assumed bounded, is not stated explicitly in [6] or [5] but (again with rate-factor $n^{2/5}/n^{\gamma}$ or $n^{2/5}/\log n$, respectively) it is implicit there. All these results can be extended to the case where the X_k's take values in $\mathbf{R}^p$, see [5].

The "kernel" h and the "unknown" density f may or may not satisfy the following conditions.

(3.2) There is an interval J, not necessarily bounded, such that f vanishes on $\mathbf{R}\setminus J$ and f has a bounded second derivative on the interior of J.

(3.3) There exist constants $C>0$ and $\alpha\in(0,1]$ such that for any real x and y, $|h(x)-h(y)|\leqq C|x-y|^{\alpha}$; furthermore, $\int_{\mathbf{R}} xh(x)\,dx=0$ and $x^4h(x)\to 0$ as $|x|\to\infty$.

For a given interval J, with endpoints a (possibly $-\infty$) and b (possibly $+\infty$), if $\varepsilon>0$ then $J_\varepsilon=\{x\in\mathbf{R}\colon a+\varepsilon\leqq x\leqq b-\varepsilon\}$.

THEOREM 5. *Let f_n be as in* (3.1). *Suppose that conditions* (3.2) *and* (3.3) *are satisfied, that $a_n\sim n^{-1/5}$, that $(r_n)_1^\infty$ is a sequence of constants such that $r_n\to\infty$, and that A is a bounded subset of* $\mathbf{R}$. *Then, for any $\varepsilon>0$ such that f does not vanish identically on $A\cap J_\varepsilon$,*

$$\lim_{n\to\infty}\left(\frac{n^{2/5}}{r_n\sqrt{\log n}}\sup_{A\cap J_\varepsilon}|f_n(x)-f(x)|\right)=0 \quad a.s.$$

PROOF. Let $\varepsilon>0$ be such that f does not vanish identically on $A\cap J_\varepsilon$, put $C_n=A\cap J_\varepsilon$ for each n, and let $\varphi_{nk}(x,y)=a_n^{-1}h\big((x-y)/a_n\big)$ for $1\leqq k\leqq n$. Let s_n, χ_n, b_n, g_n be as in § 1. Note that condition (3.2) implies that f is bounded on J_ε and condition (3.3) implies that h is bounded, say $\sup_{x\in\mathbf{R}}h(x)=H_0<\infty$.

For the case at hand, the rate of asymptotic bias is known; according to Lemma 6a of [6], (3.2) and (3.3) imply

(3.4) $$\exists D>0 \quad \text{such that}$$

$$g_n=\sup_{A\cap J_\varepsilon}|f(x)-\mathbf{E}f_n(x)|\leqq Da_n^2.$$

By Theorem 2A of [4], if $x\in A\cap J_\varepsilon$ then

$$a_n\mathbf{V}\varphi_{nk}(x,X)=na_n\mathbf{V}f_n(x)\to f(x)\int h^2;$$

i.e., $\mathbf{V}\varphi_{nk}(x,X)=t_n(x)/a_n$ with $t_n(x)\to f(x)\int h^2$; since f is bounded on J_ε and h is bounded on $\mathbf{R}$, $\sup\{t_n(x)\colon x\in J_\varepsilon\}$ is finite. Now let $\xi\in A\cap J_\varepsilon$ be such that $f(\xi)\neq 0$; then $t_n(\xi)$ converges to a finite nonzero limit. Since $s_n^2=\sup\{\mathbf{V}\varphi_{nk}(x,X)\colon x\in A\cap J_\varepsilon\}/n$, it follows that

(3.5) there exist m, M, and n_0 such that

$$n\geqq n_0\Rightarrow 0<m/na_n\leqq s_n^2\leqq M/na_n.$$

By (3.4) and (3.5), if $n\geqq n_0$ then

$$\min\left\{\frac{1}{s_n\sqrt{\log n}},\frac{1}{g_n}\right\}\geqq\min\frac{1}{(n^{-1}a_n^{-1}M)^{1/2}\sqrt{\log n}},\frac{1}{Da_n^2}\Big\}.$$

Since $a_n\sim n^{-1/5}$, the first term in braces on the right is $\sim n^{2/5}/\sqrt{\log n}$. Hence the right side of this inequality is $c_n\cdot n^{2/5}/\sqrt{\log n}$ for some sequence $(c_n)_1^\infty$ which converges to a finite nonzero limit. Put $\varrho_n=c_nr_n$; since $r_n\to\infty$, it follows that

(3.6) there exists a sequence $(\varrho_n)_1^\infty$ such that $\varrho_n \to \infty$ and

$$\frac{1}{\varrho_n} \cdot \min\left\{\frac{1}{s_n\sqrt{\log n}}, \frac{1}{g_n}\right\} \geqq \frac{n^{2/5}}{r_n\sqrt{\log n}}.$$

By (3.5), $b_n/s_n \leqq (a_n^{-1}H_0)/\sqrt{m/(na_n)}$ when $n \geqq n_0$. Since $a_n \sim n^{-1/5}$, this bound is eventually smaller than $n/\sqrt{\log n}$; thus condition (1.1) is satisfied.

Let $[c, d]$ be a bounded interval containing A and let L be an integer such that $L \geqq 1/\alpha$, where α is as in condition (3.3). Divide $[c, d]$ into n^L equal subintervals and intersect A with each subinterval. The result is a sequence $(\mathscr{P}_n)_1^\infty$ of partitions of A such that for $n \geqq n_0$

$$\chi_n \Phi_n(\mathscr{P}_n) = \left(1/(\varrho_n s_n \sqrt{\log n})\right) \Phi_n(\mathscr{P}_n) \leqq$$

$$\leqq \frac{1}{\varrho_n \sqrt{\log n}\sqrt{m/(na_n)}} \cdot a_n^{-1} \cdot C\left(\frac{d-c}{n^L}\right)^\alpha = \frac{t_n}{\varrho_n\sqrt{\log n}}$$

with $t_n \sim n^{-2/5}$; therefore $\chi_n \Phi_n(\mathscr{P}_n) \to 0$, i.e. condition (1.2) is satisfied.

In view of (3.6) and Theorem 1, the proof is complete. ∎

If $|X|^\delta$ is integrable then there is no need to introduce a bounded set A.

THEOREM 6. *Let f_n be as in* (3.1). *Suppose that conditions* (3.2) and (3.3) are *satisfied and that, for some* $\delta > 0$, $\mathbf{E}|X|^\delta < \infty$. *If* $a_n \sim n^{-1/5}$ *and* $r_n \to \infty$ *then, for any* $\varepsilon > 0$ *such that f does not vanish identically on* J_ε,

$$\lim_{n\to\infty}\left(\frac{n^{2/5}}{r_n\sqrt{\log n}} \sup_{J_\varepsilon} |f_n(x) - f(x)|\right) = 0 \quad a.s..$$

PROOF. Let δ be such that $\mathbf{E}|X|^\delta < \infty$ and let $C_n = [-n^{1/\delta}, n^{1/\delta}]$. Let $\varepsilon > 0$ be such that f does not vanish identically on J_ε and put $A = J_\varepsilon$. The arguments used in the proof of Theorem 5 show that (3.4), (3.5), (3.6) are again valid and that condition (1.1) is satisfied. To show that condition (1.2) is satisfied, divide C_n into n^L equal subintervals, with $L \geqq \alpha^{-1} + \delta^{-1}$, and argue as in Theorem 5. [Conditions (1.2) and (1.3) involve χ_n, hence ϱ_n; the ϱ_n's involved here are the terms $c_n r_n$ introduced in deriving (3.6).]

Finally, a simple calculation shows that $|x|^4 h(x) \to 0$ implies that condition (1.3) is also satisfied. — Now the desired result is a direct consequence of (3.6) and Theorem 2. ∎

The convergence rate of some kernel estimators was obtained above by applying Theorems 1 and 2. It should be noted that this technique does not apply to the "naive" estimator, obtained from $h = I_{[-1/2, +1/2]}$, because this kernel does not satisfy condition (3.3). To handle such a discontinuous kernel, one must resort to a more delicate analysis. Instead of arguing as in the forthcoming Lemma 3 (in § 4), one can argue as on pp 103—4 of [6] and demonstrate strong uniform consistency with rate-factor $n^{2/5}/n^\gamma$, $\gamma > 0$.

4. Proof of Theorems 1 and 2

LEMMA 1. *Let Y be a rv which is a.s. bounded by a finite positive b. If $0<\varepsilon\leqq 1/(2b)$ then*

$$\mathbf{E}e^{\varepsilon(Y-\mathbf{E}Y)} \leqq e^{\varepsilon^2 \mathbf{V}Y}.$$

PROOF. It is not hard to show that

$$-\infty < x \leqq 1 \Rightarrow 1+x+x^2 \geqq e^x.$$

Since $|Y|\leqq b$ and $0<\varepsilon\leqq 1/(2b)$, one has $\varepsilon(Y-\mathbf{E}Y)\leqq\varepsilon(2b)\leqq 1$ and hence

$$1+\varepsilon(Y-\mathbf{E}Y)+\varepsilon^2(Y-\mathbf{E}Y)^2 \geqq e^{\varepsilon(Y-\mathbf{E}Y)}.$$

Therefore

$$\mathbf{E}e^{\varepsilon(Y-\mathbf{E}Y)} \leqq 1+\varepsilon^2\mathbf{V}Y \leqq e^{\varepsilon^2\mathbf{V}Y}. \qquad \blacksquare$$

LEMMA 2. *Suppose that condition* (1.1) *is satisfied. Then, for given $\varepsilon>0$ and $K>0$, there exists $n_{\varepsilon K}$ (not depending on x) such that*

$$x\in A\cap C_n \quad \text{and} \quad n \geqq n_{\varepsilon K} \Rightarrow \mathbf{P}[\chi_n|f_n(x)-\mathbf{E}f_n(x)| > \varepsilon] \leqq n^{-K}.$$

PROOF. By (1.1) $\exists M$ such that

$$\frac{1}{2b_n} \geqq \frac{\sqrt{\log n}}{2Mns_n} = \frac{\chi_n \varrho_n \log n}{2Mn}.$$

If $x\in A\cap C_n$ then, by the exponential form of Chebyshev's inequality and Lemma 1,

$$\mathbf{P}[\chi_n(f_n(x)-\mathbf{E}f_n(x)) > \varepsilon] =$$

$$= \mathbf{P}\left[\left(\frac{1}{2M}\chi_n\varrho_n\log n\right)\frac{1}{n}\sum_{k=1}^{n}(\varphi_{nk}(x, X_k)-\mathbf{E}\varphi_{nk}(x, X_k)) > \frac{1}{2M}\varepsilon\varrho_n\log n\right] \leqq$$

$$\leqq e^{-\frac{1}{2M}\varepsilon\varrho_n\log n}\,\mathbf{E}\exp\left\{\left(\frac{1}{2M}\chi_n\varrho_n\log n\right)\frac{1}{n}\sum_{k=1}^{n}(\varphi_{nk}(x, X_k)-\mathbf{E}\varphi_{nk}(x, X_k))\right\} =$$

$$= \exp\left\{-\frac{1}{2M}\varepsilon\varrho_n\log n\right\}\cdot\prod_{k=1}^{n}\mathbf{E}\exp\left\{\frac{\chi_n\varrho_n\log n}{2Mn}(\varphi_{nk}(x, X_k)-\mathbf{E}\varphi_{nk}(x, X_k))\right\} \leqq$$

$$\leqq \exp\left\{-\frac{1}{2M}\varepsilon\varrho_n\log n\right\}\prod_{k=1}^{n}\exp\left\{\frac{\log n}{4M^2n^2s_n^2}\mathbf{V}\varphi_{nk}(x, X)\right\} \leqq$$

$$\leqq \exp\left\{-\frac{1}{2M}\varepsilon\varrho_n\log n\right\}\left(\exp\left\{\frac{\log n}{4M^2n}\right\}\right)^n = n^{\xi_n}$$

where $\xi_n=1/(4M^2)-\varepsilon\varrho_n/(2M)$. An identical argument shows that $\mathbf{P}[\chi_n(f_n(x)-\mathbf{E}f_n(x))<-\varepsilon]\leqq n^{\xi_n}$. The proof is completed by noting that, since $\varrho_n\to\infty$, n^{ξ_n} is eventually smaller than n^{-K}. $\blacksquare$

LEMMA 3. *Suppose that conditions* (1.1) *and* (1.2) *are satisfied. Then*

$$\lim_{n\to\infty}\left(\chi_n \sup_{A\cap C_n}|f_n(x)-\mathbf{E}f_n(x)|\right) = 0 \quad a.s..$$

PROOF. Let $(\mathscr{P}_n)_1^\infty$ be a sequence of partitions satisfying the conditions in (1.2), with $\mathscr{P}_n=(A_{n1}, \ldots, A_{nk_n})$. In each A_{nj}, select an arbitrary point y_{nj} and call these points "distinguished". If $x\in A\cap C_n$, use x_n to designate the "distinguished" point which is in the subset A_{nj} that contains x. Now, for any x,

(a) $\chi_n|f_n(x)-\mathbf{E}f_n(x)|\leqq\alpha_n(x)+\beta_n(x)+\gamma_n(x)$ where

$$\alpha_n(x) = \chi_n|f_n(x)-f_n(x_n)|,$$

$$\beta_n(x) = \chi_n|f_n(x_n)-\mathbf{E}f_n(x_n)|,$$

$$\gamma_n(x) = \chi_n|\mathbf{E}f_n(x_n)-\mathbf{E}f_n(x)|.$$

Let L be an integer such that $\#\mathscr{P}_n\leqq n^L$; by (1.2), such an L exists. Put $K=L+2$ and fix $\varepsilon>0$. By Lemma 2, if $n\geqq n_{\varepsilon K}$ then

$$\mathbf{P}\Big[\sup_{A\cap C_n}\beta_n(x)>\varepsilon\Big]=\mathbf{P}\Big[\sup_{1\leqq j\leqq k_n}\chi_n|f_n(y_{nj})-\mathbf{E}f_n(y_{nj})|>\varepsilon\Big]\leqq$$

$$\leqq\sum_{j=1}^{k_n}\mathbf{P}[\chi_n|f_n(y_{nj})-\mathbf{E}f_n(y_{nj})|>\varepsilon]\leqq k_n\cdot n^{-K}\leqq n^L\cdot n^{-K}=n^{-2}.$$

By a standard Borel—Cantelli argument,

(b) $$\mathbf{P}\Big[\lim_{n\to\infty}\sup_{A\cap C_n}\beta_n(x)=0\Big]=1.$$

Since x and x_n always belong to the same subset in the partition,

(c) $$\sup_{A\cap C_n}\gamma_n(x)=\chi_n\sup_{A\cap C_n}\left|\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\{\varphi_{nk}(x_n, X_k)-\varphi_{nk}(x, X_k)\}\right|\leqq$$

$$\leqq\sup_{A\cap C_n}\frac{1}{n}\sum_{k=1}^{n}\chi_n\mathbf{E}|\varphi_{nk}(x_n, X_k)-\varphi_{nk}(x, X_k)\leqq\chi_n\Phi_n(\mathscr{P}_n);$$

in view of (1.2), this goes to zero as $n\to\infty$. Likewise (removing "E" in the above steps),

(d) $$\sup_{A\cap C_n}\alpha_n(x)\leqq\chi_n\Phi_n(\mathscr{P}_n)\to 0.$$

The desired result now follows from (a)—(d). ∎

THEOREM 1. *If conditions* (1.1) *and* (1.2) *are satisfied then*

$$\lim_{n\to\infty}\left(\frac{1}{\varrho_n}\cdot\min\left\{\frac{1}{s_n\sqrt{\log n}}, \frac{1}{g_n}\right\}\cdot\sup_{A\cap C_n}|f_n(x)-f(x)|\right)=0\quad a.s..$$

PROOF. Since $\chi_n=1/(\varrho_n s_n\sqrt{\log n})$ and

$$\frac{1}{\varrho_n}\cdot\min\left\{\frac{1}{s_n\sqrt{\log n}}, \frac{1}{g_n}\right\}\cdot|f_n(x)-f(x)|\leqq$$

$$\leqq\frac{1}{\varrho_n s_n\sqrt{\log n}}|f_n(x)-\mathbf{E}f_n(x)|+\frac{1}{\varrho_n}\frac{|\mathbf{E}f_n(x)-f(x)|}{g_n},$$

the result follows from Lemma 3 and the definition of g_n. ∎

LEMMA 4. *If* $E\|X\|^{\delta}<\infty$ *for some* $\delta>0$ *then, for any* $c>0$, $\exists\Omega_0$ *such that* $\mathbf{P}(\Omega_0)=1$ *and*

$$(\forall\,\omega\in\Omega_0)(\exists\,n_\omega)(\forall\,n>n_\omega)\big(n_\omega<k\leqq n\Rightarrow\|X_k(\omega)\|\leqq cn^{1/\delta}\big).$$

PROOF. This is an easy consequence of the Borel—Cantelli lemma; see Lemma 12 in [6] or Lemma 6 in [5]. ∎

LEMMA 5. *Suppose that, for some* $\delta>0$, $\mathbf{E}\|X\|^{\delta}<\infty$. *Suppose also that, with this* δ *and* $C_n=\{x\in\mathbf{R}^p\colon\|x\|\leqq n^{1/\delta}\}$, *conditions* (1.1) *and* (1.3) *are satisfied. If* $T_n=\{x\colon\|x\|>n^{1/\delta}\}$ *then*

$$\lim_{n\to\infty}\big(\chi_n\sup_{T_n}|f_n(x)-\mathbf{E}f_n(x)|\big)=0\quad a.s.$$

PROOF. Let $\delta>0$ be such that $\mathbf{E}\|X\|^{\delta}<\infty$ and put $C_n=\{x\in\mathbf{R}^p\colon\|x\|\leqq n^{1/\delta}\}$; use this C_n to define s_n and χ_n as in § 1. Let $c\in(0,1)$ be such that if

$$d_n=\chi_n\max_{1\leqq k\leqq n}\sup\{|\varphi_{nk}(x,y)|\colon\|x-y\|>cn^{1/\delta}\}$$

then $\lim_{n\to\infty}d_n=0$; by (1.3) such a c exists. Having chosen δ and c, let

$$A_n(x)=\{y\in\mathbf{R}^p\colon\|x-y\|>cn^{1/\delta}\}.$$

(a) For any x and any ω,

$$I_{A_n(x)}\big(X_k(\omega)\big)=1\Leftrightarrow\|x-X_k(\omega)\|>cn^{1/\delta};$$

consequently,

$$\chi_n\cdot\frac{1}{n}\sum_{k=1}^{n}I_{A_n(x)}\big(X_k(\omega)\big)\cdot\big|\varphi_{nk}\big(x,X_k(\omega)\big)\big|\leqq\frac{1}{n}\sum_{k=1}^{n}d_n=d_n.$$

(b) Consider an arbitrary $x\in T_n=\{x\in\mathbf{R}^p\colon\|x\|>n^{1/\delta}\}$. If $y\notin A_n(x)$ then $\|y\|>(1-c)n^{1/\delta}$; indeed, if this were not the case then we would have $\|y\|\leqq(1-c)n^{1/\delta}$ and $\|x-y\|\leqq cn^{1/\delta}$, which implies $\|x\|\leqq n^{1/\delta}$, but that contradicts $x\in T_n$. In contrapositive form,

$$(*)\qquad\|y\|\leqq(1-c)^{1/\delta}\Rightarrow y\in A_n.$$

By Lemma 4, $\exists\Omega_0$ such that $\mathbf{P}(\Omega_0)=1$ and

$$(\forall\,\omega\in\Omega_0)(\exists\,n_\omega)(\forall\,n>n_\omega)\big(n_\omega<k\leqq n\Rightarrow\|X_k(\omega)\|\leqq(1-c)n^{1/\delta}\big).$$

Consider an arbitrary $\omega\in\Omega_0$. If $n_\omega<k\leqq n$ then, by $(*)$, $I_{\mathbf{R}^p\setminus A_n(x)}\big(X_k(\omega)\big)=0$. Therefore

$$\chi_n\cdot\frac{1}{n}\sum_{k=1}^{n}I_{\mathbf{R}^p\setminus A_n(x)}\big(X_k(\omega)\big)\cdot\big|\varphi_{nk}\big(x,X_k(\omega)\big)\big|=$$

$$=\chi_n\cdot\frac{1}{n}\sum_{k=1}^{n_\omega}I_{\mathbf{R}^p\setminus A_n(x)}\big(X_k(\omega)\big)\cdot\big|\varphi_{nk}\big(x,X_k(\omega)\big)\big|\leqq\frac{\chi_n b_n}{n}\cdot n_\omega.$$

(c) If $x\in T_n$ and $\omega\in\Omega_0$ then, from (a) and (b),

$$\chi_n|f_n(x)|\leqq\chi_n\cdot\frac{1}{n}\sum_{k=1}^{n}\big|\varphi_{nk}\big(x,X_k(\omega)\big)\big|\leqq d_n+\frac{\chi_n b_n}{n}\cdot n_\omega=d_n+\frac{b_n\sqrt{\log n}}{ns_n}\cdot\frac{n_\omega}{\varrho_n\log n}$$

which goes to zero by (1.3) and (1.1). Therefore

$$\lim_{n\to\infty}(\chi_n \sup_{T_n}|f_n(x)|)=0.$$

(d) Using the inequality obtained in (a),

$$\chi_n \mathbf{E}|f_n(x)| \leqq \chi_n \frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\{I_{A_n(x)}(X_k)\cdot|\varphi_{nk}(x, X_k)|\}+$$

$$+\chi_n \frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\{I_{\mathbf{R}^p\setminus A_n(x)}(X_k)\cdot|\varphi_{nk}(x, X_k)|\}\leqq$$

$$\leqq d_n+n^{-1}\chi_n b_n\sum_{k=1}^{n}\mathbf{P}[X_k\notin A_n(x)].$$

If $x\in T_n$ then, by $(*)$ and Chebyshev's inequality,

$$n^{-1}\chi_n b_n\sum_{k=1}^{n}\mathbf{P}[X_k\notin A_n(x)]\leqq n^{-1}\chi_n b_n\sum_{k=1}^{n}\mathbf{P}[\|X_k\|>(1-c)n^{1/\delta}]\leqq$$

$$\leqq n^{-1}\chi_n b_n\sum_{k=1}^{n}((1-c)n^{1/\delta})^{-\delta}\mathbf{E}\|X_k\|^{\delta}=(\text{const.})n^{-1}\chi_n b_n.$$

Therefore

$$\chi_n \sup_{T_n}|\mathbf{E}f_n(x)|\leqq d_n+(\text{const.})n^{-1}\chi_n b_n=$$

$$=d_n+(\text{const.})\cdot(b_n\sqrt{\log n}/(ns_n))\cdot(1/(\varrho_n\log n)),$$

which goes to zero by (1.3) and (1.1). In view of (c), the proof is complete. ∎

THEOREM 2. *Suppose that, for some* $\delta>0$, $\mathbf{E}\|X\|^{\delta}<\infty$. *Suppose also that, with this* δ *and* $C_n=\{x\in\mathbf{R}^p\colon \|x\|\leqq n^{1/\delta}\}$, *conditions* (1.1), (1.2), *and* (1.3) *are satisfied. Then*

$$\lim_{n\to\infty}\left(\frac{1}{\varrho_n}\cdot\min\left\{\frac{1}{s_n\sqrt{\log n}},\frac{1}{g_n}\right\}\cdot\sup_{A}|f_n(x)-f(x)|\right)=0 \quad a.s.$$

PROOF. Let $\delta>0$ be such that $\mathbf{E}\|X\|^{\delta}<\infty$ and put $C_n=\{x\in\mathbf{R}^p\colon \|x\|\leqq n^{1/\delta}\}$; use this C_n to define s_n and χ_n as in § 1. Let $T_n=\mathbf{R}^p\setminus C_n=\{x\colon \|x\|>n^{1/\delta}\}$. Since $\chi_n=1/(\varrho_n s_n\sqrt{\log n})$, we have

$$\frac{1}{\varrho_n}\cdot\min\left\{\frac{1}{s_n\sqrt{\log n}},\frac{1}{g_n}\right\}\sup_{A}|f_n(x)-f(x)|\leqq$$

$$\leqq \chi_n\sup_{A\cap C_n}|f_n(x)-\mathbf{E}f_n(x)|+\chi_n\sup_{A\cap T_n}|f_n(x)-\mathbf{E}f_n(x)|+$$

$$+\frac{1}{\varrho_n g_n}\sup_{A}|\mathbf{E}f_n(x)-f(x)|.$$

The desired result now follows from Lemmata 3 and 5 and the definition of g_n. ∎

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NOTE ON BASU'S LEMMA

by

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1. Introduction

Let $\mathbf{P}=\{P_\theta\colon \theta\in\Gamma\}$ be a family of probability measures (p.m.'s) defined on a measurable space $(\mathbf{X}, \mathbf{A})$. In [2], BASU defined two probability measures P_{θ_1} and P_{θ_2} to be *overlapping* if for each $B\in\mathbf{A}$, $P_{\theta_1}(B)=1$ implies $P_{\theta_2}(B)>0$. Two p.m.'s P_θ and P_φ are then said to be *connected* if for some $n\geqq 2$ there exist p.m.'s $P_{\theta_1}, \ldots, P_{\theta_2}$ with $\theta_1=\theta$ and $\theta_n=\varphi$ such that P_{θ_i} and $P_{\theta_{i+1}}$ overlap $(i=1, \ldots, n-1)$. It is readily seen that *connectedness* is an equivalence relation. The family $P=\{P_\theta\colon \theta\in\Gamma\}$ is called connected if for each pair $\theta, \varphi\in\Gamma$, the p.m.'s P_θ and P_φ are connected. In [2], BASU established the following theorem concerning statistics which are independent of a sufficient statistic:

THEOREM 1. *Let S be a sufficient statistic for the family* $(\mathbf{X}, \mathbf{A}, \mathbf{P})$ *and suppose that P is connected. Let V be a statistic which is distributed independently of S for each $P_\theta\in\mathbf{P}$. Then the distribution of V is free of $\theta\in\Gamma$.*

The above theorem has now been strengthened by KOEHN and THOMAS [5] as follows:

THEOREM 2. *Let S be a sufficient statistic for the family* $(\mathbf{X}, \mathbf{A}, \mathbf{P})$. *Then every statistic independent of S, for all $P_\theta\in\mathbf{P}$, is distribution free if and only if there does not exist a measurable set A such that for each $\theta\in\Gamma$, $P_\theta(A)=0$ or 1 and for at least one pair $(\theta_1, \theta_2)\in\Gamma$, $P_{\theta_1}(A)=P_{\theta_2}(A^\sim)=1$.* (Such an A is called a *measure-splitting set*).

A comparison of these theorems shows that for the nonexistence of a statistic independent of a sufficient statistic and whose distribution depends on $\theta\in\Gamma$, connectedness of $\mathbf{P}=\{P_\theta\colon \theta\in\Gamma\}$ is a sufficient condition while the nonexistence of a measure-splitting set A, i.e. $P_\theta(A)=0$ or 1 $\forall\theta$ and $P_{\theta_1}(A)=P_{\theta_2}(A^\sim)=1$ for at least one pair $(\theta_1, \theta_2)\in\Gamma$, is both necessary and sufficient.

2. Two lemmas

The object of this note is to point out that at least in two important cases, Basu's condition is both necessary and sufficient. That this is indeed so in discrete probability models with the σ-algebra of all subsets is an immediate consequence of the following lemma.

LEMMA 1. *Let $(X, \mathbf{A})$ be a measurable space in which A consists of all subsets of X.* Let $P=\{P_\theta\colon \theta\in\Gamma\}$ be a family of discrete probability measures on $(X, \mathbf{A})$. Then the family $\mathbf{P}$ is *connected* if and only if there exists no measure-splitting set $A\in\mathbf{A}$.

PROOF. We assume without loss of generality that for each $x\in\mathbf{X}$, $P_\theta(\{x\})>0$ for some $\theta\in\Gamma$. For each $\theta\in\Gamma$, let A_θ be the essential range of P_θ, i.e. $P_\theta(A_\theta)=1$ and $P_\theta(B)>0$ if $\emptyset\neq B\subset A_\theta$. That connectedness of $\mathbf{P}$ implies the nonexistence of a desired measure-splitting set is trivial. Now suppose that $\mathbf{P}$ is not connected, and for each θ, $\varphi\in\Gamma$ define the equivalence relation: $P_\theta\sim P_\theta$ if P_θ and P_φ are connected. Pick a $\varphi\in\Gamma$ and let $\mathbf{P}_1=\{P_\theta\colon P_\theta\in\mathbf{P}$ and $P_\theta\sim P_\varphi\}$. Since $\mathbf{P}$ is not connected, $\mathbf{P}_2=\mathbf{P}-\mathbf{P}_1\neq\emptyset$. Let $\Gamma_1=\{\theta\colon P_\theta\in\mathbf{P}_1\}$ and $\Gamma_2=\{\theta\colon P_\theta\in\mathbf{P}_2\}$. It is easily seen that $\Gamma_1\cup\Gamma_2=\Gamma$ and for each $\theta_1\in\Gamma_1$ and $\theta_2\in\Gamma_2$, $A_{\theta_1}\cap A_{\theta_2}=\emptyset$. Let $A=\bigcup_{\theta_1\in\Gamma_1} A_{\theta_1}$. It can now be seen that A is a measure-splitting set since $P_{\theta_1}(A)=P_{\theta_2}(A^\sim)=1$ for all $\theta_1\in\Gamma_1$ and $\theta_2\in\Gamma_2$. This completes the proof. □

It would now be interesting to inquire if Basu's condition and that of KOEHN and THOMAS [5], are indeed equivalent if the probability model is dominated by a σ-finite measure. The answer in this case is also in the affirmative.

LEMMA 2. *Let* $(X, \mathbf{A})$ *be a given measurable space. Let* $\mathbf{P}=\{P_\theta\colon \theta\in\Gamma\}$ *be a family of probability measures defined on* $(X, \mathbf{A})$ *and suppose that the family P is dominated by a* σ*-finite measure* μ*. Then the family* $\mathbf{P}$ *is connected if and only if there exists no measure-splitting set* $A\in\mathbf{A}$.

PROOF. This is analogous to that of Lemma 1. Let $f(x,\theta)=(dP_\theta/d\mu)(x)$ denote a Radon—Nikodym derivative of P_θ with respect to μ, and let $A_\theta=\{x\colon f(x,\theta)>0\}$ for each $\theta\in\Gamma$. Then the set A_θ is an essential range of the p.m. P_θ, $\theta\in\Gamma$. As in Lemma 1, it suffices to show that if $\mathbf{P}$ is not connected, $\mathbf{P}$ admits a measure-splitting set A. Now pick a $\varphi\in\Gamma$, let $\mathbf{P}_1=\{P_\theta\colon P_\theta\sim P_\varphi,\ P_\theta\in\mathbf{P}\}$. Let $\mathbf{P}_2=\mathbf{P}-\mathbf{P}_1$. Since $\mathbf{P}$ is not connected, $\mathbf{P}_2$ is nonempty. Next since the family $\mathbf{P}_1$ is dominated by the σ-finite measure μ, it follows from the HALMOS—SAVAGE Lemma [4] that there exists a countable set in $\mathbf{P}_1$, say $\mathbf{P}_0$, such that if $P_\theta(N)=0$ for each P_θ in $\mathbf{P}_0$ then $P_\theta(N)=0$ for all P_θ in $\mathbf{P}_1$. Let $\Gamma_0=\{\theta\colon P_\theta\in\mathbf{P}_0\}$, $\Gamma_1=\{\theta\colon P_\theta\in\mathbf{P}_1\}$ and $\Gamma_2=\{\theta\colon P_\theta\in\mathbf{P}_2\}$. Let $A=\bigcup_{\theta\in\Gamma_0} A_\theta$. Then A, being the union of countably many measurable sets, is measurable. Trivially $P_\theta(A)=1$ for each $\theta\in\Gamma_0$ so $P_\theta(A^\sim)=0$ for each $\theta\in\Gamma_0$. Consequently $1-P_\theta(A)=P_\theta(A^\sim)=0$ for each $\theta\in\Gamma_1$. Since none of the p.m.'s in $\mathbf{P}_2$ are connected with those of $\mathbf{P}_1$, and since $A_\theta=\{x\colon f(x,\theta)>0\}$, it follows that $P_{\theta_2}(A\cap A_{\theta_2})=0$ for all $\theta_2\in\Gamma_2$. Consequently $P_{\theta_2}(A^\sim)=1\,\forall\theta_2\in\Gamma_2$. Thus A is the desired measure splitting set. □

In conclusion we furnish an example to show that the Koehn—Thomas condition is indeed weaker than that of Basu in a general setting.

EXAMPLE. Let $\mathbf{P}$ denote the following family of discrete probability measures defined on the real line

$$\mathbf{P}=\{P\colon P(x)=P(y)=\frac{1}{2} \text{ where } y>x\geqq 0\}.$$

Let Q be the probability measure induced by the standard normal density on the real line. Consider now the probability space $(\mathbf{X}, \mathbf{A}, \mathbf{P}\cup Q)$ in which $\mathbf{X}$ is the real line and $\mathbf{A}$ the class of all Borel sets. The family $\mathbf{P}\cup Q$ of p.m.'s is clearly not con-

nected for Q is not connected to any member of **P**, but it does not possess a measure-splitting set. (Note that the half-line is the only candidate for the measure-splitting set and it carries probability 1/2 under Q.)

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SCHNITTE VON ZUFÄLLIGEN GERADEN DURCH ENDLICHE PUNKTMENGEN IN DER EBENE

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Zusammenfassung

Gegeben sei eine endliche Punktmenge P_n in der Euklid'schen Ebene. Wir untersuchen die Maße von Mengen von zufälligen ungerichteten Geraden, die die Punktmenge in zwei Teile zerlegen; l sei die Anzahl der Punkte in dem kleineren dieser beiden Teile. Eine von AMBARTZUMIAN angegebene Formel für diese Maße wird mit Hilfe des Seilliniensatzes von CROFTON neu bewiesen. Als Separationsproblem bezeichnen wir die Bestimmung der Maße der zu jedem l gehörenden Geradenmengen; sie lassen sich durch die zweiten Differenzen der „inneren Umfänge" von P_n ausdrücken. Die inneren Umfänge charakterisieren die Geometrie der Punktmenge P_n; im Anschluß geben wir für sie einige Ungleichungen an. Als zweite Anwendung betrachten wir das BUFFON-SYLVESTER-Poblem für konvexe Polygone. Gegeben seien endlich viele konvexe Polygone; wir berechnen dann die Maße der Geradenmengen, deren Geraden eine vorgegebene Anzahl der Polygone schneiden.

1. Einleitung

Gegeben sei eine endliche feste Punktmenge

$$P_n = \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\} \quad (n = 2, 3, \ldots)$$

in der Euklid'schen Ebene R^2. Der Koordinatenursprung $\mathbf{o}$ liege auf keiner Geraden, die zwei Punkte aus P_n enthält. $\mathbf{G}$ sei die Menge der zufälligen ungerichteten Geraden in der Ebene; sie können in der Form:

$$G = \{\mathbf{x};\ (\mathbf{x}, \mathbf{e}) = p\} \quad \text{mit} \quad \mathbf{e} \in R^2, \quad |\mathbf{e}| = 1 \quad \text{und} \quad p \geq 0 \tag{1}$$

geschrieben werden[1]. Weiter seien

$$H_{+,-} = \{\mathbf{x};\ (\mathbf{x}, \mathbf{e}) \gtrless p\}$$

die von G bestimmten offenen Halbebenen, sowie

$$P_{+,-} = P_{+,-}(G) = P_n \cap H_{+,-}$$

und

$$P_0 = P_0(G) = P_n \cap G$$

die Teilmengen von P_n, die in den Halbebenen H_+ und H_- bzw auf G selbst liegen. Wir definieren noch folgende Indexmengen:

$$N_{+,-,0} = N_{+,-,0}(G) = \{i \in N_n;\ \mathbf{p}_i \in P_{+,-,0}(G)\}$$

[1] Das Klammersymbol (,) kennzeichnet das Skalarprodukt.

mit $N_n=\{1, 2, \ldots, n\}$, sowie[2]

$$N = N(G) = \begin{cases} N_+ & \text{für}^{2)} \quad |N_+| < |N_-|, \\ N_- & \text{für} \quad |N_-| < |N_+|, \end{cases}$$

und für $|N_+|=|N_-|$ setzen wir N gleich der Indexmenge N_+ oder N_-, die den kleinsten Index enthält. Die Menge der N sei $\mathbf{N}$. Umgekehrt sei jedem vorgegebenen N, $N_0 \subset N_n$ mit $N \cap N_0 = \varnothing$ und $|N| \leqq \left[\frac{n}{2}\right]$ die Geradenmenge[3]

$$S(N, N_0) = \{G \in \mathbf{G};\ N = N(G),\ N_0 = N_0(G),\ \mathbf{o} \notin G\}$$

zugeordnet. Wir setzen speziell $S_N = S(N, \varnothing)$ und nennen S_N einen Schnitt durch P_n. Mit μ bezeichnen wir das rotations- und translationsinvariante Geradenmaß, [2], [3], [5]. Wir interessieren uns für die Maße $\mu_N = \mu(S_N)$, für die AMBARTZUMIAN ([1b], Seite 60, Formel (4.1)) im Jahre 1973 eine Formel angegeben hat (vergl. auch [1a]). Wir werden einen neuen Beweis dieser Formel, der auf dem Seillinien-satz von CROFTON (vergl. [2], Seite 15 und 16, Formel (77)) aufbaut, angeben.

2. Berechnung der Maße μ_N

Es sei $N_0 = \varnothing$; die Menge der Geraden mit $N_0 \neq \varnothing$ hat das Maß Null. $\mathbf{G}_0 \subset \mathbf{G}$ sei die Menge der Geraden, die $\mathbf{o}$ nicht enthalten. Wir definieren die bijektive Abbildung:

$$v\colon \mathbf{G}_0 \leftrightarrow R^2 - \{\mathbf{o}\},$$

indem wir jeder Geraden $G \in \mathbf{G}_0$ den Fußpunkt des Lotes auf $\mathbf{o}$, das ist ep, zuordnen, also $v(G) = ep$ (vergl. (1)). Sei

$$C_i = \{\mathbf{x};\ (\mathbf{x}, \mathbf{x}-\mathbf{p}_i) = 0\},$$

das ist der Kreis mit dem Mittelpunkt $\mathbf{p}_i/2$ und dem Radius $|\mathbf{p}_i|/2$. Beachte, daß $\mathbf{o} \in C_i$ für $i = 1, 2, \ldots, n$ ist. Weiter sei

$$\dot{C}_i = C_i - \{\mathbf{o}\},$$

dann gilt:

$$v(\{G;\ \mathbf{p}_i \in G \in \mathbf{G}_0\}) = \dot{C}_i.$$

D_i (D_i') sei das Innere (Äußere) von C_i. Für $G \in \mathbf{G}_0$ liegt $\mathbf{p}_i$ genau dann in der entgegensetzten (gleichen) Halbebene wie $\mathbf{o}$, wenn $v(G) \in D_i$ (bzw. $v(G) \in D_i'$) ist. Die Punkte aus $\{\mathbf{p}_i;\ i \in N\}$ und kein anderer aus P_n liegen genau dann auf der entgegensetzten (gleichen) Seite wie $\mathbf{o}$, wenn

$$v(G) \in \bigcap_{i \in N} D_i \cap \bigcap_{i \notin N} D_i' \quad \left(\text{bzw. } v(G) \in \bigcap_{i \notin N} D_i \cap \bigcap_{i \in N} D_i'\right)$$

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[2] Die senkrechten Striche bei Mengen bezeichnen die Anzahl der Elemente der Menge.

[3] Die Geraden durch $\mathbf{o}$ werden ausgeschlossen; sie haben das Maß Null.

ist. Die Lage des Ursprungs **o** soll unberücksichtigt bleiben. Wir definieren

$$B_N = \left(\bigcap_{i\in N} D_i \cap \bigcap_{i\notin N} D_i'\right) \cup \left(\bigcap_{i\notin N} D_i \cap \bigcap_{i\in N} D_i'\right)$$

für $N \in \mathbf{N}$. Es gilt

$$v(S_N) = B_N .$$

Jedem Schnitt S_N entspricht eineindeutig eine Zelle B_N in dem Kreisnetz

$$C = \bigcup_{i=1}^{n} \dot{C}_i$$

(vergl. Abb. 1), wobei noch folgendes zu beachten ist:

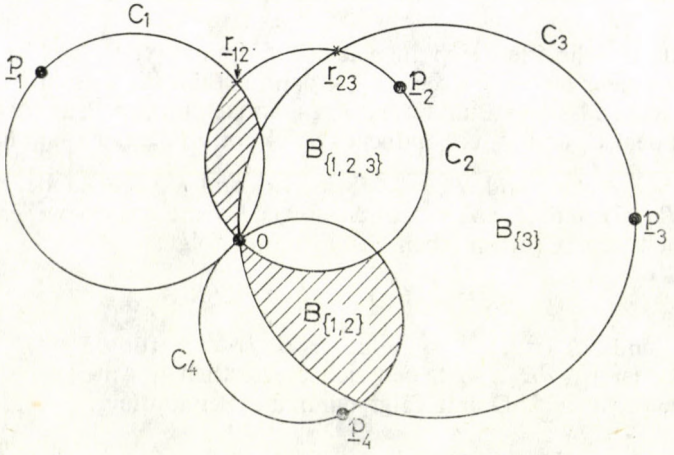

Abb. 1

Ist **o** nicht Randpunkt von B_N, so ist B_N eine einfach zusammenhängende Menge; ∂B_N sei der Rand von B_N. Ist **o** dagegen Randpunkt, so besteht B_N aus zwei disjunkten nicht zusammenhängenden Teilen. Es existiert dann eine Strecke, die **o** und Punkte aus beiden Teilen enthält und bis auf **o** ganz in B_N liegt. **o** soll nicht zu ∂B_N gezählt werden.

Wir benötigen noch einige Definitionen. K_{ij} sei die Kante zwischen $\mathbf{p}_i$ und $\mathbf{p}_j$; G_{ij} die Gerade durch $\mathbf{p}_i$ und $\mathbf{p}_j$. $\delta_{ij} = L(K_{ij})$ sei die Länge[4] von K_{ij}. $\mathbf{r}_{ij}$ sei der Schnittpunkt von $\dot{C}_i$ und $\dot{C}_j$. P_n heiße „nicht entartet", wenn niemals drei Punkte aus P_n auf einer Geraden liegen. P_n ist genau dann nicht entartet, wenn die Punkte $\mathbf{r}_{ij}$ paarweise verschieden sind. Wir definieren für $N \in \mathbf{N}$ und $i, j = 1, 2, \ldots, n$ mit $i<j$:

$$\tau_N(i,j) = \begin{cases} -1 & \text{für} \quad \varnothing \neq B_N \subset (D_i \cap D_j) \cup (D_i' \cap D_j') \\ +1 & \text{für} \quad \varnothing \neq B_N \subset (D_i \cap D_j') \cup (D_i' \cap D_j) \\ 0 & \text{für} \quad \varnothing = B_N \end{cases} \tag{2}$$

[4] $L(\ )$ bezeichnet die Länge einer Kurve.

Sei $N \neq \varnothing$ (es gilt $\mu(S_\varnothing) = \infty$) und p und φ die Polarkoordinaten von $\mathfrak{e}p$ (vergl. (1)), dann gilt, [2], [5]:

$$\mu_N = \int\int_{B_N} dp\, d\varphi.$$

Diese Integrale lassen sich nun mit Hilfe des Seilliniensatzes bestimmen. Es sind höchstens $\binom{n}{2}$ der Maße μ_N verschieden von 0 und ∞; diese obere Schranke wird genau dann angenommen, wenn P_n nicht entartet ist; [4a]. Es gilt der

SATZ 1.

$$\mu_N = \sum_{\mathfrak{r}_{ij} \in \partial B_N} \delta_{ij} \tau_N(i, j). \tag{3}$$

BEMERKUNG. Schneiden sich in einem Randpunkt von B_N (der Ursprung $\mathbf{o}$ ausgenommen) mehr als zwei Kreise $\dot{C}_i$ — dann enthält die diesem Punkt zugeordnete Gerade mehr als zwei Punkte aus P_n — so ist nur das Paar (i, j) zu berücksichtigen, bei dem $\dot{C}_i$ und $\dot{C}_j$ ein endliches Stück mit ∂B_N gemeinsam haben.

BEWEIS. Sei $N \neq \varnothing$ und $B_N \neq \varnothing$. Seien K_+ und K_- die Ränder der konvexen Hüllen von $P_+(G)$ und $P_-(G)$ für $G \in S_N$ sowie $K_\times$ die gekreutze Seillinie um K_+ und K_-. Nach dem Crofton'schen Seilliniensatz folgt:

$$\mu_N = L(K_\times) - L(K_+) - L(K_-).$$

Ist $\mathfrak{r}_{ij} \in \partial B_N$ und $K_{ij} \subset K_\times$, so folgt $\tau_N(i, j) = 1$, für $K_{ij} \subset K_+ \cup K_-$ folgt $\tau_N(i, j) = -1$. Ist $\mathfrak{r}_{ij} \notin \partial B_N$, so heben sich die restlichen Anteile aus $K_\times$ und K_+ bzw K_- gegenseitig auf. Damit folgt dann die Behauptung.

3. Funktionale und Erwartungswerte

Sei α: $\mathbf{N} \to R$ eine gegebene reelle Funktion auf $\mathbf{N}$, dann heißt

$$F_\alpha = \int_{N(G) \neq \varnothing} \alpha\big(N(G)\big)\, d\mu(G) = \sum_{\substack{N \in \mathbf{N} \\ N \neq \varnothing}} \alpha(N) \mu_N \tag{4}$$

ein Funktional. Nahezu alle uns interessierende Größen lassen sich als Funktionale ausdrücken.

Sei $\mathbf{S}$ die von den Schnitten S_N mit $N \in \mathbf{N}$ und $N \neq \varnothing$ erzeugte Algebra, dann läßt sich das Maß eines beliebigen Elementes $A \in \mathbf{S}$ als Funktional der Funktion

$$\alpha(N) = \begin{cases} 1 & \text{für} \quad S_N \subset A \\ 0 & \text{für} \quad S_N \not\subset A \end{cases}$$

schreiben.

Sei U der Umfang der konvexen Hülle von P_n. Der bedingte Erwartungswert der Zufallsvariablen $\alpha(N)$ unter der Bedingung $N \neq \varnothing$ schreibt sich dann als

$$E[\alpha(N)/N \neq \varnothing] = F_\alpha / U. \tag{5}$$

Mit Hilfe von (3) folgt aus (4)

$$F_\alpha = \sum_{\substack{N \in \mathfrak{N} \\ N \neq \emptyset}} \alpha(N) \sum_{r_{ij} \in \partial B_N} \delta_{ij} \tau_N(i, j)$$

oder, wenn:

$$\sigma_{ij} = \sum_{\substack{r_{ij} \in \partial B_N \\ N \neq \emptyset}} \alpha(N) \tau_N(i, j) \tag{6}$$

ist, folgt der

SATZ 2.

$$F_\alpha = \sum_{i<j} \delta_{ij} \sigma_{ij}. \tag{7}$$

Wir haben dabei die Summationsreihenfolge vertauscht. Wenn die Zahlen σ_{ij} bekannt sind, haben wir nur noch in sehr übersichtlicher Weise über alle Paare (i, j) mit $i<j$ zu summieren. Die Zahlen σ_{ij} sind einer Ähnlichkeitstransformation von P_n gegenüber invariant.

Zu Beginn der weiteren Diskussion wollen wir zwei einfache Beispiele betrachten.

1) Für

$$\alpha(N) = 1$$

folgt sofort $F_\alpha = U$, wie es die Formel von CROFTON

$$\int_{N(G) \neq \emptyset} d\mu(G) = U$$

(siehe [2], Seite 11, Formel (60)) verlangt.

2) Sei

$$K = \bigcup_{i<j} K_{ij},$$

außerdem sei P_n nicht entartet. Die Anzahl der Schnittpunkte einer Geraden $G \in S_N$ mit K ist dann $l(n-l)$ mit $l = |N|$. Wir wählen deshalb $\alpha(N) = l(n-l)$ und erhalten nach leichter Rechnung

$$F_\alpha = 2 \sum_{i<j} \delta_{ij} = 2L(K).$$

Das ist wiederum ein altes Ergebnis von CROFTON

$$\int_{N(G) \neq \emptyset} l(n-l)\, d\mu(G) = 2L(K)$$

(siehe [2], Seite 11, Formel (58)).

4. Das Separationsproblem

Wir wollen nun das Maß der Geradenmenge berechnen, deren Geraden l Punkte von den übrigen Punkten aus P_n trennen. Sei also

$$m_l = \mu(\{G; |N(G)| = l\}) \quad \text{für} \quad l = 1, 2, \ldots, \left[\frac{n}{2}\right].$$

Wir haben:

$$\alpha(N) = \begin{cases} 1 & \text{für} \quad |N| = l \\ 0 & \text{für} \quad |N| \neq l \end{cases}$$

zu wählen. Wir wollen uns auf nicht entartete Punktmengen P_n beschränken. G_{ij} enthält also außer $\mathbf{p}_i$ und $\mathbf{p}_j$ keinen weiteren Punkt aus P_n — im Punkt $\mathbf{r}_{ij}$ schneiden sich nur $\dot{C}_i$ und $\dot{C}_j$, $i, j = 1, 2, \ldots, n$; $i < j$. Sei $l_{ij} = |N(G_{ij})|$. $\mathbf{r}_{ij}$ ist Randpunkt der vier Zellen B_{N_1}, B_{N_2}, B_{N_3} und B_{N_4}, (vergl. Abb. 2) wobei wir ohne Einschränkung der Allgemeinheit verlangen können, daß:

$$B_{N_2} \subset D_i' \cap D_j, \quad B_{N_3} \subset D_i \cap D_j',$$

$$B_{N_1}, B_{N_4} \subset (D_i \cap D_j) \cup (D_i' \cap D_j')$$

und $|N_1| = l_{ij}$ ist.

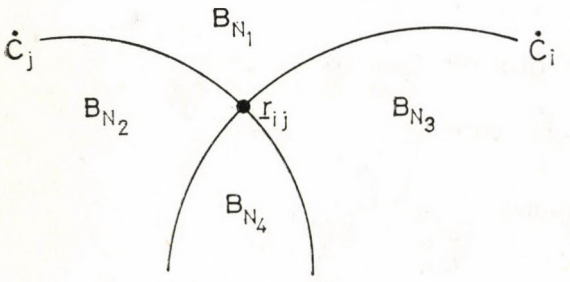

Abb. 2

(B_{N_1} und B_{N_4} können vertauscht sein)

Wir führen eine Fallunterscheidung durch. Aus (2) und (6) folgt jeweils:

1) Sei $l_{ij} = 0, 1, 2, \ldots, v-2$. Für $l = l_{ij}$ folgt $\sigma_{ij} = -1$, für $l = l_{ij}+1$ ist $\sigma_{ij} = 2$ und für $l = l_{ij}+2$ ist $\sigma_{ij} = -1$.

2) Sei $l_{ij} = v-1$ für $n = 2v$. Für $l = l_{ij}$ ist $\sigma_{ij} = -2$ und für $l = l_{ij}+1$ gilt $\sigma_{ij} = 2$.

3) Sei $l_{ij} = v-1$ für $n = 2v+1$. Für $l = l_{ij}$ ist $\sigma_{ij} = -1$ und für $l = l_{ij}+1$ ist $\sigma_{ij} = 1$.

4) Für alle Punkte $\mathbf{r}_{ij}$, die in keine der eben behandelten Kategorien fallen, gilt $\sigma_{ii} = 0$.

Bevor wir die Ergebnisse formulieren, wollen wir noch einen neuen Begriff einführen. U_k sei die Länge der Kanten K_{ij}, für die $|N(G_{ij})| = l_{ij} = k$ ist, $k = 0, 1, 2, \ldots, \left[\frac{n}{2}\right] - 1$. Wir wollen U_k den „k-ten inneren Umfang" nennen, z. B.

ist $U_0=U$ der Umfang der konvexen Hülle von P_n. Es sei noch $U_{-1}=U_{-2}=0$. Fassen wir alles zusammen, so folgt:

$$m_l=2U_{l-1}-U_l-U_{l-2} \text{ für } l=1,2,\ldots,\nu-1 \text{ für } n=2\nu+1 \text{ und für } l=1,2,\ldots,\nu-2$$

für $n=2\nu$,

$$m_{\nu-1}=2U_{\nu-2}-2U_{\nu-1}-U_{\nu-3} \text{ für } n=2\nu,$$

$$m_\nu=2U_{\nu-1}-U_{\nu-2} \text{ für } n=2\nu,$$

$$m_\nu=U_{\nu-1}-U_{\nu-2} \text{ für } n=2\nu+1.$$

Δ und Δ^2 seien die Operatoren, die einer auf den ganzen Zahlen definierten Funktion f die erste und zweite Differenz zuordnen. Also:

$$\Delta f(l)=f(l)-f(l-1)$$

und

$$\Delta^2 f(l)=f(l)-2f(l-1)+f(l-2).$$

Damit folgt der

SATZ 3.

(8) $m_l=-\Delta^2 U_{-l}$ für $l=1,2,\ldots,\nu-1$ für $n=2\nu+1$ und

für $l=1,2,\ldots,\nu-2$ für $n=2\nu$,

$$m_{\nu-1}=-\Delta^2 U_{\nu-1}-U_{\nu-1} \text{ für } n=2\nu,$$

$$m_\nu=\Delta U_{\nu-1}+U_{\nu-1} \text{ für } n=2\nu \text{ und}$$

$$m_\nu=\Delta U_{\nu-1} \text{ für } n=2\nu+1.$$

Mit Hilfe von (8) weist man nun leicht nach, daß für die bedingten Erwartungswerte (5) von I folgendes gilt:

$$E[l/l>0]=U_{\nu-1}/U \quad \text{für} \quad n=2\nu+1$$

und

$$E[l/l>0]=2U_{\nu-1}/U \quad \text{für} \quad n=2\nu.$$

Wir setzen zur Abkürzung $V=U_{\nu-1}$ für $n=2\nu+1$ und $V=2U_{\nu-1}$ für $n=2\nu$, dann gilt:

(9) $$E[l/l>0]=V/U.$$

Man beachte, daß die entsprechende Fragestellung für gerichtete Geraden auf sehr viel einfachere Weise mit $n/2$ zu beantworten ist, [1a].

5. Ungleichungen für die inneren Umfänge

Aus unseren Ergebnissen lassen sich eine Reihe von Ungleichungen über die inneren Umfänge herleiten. Man weist leicht nach, daß für beliebiges P_n immer $m_l > 0$ ist, $l = 1, 2, \ldots, \left[\frac{n}{2}\right]$ $(n \geqq 5)$. Aus (8) folgt für $l \leqq v^*$ mit $v^* = \left[\frac{n-1}{2}\right] - 1$

$$U_0 > U_{l-1} - U_{l-2} > U_l - U_{l-1} > 0,$$

d. h. U_1 als Funktion von l ist „konvex"; denn für $2 \leqq l_1 < l < l_2 \leqq v^*$ gilt

$$U_l > U_{l_1} + \frac{l - l_1}{l_2 - l_1}(U_{l_2} - U_{l_1}).$$

Weiter beweist man leicht die Monotonie von U_l:

$$U_0 < U_1 < U_2 < \ldots < U_{v^*}, \quad \text{sowie} \quad U_{v-2} < 2U_{v-1} \quad \text{für} \quad n = 2v.$$

Außerdem gilt:

$$U_l < (l+1)U_0 \quad \text{für} \quad l = 2, 3, \ldots, v^*; \quad \text{sowie}$$

$$U_{v-1} < \frac{v}{2} U_0 \quad \text{für} \quad n = 2v.$$

Wir haben im dritten Abschnitt gezeigt, daß:

$$E[l(n-l)/l > 0] = \frac{2L(K)}{U}$$

ist. Daraus folgt mit Hilfe von (9):

$$E[l^2/l > 0] = \frac{n}{U} V - \frac{2}{U} L(K).$$

Als letztes sei die Ungleichung

$$\frac{n}{2} - \sqrt{\frac{n^2}{4} - \frac{2L(K)}{U}} < \frac{V}{U}$$

notiert; diese Ungleichung folgt aus der Ungleichung

$$E[l^2/l > 0] \geqq E^2[l/l > 0].$$

Man überzeugt sich leicht davon, daß der Radikand nicht negativ wird, da

$$L(K) < \frac{U}{8} \begin{cases} n^2 - 1 & \text{für } n \text{ ungerade} \\ n^2 & \text{für } n \text{ gerade} \end{cases}$$

ist.

6. Das Buffon—Sylvester-Problem für konvexe Polygone

Als weitere Anwendung von Satz 2 untersuchen wir das BUFFON—SYLVESTER-Problem für konvexe Polygone, [6], vergl. auch [1c]. Seien $\Omega_1, \Omega_2, \ldots, \Omega_\nu$ nicht zu Punkten entartete konvexe Polygone ($\nu \geqq 2$) mit folgenden Eigenschaften:

— Je zwei Polygone Ω_i und Ω_j haben nur endlich viele Punkte, jedoch keine Eckpunkte, gemeinsam. (Beachte, daß Überlagerungen durchaus zulässig sind.)

— Die Menge der Ecken der Polygone, das ist das zu untersuchende P_n mit $n \geqq 2\nu$, sei nicht entartet.

Mit $s=s(N)$ bezeichnen wir die Anzahl der von einer Geraden $G \in S_N$ geschnittenen Polygone.

$$M_s = \sum_{s=s(N)} \mu_N$$

ist das Maß der Geradenmenge, deren Geraden s Polygone schneiden, $s=1, 2, \ldots, \nu$. Wir haben das Funktional von

$$\alpha(N) = \begin{cases} 1 & \text{für} \quad s = s(N) \\ 0 & \text{für} \quad s \neq s(N) \end{cases}$$

zu bestimmen. Sei $i, j=1, 2, \ldots, n$ und $i<j$ sowie s_{ij} die Anzahl der von G_{ij} geschnittenen Polygone, wobei wir das (die) Polygon(e), das (die) $\mathbf{p}_i$ und $\mathbf{p}_j$ als Eckpunkte besitzt (besitzen), nicht mitrechnen wollen. Wir verwenden (2) und (6) und haben dabei folgende Fälle zu unterscheiden.

1) K_{ij} sei Kante eines Polygons Ω.

a) Ω sei nicht entartet. Für $s=s_{ij}$ ist $\sigma_{ij}=-1$, für $s=s_{ij}+1$ ist $\sigma_{ij}=1$. (Abb. 3.1a)

b) Ω ist zur Strecke K_{ij} entartet. Man erhält für $s=s_{ij}$ sofort $\sigma_{ij}=-2$, für $s=s_{ij}+1$ ist $\sigma_{ij}=2$ (Abb. 3.1b).

— In den folgenden Fällen müssen wir nicht mehr zwischen entarteten und nicht entarteten Polygonen unterscheiden. Sei $\mathbf{p}_i$ Eckpunkt des Polygons Ω und $\mathbf{p}_j$ Eckpunkt des Polygons Ω'.

2) Die Polygone Ω und Ω' werden von G_{ij} durchschnitten.

a) $\Omega \neq \Omega'$. Für $s=s_{ij}+2$ — etwas anderes ist bei diesen Paaren (i, j) nicht möglich — ist $\sigma_{ij}=0$ (Abb. 3.2).

b) $\Omega=\Omega'$. $\mathbf{p}_i$ und $\mathbf{p}_j$ sind also Eckpunkte desselben Polygons. Auch dann ist $\sigma_{ij}=0$.

3) Nur eines der beiden Polygone Ω und Ω' werde von G_{ij} durchschnitten. Für $s=s_{ij}+1$ und für $s=s_{ij}+2$ ist $\sigma_{ij}=0$ (Abb. 3.3).

4) Ω und Ω' liegen auf der gleichen Seite von G_{ij}. Für $s=s_{ij}$ ist $\sigma_{ij}=-1$, für $s=s_{ij}+1$ ist $\sigma_{ij}=2$ und für $s=s_{ij}+2$ ist $\sigma_{ij}=-1$. (Abb 3.4.)

5) Ω und Ω' liegen auf entgegengesetzten Seiten von G_{ij}. Für $s=s_{ij}$ ist $\sigma_{ij}=1$, für $s=s_{ij}+1$ ist $\sigma_{ij}=-2$ und für $s=s_{ij}+2$ ist $\sigma_{ij}=1$ (Abb. 3.5).

6) In allen noch ausstehenden Fällen ist $\sigma_{ij}=0$.

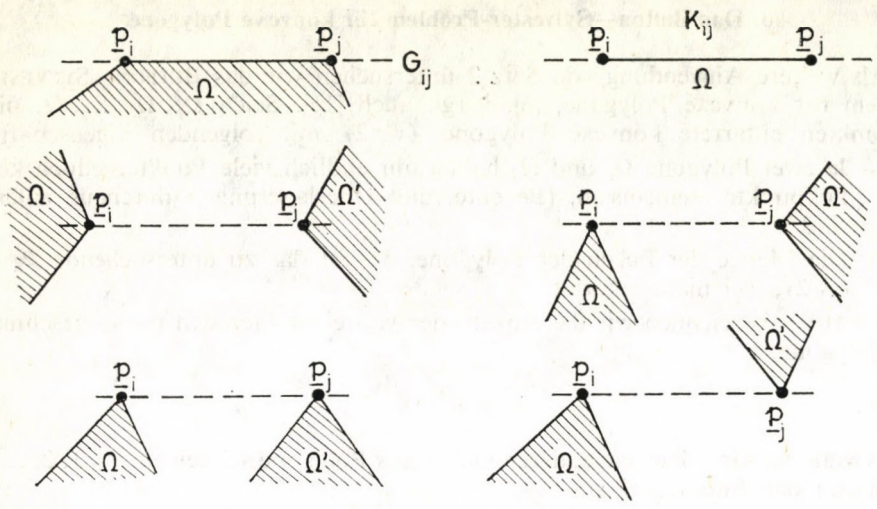

Abb. 3.1—3.5

DEFINITION. W_s^0 sei die Gesamtlänge der Polygonkanten, bei denen die durch die jeweilige Kante bestimmte Gerade s weitere Polygone schneidet, (Fall 1). Falls die entsprechende Kante ein entartetes Polygon ist, soll die Länge doppelt gezählt werden, (Fall 1b). W_s^+ (bzw. W_s^-) sei die Gesamtlänge der Kanten, bei denen die durch sie bestimmte Gerade s weitere Polygone schneidet und die angrenzenden Polygone auf der gleichen (entgegengesetzten) Seite der Geraden liegen (Fall 4, bzw. Fall 5). Wir setzen noch $W_\varkappa^0=0$ für $\varkappa\neq 0, 1, \ldots, \nu-1$ und $W_\varkappa^+=W_\varkappa^-=0$ für $\varkappa\neq 0, 1, \ldots, \nu-2$. Also:

$$W_s^0 = \sum_{\substack{s=s_{ij}\\ \text{(Fall 1a)}}} \delta_{ij} + \sum_{\substack{s=s_{ij}\\ \text{(Fall 1b)}}} 2\delta_{ij}, \quad s = 0, 1, \ldots, \nu-1$$

$$W_s^+ = \sum_{\substack{s=s_{ij}\\ \text{(Fall 4)}}} \delta_{ij} \quad \text{und} \quad W_s^- = \sum_{\substack{s=s_{ij}\\ \text{(Fall 5)}}} \delta_{ij}, \quad s = 0, 1, \ldots, \nu-2.$$

Mit diesen Bezeichnungen erhalten wir:

$$M_s = -W_s^0 + W_{s-1}^0 - W_s^+ + 2W_{s-1}^+ - W_{s-2}^+ + W_2^- - 2W_{s-1}^- + W_{s-2}^-.$$

Es gilt der

SATZ 4.

(10) $$M_s = -\Delta W_s^0 - \Delta^2 W_s^+ + \Delta^2 W_s^-.$$

Speziell ist das Maß der Geradenmenge, deren Geraden alle Polygone schneiden

(11) $$M_\nu = W_{\nu-1}^0 - W_{\nu-2}^+ + W_{\nu-2}^-,$$

sowie das Maß der Geradenmenge, deren Geraden wenigstens ein Polygon schneiden, ist

$$M_\nu^* = \sum_{s=1}^{\nu} M_s = W_0^0 + W_0^+ - W_0^- . \tag{12}$$

Durch (11) und (12) sind die gesuchten Maße des Buffon—Syvester-Problems gegeben. Für disjunkte Polygone sind diese Maße von AMBARTZUMIAN auf anderem Wege berechnet worden, [1c].

7. Schlußbemerkung

Im 4. Abschnitt haben wir gesehen, daß sich die Maße m_1 im wesentlichen als das Negative von zweiten Differenzen von U_1 schreiben lassen. Definiert man für Eilinien als „innere Umfangsfunktion" das Integral über die Sehnenlänge bei konstanter abgeschnittener Bogenlänge, so erhält man als Maßdichte der Geraden, die eine gewisse Bogenlänge abschneiden, das Negative der zweiten Ableitung der inneren Umfangsfunktion, [4b]. Die Eigenschaften der inneren Umfangsfunktion ähneln sehr denen der hier untersuchten inneren Umfänge. Dem Verfasser ist nicht bekannt, ob diese Analogie für allgemeinere unendliche Punktmengen besteht.

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ÜBER EINE VERALLGEMEINERUNG DES CROFFTON'SCHEN SEILLINIENSATZES

von

F. PIEFKE

Zusammenfassung

Wir betrachten ein System **E** von l Eilinien in der Euklid'schen Ebene ($l=1, 2, 3, \ldots$) sowie zufällige Geraden, die die Eilinien aus **E** in gewisser Weise schneiden. Die Menge der Geraden zerfällt bezüglich der verschiedenen Schnittmöglichkeiten in disjunkte Teilmengen, die die Atome einer Algebra bilden. Wir bestimmen für jedes Element dieser Algebra das translations- und rotationsinvariante Geradenmaß. Der entsprechende Satz ist eine Verallgemeinerung des Crofton'schen Seilliniensatzes auf eine beliebige endliche Anzahl von Eilinien.

Um den Satz zu beweisen, werden die Eilinien durch Polygone approximiert. Die Eckenmenge der Polygone ist endlich, es können dann Methoden aus einer früheren Arbeit verwendet werden.

Als eine Anwendung des Satzes bestimmen wir die Verteilung der Anzahl der geschnittenen Eilinien und lösen damit ein von Sylvester gestelltes Problem.

1. Einleitung

Im Jahre 1890 veröffentlichte J. SYLVESTER [11], eine viel beachtete Arbeit über eine Verallgemeinerung des klassischen „Nadelproblems von Buffon", (vergl. dazu etwa [5] oder [6]). Seine Fragestellung ist — formuliert in der Sprache der Integralgeometrie, [2], [9] — die folgende:

Gegeben seien l Eilinien in der Ebene ($l=1, 2, \ldots$). *Bestimme das Maß* M_1^* *(bzw.* M_1*) der Geradenmenge, deren Geraden wenigstens eine (bzw. alle) Eilinien schneiden.*

Für $l=1$ und $l=2$ finden wir die Lösung bei M. W. CROFTON, [4a], [4b], [4c], [7]. Man beachte, daß für $l=2$ die Lösung in Form des Seilliniensatzes gegeben ist. E_1 und E_2 seien zwei sich nicht überlappende Eilinien, dann sagt der Seilliniensatz aus:

Das Maß der Geradenmenge, deren Geraden wenigstens eine der beiden Eilinien schneiden, ist gleich der Summe der Umfänge der Eilinien und der Länge der glatten Seillinie vermindert um die Länge der gekreuzten Seillinie. Das Maß der Geradenmenge, deren Geraden beide Eilinien schneiden, ist gleich der Differenz der Längen der gekreuzten und der glatten Seillinie um beide Eilinien (vergl. Abb. 1).*

Den Fall $l=3$ behandelt SYLVESTER in der bereits erwähnten Arbeit. Es sind dabei sehr viele Fälle zu untersuchen, so daß es hoffnungslos erscheint, eine noch größere Anzahl von Eilinien zu betrachten.

Neues Interesse an diesen Problemkreisen haben einige Arbeiten von R. V. AMBARTZUMIAN [1a] bis [1d], erweckt. In [1a] wird die Verteilung der Anzahl der Schnittpunkte einer zufälligen Geraden mit einer gegebenen Kurve untersucht. Aus

* Damit bezeichnen wir den Rand der konvexen Hülle der beiden Eilinien.

diesen Ergebnissen lassen sich die Formeln für die Maße M_s der Geradenmengen, deren Geraden $s=1, 2, \ldots, l$ der Eilinien schneiden, ableiten. In [1d] werden die Maße M_l und M_l^* für nicht überlappende konvexe Polygone auf gekrümmten konvexen Oberflächen — darin ist der Fall der Ebene enthalten — angegeben.

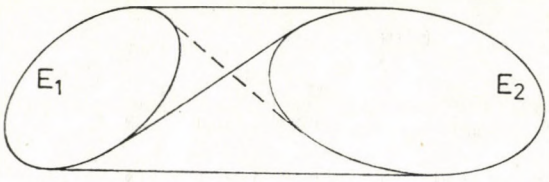

Abb. 1.

Durch die Geradenmengen, deren Geraden eine Eilinie E_i $(i=1, 2, \ldots, l)$ schneiden, wird eine Algebra $\mathscr{A}$ erzeugt, deren Elemente als Schnitte bezeichnet werden sollen. In dieser Arbeit werden wir die Maße von beliebigen Schnitten bestimmen. Unsere Methode, die ihren Ausgangspunkt beim Seilliniensatz hat, unterscheidet sich wesentlich von der von AMBARTZUMIAN verwendeten Methode der "invarianten Einbettung" in [1a] bis [1c]. Wir stützen uns hier sehr stark auf meine frühere Arbeit [8], in der die Zerlegungen von endlichen Punktmengen durch zufällige Geraden betrachtet werden. Es sei noch erwähnt, daß wir nicht die allgemeinsten Voraussetzungen, unter denen die Sätze richtig bleiben, untersuchen wollen.

2. Definitionen

DEFINITION 1. Gegeben sei ein offenes beschränktes konvexes Gebiet $\dot{E}$ in der Euklid'schen Ebene. Unter einer *Eilinie* E verstehen wir den Rand eines solchen Gebietes — also die Menge der Häufungspunkte von $\dot{E}$, die nicht innere Punkte sind. $\hat{E}=E\cup\dot{E}$ sei die abgeschlossene Hülle von $\dot{E}$, $\hat{E}$ ist kompakt. Auch eine abgeschlossene Strecke endlicher Länge wollen wir als Eilinie E bezeichnen, dann ist $\dot{E}=\varnothing$ — wir sagen: E ist *entartet,* oder auch: E ist eine Nadel.

Eine Gerade S heißt *Stützgerade* an E, wenn ein Punkt $\mathbf{b}$ existiert, der E und S gemeinsam ist — also $\mathbf{b}\in S\cap E$ —, wir sagen *Berührpunkt,* und wenn E mit einer der von S bestimmten offenen Halbebenen keine Punkte gemeinsam hat.

Eine *gemeinsame Stützgerade* ist eine Gerade, die Stützgerade an zwei oder mehr Elinien ist.

Wir sagen, eine Gerade G *durchschneidet* eine Eilinie E — im Zeichen $G\uparrow E$, bzw. $G\nmid E$ bedeutet die Verneinung davon —, wenn beide von G bestimmte offene Halbebenen Punkte mit E gemeinsam haben.

DEFINITION 2. Ein System $\mathbf{E}$ von l Eilinien $E_1, E_2, \ldots, E_l$ $(l=1, 2, \ldots)$, wobei nur solche Eilinien zugelassen sind, die nur endlich viele singuläre Punkte* enthalten, heißt *regulär*, wenn folgende Bedingungen erfüllt sind:

* Das sind Punkte, durch die mehrere Stützgeraden gehen, [3].

(A) Je zwei Eilinien aus **E** haben höchstens endlich viele gemeinsame Punkte.
(B) Je zwei Eilinien E_i und E_j aus **E** haben höchstens endlich viele gemeinsame Stützgeraden ($i \neq j$, $i, j = 1, 2, \ldots, l$). Jede Stützgerade an E_i und E_j enthält nur *einen* Berührpunkt aus E_i und nur *einen* Berührpunkt aus E_j; beide Berührpunkte sind verschieden.
(C) Keine drei Eilinien aus **E** haben eine gemeinsame Stützgerade.

Gegeben sei ein reguläres System **E** von l Eilinien, wir betrachten in dieser Arbeit ausschließlich solche Systeme. **S** sei die Menge der gemeinsamen Stützgeraden, sie ist wegen (B) endlich. Ebenso ist die Menge P der Berührpunkte von gemeinsamen Stützgeraden endlich. Für $S \in \mathbf{S}$ sei K mit $K \subset S$ die Kante zwischen den, wegen (B) verschiedenen, Berührpunkten; **K** sei die Menge dieser Kanten. Wir können zwei Typen von Stützgeraden und damit Kanten unterscheiden. $S \in \mathbf{S}$ sei gemeinsame Stützgerade von E_i und E_j. Für $K \in \mathbf{K}$ mit $K \subset S$ setzen wir:

$$\tau(K) = 1 \quad (\text{bzw.} -1),$$

wenn E_i und E_j auf gleichen (verschiedenen) Seiten von S liegen (vergl. Abb. 2).

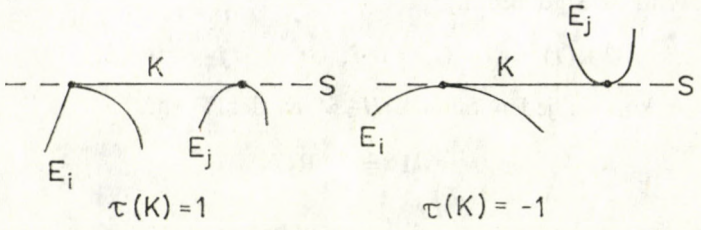

Abb. 2.

Durch die Berührpunkte aus P werden die Eilinien in Bogenstücke zerlegt. B_{ik} mit $k = 1, 2, \ldots, k_i$ bezeichne die disjunkten Bogenstücke der Eilinie E_i ($i = 1, 2, \ldots, l$; $k_i = 1, 2, \ldots$); wir rechnen die beiden B_{ik} begrenzenden Berührpunkte nicht zu B_{ik} (wir schreiben sonst $\hat{B}_{ik}$). **B** sei die Menge der Bögen B_{ik}. Weiter sei $\mathbf{N} = \mathbf{K} \cup \mathbf{B}$. Durch:

$$\bigcup_{A \in \mathbf{N}} A \cup P = \bigcup_{E \in \mathbf{E}} E \cup \bigcup_{K \in \mathbf{K}} K$$

ist ein Kurvennetz gegeben, wir nennen **N** das *Seilliniennetz*. Das hat folgenden Grund: Man trägt für je zwei Eilinien E_i und E_j aus **E** die glatte (nichtgekreuzte) Seillinie und wenn möglich die gekreuzte Seillinie ein. Überlappen sich E_i und E_j, so existiert die gekreuzte Seillinie nicht. Schließlich markieren wir die Berührpunkte von gemeinsamen Stützgeraden, das sind die Punkte, bei denen sich die Seillinien von den Eilinien lösen. Die Menge der entstehenden Kanten und Bögen ist dann **N**.

DEFINITION 3. Sei **G** die Menge der zufälligen Geraden in der Ebene. Mit $\mathscr{A}$ bezeichnen wir dann die durch die Mengen

$$\{G \in \mathbf{G};\ G \uparrow E_i\} \quad (i = 1, 2, \ldots, l)$$

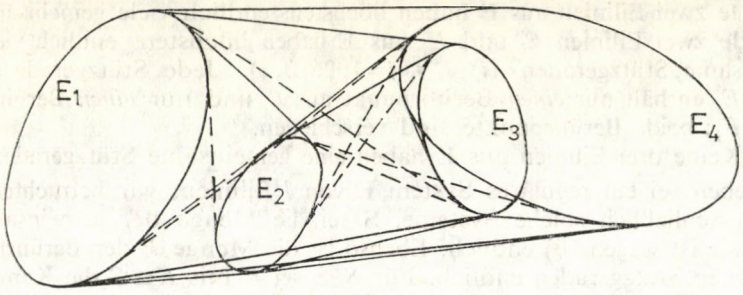

Abb. 3.
Beispiel eines Seilliniennetzes

erzeugte Algebra. Die Elemente der Algebra heißen *Schnitte,* die Algebra heißt die *Schnittalgebra.* Wir führen folgende Indexmengen ein:

$$I_l = \{1, 2, \ldots, l\}, \quad I \subset I_l \quad \text{sowie} \quad I' = I_l - I.$$

Die Atome von $\mathscr{A}$ sind durch:

$$\mathbf{R}(I) = \{G \in \mathbf{G};\ \forall i \in I\colon G \uparrow E_i,\ \forall j \in I'\colon G \overset{\uparrow}{\uparrow} E_j\}$$

gegeben. Wir können jeden Schnitt $H \in \mathscr{A}$ in der Form:

$$\mathbf{H} = \bigcup_{I \in \mathbf{I}} \mathbf{R}(I)$$

mit einer Menge $\mathbf{I}$ von Indexmengen $I \subset I_l$ schreiben.

Mit μ bezeichnen wir das translations- und rotationsinvariante Geradenmaß, (vergl. [2], [9] u. a.) Wir denken uns jede Gerade $G \in \mathbf{G}$ in der Hesse'schen Normalform

$$G = \{\mathbf{x} = (x_1, x_2);\ x_1 \cos\varphi + x_2 \sin\varphi = p\} \tag{1}$$

mit $0 \leqq \varphi < 2\pi$ (φ is mod 2π zu lesen) und $0 \leqq p$ gegeben. v sei die bijektive Abbildung, die jeder Geraden G das Tupel (p, φ) zuordnet, also $v(G) = (p, \varphi)$. Sei $H \subset \mathbf{G}$, so kann man das Maß der Geradenmenge $\mathbf{H}$ (falls diese meßbar ist) in der Form:

$$\mu(\mathbf{H}) = \iint_{v(\mathbf{H})} dp\, d\varphi \tag{2}$$

schreiben.

Wir interessieren uns hier für die Maße der Geradenmengen $\mathbf{H} \in \mathscr{A}$. Es gilt:

$$\mu(\mathbf{H}) = \infty \Leftrightarrow \mathbf{R}(\varnothing) \subset \mathbf{H},$$

wie man leicht sieht. $\mathbf{R}(\varnothing)$ hat als einziges Atom aus $\mathscr{A}$ kein endliches Maß.

Wir betrachten nun einen Schnitt $\mathbf{H} \in \mathscr{A}$ als gegeben und definieren eine Funktion $\gamma_{\mathbf{H}}$ auf $\mathbf{N}$, die die $\mathbf{H}$ kennzeichnende Funktion genannt werden soll. Mit ihrer Hilfe lassen sich die zentralen Sätze sehr übersichtlich formulieren.

Mit $\chi_{\mathbf{H}}$ bezeichnen wir wie üblich die charakteristische Funktion der Menge $\mathbf{H}$, also

$$\chi_{\mathbf{H}}(G) = \begin{cases} 1 & \text{für } G \in \mathbf{H} \\ 0 & \text{für } G \notin \mathbf{H} \end{cases} \quad (G \in \mathbf{G}).$$

DEFINITION 4. 1) Sei $K \in \mathbf{K} \subset \mathbf{N}$ mit $S \in \mathbf{S}$ und $K \subset S$. S ist also die durch die Kante K bestimmte gemeinsame Stützgerade an zwei Eilinien E_i und E_j, $\mathbf{p}_i$ und $\mathbf{p}_j$ seien die Berührpunkte. Es existiert nun eine Geradenumgebung* $\mathbf{U}(S)$ von S und eine möglicherweise leere Indexmenge $I \subset I_l$, so daß

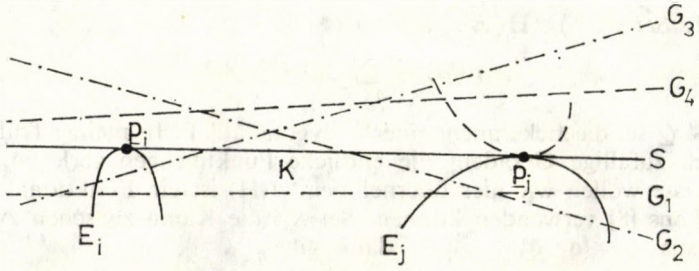

Abb. 4.

$$\mathbf{U}(S) \subset \mathbf{R}(I) \cup \mathbf{R}(I \cup \{i\}) \cup \mathbf{R}(I \cup \{j\}) \cup \mathbf{R}(I \cup \{i, j\})$$

ist, wobei jedes der vier Atome Geraden mit $\mathbf{U}(S)$ gemeinsam hat. (Beachte Forderung (C)). Aus jedem dieser Atome betrachten wir nun eine solche gemeinsame Gerade G_k $(k=1, 2, 3, 4)$, wobei G_2 und G_3 die Punkte $\mathbf{p}_i$ und $\mathbf{p}_j$ trennen und G_1 und G_4 nicht (s. Abb. 4.) Dann sei:

$$\gamma_{\mathbf{H}}(K) = -\chi_{\mathbf{H}}(G_1) + \chi_{\mathbf{H}}(G_2) + \chi_{\mathbf{H}}(G_3) - \chi_{\mathbf{H}}(G_4). \tag{3}$$

Bemerkung: Die Geraden G_1, G_2, G_3 und G_4 sind Geraden, die durch eine beliebig kleine Verschiebung und Verdrehung aus S entstehen und die bis auf E_i und E_j die gleichen Eilinien aus $\mathbf{E}$ durchschneiden. Wegen (B) und (C) gibt es nun auch gerade vier verschiedene Lagen solcher Geraden.

2) Sei $B \in \mathbf{B}$, sowie S eine Stützgerade an E_i (falls $B \subset E_i$) mit Berührpunkt in B. Es existiert dann eine Geradenumgebung $\mathbf{U}(S)$ und eine möglicherweise leere Indexmenge $I \subset I_l$, so daß

$$\mathbf{U}(S) \subset \mathbf{R}(I) \cup \mathbf{R}(I \cup \{i\})$$

ist und $\mathbf{U}(S)$ mit beiden Atomen Geraden gemeinsam hat. Aus jedem dieser Atome wähle eine solche Gerade G_k $(k=1, 2)$, wobei $G_1 \uparrow E_i$ und $G_2 \nuparrow E_i$ gelte. Dann sei:

$$\gamma_{\mathbf{H}}(B) = \chi_{\mathbf{H}}(G_1) - \chi_{\mathbf{H}}(G_2). \tag{4}$$

Bemerkung: G_1 und G_2 sind Geraden, die durch beliebig kleine Verschiebung und Verdrehung aus S entstehen und die bis auf E_i die gleichen Eilinien durchschneiden. G_1 soll E_i durchschneiden, G_2 nicht. Aufgrund der Struktur der Seilliniennetzes ist $\gamma_{\mathbf{H}}$ eindeutig bestimmt unabhängig von der Wahl von S.

* Siehe Anhang.

3. Ein Satz über konvexe Polygone

Bevor wir uns dem allgemeinen Fall von Eiliniensystemen widmen, betrachten wir speziell reguläre Systeme von konvexen Polygonen. Durch

$$\mathbf{F} = \{F_1, F_2, \ldots, F_l\} \quad (l = 1, 2, \ldots)$$

sei ein solches System gegeben. $\mathbf{N}=\mathbf{K}\cup\mathbf{B}$ sei das zugehörige Seilliniennetz, $\mathscr{A}$ die Schnittalgebra. Mit $L(\ )$ bezeichnen wir die Länge einer Kurve, wobei wir verabreden die Länge von entarteten Eilinien doppelt zu zählen. Es gilt der

SATZ 1. *Sei* $R(\varnothing)\subset\mathbf{H}\in\mathscr{A}$. *Dann ist*

$$\mu(\mathbf{H}) = \sum_{A\in N} \gamma_{\mathbf{H}}(A) L(A). \tag{5}$$

BEWEIS. Q sei die Eckenmenge der Polygone aus $\mathbf{F}$. In meiner früheren Arbeit [8] habe ich zufällige Geraden, die endliche Punktmengen zerlegen, untersucht. Die Ergebnisse wollen wir hier übernehmen. $\mu(\mathbf{H})$ ist ein Funktional, so daß wir den Satz 2 aus [8] verwenden können. Sei K_{ij} die Kante zwischen zwei Punkten $\mathbf{q}_i$ und $\mathbf{q}_j$ aus $Q=\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$; dann gilt

$$\mu(\mathbf{H}) = \sum_{\substack{i,j=1\\ i<j}}^{n} \sigma_{ij} L(K_{ij}).$$

Wir werden nun diese Formel etwas umschreiben — dazu eine Änderung der Summationsvorschrift. Wir tragen alle Geraden G^* ein, die durch die Kanten K_{ij} bestimmt werden und summieren über alle Tupel (i, j), für die die Punkte $\mathbf{q}_i$ und $\mathbf{q}_j$ auf einer Geraden G^* liegen, und summieren dann über alle solche Geraden. Also:

$$\mu(\mathbf{H}) = \sum_{G^*} \sum_{\substack{i<j\\ \mathbf{q}_i,\mathbf{q}_j\in G^*}} \sigma_{ij} L(K_{ij}) = \sum_{G^*} a_{G^*}. \tag{6}$$

Wir numerieren die Punkte aus Q auf einer Geraden G^* der Reihe nach durch — etwa $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_k \in Q\cap G^*$. Seien nun G_1, G_4, $G_2^{(\varkappa)}$ und $G_3^{(\varkappa)}\in\mathbf{U}(G^*)$ mit $(\varkappa=1, 2, \ldots, k-1)$ $2k$ Geraden, die bis auf $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_k$ die Punktmenge Q in gleicher Weise zerlegen wie G^* und wobei G_1 und G_4 die Strecke $\overline{\mathbf{r}_1\mathbf{r}_k}$ nicht schneiden und $G_2^{(\varkappa)}$ und $G_3^{(\varkappa)}$ die Strecken $\overline{\mathbf{r}_\varkappa\mathbf{r}_{\varkappa+1}}$ schneiden (vergl. Abb. 5).
Dann gilt (vergl. auch die Bemerkung nach Satz 1 in [8]) für $k=2, 3, \ldots$:

$$a_{G^*} = \sum_{\varkappa=1}^{k-1} \left(\chi_{\mathbf{H}}(G_2^{(\varkappa)}) + \chi_{\mathbf{H}}(G_3^{(\varkappa)})\right)\delta_{\varkappa,\varkappa+1} - \left(\chi_{\mathbf{H}}(G_1) + \chi_{\mathbf{H}}(G_4)\right)\delta_{1,k}, \tag{7}$$

dabei ist $\delta_{\varkappa,\lambda}$ der Abstand der Punkte $\mathbf{r}_\varkappa$ und $\mathbf{r}_\lambda$. Die Punkte $\mathbf{r}_\varkappa$ können in verschiedene Klassen eingeteilt werden. $\mathbf{r}_\varkappa$ sei Eckpunkt des Polygons $F\in\mathbf{F}$. Dann kann eintreten (vergl. Abb. 6):

(a) $G^*\uparrow F$.

(b) G^* enthält eine Kante, sagen wir $\overline{\mathbf{r}_\varkappa\mathbf{r}_\lambda}$, von F.

(c) G^* ist Stützgerade von F, enthält aber keine Kante von F.

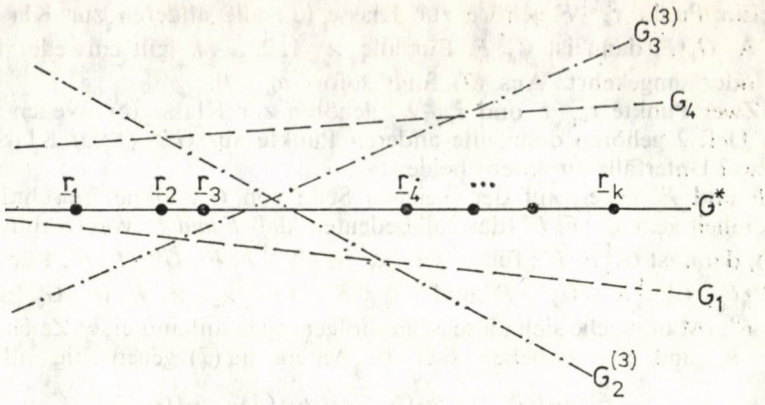

Abb. 5.

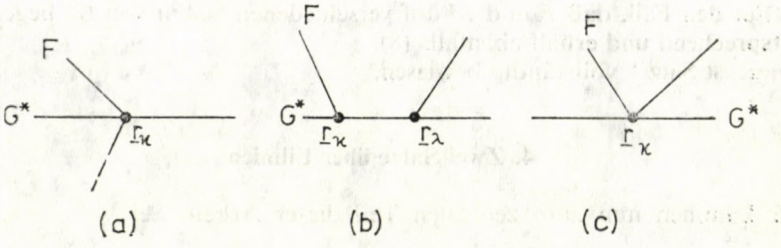

Abb. 6.

Wir müssen folgende Fälle untersuchen:

1) Alle $\mathbf{r}_\varkappa$ $(\varkappa = 1, 2, \ldots, k)$ gehören zur Klasse (a). Dann gehören entweder alle G_1, G_4, $G_2^{(\varkappa)}$ und $G_3^{(\varkappa)}$ oder keine zu $\mathbf{H}$. Es folgt sofort $a_{G^*} = 0$ in Übereinstimmung mit der Behauptung.

2) Zwei Punkte $\mathbf{r}_{\varkappa_1}$ und $\mathbf{r}_{\varkappa_2}$ gehören zur Klasse (b), dabei sei o. E. d. A. $\varkappa_1 < \varkappa_2$. Dann kann G^* wegen Forderung (B) aus Def. 2 keine gemeinsame Stützgerade sein. Das aber heißt, daß die restlichen Punkte zur Klasse (a) gehören müssen. Sei $\overline{\mathbf{r}_{\varkappa_1}, \mathbf{r}_{\varkappa_2}} \subset F \in \mathbf{F}$.

Für nichtentartetes F sei o. E. d. A. $G_1 \uparrow F$, woraus folgt $G_4 \nuparrow F$. Für $1 \leqq \varkappa < \varkappa_2$ gilt $G_2^{(\varkappa)} \uparrow F$; für $\varkappa_2 \leqq \varkappa$ ist $G_2^{(\varkappa)} \nuparrow F$; für $1 \leqq \varkappa < \varkappa_1$ ist $G_3^{(\varkappa)} \nuparrow F$ und für $\varkappa_1 \leqq \varkappa$ ist $G_3^{(\varkappa)} \uparrow F$. In (7) heben sich die Anteile für $1 \leqq \varkappa < \varkappa_1$ und $\varkappa_2 \leqq \varkappa < k$ gegenseitig auf. Wir erhalten:

$$a_{G^*} = (\chi_{\mathbf{H}}(G_1) - \chi_{\mathbf{H}}(G_4)) \sum_{\varkappa=\varkappa_1}^{\varkappa_2 - 1} \delta_{\varkappa, \varkappa+1} = (\chi_{\mathbf{H}}(G_1) - \chi_{\mathbf{H}}(G_4)) \delta_{\varkappa_1, \varkappa_2}.$$

Wir haben nun über alle Kanten eines Kantenzuges $B \in \mathbf{B}$ zu summieren und erhalten wegen (4) gerade die Behauptung des Satzes für diese Anteile.

Ist F entartet, so erhält man aufgrund entsprechender Überlegungen das gleiche Ergebnis multipliziert mit einem Faktor 2. Darum werden die Längen entarteter Eilinien doppelt gezält.

3) Ein Punkt $\mathbf{r}_{\varkappa_1} \in F$ gehöre zur Klasse (c), alle anderen zur Klasse (a). Sei o. E. d. A. $G_1 \uparrow F$, dann ist $G_4 \nmid F$. Für alle $\varkappa = 1, 2, \ldots, k$ gilt entweder $G_2^{(\varkappa)} \uparrow F$ und $G_3^{(\varkappa)} \nmid F$ oder umgekehrt. Aus (7) folgt sofort $a_{G^*} = 0$.

4) Zwei Punkte $\mathbf{r}_{\varkappa_1} \in F$ und $\mathbf{r}_{\varkappa_2} \in F'$ gehören zur Klasse (c). Wegen Forderung (C) aus Def. 2 gehören dann alle anderen Punkte aus $G^* \cap Q$ zur Klasse (a). Wir haben zwei Unterfälle zu unterscheiden:

—F und F' liegen auf der gleichen Seite von G^*. Ohne Einschränkung der Allgemeinheit gelte $G_1 \uparrow F, F'$ (das soll bedeuten, daß F *und* F' von G_1 durchschnitten werden), dann ist $G_4 \nmid F, F'$; für $1 \leqq \varkappa < \varkappa_1$ ist $G_3^{(\varkappa)} \nmid F, F'$, $G_2^{(\varkappa)} \uparrow F, F'$. Für $\varkappa_1 \leqq \varkappa < \varkappa_2$ ist $G_3^{(\varkappa)} \uparrow F$, $G_3^{(\varkappa)} \nmid F'$, $G_2^{(\varkappa)} \nmid F$ und $G_2^{(\varkappa)} \uparrow F'$. Für $\varkappa_2 \leqq \varkappa < k$ ist $G_3^{(\varkappa)} \uparrow F, F'$ und $G_2^{(\varkappa)} \nmid F, F'$. (Man mache sich diese Schlußfolgerungen anhand einer Zeichnung klar.) Für $\varkappa < \varkappa_1$ und $\varkappa_2 \leqq \varkappa$ heben sich die Anteile in (7) gegenseitig auf. Es folgt:

$$a_{G^*} = \left(\chi_{\mathbf{H}}(G_2^{(\varkappa')}) + \chi_{\mathbf{H}}(G_3^{(\varkappa')}) - \chi_{\mathbf{H}}(G_1) - \chi_{\mathbf{H}}(G_2)\right)\delta_{\varkappa_1, \varkappa_2} \tag{8}$$

wobei $\varkappa_1 \leqq \varkappa' < \varkappa_2$ beliebig ausgewählt werden kann.

— Für den Fall, daß F und F' auf verschiedenen Seiten von G^* liegen, folgert man entsprechend und erhält ebenfalls (8).

Damit ist Satz 1 vollständig bewiesen.

4. Zwei Sätze über Eilinien

Wir kommen nun zum zentralen Teil dieser Arbeit. Sei:

$$\mathbf{E} = \{E_1, E_2, \ldots, E_l\} \quad (l = 1, 2, \ldots)$$

ein reguläres System von Eilinien, $\mathbf{N}$ das zugehörige Seilliniennetz und $\mathscr{A}$ die Schnittalgebra. Dann gilt der

SATZ 2. *Sei* $\mathbf{R}(\varnothing) \subset \mathbf{H} \in \mathscr{A}$. *Dann ist*

$$\mu(\mathbf{H}) = \sum_{A \in \mathbf{N}} \gamma_{\mathbf{H}}(A) L(A). \tag{9}$$

FOLGERUNG. Das Maß jedes Schnittes schreibt sich als Linearkombination der Längen der Kurvenstücke des Seilliniennetzes, wobei die Koeffizienten nur die Werte $-2, -1, 0, 1, 2$ annehmen.

BEWEIS. Wir werden die Eilinien aus $\mathbf{E}$ durch ein Polygonsystem $\mathbf{F}$ approximieren, um dann den Satz 1 anzuwenden.

1) Eine nichtentartete Eilinie E sei in der Parameterdarstellung:

$$E = \left\{\mathbf{x};\ \mathbf{x} = \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},\ 0 \leqq t < 1\right\}$$

mit stetigen, bis auf endlich viele Stellen differenzierbaren, Funktionen $x_1(t)$ und $x_2(t)$ gegeben. Weiter sei $\mathbf{x}(0) = \lim_{t \to 1} \mathbf{x}(t)$. Eine Zerlegung des Intervalls $[0, 1)$ sei:

$$0 = t_0 < t_1 < t_2 < \ldots < t_n < 1, \quad (t_{n+1} = t_0).$$

Durch die Strecken $\overline{\mathbf{x}(t_i)\mathbf{x}(t_{i+1})}$ mit $i=0, 1, \ldots, n$ wird ein konvexes Polygon F definiert, es ist $F \subset \hat{E}$. Wir können nun die Zerlegung so fein machen, daß für jedes vorgegebene $\varepsilon>0$ gilt:

$$\hat{F} \subset \hat{E} \subset \hat{F}_\varepsilon, \tag{10}$$

dabei ist F_ε die äußere Paralleleilinie zu F. $\hat{F}_\varepsilon$ ist also die Vereinigung aller Kreisscheiben mit dem Radius ε und dem Mittelpunkt in $\hat{F}$, [3]. Es gilt:

$$L(F) \leqq L(E) \leqq L(F_\varepsilon) = L(F)+2\pi\varepsilon. \tag{11}$$

Fügen wir zu F einen weiteren Punkt aus E als Eckpunkt hinzu, d. h. verfeinern wir die Zerlegung, so bleiben (10) und (11) erhalten.

2) $\varepsilon>0$ sei vorgegeben. Wir approximieren in der eben beschriebenen Weise jede Eilinie $E_i \in \mathbf{E}$ durch ein konvexes Polygon F_i. Ist eine Eilinie E_i entartet, so setzen wir $F_i=E_i$. Es sei $\mathbf{F}=\{F_1, F_2, \ldots, F_l\}$ die Menge der approximierenden Polygone, P' sei deren Eckenmenge.

Durch Verfeinerung der Zerlegungen kann man nun erreichen, daß folgendes gilt:

(I) $P \subset P'$. Das System der gemeinsamen Stützgeraden von $\mathbf{F}$ sei gleich $\mathbf{S}$ — also gleich dem System der gemeinsamen Stützgeraden von $\mathbf{E}$.

(II) Je zwei Polygone besitzen nur endlich viele gemeinsame Punkte.

(III) Sei $\tilde{B}$ der Kantenzug, der $B \in \mathbf{B}$ mit $B \subset E \in \mathbf{E}$ approximiert. Für jede Gerade G, die durch eine Kante $\tilde{K} \subset \tilde{B}$ bestimmt wird (also $\tilde{K} \subset G$), gelte:

$$G \uparrow F_i \Leftrightarrow S \uparrow E_i \quad \text{für} \quad i = 1, 2, \ldots, l \quad \text{und} \quad E_i \neq E;$$

dabei sei S eine Stützgerade an E mit Berührpunkt in $B \subset E$. Die Gerade G durchschneide also genau die Polygone, für die S die zugehörigen Eilinien schneidet (vergl. Abb. 7).

(IV) Keine drei Polygone aus $\mathbf{F}$ besitzen eine gemeinsame Stützgerade.

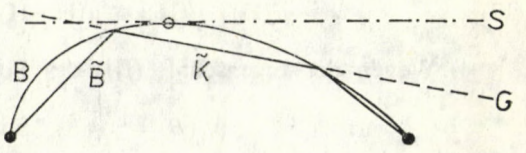

Abb. 7.

zu (I): Wir fügen die Berührpunkte gemeinsamer Stützgeraden aus $\mathbf{S}$ als Eckpunkte ein. Jede gemeinsame Stützgerade von $\mathbf{E}$ ist deshalb gemeinsame Stützgerade von $\mathbf{F}$. Sollten andererseits gemeinsame Stützgeraden von $\mathbf{F}$ nicht in $\mathbf{S}$ sein, so können sie wegen Forderung (B) aus Def. 2 durch Verfeinerung der Approximation gelöscht werden.

zu (II): Besitzen zwei Polygone unendlich viele Punkte gemeinsam, so können das nur gemeinsame Streckenstücke sein. Wegen Forderung (A) aus Def. 2 kann man dann in einem der beiden Polygone die entsprechenden Kanten ersetzen.

zu (III): Wir nehmen an, für ein $E_i \neq E$ mit $i=1, 2, \ldots, l$ gelte:

a) $S \uparrow E_i$ und $G \nuparrow F_i$.

i) Es sei $G \uparrow E_i$; dann existiert in der Halbebene, durch G gegeben, die F_i gegenüberliegt, ein Punkt aus E_i, der F_i als Eckpunkt hinzugefügt wird. Dann gilt $G \uparrow F_i$.

ii) Falls $G \overset{\uparrow}{\uparrow} E_i$ ist, müssen wir etwas weiter ausholen. Wir behaupten: es existiert ein Kantenzug $\tilde{B}$, so daß für jede Gerade G mit $\tilde{K} \subset G$ und $\tilde{K} \subset \tilde{B}$ gilt $G \uparrow E_i$. Dann verfahren wir weiter wie unter Punkt a)i). Wir beweisen die Behauptung indirekt. Bei immer weitergehender Verfeinerung der Approximation von B konvergieren* die durch die Kanten $\tilde{K} \subset \tilde{B}$ bestimmten Geraden gegen Stützgeraden S. Sei $\{G_m\}_{m=1}^{\infty}$ mit $S = \lim_{m \to \infty} G_m$ eine solche Geradenfolge mit $G_m \overset{\uparrow}{\uparrow} E_i$ für jedes $m = 1, 2, \ldots$. Aus $S \uparrow E_i$ folgt, falls E_i nicht entartet ist, daß ein Punkt $\mathbf{x} \in S \cap \dot{E}_i$ existiert. $\dot{E}_i$ ist offen, eine Umgebung $U(\mathbf{x}) \subset \dot{E}_i$ darf von keiner Geraden G_m geschnitten werden. Das ist ein Widerspruch zur Konvergenz der Geradenfolge gegen S. In ähnlicher Weise erzeugt man einem Widerspruch bei entarteten Eilinien. Also existieren Geraden aus jeder Folge von Geraden, die gegen eine Stützgerade konvergieren, für die $G \uparrow E_i$ ist. Aus den mit E gemeinsamen Abschnitten solcher Geraden erzeugen wir den Kantenzug $\tilde{B}$.

b) $S \overset{\uparrow}{\uparrow} E_i$ und $G \uparrow F_i$.

Wieder sei $\{G_m\}_{m=1}^{\infty}$ eine gegen S konvergierende Geradenfolge mit $G_m \uparrow F_i$ für alle $m = 1, 2, \ldots$; es gilt dann auch $G_m \uparrow E_i$, da $F_i \subset \hat{E}_i$ ist. Daraus aber folgt, daß S gemeinsame Stützgerade an E und E_i sein muß. Das ist ein Widerspruch zur Voraussetzung. Es muß also Geraden in der Folge geben, für die $G_m \overset{\uparrow}{\uparrow} F_i$ gilt. Aus den mit E gemeinsamen Abschnitten solcher Geraden erzeugen wir den Kantenzug $\tilde{B}$.

zu (IV): Existiert eine solche Stützgerade, so kann sie durch Verfeinerung der Zerlegungen wegen (C) aus Def. 2 gelöscht werden.

Es folgt, daß das Polygonsystem $\mathbf{F}$ regulär ist.

3) Für die Größen, die sich auf das Polygonsystem $\mathbf{F}$ beziehen, verwenden wir (wie bisher) die entsprechenden Bezeichnungen des Eiliniensystems $\mathbf{E}$ versehen mit dem Symbol „$\sim$“.

Wir schätzen ab:

$$|\mu(\mathbf{H}) - \mu(\tilde{\mathbf{H}})| \leqq \mu(\mathbf{H} - \tilde{\mathbf{H}}) + \mu(\tilde{\mathbf{H}} - \mathbf{H}) \leqq 4\pi l\varepsilon \tag{12}$$

4) Wegen der Eigenschaft (III) des Polygonsystems $\mathbf{F}$ gilt:

$$\gamma_{\mathbf{H}}(B) = \gamma_{\tilde{\mathbf{H}}}(\tilde{B}) \quad \text{für jedes} \quad B \in \mathbf{B}.$$

Außerdem ist:

$$\gamma_{\mathbf{H}}(K) = \gamma_{\tilde{\mathbf{H}}}(K) \quad \text{für jedes} \quad K \in \mathbf{K}.$$

Weiter gilt deshalb unter Verwendung von (11):

$$\Big| \sum_{A \in \mathbf{N}} \gamma_{\mathbf{H}}(A) L(A) - \sum_{\tilde{A} \in \tilde{\mathbf{H}}} \gamma_{\tilde{\mathbf{H}}}(\tilde{A}) L(\tilde{A}) \Big| = \tag{13}$$

$$= \Big| \sum_{B \in \mathbf{B}} \gamma_{\mathbf{H}}(B) \big(L(B) - L(\tilde{B})\big) \Big| \leqq \sum_{B \in \mathbf{B}} \big(L(B) - L(\tilde{B})\big) \leqq 2\pi l\varepsilon.$$

* Vergleiche Anhang.

5) Mit Hilfe der Dreiecksungleichung erhalten wir aus (12), Satz 1 und (13):

$$\left|\mu(\mathbf{H})-\sum_{A\in N}\gamma_{\mathbf{H}}(A)L(A)\right| \leqq |\mu(\mathbf{H})-\mu(\tilde{\mathbf{H}})|+$$

$$+\left|\sum_{A\in N}\gamma_{\mathbf{H}}(A)L(A)-\sum_{\tilde{A}\in\tilde{N}}\gamma_{\tilde{\mathbf{H}}}(\tilde{A})L(\tilde{A})\right| \leqq 6\pi l\varepsilon.$$

Wir können ε beliebig klein wählen; daraus folgt die Behauptung des Satzes.

Als Anwendung von Satz 2 berechnen wir nun die Maße M_s, die bereits in der Einleitung angesprochen wurden. Wir definieren auf $\mathbf{N}$ eine Funktion σ.

DEFINITION 5. Für $K\in\mathbf{K}$ sei $\sigma(K)$ die Anzahl der von der Geraden S mit $K\subset S\in\mathbf{S}$ durchschnittenen Eilinien. Für $B\in\mathbf{B}$ mit $B\subset E$ sei S eine beliebige Stützgerade an E mit Berührpunkt in B. $\sigma(B)$ sei dann die Anzahl der von S durchschnittenen Eilinien aus $\mathbf{E}$. Es gilt: $\sigma(K)=0, 1, \ldots, l-2$ und $\sigma(B)=0, 1, \ldots, l-1$.

DEFINITION 6.

$$W_s^0 = \sum_{\substack{B\in\mathbf{B}\\ \sigma(B)=s}} L(B) \quad \text{für} \quad s=0, 1, \ldots, l-1, \tag{14}$$

$$W_s^{\pm} = \sum_{\substack{K\in\mathbf{K},\ \sigma(K)=s\\ \tau(K)=\pm 1}} L(K) \quad \text{für} \quad s=0, 1, \ldots, l-2, \tag{15}$$

$$W_s^0 = W_s^+ = W_s^- = 0 \quad \text{sonst.}$$

Wir führen noch die Operatoren Δ und Δ^2 für die erste und zweite Differenz einer auf den ganzen Zahlen definierten Funktion f ein; also:

$$\Delta f(s) = f(s)-f(s-1)$$

und

$$\Delta^2 f(s) = \Delta\big(\Delta f(s)\big) = f(s)-2f(s-1)+f(s-2).$$

Sei nun*:

$$\mathbf{H}_s = \bigcup_{I\subset I_l,\ |I|=s} \mathbf{R}(I) \quad \text{für} \quad s=0, 1, \ldots, l$$

die Geradenmenge aus $\mathscr{A}$, deren Geraden s der Eilinien durchschneiden. Die Maße $M_s=\mu(\mathbf{H}_s)$ sind dann durch den Satz 3 gegeben.

SATZ 3. *Sei* $s=1, 2, \ldots, l$, *dann ist:*

$$M_s = -\Delta W_s^0 - \Delta^2 W_s^+ + \Delta^2 W_s^-. \tag{16}$$

FOLGERUNGEN. 1) Speziell ist das Maß der Geradenmenge $\mathbf{H}_l$, deren Geraden alle Eilinien durchschneiden:

$$M_l = W_{l-1}^0 - W_{l-2}^+ + W_{l-2}^-. \tag{17}$$

* Die Betragsstriche bei Mengen bezeichnen die Anzahl der Elemente.

2) Das Maß der Geradenmenge, deren Geraden wenigstens eine Eilinie durchschneiden, ist:

$$M_1^* = \sum_{s=1}^{1} M_s = W_0^0 + W_0^+ - W_0^-. \tag{18}$$

Diese letzte Formel läßt sich durch leichte Rechnung nachprüfen.

BEMERKUNG. Diese Ergebnisse lassen sich auch aus den in [1a] angegebenen Formeln ableiten.

BEWEIS. Wir verwenden den Satz 2.

1) Wir berücksichtigen zuerst die Anteile, die sich von den Bögen $B \in \mathbf{B}$ ergeben. Für $\sigma(B)=s$ ist $\gamma_{\mathbf{H}_s}(B)=1$, für $\sigma(B)=s-1$ ist $\gamma_{\mathbf{H}_s}(B)=-1$ und $\gamma_{\mathbf{H}_s}(B)=0$ sonst. Insgesamt ergeben die Bögen in Satz 2 den Anteil $-\Delta W_s^0$.

2) Sei $K \in \mathbf{K}$ mit $\tau(K)=1$. Für $\sigma(K)=s$ folgt $\gamma_{\mathbf{H}_s}(K)=-1$, für $\sigma(K)=s-1$ ist $\gamma_{\mathbf{H}_s}(K)=2$ und für $\sigma(K)=s-2$ ist $\gamma_{\mathbf{H}_s}(K)=-1$; in den übrigen Fällen ist $\gamma_{\mathbf{H}_s}(K)=0$. Für diese Kanten erhalten wir in (9) insgesamt den Beitrag $-\Delta^2 W_s^+$.

3) Schließlich summieren wir über die Kanten $K \in \mathbf{K}$ mit $\tau(K)=-1$. Für $\sigma(K)=s$ folgt $\gamma_{\mathbf{H}_s}(K)=1$, für $\sigma(K)=s-1$ ist $\gamma_{\mathbf{H}_s}(K)=-2$ und für $\sigma(K)=s-2$ ist $\gamma_{\mathbf{H}_s}(K)=-1$. Da sich sonst $\gamma_{\mathbf{H}_s}(K)=0$ ergibt, erhalten wir für diese Kanten den Beitrag $\Delta^2 W_s^-$.

Damit ist Satz 3 bewiesen.

5. Schlußbemerkungen

1) Sei E^* der Rand der konvexen Hülle der Eilinien $E_1, E_2, \ldots, E_l$. Nach CROFTON ist das Maß der Geradenmenge, deren Geraden E^* durchschneiden, durch

$$\mu(\{G;\ G \uparrow E^*\}) = L(E^*)$$

gegeben. Das Maß der Geradenmenge, deren Geraden zwar E^* aber keine der Eilinien aus $\mathbf{E}$ durchschneiden, ist wegen (18):

$$M_0 = \mu(\{G;\ G \uparrow E^*,\ G \nmid E_i \text{ für } i = 1, 2, \ldots, l\}) =$$

$$= L(E^*) - W_0^0 - W_0^+ + W_0^-.$$

Jede Kante und jedes Bogenstück aus $\mathbf{N}$, das in E^* enthalten ist, ergibt einen Beitrag zu $W_0^0 + W_0^+$, so daß folgende Ungleichung gilt:

$$L(E^*) \leqq W_0^0 + W_0^+.$$

Man beweist leicht folgende Äquivalenzen:

$$M_0 = 0 \Leftrightarrow L(E^*) = W_0^0 + W_0^+ \Leftrightarrow W_0^- = 0.$$

Betrachten wir ein Beispiel: Wenn $\mathbf{E}$ aus den waagerechten Schnitten durch Baumstämme eines Waldes besteht, so ist:

$$\frac{M_0}{L(E^*)} = 1 - \frac{W_0^0 + W_0^+ - W_0^-}{L(E^*)}$$

die Wahrscheinlichkeit dafür, daß ein ungezielter Schuß durch den Wald keinen Baum trifft, unter der Bedingung, daß der „Wald getroffen" wird.

2) Es gilt der

SATZ 4. *Sei* $s=2, 3, \ldots, 1$, *dann gilt:*

$$M_s > 0 \Rightarrow M_{s-1} > 0.$$

BEWEIS. Aus $M_s>0$ folgt die Existenz einer Geradenumgebung $\mathbf{U}(G)$, deren Geraden s der Eilinien durchschneiden. Die Menge $\mathbf{S}$ der gemeinsamen Stützgeraden ist endlich, wir können deshalb in $\mathbf{U}(G)$ eine Gerade G^* finden, deren Richtung von allen Richtungen der gemeinsamen Stützgeraden um einen positiven Winkel abweicht. Diese Gerade verschieben wir nun so weit parallel, bis wir eine Stützgerade passiert haben, und die Gerade somit eine Eilinie weniger durchschneidet. Sicher gibt es dann auch eine Umgebung dieser Geraden, deren Geraden eine Eilinie weniger durchschneiden. Daraus folgt $M_{s-1}>0$.

Sei $\bar{s}$ das maximale s, für das $M_s>0$ ist. Aus Satz 4 folgt dann, daß die Verteilung M_s immer so beschaffen ist, daß $M_s>0$ für $1\leqq s\leqq\bar{s}$ ist.

6. Anhang

Wir stellen einige Ergebnisse über Geradenumgebungen zusammen. Eine Gerade $G\in\mathbf{G}$ sei in der Hesse'schen Normalform (1) gegeben. Sei:

$$X = \{(p, \varphi);\ 0 \leqq p,\ 0 \leqq \varphi < 2\pi\},$$

so wird durch $v: \mathbf{G}\leftrightarrow X$ mit $v(G)=(p, \varphi)$ eine bijektive Abbildung definiert. Durch:

$$U_{\varepsilon,\delta}((p, \varphi)) = \{(p', \varphi');\ |p-p'| < \varepsilon,\ |\varphi-\varphi'| < \delta\}$$

mit $\varepsilon>0$ und $\delta>0$ ist eine in X offene Umgebung von (p, φ) gegeben. Durch:

$$\mathbf{U}(G) = v^{-1}(U_{\varepsilon,\delta}(v(G)))$$

wird eine offene Geradenumgebung der Geraden G definiert. Damit haben wir auf $\mathbf{G}$ eine Topologie zur Verfügung. v und v^{-1} sind stetig.

Gegeben sei eine Geradenfolge $\{G_m\}_{m=1}^{\infty}$. Wir schreiben:

$$G = \lim_{m\to\infty} G_m,$$

wenn für jede Umgebung $\mathbf{U}(G)$ eine Zahl m_0 existiert, so daß für alle $m\geqq m_0$:

$$G_m\in\mathbf{U}(G)$$

ist.

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CYCLIC MODULES OVER SOME MODULAR GROUP ALGEBRAS

by

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1. Introduction

Let G be a finite p-group and let K be a field of characteristic p. A KG-module M is almost free if its restriction M_H to any proper subgroup H of G is a free KH-module. In [2] it was shown that if G/G' is not elementary abelian, where G' is the derived subgroup of G, then every almost free KG-module is free. The purpose of this paper is to investigate cyclic KG-modules in the case where G is a 2-group and G/G' is elementary abelian. We shall show that if G has a non-abelian factor of order less than 64, every cyclic almost free KG-module is free. In addition some information is derived about the relationship between the structure of a principal left ideals KGl and the nature of the generating element l for certain 2-groups G.

Throughout this paper G will denote a finite 2 group and K a field of characteristic 2. If M is a KG-module M_H is its restriction to a KH-module for H a subgroup of G. If elements $x_1, \ldots, x_n \in G$ generate H, write $H=\langle x_1, \ldots, x_n\rangle$. Rad KH is the radical of KH and $\tilde{H}=\sum_{h\in H} h$. We will need the following well known result.

LEMMA 1.1. *A KG-module M has a free direct summand if and only if $\tilde{G}M \neq \{0\}$.*

A KG-module M is periodic if there exists an exact sequence

$$0 \to M \to F_1 \to \ldots \to F_n \to M \to 0$$

where $F_1, \ldots, F_n$ are free KG-modules. The period of such a module is the shortest length n of any such sequence.

2. Groups in which every factor is elementary abelian

We wish to consider some specific 2-groups. Let $G_1=\langle x|x^4=1\rangle$,

$$Q_2 = \langle x_1, x_2 | x_1^4 = 1,\ x_1^2 = x_2^2 = (x_1x_2)^2\rangle$$

and

$$D_2 = \langle x_1, x_2 | x_1^4 = x_2^2 = 1,\ x_2x_1x_2 = x_1^3\rangle.$$

Q_2 is a quaternion group and D_2 is a dihedral group. For n even, $n\geqq 4$ let Q_n, D_n denote the extra-special groups of order 2^{n+1}. By [2, Theorem 5.2] such a group is either a central product of ordinary quaternion groups or a central product of quaternion groups with a dihedral group. So let $Q_n=\langle x_1, \ldots, x_n\rangle$ where for each $t=1, \ldots, n/2$, $\langle x_{2t-1}, x_{2t}\rangle \cong Q_2$, $x_1^2=x_2^2=\ldots=x_n^2$, and if $i<j$, $x_ix_j=x_jx_i$ except when i is odd and $j=i+1$. Let $D_n=\langle x_1, \ldots, x_n\rangle$ with the same relations except

that $\langle x_{n-1}, x_n\rangle \cong D_2$ and $x_n^2=1$. These representations of Q_n, D_n in terms of generations and relations will be used throughout this paper.

If n is odd, let $Q_n=D_n=\langle x_1, \ldots, x_n\rangle$ where $\langle x_1, \ldots, x_{n-1}\rangle = Q_{n-1}$ represented as before, $x_n^2=x_1^2$, and $x_i x_n = x_n x_i$ for $i=1, \ldots, n-1$. It should be noted that an isomorphic group is obtained if we let $\langle x_1, \ldots, x_{n-1}\rangle = D_{n-1}$, since we can replace x_{n-1} by $x'_{n-1}=x_{n-1}x_n$.

G_n will be used to denote either Q_n or D_n. For convenience write x^2 instead of x_1^2. In every case $|G_n|=2^{n+1}$, $G'_n=\Phi(G_n)=\langle x^2\rangle$. $Z(G_n)=\langle x^2\rangle$ if n is even and $Z(G_n)=\langle x_n\rangle$ if n is odd. Let $^-: G_n \to \bar{G}_n = G_n/\langle x^2\rangle$.

LEMMA 2.1. *Let G be a non-abelian 2-group of which every non-trivial factor is elementary abelian. Then $G \cong Q_n$ or D_n for some n.*

PROOF. G must have a unique involution x in its centre. So $G'=\Phi(G)=\langle x\rangle$. The center $Z(G)$ must be cyclic of order 2 or 4. If $|Z(G)|=2$, G is extra-special. If $|Z(G)|=4$, there exists a maximal subgroup H of G with $H \cap Z(G)=\langle x\rangle$. Here H must be extral-special.

For any sequence $i_1, \ldots, i_t$ of integers with $1 \leqq i_1 < i_2 < \ldots < i_t \leqq n$ let

$$g(i_1, \ldots, i_t) = (1+x_{i_1})(1+x_{i_2})\ldots(1+x_{i_t})$$

in KG_n.

LEMMA 2.2. *The elements 1, $1+x^2$, $g(i_1, \ldots, i_t)$ and $(1+x^2)g(i_1, \ldots, i_t)$, for all such sequences $i_1, \ldots, i_t$, form a K-basis for KG_n. Moreover* $(\operatorname{Rad} KG_n)^{n+3}=\{0\}$.

PROOF. Note that $(1+x)^2 KG_n$ and $KG_n/(1+x^2)KG_n$ are free as $K\bar{G}_n$ modules. It is not difficult to show that $K\bar{G}_n$ has a basis consisting of $\bar{1}$ and all $\overline{g(i_1, \ldots, i_t)}$ (see [2, (4.1)]). The last statement follows from the fact that any product of $n+3$ of these basis elements of $\operatorname{Rad} KG_n$ is zero.

PROPOSITION 2.3. *Let $G=G_n$ for some n and let $l \in KG$. Suppose* $l \equiv \sum_{i=1}^{n} p_i(1+x_i)$ mod $(\operatorname{Rad} KG)^2$ *for $p_i \in K$. Let*

$$\alpha = \alpha(l) = \sum_{i=1}^{n/2} (p_{2i-1}^2 + p_{2i-1}p_{2i} + p_{2i}^2)$$

if n is even and $G=Q_n$,

$$\alpha = \alpha(l) = \sum_{i=1}^{(n-1)/2} (p_{2i-1}^2 + p_{2i-1}p_{2i} + p_{2i}^2) + p_{n-1}^2 + p_{n-1}p_n$$

if n is even and $G=D_n$, and

$$\alpha = \alpha(l) = \sum_{i=1}^{n/2-1} (p_{2i-1}^2 + p_{2i-1}p_{2i} + p_{2i}^2),$$

if n is odd. If $\alpha \neq 0$ the left ideal $M=KGl$ is periodic of period 2 and

$$\operatorname{Dim}_K M = \frac{3}{4} \operatorname{Dim}_K KG.$$

PROOF. A direct calculation shows that $l^2=\alpha(1+x^2)+m$ where $m\in$ $\in(1+x^2)$ Rad KG. Hence $l^2=u(1+x^2)$ where u is a unit in KG. We wish to show that the annihilator of l is exactly the left ideal $V=KG(1+x^2)l$. Clearly V is in the annihilator of l.

Now suppose $p_i\neq 0$. Let $H=\langle x_j \mid j\neq i\rangle$. Then $\tilde{H}l=p_i\tilde{G}\neq 0$, and $KH(1+x^2)l\cong$ $\cong KH/(1+x^2)KH\cong K\overline{H}$ as $K\overline{H}$ modules. So $\operatorname{Dim} V\geqq\frac{1}{4}\operatorname{Dim} KG$. Likewise $KHl/(KG(1+x^2)\cap KHl)\cong K\overline{H}$. Thus $\operatorname{Dim} M\geqq\frac{3}{4}\operatorname{Dim} KG$, since $\operatorname{Dim} KG(1+x^2)=$ $=\frac{1}{2}\operatorname{Dim} KG$ and $KG(1+x^2)\subseteqq M$. If $\theta\colon KG\to KGl$ by $\theta(\omega)=\omega l$ for all $\omega\in KG$ V must be the kernel of θ by a dimension argument. Also

$$0\to V\to KG\xrightarrow{\theta} M\to 0$$

is exact and $\operatorname{Dim} M=\frac{3}{4}\operatorname{Dim} KG$.

By a similar argument M is the annihilator of $(1+x^2)l$. So the sequence

$$0\to M\to KG\xrightarrow{\sigma} V\to 0$$

when $\sigma(\omega)=\omega(1+x^2)l$, is exact. This proves the proposition.

It can be checked that if $G=Q_2$ or D_2 the module KGl is always periodic if $l\not\equiv 0 \bmod (\operatorname{Rad} KG)^2$.

COROLLARY 2.4. *Let l be as in* (2.3) *with* $\alpha(l)\neq 0$. *Then KGl is not almost free.*

PROOF. For KGl to be almost free it would be necessary that $\operatorname{Dim} KGl=$ $=\frac{1}{2}\operatorname{Dim} KG$.

PROPOSITION 2.5. *Let $G=G_n$ and let $u_1,\ldots,u_t$ be involutions in G with $u_iu_j=$ $=u_ju_i$ for all i, j. Suppose further that they are independent in the sense that no u_i is in the subgroup of G generated by x^2 and the others. Let $p_1,\ldots,p_t$ be non-zero elements of K. If $l=\sum_{i=1}^{t}p_i(1+u_i)$ then KGl is periodic with period* 1 *and* $\operatorname{Dim} KGl=$ $=\frac{1}{2}\operatorname{Dim} KG$.

PROOF. First note that $l^2=0$. So the kernel of the homomorphism $\theta\colon KG\to KGl$ given by $\theta(\omega)=\omega l$, contains KGl. There exists a maximal subgroup $\overline{H}$ of $\overline{G}$ with $\bar{u}_1\notin\overline{H}$, $\bar{u}_2,\ldots,\bar{u}_t\in\overline{H}$. Hence there is a maximal subgroup H of G with $u_1\notin H, u_2,$ $\ldots, u_t\in H$. This implies $\tilde{H}l=p_1\tilde{G}\neq 0$ and $KHl\cong KH$. Therefore $\operatorname{Dim} KGl=$ $=\frac{1}{2}\operatorname{Dim} KG$ and the sequence

$$0\to KGl\to KG\xrightarrow{\theta} KGl\to 0$$

is exact.

3. Some principal ideal in KG_n, $n=3, 4, 5$

We wish to show in this section that for $n=3, 4, 5$ there is something of a weak converse to Proposition (2.5). First we need the following.

PROPOSITION (3.1). *Let* $G=G_n$, $n\geqq 3$. *Suppose* $l\in KG$, $l\equiv \sum_{i=1}^{n} P_i(1+x_i)\not\equiv$ $\not\equiv 0 \bmod (\text{Rad } KG)^2$. *Assume* $\text{Dim } KGl=\frac{1}{2}\text{Dim } KG$. *Then*

$$\sum (p_{2i-1}^2 p_{2i}+p_{2i-1}p_{2i}^2)=0 \tag{3.2, n}$$

where the sum is over $i=1, \ldots, (n-1)/2$ *if* n *is odd and over* $i=1, \ldots, n/2$ *if* n *is even.*

PROOF. Suppose $p_k\neq 0$. Let $H=\langle x_i | i\neq k\rangle$. Then $\tilde{H}l=p_k\tilde{G}\neq 0$. Hence $M=KGl=KHl$ since $\text{Dim } M=\frac{1}{2}\text{Dim } KG$. Similarly $(KG/M)_H\cong KH$ and in particular $\tilde{H}\notin M$. This implies that $KH\cap M=\{0\}$.

There exist unique $\alpha, \beta\in KH$ such that $l=\alpha+\beta x_k$. We claim that α is a unit in KH. For if α were not a unit there would exist a non-zero element $\gamma\in KH$ with $\gamma\alpha=0$. Since $\gamma l\neq 0$, $0\neq x_k^{-1}\gamma l=x_k^{-1}\gamma\beta x_k\in KH$, which is impossible. Let $c\in K$ and $\mu\in \text{Rad } KH$ be elements such that $\alpha^{-1}=c^{-1}(1+\mu)$. Then $(1+\mu)l=c+(1+\mu)\beta x_k=$ $=c+\nu x_k$ for $(1+\mu)\beta=\nu\in KH$. In this case $KGl=KG(1+\mu)l$ and $(1+\mu)l\equiv$ $\equiv \Sigma p_i(1+x_i) \bmod (\text{Rad } KG)^2$. Hence there is no loss of generality in substituting $(1+\mu)l$ for l, and we shall do this. It follows that $l^2=c^2+\nu x_k\nu x_k\in KH$. Therefore $l^2=0$.

By lemma (2.2) there exist $q_{ij}\in K$, $r\in K$ such that

$$l=\sum p_i(1+x_i)+\sum_{i<j} q_{ij}(1+x_i)(1+x_j)+r(1+x^2) \quad \bmod (\text{Rad } KG)^3.$$

We proceed by multiplying l times itself and computing the coefficient of l^2 on each of the basis element given in (2.2). Of course each of these coefficients must be zero. If n is even and $i=2j-1$ the coefficient on $(1+x^2)(1+x_i)$ is

$$p_i p_{i+1}+p_i q_{i,i+1}+(p_{2k-1}q_{i,2k}+p_{2k}q_{i,2k-1})=0 \tag{3.3, i, n}$$

where the sum is over all $k\neq j$, $k=1, \ldots, n/2$. If n is even and $i=2j$, the coefficient is

$$p_i p_{i-1}+p_i q_{i-1,i}+\sum (p_{2k-1}q_{i,2k}+p_{2k}q_{i,2k-1})=0, \tag{3.3, i, n}$$

where the sum is as before. We assume here $q_{ij}=q_{ji}$. If n is odd, $i<n$, we get the same equations except that the sums are taken over all $k\neq j$, $k=1, \ldots, (n-1)/2$.

If i is odd, $i<n$, multiply equation (3.3, i, n) by p_{i+1}. If i is even multiply equation (3.3, i, n) by p_{i-1}. Now sum the resulting equations from $i=1$ to $n-1$ if n is odd, and from $i=1$ to n if n is even. The result is (3.2, n).

COROLLARY 3.4. *Let* $G=G_3$ *and suppose* $l\in KG$, $l\notin(\operatorname{Rad} KG)^2$. *If* $\operatorname{Dim} KGl=\frac{1}{2}\operatorname{Dim} KG$ *there exists an involution* $u\in G$ *such that*

$$l \equiv a(1+u) \bmod (\operatorname{Rad} KG)^2$$

for some $a\in K$.

PROOF. Suppose $l\equiv\Sigma p_i(1+x_i) \bmod (\operatorname{Rad} KG)^2$. By (2.3) and (3.1)

$$p_1^2+p_1p_2+p_2^2+p_3^2=0, \quad p_1^2p_2+p_1p_2^2=0.$$

From the second equation $p_1=0$ or $p_2=0$ or $p_1=p_2$. If $p_1=0$ then $p_2=p_3$ by the first equation. In this case $l\equiv p_3(1+x_2x_3) \bmod (\operatorname{Rad} KG)^2$. Similarly if $p_2=0$, $l\equiv p_3(1+x_1x_3)$, and if $p_1=p_2$, $l\equiv p_3(1+x_1x_2x_3)$.

COROLLARY 3.5. *Let* $G=Q_4$, $l\in KG$, $l\notin(\operatorname{Rad} KG)^2$. *If* $\operatorname{Dim} KGl=\frac{1}{2}\operatorname{Dim} KG$, *there exist involutions* $u_1, u_2\in G$ *and elements* $a_1, a_2\in K$ *such that* $u_1u_2=u_2u_1$ *and*

$$l \equiv a_1(1+u_1)+a_2(1+u_2) \bmod \operatorname{Rad} KG)^2.$$

PROOF. Write $l\equiv\Sigma p_i(1+x_i) \bmod (\operatorname{Rad} KG)^2$. If $p_4=0$ (2.3) and (3.1) give the same equation and the same result as in (3.4). Hence we can assume $p_4=1$ by replacing l by $\frac{1}{p_4}l$ if necessary.

From (2.3) and (3.1) we have

$$p_1^2+p_1p_2+p_2^2+p_3^2+p_3+1=0$$

and

$$p_1^2p_2+p_1p_2^2+p_3^2+p_3=0.$$

The sum of these equations is

$$p_1^2p_2+p_1p_2^2+p_1^2+p_1p_2+p_2^2+1=0$$

or

$$(p_1+1)(p_2+1)(p_1+p_2+1)=0.$$

Hence either $p_1=1$, $p_2=1$ or $p_1=p_2+1$. If for example $p_1=1$ then $(p_2+p_3)^2=p_2+p_3$ and either $p_2=p_3$ or $p_2=p_3+1$. In the first case

$$l \equiv (l+x_1x_4)+p_2(1+x_2x_3) \bmod (\operatorname{Rad} KG)^2,$$

and in the second

$$l \equiv (1+x_1x_2x_4)+p_3(1+x_2x_3).$$

The other cases follow in a similar fashion.

It seems likely that this process can be continued indefinitely. To handle the cases $G=D_4$, G_5 we need the following lemma. Since the details of the proof are lengthy but straightforward, they are left to the reader as an exercise.

LEMMA 3.6. *Let* $G=D_4$ *or* G_5. *Suppose* $l\in KG$, $l\equiv p_i(1+x_i)\not\equiv 0 \bmod (\operatorname{Rad} KG)^2$. *If* $\operatorname{Dim} KGl=\frac{1}{2}\operatorname{Dim} KG$, *then*

$$p_1^4p_2+p_1p_2^4+p_3^4p_4+p_3p_4^4=0.$$

PROOF. The procedure is very similar that in the proof of (3.1). There exist $q_{ij}\in K$, $r_{ijk}\in K$ such that

$$l \equiv \sum p_i(1+x_i)+\sum_{i<j} q_{ij}(1+x_i)(1+x_j)+$$

$$+\sum_{i<j<k} r_{ijk}(1+x_i)(1+x^2)(1+x_k) \bmod [(\operatorname{Rad} KG)^3+(1+x^2)KG].$$

This time we compute the coefficients of $l^2=0$ on elements of the form $(1+x^2)(1+x_i)(1+x_j)$. Appropriate manipulations of the resulting equations prove the lemma.

COROLLARY 3.7. *Let* $G=D_4$ *or* G_5. *Suppose* $l\in KG$, $l\notin(\operatorname{Rad} KG)^2$ *and* $\operatorname{Dim} KGl=\frac{1}{2}\operatorname{Dim} KG$. *If* $G=D_4$ *there exist elements* $u\in G$, $a\in K$ *with* $u^2=1$ *and*

$$l \equiv a(1+u) \mod (\operatorname{Rad} KG)^2.$$

If $G=G_5$ *there exist* $u_1, u_2\in G$, $a_1, a_2\in K$ *with* $u_1^2=u_2^2=1$, $u_1u_2=u_2u_1$ *and*

$$l \equiv a_1(1+u_1)+a_2(1+u_2) \mod (\operatorname{Rad} KG)^2.$$

PROOF. Suppose $G=D_4$. As in the proof of (3.5) we can assume $p_4=1$. By (2.3) and (3.1)

$$p_1^2+p_1p_2+p_2^2+p_3^2+p_3=0, \quad p_1^2p_2+p_1p_2^2+p_3^2+p_3=0.$$

From (3.6)

$$p_3^4+p_3=p_1^4p_2+p_1p_2^4=(p_1^2p_2+p_1p_2)(p_1^2+p_1p_2+p_2^2)=p_3^4+p_3^2.$$

Hence p_3 is 0 or 1. In either case $p_1=p_2=0$. Therefore $l\equiv(1+x_4)$ or $l\equiv(1+x_3x_4) \bmod (\operatorname{Rad} KG)^2$.

Now suppose $G=G_5$. If $p_5=0$ we have the same equations as in the proof of (3.5). So again assume $p_5=1$. By (3.6), (2.3), (3.1)

$$p_1^4p_2+p_1p_2^4=(p_3^2p_4+p_3p_4^2)(p_3^2+p_3p_4+p_4^2)=$$

$$=(p_1^2p_2+p_1p_2^2)(p_1^2+p_1p_2+p_2^2+1)=p_1^4p_2+p_1p_2^4+p_1^2p_2+p_1p_2^2.$$

Hence $p_1=0$ or $p_2=0$ or $p_1=p_2$. Similarly $p_3=0$, $p_4=0$ or $p_3=p_4$. This is sufficient to prove the proposition.

4. Cyclic almost free modules

Statements (3.4), (3.5) and (3.7) imply that every cyclic almost free KG_n-module M is free when $n=3, 4, 5$. For if u_1, u_2 are commuting involutions in G_n there exists a maximal subgroup H of G with $u_1, u_2\in H$. If $M=KGl$ where $l\equiv a_1(1+u_1)+a_2(1+u_2) \bmod (\operatorname{Rad} KG)^2$ then $\tilde{H}M=\{0\}$.

THEOREM 4.1. *Let* G *be a 2-group which contains a normal subgroup* H *with* G/H *not elementary abelian and* $|G/H|\leqq 64$. *Let* K *be a field of characteristic* 2. *Then every cyclic almost free* KG*-module is free.*

PROOF. Let M be a cyclic almost free KG-module. Let H be a largest normal subgroup of G such that G/H is not elementary abelian. Then $\tilde{H}M$ is a cyclic almost free $K(G/H)$-module which is free if and only if M is free (see [2, (2.3)]). If G/H is abelian the theorem is proved by [2, (2.4)]. If G/H is not abelian, $G/H \approx G_n$ for some $n \leqq 5$. If $G = G_2$ see [1, (4.1)] or [2, 5.1]. Hence M must be free.

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SOME PROBLEMS ON EXISTENCE IN CONVEX OPTIMAZITION ON INFINITE DIMENSIONAL VECTOR SPACES

by

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1. Introduction

Let U be a subset of a convex space E and E^* the continuous dual of E. We assume that a convex close cone K (in $\sigma(E, E^*)$-topology) is chosen in E and write $x_1 \geqq x_2$ if $x_1 - x_2 \in K$. Consider the following type of problem: to find

$$\inf f(x) \tag{1.1}$$

subject to the condition that

$$g(x) \geqq 0, \quad x \in U \tag{1.2}$$

where $f(x)$ is a real function on U and $g: U \to E$.

The problem (1.1), (1.2) is known as a mathematical programming. Moreover, it is known that, under definite conditions, (1.1), (1.2) is equivalent to the problem of finding the saddle point of its lagrangian function on $E \times E^*$

$$L(x, y) = f(x) - \langle g(x), y \rangle, \quad (x \in U, \ y \geqq 0)$$

where $\langle g(x), y \rangle$ is the value of the linear functional $y \in E^*$ on the element $g(x)$.

Hence, this problem directly concerns the general question of minimax problem, i.e. to find a saddle point for a convex-concave function $h(x, y)$ on $E \times F$ where E and F are two convex spaces. It is interested to know that: for which subsets $U \times V \subseteq E \times F$ does a saddle point $(x_0, y_0) \in U \times V$ exist? Of course, we consider only infinite dimensional vector spaces. As for the finite dimensional vector spaces the problem of existence has been throughly discussed (c.f. ROCKAFELLAR [4]).

In [3], MOREAU has given the necessary and sufficient conditions for convex-concave function to have a saddle point. Of course, the author has presupposed the existence of a norm on E with respect to which the considered convex function f on E has some continuity property (i.e. continuity of convex function in the origin $0 \in E$ relative to an arbitrary convex topology on E). The assumption on f of this kind is sometimes not desirable from the point of view of application, it seems necessary to develop a theory without continuity assumptions on f.

In this paper, we are able to show the existence of a saddle point under the assumption of weak semicontinuity of convex-concave function. The methods we use here are similar to that in [5].

2. General result

Let E and F be two convex spaces and f and φ the proper convex functions defined on E and F respectively. We define $\operatorname{dom}(f)=\{x\in E: f(x)<\infty\}$ and, similarly, for $\operatorname{dom}(\varphi)$. E^*, F^* denote the continuous dual of E and F, respectively. Then, obviously, the products $E\times F$ and $E^*\times F^*$ are dual with respect to the bilinear form $\langle x,u\rangle+\langle y,v\rangle$ where $(x,y)\in E\times F$, $(u,v)\in E^*\times F^*$. Denote

$$N_{f,\alpha}=\{x\in E: f(x)\leqq\alpha\} \quad \text{for} \quad \alpha\in\mathbf{R}$$

the level set of f and, similarly, for the level set of φ. According to Fenchel we define the conjugate function f^c: $E^*\to\mathbf{R}$

$$f^c(u)=\sup\{\langle x,u\rangle-f(x): x\in\operatorname{dom}(f)\}$$

and, similarly, for $\varphi^c(v)$.

Following a similar argument as in [1, p. 46], we have

LEMMA 1. *Let f be a proper convex function then f is $\sigma(E,E^*)$-lower semicontinuous if and only if*

$$f(x)=\sup\{\langle x,u\rangle-f^c(u): u\in E^*\}.$$

Hence, in the following discussion, we shall assume that f,φ are $\sigma(E,E^*)$, $\sigma(F,F^*)$-lower semicontinuous, respectively, and $E=\operatorname{lin}(\operatorname{dom}(f))$, $F=\operatorname{lin}(\operatorname{dom}(\varphi))$ (i.e. the linear hull of $\operatorname{dom}(f)$ and linear hull of $\operatorname{dom}(\varphi)$, respectively), unless otherwise stated.

It is well-known that the $B\subseteqq E$ is $\sigma(E,E^*)$-compact iff B is $\sigma(E,E^*)$-bounded and $\sigma(E,E^*)$-complete. In general, the level sets of f need not be $\sigma(E,E^*)$-bounded, even if there are $\sigma(E,E^*)$-bounded, the $\sigma(E,E^*)$-complete may not be fulfilled. However, we have the following similar result as in [5]. Here, a nonnegative convex function f is replaced by a convex function f which is bounded below.

THEOREM 1. *Suppose that f is bounded below and $f(0)=0$. Then $N_{f,1}$ is $\sigma(E,E^*)$-bounded iff $\operatorname{dom}(f^c)=\{u\in E^*: f^c(u)<\infty\}$ is absorbing in E^*.*

PROOF. Suppose $N_{f,1}$ is $\sigma(E,E^*)$-bounded and $u\in E^*$, we have to show that there exists $\lambda_0>0$ such that $u\in\lambda\operatorname{dom}(f^c)$ whenever $|\lambda|\geqq\lambda_0$. Since $N_{f,1}$ is $\sigma(E,E^*)$-bounded, there exists $\gamma>0$ such that

$$\sup\{|\langle x,u\rangle|: x\in N_{f,1}\}\leqq\gamma. \tag{2.1}$$

Suppose $x\notin N_{f,1}$, then $1<f(x)<\infty$ or $f(x)=\infty$. For the case $1<f(x)<\infty$, then $\dfrac{1}{f(x)+1}x\in N_{f,1}$, so we have

$$\left|\left\langle\frac{1}{f(x)+1}x,u\right\rangle\right|\leqq\gamma$$

i.e.

$$|\langle x,u\rangle|\leqq\gamma(f(x)+1). \tag{2.2}$$

If $f(x)=\infty$ then (2.1) is obvious. Hence, (2.2) holds for $x\in(E\setminus N_{f,1})\cup P_1$ where $P_1=\{x\in N_{f,1}\colon f(x)\geqq 0\}$. For $x\in N_{f,1}\setminus P_1$, by (2.1) we have

$$|\langle x,u\rangle|\leqq\gamma(-f(x)+1). \tag{2.3}$$

Since $f(x)$ is bounded below, (2.2) and (2.3) imply that $f^c(\gamma^{-1}u)<\infty$ and $f^c(-\gamma^{-1}u)<\infty$. Put $\lambda_0=\gamma$, then $u\in\lambda\,\mathrm{dom}\,(f^c)$ whenever $|\lambda|\geqq\lambda_0$.

Conversely, if $\mathrm{dom}\,(f^c)$ is absorbing on E^*, then, for any $u\in E^*$, there exist real numbers $\beta,\gamma>0$ such that

$$f^c(\beta^{-1}u)\leqq\gamma \quad\text{and}\quad f^c(-\beta^{-1}u)\leqq\gamma.$$

This implies that

$$|\langle x,u\rangle|\leqq\beta(|f(x)|+\gamma) \quad\text{for all}\quad x\in N_{f,1}.$$

Hence, $N_{f,1}$ is $\sigma(E,E^*)$-bounded.

REMARK. The convex function f with $f(0)=0$ has the following property: If $N_{f,\alpha}$ is $\sigma(E,E^*)$-bounded for some $\alpha>0$ then all the level sets of f are $\sigma(E,E^*)$-bounded. This exactly follows from the fact that $f(cx)\leqq cf(x)$ for $0<c<1$. Hence the previous theorem can be restated as follows: Suppose f is bounded below and $f(0)=0$. Then all the level sets of f are $\sigma(E,E^*)$-bounded iff $\mathrm{dom}\,(f^c)$ is absorbing in E^*.

Let $\mathcal{N}_f$, $\mathcal{N}_\varphi$ be the families of sets $N_{f,\alpha}$ and $N_{\varphi,\beta}$, respectively, where $\alpha,\beta\in\mathbf{R}$. Denote

$$\mathcal{N}_{f,\varphi}=\{N_{f,\alpha}\times N_{f,\beta}\subseteqq E\times F\colon N_{f,\alpha}\in\mathcal{N}_f,\ N_{\varphi,\beta}\in\mathcal{N}_\varphi\},$$

define the set

$$\mathrm{epi}\,(f)=\{(x,t)\in E\times\mathbf{R}\colon f(x)\leqq t\}$$

and similarly for $\mathrm{epi}\,(f^{cc})$, where

$$f^{cc}(u^*)=\sup\{\langle u^*,z\rangle-f^c(z)\colon z\in\mathrm{dom}\,(f^c)\} \quad\text{for all}\quad u^*\in E^{**}.$$

Then we have

LEMMA 2. *Let M be a convex subset of $E\times\mathbf{R}$ containing* $\mathrm{epi}\,(f)$. *If M is closed under the product topology $\sigma(E^{**},E^*)\times\eta$ where η is the Euclidean topology of $\mathbf{R}$ and if $(x_0,u_0)\notin M$, then there exist $u_0\in E^*$, $\gamma\in\mathbf{R}$, and $\varepsilon>0$ such that*

$$\langle x,u_0\rangle-t\leqq\gamma-\varepsilon\leqq\gamma\leqq\langle x_0,u_0\rangle-t_0$$

whenever $(x,t)\in M$.

PROOF. We recall that $(E\times\mathbf{R})^*$ is isomorphic to $E^*\times\mathbf{R}$ under the correspondence $E^*\times\mathbf{R}\ni(u,s)\leftrightarrow\psi\in(E\times\mathbf{R})^*$ where $\psi(x,t)=\langle x,u\rangle+st$. Hence, by separation theorem, there exists $\psi_0\in(E\times\mathbf{R})^*$ such that

$$\psi_0(x,t)\leqq\gamma-\varepsilon\leqq\gamma\leqq\psi_0(x_0,t_0) \quad\text{for all}\quad (x,t)\in M.$$

Therefore, there exist $u_0\in E^*$ and $s_0\in\mathbf{R}$ such that

$$\langle x,u_0\rangle+s_0t\leqq\gamma-\varepsilon\leqq\gamma\leqq\langle x_0,u_0\rangle+s_0t_0 \quad\text{for all}\quad (x,t)\in M.$$

i.e.

$$\langle x_0,u_0\rangle+s_0t_0>\sup_{(x,t)\in M}\{\langle x,u_0\rangle+s_0t\}. \tag{2.4}$$

Obviously $s_0 \neq 0$, since otherwise it would imply $x_0 \in \overline{\text{dom}\,(f)}$, the closure is taken under $\sigma(E^{**}, E^*)$-topology. Moreover, if $s_0 > 0$, then the sup on the right-hand side of (2.4) would be infinite. So s_0 must be negative, we may assume that $s_0 = -1$. Hence, we have

$$\langle x, u_0\rangle - t \leqq \gamma - \varepsilon \leqq \gamma \leqq \langle x_0, u_0\rangle - t_0$$

whenever $(x, t) \in M$.

LEMMA 3. epi (f) *is dense in* epi (f^{cc}) *for the product topology* $\sigma(E^{**}, E^*) \times \eta$.

PROOF. Consider $E^{**} \times \mathbf{R}$ with the topology $\sigma(E^{**}, E^*) \times \eta$. It is clear that epi $(f) \subseteqq$ epi (f^{cc}). Suppose there exists $(u_0^*, t_0) \in$ epi (f^{cc}) s.t. $(u_0^*, t_0) \notin \overline{\text{epi}\,(f)}$ (under $\sigma(E^{**}, E^*) \times \eta$ topology). Then by Lemma 2, there exist $u_0 \in E^*$ and $\gamma \in \mathbf{R}$, $\varepsilon > 0$ such that

$$\langle x, u_0\rangle - t \leqq \gamma - \varepsilon \leqq \gamma \leqq \langle u_0^*, u_0\rangle - t_0 \quad \text{for all} \quad (x, t) \in \text{epi}\,(f).$$

It follows that $f^c(u_0) \leqq \gamma - \varepsilon$. On the other hand, we have $f^{cc}(u_0^*) \leqq t_0$, whence

$$\gamma \leqq \langle u_0^*, u_0\rangle - t_0 \leqq \langle u_0^*, u_0\rangle - f^{cc}(u_0^*) \leqq f^c(u_0),$$

a contradiction, which shows that epi (f) is dense in epi (f^{cc}).

COROLLARY. *Let* $\alpha < \beta$. *Then* $N_{f^{cc},\alpha} = \{u^* \in E^{**} : f^{cc}(u^*) \leqq \alpha\}$ *is contained in the* $\sigma(E^{**}, E^*)$*-closure of* $N_{f,\beta}$.

This is an immediate consequence of Lemma 3.

Let us now put $E_f^{**} = \text{lin}\,(\text{dom}\,(f^{cc}))$ and $F_\varphi^{**} = \text{lin}\,(\text{dom}\,(\varphi^{cc}))$.

DEFINITION 1 [see 5]. The dual system (E, E^*) is said to be *f-semi-reflexive* if $E = E_f^{**}$.

DEFINITION 2. The dual system $(E \times F, E^* \times F^*)$ is said to be (f, φ)*-semi-reflexive* if $E \times F = E_f^{**} \times E_\varphi^{**}$.

Then we have

THEOREM 2. *The dual system* (E, E^*) *is f-semireflexive iff all sets in* $\mathcal{N}_f$ *are* $\sigma(E, E^*)$*-complete.*

PROOF. Suppose that (E, E^*) is f-semireflexive, then it follows that $N_{f,\alpha} = N_{f^{cc},\alpha}$ for $\alpha \in \mathbf{R}$. Moreover, $N_{f^{cc},\alpha}$ is $\sigma(E^{**}, E^*)$-closed, therefore, it is $\sigma(E^{**}, E^*)$-complete. Hence, it is $\sigma(E_f^{**}, E^*)$-complete. This shows that $N_{f,\alpha}$ is $\sigma(E, E^*)$-complete.

Conversely, let $N_{f,\alpha}$ be $\sigma(E, E^*)$-complete for any $\alpha \in \mathbf{R}$. We will show that $N_{f,\alpha} = N_{f^{cc},\alpha}$. On $N_{f,\beta}$, the uniformities of $\sigma(E^{**}, E^*)$ and $\sigma(E, E^*)$ coincide. Therefore $N_{f,\beta}$ is $\sigma(E^{**}, E^*)$-complete and so $\sigma(E^{**}, E^*)$-closed for every $\beta \in \mathbf{R}$. Now by Corollary to Lemma 3, we have

$$N_{f^{cc},\alpha} \subseteqq \overline{N}_{f,\beta} \quad (\text{under } \sigma(E^{**}, E^*)\text{-topology}) \quad \text{for all} \quad \beta > \alpha.$$

This implies that

$$N_{f,\alpha} \subseteqq N_{f^{cc},\alpha} \subseteqq \bigcap_{\beta > \alpha} \overline{N}_{f,\beta} = N_{f,\alpha} \quad \text{for all} \quad \alpha \in \mathbf{R}.$$

This shows that (E, E^*) is f-semireflexive.

By virtue of Tychonoff's theorem on product topology and Theorem 2, we have the following immediate consequence.

THEOREM 3. *If E, F and $\mathcal{N}_{f,\varphi}$ are defined as above, then the following conditions are equivalent:*

(1) *the dual system $(E\times F,\ E^*\times F^*)$ is (f,φ)-semi-reflexive*

(2) *all sets in $\mathcal{N}_{f,\varphi}$ are $\sigma(E,E^*)\times\sigma(F,F^*)$-complete*

(3) *the dual systems (E,E^*) and (F,F^*) are f-semireflexive and φ-semireflexive, respectively.*

REMARK. It is obvious that the semireflexivity of E implies that E is f-semireflexive. In case E is a normed linear space and reflexive, then, obviously E is f-reflexive.

3. Convex program and minimax problem

Let E be defined as in Section 2, f a proper convex function on E and $\sigma(E,E^*)$-lower semicontinuous. The problem of minimizing f over a convex subset U of E is often called the convex program, denoted by $(U,f|U)$. It is interesting to know that for which convex subsets $U\subseteqq E$ exist optimal solutions $x_0\in U$ such that $f(x_0)=\inf\{f(x):\ x\in U\}$. To begin with, let H be a set of $\sigma(E,E^*)$-lower semicontinuous convex functions and

$$U=\bigcap_{g\in H}\{x\in E:\ g(x)\leqq 0\}.$$

Furthermore, we assume that H is chosen such that $U\neq\varnothing$ and $U\cap\mathrm{dom}\,(f)\neq\varnothing$, for otherwise the question becomes trivial. Then we have the following result.

THEOREM 4. *Suppose the dual system (E,E^*) is f-semireflexive and the level sets $N_{f,\alpha}$ $(\alpha\in\mathbf{R})$ are $\sigma(E,E^*)$-bounded. Then the convex program $(U,f|U)$ always has an optimal solution.*

PROOF. Since (E,E^*) is f-semireflexive and the level sets of f are $\sigma(E,E^*)$-bounded, the level sets of f are $\sigma(E,E^*)$-compact. Moreover, U is $\sigma(E,E^*)$-closed, since each convex function in H is $\sigma(E,E^*)$-lower semicontinuous. This implies that $N_{f,\alpha}\cap U$ is again a $\sigma(E,E^*)$-compact set, for $\alpha\in\mathbf{R}$. Hence the existence of an optimal solution on U is guaranteed by the $\sigma(E,E^*)$-lower semicontinuity of f.

REMARK. While the objective function f is nonnegative with $f(0)=0$, the $\sigma(E,E^*)$-boundedness of level sets of f can be characterized by Theorem 1, i.e. we only need to determine whether $\mathrm{dom}\,(f^c)$ is absorbing in E^*.

Now, let E and F be two convex spaces and $h(\cdot,\cdot)$ a function defined on $E\times F$ and taking values in $[-\infty,+\infty]$. Write

$$\mathrm{dom}\,(h)=\{(x,y)\in E\times F:\ |h(x,u)|<\infty\}.$$

The projections of $\mathrm{dom}\,(h)$ on E and F are denoted by $\mathrm{dom}_E\,(h)$ and $\mathrm{dom}_F\,(h)$, respectively.

We say that $h(x,y)$ is convex with respect to x if $h(x,y)$ is convex for each $y\in\mathrm{dom}_F\,(h)$. Similarly, $h(x,y)$ is concave with respect to y if $h(x,y)$ is concave for

each $x \in \mathrm{dom}_E(h)$. A function $h(x, y)$ convex with respect to x and concave with respect to y is called a convex-concave function. It is interesting to know for which convex-concave function $h(\cdot, \cdot)$ exists a solution $(x_0, y_0) \in \mathrm{dom}\,(h)$ such that

$$h(x_0, y_0) = \sup_{y \in \mathrm{dom}_F(h)} \inf_{x \in \mathrm{dom}_E(h)} h(x, y) = \inf_{x \in \mathrm{dom}_E(h)} \sup_{y \in \mathrm{dom}_F(h)} h(x, y).$$

The solution (x_0, y_0) is often called the saddle point of h.

To begin with, we assume that $E = \mathrm{lin}\,(\mathrm{dom}_E(h))$, $F = \mathrm{lin}\,(\mathrm{dom}_F(h))$ and $h(\cdot, y)$ is $\sigma(E, E^*)$-lower semicontinuous on E, for each $y \in \mathrm{dom}_F(h)$ and $h(x, \cdot)$ is $\sigma(F, F^*)$-upper semicontinuous on F, for each $x \in \mathrm{dom}_E(h)$. Define

$$f(x) = \sup_{y \in \mathrm{dom}_F(h)} h(x, y) \quad \text{and} \quad \varphi(y) = \inf_{x \in \mathrm{dom}_E(h)} h(x, y).$$

Obviously, f is a convex function on E and φ is a concave function of F. Moreover, f and φ are $\sigma(E, E^*)$-lower semicontinuous and $\sigma(F, F^*)$-upper semicontinuous, respectively. Hence, f, $-\varphi$ are convex functions and $\sigma(E, E^*)$-lower semicontinuous and $\sigma(F, F^*)$-lower semicontinuous, respectively. Then we have

THEOREM 5. *Assume that the dual system $(E \times F, E^* \times F^*)$ is $(f, -\varphi)$-semi-reflexive and all the sets $N_{f,\alpha}$, $N_{-\varphi,\beta}$ are $\sigma(E, E^*)$-bounded and $\sigma(F, F^*)$-bounded, respectively, for $\alpha, \beta \in \mathbf{R}$. Then $h(x, y)$ has a saddle point.*

PROOF. By virtue of Theorem 3, the sets $N_{f,\alpha}$, $N_{\varphi,\beta}$ are $\sigma(E, E^*)$-compact and $\sigma(F, F^*)$-compact, respectively, for $\alpha, \beta \in \mathbf{R}$. We can follow a similar argument as that in [2, Theorems 2—4, p. 85], and show that $h(x, y)$ has a saddle point.

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ÜBER GEWISSE VERSCHÄRFUNGEN DER BEIDEN KLASSISCHEN INDUKTIVEN TOPOLOGISCHEN DIMENSIONSBEGRIFFE, II. OFFEN-ERBLICH TOTAL-PARAKOMPAKTE RÄUME

von

E. DEÁK

Einleitung

Wir nennen eine Eigenschaft auf einen gewissen Raum bezogen *offen-erblich*, wenn sie jedem offenen Teilraum desselben zukommt.

In diesem Teil behandeln wir die *offen-erblich total-parakompakten* T_2-Räume. (S. den Begriff der Total-Parakompaktheit selbst etwa im § 0 des I. Teils, [1].) Der Gedanke, eben diese Klasse von Räumen zu untersuchen, entspringt den drei, von R. M. FORD und B. FITZPATRICK, JR. ermittelten Tatsachen:

(I) SATZ. *Für jeden total-parakompakten metrisierbaren Raum X gilt* $\operatorname{Ind} X = \operatorname{ind} X$.

(II) SATZ. *Jeder offene Teilraum eines total-parakompakten metrisierbaren Raumes ist total-parakompakt.*

(III) SATZ. *Es gibt total-parakompakte metrisierbare Räume (z. B. sind alle Euklidischen Räume dieser Art), deren Teilräume nicht sämtlich total-parakompakt sind.* ([4], [3].)

Unser Hauptergebnis in diesem Teil ist der Satz (2.3), der für Räume X, mit $\operatorname{ind} X = n < \infty$, der besagten Klasse die Existenz „beliebig feiner", lokal-endlicher und ind-n-hübscher offener Überdeckungen aussagt. (S. den Begriff der ind-n-hübschen Überdeckung in [1], § 0.)

Dieser Satz wird im § 2 bewiesen, und sein Beweis wird, der Übersichtlichkeit wegen, im § 1 vorbereitet. Die Sätze des § 1 dürften aber auch an sich nicht uninteressant sein; jedenfalls gibt es solche — auf metrisierbare Räume bezogene — Sätze in der Literatur der Dimensionstheorie (und das ist unser zweiter heuristischer Ausgangspunkt). Als Beispiel dafür kann etwa das folgende Ergebnis von M. KATĚTOV ([7], oder auch [11], 80) dienen:

(IV) SATZ. *Sind X ein metrisierbarer Raum, C_i ($i \in \mathbf{N}$) abgeschlossene Mengen mit* $\dim C_i \leqq n_i$ ($i \in \mathbf{N}$), *F eine abgeschlossene und U eine offene Menge mit $F \subseteq U$, so gibt es eine offene Menge G mit*

$$F \subseteq G \subseteq G^- \subseteq U$$

und mit

$$\dim (C_i \cap \operatorname{Gr} G) \leqq n_i - 1 \quad (i \in \mathbf{N}).$$

(Hier bedeutet „dim" die Lebesguesche Überdeckungsdimension.) Ein Analogon dieses Satzes beweisen wir unter (1.5).

Im § 0 führen wir die bekannten Begriffe und Sätze an, die wir als Hilfsmittel brauchen.

Was die Terminologie betrifft, halten wir uns im allgemeinen an den Wortgebrauch des Buches [8] von J. L. Kelley, die *Parakompaktheit* ausgenommen, worin wir auch die Regularität nicht implizieren wollen. Wir folgen auch bei der Benennung von Begriffen, die in [8] nicht vorkommen (wie etwa *stark parakompakt* u. dgl. m.), dem terminologischen Prinzip, daß nicht überall die Regularität oder irgendeine der T_i-Eigenschaften per definitionem miteinbegriffen werden soll.

Im übrigen benutzen wir dieselben Bezeichnungen wie in [1].

§ 0.

(0.1) SATZ. *Jeder abgeschlossene Teilraum eines total-parakompakten Raumes ist total-parakompakt.* [4].

Hieraus folgt sofort der

(0.2) SATZ. *Sind X ein offen-erblich total-parakompakter Raum, F eine abgeschlossene und G eine offene Menge in X, so ist auch der Teilraum $F \cap G$ von X total-parakompakt.*

(0.3) DEFINITION. Ein normaler Raum X heißt *total-normal,* wenn jede offene Menge G in X als Vereinigung eines Mengensystems dargestellt werden kann, welches lokal-endlich in G ist und dessen Elemente offene F_σ-Mengen bezüglich X sind. (C. H. DOWKER, [2].)

(0.4) SATZ. *Jeder offen-erblich total-parakompakte T_2-Raum ist total-normal.*

Tatsächlich ist ja jeder offen-erblich parakompakte Raum erblich parakompakt (J. DIEUDONNÉ [1a]) und jeder erblich parakompakte T_2-Raum total-normal (C. H. DOWKER, [2]).

(0.5) KOROLLAR. *Jeder offen-erblich total-parakompakte T_2-Raum ist vollständig normal.*

Die Tatsache unter (0.4) ermöglicht es, einige sehr gute dimensionstheoretische Eigenschaften der total-normalen Räume für unseren Gegenstand zu verwerten.

(0.6) SATZ. *Für einen total-normalen T_2-Raum X* gilt $\operatorname{Ind} Y \leqq \operatorname{Ind} X$ $(Y \subseteqq X)$. (C. H. DOWKER, [2].)

(0.7) SATZ. *Für einen total-normalen T_2-Raum X und ein lokal-endliches System $\mathscr{F}$ abgeschlossener Mengen in X mit*

$$\operatorname{Ind} F \leqq n \quad (F \in \mathscr{F}),$$

n eine natürliche Zahl, gilt

$$\operatorname{Ind} \bigcup \mathscr{F} \leqq n.$$

(N. KIMURA, [9].)

Wir werden noch einige weitere dimensionstheoretische Tatsachen benutzen.

(0.8) SATZ. *Für einen offen-erblich total-parakompakten T_2-Raum X gilt* $\operatorname{Ind} X = \operatorname{ind} X$.

(Das gilt sogar unter allgemeineren Voraussetzungen, s. etwa [5].)

Durch (0.8) sieht man auch (0.6) in anderem Licht. Eine sehr einfache und wohlbekannte Tatsache ist ja der

(0.9) SATZ. *Für einen beliebigen Raum X gilt*

$$\operatorname{ind} Y \leqq \operatorname{ind} X \quad (Y \subseteq X).$$

(0.10) SATZ. *Für einen vollständig normalen Raum X und eine abgeschlossene Menge $A \subseteq X$ mit* $\operatorname{Ind} A \leqq n$ *und* $\operatorname{Ind}(X \setminus A) \leqq n$ *gilt* $\operatorname{Ind} X \leqq n$. (C. H. DOWKER [2].)

Wir berufen uns noch auf den bekannten

(0.11) SATZ. *Zu jeder punktal-endlichen offenen Überdeckung U eines normalen T_1-Raumes gibt es eine offene Überdeckung $\{V_U : U \in \mathfrak{U}\}$ mit $V_U^- \subseteq U$ $(U \in \mathfrak{U})$.*

(0.12) DEFINITION. Ein Raum heißt *stark parakompakt,* wenn jedes seiner offenen Überdeckungssysteme zu einem stern-endlichen offenen Überdeckungssystem verfeinert werden kann. (J. M. SMIRNOV, [12].)

§ 1.

(1.1) HILFSSATZ. *Es seien $Y \neq \emptyset$ ein offen-erblich total-parakompakter T_2-Raum, H eine abgeschlossene und U eine offene Menge in Y, Γ eine in H offene und Φ eine in H abgeschlossene Menge mit*

$$\Gamma \subseteq \Phi \subseteq H \cap U. \tag{1.1.1}$$

Dann gibt es eine offene Menge V mit

$$\Gamma \subseteq V, \tag{1.1.2}$$

$$V^- \subseteq U, \tag{1.1.3}$$

$$H \cap \operatorname{Gr} V \subseteq \Phi \setminus \Gamma \tag{1.1.4}$$

und — im Falle $\operatorname{ind} Y < \infty$ —

$$\operatorname{ind}[(Y \setminus H) \cap \operatorname{Gr} V] \leqq \operatorname{ind} Y - 1. \tag{1.1.5}$$

BEWEIS. 1° Da die Mengen Γ und $H \setminus \Phi$ im Raum H separiert sind, sind sie es auch im Raum Y, der nach (0.5) vollständig normal ist. Es gibt also in Y eine offene Menge G mit

$$\Gamma \subseteq H \cap G \subseteq H \cap G^- \subseteq \Phi, \tag{1.1.6}$$

und es darf

$$G \subseteq U \tag{1.1.7}$$

angenommen werden, weil ja anstelle von G — nach (1.1.1) — ohne weiteres $G \cap U$ gesetzt werden kann.[1]

Es sei nun

$$Z = Y \setminus (H \cap \operatorname{Gr} G), \tag{1.1.8}$$

oder anders ausgedrückt,

$$Z = G \cup (Y \setminus H) \cup (Y \setminus G^-). \tag{1.1.9}$$

2° Nehmen wir jetzt für jeden Punkt $x \in Z$ eine offene Umgebungsbasis $\mathscr{W}(x)$ in Y an, u. zw. soll für $W \in \mathscr{W}(x)$

$$W^- \subseteqq \begin{cases} G & (x \in G), \quad \text{(1.1.10a)} \\ Y \setminus H & (x \in Y \setminus H), \quad \text{(1.1.10b)} \\ Y \setminus G^- & (x \in Y \setminus G^-) \quad \text{(1.1.10c)} \end{cases}$$

(auf Grund der Regularität von X) und im Falle $\operatorname{ind} Y < \infty$ obendrein

$$\operatorname{ind} \operatorname{Gr} W \leqq \operatorname{ind} Y - 1 \quad (W \in \mathscr{W}(x),\ x \in Z) \tag{1.1.11}$$

sein. (Wir haben

$$W^{-(Z)} = W^- \tag{1.1.12a}$$

$$\operatorname{Gr}_Z W = \operatorname{Gr} W \qquad (W \in \mathscr{W}(x),\ x \in Z), \tag{1.1.12b}$$

da jedes $W \in \mathscr{W}(x)$ offen bzw. die entsprechende Menge W^- abgeschlossen auch in Z ist.) Das Mengensystem

$$\mathscr{W} = \bigcup \{\mathscr{W}(x) \colon x \in Z\}$$

gibt dann eine Basis des Raumes Z ab, und da dieser — als offener Teilraum von Y — total-parakompakt ist, existiert ein im Raum Z lokal-endliches System

$$\mathscr{W}^* \subseteqq \mathscr{W}$$

mit

$$Z = \bigcup \{W \colon W \in \mathscr{W}^*\}. \tag{1.1.13}$$

3° Für das — ebenfalls lokal-endliche — Teilsystem

$$\mathscr{V}^* = \{W \in \mathscr{W}^* \colon H \cap G \cap W \neq \varnothing\}^2 \tag{1.1.14}$$

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[1] Das bisher Gesagte ist eigentlich nichts weiter als der Gedankengang des üblichen Beweises für den folgenden elementaren Satz, der zum klassischen Bestand der Dimensionstheorie gehört:

Für einen Teilraum M eines vollständig normalen Raumes X und für eine ganze Zahl $k \geqq 0$ gilt $\operatorname{ind} M \leqq k$ *dann und nur dann, wenn jeder Punkt $x \in M$ eine Umgebungsbasis $\mathscr{B}$ bezüglich X besitzt, die aus offenen Mengen B mit*

$$\operatorname{ind} (M \cap \operatorname{Gr} B) \leqq k - 1 \quad (B \in \mathscr{B})$$

besteht. ([6] 27.)

[2] Man könnte etwa mit dem System

$$\{W \in \mathscr{W}^* \colon \Gamma \cap W \neq \varnothing\}$$

anstelle von $\mathscr{V}^*$ ebensogut weiterarbeiten.

gilt nun, wegen (1.1.10b) und (1.1.10c),

$$\mathscr{V}^* \subseteqq \cup\{\mathscr{W}(x): x \in G\}, \tag{1.1.15}$$

und daher ist

$$V = \cup\{W: W \in \mathscr{V}^*\} \tag{1.1.16}$$

eine offene Menge, für welche — wegen (1.1.10a) —

$$V \subseteqq G \tag{1.1.17}$$

und — wegen (1.1.13) und (1.1.14) —

$$H \cap G \subseteqq V \tag{1.1.18}$$

gilt. (Übrigens haben wir sogar $H \cap G = H \cap V$.)

Jetzt können wir darangehen, zu zeigen, daß die Menge V den Forderungen des Satzes genügt.

4° Aus (1.1.17) und (1.1.7) folgt $V \subseteqq U$, und (1.1.18) zieht, wegen (1.1.6), (1.1.2) nach sich.

5° Wegen (1.1.17), (1.1.6) und (1.1.1) gilt

$$V^- \cap H \subseteqq U. \tag{1.1.19}$$

Weiter folgt aus (1.1 17) und (1.1.9) $V \subseteqq Z$, also gilt

$$V^- \cap Z = V^{-(Z)}; \tag{1.1.20}$$

wegen (1.1.9) haben wir

$$V^- \cap (Y \setminus H) \subseteqq V^- \cap Z \tag{1.1.21}$$

und wegen (1.1.12a) und der Lokal-Endlichkeit von $\mathscr{V}^*$

$$V^{-(Z)} = \cup\{W^-: W \in \mathscr{V}^*\},$$

was nach (1.1.14) und (1.1.10)

$$V^{-(Z)} \subseteqq G \tag{1.1.22}$$

nach sich zieht.

Aus (1.1.20), (1.1.21) und (1.1.22) folgert man

$$V^- \cap (Y \setminus H) \subseteqq G$$

und daher, nach (1.1.7),

$$V^- \cap (Y \setminus H) \subseteqq U. \tag{1.1.23}$$

Nun ergibt sich (1.1.3) aus (1.1.9) und (1.1.23).

6° Auf Grund von (1.1.6) und (1.1.17) gilt

$$H \cap V^- \subseteqq \Phi, \tag{1.1.24}$$

und wegen (1.1.1) und (1.1.2) haben wir

$$\Gamma \subseteq H \cap V; \tag{1.1.25}$$

aus (1.1.24) und (1.1.25) folgt nun aber (1.1.4).

7° Was endlich (1.1.5) betrifft:

Nach (1.1.12b) und wegen der Lokal-Endlichkeit von $\mathscr{V}^*$ bezüglich Z gilt

$$\mathrm{Gr}_Z V \subseteq \cup\{\mathrm{Gr} W\colon W \in \mathscr{V}^*\}$$

und daher — auf Grund von (0.4), (0.8), (0.7), (1.1.11) und nach (0.6) oder (0.9) —

$$\mathrm{ind}\, \mathrm{Gr}_Z V \leqq \mathrm{ind}\, Y - 1. \tag{1.1.26}$$

Auf den Ausdruck

$$Z = (H \cap G) \cup (Y \setminus H) \cup (Y \setminus G^-)$$

von (1.1.9) angewandt ergeben (1.1.18) und (1.1.17)

$$(Y \setminus H) \cap \mathrm{Gr}\, V = Z \cap \mathrm{Gr}\, V,$$

und daher gilt — nach (1.1.26) und da (wegen $V \subseteq Z$) $Z \cap \mathrm{Gr}\, V = \mathrm{Gr}_Z V$ ist — im Falle $\mathrm{ind}\, Y < \infty$ tatsächlich auch (1.1.5). □

(1.2) BEMERKUNG. Der Beweis dieses Hilfssatzes bleibt auch im Falle $H = Y$ stichhaltig, und die Behauptung selbst ist dann sogar trivial. Das ist zu betonen, weil bei der Anwendung — über (1.3), (1.4) und (1.5) — im Beweis des Satzes (2.2) auch dieser Fall berücksichtigt werden muß.

(1.3) HILFSSATZ. *Es seien X ein offen-erblich total-parakompakter T_2-Raum, i eine natürliche Zahl und*

$$H_1 \subseteq H_2 \subseteq \ldots \subseteq H_i \subseteq X \tag{1.3.1}$$

abgeschlossene Mengen mit

$$\mathrm{ind}\, H_j < \infty \quad (j = 1, 2, \ldots, i).$$

Für einen beliebigen Punkt $x \in H_1$ und eine beliebige Umgebung U von x in X gibt es dann eine offene Menge B mit

$$x \in B \subseteq B^- \subseteq U, \tag{1.3.2}$$

$$\mathrm{ind}\, (H_j \cap \mathrm{Gr}\, B) \leqq \mathrm{ind}\, H_j - 1 \quad (j = 1, 2, \ldots, i). \tag{1.3.3}$$

BEWEIS. 1° Es ist eingangs zu bemerken, daß — nach (0.2) — die Teilräume von X unter (1.3.1) sämtlich offen-erblich total-parakompakt sind.

2° Setzen wir

$$H_0 = \varnothing, \quad H_{i+1} = X$$

und

$$B_0 = \varnothing.$$

3° Es sei B_1 eine offene Menge des Teilraumes H_1 von X mit

$$x \in B_1 \subseteq B_1^{-(H_1)} \subseteq H_1 \cap U \tag{1.3.4}$$

und mit

$$\text{ind}\,\mathrm{Gr}_{H_1}B_1 \leqq \text{ind}\,H_1 - 1. \tag{1.3.5}$$

4° Nehmen wir nun an, daß für eine natürliche Zahl

$$1 \leqq j \leqq i$$

schon offene Mengen

$$B_1 \subseteq B_2 \subseteq \ldots \subseteq B_j \tag{1.3.6}$$

der entsprechenden Teilräume

$$H_1, H_2, \ldots, H_j$$

von X mit

$$B_l^- (= B_l^{-(H_l)}) \subseteq U, \tag{1.3.7}$$

$$H_{l-1} \cap \mathrm{Gr}_{H_l} B_l \subseteq \mathrm{Gr}_{H_{l-1}} B_{l-1} \tag{1.3.8}$$

und

$$\text{ind}\,[(H_l \setminus H_{l-1}) \cap \mathrm{Gr}_{H_l} B_l] \leqq \text{ind}\,H_l - 1 \quad (l = 1, 2, \ldots, j) \tag{1.3.9}$$

gefunden worden sind. (Für $j=1$ ist diese Voraussetzung mit der unter 3° definierten Menge B_1 — wegen (1.3.1), (1.3.4) und (1.3.5) — erfüllt.) Durch Anwendung des vorigen Hilfssatzes auf

$$Y = H_{j+1}, \quad H = H_j, \quad \Gamma = B_j, \quad \Phi = B_j^{-(H_j)}$$

(wobei die Voraussetzung (1.1.1) nach (1.3.7) erfüllt ist) ergibt sich dann die Existenz einer offenen Menge $V = B_{j+1}$ des Teilraumes H_{j+1} mit

$$B_j \subseteq B_{j+1} \subseteq B_{j+1}^- \subseteq U,$$

$$H_j \cap \mathrm{Gr}_{H_{j+1}} B_{j+1} \subseteq \mathrm{Gr}_{H_j} B_j$$

und mit

$$\text{ind}\,[(H_{j+1} \setminus H_j) \cap \mathrm{Gr}_{H_{j+1}} B_{j+1}] \leqq \text{ind}\,H_{j+1} - 1 \quad (j \neq i)$$

(letzteres könnte für $j=i$ nur auf den Fall $\text{ind}\,X < \infty$ bezogen werden, aber auch das nutzen wir hier nicht aus).

5° Mittels dieses (endlichen) Induktionsverfahrens erhalten wir also offene Mengen

$$B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots \subseteq B_{i+1} \tag{1.3.10}$$

der entsprechenden Räume

$$H_0, H_1, H_2, \ldots, H_{i+1},$$

und es gelten

$$B_l^- \subseteq U, \tag{1.3.11}$$

$$H_{l-1} \cap \mathrm{Gr}_{H_l} B_l \subseteq \mathrm{Gr}_{H_{l-1}} B_{l-1} \tag{1.3.12}$$

$$(l = 1, 2, \ldots, i+1)$$

und

$$\text{ind}\,[(H_l \setminus H_{l-1}) \cap \text{Gr}_{H_l} B_l] \leqq \text{ind}\, H_l - 1 \tag{1.3.13}$$

$$(l = 1, 2, \ldots, i).$$

6° Wir definieren jetzt

$$B = B_{i+1}, \tag{1.3.14}$$

und zeigen, daß diese offene Menge B den Forderungen des Satzes genügt.

Erstens gilt (1.3.2) nach (1.3.11), (1.3.10) und (1.3.4).

7° Zweitens folgt aus (1.3.12) und (1.3.1)

$$\begin{aligned} H_j \cap \text{Gr}_{H_k} B_k &\subseteq H_j \cap \text{Gr}_{H_{k-1}} B_{k-1} \subseteq \\ &\subseteq H_j \cap \text{Gr}_{H_{k-2}} B_{k-2} \subseteq \ldots \subseteq H_j \cap \text{Gr}_{H_j} B_j = \\ &= \text{Gr}_{H_j} B_j \quad (1 \leqq j < k \leqq i+1), \end{aligned} \tag{1.3.15}$$

also gilt insbesondere für (1.3.14), den trivialen Fall $j = i+1$ hinzugenommen,

$$H_j \cap \text{Gr}\, B \subseteq \text{Gr}_{H_j} B_j \quad (j = 1, 2, \ldots, i+1), \tag{1.3.16}$$

und mit der Bezeichnung

$$H_j^* = \bigcup_{l=1}^{j} [(H_l \setminus H_{l-1}) \cap \text{Gr}_{H_l} B_l] \quad (j = 1, 2, \ldots, i+1) \tag{1.3.17}$$

folgt aus (1.3.15) weiter

$$\text{Gr}_{H_j} B_j = \bigcup_{l=1}^{j} [(H_l \setminus H_{l-1}) \cap \text{Gr}_{H_j} B_j] \subseteq H_j^* \tag{1.3.18}$$

$$(1 \leqq j \leqq i+1).$$

Nun haben wir laut (1.3.5)

$$\text{ind}\, H_1^* \leqq \text{ind}\, H_1 - 1.$$

Nehmen wir also an, daß $i > 1$ und daß für irgendein j mit $1 \leqq j < i$

$$\text{ind}\, H_j^* \leqq \text{ind}\, H_j - 1 \tag{1.3.19}$$

gilt. Auf den — nach (0.5) vollständig normalen — Raum

$$H_{j+1}^* = H_j^* \cup [(H_{j+1} \setminus H_j) \cap \text{Gr}_{H_{j+1}} B_{j+1}]$$

angewendet, woselbst H_j^* eine abgeschlossene Menge des Raumes H_{j+1}^* (weil aus (1.3.17) $H_j^* = H_{j+1}^* \cap H_j$ folgt) und die beiden Summanden disjunkt sind, ergibt sich aus (0.10) und (0.8)

$$\text{ind}\, H_{j+1}^* \leqq \max \{\text{ind}\, H_j^*,\ \text{ind}\, [(H_{j+1} \setminus H_j) \text{Gr}_{H_{j+1}} B_{j+1}]\},$$

woraus auf Grund von (1.3.19) und (1.3.1)

$$\text{ind}\, H_{j+1}^* \leqq \text{ind}\, H_{j+1} - 1$$

gefolgert werden kann. (Es ist hier unumgänglich, sich auch auf einen geeigneten Monotoniesatz zu berufen; es kann entweder (0.9) oder — mit Hinsicht auf (0.4) — (0.6) sein.) Es gilt also ganz allgemein

$$\text{ind}\, H_j^* \leqq \text{ind}\, H_j - 1 \quad (j = 1, 2, \ldots, i). \tag{1.3.20}$$

Aus (1.3.18), (1.3.16) und (1.3.20) folgt nun tatsächlich auch (1.3.3), womit der Hilfssatz vollständig bewiesen ist. □

(1.4) SATZ. *Es seien X ein offen-erblich total-parakompakter T_2-Raum, k eine natürliche Zahl und*

$$H_1 \subseteqq H_2 \subseteqq \ldots \subseteqq H_k \subseteqq X \tag{1.4.1}$$

abgeschlossene Mengen in X mit

$$\text{ind}\, H_r < \infty \quad (r = 1, 2, \ldots, k).$$

Für eine beliebigen Punkt $x_0 \in X$ und eine beliebige Umgebung U desselben existiert dann eine offene Menge V mit

$$x_0 \in V \subseteqq V^- \subseteqq U \tag{1.4.2}$$

und mit

$$\text{ind}\,(H_r \cap \text{Gr}\, V) \leqq \text{ind}\, H_r - 1 \quad (r = 1, 2, \ldots, k). \tag{1.4.3}$$

BEWEIS. 1° Sind $H_0 = \varnothing$, $H_{k+1} = X$ und $r_0 \in \{0, 1, 2, \ldots, k\}$ der Index, für welchen

$$x_0 \in H_{r_0+1} \setminus H_{r_0}$$

(wobei eigentlich nur der Fall $r_0 > 0$ von Belang ist, weil ja für $r_0 = 0$ die Behauptung des Satzes in jene des vorigen Satzes übergeht), so darf man

$$U \subseteqq X \setminus H_{r_0} \tag{1.4.4}$$

annehmen, da anstelle von U ohne weiteres $U \setminus H_{r_0}$ gesetzt werden kann.

2° Nach (1.3) — dem dortigen H_1 entspricht das hiesige H_{r_0+1} — existiert eine offene Menge V in X mit (1.4.2) und mit

$$\text{ind}\,(H_r \cap \text{Gr}\, V) \leqq H_r - 1 \quad (r = r_0+1, r_0+2, \ldots, k); \tag{1.4.5}$$

wegen (1.4.4), (1.4.2) und (1.4.1) haben wir weiter

$$\text{ind}\,(H_r \cap \text{Gr}\, V) \quad (= -1) \leqq \text{ind}\, H_r - 1. \tag{1.4.6}$$

$$(r = 1, 2, \ldots, r_0).$$

Mit (1.4.5) und (1.4.6) ist auch (1.4.3) erfüllt. □

(1.5) SATZ. *Es seien X ein offen-erblich total-parakompakter T_2-Raum mit* $\text{ind}\, X < \infty$, *k eine nichtnegative ganze Zahl und*

$$\varnothing = H_0 \subseteqq H_1 \subseteqq H_2 \subseteqq \ldots \subseteqq H_k \subseteqq H_{k+1} = X$$

abgeschlossene Mengen.

Für eine beliebige nichtleere abgeschlossene Menge F in X und eine beliebige Umgebung U derselben gibt es dann eine offene Menge W mit

$$F \subseteqq W \subseteqq W^- \subseteqq U \tag{1.5.1}$$

und mit

$$\operatorname{ind}(H_r \cap \operatorname{Gr} W) \leqq \operatorname{ind} H_r - 1 \tag{1.5.2}$$

$$(r = 1, 2, \ldots, k+1,\ H_r \neq \varnothing).$$

BEWEIS. Für jeden Punkt $x \in X$ sei — auf Grund von (1.4) — $\mathscr{V}(x)$ eine offene Umgebungsbasis mit

$$V^- \subseteqq U \quad (V \in \mathscr{V}(x),\ x \in F), \tag{1.5.3}$$

$$V^- \subseteqq X \setminus F \quad (V \in \mathscr{V}(x),\ x \in X \setminus F)$$

und mit

$$\operatorname{ind}(H_r \cap \operatorname{Gr} V) \leqq \operatorname{ind} H_r - 1 \tag{1.5.4}$$

$$(r = 1, 2, \ldots, k+1,\ H_r \neq \varnothing,\ V \in \mathscr{V}(x),\ x \in F).$$

Es gibt dann einen lokal-endlichen Teil $\mathscr{W}$ der Basis $\cup\{\mathscr{V}(x): x \in X\}$ von X mit

$$\cup\{V: V \in \mathscr{W}\} = X, \tag{1.5.5}$$

und für die offene Menge

$$W = \cup\{V \in \mathscr{W}: F \cap V \neq \varnothing\}$$

gelten tatsächlich (1.5.1) (wegen (1.5.3), (1.5.5) und der Lokal-Endlichkeit von $\mathscr{W}$) und

$$\operatorname{ind}(H_r \cap \operatorname{Gr} W) = \operatorname{ind} \cup\{H \cap \operatorname{Gr} V: V \in \mathscr{W},\ F \cap V \neq \varnothing\} \leqq \operatorname{ind} H_r - 1$$

$$(r = 1, 2, \ldots, k+1,\ H_r \neq \varnothing)$$

(wegen (1.5.4) und (0.7)), also auch (1.5.2). □

§ 2.

(2.1) BEMERKUNG. *Eine beliebige offene Überdeckung $\mathscr{U}$ eines total-parakompakten bzw. total-stark-parakompakten Raumes X mit* $\operatorname{ind} X = 0$ *kann zu einer* ind-ind *X-hübschen offenen Überdeckung verfeinert werden, die obendrein lokal-endlich bzw. Stern-endlich ist.*

Tatsächlich: verfeinert man $\mathscr{U}$ zu einer offen-abgeschlossenen Basis von X, so kann aus dieser ein lokal-endliches bzw. Stern-endliches Überdeckungssystem von X ausgewählt werden, das dem genannten Zweck entspricht. □

Von dieser elementaren Feststellung ausgehend erhebt sich nun aber die Frage nach sinngemäßen Abänderungen der Voraussetzung der Total-Parakompaktheit bzw. der Total-Stark-Parakompaktheit von X, unter denen ähnliches auch in den Fällen $0 < \operatorname{ind} X < \infty$ bewiesen werden kann.

Es folgen Ergebnisse dieser Art.

(2.2) SATZ. *Es seien n eine nichtnegative ganze Zahl, X ein offen-erblich total-parakompakter T_2-Raum mit*

$$\text{ind } X = n, \tag{2.2.1}$$

ϱ eine Ordnungszahl, $\mathfrak{U}=\{U_\alpha\colon \alpha<\varrho\}$ ein lokal-endliches System offener Mengen und $\mathfrak{B}=\{B_\alpha\colon \alpha<\varrho\}$ ein offenes Überdeckungssystem von X mit

$$B_\alpha^- \subseteq U_\alpha \tag{2.2.2}$$

$$\text{ind Gr } B_\alpha \leqq n-1 \quad (\alpha < \varrho). \tag{2.2.3}$$

Dann gibt es ein ind-*n-hübsches System*

$$\mathfrak{G} = \{G_\alpha\colon \alpha < \varrho\} \tag{2.2.4}$$

mit

$$B_\alpha^- \subseteq G_\alpha \subseteq G_\alpha^- \subseteq U_\alpha \quad (\alpha < \varrho). \tag{2.2.5}$$

BEWEIS. 1° Für $n=0$ ist unsere Aufgabe schon mit der Wahl $G_\alpha=B_\alpha$ $(\alpha<\varrho)$ mit der Wahl $G_\alpha=B_\alpha$ $(\alpha<\varrho)$ gelöst; es handelt sich also um die Fälle, daß

$$n \geqq 1.$$

2° Setzen wir

$$B(0,\alpha) = B_\alpha \quad (\alpha < \varrho), \tag{2.2.6}$$

und es sei $\prec$ (verst.: $\succcurlyeq$) das Zeichen für die lexikographische Ordnung der Menge

$$I = \{(k,\alpha)\colon k = 0, 1, 2, \ldots, n, \alpha < \varrho\} \tag{2.2.7}$$

von geordneten Paaren.

Wir werden jetzt, mittels eines — auf die durch $\prec$ wohlgeordnete Menge I bezogenen — transfiniten Induktionsverfahrens, auch für jedes $(k,\alpha)\in I$ mit $k>0$ eine entsprechende Menge $B(k,\alpha)$ definieren.

3° Nehmen wir also an, daß für ein festgesetztes $(k,\alpha)\in I$ mit $k>0$ und für jedes $(k',\alpha')\in I$ mit

$$(k',\alpha') \prec (k,\alpha) \tag{2.2.8}$$

schon eine entsprechende offene Menge $B(k',\alpha')$ definiert worden ist, u. zw. derart, daß einerseits

$$B(0,\alpha') \subseteq B(k',\alpha') \subseteq B(k',\alpha')^- \subseteq U_{\alpha'}, \tag{2.2.9}$$

andererseits für ein beliebiges

$$i\in\{1, 2, \ldots, k'+1\}$$

und für jedes i-Tupel

$$\{(k_1,\alpha_1), (k_2,\alpha_2), \ldots, (k_i,\alpha_i)\} \subseteq I$$

mit

$$\begin{aligned}(\max[0, k'-1], \alpha') &\prec (k_1,\alpha_1) \prec \\ &\prec (k_2,\alpha_2) \prec \ldots \prec (k_i,\alpha_i) = (k',\alpha')\end{aligned} \tag{2.2.10}$$

die Beziehung

$$\text{(2.2.11)} \qquad \operatorname{ind} \bigcap_{j=1}^{i} \operatorname{Gr} B(k_j, \alpha_j) \leqq n-i$$

besteht.

4° Zu dieser Induktionsannahme bemerken wir, daß

(a) sie zulässig ist, weil sie ja für $(k, \alpha)=(1, 0)$, d. i. das erste Element von I mit $k>0$, durch die Wahl unter (2.2.6) im vorhinein erfüllt ist (es gelten (2.2.9) wegen (2.2.2) und (2.2.11) wegen (2.2.3)), und daß

(b) aus (2.2.10)

$$\text{(2.2.12)} \qquad \alpha_{j_1} \neq \alpha_{j_2}$$

$$(j_1, j_2 \in \{1, 2, \ldots, i\},\ j_1 \neq j_2)$$

folgt (weil nämlich, bei etwa $j_1<j_2$ und daher $k_{j_1} \leqq k_{j_2}$ im Falle

$$k_{j_1} = k_{j_2}$$

selbstverständlich

$$\alpha_{j_1} < \alpha_{j_2},$$

im Falle

$$(k'-1 =) k_{j_1} < k_{j_2} (= k')$$

aber, woselbst auf Grund von (2.2.7) $\max [0, k_{j_1}]=k_{j_1}$ ist,

$$(k_{j_1}, \alpha') \prec (k_{j_1}, \alpha_{j_1}), \quad (k_{j_2}, \alpha_{j_2}) \leqq (k_{j_2}, \alpha')$$

und demzufolge

$$\alpha_{j_2} \leqq \alpha' < \alpha_{j_1}$$

besteht).

5° Um nachher $B(k, \alpha)$ zweckdienlich definieren zu können, setzen wir zunächst

$$\text{(2.2.13)} \qquad \Phi_0(k, \alpha) = X$$

und

$$\text{(2.2.14)} \qquad \Phi_i(k, \alpha) = \bigcup \Big\{ \bigcap_{j=1}^{i} \operatorname{Gr} B(k_j, \alpha_j) \colon (k_j, \alpha_j) \in I$$

$$(j = 1, 2, \ldots, i), \quad (k-1, \alpha) \prec (k_1, \alpha_1) \prec (k_2, \alpha_2) \prec \ldots \prec (k_i, \alpha_i) \prec (k, \alpha) \Big\}$$

$$(i = 1, 2, \ldots, k)$$

— insbesondere also

$$\text{(2.2.15)} \qquad \Phi_1(k, \alpha) = \bigcup \{\operatorname{Gr} B(k', \alpha') \colon (k', \alpha') \in I,$$

$$(k-1, \alpha) \prec (k', \alpha') \prec (k, \alpha)\} \text{ —}$$

wobei selbstverständlich

$$\text{(2.2.16)} \qquad \Phi_k(k, \alpha) \subseteq \Phi_{k-1}(k, \alpha) \subseteq \ldots \subseteq \Phi_1(k, \alpha) \subseteq \Phi_0(k, \alpha),$$

und bemerken, daß für diese Mengen $\Phi_i(k, \alpha)$, die ja — wegen (2.2.2) und der Lokal-Endlichkeit von $\mathcal{U}$ — Vereinigungen je eines lokal-endlichen Systems ab-

geschlossener Mengen vom Typ unter (2.2.11) sind (man beachte auch (2.2.10) und (2.2.8)), eben nach (2.2.11) und auf Grund von (0.7) und (0.8) die Beziehungen

$$\text{ind}\, \Phi_i(k, \alpha) \leqq n-i \quad (i = 0, 1, \ldots, k) \tag{2.2.17}$$

gelten.

Wir werden auch noch die Bezeichnung

$$H(k, \alpha) = \Phi_1(k, \alpha) \cap \text{Gr}\, B(k-1, \alpha) \tag{2.2.18}$$

gebrauchen (s. (2.2.15)).

Jetzt können wir die gewünschte Menge $B(k, \alpha)$ definieren.

6° (a) Im Falle

$$H(k, \alpha) = \varnothing$$

nehmen wir einfach

$$B(k, \alpha) = B(k-1, \alpha).$$

(b) Wenn aber

$$H(k, \alpha) \neq \varnothing$$

ist, so ergibt sich durch Anwendung des Satzes (1.5) — mit $H(k, \alpha)$ für das dortige F und mit den $\Phi_{k-r+1}(k, \alpha)$ für die dortigen H_r $(r=1, 2, \ldots, k+1)$ — die Existenz einer offenen Menge $W(k, \alpha)$ mit

$$H(k, \alpha) \subseteq W(k, \alpha) \subseteq W(k, \alpha)^- \subseteq U_\alpha \tag{2.2.19}$$

und

$$\text{ind}\, [\Phi_i(k, \alpha) \cap \text{Gr}\, W(k, \alpha)] \leqq n-i-1 \quad (i = 0, 1, \ldots, k) \tag{2.2.20}$$

(man beachte, daß, nach (2.2.7), $k \leqq n$ ist), und wir setzen

$$B(k, \alpha) = B(k-1, \alpha) \cup W(k, \alpha).^* \tag{2.2.21}$$

(c) Mit der Vereinbarung

$$W(k, \alpha) = \varnothing \quad (H(k, \alpha) = \varnothing)$$

erreichen wir, daß die Bestimmungsformeln unter (2.2.19) und (2.2.20) allgemeingültig werden.

7° Es gilt jetzt zu zeigen, daß die so definierte Menge $B(k, \alpha)$ Bedingungen genügt, die jenen in der Induktionsvoraussetzung unter 3° sinngemäß entsprechen.

Erstens haben wir nach (2.2.9), (2.2.21) und (2.2.19)

$$B(0, \alpha) \subseteq B(k-1, \alpha) \subseteq B(k, \alpha) \subseteq B(k, \alpha)^- \subseteq U_\alpha. \tag{2.2.22}$$

---

* Anstelle von $H(k, \alpha)$ könnte man hier ebensogut die ganze Menge $\text{Gr}\, B(k-1, \alpha)$ oder sogar $B(k-1, \alpha)^-$ zugrunde legen (d. h. mit einer offenen Menge W der Art unter (2.2.19) und (2.2.20) überdecken). Wir möchten aber mit der gewählten Variante den Kern der Sache klar zum Ausdruck bringen, daß uns nämlich der Teil $\text{Gr}\, B(k-1, \alpha) \setminus H(k, \alpha)$ von $\text{Gr}\, B(k-1, \alpha)$ keine Sorgen macht; die Menge $B(k-1, \alpha)$ muß evtl. eben wegen der Dimensionsverhältnisse von $\text{Gr}\, B(k-1, \alpha)$ *innerhalb* $H(k, \alpha)$ durch eine andere Menge $B(k, \alpha)$ ersetzt werden.

Zweitens gilt auf Grund von (2.2.21)

$$\text{(2.2.23)} \qquad \operatorname{Gr} B(k,\alpha) \subseteq [\operatorname{Gr} B(k-1,\alpha) \setminus W(k,\alpha)] \cup \operatorname{Gr} W(k,\alpha);$$

weiter haben wir (vgl. (2.2.17))

$$\text{(2.2.24)} \qquad \operatorname{ind}\left[\Phi_{i-1}(k,\alpha) \cap (\operatorname{Gr} B(k,\alpha) \setminus W(k-1,\alpha))\right] \leqq n-i$$
$$(i = 1, 2, \ldots, k+1)$$

— was für $i=1$ aus (2.2.13) und (2.2.11) folgt (man wende 4° auf $k'=k-1$, $\alpha'=\alpha$ und $i=1$ an), für $i=2, 3, \ldots, k+1$ aber eine Trivialität ist (handelt es sich ja dann, weil nach (2.2.19), (2.2.18) und (2.2.16)

$$\Phi_{i-1}(k,\alpha) \cap \operatorname{Gr} B(k-1,\alpha) \subseteq W(k,\alpha)$$

gilt, um die Dimension der leeren Menge) — und daher, nach (2.2.23), (2.2.24), (2.2.20) und den Sätzen (0.7) und (0.8)

$$\operatorname{ind}[\Phi_{i-1}(k,\alpha) \cap \operatorname{Gr} B(k,\alpha)] \leqq n-i$$
$$(i = 1, 2, \ldots, k+1),$$

was in Anlehnung an 3° und nach (2.2.13) auch so formuliert werden kann:

(A) Für ein beliebiges

$$i \in \{1, 2, \ldots, k+1\}$$

und für jedes i-Tupel

$$\{(k_1,\alpha_1), (k_2,\alpha_2), \ldots, (k_i,\alpha_i)\} \subseteq I$$

mit

$$(k-1,\alpha) \prec (k_1,\alpha_1) \prec (k_2,\alpha_2) \prec \ldots \prec (k_i,\alpha_i) = (k,\alpha)$$

gilt

$$\operatorname{ind} \bigcap_{j=1}^{i} \operatorname{Gr} B(k_j,\alpha_j) \leqq n-i.$$

8° Unser Induktionsverfahren ergibt also für jedes $(k,\alpha) \in I$ eine offene Menge $B(k,\alpha)$ mit den Eigenschaften (2.2.22) und 7° (A).

Setzen wir endlich

$$G_\alpha = B(n,\alpha) \quad (\alpha < \varrho).$$

Bei dieser Wahl genügt das System $\mathcal{G}$ unter (2.2.4) tatsächlich den Forderungen des Satzes:

(a) Nach (2.2.22) ist (2.2.5) erfüllt.

(b) Auf den Fall

$$k_1 = k_2 = \ldots = k_i = k = n$$

angewendet ergibt 7° (A) eben (vgl. 4° (b)), daß das System $\mathcal{G}$ ind-n-hübsch ist. □

(2.3) SATZ. *Es seien n eine nichtnegative ganze Zahl und X ein offen-erblich total-parakompakter (bzw. obendrein stark-parakompakter) T_2-Raum mit*

$$\text{(2.3.1)} \qquad \operatorname{ind} X = n.$$

Eine beliebige offene Überdeckung $\mathscr{U}$ von X kann dann zu einer lokal-endlichen und ind-*n-hübschen offenen Überdeckung $\mathscr{G}$ verfeinert werden (so daß, im stern-endlichen Fall, sogar die Überdeckung $\{G^-: G\in\mathscr{G}\}$ stern-endlich ist).*

BEWEIS. Wegen der Parakompaktheit (bzw. Stark-Parakompaktheit) des Raumes darf ohne weiteres angenommen werden, daß $\mathscr{U}$ lokal-endlich (bzw. Stern-endlich) ist, und es sei — auf Grund von (0.11) —

$$\mathscr{F} = \{F_U: U\in\mathscr{U}\}$$

eine abgeschlossene Überdeckung von X mit

$$F_U \subseteqq U \quad (U\in\mathscr{U}).$$

(Daß jede der Mengen F_U als die abgeschlossene Hülle einer offenen Menge gewählt werden kann, ist hier ohne Bedeutung.) Nach (2.3.1) und (0.8) gibt es dann weiter offene Mengen B_U $(U\in\mathscr{U})$ mit

$$F_U \subseteqq B_U \subseteqq B_U^- \subseteqq U \quad (U\in\mathscr{U}) \tag{2.3.2}$$

und mit

$$\operatorname{ind}(B_U^- \setminus B_U) \leqq n-1 \quad (U\in\mathscr{U}).$$

Auf die offenen Mengensysteme $\mathscr{U}$ und

$$\mathscr{B} = \{B_U: U\in\mathscr{U}\}$$

angewandt (man beachte dabei, daß nach (2.3.2) auch $\mathscr{B}$ eine Überdeckung von X ist) ergibt nun Satz (2.2) die Existenz eines ind-n-hübschen offenen Systems

$$\mathscr{G} = \{G_U: U\in\mathscr{U}\},$$

derart, daß

$$B_U^- \subseteqq G_U \subseteqq G_U^- \subseteqq U \quad (U\in\mathscr{U}). \tag{2.3.3}$$

Da das System $\{G^-: G\in\mathscr{G}\}$ wegen (2.3.3) lokal-endlich (bzw. Stern-endlich) und wegen (2.3.2) und (2.3.3) auch eine Überdeckung von X ist, entspricht $\mathscr{G}$ den Forderungen des Satzes. □

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A FIXED POINT THEOREM IN COMPLETE METRIC SPACES

by

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1. Introduction

It is possible to give a variety of generalizations of the classical fixed point theorem of BANACH [1] simply by using modified contraction type conditions. Fixed point theorems where the contractive type conditions involve two points of the underlying metric space have been established by a number of authors. See e.g. RHOADES [6] and the many references cited therein. Only recently RHOADES [7] and PITTNAUER [4] have established fixed point theorems for mappings which satisfy contractive type conditions involving three points of the space.

In this paper we establish a fixed point theorem whose contractive type condition involves four points of the space. This condition includes as special cases that of BANACH [1], BROWDER [2], KANNAN [3], ZAMFIRESCU [8], and as a consequence, applies even to various discontinuous mappings. It includes partly the contractive type condition of PITTNAUER [4], but not that of RHOADES [7].

To prove our theorem we follow the methods employed by BROWDER [2] and ZAMFIRESCU [8].

2. A fixed point theorem

Let M be a bounded closed subset of a complete metric space with distance function d. Let φ be a monotone nondecreasing self-mapping of the set of real numbers $r \geqq 0$, which is continuous on the right and satisfies the condition

$$\varphi(r) < \frac{r}{2} \quad \text{for} \quad r > 0. \tag{1}$$

Suppose further, that f is a mapping of M into itself, and consider the following contractive type condition:

$$\begin{cases} d[f(z_1), f(z_2)] \leqq \varphi(d[z_1, f^k(z_3)] + d[z_2, f^l(z_4)]) \\ \text{for each} \quad z_1, z_2, z_3, z_4 \in M \quad \text{and fixed} \\ \text{integers} \quad 0 \leqq k \leqq l. \end{cases} \tag{2}$$

THEOREM. *Let f be a mapping of M into itself satisfying condition* (2). *Then f has a unique fixed point in M.*

PROOF. We proceed in four steps.

a) First we show that under our hypothesis f has at most one fixed point. Let $\tilde{x}, \tilde{y} \in M$, $\tilde{x} \neq \tilde{y}$, and $\tilde{x} = f(\tilde{x})$, $\tilde{y} = f(\tilde{y})$. Then conditions (1) and (2) yield for

$$z_1 = z_4 = \tilde{x}, \quad z_2 = z_3 = \tilde{y}$$

the contradiction

$$0 < d(\tilde{x}, \tilde{y}) = d[f(\tilde{x}), f(\tilde{y})] \leqq \varphi[2 \cdot d(\tilde{x}, \tilde{y})] < d(\tilde{x}, \tilde{y}).$$

b) We now construct a suitable iteration sequence. For this purpose we define for given $x, y \in M$

$$z_1 = f^l(x), \quad z_2 = f^k(y), \quad z_3 = y, \quad z_4 = x,$$

and obtain from (2)

$$d[f(z_1), f(z_2)] \leqq \varphi(2 \cdot d[f^l(x), f^k(y)]). \tag{3}$$

If we choose an arbitrary point $x_0 \in M$ and define

$$x = f^{n+k}(x_0), \; y = f^{n+l+m}(x_0)$$

for some fixed positive integers n, m then inequality (3) implies

$$d[f^{n+k+l+m+1}(x_0), f^{n+k+l+1}(x_0)] \leqq \varphi(2 \cdot d[f^{n+k+l+m}(x_0), f^{n+k+l}(x_0)]). \tag{4}$$

c) We show that the iteration sequence

$$I(f, x_0) = \{x_n | x_n = f^n(x_0); \; n = 0, 1, 2, \ldots\}$$

is a *Cauchy* sequence.

First we observe that inequality (4) implies

$$d(x_{n+m+2}, x_{n+1}) \leqq \varphi[2 \cdot d(x_{n+m}, x_n)] \quad \text{for} \quad n \geqq k+l. \tag{5}$$

Next, following BROWDER [2], we define

$$\lambda_n = \sup_{m \geqq 0} d(x_n, x_{n+m}) \quad \text{for} \quad n \geqq k+l+1$$

and note that $\lambda_n < D < \infty$, where D denotes the diameter of M. Considering (5) and the monotonicity of φ we obtain

$$\lambda_n \leqq \sup_{m \geqq 0} (\varphi[2 \cdot d(x_{n-1}, x_{n+m-1})]) \leqq \varphi[2 \cdot \sup_{m \geqq 0} d(x_{n-1}, x_{n+m-1})] = \varphi(2 \cdot \lambda_{n-1}). \tag{6}$$

Let us consider the case where $\lambda_{n-1} = 0$ for some $n \geqq k+l+1$. Then we obtain immediately

$$x_n = x_{n+1} = f(x_n).$$

Therefore x_n is a fixed point of f, and $I(f, x_0)$ is, in fact, a *Cauchy* sequence. Next let $\lambda_{n-1} \neq 0$ for all $n \geqq k+l+1$. Then we obtain from (6) and (1)

$$\lambda_n < \lambda_{n-1} \quad \text{for} \quad n \geqq k+l+1,$$

and so the limit

$$0 \leqq \lambda_\infty = \lim_{n \to \infty} \lambda_n < D$$

obviously exists.

Because φ is continuous on the right we obtain

$$\lim_{n\to\infty} \varphi(\lambda_n) = \varphi(\lambda_\infty),$$

and considering (6) we finally obtain

$$\lambda_\infty \leqq \varphi(2\cdot\lambda_\infty).$$

If $\lambda_\infty > 0$, this inequality contradicts (1). Therefore we have $\lambda_\infty = 0$, which means that $I(f, x_0)$ is again a *Cauchy* sequence. Since M is a closed subset of a complete space, the limit

$$\lim_{n\to\infty} x_n = \tilde{x} \in M \tag{7}$$

exists.

d) We show that the assumption $\tilde{x} \neq f(\tilde{x})$ is contradictory.

For this purpose, following ZAMFIRESCU [8], we consider the ball

$$B = \left\{x \mid x \in M,\ d(x, \tilde{x}) \leqq \frac{2}{5}\cdot d[\tilde{x}, f(\tilde{x})]\right\}. \tag{8}$$

We observe that

$$d[x, f(\tilde{x})] \geqq \left(\frac{3}{5}\right)\cdot d[\tilde{x}, f(\tilde{x})] \quad (x \in B). \tag{9}$$

Now let us choose the smallest integer $N > l+1$ for which $f^N(x_0) \in B$. Taking

$$z_1 = f^N(x_0),\ z_2 = \tilde{x},\ z_3 = f^{N+l}(x_0),\ z_4 = f^{N+k}(x_0)$$

and considering (2), (8) and (9) we obtain

$$d[f^{N+1}(x_0), f(\tilde{x})] \leqq \varphi\big(d[f^N(x_0), f^{N+k+l}(x_0)] + d[\tilde{x}, f^{N+k+l}(x_0)]\big) \leqq$$

$$\leqq \varphi\big(d[f^N(x_0), \tilde{x}] + d[\tilde{x}, f^{N+k+l}(x_0)] + d[\tilde{x}, f^{N+k+l}(x_0)]\big) \leqq$$

$$\leqq \varphi\left(\frac{6}{5}\, d[\tilde{x}, f(\tilde{x})]\right) \leqq \varphi\big(2\cdot d[f^{N+1}(x_0), f(\tilde{x})]\big).$$

If $f^{N+1}(x_0) \neq f(\tilde{x})$, this contradicts (1). Next let $f^{N+1}(x_0) = f(\tilde{x})$. Taking $z_1 = \tilde{x}$, $z_2 = f(\tilde{x})$, $z_3 = f^m(x_0)$, $z_4 = f^{N+1-l}(x_0)$ with a positive integer m, we obtain from (2)

$$d[f(\tilde{x}), f^2(\tilde{x})] \leqq \varphi\big(d[\tilde{x}, f^{m+k}(x_0)]\big) = \varphi[d(\tilde{x}, x_{m+k})].$$

Using (7), for $m \to \infty$ we find

$$f(\tilde{x}) = f^2(\tilde{x}). \tag{10}$$

As we are considering the case $f(\tilde{x}) = f^{N+1}(x_0) = x_{N+1}$, we obtain from (7) and (10) that $\tilde{x} = f(\tilde{x})$ in contradiction to our origional assumption.

Hence our origional assumption on $\tilde{x}$ is always contradictory and $\tilde{x}$ is, in fact, a fixed point of f. This, of course, completes the proof.

3. Some special cases

Our theorem includes the fixed point theorem of

(a) BROWDER [2] for $k=l=0$, $z_3=z_2$, $z_4=z_1$.

Moreover, if we specialize the function φ by defining

$$\varphi(r) = \alpha \cdot r \quad \text{for} \quad 0 \leqq r < \infty \quad \text{with fixed} \quad 0 \leqq \alpha < \frac{1}{2}, \tag{11}$$

we can, in a somewhat simpler way, prove our theorem under the weaker assumption that M is only a complete metric space. This theorem includes as special cases the fixed point theorems of

(b) BANACH [1] for $k=l=0$, $z_3=z_2$, $z_4=z_1$,

(c) ZAMFIRESCU [8] for $k=l=1$, $z_3=z_2$, $z_4=z_1$,

(d) KANNAN [3] for $k=l=1$, $z_3=z_1$, $z_4=z_2$,

(e) PITTNAUER [4] (excluding the case $\frac{1}{2} \leqq \alpha < 1$ also allowed there) for $k=l \geqq 1$, $z_3=z_4$.

It should be mentioned that the fixed point theorem of RHOADES [7] is not contained in our theorem.

4. Concluding remarks

Our contractive type condition (2) simplifies somewhat for self-mappings of M which satisfy the following restriction

$$f^k(x) = a \quad \text{for every} \quad x \in M. \tag{12}$$

Of course, such a mapping cannot have fixed points other than $\tilde{x}=a$. This, however, leads immediately to the question whether it is possible to characterize by a contraction type condition alone the set of mappings f satisfying (12) with some a and possessing a fixed point.

Naturally, a similar problem arises for mappings f whose k-th iterates have a finite set for range. Finally it should be noted that it is not possible to extend our theorem to locally contractive mappings, even if we restrict φ to satisfy (11). This follows from REICH [5] and RHOADES [6]. As Professor DR. B. E. RHOADES, Bloomington, kindly pointed out to us, this argument applies also to corollary 7 in [4].

Last but not least we want the thank the referee, Dr. J. Bognár for many helpful suggestions.

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P-NEAT EXACT SEQUENCES

by

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1. Introduction

The concept of purity is of great importance in the theory of abelian groups, and much work has been done on pure subgroups and pure exact sequences. This concept generalizes naturally to that of P-purity, where P is a family of primes. Of course, if P is the set of all primes a P-pure subgroup is indeed a pure subgroup. We, therefore, preferred to consider P-pure exact sequences in [5] instead of pure exact sequences. A concept, which is closely allied to purity and is of growing significance, is that of neatness introduced by HONDA in [3]. A subgroup A' of an abelian group A is neat in A if $A' \cap pA = pA'$, where p is any prime. We can generalize also this concept and say that A' is *P-neat* in A if $A' \cap pA = pA'$ for each $p \in P$. Equivalently, A' is P-neat in A if the equation $a' = ma$, $a' \in A'$, $a \in A$, $m \in \overline{P}$ implies that $a' = ma_1'$ for some $a_1' \in A'$, where $\overline{P}$ is the set of finite products of distinct primes from P. It is clear that a P-neat subgroup is a neat subgroup if P is the family of all primes. If A' is P-neat in A we call the corresponding quotient group $A/_{A'}$ a *P-neat quotient group*. Also we call the exact sequence $A' \overset{\lambda}{\rightarrowtail} A \overset{\mu}{\twoheadrightarrow} A''$ *P-neat exact* if $A'\lambda$ is P-neat in A.

The object of this paper is to present systematically some properties of P-neat exact sequences. In section 2 we discuss their elementary properties, including some conditions under which a P-neat exact sequence splits. In section 3 we examine the effect of some functors on P-neat exact sequences, and in section 4 we describe P-neat projective and injective groups. Section 5 is devoted to the study of coneat (co-P-neat) subsets. The concept of a coneat subset is parallel to that of a copure subset introduced by the author in [4]. We call a subset S of an abelian group A a *coneat subset* of A if every subgroup H_S of A maximal with respect to missing S is a neat subgroup of A, i.e., if the cosets corresponding to S cogenerate a neat quotient group of A. Since we have considered copure subsets in detail in [4] we content ourselves by just stating there the corresponding results relating to coneat subsets. In fact, for the sake of brevity, we have omitted all those proofs which are either trivial or otherwise evident.

We follow [1] for notation and terminology, and by a group in the rest of the paper we mean an abelian group.

2. Elementary Properties

(2.1) PROPOSITION. *If A_1' is a P-neat subgroup of A' and A' a P-neat subgroup of A, then A_1' is a P-neat subgroup of A.*

(2.2) PROPOSITION. *A P-neat subgroup of a P-torsion-free* (see [2]) *group A is P-pure in A.*

(2.3) THEOREM. *The exact sequence*

$$0 \to A' \xrightarrow{\lambda} A \xrightarrow{\mu} A'' \to 0 \tag{2.4}$$

is P-neat exact iff for each element $a'' \in A''$ of order $n \in \bar{P}$, there is an element $a \in A$ of the same order such that $a\mu = a''$.

(2.5) COROLLARY. *If A_1'' is a P-neat quotient group of A'', and A'' is a P-neat quotient group of A, then A_1'' is a P-neat quotient group of A, i.e., there is a P-neat subgroup A_1' of A such that $A/A_1' = A_1''$.*

(2.6) DEFINITION. We call a group A a *$\bar{P}$-group* if the order of each of its elements belongs to $\bar{P}$.

We note that a $\bar{P}$-group is elementary, and so its every subgroup is a direct summand.

(2.7) PROPOSITION. *Let the sequence* (2.4) *be P-neat exact. If the torsion subgroup of A'' is a $\bar{P}$-group, then the sequence is pure exact. If A'' has no elements of order p^m, $m>1$, $p \in P$, then the sequence is P-pure exact.*

PROOF. This follows from Theorem 2.3 and the corresponding results for pure exact and P-pure exact sequences.

(2.8) DEFINITION. We call a group *$\bar{P}$-cyclic* if it is either an infinite cyclic group or a cyclic group of order n, $n \in \bar{P}$.

(2.9) DEFINITION. We say that a finitely generated group A is *$\bar{P}$-finitely generated* if its torsion subgroup is a $\bar{P}$-group.

Clearly, a $\bar{P}$-finitely generated group is a direct sum of a finite number of $\bar{P}$-cyclic groups.

(2.10) DEFINITION. We call a group *$\bar{P}$-cocyclic* if it is either $Z(p)$, $p \in P$, or any $Z(p^\infty)$. We similarly define a $\bar{P}$-finitely cogenerated group.

(2.11) THEOREM. *Let the sequence*

$$0 \to A' \xrightarrow{\lambda} A \xrightarrow{\mu} A'' \to 0 \tag{2.12}$$

be P-neat exact. Then the sequence splits if A'' is a direct sum of $\bar{P}$-cyclic groups. In particular, if A'' is a $\bar{P}$-group the sequence splits.

PROOF. Let $A'' = \bigoplus_{i \in I} \langle a_i'' \rangle$. Let a_i be a pre-image (w.r.t. μ) of a_i'' of the same order as that of a_i'' (see Theorem 2.3). Then the mapping $a_i'' \to a_i$ induces a well-defined homomorphism $\eta: A'' \to A$ such that $\eta\mu = 1$.

(2.13) COROLLARY. *The P-neat exact sequence* 2.12 *splits if A'' is $\bar{P}$-finitely generated.*

(2.14) THEOREM. *The P-neat exact sequence* 2.12 *splits if* $mA'=0$ *for some* $m\in\overline{P}$.

PROOF. Let $m=p_1p_2\dots p_k$, each $p_i\in P$, and let $m_i=p_1\dots p_{i-1}p_{i+1}\dots p_k$, so that $m=p_im_i$. Since $A'\lambda$ is P-neat in A, $A'\lambda\cap mA=mA'\lambda=0$. Let B be a subgroup of A containing mA and maximal with respect to $A'\lambda\cap B=0$. We shall show that $A'\lambda\oplus B=A$. Let $a\in A$, then $ma\in B$. If $a_i=m_ia\notin B$, then $B+\langle a_i\rangle$ contains a non-zero element, say a', of $A'\lambda$. Hence $a'=b+ka_i$, where $(k,p_i)=1$, so $ka_i=$ $=a'-b\in A'\lambda\oplus B$. Since $p_ia_i\in B$ and $(k,p_i)=1$, it follows that $a_i\in A'\lambda\oplus B$. Since $(m_1,m_2,\dots,m_k)=1$ and $m_ia\in A'\lambda\oplus B$ for each i, we infer that $a\in A'\lambda\oplus B$.

(2.15) COROLLARY. *The P-neat sequence* 2.12 *splits if* A' *is a* $\overline{P}$*-cocyclic group.*

(2.16) COROLLARY. *The P-neat sequence* 2.12 *splits if* A' *is* $\overline{P}$*-finitely cogenerated.*

(2.17) THEOREM. *The P-neat exact sequence* 2.12 *splits if A is a P-group (see* [2]*) such that its p-component* A_p, *for each* $p\in P$, *is a direct sum of cyclic groups, each of which has order* p^m or p^{m+1} for some $m\geqq 0$.

PROOF. By Theorem 4.1 [4] the sequence is pure exact, and so it splits, for A'' is a direct sum of cyclic groups.

In [2] we have discussed the concepts of P-divisibility, P-injectivity, and that of a P-extension. In the sequel we use these terms in the same sense.

(2.18) PROPOSITION. A' *is a P-neat subgroup of A iff* A' *has no proper essential P-extensions in A.*

(2.19) THEOREM. *Let* A'' *be a P-group in the exact sequence* (2.4). *Then the sequence is P-neat exact iff* $A'\lambda=A\cap D$ *for some P-injective (P-divisible) subgroup D of B, where B is a minimal P-injective P-extension of A.*

PROOF. Let $A'\lambda=A\cap D$, and let $a'\in A'\lambda$ with $a'=pa$ for some $a\in A$, where $p\in P$. Since D is P-divisible, $a'=pd$ for some $d\in D$. Hence $p(a-d)=0$, and so $a-d$ is of order p if $a\neq d$. If $a-d\notin A$, then $\langle a-d\rangle\cap A=0$, so $a-d=0$, for B is an essential extension of A (see Theorem 4.6 [2]). This contradiction implies that $a-d\in A$, and so $d\in A'\lambda$. Conversely, suppose that the sequence is P-neat exact. Since B is a P-injective P-extension of $A'\lambda$, it contains a minimal P-injective P-extension of $A'\lambda$, say D. We shall show that $A'\lambda=A\cap D$. We note that $A\cap D$ is an essential P-extension of $A'\lambda$ and that $A'\lambda$ is P-neat in $A\cap D$. Hence $A'\lambda=A\cap D$ by Proposition (2.18).

(2.20) COROLLARY. *If A is a P-extension of* A', *then it contains a minimal P-neat subgroup containing* A'.

PROOF. Let B be a minimal P-injective P-extension of A (see [2]), and let D be a minimal P-injective subgroup of B containing A'. Then $N=D\cap A$ is a minimal P-neat subgroup of A containing A'.

3. Action of some functors on *P*-neat exact sequences

We first prove a lemma.

(3.1) LEMMA. *Let* A' *be a subgroup of a group A and let B be any group such that its torsion subgroup is a* $\overline{P}$*-group. Then the homomorphism* $A'\otimes B\to A\otimes B$ *is a monomorphism for all such B iff* A' *is P-neat in A.*

PROOF. This is a consequence of the observation that the kernel of $A'\otimes B\to A\otimes B$ is generated by elements of the form $a'\otimes b, a'\in A', b\in B$, with the property that there is a positive integer n such that a' is divisible by n in A and $nb=0$.

(3.2.) COLLOARY. A' is a P-neat subgroup of A iff $A'\otimes Z(n)\to A\otimes Z(n)$ is monic for all $n\in\bar{P}$.

(3.3) THEOREM. *Let the exact sequence*

$$0\to A'\xrightarrow{\lambda} A\xrightarrow{\mu} A''\to 0 \tag{3.4}$$

be P-neat exact, and let B be a group such that its torsion subgroup $T(B)$ is a $\bar{P}$-group, then the sequences

$$0\to A'\otimes B\xrightarrow{\lambda\otimes 1} A\otimes B\xrightarrow{\mu\otimes 1} A''\otimes B\to 0, \tag{3.5}$$

$$0\to \operatorname{Tor}(A', B)\xrightarrow{\lambda_*}\operatorname{Tor}(A,B)\xrightarrow{\mu_*}\operatorname{Tor}(A'', B)\to 0 \tag{3.6}$$

are exact. Moreover, the sequence 3.6 *splits.*

PROOF. The exactness of sequence 3.5, and so also of 3.6, follows immediately from Lemma 3.1. Since $T(B)$ is a $\bar{P}$-group, $\operatorname{Tor}(A, B)$ is also a $\bar{P}$-group and its every subgroup is a direct summand.

(3.7) COROLLARY. *If B is a $\bar{P}$-group, then the sequence* 3.5 *also splits.*

PROOF. In this case $A\otimes B$ is also a $\bar{P}$-*group.*

(3.8) THEOREM. *The direct limit of a direct system of P-neat exact sequences is a P-neat exact sequence.*

PROOF. The proof is similar to that of Theorem 3.10 in [5].

We shall now show that, in fact, the sequence 3.5 is a P-neat exact sequence.

(3.9) THEOREM. *The exact sequence* 3.5 *is P-neat exact.*

PROOF. Let $B=\varinjlim B_\alpha$, where B_α runs through all finitely generated subgroups of B. Now $B_\alpha=F_\alpha\oplus T_\alpha$, where F_α is free and T_α is a $\bar{P}$-group. It is easy to see that the sequence $0\to A'\otimes F_\alpha\to A\otimes F_\alpha\to A''\otimes F_\alpha\to 0$ is P-neat exact. Also since the sequence $0\to A'\otimes T_\alpha\to A\otimes T_\alpha\to A''\otimes T_\alpha\to 0$ splits by Cor. 3.7, it follows that $0\to A'\otimes\otimes B_\alpha\to A\otimes B_\alpha\to A''\otimes B_\alpha\to 0$ is a P-neat exact sequence. Hence the sequence 3.5 is P-neat exact by Theorem 3.8.

(3.10) PROPOSITION. *The sequence* 3.4 *is P-neat exact iff* $\operatorname{Hom}(Z(n), A)\to\operatorname{Hom}(Z(n), A'')$ *is epic for all* $n\in\bar{P}$.

PROOF. This follows from Theorem 2.3.

(3.11) THEOREM. *If the sequence* $0\to B'\xrightarrow{\lambda} B\xrightarrow{\mu} B''\to 0$ *is P-neat exact, then for all groups A the sequence*

$$0\to\operatorname{Hom}(A, B')\xrightarrow{\lambda_*}\operatorname{Hom}(A, B)\xrightarrow{\mu_*}\operatorname{Hom}(A, B'') \tag{3.12}$$

is P-neat exact.

PROOF. Let $\theta \in \mathrm{Hom}(A, B'')$ be of order m, $m \in \bar{P}$, and let $\theta = \varphi\mu_*$ for some $\varphi \in \mathrm{Hom}(A, B)$. We shall show that there is a homomorphism $\psi \in \mathrm{Hom}(A, B)$ of order m with $\psi\mu_* = \theta$, thus proving that the sequence 3.12 is P-neat exact. Since $m\theta = 0$, we have $m(A\theta) = 0$. Let us consider the exact sequence $0 \to B' \xrightarrow{\lambda} \bar{B} \xrightarrow{\mu} A\theta \to 0$, where $B = \mu^{-1}(A\theta)$. The sequence splits by Theorem 2.11, so there exists a homomorphism $\eta\colon A\theta \to \bar{B}$ such that $\eta\mu = 1$. We take $\psi = \theta\eta$.

(3.13) COROLLARY. *If* $A = Z(n)$, $n \in \bar{P}$, *then* μ_* *is epic and sequence* 3.12 *splits.*

PROOF. μ_* is epic by Proposition 3.10. The sequence splits, for $\mathrm{Hom}(A, B) \cong \cong B[n]$ is a $\bar{P}$-group.

(3.14) COROLLARY. *If* $nA = 0$, $n \in \bar{P}$, *then* μ_* *is epic, and sequence* 3.12 *splits.*

PROOF. Note that $n\,\mathrm{Hom}(A, B) = 0$.

(3.15) COROLLARY. *If* A *is a torsion* $\bar{P}$*-finitely generated group, then* μ_* *is epic and sequence* 3.12 *splits.*

(3.16) THEOREM. *If the sequence* $0 \to A' \xrightarrow{\lambda} A \xrightarrow{\mu} A'' \to 0$ *is P-neat exact, then the sequence*

$$0 \to \mathrm{Hom}(A'', B) \xrightarrow{\mu^*} \mathrm{Hom}(A, B) \xrightarrow{\lambda^*} \mathrm{Hom}(A', B) \tag{3.17}$$

is P-neat exact, where B is any abelian group.

PROOF. Let $\theta \in \mathrm{Hom}(A', B)$ be of order m, $m \in \bar{P}$, and let $\theta = \varphi\lambda^*$ for some $\varphi \in \mathrm{Hom}(A, B)$. We shall show that there is a homomorphism $\psi \in \mathrm{Hom}(A, B)$ of order m with $\psi\lambda^* = \theta$. Since $m\theta = 0$, we have $(mA')\theta = 0$, so $mA' \subseteq \ker\theta$. Hence $A'\theta \cong A'/\ker\theta \cong \dfrac{A'/mA'}{\ker\theta/mA'}$, so there is a homomorphism η and an isomorphism α such that $A'/mA' \xrightarrow{\eta} A'/\ker\theta \xrightarrow{\alpha} A'\theta \subseteq B$. Also $A'\lambda/mA'\lambda$ is P-neat in $A/mA'\lambda$, and so a direct summand by Theorem 2.14. Hence we have the following exact commutative diagram

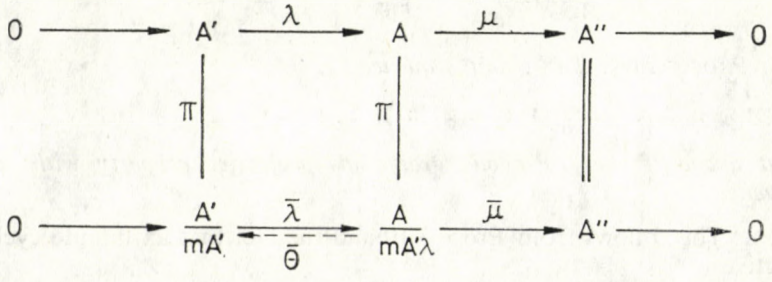

Fig. 1

where $\bar{\lambda}\bar{\theta} = 1$. We set $\psi = \pi\bar{\theta}\eta\alpha$.

(3.18) COROLLARY. *If* $nB = 0$, $n \in \bar{P}$, *then* λ^* *is epic and sequence* 3.17 *splits.*
We have similar results for the functor Ext.

4. *P*-neat projectivity and *P*-neat injectivity

(4.1) DEFINITION. A group G is *P-neat projective* if it has the projective property with respect to every P-neat exact sequence, i.e., given any P-neat exact sequence

$$0 \to A' \xrightarrow{\lambda} A \xrightarrow{\mu} A'' \to 0 \tag{4.2}$$

and a homomorphism $\varphi: G \to A''$, then there exists a homomorphism $\psi: G \to A$ such that the diagram

(4.3)

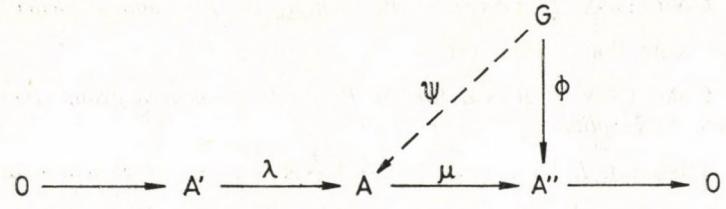

Fig. 2

commutes.

A *P-neat injective* group G is defined by the dual situation:

(4.4)

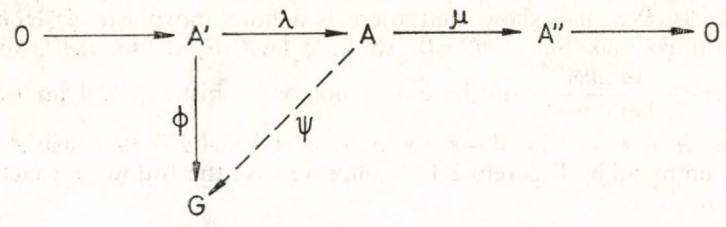

Fig. 3

(4.5) PROPOSITION. *The exact sequence*

$$0 \to A' \xrightarrow{\lambda} A \xrightarrow{\mu} A'' \to 0 \tag{4.6}$$

is P-neat exact iff every $\bar{P}$-cyclic group has projective property with respect to this sequence.

PROOF. This follows from Prop. 3.10 and the fact that an infinite cyclic group is projective.

(4.7) PROPOSITION. *The exact sequence 4.6 is P-neat exact iff every $\bar{P}$-cocyclic group has injective property for it.*

PROOF. Let the sequence be P-neat exact. Since $Z(p^\infty)$ is injective and $Z(p)$, $p \in P$, has injective property for the sequence by Corollary 3.18, the result follows.

Conversely, suppose that $G=Z(p)$, $p\in P$, has injective property for the sequence 4.6 (see 4.4). Let $a'\lambda$, $a'\in A'$, be divisible by p in A but not in $A'\lambda$. Then $a'\notin pA'$. Let H be a subgroup of A' containing pA' and maximal with respect to missing a'. By taking $G=A'/H$ we arrive at a contradiction.

Slight modifications in the proofs of some well-known results on purity lead to the following theorems.

(4.8) THEOREM. (i) *A pushout of a P-neat exact sequence is P-neat exact.*
(ii) *A pullback of a P-neat exact sequence is P-neat exact.*

(4.9) THEOREM. *Every group can be expressed as a P-neat quotient group of a direct sum of $\bar{P}$-cyclic groups.*

(4.10) THEOREM. *Every group can be embedded as a P-neat subgroup of a direct product of $\bar{P}$ cocyclic groups.*

We characterize P-neat projective and injective groups in the following theorems.

(4.11) THEOREM. *For a group G the following statements are equivalent:*
(i) *G is P-neat projective.*
(ii) *Every P-neat exact sequence* $0\to A\xrightarrow{\lambda'} B\xrightarrow{\mu'} G\to 0$ *splits.*
(iii) *G is a direct sum of $\bar{P}$-cyclic groups.*

PROOF. (i)⇒(ii). There exists a homomorphism $\psi\colon G\to B$: such that $\psi\mu'=1\colon G\to G$, so the sequence splits.
(ii)⇒(iii). This follows from Theorem 4.9.
(iii)⇒(i). Let $0\to A'\xrightarrow{\lambda} A\xrightarrow{\mu} A''\to 0$ be a P-neat exact sequence, and let $\varphi\colon G\to A''$ be a homomorphism. We have the pullback diagram

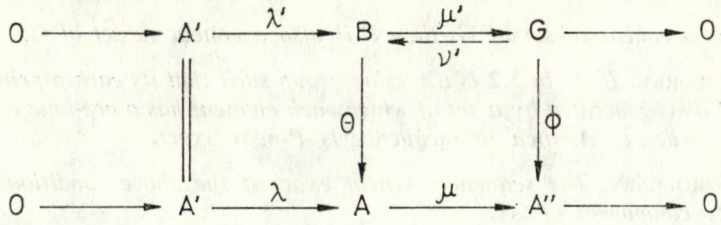

Fig. 4

By Theorem 4.8 (ii) the upper sequence is P-neat exact, and so it splits by Theorem 2.11. Hence there is a homomorphism $\nu'\colon G\to B$ such that $\nu'\mu'=1$. Then $\psi=\nu'\theta\colon G\to A$ is such that $\psi\mu=\varphi$.

(4.12) THEOREM. *For a group G the following are equivalent:*
(i) *G is P-neat injective.*
(ii) *Every P-neat exact sequence* $0\to G\xrightarrow{\lambda'} B\xrightarrow{\mu'} C\to 0$ *splits.*
(iii) *G is a direct summand of a direct product of $\bar{P}$-cocyclic groups.*

PROOF. (i)⇒(ii). There is a homomorphism $\psi\colon B\to G$ such that $\lambda'\psi=1\colon G\to G$, so the sequence splits.

(ii)⇒(iii). This follows from Theorem 4.10.

(iii)⇒(i). Let G be a direct summand of a direct product $H=\prod_i C_i$ of $\bar{P}$-cocyclic groups C_i. Let π_i, ε_i denote the coordinate projections and injections associated with this direct product, and let $\pi: H\to G$ and $\varepsilon: G\to H$ satisfy $\varepsilon\pi=1: G\to G$. Let $0\to A'\xrightarrow{\lambda} A\xrightarrow{\mu} A''\to 0$ be a P-neat exact sequence, and let $\varphi: A'\to G$ be a homomorphism. Then by Proposition 4.7 there exists a homomorphism $\psi_i: A\to C_i$ such that $\lambda\psi_i=\varphi\varepsilon\pi_i$. These homomorphisms give rise to a homomorphism $\psi_1: A\to H=\prod_i C_i$ such that $\psi_1\pi_i=\psi_i$. Then $\psi=\psi_1\pi: A\to G$ is such that $\lambda\psi=\varphi$.

5. Coneat subsets

(5.1) PROPOSITION. *Let*

$$0\to A'\xrightarrow{\lambda} A\xrightarrow{\mu} A''\to 0 \tag{5.2}$$

be an exact sequence, and let A'' be a direct sum of cocyclic groups. If a cogenerator of each p-cocyclic group, $p\in P$, has a pre-image (w.r.t. μ) of the same order in A, then the sequence is P-neat exact.

(5.3) COROLLARY. *If a cogenerator of each cocyclic group in A'' has a pre-image of the same order, then the sequence is neat exact.*

(5.4) COROLLARY. *Let x be any element of order p in a group G, where p is a prime, then $\{x\}$ is a coneat subset.*

(5.5) THEOREM. *Let G be a group and let $S=\bigcup_{p\in P} S_p$, where S_p is a subset of G_p. If each S_p is a coneat subset of G, then S is also a coneat subset of G.*

(5.6) THEOREM. *If A'' in 5.2 is a torsion group such that its each p-primary component, $p\in P$, is cogenerated by a set of which each element has a pre-image (w.r.t. μ) of the same order in A, then the sequence is P-neat exact.*

(5.7) COROLLARY. *The sequence is neat exact if the above condition holds for each primary component of A''.*

(5.8) COROLLARY. *Let S be a set of elements of order p in a group G, where p is a prime, then S is a coneat subset of G.*

In fact, we have

(5.9) THEOREM. *Any subset of non-zero elements in the socle $S(G)$ of a group is a coneat subset of G.*

(5.10) THEOREM. *Let S be a subset of a group G such that if $x\in S$, then $nx\neq 0$ and $nx\in S$ for all $n\in\bar{P}$. Then S is a co-P-neat subset of G.*

(5.11) THEOREM. *Let S_1 be a subset of the socle $S(G)$ of a group G and let S_2 be a subset of G such that if $x\in S_2$, then $nx\neq 0$ and $nx\in S_2$, where n is any square free integer. Then $S=S_1\cup S_2$ is a coneat subset of G.*

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ON THE FIRST EMPTY CELL

by

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Introduction

Let $\{a_k(\omega)\}_{k=1}^{\infty}$ be a sequence of independent identically distributed random variables on a given probability space $(\Omega, \mathcal{F}, P)$ and suppose, that

$$P(a_k(\omega) = i) = p_i, \quad i = 1, 2, \ldots .$$

Assume that the p_i's are decreasing, i.e. $p_i \geqq p_{i+1}$ for $i = 1, 2, \ldots$ and that $p_i > 0$ for $i = 1, 2, \ldots$.

Let $Z_n(\omega)$ be the smallest natural number for which

$$a_k(\omega) \neq Z_n(\omega) \quad \text{for} \quad 1 \leqq k \leqq n.$$

Z_n can be interpreted as the first empty cell in an occupancy problem with infinite number of cells, p_i being the probability of hitting the i-th cell. This cell occupancy-model has been investigated by KARLIN [2] who proved central limit theorems and a.s. convergence for the number of occupied cells and related variables.

In the present paper we prove some theorems concerning the order of magnitude of Z_n in case $p_i \sim \frac{c}{i^{\alpha}}$ $i \to \infty$.

First we give a limit distribution theorem.

THEOREM 1. *Let* $p_i \sim \frac{c}{i^{\alpha}}$ $(i \to \infty)$ *and for any real* y *put*

$$N(n, y) = \left[\frac{\alpha c n}{\log n}\left(1 + \frac{(\alpha+1)\log\log n}{\log n} + \frac{y}{\log n}\right)\right]^{\frac{1}{\alpha}}. \tag{1}$$

Then

$$\lim_{n\to\infty} P(Z_n \geqq N(n, y)) = e^{-e^{\frac{y}{\alpha}}}.$$

The next two theorems concern the almost sure behavior of Z_n.

THEOREM 2. *Let* $p_i \sim \frac{c}{i^{\alpha}}$ $(i \to \infty)$,

$$y_U^{(1)}(n) = \alpha \log\log\log n - \log(\alpha c) + \alpha \log(1+\varepsilon) + \alpha \log 2 \tag{2}$$

and

$$y_U^{(2)}(n) = \alpha \log\log\log n - \log(\alpha c) + \alpha \log(1-\varepsilon). \tag{3}$$

Then with probability 1 *only a finite number of the events*

$$\{Z_n > N(n, y_U^{(1)}(n))\}, \quad n = 1, 2, \ldots \tag{4}$$

occur. Moreover with probability 1 *infinitely many of the events*

$$\{Z_n > N(n, y_U^{(2)}(n))\}, \quad n = 1, 2, \ldots \tag{5}$$

occur.

THEOREM 3. *Let* $p_i \sim \frac{c}{i^\alpha}$ $(i \to \infty)$, *and*

$$y_L^{(1)}(n) = -2\alpha(1+\varepsilon)\log\log n, \quad n = 1, 2, \ldots, \tag{6}$$

$$y_L^{(2)}(n) = -\alpha \log\log n, \quad n = 1, 2, \ldots. \tag{7}$$

Then with probability 1 *only a finite number of the events*

$$\{Z_n < N(n, y_L^{(1)}(n))\}, \quad n = 1, 2, \ldots \tag{8}$$

occur. Moreover with probability 1 *infinitely many of the events*

$$\{Z_n < N(n, y_L^{(2)}(n))\}, \quad n = 1, 2, \ldots \tag{9}$$

occur.

An immediate consequence of Theorems 2 and 3 is the following

COROLLARY.

$$P\left(\lim_{n\to\infty} Z_n \left(\frac{\alpha c n}{\log n}\right)^{-\frac{1}{\alpha}} = 1\right) = 1. \tag{10}$$

1. §. Preliminary lemmas

In this section we prepare the proof of Theorems by presenting a few lemmas. We use the notations given in the Introduction. The first lemma gives an explicit expression for the exact distribution of Z_n.

LEMMA 1.1.

$$P(Z_n \leqq N) = \sum_{i=1}^{N} (1-p_i)^n - \sum_{1 \leqq i_1 < i_2 \leqq N} (1-p_{i_1}-p_{i_2})^n + \ldots$$
$$+ (-1)^{k+1} \sum_{1 \leqq i_1 < i_2 < \ldots < i_k \leqq N} (1-p_{i_1}-p_{i_2}-\ldots-p_{i_k})^n + \ldots. \tag{11}$$

PROOF. Let $B_i^{(n)}$ denote the event that $a_k \neq i$ for $k = 1, 2, \ldots, n$. Then obviously $P(B_i^{(n)}) = (1-p_i)^n$ and in general for $1 \leqq i_1 < i_2 < \ldots < i_k$

$$P(B_{i_1}^{(n)} B_{i_2}^{(n)} \ldots B_{i_k}^{(n)}) = (1-p_{i_1}-p_{i_2}-\ldots-p_{i_k})^n. \tag{12}$$

Since

$$P(Z_n > N) = P(\bar{B}_1^{(n)} \bar{B}_2^{(n)} \ldots \bar{B}_N^{(n)})$$

using inclusion and exclusion and (12) we have our Lemma.

The next lemma gives an asymptotic value of (11), together with a remainder. Define $\lambda=\lambda(n, N)$ by

$$\lambda = \sum_{i=1}^{N} e^{-np_i} \tag{13}$$

LEMMA 1.2.

$$P(Z_n > N) = e^{-\lambda} + R(n, N, \lambda) \tag{14}$$

where

$$|R(n, N, \lambda)| \leqq e^{\lambda}\left(3Ne^{-n^{\frac{1}{4}}} + e^{-np_N} + \frac{\lambda^2+2\lambda}{\sqrt{n}} + \frac{\lambda^{N+1}}{(N+1)!}\right). \tag{15}$$

PROOF. In the proof of Lemma 1.2 we follow SEVAST'YANOV [4], making his remainders explicit.

From Lemma 1.1, with the notations $S_0=1$,

$$S_k = \sum_{1\leqq i_1<i_2<\ldots<i_k\leqq N} (1-p_{i_1}-p_{i_2}-\ldots-p_{i_k})^n, \quad 1 \leqq k \leqq N, \tag{16}$$

$$\lambda_1 = \sum_{i=1}^{N} (1-p_i)^n \tag{17}$$

we have

$$P(Z_n > N) = \sum_{k=0}^{N} (-1)^k S_k = \sum_{k=0}^{N} \frac{(-\lambda_1)^k}{k!} + \sum_{k=1}^{N} (-1)^k \left(S_k - \frac{\lambda_1^k}{k!}\right) \tag{18}$$

hence

$$|P(Z_n > N) - e^{-\lambda}| \leqq |e^{-\lambda_1} - e^{-\lambda}| + \sum_{k=N+1}^{\infty} \frac{\lambda_1^k}{k!} + \sum_{k=1}^{N} \left|S_k - \frac{\lambda_1^k}{k!}\right|. \tag{19}$$

Define $I_1(n)$ as the subset of $\{1, 2, \ldots N\}$ such that $i \in I_1(n)$ whenever $p_i > n^{-3/4}$. Let $I_k(n)$ be the subset of all possible $\{i_1, i_2, \ldots i_k\}$ with different i_j and for at least one j $(1 \leqq j \leqq k)$ $p_{i_j} > n^{-3/4}$. Denote by $\bar{I}_k(n)$ the set of those $\{i_1, \ldots i_k\}$ with different i_j which do not belong to $I_k(n)$. Then SEVAST'YANOV [4] gives the following estimations:

$$\lambda_1 \leqq \lambda \leqq e^{1/\sqrt{n}} \lambda_1 + Ne^{-n^{\frac{1}{4}}}, \tag{20}$$

$$\sum_{I_k(n)} (1-p_{i_1})^n \ldots (1-p_{i_k})^n \leqq k\lambda_1^{k-1} Ne^{-n^{\frac{1}{4}}}, \tag{21}$$

$$\sum_{I_k(n)} (1-p_{i_1}-\ldots-p_{i_k})^n \leqq k\lambda^{k-1} Ne^{-n^{\frac{1}{4}}}, \tag{22}$$

and if $(i_1, \ldots, i_k) \notin I_k(n)$ and $n \geqq 3$, then

$$(1-p_{i_1})^n \ldots (1-p_{i_k})^n e^{-k^2/\sqrt{n}} \leqq (1-p_{i_1}-\ldots-p_{i_k})^n \leqq (1-p_i)^n \ldots (1-p_{i_k})^n. \tag{23}$$

From these we obtain immediately the following estimations:

$$|e^{-\lambda_1}-e^{-\lambda}| \leqq \lambda-\lambda_1 \leqq \frac{\lambda}{\sqrt{n}}+Ne^{-n^{\frac{1}{4}}} \leqq e^{\lambda}\left(\frac{\lambda}{\sqrt{n}}+Ne^{-n^{\frac{1}{4}}}\right), \tag{24}$$

$$\sum_{k=N+1}^{\infty}\frac{\lambda_1^k}{k!} < e^{\lambda_1}\frac{\lambda_1^{N+1}}{(N+1)!} < e^{\lambda}\frac{\lambda^{N+1}}{(N+1)!}. \tag{25}$$

On the other hand

$$\lambda_1^k = \sum_{I_k(n)}(1-p_{i_1})^n \dots (1-p_{i_k})^n + \tag{26}$$

$$+\sum_{I_k(n)}(1-p_{i_1})^n \dots (1-p_{i_k})^n + \sum{}'(1-p_{i_1})^n \dots (1-p_{i_k})^n$$

where Σ' denotes summation over those indices $(i_1, \dots, i_k)$ for which at least two of them are equal. Furthermore

$$k!\,S_k = \sum_{(i_1,\dots,i_k)}(1-p_{i_1}-\dots-p_{i_k})^n =$$

$$= \sum_{I_k(n)}(1-p_{i_1}-\dots-p_{i_k})^n + \sum_{I_k(n)}(1-p_{i_1}-\dots-p_{i_k})^n.$$

Hence

$$|\lambda_1^k - k!\,S_k| \leqq \sum{}'(1-p_{i_1})^n \dots (1-p_{i_k})^n + \tag{27}$$

$$+\sum_{I_k(n)}(1-p_{i_1})^n \dots (1-p_{i_k})^n + \sum_{I_k(n)}(1-p_{i_1}-\dots-p_{i_k})^n +$$

$$+\sum_{I_k(n)}[(1-p_{i_1})^n \dots (1-p_{i_k})^n - (1-p_{i_1}-\dots-p_{i_k})^n] = \Sigma^{(1)}+\Sigma^{(2)}+\Sigma^{(3)}+\Sigma^{(4)}.$$

Using (20)—(23) it can be seen that

$$\Sigma^{(1)} \leqq (1-p_N)^n k\lambda^{k-1}, \tag{28}$$

$$\Sigma^{(2)} \leqq k\lambda^{k-1}Ne^{-n^{\frac{1}{4}}}, \tag{29}$$

$$\Sigma^{(3)} \leqq k\lambda^{k-1}Ne^{-n^{\frac{1}{4}}}, \tag{30}$$

$$\Sigma^{(4)} \leqq \sum_{I_k(n)}(1-p_{i_1})^n \dots (1-p_{i_k})^n(1-e^{-k^2/\sqrt{n}}) \leqq \frac{k^2}{\sqrt{n}}\lambda^k. \tag{31}$$

Summing up for k we finally obtain

$$\sum_{k=1}^{N}\left|\frac{\lambda_1^k}{k!}-S_k\right| \leqq \sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}\left((1-p_N)^n+2Ne^{-n^{\frac{1}{4}}}\right)+\frac{1}{\sqrt{n}}\sum_{k=1}^{\infty}\frac{k\lambda^k}{(k-1)!} \leqq \tag{32}$$

$$\leqq (e^{-np_N}+2Ne^{-n^{\frac{1}{4}}})e^{\lambda}+\frac{\lambda^2+\lambda}{\sqrt{n}}e^{\lambda}.$$

Combining (19), (24), (25), (31), we get (14) and (15).

LEMMA 1.3. *Suppose that*

$$p_i \sim \frac{c}{i^\alpha}, \quad i \to \infty, \tag{33}$$

and let $\varrho = \frac{n}{N^\alpha}$. *Assume further that*

$$\varrho \to \infty \quad \text{and} \quad \frac{\varrho^2}{N} \to 0. \tag{34}$$

Then we have

$$\lambda = \lambda(n, N) = \frac{N^{\alpha+1}}{\alpha c n} e^{-\frac{cn}{N^\alpha}} \left(1 + O\left(\frac{N^\alpha}{n}\right)\right). \tag{35}$$

PROOF. Since $e^{-\frac{cn}{x^\alpha}}$ is increasing it is easily seen that

$$\left| \sum_{i=1}^{N} e^{-\frac{cn}{i^\alpha}} - \int_0^N e^{-\frac{cn}{x^\alpha}} dx \right| \leqq e^{-\frac{cn}{N^\alpha}} \tag{36}$$

By substituting $\frac{cn}{x^\alpha} = u$ in the integral, we get

$$\int_0^N e^{-\frac{cn}{x^\alpha}} dx = \frac{(nc)^{\frac{1}{\alpha}}}{\alpha} \int_{\frac{cn}{N^\alpha}}^{\infty} u^{-\left(1+\frac{1}{\alpha}\right)} e^{-u} du. \tag{37}$$

Using the asymptotic expansions for incomplete Gamma function (see ERDÉLYI et al. [1], pp. 266, (21) and pp. 278 (1)),

$$\int_{\frac{cn}{N^\alpha}}^{\infty} u^{-\left(1+\frac{1}{\alpha}\right)} e^{-u} du = e^{-\frac{cn}{N^\alpha}} \left[\left(\frac{cn}{N^\alpha}\right)^{-\left(1+\frac{1}{\alpha}\right)} + O\left(\left(\frac{cn}{N^\alpha}\right)^{-\left(2+\frac{1}{\alpha}\right)} \right) \right] \tag{38}$$

proving our lemma.

2. §. Proofs of the theorems

PROOF OF THEOREM 1. Putting (1) in (35), we immediately obtain that

$$\lim_{n \to \infty} \lambda(n, N(n)) = e^{\frac{y}{\alpha}}, \tag{39}$$

hence an application of Lemma 1.2 gives our statement.

PROOF OF THEOREM 2. Now let $N(n, y)$ be defined by (1) and put $y = y_U^{(1)}(n)$, given by (2). Since the random variables Z_n are increasing ($n = 1, 2, \ldots$), in order to prove (4), it suffices to show that the series

$$\sum_k P\left(Z_{n_k+1} > N(n_k, y_U^{(1)}(n_k))\right) \tag{40}$$

converges for suitably chosen sequence n_k. Now let $n_k = e^{\sqrt{\frac{k}{\log k}}}$. Then $\frac{n_{k+1}}{n_k} = 1+\theta_k$ with $\theta_k \sim \frac{1}{2\sqrt{k \log k}}$.

An application of Lemma 1.3 gives that

$$\lambda = \frac{(\alpha c)^{\frac{1}{\alpha}} n_k^{\frac{1}{\alpha}}}{(\log n_k)^{\frac{\alpha+1}{\alpha}}}(1+o(1))\times \tag{41}$$

$$\times \exp\left\{-\frac{n_{k+1}\log n_k}{\alpha n_k}\left(1-\frac{(\alpha+1)\log\log n_k}{\log n_k}-\frac{y_U^{(1)}(n_k)}{\log n_k}\right)\right\} =$$

$$= (1+o(1))\exp\left(\frac{1}{\alpha}y_U^{(1)}(n_k)+\frac{1}{\alpha}\log(\alpha c)\right).$$

From (2) we obtain

$$\frac{1}{\alpha}y_U^{(1)}(n_k)+\frac{1}{\alpha}\log(\alpha c) = \tag{42}$$

$$= \log\log\log n_k + \log 2 + \log(1+\varepsilon) =$$

$$= \log(\log k - \log\log k) + \log(1+\varepsilon)$$

i.e.

$$\lambda = (1+o(1))(1+\varepsilon)(\log k - \log\log k). \tag{43}$$

By using Lemma 1.2, we have

$$P(Z_{n_{k+1}} > N(n_k, y_U^{(1)}(n_k))) = O\left(\frac{\log k}{k}\right)^{(1+\varepsilon)(1+o(1))}. \tag{44}$$

Hence (40) is convergent. This proves the first part of Theorem 2.

To prove the divergent part of our theorem we consider the variables $Z^*_{(n_k+1, n_{k+1})}$, where $\{n_k\}_{k=1}^{\infty}$ is a convenient sequence of indices. We define $Z^*_{(n_k+1, n_{k+1})}$ as the smallest natural number for which

$$a_l(\omega) \neq Z^*_{(n_k+1, n_{k+1})}(\omega) \quad \text{for} \quad n_k+1 \leqq l \leqq n_{k+1}. \tag{45}$$

Clearly $Z^*_{(n_k+1, n_{k+1})}$ has the same distribution as $Z_{n_{k+1}-n_k}$ and the variables $Z^*_{(n_k+1, n_{k+1})}$ for $k=0, 1, 2, \ldots$ are independent. Since $Z_{n_{k+1}} \geqq Z^*_{(n_k+1, n_{k+1})}$ we have to prove only that

$$\sum_{k=0}^{\infty} P(Z_{n_{k+1}-n_k} > N(n_{k+1}, y_U^{(2)}(n_{k+1}))) = \infty. \tag{46}$$

Now let $n_k = k^{\beta k}$ where $\beta > 1$, then

$$\frac{n_{k+1}}{n_k} = 1+\theta_k \quad \text{with} \quad 1+\theta_k \sim e^{\beta}(k+1)^{\beta}. \tag{47}$$

Using (1) with (3) and applying Lemma 1.3 we obtain, that

$$\lambda_k = \frac{\frac{1+\theta_k}{\theta_k}(\alpha c)^{\frac{1}{\alpha}}(n_{k+1})^{\frac{1}{\alpha}}}{(\log n_{k+1})^{1+\frac{1}{\alpha}}}(1+o(1)) * \tag{48}$$

$$* \exp\left\{-\frac{\theta_k}{1+\theta_k}\left(\frac{\log n_{k+1}}{\alpha} - \frac{3\log\log n_{k+1}}{\alpha} - \frac{y_U^{(2)}(n_{k+1})}{\alpha}\right)\right\}.$$

Consequently

$$\lambda_k = (1+o(1))(\alpha c)^{\frac{1}{\alpha}} * \tag{49}$$

$$* \exp\left\{\frac{y_U^{(2)}(n_{k+1})}{\alpha} + \frac{1}{1+\theta_k}\left(\frac{\log n_{k+1}}{\alpha} - \frac{\alpha+1}{\alpha}\log\log n_{k+1}\right)\right\}.$$

Using (47) we clearly have that

$$\lambda_k = (1+o(1))(\alpha c)^{\frac{1}{\alpha}} \exp\left\{\frac{y_U^{(2)}(n_{k+1})}{\alpha} + o(1)\right\} = \tag{50}$$

$$= (1+o(1))(\alpha c)^{\frac{1}{\alpha}} \exp\left\{\frac{y_U^{(2)}(n_{k+1})}{\alpha}\right\}.$$

Let us choose (see (3))

$$y_U^{(2)}(n_{k+1}) = \alpha\log\log\log n_{k+1} + \alpha\log(1-\varepsilon) - \log(\alpha c).$$

Then we have

$$\lambda_k = (1+o(1))\exp\{\log\log\log n_{k+1} + \log(1-\varepsilon)\} = \tag{51}$$

$$= (1+o(1))\exp\{\log((1-\varepsilon)\log\log n_{k+1})\} =$$

$$= (1+o(1))(1-\varepsilon)\log\log n_{k+1}.$$

As $n_{k+1} = (k+1)^{\beta(k+1)}$ we have

$$\lambda_{k-1} = (1+o(1))\log(\beta k\log k)^{1-\varepsilon}.$$

Thus

$$\sum_k e^{-\lambda_k} \sim \sum_k \frac{1}{(\beta k\log k)^{1-\varepsilon} e^{1+o(1)}} = \infty. \tag{52}$$

It can be seen further that the series corresponding to the remainders (15) is convergent, hence (46) follows from Lemma 1.2 and (52). From (46) and the Borel—Cantelli lemma we get our theorem.

PROOF OF THEOREM 3. To prove (8) it suffices to show that

$$\sum_k P(Z_{n_k} < N(n_{k+1}, y_L^{(1)}(n_{k+1}))) \tag{53}$$

converges for suitably chosen sequence n_k.

We again use Lemma 1.2, and the inequality

$$1-e^{-\lambda}<\lambda \quad \text{for} \quad \lambda>0.$$

Thus

$$P\left(Z_{n_k}<N(n_{k+1}, y_L^{(1)}(n_{k+1}))\right)=1-e^{-\lambda}+R<\lambda+R. \tag{54}$$

Choose $n_k=e^{\sqrt{k}}$. Then $\frac{n_{k+1}}{n_k}=1+\theta_k$ with $\theta_k\sim\frac{1}{2\sqrt{k}}$.

By applying Lemma 1.3 we obtain

$$\lambda=\lambda(k)=\frac{(\alpha c)^{\frac{1}{\alpha}} n_{k+1}^{\frac{1}{\alpha}}}{(\log n_{k+1})^{\frac{\alpha+1}{\alpha}}}(1+o(1))\times \tag{55}$$

$$\times\exp\left\{-\frac{n_k\log\frac{n_{k+1}}{\alpha n_{k+1}}}{}\left(1-\frac{(\alpha+1)\log\log n_{k+1}}{\log n_{k+1}}-\frac{y_L^{(1)}(n_{k+1})}{\log n_{k+1}}\right)\right\}=$$

$$=(1+o(1))\exp\left\{\frac{1}{\alpha}y_L^{(1)}(n_{k+1})+\frac{1}{\alpha}\log(\alpha c)+\frac{1}{2\alpha}+o(1)\right\}\leqq$$

$$\leqq C\exp\left\{\frac{1}{\alpha}y_L^{(1)}(n_{k+1})\right\}.$$

By choosing (see (6))

$$y_L^{(1)}(n_{k+1})=-2\alpha(1+\varepsilon)\log\log n_{k+1}$$

$$\frac{y_L^{(1)}(n_{k+1})}{\alpha}=-(1+\varepsilon)\log(k+1),$$

hence

$$\lambda(k)\leqq C\frac{1}{k^{1+\varepsilon}}$$

i.e. (53) converges.

To prove the second part of our Theorem 3 we use the Erdős—Rényi version of the Borel—Cantelli lemma ([3]), which states that $\sum_k P(A_k)=\infty$ and (58) implies that with probability 1 infinitely many of A_k's occur.

Let $\{n_k\}$ as in the second part of the proof of Theorem 2; $n_k=k^{\beta k}$, where $\beta>1$.

Define the following events:

$$A_k=\left\{Z_{n_k}<N(n_k, y(n_k))\right\}, \quad k=1, 2, \ldots \tag{56}$$

and

$$C_{j,k}=\left\{Z^*_{(n_j+1,\, n_k)}<N(n_k, y(n_k))\right\}_{k=j+1,\, j+2\ldots}^{j=1,2\ldots} \tag{57}$$

where $j<k$ and Z^* is the same as in the proof of the previous theorem.

To prove that

$$\lim_K \frac{\sum_{j,k=1}^{K} P(A_j A_k)}{\left(\sum_{j=1}^{K} P(A_j)\right)^2} = 1 \tag{58}$$

it suffices to show that given any $\varepsilon>0$, the inequality

$$P(A_j A_k) \leqq (1+\varepsilon) P(A_j) P(A_k) \tag{59}$$

holds true for $k \geqq k_0(\varepsilon)$ and $1 \leqq j < k$

Observe that if $j<k$, then

$$A_j A_k \subseteq A_j C_{j,k}.$$

Since A_j and $C_{j,k}$ are independent,

$$P(A_j A_k) \leqq P(A_j) P(C_{j,k}) \leqq P(A_j) P(C_{k-1,k}). \tag{60}$$

From the proof of the second part of Theorem 2 one can easily see that

$$P(C_{k-1,k}) = (1+o(1))(\alpha c)^{\frac{1}{\alpha}} e^{\frac{y_L^{(2)}(n_k)}{\alpha}}. \tag{61}$$

Moreover

$$P(A_k) = (1+o(1))(\alpha c)^{\frac{1}{\alpha}} e^{\frac{y_L^{(2)}(n_k)}{\alpha}} \tag{62}$$

From (61) and (62),

$$P(C_{k-1,k}) \leqq (1+\varepsilon) P(A_k) \quad \text{if} \quad k > k_0(\varepsilon). \tag{63}$$

If we choose (see (7))

$$y_L^{(2)}(n_k) = -\alpha \log\log n_k = -\alpha \log \beta - \alpha \log k - \alpha \log\log k \tag{64}$$

then (using $1-e^{-x}>x-x^2$ if x small) we get

$$\sum_k P(A_k) = C_0 \sum_k \frac{1}{k \log k} = \infty, \tag{65}$$

which proves Theorem 3.

REMARK. Define the random variables $Y_{i,n}$ by

$$Y_{i,n} = \begin{cases} 1 & \text{if} \quad a_k \neq i \quad \text{for} \quad k = 1, 2, \ldots, n, \\ 0 & \text{otherwise} \end{cases} \tag{66}$$

and put

$$\varrho(n, N) = \sum_{i=1}^{N} Y_{i,n}. \tag{67}$$

Then by using Sevast'yanov's theorem it can be seen that

$$\lim_{n\to\infty} P\big(\varrho(n, N) = k\big) = \frac{(e^{\frac{y}{\alpha}})^k}{k!} e^{-e^{\frac{y}{\alpha}}}, \quad k = 1, 2, \ldots \tag{68}$$

where $N=N(n, y)$ is defined by (1).

If $Z_n(k)$ denotes the k-th empty cell after n trials, then as

$$P\big(Z_n(k) > N\big) = P\big(\varrho(n, N) \leqq k\big), \tag{69}$$

from (68) we can obtain the following generalization of Theorem 1:

Under the conditions of Theorem 1, *we have for* $k=1, 2 \ldots$

$$\lim_{n\to\infty} P\big(Z_n(k) > N(n, y)\big) = \sum_{j=0}^{k} \frac{(e^{\frac{y}{\alpha}})^j}{j!} e^{-e^{\frac{y}{\alpha}}}. \tag{70}$$

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ÜBER DIE DURCHLÄSSIGKEIT EINER AUS SCHEIBEN KONSTANTER BREITE BESTEHENDEN SCHICHT

von

I. HORTOBÁGYI

Unter einer Scheibenschicht der Breite D verstehen wir eine Menge von disjunkten, offenen, konvexen Scheiben, die in einem Parallelstreifen der Breite D liegen. Wir betrachten eine Kurve, die die Schicht kreuzt, d. h. eine stetige, rektifizierbare Kurve, die die Randgeraden des Parallelstreifens so miteinander verbindet, daß sie keine Scheibe trifft. Die Durchlässigkeit der Schicht wird durch $p=\frac{D}{\inf L}$ definiert, wo L die Länge einer die Schicht kreuzenden Kurve bezeichnet.

In den Arbeiten [1], [2], [3] wurde die Durchlässigkeit aus Kreisen bzw. aus Parallelogrammen bestehender Schichten untersucht.

L. FEJES TÓTH [1] hat gezeigt, daß eine aus kongruenten Kreisen vom Durchmesser d bestehende Schicht der Breite D sich stets durch eine Kurve der Länge

$$L \leqq \frac{2\pi}{\sqrt{27}}(D-d)+d$$

kreuzen läßt. Daraus folgt, daß die Durchlässigkeit einer aus kongruenten Kreisen bestehenden Schicht stets größer ist als $\frac{\sqrt{27}}{2\pi}=0{,}82\,699\ldots$.

Wir beweisen hier den folgenden allgemeineren

SATZ. *Es sei eine Scheibenschicht der Breite D vorgegeben, die aus verschobenen Exemplaren einer Scheibe konstanter Breite besteht. Ist d die Breite einer Scheibe, so läßt sich die Schicht stets durch eine Kurve der Länge*

$$L \leqq \frac{2\pi}{\sqrt{27}}(D-d)+d$$

kreuzen.

Der Beweis beruht auf zwei Hilfssätzen.

HILFSSATZ 1. *Es seien K eine Kurve konstanter Breite d und A_1A_2 und B_1B_2 zwei beliebige Durchmesser von K. Schließen A_1A_2 und B_1B_2 einen Winkel α ein, so ist die Summe der Längen der Bogen $\overset{\frown}{A_1B_1}$ und $\overset{\frown}{A_2B_2}$ von K gleich $d\alpha$* (Fig. 1).

Der Beweis erfolgt mit bekannten Methoden (s. etwa [4]), die auf Annäherung von K durch Kreisbogenpolygone beruhen.

HILFSSATZ 2. *Es seien K eine beliebige Scheibe konstanter Breite d, P_1 und P_2 die gemeinsamen Punkte von K mit ihren vertikalen Stützgeraden und h eine die Strecke P_1P_2 senkrecht schneidende Gerade. Die verschobenen Exemplare K_1 und K_2 von K sollen K und h der Figur 2 entsprechend in den Punkten Q_1, Q_2 und R_1, R_2*

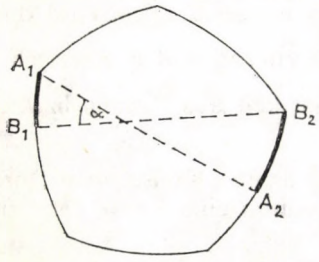

Fig. 1.

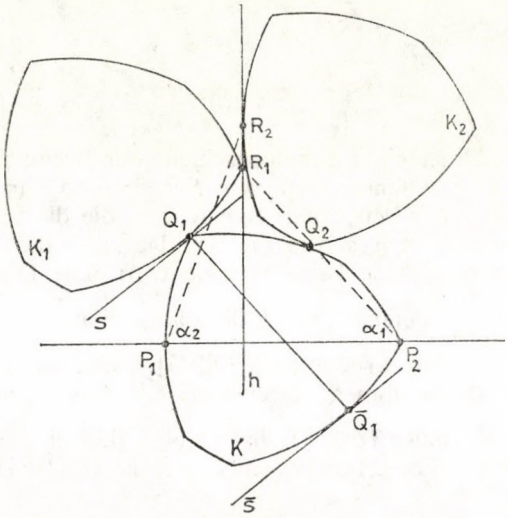

Fig. 2.

berühren. Bezeichnen λ_i $(i=1,2)$ die Länge der aus Randpunkten von K_i und K bestehenden Kurve $R_iQ_iP_i$ und δ_i die Niveaudifferenz zwischen R_i und P_i, so ist

$$\min\left(\frac{\lambda_1}{\delta_1}, \frac{\lambda_2}{\delta_2}\right) \leqq \frac{2\pi}{\sqrt{27}}.$$

Gleichheit gilt dann und nur dann, wenn h die Mittelsenkrechte der Strecke P_1P_2 ist.

BEWEIS. Wir bezeichnen mit s die durch Q_1 hindurchgehende gemeinsame Stützgerade von K und K_1, mit $\bar{s}$ die zu s parallele andere Stützgerade von K, weiterhin mit $\bar{Q}_1$ den gemeinsamen Punkt von K und $\bar{s}$ (Fig. 2). Man sieht leicht ein, daß die Länge des Verschiebungsvektors, der K in K_1 überführt, d ist. Diese Verschiebung führt die Punkte $\bar{Q}_1$ und P_2 in die Punkte Q_1 und R_1 über. So haben wir $P_2R_1=d$, $P_2R_1 \| \bar{Q}_1Q_1$, $\overset{\frown}{\bar{Q}_1P_2}=\overset{\frown}{Q_1R_1}$ und folglich $\lambda_1=\overset{\frown}{R_1Q_1}+\overset{\frown}{Q_1P_1}=\overset{\frown}{\bar{Q}_1P_2}+$ $+\overset{\frown}{Q_1P_1}$. Bezeichnen wir den Winkel $P_1P_2R_1$ mit α_1, so ist $\delta_1=d\sin\alpha_1$. Auf Grund des Hilfssatzes 1 ist $\overset{\frown}{\bar{Q}_1P_2}+\overset{\frown}{Q_1P_1}=d\alpha_1$ und folglich

$$\frac{\lambda_1}{\delta_1}=\frac{\alpha_1}{\sin\alpha_1}.$$

Auf ähnliche Weise gilt $\frac{\lambda_2}{\delta_2}=\frac{\alpha_2}{\sin\alpha_2}$, wo α_2 den Winkel $P_2P_1R_2$ bezeichnet. Da $\frac{\alpha}{\sin\alpha}$ im Intervall $\left[0,\frac{\pi}{2}\right]$ eine monoton zunehmende Funktion von α und $\min(\alpha_1,\alpha_2)\leqq\frac{\pi}{3}$ sind, haben wir

$$\min\left(\frac{\lambda_1}{\delta_1},\frac{\lambda_2}{\delta_2}\right)=\min\left(\frac{\alpha_1}{\sin\alpha_1},\frac{\alpha_2}{\sin\alpha_2}\right)\leqq\frac{2\pi}{\sqrt{27}}.$$

Gleichheit gilt dann und nur dann, wenn $\min(\alpha_1,\alpha_2)=\frac{\pi}{3}$ ist, das heißt, wenn h die Mittelsenkrechte der Strecke P_1P_2 ist.

Jetzt wenden wir uns dem Beweis des Satzes zu.

Es sei Σ die im Satz betrachtete Schicht. Wir konstruieren eine Kurve, die die Schicht Σ kreuzt. Wir setzen voraus, daß die Randgeraden von Σ horizontal sind. Wir sagen, daß ein Punkt irgendeiner Scheibe von Σ ein Extrempunkt ist, wenn die Scheibe in diesem Punkt eine vertikale Stützgerade hat.

Wir gehen von einem Punkt U der oberen Randgerade von Σ aus, und fahren vertikal abwärts bis wir in einem Punkt B_1 an eine Scheibe K_1 stoßen. Wir können voraussetzen, daß der Punkt U so gewählt wurde, daß B_1 ein Extrempunkt von K_1 ist. Durch den Punkt B_1 gehen wir weiter vertikal abwärts bis wir in A_2 an die folgende Scheibe K_2 stoßen. Dann gehen wir entlang der Scheibe K_2 zum nächstgelegenen Extrempunkt B_2 von K_2. Im Punkt B_2 verlassen wir die Scheibe K_2 und gehen weiter vertikal abwärts bis wir in A_3 an die folgende Scheibe K_3 stoßen und so weiter. Es ist nicht schwer einzusehen, daß wir nach endlich vielen Schritten in einen Punkt V der unteren Randgerade von Σ gelangen. Die Kurve $U\ldots B_{i-1}A_iB_i\ldots V$ kreuzt die Schicht Σ (Fig. 3).

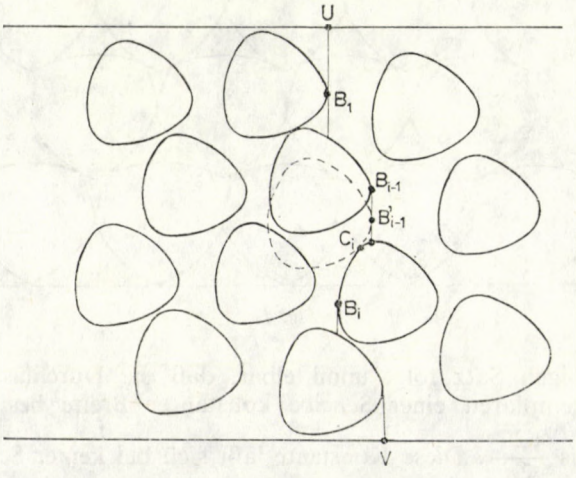

Fig. 3.

Wir können die Länge der Kurve $U\ldots B_{i-1}A_iB_i\ldots V$ abkürzen, wenn wir die Ecken bei den Punkten abrunden. Deswegen verschieben wir die Scheibe K_{i-1} vertikal abwärts bis sie in einem Punkt C_i an die Scheibe K_i stößt. Diese Verschiebung führt den Punkt B_{i-1} in den Punkt B'_{i-1} über.

Jetzt ersetzen wir den Weg $B_{i-1}A_iB_i$ durch den Weg $B_{i-1}B'_{i-1}C_iB_i$ (Fig. 3). So haben wir eine kürzere Kurve $U\ldots B_{i-1}B'_{i-1}C_iB_i\ldots V$ der Länge L, die die Schicht Σ kreuzt.

Wir bezeichnen die Länge der Kurve $B'_{i-1}C_iB_i$ mit λ_i und die Niveaudifferenz zwischen den Punkten B'_{i-1}, B_i mit δ_i. Auf Grund des Hilfssatzes 2 ist $\lambda_i \leqq \frac{2\pi}{\sqrt{27}}\delta_i$.

Summieren wir diese Ungleichungen und addieren zu beiden Seiten die Gesamtlänge s der eventuell auftretenden vertikalen Strecken, so ergibt sich die Ungleichung

$$L \leqq \frac{2\pi}{\sqrt{27}}(D-s)+s.$$

Der erste und der letzte Teil der Kurve $U\ldots B_{i-1}B'_{i-1}C_iB_i\ldots V$ sind wegen der Wahl des Punktes U Strecken, deren Gesamtlänge mindestens gleich der Breite d der Scheibe ist. Deshalb ist $s \geqq d$. Damit haben wir bewiesen, daß sich die Schicht durch eine Kurve der Länge $L \leqq \frac{2\pi}{\sqrt{27}}(D-d)+d$ kreuzen läßt.

Die Abschätzung ist genau, wenn entweder $D=d$ ist oder die Scheiben so in horizontalen Reihen angeordnet sind, daß jede Scheibe zwei Scheiben ihrer eigenen Reihe und je zwei Scheiben jeder Nachbarreihe berührt (Fig. 4). Man sieht leicht ein, daß in diesem Fall die konstruierte Kurve unter sämtlichen die Schicht kreuzenden Kurven die kürzeste ist.

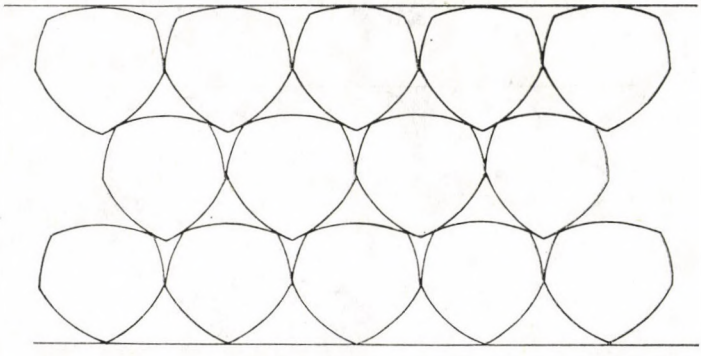

Fig. 4.

Aus dem obigen Satz folgt unmittelbar, daß die Durchlässigkeit einer aus verschobenen Exemplaren einer Scheibe konstanter Breite bestehenden Schicht stets größer ist als $\frac{\sqrt{27}}{2\pi}$. Diese Konstante läßt sich bei keiner Scheibe durch eine größere ersetzen.

Zum Schluß konstruieren wir eine Schicht aus kongruenten Reuleaux-Dreiecken mit einer kleineren Durchlässigkeit als $\frac{\sqrt{27}}{2\pi}$.

Jede Scheibe, d. h. jedes Reuleaux-Dreieck soll eine vertikale Symmetrieaxe haben. Die Schicht ist aus horizontalen Reihen aufgebaut, und zwar so, daß jede Scheibe zwei Scheiben ihrer eigenen Reihe und je zwei Scheiben jeder benachbarten Reihe berührt. Die benachbarten Reihen entstehen aus einander durch eine Gleitspiegelung.

Wir betrachten die in Fig. 5 dargestellten Berührungspunkte P, Q und R einer Scheibe mit den benachbarten Scheiben. Sind λ die Bogenlängensumme von $\widehat{PQ}$

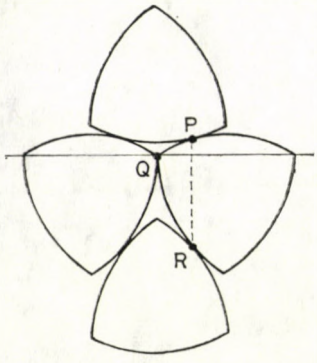

Fig. 5.

und $\widehat{QR}$ und δ die Niveaudifferenz zwischen P und R, so gilt für die Durchlässigkeit p der Schicht

$$p > \frac{\delta}{\lambda} = \frac{\sqrt{15}+\sqrt{7}-2\sqrt{3}}{\frac{8\pi}{3}-4\left(\arcsin\frac{1}{4}+\arcsin\frac{3}{4}\right)} = 0{,}768\,55\ldots.$$

Besteht die Schicht aus genügend vielen Reihen, so kommt p dieser Schranke beliebig nahe.

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SUBBASE AND DIMENSION, I

by

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This paper deals with some (partly known, partly new) inequalities between topological dimension functions. As the theorems in question (see § 2.) can be proved in a very similar way, we have searched for their common background. After introducing two new dimension functions, we shall prove a theorem yielding these inequalities as corollaries (see § 3.).

§ 1.
Notations and terminology

A *space* is a topological space. A space X is an S_i-space ($i \geqq 1$) iff the T_0-reflexion of X is a T_i-space. (For a systematic treatment of axioms S_i, see the book [1] of Á. CSÁSZÁR.)

The large inductive dimension Ind means here the one defined by "Ind $X \leqq n$ iff for each $S \subset G \subset X$, G open and S closed, there is an open set V with $S \subset V \subset G$ and Ind Fr $V \leqq n-1$" (i.e. $\overline{V} \subset G$ is not required). The small inductive dimension ind is similarly defined with "$|S|=1$" substituted for "S closed".

MANCUSO [8] introduced a third inductive dimension indc defined as Ind with "S compact" substituted for "S closed".

Two other dimension functions we are interested in are the directional dimension Dim of E. DEÁK [2, 3] and the half-directional dimension DIM introduced by the author [5]. For the convenience of the reader, both will be defined here.

A family $\mathscr{H}$ of open subsets of a space X is a *halfdirection* if (i) $H_1, H_2 \in \mathscr{H}$, $H_1 \neq H_2 \Rightarrow \overline{H}_1 \subset H_2$ or $\overline{H}_2 \subset H_1$ and (ii) $\mathscr{H}_0 \subset \mathscr{H} \Rightarrow \bigcup \mathscr{H}_0 \in \mathscr{H}$. A *halfdirectional structure* is a non-void system of halfdirections on a space. For a space X, DIM X is defined by DIM $X+1$ = the minimum of the cardinalities of halfdirectional structures $\mathfrak{H}$ with $\bigcup \mathfrak{H}$ a subbase of X.

Let $\mathscr{R}$ be a family of ordered pairs of subsets of a space X. Set $\mathscr{G}(\mathscr{R}) = \{G: \exists F, (G, F) \in \mathscr{R}\}$; $\mathscr{H}(\mathscr{R}) = \{H: \exists G, (G, X-H) \in \mathscr{R}\}$. $\mathscr{R}$ is a *direction* if (i) $(G, F) \in \mathscr{R} \Rightarrow G \subset F$, (ii) $(G, F), (G', F') \in \mathscr{R}$, $(G, F) \neq (G', F') \Rightarrow F \subset G'$ or $F' \subset G$ and (iii) $\mathscr{G}(\mathscr{R})$ and $\mathscr{H}(\mathscr{R})$ are halfdirections. A *directional structure* is a system of directions on X. Dim X is the minimum of the cardinalities of directional structures $\mathfrak{R}$ with $\bigcup \{\mathscr{G}(\mathscr{R}) \cup \mathscr{H}(\mathscr{R}): \mathscr{R} \in \mathfrak{R}\}$ a subbase of X.

If $\mathscr{S}$ is a set and n is a natural number, put

$$[\mathscr{S}]^n = \{\mathscr{A}: \mathscr{A} \subset \mathscr{S}, |\mathscr{A}| = n\}$$

and

$$\mathrm{B}\mathscr{S} = \{\bigcap \mathscr{A}: \mathscr{A} \subset \mathscr{S}, |\mathscr{A}| < \omega\}.$$

Let d be a dimension function. A system $\mathscr{S}$ of subsets of a space X is called d-*nice* if $\mathrm{d}X$ is finite and $\mathscr{A}\in[\mathscr{S}]^k$ implies $\mathrm{d}\bigcap\{\mathrm{Fr}\,A\colon A\in\mathscr{A}\}\leqq\mathrm{d}X-k$ for $1\leqq k\leqq\mathrm{d}X+1$ (see E. DEÁK [4]).

Let d_1 and d_2 be dimension functions. "$\mathrm{d}_1X\leqq\mathrm{d}_2X$" will always mean: "if d_2X is finite, then $\mathrm{d}_1X\leqq\mathrm{d}_2X$".

REMARKS. a) When defining Dim and DIM, we have begun with the more recent notion of halfdirection in order to keep the definitions as short as possible. The meaning of "direction" and "directional structure" is here not exactly the same as in [2] — our definition of Dim is, however, equivalent to E. DEÁK's original one; cf. Remark (0.5) in [5].

b) Our terminology concerning nice systems is somewhat different from that of [4]. There is, however, no discrepancy in the special case of *nice open bases*.

§ 2.

E. DEÁK [3, 4] has proved the following theorems:

(2.1) THEOREM. *For an arbitrary space* X,

$$\operatorname{ind}X\leqq\operatorname{Dim}X.$$

(2.2) THEOREM. *For a totally paracompact* space* X,

$$\operatorname{Ind}X\leqq\operatorname{Dim}X.$$

(2.3) THEOREM. *If a totally paracompact* T_2*-space* X *has an* ind*-nice base, then*

$$\operatorname{ind}X=\operatorname{Ind}X.$$

In the next paragraph we prove a theorem generalizing these theorems and yielding inequalities between DIM and the inductive dimensions as well.

§ 3.

Now we introduce two (as far as we know new) dimension functions well-fitted for our purpose stated above.

(3.1) DEFINITION. For a space X, $\mathrm{sbd}X\leqq n$ iff X has a subbase $\mathscr{S}$ with $\mathscr{A}\in[\mathscr{S}]^{n+1}$ implying $\overline{\bigcap\mathscr{A}}\subset\bigcup\mathscr{A}$ $(n=-1,0,1,\ldots)$; $\mathrm{sbd}X=n$ iff $\mathrm{sbd}\,X\leqq n$ but $\mathrm{sbd}\,X\leqq n-1$ is not true.

Note that $\mathrm{sbd}\,X=-1$ iff $X=\varnothing$. (Since $\mathscr{A}\in[\mathscr{S}]^0$ means $\mathscr{A}=\varnothing$, and $\overline{\bigcap\mathscr{A}}\subset\bigcup\mathscr{A}$ holds iff the fundamental set is empty.)

(3.2) DEFINITION. For a space X, $\mathrm{ind}^+X=-1$ iff $X=\varnothing$; $\mathrm{ind}^+X\leqq n$ iff X has a base $\mathscr{B}$ with

$$\mathrm{ind}^+\,\mathrm{Fr}\bigcup\mathscr{V}\leqq n-1$$

* A space is totally paracompact if for an arbitrary open base $\mathscr{B}$ and an arbitrary open covering $\mathscr{U}$ there is a locally finite open covering $\mathscr{V}$ refining $\mathscr{U}$ with $\mathscr{V}\subset\mathscr{B}$ (cf. FORD [6]).

for an arbitrary locally finite subsystem $\mathscr{V}$ of $\mathscr{B}$ $(n=0, 1, 2, \ldots)$; $\operatorname{ind}^+ X=n$ iff $\operatorname{ind}^+ X\leqq n$ but $\operatorname{ind}^+ X\leqq n-1$ is not true.

The theorem below is the main result of this paper.

(3.3) THEOREM. *For an arbitrary space X,*

$$\operatorname{ind}^+ X \leqq \operatorname{sbd} X. \tag{1}$$

The idea of the proof is the same as that of E. Deák's proof of Theorems (2.1) and (2.2).

PROOF* To prove (1), one has to show that

$$\operatorname{sbd} X \leqq n \Rightarrow \operatorname{ind}^+ X \leqq n \quad (n=-1, 0, 1, \ldots). \tag{2}$$

Let n be a fixed integer, $n\geqq -1$; suppose $\operatorname{sbd} X\leqq n$ and let $\mathscr{S}$ be an open subbase of X with

$$\mathscr{A}\in[\mathscr{S}]^{n+1} \Rightarrow \overline{\bigcap \mathscr{A}} \subset \bigcap \mathscr{A} \tag{3}$$

(cf. Definition (3.1)).

We consider the propositions $(*k)$ for $0\leqq k\leqq n+1$:

$(*k)$ *if $\mathscr{F}$ is a locally finite collection of closed sets in X and if for each $F\in\mathscr{F}$ there exists a system $\mathscr{A}=\mathscr{A}(F)\in[\mathscr{S}]^k$ with $F\subset\overline{\bigcap\mathscr{A}}-\bigcup\mathscr{A}$, then* $\operatorname{ind}^+\bigcup\mathscr{F}\leqq n-k$.

To prove the theorem, it is enough to show that

$$(*k+1)\Rightarrow(*k) \quad (0\leqq k\leqq n). \tag{4}$$

Indeed, $(*n+1)$ is a trivial consequence of (3), and, on the other hand, $(*\ 0)$ implies $\operatorname{ind}^+ X\leqq n$ (take the collection $\mathscr{F}=\{X\}$ and set $\mathscr{A}(X)=\varnothing$), thus (2) follows from (4).

To prove (4), suppose $(*k+1)$ and let $\mathscr{F}$ be a collection satisfying the premise of the implication in $(*k)$. Now (4) means

$$\operatorname{ind}^+\bigcup\mathscr{F} \leqq n-k. \tag{5}$$

Further, (5) is proved if

$$\operatorname{ind}^+ \operatorname{Fr}_{\cup\mathscr{F}}\bigcup\mathscr{V} \leqq n-k-1 \tag{6}$$

is valid for an arbitrary locally finite subcollection $\mathscr{V}$ of

$$\mathscr{B}=\mathbf{B}\mathscr{S}|\bigcup\mathscr{F} \tag{7}$$

(cf. Definition (3.2); "locally finite" is to be understood in $\cup\mathscr{F}$). Only (6) has remained to be proved.

For each $V\in\mathscr{V}$ there is a *finite* system $\mathscr{W}_V\subset\mathscr{S}$ with

$$V=\bigcap\mathscr{W}_V\cap\bigcup\mathscr{F}, \tag{8}$$

* Observe that the definition of ind^+ (except for $\operatorname{ind}^+\varnothing=-1$) will not be used before (6), thus the beginning of this proof could be told for an arbitrary modification of ind^+. — This fact will be made use of in the second part of this series, to be published in the next volume of this journal. (*Added in proof.*)

since $V \in \mathscr{B}$ (cf. (7)). We may suppose without loss of generality that $\mathscr{W}_V \neq \emptyset$ (otherwise we had $V = \bigcup \mathscr{F}$, thus $\bigcup \mathscr{V} = \bigcup \mathscr{F}$, and (6) were trivially true). Now we define a collection $\mathscr{K}$ that $(*k+1)$ can be applied to:

$$\mathscr{K} = \{K(F,V,W) = \mathrm{Fr}_F(W \cap F) \cap \mathrm{Fr}_F(V \cap F) \cap \mathrm{Fr}_{\bigcup \mathscr{F}} \bigcup \mathscr{V} : \tag{9}$$

$$F \in \mathscr{F}, V \in \mathscr{V}, \quad W \in \mathscr{W}_V, W \cap F \neq \emptyset\}.$$

The proof of the theorem is complete if we show

$$\mathrm{Fr}_{\bigcup \mathscr{F}} \bigcup \mathscr{V} = \bigcap \mathscr{K}, \tag{a}$$

(b) *$\mathscr{K}$ is a locally finite collection of closed sets* and

(c) *for each $K = K(F,V,W) \in \mathscr{K}$ there exists a system $\mathscr{E} = \mathscr{E}(F,V,W) \in [\mathscr{S}]^{k+1}$ with* $K \subset \overline{\bigcap \mathscr{E} - \bigcup \mathscr{E}}$.

Indeed: (b), (c), $(*k+1)$ and (a) imply (6). The statements (a), (b) and (c) will be proved in the points a), b) and c) below.

a) As each $K(F,V,W)$ is a subset of $\mathrm{Fr}_{\bigcup \mathscr{F}} \bigcup \mathscr{V}$,

$$\mathrm{Fr}_{\bigcup \mathscr{F}} \bigcup \mathscr{V} \supset \bigcup \mathscr{K}. \tag{a1}$$

On the other hand, let x be an arbitrary point with

$$x \in \mathrm{Fr}_{\bigcup \mathscr{F}} \bigcup \mathscr{V}. \tag{a2}$$

Set

$$\mathscr{C} = \{F : x \notin F \in \mathscr{F}\}, \quad G_1 = X - \bigcup \mathscr{C}. \tag{a3}$$

G_1 is an open set in X, since $\mathscr{C}$ is a locally finite collection of closed sets. Clearly, $x \in G_1$. Let G_2 be an open neighbourhood of x in X intersecting only a finite number of elements of the (locally finite) collection $\mathscr{V}$. $G_1 \cap G_2$ is an open neighbourhood of x in X. Now let G be an arbitrary open neighbourhood of x in X with

$$G \subset G_1 \cap G_2. \tag{a4}$$

In consequence of (a2), there is a point y_G with

$$y_G \in G \cap \bigcup \mathscr{F} \cap \bigcup \mathscr{V},$$

so there are sets $F = F(G) \in \mathscr{F}$ and $V = V(G) \in \mathscr{V}$ with

$$y_G \in G \cap F \cap V. \tag{a5}$$

From (a2), (a3) and (a4) we have

$$x \in G \cap F, \quad x \notin V \tag{a6}$$

(recall that the elements of $\mathscr{V}$ are open in $\bigcup \mathscr{F}$). The sets F and V we have defined depend on G but the subsets of $\mathscr{F}$ and $\mathscr{V}$ we can choose from are finite, so there is a neighbourhood base $\mathscr{G}$ of x in X, such that (a5) and (a6) are true for each $G \in \mathscr{G}$ with F and V independent of G. Thus (a5) and (a6) mean that

$$x \in \mathrm{Fr}_F(V \cap F). \tag{a7}$$

Because of (8) and (a6), there is a set $W \in \mathscr{W}_V$ with

(a8) $$x \in G \cap F, \quad x \notin W \quad (G \in \mathscr{G}).$$

Further on, (a5) implies

(a9) $$y_G \in G \cap F \cap W \quad (G \in \mathscr{G}).$$

From (a8) and (a9) we have

(a10) $$x \in \mathrm{Fr}_F (W \cap F).$$

Now $x \in \bigcup \mathscr{K}$ follows from (9), (a10), (a7) and (a2). Thus we have proved

(a11) $$\mathrm{Fr}_{\bigcup \mathscr{F}} \bigcup \mathscr{V} \subset \bigcup \mathscr{K}.$$

(a1) and (a11) give us (a).

b) $\mathscr{F}$ is locally finite in X; $\mathscr{V}$ is locally finite in $\bigcup \mathscr{F}$ and so is it in X, since $\bigcup \mathscr{F}$ is closed. Thus the system

$$\{\overline{V}\colon V \in \mathscr{V}\}$$

is locally finite as well and so is the system

$$\{\mathrm{Fr}_F (V \cap F)\colon V \in \mathscr{V},\ F \in \mathscr{F}\},$$

because

$$\mathrm{Fr}_F (V \cap F) \subset \overline{V} \cap F.$$

For V fixed, $\mathscr{W}_V$ is finite, thus $\mathscr{K}$ is locally finite (see (9)). The elements of $\mathscr{K}$ are evidently closed in X.

c) Let $K = K(F, V, W) \in \mathscr{K}$ be fixed. $\mathscr{F}$ satisfies the premise of the implication in $(*k)$, i.e. there is a system $\mathscr{A} = \mathscr{A}(F) \in [\mathscr{S}]^k$ with

(c1) $$F \subset \overline{\bigcap \mathscr{A}} - \bigcup \mathscr{A}.$$

The condition $F \cap W \neq \varnothing$ in (9) implies $W \notin \mathscr{A}$, so

(c2) $$\mathscr{E} = \mathscr{A} \cup \{W\} \in [\mathscr{S}]^{k+1}.$$

The set W is open, so

(c3) $$K \subset \mathrm{Fr}_F (W \cap F) \subset F \cap (\overline{W} - W)$$

(cf. (9)). From (c2), (c3) and (c1) we have

(c4) $$K \cap \bigcup \mathscr{E} = (K \cap \bigcup \mathscr{A}) \cup (K \cap W) = \varnothing.$$

According to (c3), for an arbitrary point $x \in K$ and for an arbitrary open neighbourhood G of x in X, there is a point $y \in W \cap F \cap G$. W is open, so there is an open neighbourhood G_1 of y in F with $G_1 \subset W \cap G$. Because of (c1) and $y \in F$, G_1 intersects $\bigcap \mathscr{A}$. Taking a point z from $G_1 \cap \bigcap \mathscr{A}$, we have

$$z \in G \cap W \cap \bigcup \mathscr{A} = G \cap \bigcap \mathscr{E}.$$

As G was an arbitrary open neighbourhood of $x \in K$,

(c5) $$K \subset \overline{\bigcap \mathscr{E}}.$$

(c5) and (c4) mean that

(c6) $$K \subset \overline{\bigcap \mathscr{E}} - \bigcup \mathscr{E},$$

thus (c) is proved with (c2) and (c6), and so is the whole theorem.

§ 4.

Now we are going to show that the theorems mentioned in § 2 are corollaries to Theorem (3.3).

(4.1) PROPOSITION. *For an arbitrary space X,*

$$\operatorname{indc} X \leqq \operatorname{ind}^+ X.$$

PROOF. A trivial induction.

(4.2) PROPOSITION. *For a totally paracompact space X,*

$$\operatorname{Ind} X \leqq \operatorname{ind}^+ X.$$

PROOF. The inequality is evidently true if $\operatorname{ind}^+ X = -1$. Let now $\operatorname{ind}^+ X = n \geqq 0$ and suppose that $\operatorname{Ind} Y \leqq \operatorname{ind}^+ Y$ holds for totally paracompact spaces with $\operatorname{ind}^+ Y \leqq n-1$. Let $\mathscr{B}$ be a base of X as described in Definition (3.2). Let F be a closed and G on open subset of X with $F \subset G$. Since X is totally paracompact, there is a locally finite open covering $\mathscr{V} \subset \mathscr{B}$ of X refining the covering $\{G, X-F\}$.

$$\mathscr{V}_0 = \{V: V \in \mathscr{V},\ V \cap F \neq \varnothing\}$$

is a locally finite system, so

$$\operatorname{ind}^+ \operatorname{Fr} \bigcup \mathscr{V}_0 \leqq n-1.$$

Here $\operatorname{Fr} \bigcup \mathscr{V}_0$ is a closed subset of a totally paracompact space, so it is totally paracompact as well, thus

$$\operatorname{Ind} \operatorname{Fr} \bigcup \mathscr{V}_0 \leqq n-1$$

by the induction assumption. Further $F \subset \bigcup \mathscr{V}_0 \subset G$, so $\operatorname{Ind} X \leqq n$, for the sets F and G were arbitrary.

(4.3) PROPOSITION. *For an arbitrary space X,*

$$\operatorname{sbd} X \leqq \operatorname{Dim} X.$$

PROOF. The inequality is trivially true if $\operatorname{Dim} X = 0$ (i.e. if X is indiscrete). Suppose $\operatorname{Dim} X = n \geqq 1$. Let $\mathfrak{R}$ be a directional structure on X with $|\mathfrak{R}| = n$ and

$$\mathscr{S} = \bigcup \{\mathscr{G}(\mathscr{R}) \cup \mathscr{H}(\mathscr{R}): \mathscr{R} \in \mathfrak{R}\}$$

a subbase of X. If $\mathscr{A}\in[\mathscr{S}]^{n+1}$, there is a direction $\mathscr{R}\in\mathfrak{R}$ with

$$\left|\mathscr{A}\cap(\mathscr{G}(\mathscr{R})\cup\mathscr{H}(\mathscr{R}))\right|\geqq 2.$$

Let A and B be two elements of this intersection. If A and B are both elements of $\mathscr{G}(\mathscr{R})$ or $\mathscr{H}(\mathscr{R})$, say of $\mathscr{G}(\mathscr{R})$, we have $\bar{A}\subset B$ or $\bar{B}\subset A$, say $\bar{A}\subset B$, so

$$\overline{\bigcap\mathscr{A}}\subset\bar{A}\subset B\subset\bigcup\mathscr{A}.$$

On the other hand, if $A\in\mathscr{G}(\mathscr{R})$ and $B\in\mathscr{H}(\mathscr{R})$, there are sets F and G with $(A,F)\in\mathscr{R}$ and $(G,X-B)\in\mathscr{R}$. Now, according to (ii) in the definition of the direction, either $(A,F)=(G,X-B)$, or $F\subset G$, or $X-B\subset A$. In the first and the second case $A\cap B=\varnothing$, so

$$\overline{\bigcap\mathscr{A}}=\varnothing\subset\bigcup\mathscr{A};$$

in the third case $A\cup B=X$, so

$$\overline{\bigcap\mathscr{A}}\subset X=\bigcup\mathscr{A}.$$

Thus the relation $\overline{\bigcap\mathscr{A}}\subset\bigcup\mathscr{A}$ has been proved in all cases and $\operatorname{sbd}X\leqq n$.

(4.4) PROPOSITION. *For an S_1-space X,*

$$\operatorname{sbd}X\leqq\operatorname{Dim}X.$$

PROOF. Suppose $\operatorname{Dim}X=n$ $(n=0,1,2,\ldots)$ and let $\mathfrak{H}$ be a halfdirectional structure with $|\mathfrak{H}|=n+1$ and $\mathscr{S}=\bigcup\mathfrak{H}$ a subbase of X. If $\mathscr{A}\in[\mathscr{S}]^{n+1}$, then $|\mathscr{A}|=|\mathfrak{H}|$, so either

(i) there is a halfdirection $\mathscr{H}_0\in\mathfrak{H}$ with $|\mathscr{H}_0\cap\mathscr{A}|\geqq 2$, or

(ii) $|\mathscr{H}\cap\mathscr{A}|=1$ for each $\mathscr{H}\in\mathfrak{H}$.

In the first case let A and B be two different elements of $\mathscr{H}_0\cap\mathscr{A}$. Now $\bar{A}\subset B$ or $\bar{B}\subset A$, say $\bar{A}\subset B$, so $\overline{\bigcap\mathscr{A}}\subset\bar{A}\subset B\subset\bigcup\mathscr{A}$. In the second case set $\{A_{\mathscr{H}}\}=\mathscr{H}\cap\mathscr{A}$ $(\mathscr{H}\in\mathfrak{H})$. Suppose $\overline{\bigcap\mathscr{A}}\not\subset\bigcup\mathscr{A}$ and let $x\in\overline{\bigcap\mathscr{A}}-\bigcup\mathscr{A}$. Since $\overline{\bigcap\mathscr{A}}\neq\varnothing$, there is a point $y\in\bigcap\mathscr{A}$. $\bigcap\mathscr{A}$ is a neighbourhood of y not containing x. On the other hand, if H is a neighbourhood of x from the *subbase* $\bigcup\mathfrak{H}$, then there is a halfdirection $\mathscr{H}$ with $H\in\mathscr{H}\in\mathfrak{H}$. $x\in H$, $x\notin A_{\mathscr{H}}$, so $A_{\mathscr{H}}\subset H$ and $y\in H$. Thus an arbitrary neighbourhood of x contains y, in contradiction to axiom S_1. The assumption $\overline{\bigcap\mathscr{A}}\not\subset\bigcup\mathscr{A}$ has led to a contradiction, so the relation $\overline{\bigcap\mathscr{A}}\subset\bigcup\mathscr{A}$ has been proved in the second case too, thus $\operatorname{sbd}X\leqq n$.

(4.5) PROPOSITION. *If a space has an* ind*-nice subbase, or an* Ind*-nice subbase, then*

$$\operatorname{sbd}X\leqq\operatorname{ind}X\quad\textit{or}\quad\operatorname{sbd}X\leqq\operatorname{Ind}X,$$

respectively.

PROOF. These inequalities are trivial consequences of the definitions and of the relation

$$\overline{\bigcap\mathscr{A}}-\bigcup\mathscr{A}\subset\bigcap\{\operatorname{Fr}A\colon A\in\mathscr{A}\}\quad(\mathscr{A}\in[\mathscr{S}]^{n+1}),$$

where $\mathscr{S}$ is an ind-nice or Ind-nice subbase of X and $n=\operatorname{ind}X$ or $n=\operatorname{Ind}X$, respectively.

(4.6) THEOREM. *For an arbitrary space* X,

$$\operatorname{indc} X \leqq \operatorname{Dim} X.$$

PROOF. Proposition (4.1), Theorem (3.3) and Proposition (4.3).

Since $\operatorname{ind} X \leqq \operatorname{indc} X$, Theorem (4.6) implies Theorem (2.1). (A slight modification of E. DEÁK's proof of Theorem (2.1) would also give the stronger Theorem (4.6.). Theorem (2.2) follows from Proposition (4.2), Theorem (3.3) and Proposition (4.3). Further, Proposition (4.2), Theorem (3.3) and Proposition (4.5) imply Theorem (2.3). In fact, since to have $\operatorname{ind} X \leqq \operatorname{Ind} X$ it is enough to suppose axiom S_1, and Proposition (4.5) holds for subbases, we have so proved the stronger

(4.7) THEOREM. *If a totally paracompact* S_1*-space* X *has an* ind*-nice subbase, then*

$$\operatorname{ind} X = \operatorname{Ind} X.$$

(4.8) THEOREM. *For an* S_1*-space* X,

$$\operatorname{indc} X \leqq \operatorname{DIM} X.$$

PROOF. Proposition (4.1), Theorem (3.3) and Proposition (4.4).

(4.9) THEOREM. *For a totally paracompact* S_1*-space* X,

$$\operatorname{Ind} X \leqq \operatorname{DIM} X.$$

PROOF. Proposition (4.2), Theorem (3.3) and Proposition (4.4).

REMARK. Since $\operatorname{Dim} X \leqq \operatorname{DIM} X+1$ ([5, Theorem (3.1)]), Theorems (2.1) and (2.2) give the weaker inequalities

$$\operatorname{ind} X \leqq \operatorname{DIM} X+1$$

for an arbitrary space and

$$\operatorname{Ind} X \leqq \operatorname{DIM} X+1$$

for a totally paracompact space. These inequalities are true without axiom S_1.

§ 5.

(5.1) PROBLEM. Theorem (4.8) does not hold for an arbitrary space: the simplest counterexample is the space $X=\{0, 1\}$ with the base $\{\{0\}, X\}$, ($\operatorname{ind} X=1$, $\operatorname{DIM} X=0$). *Does Theorem* (4.9) *hold without axiom* S_1?

(5.2) PROPOSITION. *For a totally paracompact metric space* M,

$$\operatorname{ind}^+ M = \operatorname{sbd} M = \operatorname{Ind} M,$$

if any of these dimensions is finite.

PROOF. According to Proposition (4.2) and Theorem (3.3), $\operatorname{Ind} M \leqq \operatorname{ind}^+ M \leqq \operatorname{sbd} M$, so $\operatorname{Ind} M$ is finite. Since M is metric, it has an Ind-nice base (KATĚTOV [7], see also [4]), thus $\operatorname{sbd} M \leqq \operatorname{Ind} M$ (Proposition (4.5)).*

* The inequality $\operatorname{sbd} M \leqq \operatorname{Ind} M$ can also be proved directly, without making use of Katětov's theorem and Proposition (4.5).

(5.3) PROBLEM. The dimension function sbd is trivially monotone and satisfies a general product theorem (see below). *Are similar (perhaps weaker) statements true for* ind^+?

(5.4) PROPOSITION. *If at least one of the spaces X and Y is non-empty, then*

$$\mathrm{sbd}\,(X\times Y) \leqq \mathrm{sbd}\,X+\mathrm{sbd}\,Y.$$

The proof is left to the reader.

(5.5) There is no general sum theorem for ind^+ or sbd: there is a compact T_2-space X which is the union of two closed subspaces F_1 and F_2 with

$$\mathrm{ind}^+ F_1 = \mathrm{sbd}\,F_1 = \mathrm{ind}^+ F_2 = \mathrm{sbd}\,F_2 = 1,$$

$$\mathrm{ind}^+ X = \mathrm{sbd}\,X = 2.$$

O. V. LOKUCIEVSKII's example of a space failing to satisfy the sum theorem for ind and Ind ([8], see also [10, pp. 99—102]) has these properties.

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ON THE ARITHMETIC PROPERTIES OF CERTAIN VALUES OF THE EXPONENTIAL FUNCTION

by

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1. Introduction and results

In [6] the Siegel—Šidlovskiĭ method is used to estimate simultaneously the absolute values of

$$\alpha-\theta, \quad \beta-\gamma, \quad P(\alpha, f_1(\beta), \ldots, f_s(\beta)),$$

where α, β are some complex numbers, θ, γ are some algebraic numbers, $P(x, x_1, \ldots, x_s) \not\equiv 0$ is a polynomial with algebraic coefficients, and $f_1(z), \ldots, f_s(z)$ are certain Siegel E-functions. As a corollary we obtain simultaneous estimates for the absolute values of $\alpha-\theta$, $\beta-\gamma$ and $P(\alpha, e^{\beta})$. In this case, however, better results can be obtained by using the ideas of [6] together with a lemma by MAHLER [5]. The purpose of the present paper is to give such results.

We shall prove the following theorems (by $(H_0, H_1, H_2) \gg (x_0, x_1, x_2)$ we mean $H_0 \geqq x_0$, $H_1 \geqq x_1$ and $H_2 \geqq x_2$).

THEOREM 1. *Let σ, d, g be natural numbers and let α, β be complex numbers. Then there exists a constant $c_0 > 0$, depending only on $\sigma, d, g, |\alpha|$ and $|\beta|$, such that if θ and $\gamma \neq 0$ are any algebraic numbers of degrees $\leqq \sigma$ and heights $\leqq H_0, H_1$, respectively, and $P(x, y) \not\equiv 0$ is any polynomial in $\mathbf{Z}[x, y]$ of height $\leqq H_2$ and degree $\leqq d, g$ with respect to x, y, respectively, satisfying $P(\theta, y) \not\equiv 0$, then the inequality*

$$\max\{|\alpha-\theta|, |\beta-\gamma|, |P(\alpha, e^{\beta})|\} > \exp\left\{-\sigma^2 \prod_{i=0}^{2} \log H_i\right\} \tag{1}$$

is satisfied for all $(H_0, H_1, H_2) \gg (3, c_0, c_0)$.

THEOREM 2. *Let the notations and assumptions be as in Theorem 1. Then there exists a constant $c_1 > 0$, depending only on $\sigma, d, g, |\alpha|$ and $|\beta|$, such that*

$$\max\{|\alpha-\theta|, |\beta-\gamma|, |P(\alpha, e^{\beta})|\} > \exp\{-70\sigma^4 dg(\log H)^2/\log\log H\} \tag{2}$$

for all $H = \max\{H_0, H_1, H_2\} \geqq c_1$.

These theorems immediately give some consequences. First let $\varepsilon > 0$ be given, and let $\{h_i\}$ be an infinite sequence of natural numbers h_i, $h_{i+1} > h_i$ $(i = 1, \ldots)$. Assume that α and $\beta \neq 0$ are complex numbers such that there exist two infinite sequences $\{\theta_i\}$ and $\{\gamma_i\}$ of algebraic numbers $\theta_i \neq \alpha$ and γ_i of bounded degrees and of heights $\leqq h_i$ satisfying

$$\max\{|\alpha-\theta_i|, |\beta-\gamma_i|\} < \exp\{-(\log h_i)^2(\log\log h_i)^{\varepsilon-1}\}, \quad i = 1, 2, \ldots .$$

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[1] This work was carried out while the autor was a research fellow of the Alexander von Humboldt Foundation.

Then Theorem 2 implies the algebraic independence of the numbers α and e^{β}. In the case $\alpha=\beta$ see [4], Theorem 3.

Choose then $\alpha=\theta=1$ and $P(x,y)\equiv P(y)\not\equiv 0$ in Theorems 1 and 2. Using the notations of these theorems we obtain the following corollaries. There exist positive constants c_0' and c_1', depending only on σ, g and $|\beta|$, such that

$$\max\{|\beta-\gamma|,\ |P(e^{\beta})|\} > \exp\{-2\sigma\log H_1\log H_2\} \tag{1}'$$

for all (H_1, H_2) satisfying $H_1\geqq c_0'$ and $H_2\geqq c_0'$, and

$$\max\{|\beta-\gamma|,\ |P(e^{\beta})|\} > \exp\{-70\sigma^2 g(\log H)^2/\log\log H\} \tag{2}'$$

for all $H=\max\{H_1, H_2\}\geqq c_1'$. (We can replace here σ^2 by σ; see the proof of the theorems.) In these results the dependence on the heights is analogous to the recent estimates of CIJSOUW [2], whose proof uses Gel'fond's method with some refinements. We note, however, that our results have the restriction $H_i\geqq c_0'$ $(i=1,2)$ or $H\geqq c_1'$, where c_0' and c_1' depend on the degrees.

Our considerations also provide a lower estimate for $P(\alpha, e^{\alpha})$ in terms of the height of P, if the polynomial $P(x,y)\not\equiv 0$ satisfies certain conditions and α is a transcendental number, which can be suitably approximated by algebraic numbers. As an example we give the following theorem.

THEOREM 3. *Let σ, d, g be natural numbers, and let τ_1, $\tau_2\geqq\tau_1$, $\varkappa$ and $\lambda\geqq 1$ be positive numbers. Assume that α is a complex number such that there exists an infinite sequence $\{\theta_i\}$ of non-zero algebraic numbers $\theta_i\neq\alpha$ of degree $\leqq\sigma$ and height $\leqq h_i$ satisfying the conditions*

$$h_1 > \max\{20, (2/\tau_1)^{1/\varkappa}\}, \tag{3}$$

$$\exp(\tau_1 q(h_i)) \leqq h_{i+1} \leqq \exp(\tau_2 q(h_i)), \quad i=1,2,\dots, \tag{4}$$

$$|\alpha-\theta_i| < 2\exp(-\tau_1 q(h_i)), \quad i=1,2,\dots, \tag{5}$$

where $q(h)=h^{\varkappa}(\log h)^{\lambda}$. There then exists a constant $C_0>0$, depending only on $\sigma, d, g, \tau_1, \tau_2, \varkappa, \lambda$ and $|\alpha|$, such that

$$|P(\alpha, e^{\alpha})| > \exp\{-\log H)^4\} \tag{6}$$

for all polynomials $P(x,y)\not\equiv 0$ in $\mathbf{Z}[x,y]$ of height $\leqq H\geqq C_0$ and degree $\leqq d, g$ with respect to x, y, respectively, which satisfy

$$P(\theta_i, y)\not\equiv 0, \quad i=1,2,\dots. \tag{7}$$

We note that analogous results can be obtained by using some other choices of $q(h)$, e.g. $q(h)=(\log h)^{\lambda}$, $\lambda\geqq 2$. Theorem 3 gives immediately the following transcendence measure of e^{α} (α given in Theorem 3). For all polynomials $P(y)\not\equiv 0$ in $\mathbf{Z}[y]$ of degree $\leqq d$ and height $\leqq H$ we have

$$|P(e^{\alpha})| > C_1\exp\{-(\log H)^4\}, \tag{8}$$

where $C_1>0$ is a constant depending only on $\sigma, d, \tau_1, \tau_2, \varkappa, \lambda$ and $|\alpha|$. In fact, by using (1)′ and the proof of Theorem 3 we are able to replace here $(\log H)^4$ by $(\log H)^2(\log\log H)^2$.

The approximation properties of α in Theorem 3 are formulated in such a way that if $\varkappa>1$, $2\varkappa^{\lambda}\leqq\tau_1(\log 2)^{\lambda-1}$ and there exists a sequence $\{h_n\}$ of natural numbers satisfying (4) and $h_1\geqq 2$, $h_n|h_{n+1}$ $(n=1,2,\ldots)$, then we have (8) for e^{α}, where

$$\alpha=\sum_{n=1}^{\infty}h_n^{-1}.$$

Thus we also have an improvement of the irrationality measure of e^{α} obtained by BUNDSCHUH ([1], Satz 6).

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2. Proof of Theorems 1 and 2

First we give a lemma, which is proved by MAHLER [5].

LEMMA. *Let m and n be arbitrary natural numbers. There exists a system of linear forms*

$$R_j(z)=P_{j0}(z)+P_{j1}(z)e^{z}+\ldots+P_{jm}(z)e^{mz},\quad j=0,\ldots,m,$$

satisfying the following conditions:

(i) *$P_{ji}(z)$ are polynomials of degree $\leqq n$, the coefficients of which are rational integers,*

(ii) *the linear forms $R_j(\gamma)$ $(j=0,\ldots,m)$ are linearly independent for all $\gamma\neq 0$,*

(iii) *if m and n are sufficiently large, say $m\geqq m_0$, $n\geqq n_0$, then*

$$|\mathrm{coeff}\,P_{ji}(z)|<e^{a_0 m(n+1)}n^{n},\tag{9}$$

$$|R_j(z)|<e^{2mn}n^{-mn}|z|^{(m+1)n}e^{m|z|},\tag{10}$$

where a_0, $0<a_0<1{,}75$, is an absolute constant.

Now assume that

$$\max\{|\alpha-\theta|,|\beta-\gamma|,|P(\alpha,e^{\beta})|\}\leqq\exp\Big\{-\sigma^2\prod_{i=0}^{2}\log H_i\Big\},\tag{11}$$

where $P(x,y)$, θ and γ satisfy the conditions given in Theorem 1. We shall prove that a contradiction follows for all $(H_0,H_1,H_2)\gg(3,c_0,c_0)$, where c_0 satisfies the required properties.

Let $m\geqq m_0$ and $n\geqq n_0$ be natural numbers such that $m>g$, and denote $v=m-g$. Beginning from

$$P(x,e^{z})=a_0(x)+a_1(x)e^{z}+\ldots+a_g(x)e^{gz}$$

we form linear forms of the functions e^{iz} $(i=0,\ldots,m)$

$$T_j(x,z)=e^{jz}P(x,e^{z})=\sum_{i=0}^{m}a_{ji}(x)e^{iz},\quad j=0,\ldots,v.$$

Here we have

$$\begin{cases}a_{j,i+j}(x)=a_i(x), & j=0,\ldots,v;\ i=0,\ldots,g,\\ a_{ji}(x)\equiv 0 & \text{otherwise.}\end{cases}\tag{12}$$

The linear forms $T_j(\theta, \gamma)$ $(j=0, \ldots, v)$ are linearly independent. Thus, if the forms $R_j(\gamma)$ are constructed by the Lemma, we can find g forms, say $R_{j_1}(\gamma), \ldots, \ldots, R_{j_g}(\gamma)$, that together with the forms $T_j(\theta, \gamma)$ $(j=0, \ldots, v)$ are linearly independent.

We consider the determinant

$$D(x, z) = \begin{vmatrix} a_{00}(x) & \ldots a_{0m}(x) \\ \cdot & \cdots \cdot \\ a_{v0}(x) & \ldots a_{vm}(x) \\ P_{j_10}(z) & \ldots P_{j_1m}(z) \\ \cdot & \cdots \cdot \\ P_{j_g0}(z) & \ldots P_{j_gm}(z) \end{vmatrix}$$

as in the proof of the theorem of [6]. This determinant can be expressed in the form

$$D(x, z) = \sum_{i=0}^{d(v+1)} \sum_{j=0}^{gn} D_{ij} x^i z^j, \tag{13}$$

where D_{ij} are rational integers satisfying

$$\max_{i,j} |D_{ij}| \leqq (g+1)^m H_2^{v+1} e^{a_0 gm(n+1)} n^{gn} 2^{d(v+1)+gn} \equiv B_1, \tag{14}$$

by (9) and (12),

Since the forms $T_j(\theta, \gamma)$ $(j=0, \ldots, v)$ and $R_{j_i}(\gamma)$ $(i=1, \ldots, g)$ are linearly independent, we have $D(\theta, \gamma) \neq 0$. If a and b are the smallest natural numbers such that $a\theta$ and $b\gamma$ respectively are algebraic integers, then the number $a^{d(v+1)} b^{gn} D(\theta, \gamma) \neq 0$ is an integer in the field $\mathbf{K}=\mathbf{Q}(\theta, \gamma)$, which means that

$$H_0^{dN(v+1)} H_1^{gNn} |D(\theta, \gamma)| \left(|\overline{D(\theta, \gamma)}|\right)^{N-1} \geqq 1,$$

where N $(\leqq \sigma^2)$ is the degree of the field $\mathbf{K}$ over $\mathbf{Q}$. From (9) and (12) we get

$$|\overline{D(\theta, \gamma)}| \leqq (g+1)^m H_2^{v+1} e^{a_0 gm(n+1)} n^{gn} (H_0+2)^{d(v+1)} (H_1+2)^{gn} \equiv B_2,$$

which implies

$$|D(\theta, \gamma)| \geqq H_0^{-dN(v+1)} H_1^{-gNn} B_2^{1-N} \equiv A_1. \tag{15}$$

We denote $\alpha=\theta+\delta_1$, $\beta=\gamma+\delta_2$. From (13) it follows that $D(\alpha, \beta)=D(\theta, \gamma)+$ $+R(\delta_1, \delta_2, \theta, \gamma)$, where $R(\delta_1, \delta_2, \theta, \gamma)$ can be easily estimated by using (11) and (14) (see [6]):

$$|R(\delta_1, \delta_2, \theta, \gamma)| \leqq A_2 \equiv \max\{|\delta_1|, |\delta_2|\} \times$$

$$\times (2d(v+1)gn)^2 B_1 2^{d(v+1)+gn} (1+|\alpha|)^{d(v+1)} (1+|\beta|)^{gn}.$$

From this estimate and (15) we immediately obtain the inequalities

$$|D(\alpha, \beta)| \geqq |D(\theta, \gamma)| - |R(\delta_1, \delta_2, \theta, \gamma)| \geqq A_1 - A_2. \tag{16}$$

We now form an upper bound for $|D(\alpha, \beta)|$. Since

$$|T_j(\alpha, \beta)| \leqq e^{m|\beta|} |P(\alpha, e^\beta)|, \quad j = 0, \ldots, v,$$

we obtain, by (9), (10) and (12)

$$|D(\alpha,\beta)| \leqq m(g+1)^m H_2^v e^{a_0 gm(n+1)} n^{gn}(1+|\alpha|)^{dv} \times \tag{17}$$
$$\times(1+|\beta|)^{gn} e^{m|\beta|}|P(\alpha, e^\beta)|+$$
$$+(g+1)^m H_2^{v+1} e^{a_0(g-1)m(n+1)} n^{(g-1)n}(1+|\alpha|)^{d(v+1)}\times$$
$$\times(1+|\beta|)^{(g-1)n+(m+1)n} e^{2mn} n^{-mn} e^{m|\beta|} \equiv A_3+A_4.$$

Using (16) and (17) we immediately obtain

$$A_3 \geqq A_1-A_2-A_4. \tag{18}$$

Let $c_2, c_3, \ldots$ be positive constants depending only on $\sigma, d, g, |\alpha|$ and $|\beta|$. Now there exists a constant c_2 such that we have

$$A_1 > 4A_2, \quad A_1 > 4A_4, \tag{19}$$

if

$$-\log(\max\{|\delta_1|, |\delta_2|\}) > (2N-1)dm\log H_0+Nm\log H_2+ \tag{20}$$
$$+(2N-1)gn\log H_1+Ngn\log n+c_2 mn,$$

$$(m+1)n\log n > (2N-1)dm\log H_0+Nm\log H_2+ \tag{21}$$
$$+(2N-1)gn\log H_1+Ngn\log n+c_2 mn.$$

We choose m and n to be integers satisfying

$$m < \log H_1 \leqq m+1,$$
$$(n-1)\log(n-1) < N\log H_0 \log H_2 \leqq n\log n,$$

and assume that H_1, H_2 are of sufficient magnitude to give $m \geqq m_0$, $n \geqq n_0$. Then we have

$$(2N-1)dm\log H_0+(2N-1)gn\log H_1+Nm\log H_2+Ngn\log n+$$
$$+c_2 mn < \left\{\frac{2d}{\log H_2}+\frac{4Ng+2c_2}{\log n}+\frac{1}{\log H_0}+\frac{2Ng}{\log H_1}\right\} N \prod_{i=1}^{2}\log H_i$$
$$< N\prod_{i=0}^{2}\log H_i \leqq (m+1)n\log n$$

for all $(H_0, H_1, H_2) \gg (3, c_3, c_3)$. Thus the inequalities (20) and (21) are, by (11), satisfied. Then the use of (18) and (19) gives $A_3 > A_1/2$, by which

$$|P(\alpha, e^\beta)| > \exp\{-((2N-1)dm\log H_0+(2N-1)gn\log H_1+$$
$$+Nm\log H_2+Ngn\log n+c_4 mn)\} > \exp\left\{-N\prod_{i=0}^{2}\log H_i\right\}$$

for all $(H_0, H_1, H_2) \gg (3, c_0, c_0)$. This contradicts our assumption (11). Thus Theorem 1 is true.

To prove Theorem 2 assume that

$$\max\{|\alpha-\theta|,\ |\beta-\gamma|,\ |P(\alpha, e^{\beta})|\} \leqq \exp\{-70\sigma^4\, dg(\log H)^2/\log\log H\}. \tag{22}$$

We can use the above proof, but we now choose m and n to be integers satisfying

$$m-1 < 10Ng\log H(\log\log H)^{-1} \leqq m,$$

$$n-1 < 10Nd\log H(\log\log H)^{-1} \leqq n.$$

We then obtain

$$(2N-1)dm\log H_0+(2N-1)gn\log H_1+Nm\log H_2+$$

$$+Ngn\log n+c_2mn < 70N^2dg(\log H)^2/\log\log H < (m+1)n\log n$$

for all $H\geqq c_5$. These inequalities imply (20) and (21). We then have, as before,

$$|P(\alpha, e^{\beta})| > \exp\{-70\sigma^4\, dg(\log H)^2/\log\log H\}$$

for all $H\geqq c_1$. This is impossible by (22). The contradiction obtained proves Theorem 2.

3. Proof of Theorem 3

Let α and $P(x, y)$ satisfy the conditions of Theorem 3, and let $C_2, C_3, \ldots$ denote positive constants depending only on $\sigma, d, g, \tau_1, \tau_2, \varkappa, \lambda$ and $|\alpha|$. From Theorem 2 we then obtain the estimate

$$\max\{|\alpha-\theta_i|,\ |P(\alpha, e^{\alpha})|\} > \exp\{-C_2(\log H')^2/\log\log H'\} \tag{23}$$

for all $H'=\max\{h_i, H\}\geqq c_1$.

For all $H>h_1$ let $i(H)$ be the index for which $h_{i(H)}<H\leqq h_{i(H)+1}$. Further let C_3 be a constant such that $\tau_1-C_2C_3^2>0$. Assume that H is of sufficient magnitude to give $h_{i(H)}\geqq c_1$.

If

$$h_{i(H)} < H \leqq \exp\{C_3 h_{i(H)}^{\varkappa/2}(\log h_{i(H)})^{\lambda/2}\},$$

then we choose $i=i(H)$ in (23). For $H\geqq C_4$ we then have, by (5),

$$|\alpha-\theta_i| < 2\exp\{-\tau_1 h_i^{\varkappa}(\log h_i)^{\lambda}\} \leqq \exp\{-C_2(\log H)^2/\log\log H\}.$$

Thus (23) gives

$$|P(\alpha, e^{\alpha})| > \exp\{-C_2(\log H)^2/\log\log H\} \tag{24}$$

for all $H\geqq C_4$, $h_{i(H)}<H\leqq\exp\{C_3 h_{i(H)}^{\varkappa/2}(\log h_{i(H)})^{\lambda/2}\}$.

Next let

$$\exp\{C_3 h_{i(H)}^{\varkappa/2}(\log h_{i(H)})^{\lambda/2}\} < H \leqq h_{i(H)+1}.$$

We choose $i=i(H)+1$ in (23). Since, by (5),

$$|\alpha-\theta_i| < 2\exp\{-\tau_1 h_i^{\varkappa}(\log h_i)^{\lambda}\} \leqq \exp\{-C_2(\log h_i)^2/\log\log h_i\}$$

for all $H\geqq C_5$, the inequality (23) implies

$$|P(\alpha, e^{\alpha})| > \exp\{-C_2(\log h_i)^2/\log\log h_i\} \tag{25}$$

for all $H \geqq C_5$, $\exp\{C_3 h_{i(H)}^{\varkappa/2} (\log h_{i(H)})^{\lambda/2}\} < H \leqq h_{i(H)+1}$. By (4) we have

$$h_i = h_{i(H)+1} \leqq \exp\{\tau_2 h_{i(H)}^{\varkappa} (\log h_{i(H)})^{\lambda}\}.$$

Thus we obtain

$$\log h_i \leqq \tau_2 h_{i(H)}^{\varkappa} (\log h_{i(H)})^{\lambda} = \tau_2 C_3^{-2} \left(C_3 h_{i(H)}^{\varkappa/2} (\log h_{i(H)})^{\lambda/2}\right)^2 < C_6 (\log H)^2.$$

The use of (25) gives now the estimate

$$|P(\alpha, e^{\alpha})| > \exp\{-C_2 C_6^2 (\log H)^4 / \log\log H\} \tag{26}$$

for all $H \geqq C_5$, $\exp\{C_3 h_{i(H)}^{\varkappa/2} (\log h_{i(H)})^{\lambda/2}\} < H \leqq h_{i(H)+1}$.

The inequalities (24) and (26) prove Theorem 3.

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ON THE LOCALIZATION PROPERTY OF THE DOUBLE WALSH—FOURIER SERIES

by

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1.

Let I_2 be the unit cube in the 2-dimensional Euclidean space E_2, that is

$$I_2 = \{x = (x_1, x_2): 0 \leqq x_j < 1, j = 1, 2\}.$$

Denote by $x=(x_1, x_2)$, $y=(y_1, y_2)$ points of I_2, and by $m=(m_1, m_2)$ a lattice point of E_2 with non-negative coordinates. Let $w_{m_j}(x)$ $(j=1, 2)$ be the m_j-th Walsh function. For a function f of $L'(I_2)$, its Walsh—Fourier series is defined by

$$S(x; f) = \sum_m \hat{f}_m w_m(x),$$

where

$$\hat{f}_m = \int_{I_2} f(x) w_m(x)\, dx, \; w_m(x) = w_{m_1}(x_1) w_{m_2}(x_2), \; dx = dx_1\, dx_2$$

and $\sum$ runs over all lattice points with non-negative coordinates.

In [2], IGARI has shown that square $(C, 1)$-sums of multiple Fourier series have localization property (for abbreviation, L.P.) for $L^p(T_n)$ if $p \geqq n-1$ but do not have it for $L^p(T_n)$ if $n-1 > p \geqq 1$. In the present paper, we will show that square Abel means of the double Walsh—Fourier series have L. P. for $L^p(I_2)$ if $p \geqq 2$ but do not have it for $L^p(I_2)$ if $2 > p \geqq 1$.

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2.

Let $n=(n_1, n_2)$ be a lattice point with non-negative coordinates, and the n-th partial sum of the Walsh—Fourier series of f be

$$S_n(x; f) = \sum \hat{f}_m w_m(x),$$

where $m=(m_1, m_2)$ runs over all m such that $m_j \leqq n_j$, $j=1, 2$. Then we have

$$S_n(x; f) = \int_{I_2} f(y) D_n(x-y)\, dy,$$

where $D_n(x) = D_{n_1}(x_1) D_{n_2}(x_2)$ and $D_{n_j}(x_j) = \sum_{k=0}^{n_j-1} w_k(x_j)$, $n_j \geqq 1$.

Let $r=(r_1, r_2)\in I_2$. The Abel means are defined by

$$f(x; r) = \sum_m \hat{f}_m r^m w_m(x) = \int_{I_2} f(y) P(x-y; r)\, dy,$$

where $P(x; r)=P(x_1; r_1)P(x_2; r_2)$ and

$$P(x_j; r_j) = \sum_{k=0}^{\infty} r_j^k w_k(x_j) = \sum_{k=0}^{\infty} \{1+r_j^{2^k} w_{2^k}(x)\}.$$

We call $S_n(x; f)$ (or $f(x; r)$) a square partial sum (or a square Abel mean) if $n_1=n_2$ (or $r_1=r_2$), and a rectangular partial sum (or a rectangular Abel mean) for an arbitrary n_j (or r_j).

3.

We prove the following:

THEOREM. (a) *Square partial sums do not have L. P. for* $C(I_2)$.

(b) *Square Abel means have L. P. for* $L^p(I_2)$ *if* $p\geqq 2$ *but do not have it for* $L^p(I_2)$ *if* $1\leqq p<2$.

(c) *Rectangular Abel means have L. P. for* $C(I_2)$ *but do not have it for* $L^p(I_2)$ *if* $p>1$.

We start with two elementary lemmas.

LEMMA 1. *Let* $L_n=\int_0^1 |D_n(x)|\, dx$. *Then* $L_n=O(\log n)$ *and there is a positive number* C *such that*

$$\limsup_{n\to\infty} \frac{L_n}{\log n} \geqq C.$$

PROOF. The first part is well known, see for example [1, p. 387]. For the second part, write $n=\sum_{t=0}^{t_0} 2^{2t}$. For $2^{-2s-2}\leqq x<2^{-2s-1}$, where s is non-negative integer less than t_0, we get

$$|D_n(x)| = 2^{2s}+2^{2s-2}+\ldots+2^2+1 = \frac{1}{3}(2^{2s+2}-1).$$

For $2^{-2s-1}\leqq x<2^{-2s}$,

$$|D_n(x)| \geqq |D_{2^{2s}}(x)| - \sum_{t=1}^{s-1} |D_{2^{2t}}(x)| \geqq 2^{2s} - \sum_{t=0}^{s-1} 2^{2t} = \frac{1}{3}(2^{2s+1}+1).$$

Thus

$$L_n \geqq \sum_{s=0}^{t_0} \int_{2^{-2s-2}}^{2^{-2s}} |D_n(x)|\, dx = \sum_{S=0}^{t_0} \Big(\int_{2^{-2s-2}}^{2^{-2s-1}} + \int_{2^{-2s-1}}^{2^{-2s}} \Big) |D(x)|\, dx \geqq$$

$$\geqq \frac{2}{3}(t_0+1) = \frac{1}{3}\log(3n+1).$$

This proves the second part. Q. E. D.

Next we list several elementary properties concerning the Abel kernel $P(x; r)$. The partial sum of $P(x; r)$ is denoted by $P_{k_0}(x; r)$ and is defined as

$$P_{k_0}(x; r) = \prod_{k=0}^{k_0} (1 + w_{2^k}(x) r^{2^k}).$$

LEMMA 2. (a) $P(x; r) \geqq 0$ $(0 \leqq r < 1)$.

(b) $P(x; r) < \dfrac{2}{x}$ $(0 \leqq r < 1, 0 < x < 1)$.

(c) $P(x; r) < \dfrac{1}{1-r}$ $(0 \leqq r < 1, 0 < x < 1)$.

(d) $\int_0^1 P(x; r)\, dx = 1$.

(e) $\|P(\cdot\,; r)\|_g^g \approx \left(\dfrac{1}{1-r}\right)^{g-1}$ as $r \to 1-0$ if $q \geqq 1$.

PROOF. For (e),

$$\int_0^1 \{P(y; r)\}^g\, dy \leqq \int_0^{1-r} \left(\frac{1}{1-r}\right)^g dy + \int_{1-r}^1 \left(\frac{2}{y}\right)^g dy = 0\left(\left(\frac{1}{1-r}\right)^{g-1}\right).$$

For the converse, let $0 < 1-r < 2^{-l}$, where l is positive integer less than k_0,

$$\int_0^1 \{P_{k_0}(y; r)\}^g\, dy \geqq \int_0^{2^{-l}} \left\{\prod_{k=0}^{l-1} (1+r^{2^k}) \prod_{k=l}^{k_0} (1-r^{2^k})\right\}^g dy \geqq$$

$$\geqq (1-r)\left(\frac{1-r^{2^{l-1}}}{1-r}\right)^g \left(\prod_{k=l}^{k_0} (1-r^{2^k})\right)^g \geqq \left(\frac{1}{1-r}\right)^{g-1} \left(\prod_{k=l-1}^{k_0} (1-r^{2^k})\right)^g.$$

This implies

$$\|P(\cdot\,; r)\|_g^g \geqq C\left(\frac{1}{1-r}\right)^{g-1} \quad \text{as} \quad r \to 1-0.$$

For (f), we observe that

$$\int_{2^{-1}}^1 \{P_{k_0}(y; r)\}^2\, dy = \int_{2^{-1}}^{2^{-1}+2^{-(k_0+1)}} + \ldots + \int_{1-2^{-(k_0+1)}}^1 =$$

$$= 2^{-(k_0+1)}[\{(1-r)(1-r^2) \ldots (1-r^{2^{k_0}})\}^2 + \{(1-r)(1-r^2) \ldots (1-r^{2^{k_0}-1})(1+r^{2^{k_0}})\}^2 +$$

$$+ \ldots + \{(1-r)(1+r^2) \ldots (1+r^{2^{k_0}})\}^2] = 2^{-1}(1-r)(1+r^4)(1+r^8) \ldots (1+r^{2^{k_0+1}}).$$

Similarly

$$\int_{2^{-2}}^{2^{-1}} \{P_{k_0}(y; r)\}^2\, dy = 2^{-2}(1-r)^2(1+r)^4(1+r)^8 \ldots (1+r^{2^{k_0+1}}).$$

Thus

$$\int_{2^{-k}}^{1} \{P_{k_0}(y;r)\}^2\,dy = (1-r)^2\{2^{-1}(1+r^4)(1+r^8)\dots(1+r^{2^{k_0+1}})+$$

$$+2^{-2}(1+r)^4(1+r)^8\dots(1+r^{2^{k_0+1}})+\dots+$$

$$+\dots+2^{-k}(1+r)^4(1+r^2)^4\dots(1+r^{2^{k-1}})^4(1+r^{2^{k+2}})\dots(1+r^{2^{k_0+1}}) \leqq$$

$$\leqq (1-r)^2(1+r)^4\dots(1+r^{2^{k-1}})^4(1+r^{2^{k+2}})\dots(1+r^{2^{k_0+1}}) < C(1-r).$$

The converse inequality is obvious. For $q=1$ or 3, it is proved by a similar calculation. Q. E. D.

Now we are in the position to prove our theorem.

PROOF OF THE THEOREM. For (a), let $0<\varepsilon<\delta<1$ and let φ be a function of $C(I_2)$ such that $\varphi(x)=0$ for $|x|<\varepsilon$ and $\varphi(x)=1$ for $|x|>\delta$. Then

$$\int_{I_2} |\varphi(y)D_{(nn)}(y)|\,dy \geqq \int_{\delta}^{1} |D_n(y_1)|\,dy_1 \int_{0}^{1} |D_n(y_2)|\,dy_2.$$

Suppose that $\delta=2^{-2k}$, $k=1,2,\dots,$ and write n in the form

$$n = \sum_{s=0}^{s_0} 2^{2s} = r\cdot 2^{2k}+t \quad (0<t<2^{2k}).$$

Applying Lemma 1,

$$\int_{\delta}^{1} |D_n(y_1)|\,dy_1 = \int_{2^{-2k}}^{1} |D_t(y_1)|\,dy_1 = L_t - \int_{0}^{2^{-2k}} |D_t(y_1)|\,dy_1 \geqq$$

$$\geqq \frac{1}{3}\log t - t\cdot 2^{-2k} \geqq \frac{1}{3}(k2-1+2^{-2k})$$

and

$$\int_{0}^{1} |D_n(y_2)|\,dy_2 \geqq \frac{1}{3}\log(3n+1).$$

Thus

$$\int_{I_2} |\varphi(y)D_n(y)|\,dy \geqq \frac{1}{9}(2k-1+2^{-2k})\log(3n+1) \to \infty \quad \text{as} \quad n\to\infty.$$

(a) follows from the uniform boundedness theorem.

To prove the first part of (b), it is sufficient to show that

$$I_r = \int_{I_2} |\varphi(y)P(y;r,r)|^g\,dy$$

is unbounded as $r\to 1-0$, where $\frac{1}{p}+\frac{1}{g}=1$ $(1\leqq p<2)$. We assume $g<\infty$ and $\delta=2^{-k}$, $k=1,2,\dots$. Then

$$I_r \geqq \int_{2^{-k}}^{1} \{P(y_1;r)\}^g\,dy_1 \int_{0}^{1} \{P(y_2;r)\}^g\,dy_2.$$

Using Lemma 2(*e*), (*f*) and Hölder's inequality, we get a minorant of I_r:

$$I_r \geqq C(1-r)^{\frac{g}{2}}\left(\frac{1}{1-r}\right)^{g-1} = C(1-r)^{1-\frac{g}{2}}$$

with C independent of r, from which we conclude that $I_r \to \infty$ as $r \to 1-0$ if $1 \leqq p < 2$.

To prove the second part of (b), it is sufficient to show that

$$\sup_{r,\,x} |(\varphi f)(x; r, r)| \leqq C\|f\|_2,$$

where $|x| \leqq \varepsilon'$, ε' being fixed with $0 < \varepsilon' < \varepsilon$. By Hölder's inequality

$$|(\varphi f)(x; r, r)| \leqq \|f\|_2 J_r^{\frac{1}{2}},$$

where

$$J_r = \int_{I_2} |\varphi(y) P(x-y; r, r)|^2 \, dy.$$

Thus

$$J \leqq \int_{\eta}^{1} \{P(y_1; r)\}^2 \, dy_1 \int_{0}^{1} \{P(y_2; r)\}^2 \, dy_2,$$

where $\eta = \frac{1}{\sqrt{2}}(\varepsilon - \varepsilon')$. We assume $\eta = 2^{-k}$, $k = 1, 2, \ldots$. Using Lemma 2(*e*) and (*f*), we get $J_r \leqq C$ with C independent of r. Thus, (b) is proved.

For (c), let $f(y) = f_1(y_1) f_2(y_2)$, where f_1 is bounded on $(0, 1)$ and vanishes in a neighbourhood of 0 and $f_2(y_2) = \sum_{k=1}^{\infty} \frac{w_k(y_2)}{k}$. Suppose f_1 is not identically zero. Then $f_1(0; r_1) \neq 0$ for infinitely many r_1 and $f_2(0; r_2) = \sum \frac{r_k^2}{k} \to \infty$ as $r_2 \to 1-0$. Therefore for suitable choice of $r = (r_1, r_2)$, $f(0; r) = f_1(0; r_1) f_2(0; r_2) \to \infty$ as $r \to 1-0$. The second part of (c) is obvious. This completes the proof. Q. E. D.

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A UNIFIED THEORY OF CYCLICALLY GENERATED MODULAR LATTICES AND *AC*-LATTICES

by

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Dedicated to the memory of Professor László Kalmár

1. Introduction

In the theory of modular lattices those lattices play an important role in which every element is a union of cycles (by a cycle z we mean an element for which the interval $z/0$ is a chain, and if $0<x\leqq z$, then z/x is a finite chain). Such lattices are called by some authors cyclically generated modular lattices; however, in consistence with another generally adopted terminology, we prefer to call them modular cyclistic lattices. Several classes of these lattices are used to investigate lattice-theoretic properties of the subgroup lattices of primary abelian groups and generalized primary abelian groups.

Projective geometries can also be considered as modular cyclistic lattices, the cycles $\neq 0$ being just the points (atoms). Subsequently the lattice-theoretic properties of more general geometries have been investigated. These investigations led to the such-called *AC*-lattices, that is, atomistic lattices with a kind of covering property. These are cyclistic (the cycles $\neq 0$ are exactly the atoms) but not modular in general. On the other hand, the above mentioned covering property holds in every modular lattice.

It can be observed that many notions and results on modular cyclistic lattices are similar to results on *AC*-lattices. These facts suggest that there is a common underlying theory.

It is the aim of this paper to begin a development of what may be called a unified theory of modular cyclistic lattices and *AC*-lattices.

2. Preliminary notions

If $a\leqq b$ in a lattice L, we write b/a for $\{x\in L: a\leqq x\leqq b\}$. Instead of e.g. $(e\vee f)/(c\wedge d)$ we write simply $e\vee f/c\wedge d$, that is, the lattice operations are regarded as more strongly binding than / . By $\leftarrow/a$ we mean the principal dual ideal generated by $a\in L$.

An element $z\in b/a$ will be called a *cycle* in b/a if z/a is a chain and z/x is a finite chain for every x with $a<x\leqq z$. In particular, we regard $a\in L$ as a cycle in b/a.

By $d(z/a)$ we denote the finite or infinite length ("dimension") of the chain z/a. We remark that $d(z/a)$ is also called the *order* of z with respect to $a\in L$ and denoted by some authors by $O_a(z)$.

By $Z(b/a)$ we mean the set of all cycles of b/a. We write $a\prec b$ if $a<b$ and if $a\leqq x\leqq b$ implies that either $x=a$ or $x=b$. If $a\prec b$ then $a\in L$ is called a *lower neighbour* of $b\in L$.

To every cycle $z \in Z(b/a)$ with $z > a$ there exists in b/a exactly one lower neighbour which will be denoted by $z_a^{(1)}$ or z_a'. We define yet $z_a^{(0)} = z$ and $z_a^{(n)} = (z_a^{(n-1)})_a'$.

If the lattice L has a zero element 0 we write briefly $Z(L)$ for the set of all cycles of L, $d(z)$ for $d(z/0)$, z' for z_0' and $z^{(n)}$ for $z_0^{(n)}$. Moreover, a cycle of L will be called shortly a cycle.

DEFINITION 1. A lattice L with 0 is called *cyclic* if for every non-zero element $a \in L$ there exists a cycle z in L such that $z \leqq a$. A lattice L with 0 is called *cyclistic* if every non-zero element $a \in L$ is the join of cycles contained in $a/0$. A cyclistic lattice is briefly called a *Z-lattice.*

LEMMA 2. *A lattice L with 0 is cyclistic if and only if L satisfies the following condition: if $a < b$ in L then there exists a cycle z in L such that $z \not\leqq a$ and $z \leqq b$ (that is, L is "relatively cyclic").*

The proof is similar to that of [19, Lemma 7.2, p. 31] and will therefore be omitted.

In a lattice L with 0, the following property will be called *the isomorphism property for cycles:*

(I) If $z \in L$ is a cycle and $a \in L$ is an arbitrary element, then the interval $H = z/a \wedge z$ is isomorphic to the interval $K = a \vee z/a$ (in symbols: $H \cong K$) in such a way that the mapping

$$\varphi\colon x \to a \vee x \quad (x \in H)$$

is an isomorphism of H onto K and the mapping

$$\psi\colon y \to z \wedge y \quad (y \in K)$$

is an isomorphism of K onto H. Moreover

$$\psi[\varphi(x)] = x \quad \text{and} \quad \varphi[\psi(x)] = y,$$

that is, φ and ψ are inverse mappings.

Before giving examples, we prove two statements to which we introduce the following notions (cf. [19]):

If L is a lattice and $a, b \in L$, we say that (a, b) is a *modular pair* and write $(a, b)M$ if

$$(c \vee a) \wedge b = c \vee (a \wedge b) \quad \text{for every} \quad c \leqq b.$$

We say that (a, b) is a *dual-modular pair,* and we write $(a, b)M^*$ if

$$(c \wedge a) \vee b = c \wedge (a \vee b) \quad \text{for every} \quad c \geqq b.$$

LEMMA 3. *Let L be a lattice with 0 and let* (I) *be satisfied. If z is a cycle in L, then $(z, b)M$ for every $b \in L$ (and hence $(z, b)M^*$ by* [19, Lemma 1.2, p. 1]*).*

PROOF. We have to show that

$$a \leqq b \quad \text{implies} \quad (a \vee z) \wedge b = a \vee (z \wedge b).$$

Assume that $a \leqq b$ in L and $z \in Z(L)$. We have

$$a \leqq (a \vee z) \wedge b \leqq a \vee z$$

and by (I) it follows that

$$(a \vee z) \wedge b = \varphi\psi[(a \vee z) \wedge b] = \varphi[(a \vee z) \wedge b \wedge z] =$$
$$= \varphi(b \wedge z) \qquad = a \vee (b \wedge z)$$

which was to be proved.

If the length of a cycle is finite, then the assertion of the previous lemma can be proved under a seemingly weaker condition that (I).

THEOREM 4. *Let L be a lattice with 0. If for a cycle $z \in Z(L)$ the condition*

(I*) $$z/z \wedge a \cong a \vee z/a \quad (a \in L \text{ arbitrary})$$

holds, then $d(z) < \infty$ implies $(z, b)M$ for every $b \in L$.

PROOF. We have to show that

(1) $$a \leqq b \quad \text{implies} \quad (a \vee z) \wedge = a \vee (z \wedge b).$$

By (I*) we have

(2) $$d(a \vee z/(a \vee z) \wedge b) = d(z/z \wedge b).$$

Since

$$a \leqq (a \vee z) \wedge b \leqq a \vee z$$

it follows that

$$a \vee z = [(a \vee z) \wedge b] \vee z.$$

This together with (2) yields

$$d(z/b \wedge z) = d([(a \vee z) \wedge b] \vee z/(a \vee z) \wedge b).$$

On the other hand, we obtain by (I*)

(3) $$d(a \vee z/a \vee (z \wedge b)) = d(a \vee (z \wedge b) \vee z/a \vee (z \wedge b)) =$$
$$= d(z/[a \vee (z \wedge b)] \wedge z),$$

where

(4) $$z \wedge b \leqq [a \vee (z \wedge b)] \wedge z \leqq z.$$

Since

$$a \vee (z \wedge b) \leqq (a \vee z) \wedge b,$$

it follows that

(5) $$a \vee z/(a \vee z) \wedge b \subseteq a \vee z/a \vee (z \wedge b).$$

On the other hand, we get by (4) that

(6) $$z/[a \vee (z \wedge b)] \wedge z \subseteq z/z \wedge b.$$

By (5), (3), (6) and (2) we have

$$d(a\vee z/a\vee(z\wedge b)) = d(a\vee z/(a\vee z)\wedge b). \tag{7}$$

Since $d(z)<\infty$, it follows by (I*) that $a\vee z/a$ is a finite chain. Therefore (7) implies $(a\vee z)\wedge b=a\vee(z\wedge b)$ which proves (1) and thereby the theorem.

In the sequel we investigate mainly cyclistic lattices which satisfy the isomorphism property (I) for cycles. These lattices will be called briefly *ZI*-lattices.

3. Examples of *ZI*-lattices

By the isomorphism principle for modular lattices (cf. e.g. [28, Satz 26, pp. 106—108]) every modular cyclistic lattice is a *ZI*-lattice. In chronological order we list authors who have studied several classes of modular cyclistic lattices: R. BAER [5], F. KLEIN-BARMEN [18], E. INABA [13], [14], [15], M. I. GRAEV [11], G. S. MONK [20], [21], B. JÓNSSON—G. S. MONK [17], S. A. ANIŠČENKO [1], [2], [3], B. ARTMANN [4][1], R. FRITZSCHE [7], [8], [9], R. FRITZSCHE—G. RICHTER [10], G. RICHTER [22], [23], [24]. We remark that L. A. SKORNJAKOV in his book [25] poses the problem (Problem 25, p. 183) of investigating a certain class of modular cyclistic lattices. Note also that M. I. Graev, S. A. Aniščenko and L. A. Skornjakov use a more general notion of cycle than we do.

As we observed already in the introduction, most of these works have their origin in the investigation of the subgroup lattices of certain abelian groups.

Further examples for *ZI*-lattices are provided by *AC*-lattices. An *AC-lattice* is an atomistic lattice (i.e. every element $\neq 0$ is a join of atoms) with the following covering property

(C) p atom, b arbitrary element and $p\not\leqq b$ imply $b\prec b\vee p$.

The cycles ($\neq 0$) of an *AC*-lattice are exactly its atoms and the covering property (C) ensures that the isomorphism property (I) is satisfied. Important examples of *AC*-lattices are partition lattices, Wille's incidence geometries of grade n and the lattice of closed subspaces of a Hilbert space. For a general theory of *AC*-lattices we refer to F. MAEDA—S. MAEDA [19].

Examples of *ZI*-lattices which are neither modular cyclistic lattices nor *AC*-lattices arise, for instance, in the following manner: to the zero element of a non-modular *AC*-lattice we paste a finite or countable chain.

4. Properties of *ZI*-lattices

In this section we study the most fundamental properties of *ZI*-lattices.

LEMMA 5. *Let L be a ZI-lattice and $a\leqq b$ $(a, b\in L)$. Then $u\in L$ is a cycle in b/a (or in $\leftarrow/a$) if and only if there exists a cycle $z\in L$ such that $u=a\vee z$.*

PROOF. Let $u=a\vee z$ with $z\in Z(L)$. By (I) it follows that u/a is a chain and that u/x is a finite chain for all $x\in b/a$ with $a<x\leqq u$. This means that $u\in Z(b/a)$.

[1] This paper and other works by B. Artmann as well as reference [3] belong to the theory of Hjelmslev planes. We found it impossible here to list all the references dealing with the connections between Hjelmslev planes an primary lattices (which are modular cyclistic lattices).

Conversely, let $u \in Z(b/a)$. Then $a \leqq u$ and

$$u = \bigvee(z: z \in Z(u/0)) = a \vee \bigvee(z: z \in Z(u/0)) =$$
$$= \bigvee(a \vee z: z \in Z(u/0)).$$

Assume that $a \vee z \prec u$ for every element $z \in Z(u/0)$. Then $a \vee z \leqq u'_a$ and thus $u \leqq u'_a$ contradicting $u'_a \prec u$. It follows that there exists a cycle $z \in Z(u/0) \subseteq Z(L)$ such that $u = a \vee z$, and the lemma is proved.

LEMMA 6. *Let L be a ZI-lattice and $a \leqq b$ $(a, b \in L)$. Then b/a (and $\leftarrow/a$) is also a ZI-lattice.*

PROOF. Let u be a cycle in b/a and let c be an arbitrary element in b/a. We have to show that

$$c \vee u/c \cong u/c \wedge u, \tag{8}$$

this isomorphism being realized by the two mappings defined by (I). Since $u \in Z(b/a)$, it follows by Lemma 5 that there exists a cycle $z \in Z(L)$ such that $u = z \vee a$. Then

$$c \vee u = c \vee z \vee a = c \vee z$$

since $c \geqq a$. This implies by (I)

$$c \vee u/c = c \vee z/c \cong z/c \wedge z \wedge u = z/z \wedge c. \tag{9}$$

Observing that $u \geqq z \vee (c \wedge u) \geqq z \vee a = u$ it follows that $u = z \vee (c \wedge u)$. This yields by (I)

$$u/c \wedge u = z \vee (c \wedge u)/c \wedge u \cong z/z \wedge c \wedge u = z/z \wedge c. \tag{10}$$

Now (9) and (10) together give (8) where the isomorphisms are obtained by composing the isomorphisms of (9) and (10). For $\leftarrow/a$ the reasoning is similar, and the proof is finished.

We remark that for AC-lattices the corresponding assertion appears in [19, Lemma 8.18, p. 39]. For modular cyclistic lattices we refer to [10].

COROLLARY 7. *Let L be a ZI-lattice and $a \leqq b$ $(a, b \in L)$. If $d(z) < \infty$ for all $z \in Z(L)$ and $u \in Z(b/a)$, then $d(u/a) < \infty$.*

The proof of this corollary is obvious.

THEOREM 8. *Let L be a ZI-lattice and $a \leqq b$ $(a, b \in L)$. If $u = a \vee z$ is in $Z(b/a)$ with $z \in Z(L)$, then*

$$u_a^{(n)} = a \vee z^{(n)} \quad (n = 0, 1, 2, \ldots).$$

PROOF. We prove the assertion by induction on n. By definition, we have

$$u_a^{(0)} = (a \vee z)_a^{(0)} = a \vee z = a \vee z^{(0)}.$$

Assume now that

$$u_a^{(n-1)} = (a \vee z)_a^{(n-1)} = a \vee z^{(n-1)} \tag{11}$$

is true $(n \geqq 1)$. From

$$z^{(n)} \prec z^{(n-1)} \tag{12}$$

it follows that $a \vee z^{(n)} \leqq a \vee z^{(n-1)}$. Assume that $a \vee z^{(n)} = a \vee z^{(n-1)}$. By (I), this implies $z^{(n)} = \psi\varphi(z^{(n)}) = \psi(a \vee z^{(n)}) = \psi(a \vee z^{(n-1)}) = \psi\varphi(z^{(n-1)}) = z^{(n-1)}$ which contradicts (12) (φ, ψ denote the isomorphisms of condition (I)). Thus

$$a \vee z^{(n)} \prec a \vee z^{(n-1)}.$$

From this we obtain

$$z^{(n-1)} \not\leqq a \vee z^{(n)}$$

since otherwise $a \vee z^{(n-1)} \leqq a \vee z^{(n)}$. Therefore

$$(a \vee z^{(n)}) \vee z^{(n-1)} = a \vee z^{(n-1)} \tag{13}$$

and

$$(a \vee z^{(n)}) \wedge z^{(n-1)} = z^{(n)}. \tag{14}$$

From (13) and (14) we get by (12) and (I)

$$(a \vee z^{(n)}) \prec (a \vee z^{(n-1)}).$$

Because of (11), this implies

$$(a \vee z^{(n)}) = u_a^{(n-1)}$$

which means that

$$a \vee z^{(n)} = u_a^{(n)}$$

and the proof is finished.

DEFINITION. In a lattice L with 0, the following property is called the *exchange property for cycles:* if z_1 and z_2 are cycles of L and if

$$z_1 \leqq a \vee z_2 \quad (a \in L)$$

but

$$z_1 \not\leqq a \vee z_2'$$

then $z_1 \vee a = z_2 \vee a$ and hence

$$z_2 \leqq z_1 \vee a.$$

Note that if z_1 and z_2 are atoms, the above definition gives the classical STEINITZ—MACLANE Exchange Property (cf. e.g. [19, Definition 7.8, p. 32]).

THEOREM 9. *A ZI-lattice satisfies the exchange property for cycles.*

PROOF. Let L be a *ZI*-lattice, $z_1, z_2 \in Z(L)$ and $a \in L$ with

$$z_1 \leqq z_2 \vee a \tag{15}$$

but

$$z_1 \not\leqq z_2' \vee a. \tag{16}$$

From (15) it follows by Lemma 5 that

$$a \vee z_1 \in Z(a \vee z_2/a). \tag{17}$$

Note that

$$a \vee z_1 \succ a. \tag{18}$$

For, if $a \vee z_1 = a$, then $z_1 \leqq a \leqq a \vee z_2'$ which contradicts (16). Now since $a \vee z_2/a$ is also a *ZI*-lattice by Lemma 6, it follows from (17) and (18) according to (15), the definition of a cycle and Corollary 7 that

$$a \vee z_1 = (a \vee z_2)_a^{(n)} \tag{19}$$

for some finite n ($n=0, 1, 2, \ldots$). From (19) we obtain by Theorem 8

$$a \vee z_1 = a \vee z_2^{(n)}.$$

From this it follows by (16) that $n=0$, that is,

$$a \vee z_1 = a \vee z_2$$

and hence

$$z_2 \leqq z_1 \vee a$$

which was to be proved.

We remark that the exchange property for cycles was used, for instance, by R. FRITZSCHE [7] in modular cyclistic lattices. For these lattices, G. RICHTER [22] showed that the exchange property for cycles always holds.

In a lattice L, the following condition will be called the *neighbourhood condition*:

$$x \wedge y \prec x \quad \text{implies} \quad y \prec x \vee y. \tag{N}$$

The reverse implication

$$y \prec x \vee y \quad \text{implies} \quad x \wedge y \prec x \tag{N*}$$

will be called the *dual neighbourhood condition.*

It is well-known that every modular lattice satisfies (N) as well as (N^*). It is likewise well-known that every *AC*-lattice satisfies (N) (cf. [19, Theorem 7.10, p. 32]) but not (N^*) in general (cf. [19]). For *ZI*-lattices we are able to prove

THEOREM 10. *A ZI-lattice satisfies the neighbourhood condition* (N).

PROOF. Without loss of generality we may assume that x and y are incomparable elements of a *ZI*-lattice L and

$$x \wedge y \prec x. \tag{20}$$

Then

$$x \in Z(x/x \wedge y).$$

By Lemma 5 there exists a cycle $z \in Z(L)$ with

$$x = (x \wedge y) \vee z \tag{21}$$

and

$$z \not\leqq x \wedge y. \tag{22}$$

By (20), (21) and (I) we have

$$z/z \wedge (x \wedge y) \cong (x \wedge y) \vee z/x \wedge y = x/x \wedge y$$

and thus

$$z\wedge(x\wedge y) = z',$$

that is

$$z' \leqq x\wedge y \leqq y. \tag{23}$$

It follows now from (21) that

$$x\vee y = (x\wedge y)\vee z\vee y = z\vee y. \tag{24}$$

From this we conclude that $x\vee y\in Z(x\vee y/y)$. Finally we obtain by (24), Theorem 8 and (23) that

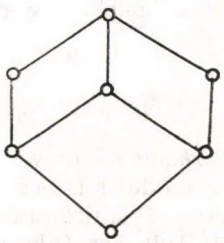
$$(x\vee y)'_y = (z\vee y)'_y = y\vee z' = y.$$

This means

$$y \prec x\vee y$$

and the proof is finished.

Since (N) implies (C) in an arbitrary lattice, it follows from our remarks before Theorem 10 that in atomistic lattices the covering property (C) is equivalent to the neighbourhood condition (N) (cf. [19, Theorem 7.10, p. 32]). The corresponding statement for general cyclistic lattices, however, is not true. Consider for instance the following lattice:



This lattice is cyclistic and satisfies (N), but it does not satisfy (I).

As in the case of AC-lattices, it is also easy to see that a ZI-lattice does not satisfy, in general, the dual neighbourhood condition (N^*). In the following section we consider ZI-lattices in which (N^*) holds.

5. ZI-lattices satisfying the dual neighbourhood condition

We give a sufficient condition which ensures that a ZI-lattice satisfies (N^*).

THEOREM 11. *Let L be a ZI-lattice and for all $z\in Z(L)$ let $(b, z)M^*$ hold, that is, $a\wedge(b\vee z)=(a\wedge b)\vee z$ for all $b\in L$ and all $a\geqq z$. Then L satisfies the dual neighbourhood condition (N^*).*

PROOF. Let

$$x \prec x\vee y. \tag{25}$$

Assume that for all $z\in Z(y/0)$ one has $z\leqq x$. Then we obtain $y=\vee(z: z\in Z(y/0))\leqq x$, a contradiction. Thus there exists a $z\in Z(y/0)$ with $z\not\leqq x$ and therefore $z\not\leqq x\wedge y$.

It follows that

$$x \prec x \vee z \leqq x \vee y$$

and by (25) we obtain

$$x \vee z = x \vee y. \tag{26}$$

From (26) and Theorem 8 we get

$$x = (x \vee y)'_x = (x \vee z)'_x = x \vee z'$$

and therefore

$$z' \leqq x.$$

This implies

$$z' \leqq x \wedge y$$

because of $z \in Z(y/0)$.

Thus we have $z \wedge (x \wedge y) = z' \prec z$ and obtain from this by (I)

$$x \wedge y \prec (x \wedge y) \vee z \leqq y. \tag{27}$$

Because of $(x, z) M^*$ it follows from (27) and (26) that

$$(x \wedge y) \vee z = (x \vee z) \wedge y = (x \vee y) \wedge y = y. \tag{28}$$

Hence we get by (27) that

$$x \wedge y \prec y,$$

that is, (N^*) holds in L and the theorem is proved.

We remark that for AC-lattices the corresponding statement forms a part of [19, Theorem 9.5, p. 42].

COROLLARY 12. *Let L be an algebraic ZI-lattice with $d(z) < \infty$ for all $z \in Z(L)$. If L satisfies (N^*), then L is modular.*

PROOF. $d(z) < \infty$ for all $z \in Z(L)$ implies by Corollary 7 that L is strongly atomic in the sense of [6]. Moreover (N) is satisfied by Theorem 10. The assertion follows now from [6, Theorem 3.6, p. 24].

Note that for AC-lattices the corresponding statement appears in [19, Theorem 14.1, p. 58].

6. Finite joins of cycles

We consider now finite joins of cycles. Let in a ZI-lattice L

$$a = z_1 \vee \ldots \vee z_n \quad (z_i \in Z(L);\ i = 1, \ldots, n) \tag{29}$$

be such a join. If

$$\bar{z}_i \overset{\text{def}}{=} z_1 \vee \ldots \vee z_{i-1} \vee z_{i+1} \vee \ldots \vee z_n \prec a$$

for every i $(i = 1, \ldots, n)$. then we say that the join representation (29) is irredundant. For irredundant join representations of the above kind, the Theorem of KUROŠ-ORE holds:

THEOREM 13. *Let L be a ZI-lattice and let $u \in L$ be an element which can be represented as a join of a finite number of cycles. Then all irredundant representations of u as a join of cycles have the same number of components.*

PROOF. The proof goes along the same lines as in the modular case (cf. e.g. [28, Satz 29, pp. 111—113]). Instead of modularity, of course, we use the isomorphism property (I) for cycles.

Let L be a ZI-lattice and let

$$u = x_1 \vee \ldots \vee x_r \quad (x_i \in Z(L);\ i = 1, \ldots, r) \tag{30}$$

and

$$u = y_1 \vee \ldots \vee y_s \quad (y_k \in Z(L);\ k = 1, \ldots, s) \tag{31}$$

be two irredundant join representations of an element $u \in L$. We show first that every cycle x_j in (30) can be replaced by a suitable y_k such that

$$x_1 \vee \ldots \vee x_{j-1} \vee y_k \vee x_{j+1} \vee \ldots \vee x_r = u.$$

For the sake of brevity we define

$$\bar{x}_j \overset{\text{def}}{=} x_1 \vee \ldots \vee x_{j-1} \vee x_{j+1} \vee \ldots \vee x_r$$

and

$$v_{jk} \overset{\text{def}}{=} \bar{x}_j \vee y_k \quad (k = 1, \ldots, s).$$

Because of (30) and (31), we obtain

$$\bar{x}_j \vee x_j = u, \tag{32}$$

$$v_{j1} \vee \ldots \vee v_{js} = u, \tag{33}$$

and

$$\bar{x}_j \leqq v_{jk} \leqq u \quad (k = 1, \ldots, s). \tag{34}$$

Since x_j is a cycle, we obtain by (I) and by (32)

$$x_j / x_j \wedge \bar{x}_j \cong x_j \vee \bar{x}_j / \bar{x}_j = u / \bar{x}_j. \tag{35}$$

The cycle x_j is join-irreducible in $x_j / x_j \wedge \bar{x}_j$ and hence by (35) the element u must be join-irreducible in $u / \bar{x}_j$. On the other hand, (33) is because of (34) a join-representation of u in $u / \bar{x}_j$. Therefore some v_{jk} must be equal to u which was to be proved. By continuing the replacement procedure, we obtain $r = s$, and the theorem is proved.

LEMMA 14. *In a ZI-lattice a join-representation*

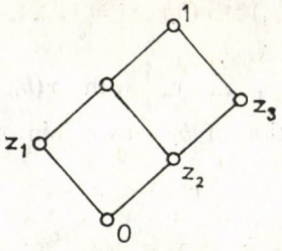
$$u = z_1 \vee \ldots \vee z_n \quad (z_i \in Z(L);\ i = 1, \ldots, n)$$

is irredundant if and only if

$$z_1 \vee \ldots \vee z_{i-1} \vee z_i' \vee z_{i+1} \vee \ldots \vee z_n \prec u.$$

PROOF. The "if" part of the proof is trivial and the "only if" part follows from Theorem 8.

We note that in *AC*-lattices L even $(0=)\ z_i' \leqq \bar{z}_i$ $(z_i \in Z(L))$ holds for all i $(i=1, \ldots, n)$, This is not the case in general *ZI*-lattices, as the following example shows:



Here we have $1=z_1 \vee z_3$ but $z_3'=z_2 \not\leqq z_1$. The property $z_i' \leqq \bar{z}_i$, however, is not characteristic of *AC*-lattices which is easy to see.

7. The ideal of the finite elements

Let now $F(L)$ denote the set of those elements of a *ZI*-lattice L which can be represented as a join of finitely many cycles. By Theorem 13, we are able to assign to every element $(0\neq)\ a \in F(L)$ a uniquely determined natural number. This number will be called the rank of the element $a \in F(L)$ and it will be denoted by $r(a)$. We put $r(0)=0$ and $r(a)=\infty$ if $a \notin F(L)$.

Note that $r(z)=1$ for every cycle $(0\neq)\ z \in Z(L)$.

We remark that for *AC*-lattices the notion of rank given here coincides with the notion of height as defined in [19, Definition 8.5, p. 36]. Let now $a \geqq b$ in an *ZI*-lattice. By Lemma 6 it is clear what we mean by the rank $r(a/b)$ of a with respect to b.

We are now going to prove some properties of the rank. We remark that corresponding properties for *AC*-lattices can be found in F. MAEDA—S. MAEDA [19]; for modular cyclistic lattices we refer to S. A. ANIŠČENKO [1], [2].

LEMMA 15. *Let L be a ZI-lattice. Then*

$$r(a \vee b) \leqq r(a)+r(b).$$

The proof is obvious and will therefore be omitted.

Generalizing a result of S. A. ANIŠČENKO [1] we prove

THEOREM 16. *Let L be a ZI-lattice. If $a \geqq b$, $a \in F(L)$ and $b \in L$, then $r(a) \geqq r(b)$.*

PROOF. Let

$$a = x_1 \vee \ldots \vee x_n$$

where the x_i $(i=1, \ldots, n)$ are cycles. Consider the elements.

$$b_1 = x_1 \vee b$$
$$b_2 = x_2 \vee b_1$$
$$\vdots$$
$$b_n = x_n \vee b_{n-1}.$$

Evidently

$$b = b_0 \leqq b_1 \leqq b_2 \leqq \ldots \leqq b_n = a.$$

We show that

$$r(b_{i-1}) \leqq r(b_i) \quad (i = 1, 2, \ldots, n).$$

Let

$$b_{i-1} = y_1 \vee \ldots \vee y_m \quad \text{with} \quad r(b_{i-1}) = m \tag{36}$$

be an irredundant representation of b_{i-1} as a join of cycles y_i $(i=1, \ldots, m)$. Assume that

$$r(b_i) < m.$$

Then

$$b_i = b_{i-1} \vee x_i = y_1 \vee \ldots \vee y_m \vee x_i$$

must be a redundant representation of b_i, and at least two cycles of this representation must be contained in the join of the others. Without loss of generality we may assume that

$$y_1, y_2 \leqq y_3 \vee \ldots \vee y_m \vee x_i.$$

Then

$$b_{i-1} = b_i \wedge b_{i-1} = (y_3 \vee \ldots \vee y_m \vee x_i) \wedge b_{i-1}. \tag{37}$$

Since x_i is a cycle and $y_3 \vee \ldots \vee y_m \leqq b_{i-1}$, we have by Lemma 3

$$(y_3 \vee \ldots \vee y_m \vee x_i) \wedge b_{i-1} = (y_3 \vee \ldots \vee y_m) \vee (x_i \wedge b_{i-1}). \tag{38}$$

Since $x_i \wedge b_{i-1}$ is a cycle, we obtain by (37) and (38) that

$$b_{i-1} = y_3 \vee \ldots \vee y_m \vee (x_i \wedge b_{i-1})$$

is a join of cycles, and thus

$$r(b_{i-1}) \leqq m-1$$

which is a contradiction to (36). Therefore $r(b_i) \geqq r(b_{i-1})$ for all i $(i=1, \ldots, n)$ which implies $r(a) \geqq r(b)$, and the theorem is proved.

COROLLARY 17. *In a ZI-lattice L, she set $F(L)$ is an ideal.*

PROOF. The assertion follows from Lemma 15 and from Theorem 16.

LEMMA 18. *Let L be a ZI-lattice. If $a, b, c \in L$, $a \in F(L)$ and $a \geqq c \geqq b$, then $r(a/c) + r(c/b) \geqq r(a/b)$.*

PROOF. The proof given by S. A. ANIŠČENKO [2, Theorem 1] for the modular case can be applied without alteration and will therefore be omitted.

In the remaining part of the paper we shall give necessary and sufficient conditions for $F(L)$ to be a standard ideal in the sense of G. GRÄTZER—E. T. SCHMIDT [12]. It turns out that the results of M. STERN [26], [27] can be generalized to *ZI*-lattices.

An ideal S of a lattice L is called *standard* (cf. [12, p. 30]) if $J \wedge (S \vee K) = (J \wedge S) \vee (J \wedge K)$ holds for any pair of ideals J, K of L. We shall need the following characterization of standard ideals:

THEOREM 19 ([12, Theorem 2, p. 30]). *An ideal S of a lattice L is standard if and only if*

$$S\vee(\leftarrow/x) = \{s\vee x_1\colon\ s\in S \text{ and } x_1\leqq x\}$$

for every principal ideal $\leftarrow/x$ of L.

Moreover we need

LEMMA 20 ([12, Lemma 10, p. 40 and Corollary 4, p. 45]). *A neutral ideal of a lattice is standard. In a modular lattice any standard ideal is neutral.*

Now we are able to give the following characterization of $F(L)$ to be a standard ideal of a ZI-lattice. For the corresponding result on AC-lattices we refer to [26].

THEOREM 21. *Let L be a ZI-lattice. Then the following two conditions are equivalent:*

(i) *$F(L)$ is a standard ideal;*

(ii) *$r(b\vee x/x)<\infty$ implies $r(b/b\wedge x)<\infty$ for any $b, x\in L$.*

PROOF. (i)$\Rightarrow$(ii): Let $F(L)$ be a standard ideal and let

$$r(b\vee x/x) < \infty.$$

Applying Lemma 5 and Lemma 18 it follows that there exists an $a\in F(L)$ with

$$x \leqq x\vee b = x\vee a.$$

This implies

$$b\in F(L)\vee(\leftarrow/x).$$

By Theorem 19 there exists an $x_1\leqq x$ and an $a_1\in F(L)$ such that

$$b = x_1\vee a_1$$

and hence $r(b/x_1)=r(x_1\vee a_1/x_1)<\infty$.

Since $x_1\leqq x\wedge b\leqq b$, it follows that

$$r(b/x\wedge b) < \infty$$

which was to be proved.

(ii)$\Rightarrow$(i): Let

$$b\in(\leftarrow/x)\vee F(L).$$

Then there exists an $a\in F(L)$ with

$$b \leqq x\vee a.$$

It follows that

$$r(x\vee a/x) < \infty$$

which implies

$$r(x\vee b/x) < \infty$$

by Theorem 16 since $x\leqq x\vee b\leqq x\vee a$. By (ii) we obtain

$$r(b/b\wedge x) < \infty$$

which means that there exists an element $a_1 \in F(L)$ such that

$$b = (x \wedge b) \vee a_1.$$

Theorem 19 yields now that $F(L)$ is a standard ideal, and the proof is finished.

COROLLARY 22. *If L is a modular cyclistic lattice, then $F(L)$ is a neutral ideal.*

PROOF. By the isomorphism principle for modular lattices, condition (ii) of the preceding theorem is satisfied. By condition (i) of this theorem it follows that $F(L)$ is standard. Finally Lemma 20 yields that $F(L)$ is neutral.

Theorem 21 can be strengthened in the following manner which is an extension of the corresponding statement on AC-lattices appearing in [27].

THEOREM 23. *In a ZI-lattice L, the following two conditions are equivalent:*
(i) *$F(L)$ is a standard ideal*
(ii) *$r(c \vee d/c) = 1$ implies $r(d/c \wedge d) < \infty$ for every $c, d \in L$.*

PROOF. (i)$\Rightarrow$(ii): this implication follows immediately from Theorem 21.
(ii)$\Rightarrow$(i): by Theorem 21 it suffices to prove that

$$r(x \vee b/x) < \infty \quad \text{implies} \quad r(b/b \wedge x) < \infty. \tag{39}$$

Let therefore

$$r(x \vee b/x) < \infty. \tag{40}$$

This means that there exist cycles $x_1, \ldots, x_n$ in $x \vee b/x$ such that

$$x = x_0 < x_1 < x_1 \vee x_2 < \ldots < x_1 \vee \ldots \vee x_i < \tag{41}$$
$$< x_1 \vee \ldots \vee x_{i+1} < \ldots < x_1 \vee \ldots \vee x_n = x \vee b.$$

We put $m_i \overset{\text{def}}{=} x_1 \vee \ldots \vee x_i$ $(i = 0, 1, \ldots, n)$ and denote by M the number of the intervals m_{i+1}/m_i. By (41) we have

$$M < \infty. \tag{42}$$

Consider now the intervals of the form

$$b \wedge m_{i+1}/b \wedge m_i$$

and denote by N the number of these intervals. Since

$$b \wedge m_{i+1} = b \wedge m_i \tag{43}$$

is possible, we get

$$N \leqq M < \infty. \tag{44}$$

What can be said of $r(b \wedge m_{i+1}/b \wedge m_i)$? We have

$$(b \wedge m_{i+1}) \wedge m_i = b \wedge m_i \tag{45}$$

and

$$m_i \leqq (b \wedge m_{i+1}) \vee m_i \leqq m_{i+1}. \tag{46}$$

Since $r(m_{i+1}/m_i)=1$ by construction, we obtain from (46), Lemma 6 and Theorem 16

$$r\big(m_i\vee(b\wedge m_{i+1})/m_i\big)\leqq r(m_{i+1}/m_i)=1,$$

and hence either

$$r\big(m_i\vee(b\wedge m_{i+1})/m_i\big)=0 \tag{47}$$

or

$$r\big(m_i\vee(b\wedge m_{i+1})/m_i\big)=1. \tag{48}$$

If (47) holds, then $b\wedge m_{i+1}\leqq m_i$ and thus $b\wedge m_{i+1}=b\wedge m_i$ which means that

$$r(b\wedge m_{i+1}/b\wedge m_i)=0.$$

On the other hand, if (48) holds we are able to apply assumption (ii) which yields that

$$r(b\wedge m_{i+1}/b\wedge m_i)<\infty. \tag{49}$$

By Lemma 5 this implies that there exists a finite number of cycles $z_1, z_2, \ldots, z_t\in Z(L)$ such that

$$b\wedge m_{i+1}=(b\wedge m_i)\vee z_1\vee z_2\vee\ldots\vee z_t \tag{50}$$

(It may be that some or all of these cycles are 0.) Therefore we obtain from (41) and (50)

$$b=(b\wedge x)\vee\bigvee(z_j\colon\ z_j\in Z(L),\ j\in B,\ B\text{ finite}).$$

Since $r(z_j)=1$ for all $j\in B$, it follows by Lemma 5 and Corollary 17 that

$$r(b/b\wedge x)<\infty.$$

This proves (39) and thereby the theorem.

COROLLARY 24. *Let L be a ZI-lattice and let $d(z)<\infty$ for all cycles $z\in Z(L)$. If the dual neighbourhood condition (N^*) holds in L, then $F(L)$ is a standard ideal.*

PROOF. We need only to show that condition (ii) of Theorem 23 is satisfied. Let therefore

$$r(c\vee d/c)=1.$$

This means by Lemma 5 that there exists a cycle z such that

$$c\vee z=c\vee d.$$

By (I) we get that

$$z/z\wedge c\cong c\vee z/c=c\vee d/c.$$

Since by assumption $d(z)<\infty$, it follows that $c\vee d/c$ is a chain of finite length. Hence we can write

$$c=c_0\prec c_1\prec\ldots\prec c_j\prec c_{j+1}\prec\ldots\prec c_m=c\vee d.$$

Consider now the intervals $d\wedge c_{j+1}/d\wedge c_j$. For every such interval we have either $d\wedge c_{j+1}=d\wedge c_j$ or $d\wedge c_j \prec d\wedge c_{j+1}$ since (N^*) holds in L. Thus

$$r(d\wedge c_{j+1}/d\wedge c_j) \leqq 1$$

and the same reasoning as in the preceding theorem (from (49) on) yields that $r(d/c\wedge d)<\infty$. This proves the corollary.

COROLLARY 25 ([16, Theorem 4.6]). *Let L be a finite-modular AC-lattice. Then $F(L)$ is a standard ideal.*

PROOF. An AC-lattice is a ZI-lattice with $d(z)=1$ for all its cycles $z\neq 0$. Moreover, in an AC-lattice the property of being "finite-modular" is equivalent to (M^*) (cf. [19, Theorem 9.5, p. 42]). The assertion follows now from Corollary 24.

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ON DEGREE OF APPROXIMATION BY EULER (E, q)-MEANS

by

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1.

Let $f(x) \in L[0, 2\pi]$ and be periodic with period 2π outside this range. Let the Fourier series associated with $f(x)$ be given by

$$f(x) \sim \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x). \tag{1.1}$$

As usual, let us write

$$\varphi_x(t) = \frac{1}{2}\{f(x+t)+f(x-t)-2f(x)\}. \tag{1.2}$$

DEFINITION. Let $\sum_{m=0}^{\infty} c_m$ be a given series and let $s_n = \sum_{m=0}^{n} c_m$. We say that the series $\sum_{m=0}^{n} c_m$ is (E, q) or Euler summable to s, if $\frac{1}{(1+q)^n} \sum_{m=0}^{n} \binom{n}{m} q^{n-m} s_m \to s$ as $n \to \infty$.

Several simple properties of (E, q) summability and formal relations between this method and Borel's method have been discussed by HARDY [2]. Euler summability has been investigated by KNOPP [5] and PRACHER and SCHMETTERER [7]. LORCH [6], discussed the Lebesgue constants for $(E, 1)$ summability of Fourier series in 1950, and BOLLINGER [1] studied asymptotic versions of Abelian Theorems for Euler summability in 1971. Several sufficient conditions for Fourier series to be summable by Euler means are known and in 1975, HOLLAND, SAHNEY and TZIMBALARIO [4] discussed a new criterion for Euler $(E, 1)$ summability of Fourier series. In this paper we study the degree of approximation of Fourier series by Euler (E, q)-means.

2.

Let $C^*[0, 2\pi]$ denote the class of all continuous functions on $[0, 2\pi]$, periodic and of period 2π. We prove the following theorem and two corollaries.

THEOREM. *If $w(t)$ is the modulus of continuity of $f \in C^*[0, 2\pi]$, then the degree of approximation of f by the (E, q)-means of the Fourier series for f, is given by*

$$E_n(f) \equiv \max_{0 \leq x \leq 2\pi} |f(x) - T_n(x)| = O\left\{\sum_{k=1}^{n} \frac{w\left(\frac{1}{k}\right)}{k}\right\} \tag{2.1}$$

where $T_n(x)$ are the (E, q)-means of the Fourier series for f.

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COROLLARY I. *If* $w(t)=O(t^\alpha)$, $0<\alpha<1$, *as* $t\to 0^+$ *then we have,*

$$E_n(f) = O\left(\frac{1}{n^\alpha}\right)+C, \quad C \text{ a constant.} \tag{2.2}$$

COROLLARY II. *If* $w(t)=o(t^\alpha)$, $0<\alpha<1$, *as* $t\to 0^+$, *then the* (E, q)*-means of the Fourier series for* f, *converge uniformly to* f, *i.e.*

$$E_n(f) = o(1). \tag{2.3}$$

3. Proof of the Theorem

The partial sums of order m of the Fourier series for f are given by

$$s_m(x) = f(x)+\frac{2}{\pi}\int_0^\pi \frac{\varphi_x(t)}{t}\sin mt\, dt+o(1). \tag{3.1}$$

If we invoke the Riemann—Lebesgue theorem, then we may write

$$s_m = f(x)+\frac{2}{\pi}\int_0^\pi \frac{\varphi_x(t)}{t}\sin(m+1)t\, dt+o(1). \text{ [8]} \tag{3.2}$$

Using the Euler (E, q) transformation, T_n of s_n, we obtain

$$|f(x)-T_n(x)| \leqq \frac{2}{\pi(1+q)^n}\int_0^\pi \frac{|\varphi_x(t)|}{t}\left\{\left|\sum_{m=0}^n \binom{n}{m} q^{n-m}\sin(m+1)t\right|\right\} dt+o(1). \tag{3.3}$$

Now clearly

$$|\varphi_x(t)| \leqq w(t)$$

therefore

$$|f(x)-T_n(x)| \leqq \frac{1}{(1+q)^n}\frac{2}{\pi}\left[\int_0^{\pi/n}+\int_{\pi/n}^\pi\right]\frac{w(t)}{t}\cdot$$

$$\cdot\left|\sum_{k=0}^n \binom{n}{k} q^{n-k}\sin(k+1)t\right| dt+o(1) = \{I_1+I_2\}+o(1),$$

say. Now

$$I_1 = O\left\{\frac{1}{(1+q)^n}\right\}\int_0^{\pi/n} w(t)\sum_{k=0}^n \binom{n}{k} q^{n-k}(k+1)\, dt = \tag{3.4}$$

$$= O\left\{\frac{1}{(1+q)^n}\right\}\{n(1+q)^{n-1}+(1+q)^n\}O\left(\frac{1}{n}\right)O\left\{w\left(\frac{1}{n}\right)\right\} =$$

$$= O\left\{w\left(\frac{1}{n}\right)\right\}.$$

However,

$$w\left(\frac{1}{n}\right) = n\frac{w\left(\frac{1}{n}\right)}{n} = \sum_{k=1}^{n}\frac{w\left(\frac{1}{n}\right)}{n} < \sum_{k=1}^{n}\frac{w\left(\frac{1}{k}\right)}{k},$$

since $w(k)$ is an increasing function, thus

$$I_1 = O\left\{\sum_{k=1}^{n}\frac{w\left(\frac{1}{k}\right)}{k}\right\}.$$

Further,

$$I_2 = O\left\{\frac{1}{(1+q)^n}\right\}\int_{\pi/n}^{\pi}\frac{w(t)}{t}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\sin(k+1)t\right|dt = \tag{3.5}$$

$$= O\left\{\frac{1}{(1+q)^n}\right\}O\left\{\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\int_{\pi/n}^{\pi}\frac{w(t)}{t}\,dt\right\} =$$

$$= O\left\{\int_{\pi/n}^{\pi}\frac{w(t)}{t}\,dt\right\} =$$

$$= O\left\{\int_{1/\pi}^{n/\pi}\frac{w\left(\frac{1}{t}\right)}{t}\,dt\right\}$$

by a change of variable,

$$= O\left\{\sum_{k=1}^{n}\frac{w\left(\frac{1}{k}\right)}{k}\right\}.$$

Consequently $I_1+I_2=O\left\{\sum_{k=1}^{n}\frac{w\left(\frac{1}{k}\right)}{k}\right\}$ and the theorem is proved.

PROOF OF COROLLARY I. Writing $w(t)=O(t^{\alpha})$ we have,

$$I_1+I_2 = O\left\{\sum_{k=1}^{n}\frac{1}{k^{\alpha+1}}\right\} = O\left(\frac{1}{n^{\alpha}}\right)+C, \quad C \text{ a constant.}$$

We note further, that corollary I may be reworded to say:

If $f \in \text{lip } \alpha$, $0 < \alpha < 1$, then the degree of approximation of f by the Euler (E, q)-means of the Fourier series for f, is given by

$$E_n(f) = O\left(\frac{1}{n^\alpha}\right) + C, \quad C \text{ a constant.}$$

PROOF OF COROLLARY II.

$$\text{Since } I_1 = O\left\{\frac{1}{(1+q)^n}\right\} \int_0^{\pi/n} w(t) \sum_{k=0}^{n} \binom{n}{k} (k+1)\, dt =$$

$$= O(n) \int_0^{\pi/n} w(t)\, dt$$

and

$$w(t) = o(t^\alpha)$$

then

$$I_1 = O(n) \int_0^{\pi/n} o(t^\alpha)\, dt =$$

$$= o(1) \quad \text{as} \quad n \to \infty.$$

Also,

$$I_2 = O\left(\frac{1}{(1+q)^n}\right) \int_{\pi/n}^{\pi} \frac{w(t)}{t} \left| \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sin(k+1)t \right| dt =$$

$$= O\left(\frac{1}{(1+q)^n}\right) \int_{\pi/n}^{\pi} (1+q)^n o(t^{\alpha-1})\, dt =$$

$$= o(1) \quad \text{as} \quad n \to \infty.$$

Thus $I_1 + I_2 = o(1)$ and this proves the corollary.

4. Remarks

It is well known that (E, q) summability and Hausdorff means are related, thus it would be interesting to see the above results for Hausdorff means. It would also be interesting to see if the above order for $I_1 + I_2$ could be improved.

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UNTERSUCHUNGEN ÜBER RICHTUNGSSTRUKTUREN, III. ÜBER DIE REICHWEITE DES $\mathfrak{R}$-KOMPAKTIFIZIERUNGSVERFAHRENS

von

E. DEÁK

In diesem Teil der Serie untersuchen wir das Kompaktifizierungsverfahren (1.1), das auf dem Einbettungssatz (0.6) über kompatible ordentliche Richtungsstrukturen beruht, auf die Reichweite seiner Anwendbarkeit: kann jede T_2-Kompaktifizierung mit dieser Methode erhalten werden? Die Antwort lautet (Satz (3.1)), daß dies — auf eine gewisse natürliche Weise — tatsächlich der Fall ist.

Wir stützen uns auf die Begriffe und Bezeichnungen bezüglich Richtungsstrukturen, die im Teil I [1] und im Teil II [2] angeführt worden sind; im übrigen verweisen wir wieder auf den Bericht [3] über die Theorie der Richtungsstrukturen und auf [4].

Im § 0 werden einige weitere Begriffe, Bezeichnungen und Sätze aus dieser Theorie angeführt. Im § 1 wird das besagte Problem erläutert. Im § 2 beweisen wir einige Hilfssätze, die zur Lösung des Problems im § 3 nötig sind.

§ 0. Aus der Theorie der Richtungsstrukturen

(0.1) DEFINITIONEN. Es sei X eine nichtleere Menge.

(a) Eine *Pseudorichtung* bzw. *ordentliche Pseudorichtung der Menge* X ist ein System $\mathscr{R}$ von geordneten Paaren (G, F) mit $G \subseteq F \subseteq X$ und mit

$$(\varnothing, \varnothing),\ (X, X) \in \mathscr{R},$$

$$(G_1, F_1),\ (G_2, F_2) \in \mathscr{R},\quad (G_1, F_1) \neq (G_2, F_2) \Rightarrow F_1 \subseteq G_2 \vee F_2 \subseteq G_1$$

bzw. obendrein

$$\bigcup \{F \setminus G\colon (G, F) \in \mathscr{R}\} = X.$$

(b) Die Familie der ersten bzw. zweiten Komponenten der Elemente von $\mathscr{R}$ wird mit $\mathscr{G}(\mathscr{R})$ bzw. $\mathscr{F}(\mathscr{R})$ bezeichnet.

(c) Ist die Menge X mit einer Topologie versehen, so wird ein solches $\mathscr{R}$ eine *Pseudorichtung* bzw. *ordentliche Pseudorichtung des Raumes* X genannt, wenn sämtliche Elemente von $\mathscr{G}(\mathscr{R})$ offene und sämtliche Elemente von $\mathscr{F}(\mathscr{R})$ abgeschlossene Mengen sind.

(0.2) BEMERKUNGEN. (a) Eine Richtung bzw. ordentliche Richtung einer Menge (eines Raumes) ist offensichtlich eine Pseudorichtung bzw. ordentliche Pseudorichtung $\mathscr{R}$ der Menge (des Raumes) mit

$$\bigcup \{G\colon G \in \mathscr{G}^*\} \in \mathscr{G}(\mathscr{R}) \quad (\varnothing \neq \mathscr{G}^* \subseteq \mathscr{G}(\mathscr{R})),$$

$$\bigcap \{F\colon F \in \mathscr{F}^*\} \in \mathscr{F}(\mathscr{R}) \quad (\varnothing \neq \mathscr{F}^* \subseteq \mathscr{F}(\mathscr{R})).$$

(b) Die Definition der *natürlichen Ordnung* $\prec$ einer Pseudorichtung $\mathscr{R}$ stimmt mit der entsprechenden Definition für Richtungen ([2] (0.1)) überein.

(c) Für diese Ordnung einer Pseudorichtung gilt

$$(G_1, F_1), (G_2, F_2) \in \mathscr{R}, \quad (G_1, F_1) \prec (G_2, F_2) \Rightarrow F_1 \subseteqq G_2.$$

(d) Ohne der „Pseudorichtung" genannten Abschwächung des Begriffs der Richtung müßte man die Hilfssätze im § 2 durch unnötige Voraussetzungen belasten. In der Theorie der RS-en kommt aber dem Begriff der Pseudorichtung auch eine inhaltlich unentbehrliche Rolle zu; man muß nämlich der Tatsache Rechnung tragen, daß bei gewissen, auf Richtungen angewandten Operationen die Vereinigungs- und die Durchschnittseigenschaft unter (a) verlorengehen kann (wie etwa bei dem Beweisverfahren unter (1.1),2° im Artikel [5]).

(0.3) BEMERKUNGEN. Es sei $\mathscr{R}$ eine ordentliche Pseudorichtung einer Menge $X \neq \emptyset$.

(a) Die Symbole $G_x(R)$ und $F_x(\mathscr{R})$ werden in demselben Sinn wie bei Richtungen $\mathscr{R}$ gebraucht (vgl. [2] (0.5),(b)).

(b) Für

$$G \in \mathscr{G}(\mathscr{R}), \quad F \in \mathscr{F}(\mathscr{R}), \quad F \subset G$$

gilt

$$F \subseteqq G_x(\mathscr{R}), \quad F_x(\mathscr{R}) \subseteqq G \quad (x \in G \setminus F).$$

(c) Es gelten

$$x \in G \Leftrightarrow F_x(\mathscr{R}) \subseteqq G \quad (G \in \mathscr{G}(\mathscr{R})),$$

$$x \notin F \Leftrightarrow F \subseteqq G_x(\mathscr{R}) \quad (F \in \mathscr{F}(\mathscr{R})),$$

also insbesondere

$$x \in G_y(\mathscr{R}) \Rightarrow (G_x(\mathscr{R}), F_x(\mathscr{R})) \prec (G_y(\mathscr{R}), F_y(\mathscr{R})),$$

$$x \notin F_y(\mathscr{R}) \Rightarrow (G_x(\mathscr{R}), F_x(\mathscr{R})) \succ (G_y(\mathscr{R}), F_y(\mathscr{R})) \quad (x, y \in X).$$

(0.4) BEMERKUNG. Für eine ordentliche Pseudorichtung eines Raumes X ist $\vartheta: X \to \mathscr{R}$ mit

$$\vartheta(x) = (G_x(\mathscr{R}), F_x(\mathscr{R})) \quad (x \in X)$$

eine stetige Abbildung von X in die Menge $\mathscr{R}$ die wir als mit der Ordnungstopologie bezüglich der natürlichen Ordnung versehen betrachten.

(0.5) DEFINITION. Es seien X ein schwach ordentlicher T_0-Raum, $\mathfrak{R}$ eine KORS von X und $R = \Pi\{\mathscr{R}: \mathscr{R} \in \mathfrak{R}\}$ das topologische Produkt der als ordnungstopologische Räume (bezüglich der natürlichen Ordnung) betrachteten Elemente von $\mathfrak{R}$. (R ist ein kompakter T_2-Raum, weil das für jede Richtung zutrifft.)

Die *kanonische Abbildung* $f: X \to R$ ist jene mit

$$f(x)_{\mathscr{R}} = (G_x(\mathscr{R}), F_x(\mathscr{R})) \quad (x \in X, \mathscr{R} \in \mathfrak{R}).$$

(0.6) EINBETTUNGSSATZ. *Unter den Bedingungen von* (0.5) *ist die kanonische Abbildung eine topologische Einbettung.*

(0.7) BEZEICHNUNGEN. Sind $\emptyset \neq X^* \subseteqq X$, so soll ein * im allgemeinen die Spur auf X^* bedeuten:

$$M^* = M \cap X^* \quad (M \subseteqq X),$$

$$\mathscr{R}^*(= \mathscr{R}|X^*) = \{(G^*, F^*): (G, F) \in \mathscr{R}\},$$

$$\mathfrak{R}^*(= \mathfrak{R}|X^*) = \{\mathscr{R}^*: \mathscr{R} \in \mathfrak{R}\}.$$

(0.8) BEZEICHNUNGEN. Für einen Raum X bedeute

(a) $\mathscr{A}(X)$ die Familie aller offen-abgeschlossenen nichttrivialen Teilmengen von X,

(b) $\mathscr{R}(A; X)$ (oder kurz $\mathscr{R}_A$) die (offensichtlich ordentliche) Richtung

$$\{(\emptyset, \emptyset), (\emptyset, A), (A, X), (X, X)\}$$

von $X (A \in \mathscr{A}(X))$ und

(c) $\mathfrak{R}(\mathscr{A}(X)) = \{\mathscr{R}_A: A \in \mathscr{A}(X)\}$.

(0.9) BEMERKUNG. Für eine KRS $\mathfrak{R}$ eines T_0-Raumes X ist jede Menge

$$\bigcap \{F(\mathscr{R}) \setminus G(\mathscr{R}): \mathscr{R} \in \mathfrak{R}\}$$

mit

$$(G(\mathscr{R}), F(\mathscr{R})) \in \mathscr{R} \quad (\mathscr{R} \in \mathfrak{R})$$

höchstens einelementig.

(0.10) DEFINITION. *Die natürliche Richtung der Zahlengeraden* $\mathbf{R}$ ist das System

$$\{(\emptyset, \emptyset)\} \cup \{((-\infty, t), (-\infty, t]): t \in \mathbf{R}\} \cup \{(\mathbf{R}, \mathbf{R})\}$$

(offensichtlich eine ordentliche Richtung).

§ 1. Die Fragestellung

(1.1) DEFINITION. Es seien Y ein schwach ordentlicher T_0-Raum, $\mathfrak{R}$ eine KORS, $R = \Pi\{\mathscr{R}: \mathscr{R} \in \mathfrak{R}\}$, $f: Y \to R$ die kanonische Einbettung und

$$X = f[Y].^{-(R)}.$$

Der (kompakte T_2-) Raum X, dessen dichter Teilraum $f[Y]$ zu Y homöomorph ist, soll *die* $\mathfrak{R}$-*Kompaktifizierung* von Y genannt werden; in Zeichen

$$c(Y, \mathfrak{R}) = X.$$

(1.2) PROBLEM. (a) Es erhebt sich sofort die Frage, ob zu einem beliebigen schwach ordentlichen T_0-Raum Y und einer beliebig vorgegebenen T_2-Kompaktifizierung X von Y auch tatsächlich eine KORS $\mathfrak{R}$ für Y mit $c(Y; \mathfrak{R}) = X$ existiert.

(b) Es liegt nahe, diese Frage in der folgenden Fassung zu untersuchen:

Es seien X ein kompakter T_2-Raum, X^* ein dichter Teilraum von X, $\mathfrak{R}$ eine KORS von X und $\mathfrak{R}^* = \mathfrak{R}|X^*$. Gilt dann

$$c(X^*; \mathfrak{R}^*) \cong X?$$

Daß dies, bei festgesetzten X und X^* nicht für jede beliebige KORS $\mathfrak{R}$ des Raumes X der Fall zu sein braucht, zeigen wir durch das folgende einfache

(1.3) BEISPIEL. Es seien

$$X = [0, 1] \cup [2, 3], \quad X^* = [0, 1) \cup (2, 3]$$

mit der Spurtopologie bezüglich der Zahlengeraden versehen und $\mathscr{R}$ die „natürliche Richtung" der Menge X (d. h. die Spur der natürlichen Richtung der Zahlengeraden).

(a) Setzen wir

$$\mathfrak{R}^* = \{\mathscr{R}^*\}$$

wobei

$$\mathscr{R}^* = \{(\varnothing, \varnothing)\} \cup \{(G_t, F_t)\colon\ 0 \leqq t < 1\} \cup$$

$$\cup \{([0, 1), [0, 1))\} \cup \{G_t, F_t)\colon\ 2 < t \leqq 3\} \cup \{(X^*, X^*)\}$$

mit

$$G_t = \{x \in X^*\colon\ x < t\}, \quad F_t = \{x \in X^*\colon\ x \leqq t\}$$

für alle in Betracht kommenden Werte von t. ($\mathscr{R}^*$ ist also die Spur von $\mathscr{R}$.)

Hier gilt

$$c(X^*; \mathfrak{R}^*) \cong \mathscr{R}^* \setminus \{(\varnothing, \varnothing), (X^*, X^*)\};$$

von den insgesamt drei Elementen,

$$(\varnothing, \varnothing), ([0, 1), [0, 1)), (X^*, X^*)$$

der Menge $\mathscr{R}^* \setminus f[X^*]$, f die kanonische Einbettung, sind ja $(\varnothing, \varnothing)$ und (X^*, X^*) isolierte Punkte im Raum $\mathscr{R}^*$ (weil nämlich $G_0 = \varnothing$ und $F_3 = X^*$), indessen ist

$$([0, 1), [0, 1)) \in f[X^*]^{-(\mathscr{R}^*)}.$$

Man sieht also, daß $c(X^*; \mathfrak{R}^*)$ zu einem abgeschlossenen Zahlenintervall — und nicht zum Raum X — homöomorph ist: wir haben die Alexandroffsche Kompaktifizierung von X^* erhalten.

(b) Man sieht auch leicht, woran das liegt: in $\mathscr{R}$ gibt es drei benachbarte Elemente,

$$([0, 1), [0, 1]), ([0, 1], [0, 1]), ([0, 1], [0, 1] \cup \{2\}),$$

deren Spur auf X^* zusammenfällt (schon die Existenz von zwei solchen „äquivalenten" Paaren in $\mathscr{R}$ würde $\mathscr{R}^*$ inadequat machen).

(c) In diesem Fall kann dem Übel folgendermaßen abgeholfen werden.

Es seien $A = [0, 1)$ (also auch $A^* = [0, 1)$) und $\mathfrak{R} = \{\mathscr{R}, \mathscr{R}(A; X)\}$ im Sinne von (0.8), $\mathscr{R}$ wieder die natürliche Richtung von X. Dann haben wir $\mathfrak{R}^* = \{R^*, \mathscr{R}(A; X)^*\}$, wobei

$$\mathscr{R}(A; X)^* = \{(\varnothing, \varnothing), (\varnothing, A^*), (A^*, X^*), (X^*, X^*)\},$$

und es bedeute jetzt $R^* = \mathscr{R}^* \times \mathscr{R}(A; X)^*$. Dann gilt

$$c(X^*; \mathfrak{R}^*) = \{(([0, t), [0, t]), (\varnothing, A^*))\colon\ 0 \leqq t \leqq 1\} \cup$$
$$\cup \{((A^*, A^*), (\varnothing, A^*))\} \cup \{((A^*, A^*), (A^*, X^*))\} \cup$$
$$\cup \{(([0, t), [0, t]), (A^*, X))\colon\ 2 \leqq t \leqq 3\}$$

(bei der Abschließung von $f[X^*]$ in R^*, f die kanonische Abbildung, kommen genau zwei Elemente, $((A^*, A^*), (\varnothing, A^*))$ und $((A^*, A^*), (A^*, X^*))$ hinzu), und man sieht leicht, daß jetzt $c(X^*; \mathfrak{R}^*) \cong X$.

Die Frage unter (1.2),(b) sollte demnach offensichtlich so gefaßt werden:

(1.4) PROBLEM. *Es seien X ein kompakter T_2-Raum, X^* ein dichter Teilraum von X und $\mathfrak{R}$ eine KORS des Raumes X mit*

$$\mathscr{R}(A; X) \in \mathfrak{R} \quad (A \in \mathscr{A}(X)).$$

Gilt dann bestimmt

$$c(X^*; \mathfrak{R}^*) \cong X?$$

Unser Ergebnis wird die bejahende Antwort auf diese Frage sein (Satz (3.1)).

§ 2. Hilfssätze

(2.1) BEMERKUNG. Es seien $\mathscr{R}_1, \mathscr{R}_2$ ordentliche Pseudorichtungen einer Menge $X \neq \varnothing$ und $x_1, x_2 \in X$.

$$x_2 \in G_{x_1}(\mathscr{R}_1) \setminus F_{x_1}(\mathscr{R}_2)$$

gilt dann und nur dann, wenn

$$x_1 \in G_{x_2}(\mathscr{R}_2) \setminus F_{x_2}(\mathscr{R}_1).$$

(Auf Grund von (0.3),(c) sind nämlich $x_2 \in G_{x_1}(\mathscr{R}_1)$ mit $F_{x_2}(\mathscr{R}_1) \subseteqq G_{x_1}(\mathscr{R}_1)$ und dies mit $x_1 \notin F_{x_2}(\mathscr{R}_1)$ äquivalent, und das genügt schon wegen der Symmetrie der Behauptung.)

(2.2) BEMERKUNGEN. Es seien $X \neq \varnothing$ eine Menge, $\varnothing \neq X^* \subseteqq X$, $\mathscr{R}$ eine Pseudorichtung von X und $(G_1, F_1), (G_2, F_2) \in \mathscr{R}$.

(a) Ist $(G_1^*, F_1^*) \prec (G_2^*, F_2^*)$ in der natürlichen Ordnung von $\mathscr{R}^*$, so auch $(G_1, F_1) \prec (G_2, F_2)$ in der natürlichen Ordnung von $\mathscr{R}$.

Nach (0.2),(c) gilt nämlich $G_1^* \subset F_2^*$, also $G_1 \subset F_2$, und da offensichtlich $(G_1, F_1) \neq (G_2, F_2)$ ist, folgt die Behauptung auf Grund von (0.2),(b).

(b) Aus $(G_1, F_1) \prec (G_2, F_2)$ folgt also $(G_1^*, F_1^*) \preceqq (G_2^*, F_2^*)$, aber die Umkehrung der Aussage unter (a) gilt nicht allgemein: es kann auch

$$(G_1, F_1) \prec (G_2, F_2) \tag{2.2.1}$$

und gleichzeitig

$$(G_1^*, F_1^*) = (G_2^*, F_2^*) \tag{2.2.2}$$

sein; in jedem solchen Fall haben wir wegen (2.2.1) $G_1 \subseteqq F_1 \subseteqq G_2 \subseteqq F_2$, also $G_1^* \subseteqq F_1^* \subseteqq G_2^* \subseteqq F_2^*$ und daher nach (2.2.2)

$$G_1^* = F_1^* = G_2^* = F_2^*. \tag{2.2.3}$$

(c) Aus $(G_1, F_1) \prec (G_2, F_2)$ folgt daher $(G_1^*, F_1^*) \prec (G_2^*, F_2^*)$ dann (und offensichtlich nur dann), wenn

$$(F_2 \setminus G_1) \cap X^* \neq \varnothing,$$

oder, anders ausgedrückt, wenn eine der Mengen $F_1 \setminus G_1$, $G_2 \setminus F_1$ und $F_2 \setminus G_1$ die Menge X^* trifft (was (2.2.3) widerspricht).

(2.3) BEMERKUNGEN. Es seien X ein Raum, X^* eine dichte Teilmenge von X und $\mathscr{R}$ eine Pseudorichtung des Raumes.

(a) Sind $(G_1, F_1), (G_2, F_2) \in \mathscr{R}$ mit $F_1 \subset G_2$, so ist $(G_1^*, F_1^*) \prec (G_2^*, F_2^*)$ in der natürlichen Ordnung von $\mathscr{R}^*$.

($F_1 \subset G_2$ ist nämlich mit $F_1^* \subset G_2^*$ äquivalent.)

(b) Sind $(G_1, F_1), (G_2, F_2) \in \mathscr{R}$ und

$$(G_1, F_1) \prec (G_2, F_2), \ (G_1^*, F_1^*) = (G_2^*, F_2^*),$$

so ist

$$F_1 = G_2.$$

(Das gilt wegen $F_1 \subseteqq G_2$ nach (2.2),(a)).

(c) Sind $(G_1, F_1), (G_2, F_2), (G_3, F_3) \in \mathscr{R}$ und

$$(G_1, F_1) \prec (G_3, F_3) \prec (G_2, F_2),$$

$$(G_1^*, F_1^*) = (G_3^*, F_3^*) = (G_2^*, F_2^*),$$

so ist wegen $F_1 \subseteqq G_3 \subseteqq F_3 \subseteqq G_2$ und nach (b)

$$F_1 = G_3 = F_3 = G_2.$$

(d) Für ein $(G, F) \in \mathscr{R}$ gibt es höchstens drei verschiedene $(\Gamma, \Phi) \in \mathscr{R}$ mit $(\Gamma^*, \Phi^*) = (G^*, F^*)$.

Sind nämlich $(G_1, F_1), (G_2, F_2), (G_3, F_3), (G_4, F_4) \in \mathscr{R}$ und etwa

$$(G_1, F_1) \prec (G_2, F_2) \prec (G_3, F_3) \prec (G_4, F_4),$$

$$(G_1^*, F_1^*) = (G_2^*, F_2^*) = (G_3^*, F_3^*) = (G_4^*, F_4^*),$$

so gilt nach (c)

$$(F_1 =) G_2 = F_2 = G_3 = F_3 (= G_4)$$

und daher

$$(G_2, F_2) = (G_3, F_3).$$

(e) Wir werden uns im folgenden auf eine Abschwächung der Aussage unter (d) berufen: für $(G, F) \in \mathscr{R}$ gibt es unter den Elementen (Γ, Φ) von $\mathscr{R}$ mit $(\Gamma^*, \Phi^*) = (G^*, F^*)$ ein erstes und ein letztes (in der natürlichen Ordnung von $\mathscr{R}$).

(2.4) HILFSSATZ. *Es seien X ein Raum, X^* eine dichte Teilmenge von X und $\mathscr{R}$ eine Pseudorichtung des Raumes. Dann ist — $\mathscr{R}$ und $\mathscr{R}^*$ als ordnungstopologische Räume betrachtet — die Abbildung $\varrho: \mathscr{R} \to \mathscr{R}^*$ mit*

$$\varrho((G, F)) = (G^*, F^*) \quad ((G, F) \in \mathscr{R})$$

stetig.

BEWEIS. 1° Es seien $(G, F) \in \mathscr{R}$ und — im Sinne von (2.3),(e) — (G_1, F_1) bzw. (G_2, F_2) das erste bzw. letzte Element (Γ, Φ) von $\mathscr{R}$ mit $(\Gamma^*, \Phi^*) = (G^*, F^*)$.

2° Es gilt

$$\varrho^{-1}[\{(\Gamma^*, \Phi^*) \in \mathscr{R}^*: (\Gamma^*, \Phi^*) \prec (G^*, F^*)\}] =$$
$$= \{(\Gamma, \Phi) \in \mathscr{R}: (\Gamma, \Phi) \prec (G_1, F_1)\}.$$

Für $(\Gamma, \Phi)\in\mathscr{R}$ mit $(\Gamma^*, \Phi^*)\prec(G^*, F^*)$ ist nämlich nach (2.2),(a) und 1° $(\Gamma, \Phi)\prec(G_1, F_1)$; ist anderseits $(\Gamma, \Phi)\prec(G_1, F_1)$, so haben wir $(\Gamma^*, \Phi^*)\neq(G^*, F^*)$ und — da nach (2.2),(a) nicht $(\Gamma^*, \Phi^*)\prec(G^*, F^*)$ sein kann — $(\Gamma^*, \Phi^*)\prec(G^*, F^*)$.

3° Ähnlich gilt auch

$$\varrho^{-1}[\{(\Gamma^*, \Phi^*)\in\mathscr{R}^*\colon (\Gamma^*, \Phi^*)\succ(G^*, F^*)\}] =$$
$$= \{(\Gamma, \Phi)\in\mathscr{R}\colon (\Gamma, \Phi)\succ(G_2, F_2)\}.$$

Für $(\Gamma, \Phi)\in\mathscr{R}$ mit $(\Gamma^*, \Phi^*)\succ(G^*, F^*)$ ist nämlich nach (2.2),(a) und 1° $(\Gamma, \Phi)\succ(G_2, F_2)$; ist anderseits $(\Gamma, \Phi)\succ(G_2, F_2)$, so haben wir $(\Gamma^*, \Phi^*)\neq(G^*, F^*)$ und — da nach (2.2), (a) nicht $(\Gamma^*, \Phi^*)\prec(G^*, F^*)$ sein kann — $(\Gamma^*, \Phi^*)\succ(G^*, F^*)$.

4° Da nach 2° und 3° die Urbilder bezüglich ϱ der Mengen aus einer offenen Subbasis der Topologie von $\mathscr{R}^*$ offene Mengen in $\mathscr{R}$ sind, ist die Abbildung ϱ tatsächlich stetig.

(2.5) BEMERKUNGEN. Es seien $X\neq\emptyset$ eine Menge und $\emptyset\neq X^*\subseteq X$.

(a) Für eine beliebige Pseudorichtung $\mathscr{R}$ von X und für $x\in X^*$ sind

$$G_x(\mathscr{R})^* = X^*\cap\big(\bigcup\{G\in\mathscr{G}(\mathscr{R})\colon x\notin G\}\big) =$$
$$= \bigcup G^*\in\mathscr{G}(\mathscr{R}^*)\colon x\notin G^*\} =$$
$$= G_x(\mathscr{R}^*),$$
$$F_x(\mathscr{R})^* = X^*\cap\big(\bigcap\{F\in\mathscr{F}(\mathscr{R})\colon x\in F\}\big) =$$
$$= \bigcap\{F^*\in\mathscr{F}(\mathscr{R}^*)\colon x\in F^*\} =$$
$$= F_x(\mathscr{R}^*).$$

(b) Die äquivalente Fassung

$$(2.5.1)\qquad \big(G_x(\mathscr{R})^*, F_x(\mathscr{R})^*\big) = \big(G_x(\mathscr{R}^*), F_x(\mathscr{R}^*)\big)\quad (x\in X^*)$$

der vorigen Tatsache ist zweckmäßiger, wenn die Pseudorichtung $\mathscr{R}^*$ (also, insbesondere, wenn sogar $\mathscr{R}$) ordentlich ist und daher $\big(G_x(\mathscr{R}^*), F_x(\mathscr{R}^*)\big)\in\mathscr{R}^*$ $(x\in X^*)$ gilt.

(c) Hieraus folgt, da $\big(G_x(\mathscr{R}^*), F_x(\mathscr{R}^*)\big)$ eindeutig zu x gehört, daß für ordentliche Pseudorichtungen $\mathscr{R}_1$ und $\mathscr{R}_2$ mit $\mathscr{R}_1^*=\mathscr{R}_2^*$,

$$(2.5.2)\qquad \big(G_x(\mathscr{R}_1)^*, F_x(\mathscr{R}_1)^*\big) = \big(G_x(\mathscr{R}_2)^*, F_x(\mathscr{R}_2)^*\big)\quad (x\in X^*).$$

(2.6) BEMERKUNGEN. *Es seien $X\neq\emptyset$ eine Menge, $\emptyset\neq X^*\subseteq X$ und $\mathscr{R}_1, \mathscr{R}_2$ ordentliche Pseudorichtungen mit*

$$(2.6.1)\qquad \mathscr{R}_1^* = \mathscr{R}_2^*$$

(a) *Es gilt*

$$X^*\cap\big(G_{x_1}(\mathscr{R}_1)\setminus F_{x_1}(\mathscr{R}_2)\big) = \emptyset\quad (x_1\in X^*).$$

BEWEIS. Ist $x_1\in X^*$, und gäbe es wider die Behauptung einen Punkt

$$(2.6.2)\qquad x_2\in X^*\cap\big(G_{x_1}(\mathscr{R}_1)\setminus F_{x_1}(\mathscr{R}_2)\big),$$

so wären nach (0.3),(c) und (2.6.2)

$$F_{x_2}(\mathscr{R}_1)^* \subseteqq G_{x_1}(\mathscr{R}_1)^*, \tag{2.6.3}$$

$$F_{x_1}(\mathscr{R}_2)^* \subseteqq G_{x_2}(\mathscr{R}_2)^*. \tag{2.6.4}$$

Da nun aber nach (2.5),(c), wegen (2.6.1),

$$G_{x_1}(\mathscr{R}_1)^* = G_{x_1}(\mathscr{R}_2)^*,$$

$$G_{x_2}(\mathscr{R}_2)^* = G_{x_2}(\mathscr{R}_1)^*$$

gilt, folgt, wegen (0.3),(c) und (2.5),(b), aus (2.6.3)

$$F_{x_2}(\mathscr{R}_1)^* \subset F_{x_1}(\mathscr{R}_2)^*$$

und aus (2.6.4)

$$F_{x_1}(\mathscr{R}_2^*) \subset F_{x_2}(\mathscr{R}_1)^*;$$

mit diesem Widerspruch ist die Behauptung bewiesen.

(b) Zur Klärung der vorigen Bemerkung sei noch (ohne Beweis) erwähnt, daß es sich eigentlich um folgendes handelt:

Für zwei Punkte $x_1, x_2 \in X^*$ gilt

$$\left(G_{x_1}(\mathscr{R}_1), F_{x_1}(\mathscr{R}_1)\right) \prec \left(G_{x_2}(\mathscr{R}_1), F_{x_2}(\mathscr{R}_1)\right)$$

dann und nur dann, wenn

$$\left(G_{x_1}(\mathscr{R}_2), F_{x_1}(\mathscr{R}_2)\right) \prec \left(G_{x_2}(\mathscr{R}_2), F_{x_2}(\mathscr{R}_2)\right).$$

Wahrhaftig ist das mit der Tatsache unter (a) äquivalent (und es kann auch unmittelbar, auf Grund von (2.5),(c) und (2.2),(c), sehr einfach bewiesen werden).

(2.7) BEMERKUNG. Es seien $X \neq \varnothing$ eine Menge und $\varnothing \neq X^* \subseteqq X$.

Für $x \in X \setminus X^*$ ist weder die rechte Seite der Gleichung unter (2.5.1) sinnvoll noch die Gleichung unter (2.5.2) unbeschränkt gültig. Das Letztere ersieht man etwa an folgendem Beispiel. (Es ist die einfachste überhaupt mögliche Illustration dieser negativen Tatsache.)

Es seien

$$X = \{x, x^*\}, \quad X^* = \{x^*\}, \quad x \neq x^*,$$

$$\mathscr{R}_1 = \{(\varnothing, \varnothing), (\varnothing, \{x\}), (\{x\}, X), (X, X)\},$$

$$\mathscr{R}_2 = \{(\varnothing, \varnothing), (\varnothing, \{x^*\}), (\{x^*\}, X), (X, X)\}.$$

(Allerdings ist hier $\mathscr{R}_2$ der „Dual" von $\mathscr{R}_1$, d. h. $\mathscr{R}_2 = \{(x \setminus F, x \setminus G) : (G, F) \in \mathscr{R}\}$, aber nicht das ist es, worauf es ankommt.) Dann gilt

$$\mathscr{R}_1^* = \mathscr{R}_2^* \left(= \{(\varnothing, \varnothing), (\varnothing, X^*), (X^*, X^*)\}\right)$$

aber

$$\left(G_x(\mathscr{R}_1)^*, F_x(\mathscr{R}_1)^*\right) = (\varnothing, \varnothing) \neq$$

$$\neq (X^*, X^*) = \left(G_x(\mathscr{R}_2)^*, F_x(\mathscr{R}_2)^*\right).$$

Der folgende Hilfssatz wird nun aber zeigen, daß unter einer Bedingung, die durchaus dem hiesigen Zweck angepaßt ist, diese Diskrepanz nicht auftreten kann.

Die Bedingung lautet, in einer äquivalenten Fassung, so: die Teilmenge X^* von X soll dicht sein, u. zw. bezüglich der Topologie, die durch die beiden ordentlichen Pseudorichtungen $\mathscr{R}_1$ und $\mathscr{R}_2$ auf X induziert wird. (Im vorigen Beispiel, wo es sich um die diskrete Topologie auf X handelt, ist diese Bedingung offensichtlich nicht erfüllt.)

(2.8) HILFSSATZ. *Es seien X ein Raum, X^* eine dichte Teilmenge von X, $\mathscr{R}_1$ und $\mathscr{R}_2$ ordentliche Pseudorichtungen des Raumes mit*

$$\mathscr{R}_1^* = \mathscr{R}_2^*. \tag{2.8.1}$$

Dann gilt

$$\big(G_x(\mathscr{R}_1)^*, F_x(\mathscr{R}_1)^*\big) = \big(G_x(\mathscr{R}_2)^*, F_x(\mathscr{R}_2)^*\big) \quad (x\in X). \tag{2.8.2}$$

BEWEIS. 1° Nehmen wir an, daß (2.8.2) nicht gilt, daß also für ein $x\in X$ (genaugenommen: $x\in X\setminus X^*$) etwa

$$\big(G_x(\mathscr{R}_1)^*, F_x(\mathscr{R}_1)^*\big) \prec \big(G_x(\mathscr{R}_2)^*, F_x(\mathscr{R}_2)^*\big) \tag{2.8.3}$$

in der natürlichen Ordnung von $\mathscr{R}_1^*$ $(=\mathscr{R}_2^*)$.

2° Wegen (2.8.1) gibt es dann Paare

$$(G_1, F_1)\in\mathscr{R}_1, \quad (G_2, F_2)\in\mathscr{R}_2$$

mit

$$(G_1^*, F_1^*) = \big(G_x(\mathscr{R}_2)^*, F_x(\mathscr{R}_2)^*\big), \tag{2.8.4}$$

$$(G_2^*, F_2^*) = \big(G_x(\mathscr{R}_1)^*, F_x(\mathscr{R}_1)^*\big),$$

wobei nach (2.8.3) und (2.2), (a)

$$(G_1, F_1) \succ \big(G_x(\mathscr{R}_1), F_x(\mathscr{R}_1)\big), \tag{2.8.5}$$

$$(G_2, F_2) \prec \big(G_x(\mathscr{R}_2), F_x(\mathscr{R}_2)\big)$$

in der natürlichen Ordnung von $\mathscr{R}_1$ bzw. $\mathscr{R}_2$.

Nun folgt einerseits aus (2.8.4)

$$[G_1\setminus G_x(\mathscr{R}_2)]^* = \varnothing, \quad [F_x(\mathscr{R}_1)\setminus F_2]^* = \varnothing, \tag{2.8.6}$$

andererseits aus (2.8.5)

$$x\in G_1\setminus F_2$$

und daher, weil X^* dicht in X ist,

$$(G_1\setminus F_2)^* \neq \varnothing; \tag{2.8.7}$$

da aber (man beachte, daß wegen (2.8.5) $F_x(\mathscr{R}_1)\subseteq G_1$ und $F_2\subseteq G_x(\mathscr{R}_2)$ sind)

$$G_1\setminus F_2 = [G_1\setminus G_x(\mathscr{R}_2)]\cup[F_x(\mathscr{R}_1)\setminus F_2]\cup$$

$$\cup\big(G_1\setminus F_x(\mathscr{R}_1)\big)\cap\big(G_x(\mathscr{R}_2)\setminus F_2\big),$$

muß wegen (2.8.7) und (2.8.6)

$$[(G_1 \setminus F_x(\mathscr{R}_1)) \cap (G_x(\mathscr{R}_1) \setminus F_2)]^* \neq \varnothing$$

sein.

3° Nehmen wir also

$$x_1 \in X^* \cap (G_1 \setminus F_x(\mathscr{R}_1)) \cap (G_x(\mathscr{R}_2) \setminus F_2),$$

sonst beliebig, und es genügt zu beachten, daß

$$x_1 \in X^* \cap (G_x(\mathscr{R}_2) \setminus F_x(\mathscr{R}_1)) \tag{2.8.8}$$

ist, was nämlich zu einem Widerspruch führt:

Tatsächlich folgt aus (2.8.8) auf Grund der Bemerkung (2.1)

$$x \in G_{x_1}(\mathscr{R}_1) \setminus F_{x_1}(\mathscr{R}_2),$$

und da X^* in X dicht ist, müßte

$$X^* \cap (G_{x_1}(\mathscr{R}_1) \setminus F_{x_1}(\mathscr{R}_2)) \neq \varnothing$$

sein, was jedoch (2.6),(a) widerspricht.

Zur weiteren Klärung dieser Sache machen wir noch die folgende

(2.9) BEMERKUNG. Unter den Bedingungen des vorigen Hilfssatzes gilt

$$(G_x(\mathscr{R}_1), F_x(\mathscr{R}_1)) = (G_x(\mathscr{R}_2), F_x(\mathscr{R}_2)) \quad (x \in X^*). \tag{2.9.1}$$

Ist nämlich $x \in X^*$ und wäre etwa $G_x(\mathscr{R}_1) \neq G_x(\mathscr{R}_2)$, gäbe es also ein

$$x_1 \in G_x(\mathscr{R}_1) \setminus G_x(\mathscr{R}_2)$$

(der Fall $G_x(\mathscr{R}_2) \setminus G_x(\mathscr{R}_1) \neq \varnothing$ kann analog behandelt werden), so müßten nach (0.3),(c) bzw. nach (0.2),(c) und (0.3),(c)

$$(G_{x_1}(\mathscr{R}_1), F_{x_1}(\mathscr{R}_1)) \prec (G_x(\mathscr{R}_1), F_x(\mathscr{R}_1)),$$

$$(G_x(\mathscr{R}_2), F_x(\mathscr{R}_2)) \leqq (G_{x_1}(\mathscr{R}_2), F_{x_1}(\mathscr{R}_2))$$

in der natürlichen Ordnung von $\mathscr{R}_1$ bzw. $\mathscr{R}_2$, also nach (2.2),(b)

$$(G_{x_1}(\mathscr{R}_1)^*, F_{x_1}(\mathscr{R}_1)^*) \prec (G_x(\mathscr{R}_1)^*, F_x(\mathscr{R}_1)^*),$$

$$(G_x(\mathscr{R}_2)^*, F_x(\mathscr{R}_2)^*) \leqq (G_{x_1}(\mathscr{R}_2)^*, F_{x_1}(\mathscr{R}_2)^*)$$

in der natürlichen Ordnung von $\mathscr{R}_1^* = \mathscr{R}_2^*$ sein, was aber unmöglich ist, weil ja nach dem Hilfssatz (2.8)

$$(G_x(\mathscr{R}_1)^*, F_x(\mathscr{R}_1)^*) = (G_x(\mathscr{R}_2)^*, F_x(\mathscr{R}_2)^*),$$

$$(G_{x_1}(\mathscr{R}_1)^*, F_{x_1}(\mathscr{R}_1)^*) = (G_{x_1}(\mathscr{R}_2)^*, F_{x_1}(\mathscr{R}_2)^*)$$

(die erste Gleichheit wurde schon unter (2.5),(c) festgestellt). Es ist also $G_x(\mathscr{R}_1) = G_x(\mathscr{R}_2)$ und ganz ähnlich erhält man auch $F_x(\mathscr{R}_1) = F_x(\mathscr{R}_2)$, womit (2.9.1) bewiesen ist.

Diese Tatsache kann auch so ausgedrückt werden: Für einen Raum X, eine dichte Teilmenge X^* und ordentliche Pseudorichtungen $\mathscr{R}_1, \mathscr{R}_2$ auf X mit $\mathscr{R}_1^* = \mathscr{R}_2^*$ gilt

$$\{(G, F) \in \mathscr{R}_1 : (F \setminus G)^* \neq \varnothing\} = \{(G, F) \in \mathscr{R}_2 : (F \setminus G)^* \neq \varnothing\},$$

d. h. die beiden Pseudorichtungen $\mathscr{R}_1$ und $\mathscr{R}_2$ haben „viele" gemeinsame Elemente. (Daß überhaupt $\mathscr{R}_1 \neq \mathscr{R}_2$ sein kann, könnte anhand sehr einfacher Beispiele gezeigt werden.)

Wir wollen diese Frage nicht weiter verfolgen, da sie für unser hiesiges Ziel belanglos ist; unumgänglich war nur, die Tatsache unter (2.8) festzustellen.

§ 3. Die Lösung des Problems

(3.1) SATZ. *Für eine beliebige dichte Teilmenge X^* eines kompakten T_2-Raumes X und eine beliebige KORS $\mathfrak{R}$ von X mit $\mathfrak{R}(\mathscr{A}(X)) \subseteq \mathfrak{R}$ ist X mit $c(X^*; \mathfrak{R}^*)$, $\mathfrak{R}^* = \mathfrak{R}|X^*$, homöomorph.*

BEWEIS. 1° Es sei $\psi : X \to R^*$ jene Abbildung, die durch

$$\psi(x)_{\mathscr{R}^*} = \big(G_x(\mathscr{R})^*, F_x(\mathscr{R})^*\big) \quad (x \in X, \mathscr{R} \in \mathfrak{R}) \tag{3.1.1}$$

bestimmt ist; $\psi(x)_{\mathscr{R}^*}$ bedeutet die $\mathscr{R}^*$-Koordinate von $\psi(x)$ im Produktraum R^*. Es versteht sich zwar nicht von selbst, daß diese Definition sinnvoll ist, weil ja ein $\mathscr{R}^* \in \mathfrak{R}^*$ die Spur von mehreren verschiedenen $\mathscr{R} \in \mathfrak{R}$ sein kann; man sieht jedoch auf Grund von (2.8), daß dies nur eine scheinbare Schwierigkeit ist: für ein gegebenes $\mathscr{R}^*$ ist das $\mathscr{R}$ unter (3.1.1) als eines der Elemente von $\mathfrak{R}$, deren Spur auf X^* das vorgegebene $\mathscr{R}^*$ ist, zu betrachten.

Da eine eineindeutige und stetige Abbildung eines kompakten Raumes auf einen T_2-Raum ein Homöomorphismus ist, genügt es nun zu zeigen, daß ψ eineindeutig und stetig ist, und daß $\psi[X] = c(X^*; \mathfrak{R}^*)$.

2° Sind

$$x_1, x_2 \in X, \quad x_1 \neq x_2,$$

so gibt es nach (0.9) ein $\mathscr{R} \in \mathfrak{R}$ mit $\big(G_{x_1}(\mathscr{R}), F_{x_1}(\mathscr{R})\big) \neq \big(G_{x_2}(\mathscr{R}), F_{x_2}(\mathscr{R})\big)$, etwa

$$\big(G_{x_1}(\mathscr{R}), F_{x_1}(\mathscr{R})\big) \prec \big(G_{x_2}(\mathscr{R}), F_{x_2}(\mathscr{R})\big).$$

Ist nun

$$\big(G_{x_1}(\mathscr{R})^*, F_{x_1}(\mathscr{R})^*\big) \neq \big(G_{x_2}(\mathscr{R})^*, F_{x_2}(\mathscr{R})^*\big),$$

also $\psi(x_1)_{\mathscr{R}^*} \neq \psi(x_2)_{\mathscr{R}^*}$, so ist deswegen $\psi(x_1) \neq \psi(x_2)$.

Wenn aber

$$\big(G_{x_1}(\mathscr{R})^*, F_{x_1}(\mathscr{R})^*\big) = \big(G_{x_2}(\mathscr{R})^*, F_{x_2}(\mathscr{R})^*\big)$$

ist, so muß nach (2.3),(b)

$$F_{x_1}(\mathscr{R}) = G_{x_2}(\mathscr{R})$$

sein, und da $x_1 \in F_{x_1}(\mathscr{R})$, $x_2 \notin G_{x_2}(\mathscr{R})$ sind, gehört die Menge $A = F_{x_1}(\mathscr{R})$ zur Familie $\mathscr{A}(X)$. Nach (0.8) ist daher

$$\big(G_{x_1}(\mathscr{R}(A; X)^*),\ F_{x_1}(\mathscr{R}(A; X)^*)\big) = (\varnothing, A^*) \neq$$

$$\neq (A^*, X^*) = \big(G_{x_2}(\mathscr{R}(A; X)^*),\ F_{x_2}(\mathscr{R}(A; X)^*)\big),$$

und wir haben also nach (3.1.1) wieder $\psi(x_1) \neq \psi(x_2)$.

Damit ist die Eindeutigkeit der Abbildung ψ bewiesen.

3° Um die Stetigkeit von ψ zu beweisen, genügt es zu zeigen, daß für jeden Koordinatenraum $\mathscr{R}^* \in \mathfrak{R}^*$ des Produkts R^* die Komposition

$$\pi_{\mathscr{R}^*} \circ \psi \colon\ X \to \mathscr{R}^*$$

von ψ mit der Projektion $\pi_{\mathscr{R}^*} \colon R^* \to \mathscr{R}^*$ stetig ist.

Für ein $\mathscr{R}^* \in \mathfrak{R}^*$ ist nun aber

$$\pi_{\mathscr{R}^*} \circ \psi(x) = \big(G_x(\mathscr{R})^*,\ F_x(\mathscr{R})^*\big) \quad (x \in X),$$

wobei $\mathscr{R}$ wieder eine der Richtungen aus $\mathfrak{R}$ bedeutet, deren Spur auf X^* das vorgegebene $\mathscr{R}^*$ ist, und wir haben

$$\pi_{\mathscr{R}^*} \circ \psi = \varrho \circ \vartheta,$$

mit der Bezeichnung ϑ für die Abbildung $X \to \mathscr{R}$, die durch

$$\vartheta(x) = \big(G_x(\mathscr{R}),\ F_x(\mathscr{R})\big) \quad (x \in X)$$

bestimmt ist, und der Bezeichnung ϱ für die Abbildung $\mathscr{R} \to \mathscr{R}^*$ mit

$$\varrho\big((G, F)\big) = (G^*, F^*) \quad \big((G, F) \in \mathscr{R}\big).$$

Nach (0.4) bzw. (2.4) sind aber die Abbildungen ϑ und ϱ stetig, und dasselbe gilt dann auch tatsächlich für ihre Zusammensetzung.

4° Aus dem bisherigen folgt, daß ψ ein Homöomorphismus von X auf $\psi[X]$ ist. Es bleibt zu zeigen, daß $\psi[X] = c(X^*; \mathfrak{R})$ ist.

Auf Grund von 1° und (2.5), (b) gilt, mit der Bezeichnung f^* für die kanonische Abbildung $X^* \to R^*$,

$$\psi(x)_{\mathscr{R}^*} = f^*(x)_{\mathscr{R}^*} \quad (\mathscr{R} \in \mathfrak{R},\ x \in X^*),$$

also $\psi(x) = f^*(x)$ $(x \in X^*)$, und daher ist

$$\psi[X^*] = f^*[X^*]. \tag{3.1.2}$$

Aus der Stetigkeit von ψ folgt nun einerseits, daß die Menge $\psi[X^*]$ dicht in $\psi[X]$ ist, und andererseits, daß $\psi[X]$ als Teilraum von R^* kompakt, also als Teilmenge des T_2-Raumes R^* abgeschlossen ist. Es gilt daher $\psi[X] = \psi[X^*]^{-(R^*)}$, und wegen (3.1.2) bedeutet das eben, daß $\psi[X] = c(X^*; \mathfrak{R}^*)$.

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FORMULAS FOR ORTHOMODULAR LATTICES

by

L. BERAN

Abstract

In this paper, a recurrence formula for commutators in n elements and a formula generalizing the distributive identities is given, for any orthmodular lattice.

§ 1

Throughout the paper $\mathscr{L}$ will be an orthomodular lattice; the unexplained notation is as in [1]. By $\overline{\mathrm{com}}\,(x_1, x_2, \ldots, x_n)$ we denote the upper commutator,

$$\overline{\mathrm{com}}\,(x_1, x_2, \ldots, x_n) = \bigwedge (x_1^{\varepsilon_1} \vee x_2^{\varepsilon_2} \vee \ldots \vee x_n^{\varepsilon_n})$$

where the meet runs over all n-tuples $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ such that $\varepsilon_i \in \{-1, 1\}$ for every $i = 1, 2, \ldots, n$ and $x_i^1 = x_i$, $x_i^{-1} = x_i^{\perp}$. The lower commutator $\underline{\mathrm{com}}\,(x_1, x_2, \ldots, x_n)$ is defined dually.

§ 2

The problem of finding a recurrence formula for the commutators, which we will now solve, was posed in [3].

THEOREM 1. *Let $\mathscr{L}$ be a lattice, $x_1, x_2, \ldots, x_n \in L$, $n \geqq 2$. Then*

$$\overline{\mathrm{com}}\,(x_1, x_2, \ldots, x_n) = \overline{\mathrm{com}}\,(x_1, x_2, \ldots, x_{n-1}) \vee$$

$$\vee \bigvee \overline{\mathrm{com}}\,(x_1^{\varepsilon_1} \vee x_2^{\varepsilon_2} \vee \ldots \vee x_{n-1}^{\varepsilon_{n-1}}, x_n)$$

where the join is over all $(n-1)$-tuples $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1})$, $\varepsilon_i = \pm 1$.

PROOF. We note that

$$\overline{\mathrm{com}}\,(x_1, x_2, \ldots, x_n) \geqq \overline{\mathrm{com}}\,(x_1, x_2, \ldots, x_{n-1}),$$

$$\overline{\mathrm{com}}\,(x_1, x_2, \ldots, x_n) = \overline{\mathrm{com}}\,(x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, \ldots, x_n^{\varepsilon_n})$$

and that

$$\overline{\mathrm{com}}\,(x_1, x_2, x_3, \ldots, x_n) \geqq \overline{\mathrm{com}}\,(x_1 \vee x_2, x_3, \ldots, x_n).$$

Hence, it becomes evident that

$$s = \overline{\mathrm{com}}\,(x_1, x_2, \ldots, x_n) \geqq t^{\perp} = \overline{\mathrm{com}}\,(x_1, x_2, \ldots, x_{n-1}) \vee$$

$$\vee \bigvee \overline{\mathrm{com}}\,(x_1^{\varepsilon_1} \vee x_2^{\varepsilon_2} \vee \ldots \vee x_{n-1}^{\varepsilon_{n-1}}, x_n).$$

Now, let $W(x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, \ldots, x_{n-1}^{\varepsilon_{n-1}})$ denote the element

$$(x_1^{-\varepsilon_1} \vee x_2^{-\varepsilon_2} \vee \ldots \vee x_{n-1}^{-\varepsilon_{n-1}} \vee x_n) \wedge (x_1^{-\varepsilon_1} \vee x_2^{-\varepsilon_2} \vee \ldots \vee x_{n-1}^{-\varepsilon_{n-1}} \vee x_n^{\perp}) \wedge$$

$$\wedge \underline{\mathrm{com}}\,(x_1^{-\varepsilon_1} \vee x_2^{-\varepsilon_2} \vee \ldots \vee x_{n-1}^{-\varepsilon_{n-1}}, x_n) =$$

$$= A \wedge B \wedge (C \vee D \vee A^{\perp} \vee B^{\perp})$$

where

$$A = x_1^{-\varepsilon_1} \vee x_2^{-\varepsilon_2} \vee \ldots \vee x_{n-1}^{-\varepsilon_{n-1}} \vee x_n,$$

$$B = x_1^{-\varepsilon_2} \vee x_2^{-\varepsilon_1} \vee \ldots \vee x_{n-1}^{-\varepsilon_{n-1}} \vee x_n^{\perp},$$

$$C = (x_1^{-\varepsilon_1} \vee x_2^{-\varepsilon_2} \vee \ldots \vee x_{n-1}^{-\varepsilon_{n-1}}) \wedge x_n,$$

$$D = (x_1^{-\varepsilon_1} \vee x_2^{-\varepsilon_2} \vee \ldots \vee x_{n-1}^{-\varepsilon_{n-1}}) \wedge x_n^{\perp}.$$

Since A commutes with the elements $C, D, B^{\perp}$, we have $A \wedge (C \vee D \vee A^{\perp} \vee B^{\perp}) = A \wedge (C \vee D \vee B^{\perp}) = C \vee D \vee B^{\perp}$, and so

$$W(x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, \ldots, x_{n-1}^{\varepsilon_{n-1}}) = B \wedge (C \vee D \vee B^{\perp}).$$

Now $C \vee D \leqq B$ so that B commutes with $C \vee D$ and, trivially, B commutes with $B^{\perp}$. Accordingly, $B \wedge (C \vee D \vee B^{\perp}) = C \vee D$ and

$$W(x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, \ldots, x_{n-1}^{\varepsilon_{n-1}}) = C \vee D \leqq x_1^{-\varepsilon_1} \vee x_2^{-\varepsilon_2} \vee \ldots \vee x_{n-1}^{-\varepsilon_{n-1}}.$$

Thus we have

$$s \wedge t \leqq \overline{\mathrm{com}}\,(x_1, x_2, \ldots, x_n) \wedge \wedge \underline{\mathrm{com}}\,(x_1^{-\varepsilon_1} \vee x_2^{-\varepsilon_2} \vee \ldots \vee x_{n-1}^{-\varepsilon_{n-1}}, x_n) =$$

$$= \bigwedge_{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1})} W(x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, \ldots, x_{n-1}^{\varepsilon_{n-1}}) \leqq$$

$$\leqq \bigwedge_{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1})} (x_1^{-\varepsilon_1} \vee x_2^{-\varepsilon_2} \vee \ldots \vee x_{n-1}^{-\varepsilon_{n-1}}) = \overline{\mathrm{com}}\,(x_1, x_2, \ldots, x_{n-1}).$$

On the other hand,

$$s \wedge t \leqq \underline{\mathrm{com}}\,(x_1, x_2, \ldots, x_{n-1}).$$

Consequently, $s \wedge t = 0$ and, hence, $s = t^{\perp}$.

COROLLARY 2. *If $x_1, x_2, \ldots, x_n$, $n \geqq 2$, are elements of a lattice $\mathscr{L}$, then*

$$\overline{\mathrm{com}}\,(x_1, x_2, \ldots, x_n) = \bigvee_{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1})} \overline{\mathrm{com}}\,(x_1^{\varepsilon_1} \vee x_2^{\varepsilon_2} \vee \ldots \vee x_{n-1}^{\varepsilon_{n-1}}, x_n) \vee$$

$$\vee \bigvee_{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-2})} \overline{\mathrm{com}}\,(x_1^{\varepsilon_1} \vee x_2^{\varepsilon_2} \vee \ldots \vee x_{n-2}^{\varepsilon_{n-2}}, x_{n-1}) \vee$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\vee \bigvee_{(\varepsilon_1, \varepsilon_2)} \overline{\mathrm{com}}\,(x_1^{\varepsilon_1} \vee x_2^{\varepsilon_2}, x_3) \vee \overline{\mathrm{com}}\,(x_1, x_2).$$

As usual, for $a, b \in L$ we write $a \perp b$ if and only if $a^{\perp} \geqq b$. Likewise for $a, b, c, d \in L$ we write $(a, b) \perp (c, d)$ if and only if $a \perp b$, $c \perp d$, $a \perp d$ and $b \perp c$.

The following lemma reveals a special property of the just defined relation $\perp$:

LEMMA 3. *In any lattice $\mathscr{L}$, the relations $(a,b)\perp(c,d)$ and $(c,d)\perp(e,f)$ imply $(a\wedge c,\ b\wedge d)\perp(e,f)$.*

PROOF. Trivial.

THEOREM 4. *Let a_i, b_i, $i=1,2,\ldots,n$, be elements of a lattice $\mathscr{L}$ such that $(a_i,b_i)\perp(a_{i+1},b_{i+1})$ for every $i=1,2,\ldots,n-1$.*
Then

$$\bigwedge_{i=1}^{n}(a_i\vee b_i)=\left(\bigwedge_{i=1}^{n}a_i\right)\vee\left(\bigwedge_{i=1}^{n}b_i\right).$$

PROOF. By Lemma 3 it suffices to investigate the case $n=2$.
Clearly, $b_1\vee b_2\leqq a_1^{\perp}\wedge a_2^{\perp}$ and so

$$(a_1\vee a_2)\wedge(b_1\vee b_2)\leqq(a_1\vee a_2)\wedge a_1^{\perp}\wedge a_2^{\perp}=0.$$

Now observe that $b_2\perp a_1$ and $b_2\perp a_2$. Therefore b_2 commutes with a_1 and a_2 and so we can see that $a_1\vee a_2 C b_2$. But $a_1\leqq a_1\vee a_2$ and it gives $a_1\vee a_2 C a_1$. Combining $a_1\vee a_2 C a_1$ and $a_1\vee a_2 C b_2$, we get $a_1\vee a_2 C a_1\vee b_2$. In a similar manner, we have $b_1\vee b_2 C a_1\vee b_2$. Thus,

$$\begin{aligned}a_1\vee b_2&=(a_1\vee b_2)\vee[(a_1\vee a_2)\wedge(b_1\vee b_2)]=\\&=(a_1\vee a_2\vee b_2)\wedge(a_1\vee b_1\vee b_2)\geqq\\&\geqq(a_1\vee b_1)\wedge(a_2\vee b_2),\end{aligned}$$

and hence

$$L=(a_1\vee b_1)\wedge(a_2\vee b_2)=(a_1\vee b_1)\wedge(a_1\vee b_2)\wedge(a_2\vee b_2)=[a_1\vee(b_1\wedge b_2)]\wedge(a_2\vee b_2)$$

by the fact that $a_1 C b_1$ and $a_1 C b_2$. In view of $b_1\wedge b_2 C a_1$, $b_1\wedge b_2 C a_2\vee b_2$ we have

$$L=[a_1\wedge(a_2\vee b_2)]\vee(b_1\wedge b_2).$$

Now, $a_2 C b_2$, $a_1 C b_2$ and $a_1\wedge b_2\leqq b_2^{\perp}\wedge b_2=0$. It follows $L=(a_1\wedge a_2)\vee(b_1\wedge b_2)$.
The proof of the following result may also be useful for the light it sheds on the role of commutators in the proofs of formulas in orthomodular lattices.

COROLLARY 5 [4, p. A—51]. *Let a,b,c,d be elements of a lattice $\mathscr{L}$ such that aCb, cCd, aCd and bCc. Then*

$$(a\vee b)\wedge(c\vee d)=(a\wedge c)\vee(a\wedge d)\vee(b\wedge c)\vee(b\wedge d).$$

PROOF. Let $\lambda=(a\vee b)\wedge(c\vee d)$, $\varrho=(a\wedge c)\vee(a\wedge d)\vee(b\wedge c)\vee(b\wedge d)$. Then

$$\lambda\wedge\varrho^{\perp}\leqq[(a\vee b)\wedge(a^{\perp}\vee d^{\perp})]\wedge[(c\vee d)\wedge(b^{\perp}\vee c^{\perp})].$$

Since $a\vee b C a^{\perp}$ and $a^{\perp}Cd^{\perp}$,

$$Z_1=(a\vee b)\wedge(a^{\perp}\vee d^{\perp})=[a^{\perp}\wedge(a\vee b)]\vee[d^{\perp}\wedge(a\vee b)].$$

Since $a^{\perp}Ca$, aCb and $d^{\perp}Ca$,

$$Z_1=(a^{\perp}\wedge b)\vee(a\wedge d^{\perp})\vee(b\wedge d^{\perp}).$$

As in the previous case, since $c \vee dCc^{\perp}$, $b^{\perp}Cc^{\perp}$, $b^{\perp}Cc$, cCd, $c^{\perp}Cc$, we have

$$Z_2 = (c \vee d) \wedge (b^{\perp} \vee c^{\perp}) = [b^{\perp} \wedge (c \vee d)] \vee [c^{\perp} \wedge (c \vee d)] =$$
$$= (b^{\perp} \wedge c) \vee (c^{\perp} \wedge d) \vee (b^{\perp} \wedge d).$$

It is easy to see that the elements $d^{\perp} \wedge b$, $b^{\perp} \wedge d$ commute with the elements a, b, c, d. Thus

$$[(c^{\perp} \wedge d) \vee (b^{\perp} \wedge c)] \wedge (b \wedge d^{\perp}) = 0$$

and

$$[(a^{\perp} \wedge b) \vee (a \wedge d^{\perp})] \wedge (b^{\perp} \wedge d) = 0.$$

This yields

$$Z_1 \wedge Z_2 = [(a^{\perp} \wedge b) \vee (a \wedge d^{\perp})] \wedge [(c^{\perp} \wedge d) \vee (b^{\perp} \wedge c)].$$

Let $a_1 = a^{\perp} \wedge b$, $b_1 = a \wedge d^{\perp}$, $a_2 = c^{\perp} \wedge d$, $b_2 = b^{\perp} \wedge c$. Thus in view of Theorem 4 we have

$$Z_1 \wedge Z_2 = (a^{\perp} \wedge b \wedge c^{\perp} \wedge d) \vee (a \wedge b^{\perp} \wedge c \wedge d^{\perp}).$$

Therefore,

$$\lambda \wedge \varrho^{\perp} \leqq \underline{\mathrm{com}}\,(a, b, c, d).$$

Clearly, $\lambda = \varrho$ in any distributive lattice $\mathscr{L}$. By [2; p. 334] and [3; Theorem 4.14] we have

$$\lambda \wedge \varrho^{\perp} \leqq \overline{\mathrm{com}}\,(a, b, c, d).$$

Consequently, $\lambda \wedge \varrho^{\perp} = 0$. Since $\lambda \geqq \varrho$, it follows $\lambda = \varrho$.

We are now in a position to give a generalization of results due to S. HOLLAND, D. J. FOULIS (cf. [1; Theorem 23, p. 53]) and E. L. MARSDEN (Corollary 5 above).

THEOREM 6. *Let $\mathscr{L}$ be a lattice, $R = \{x_{r_1}, x_{r_2}, \ldots, x_{r_h}, y_{\varrho_1}, y_{\varrho_2}, \ldots, y_{\varrho_m}\}$, $S = \{x_{s_1}, x_{s_2}, \ldots, x_{s_k}, y_{\sigma_1}, y_{\sigma_2}, \ldots, y_{\sigma_n}\}$, $h \geqq 0$, $k \geqq 0$, $m \geqq 0$, $n \geqq 0$, $h+m > 0$, $k+n > 0$, be subsets of L such that*

$$R \cup S = \{x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q\}$$

and uCv for each $u \in R$, $v \in S$.

If $\Lambda = (x_1 \vee x_2 \vee \ldots \vee x_p) \wedge (y_1 \vee y_2 \vee \ldots \vee y_q)$, then

$$\Lambda = [(x_{r_1} \vee x_{r_2} \vee \ldots \vee x_{r_n}) \wedge (y_{\varrho_1} \vee y_{\varrho_2} \vee \ldots \vee y_{\varrho_m})] \vee$$
$$\vee [(x_{s_1} \vee x_{s_2} \vee \ldots \vee x_{s_k}) \wedge (y_{\sigma_1} \vee y_{\sigma_2} \vee \ldots \vee y_{\sigma_n})] \vee$$
$$\vee \bigvee_{\substack{1 \leqq i \leqq h \\ 1 \leqq j \leqq n}} (x_{r_i} \wedge y_{\sigma_j}) \vee \bigvee_{\substack{1 \leqq i \leqq k \\ 1 \leqq j \leqq m}} (x_{s_i} \wedge y_{\varrho_j}).$$

PROOF. Let $a = x_{r_1} \vee x_{r_2} \vee \ldots \vee x_{r_h}$, $b = x_{s_1} \vee x_{s_2} \vee \ldots \vee x_{s_k}$, $c = y_{\varrho_1} \vee y_{\varrho_2} \vee \ldots \vee y_{\varrho_m}$, $d = y_{\sigma_1} \vee y_{\sigma_2} \vee \ldots \vee y_{\sigma_n}$. By [1; Theorem 22, p. 53] aCb, cCd, aCd and bCc. Therefore, by Corollary 5, we have

$$\Lambda = (a \wedge c) \vee (a \wedge d) \vee (b \wedge c) \vee (b \wedge d).$$

Since aCy_{σ_j} for every $j = 1, 2, \ldots, n$,

$$a \wedge d = \bigvee_{1 \leqq j \leqq n} a \wedge y_{\sigma_j}.$$

Moreover, $x_{r_i} C y_{\sigma_j}$ implies

$$a \wedge y_{\sigma_j} = \bigvee_{1 \leqq i \leqq h} x_{r_i} \wedge y_{\sigma_j}.$$

Hence,

$$a \wedge d = \bigvee_{\substack{1 \leqq i \leqq h \\ 1 \leqq j \leqq n}} x_{r_i} \wedge y_{\sigma}.$$

Similarly,

$$b \wedge c = \bigvee_{\substack{1 \leqq i \leqq k \\ 1 \leqq j \leqq m}} x_{s_i} \wedge y_{\varrho_j}.$$

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KANTENKRÜMMUNG UND UMKUGELRADIUS KONVEXER POLYEDER

von

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Sei P ein konvexes Polyeder im $\mathbf{R}^3$ mit Kantenlängen $l_1, \ldots, l_k$. α_i sei der Außenwinkel bei der i-ten Kante. Dann heißt $M=\frac{1}{2}\Sigma\alpha_i l_i$ die Kantenkrümmung von P.

SATZ. *Für jedes konvexe Polyeder mit f Flächen, k Kanten, e Ecken und Umkugelradius R gilt*

$$M/R \leqq 2k \sin\frac{\pi f}{2k}\left(1-\cot^2\frac{\pi f}{2k}\cot^2\frac{\pi e}{2k}\right)^{1/2}\arccos\left(\cos\frac{\pi e}{2k}\operatorname{cosec}\frac{\pi f}{2k}\right),$$

mit Gleichheit genau für die fünf regulären Polyeder.

BEMERKUNG. Dieser Satz wurde von A. FLORIAN [2] unter der zusätzlichen Voraussetzung bewiesen, daß die Fußpunkte der vom Umkugelmittelpunkt auf die Flächen und Kanten von P gefällten Lote auf den entsprechenden Flächen bzw. Kanten liegen. Im folgenden wird mit Hilfe einer allgemeinen Ungleichung von L. FEJES TÓTH [1] ein relativ kurzer Beweis angegeben, der ohne diese „Fußpunktbedingung" auskommt. Es sei weiters bemerkt, daß M/R in [5] mit einer ähnlichen Methode nach unten abgeschätzt wird: $M \geqq 2\pi R$, mit Gleichheit genau dann, wenn P zu einer Strecke ausgeartet ist.

BEWEIS. Wir wählen den Umkugelmittelpunkt O als Ursprung und identifizieren Punkte und Ortsvektoren. Ohne Beschränkung der Allgemeinheit sei $R=1$. Es ist

$$M = \int_U t(u)\,du,$$

wobei $t(u)=\max_{x\in P}\langle u, x\rangle$ die Stützfunktion von P, und U die Einheitssphäre (mit Mittelpunkt O) ist [4]. Sind $v_1, \ldots, v_e$ die Ecken von P, so lassen sich die Flächen des polaren Polyeders $P^*=\{y|\langle y, x\rangle \leqq 1 \text{ für alle } x\in P\}$ so darstellen [3]:

$$F_i = \{y\in P^*|\langle y, v_i\rangle = 1\} \quad (1 \leqq i \leqq e).$$

Wenn O nicht im Inneren von P liegt, so ist O Randpunkt von P, und P^* ist unbeschränkt. O kann jedoch keine Ecke von P sein, daher enthält dann P^* einen höchstens zweidimensionalen Kegel mit Spitze O. $F_i'=\left\{u\in U|u=\frac{y}{\|y\|} \text{ für ein } y\in F_i\right\}$ ist die Projektion von F_i auf U. Für $u\in F_i'$ ist $t(u)=\langle u, v_i\rangle \leqq \cos\sphericalangle(u, v_i)$, da $\|v_i\| \leqq 1$.

Die F_i' sind konvexe sphärische Polygone, die auf U ein Mosaik mit e Flächen, k Kanten und f Ecken bilden. Da *cos* in $\left[0, \frac{\pi}{2}\right]$ streng abnehmend ist, erhalten wir auf Grund einer Ungleichung von L. FEJES TÓTH ([1], S. 137):

$$M \leqq \sum_i \int_{F_i'} \cos \sphericalangle (u, v_i)\, du \leqq 4k \int_{\triangle} \cos \sphericalangle (u, A)\, du,$$

wobei $\triangle = ABC$ ein rechtwinkeliges sphärisches Dreieck mit den Winkeln $\alpha = \frac{\pi e}{2k}$ und $\beta = \frac{\pi f}{2k}$ bei A bzw. B bedeutet. a, b, c seien die Seitenlängen von $\triangle$ ($a = \widehat{BC} = \sphericalangle(B, C)$, usw.). Das oben rechts stehende Integral ist gleich dem Inhalt der Normalprojektion $\triangle'$ von $\triangle$ auf eine zu A orthogonale Ebene. $\triangle'$ ist ein Sektor einer Ellipse, dessen Inhalt $|\triangle'|$ leicht berechnet werden kann:

$$|\triangle'| = \frac{a}{2} \sin b = \frac{1}{2} \sin \beta (1 - \cot^2 \beta \cot^2 \alpha)^{\frac{1}{2}} \arccos (\cos \alpha \operatorname{cosec} \beta).$$

Damit folgt sofort die Richtigkeit des Satzes, da der Fall der Gleichheit schon bei der zitierten Ungleichung von Fejes Tóth dargelegt ist.

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DAS GESETZ DER GROSSEN ZAHLEN VON ERDŐS—RÉNYI IM WARTEMODELL $G/G/1$

von

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Die vorliegende Arbeit beschäftigt sich mit Wartezeiten in einem Bedienungssystem vom Typ $G/G/1$. Bezeichne X_n die Differenz zwischen der Bedienungszeit B_n des n-ten Kunden und dem Zeitintervall A_n zwischen der Ankunft des n-ten und $(n+1)$-ten Kunden, so gilt für die Wartezeit W_n des $(n+1)$-ten Kunden die Rekursionsformel

$$W_n = [W_{n-1} + X_n]^+, \quad n = 1, 2, \dots, \tag{1}$$

wobei $W_0 = 0$ gesetzt wird. Mit $S_n := \sum_{i=1}^{n} X_i$, $S_0 := 0$, ergibt sich aus (1) sofort

$$W_n = \max_{k=0,\dots,n} (S_n - S_k), \tag{2}$$

und bei Unabhängigkeit und identischer Verteilung der X_i stimmt die Verteilung von W_n mit der von

$$W_n^* = \max_{k=0,\dots,n} S_k$$

überein. Eine Vielzahl von Untersuchungen sind durchgeführt worden über das Konvergenzverhalten der Verteilungen von W_n bzw. W_n^*, das unterschiedlich ist je nach der Gültigkeit von $EX_i < 0$, $EX_i = 0$, $EX_i > 0$ (vgl. IGLEHARDT (1973)).

BÁRTFAI (1970) hat in Analogie zum Satz vom iterierten Logarithmus bei der Irrfahrt einige ‚Limes superior Sätze' für die Wartemodelle bewiesen, wie z. B.

$$P\left(\limsup_{n\to\infty} \frac{W_n}{\log n} = \frac{1}{t_0}\right) = 1$$

und

$$P\left(\lim_{n\to\infty} \frac{\max_{k\leqq n} W_k}{\log n} = \frac{1}{t_0}\right) = 1,$$

wobei $t_0 > 0$ eine Lösung der Gleichung

$$\varphi(t) = 1$$

ist und $\varphi(t)$ die momenterzeugende Funktion der Verteilung der X_i bezeichnet.

In dieser Arbeit soll in Analogie zum Gesetz der großen Zahlen von ERDŐS und RÉNYI (1970) bei der Irrfahrt das Konvergenzverhalten von

$$D(N, K) := \max_{0\leqq n\leqq N-K} \frac{W_{n+K} - W_n}{K}, \quad 1 \leqq K \leqq N, \tag{3}$$

für bestimmte Wahlen von $K=K(N)$ untersucht werden. Erdős und Rényi haben für die Irrfahrt gezeigt, daß

$$\lim_{N\to\infty} \max_{0\leqq n\leqq N-[C\log N]} \frac{S_{n+[C\log N]}-S_n}{[C\log N]} = a \tag{4}$$

mit Wahrscheinlichkeit 1 gilt, falls eine Wahl von $C=C(a)>0$ möglich ist, derart daß

$$\exp(-1/C) = \inf_t \varphi(t)\exp(-ta) =: \varrho(a). \tag{5}$$

$[x]$ bezeichnet hier den ganzzahligen Anteil von x. Notwendig *und* hinreichend ist hierfür die Existenz der momenterzeugenden Funktion $\varphi(t)$ in einem Intervall $(0, T)$, $T>0$ (vgl. ERDŐS, RÉNYI (1970), STEINEBACH (1977)).

Im folgenden seien die Zufallsvariablen $X_1, X_2, \ldots$ nicht-degeneriert und unabhängig, identisch verteilt (i.i.d.) mit endlicher momenterzeugender Funktion

$$\varphi(t) = E\exp(tX_1) < \infty \quad \text{für} \quad t\in(0,T).$$

Dann gilt:

SATZ 1. *Sei* $EX_1\geqq 0$. *Ferner nehme für eine reelle Zahl* a *die Funktion* $\varphi(t)e^{-ta}$ *ihr Minimum in* $(0, T)$ *an. Wählt man dann eine positive Konstante* $C=C(a)$ *wie in* (5), *so folgt mit der Bezeichnung* (3)

$$\lim_{N\to\infty} D(N, [C\log N]) = a \tag{6}$$

mit Wahrscheinlichkeit 1.

BEMERKUNG 1. Bei nichtnegativer Drift liegt somit für Irrfahrt und Wartemodell dasselbe Konvergenzverhalten im Sinne des Gesetzes der großen Zahlen von Erdős—Rényi vor.

SATZ 2. *Sei* $EX_1<0$. *Ferner existiere eine positive Zahl* $t_0\in(0, T)$ *mit* $\varphi(t_0)=1$. *Man setze*

$$a_0 := \varphi'(t_0), \quad a_1 := \sup_{t<T} \varphi'(t)/\varphi(t)$$

und

$$\varrho^*(a) := \begin{cases} e^{-t_0 a}, & a\in(0, a_0], \\ \inf\limits_{t\in(0,T)} \varphi(t)e^{-ta}, & a\in(a_0, a_1). \end{cases}$$

Wählt man eine Konstante $C^*=C^*(a)$, *so daß*

$$\exp(-1/C^*) = \varrho^*(a)$$

gilt, so folgt mit der Bezeichnung (3)

$$\lim_{N\to\infty} D(N, [C^*\log N]) = a$$

mit Wahrscheinlichkeit 1.

BEMERKUNG 2. Bei negativer Drift unterscheidet sich also das Konvergenzverhalten der Irrfahrt von dem des Wartemodells nur für Werte $a\in(0, a_0)$.

BEMERKUNG 3. Die Existenz von EX_1 ist nicht notwendig. Die Bedingungen $EX_1 \geqq 0$, $EX_1 < 0$ können ersetzt werden durch $\varphi'(0+) \geqq 0$ bzw. $\varphi'(0+) < 0$, wobei $\varphi'(0+) := \lim_{t \searrow 0} \varphi'(t)$.

Zum Beweis der Sätze 1 und 2 benötigen wir das folgende Ergebnis über Wahrscheinlichkeiten großer Abweichungen bei Wartezeiten (vgl. BÁRTFAI (1972), PLACHKY und STEINEBACH (1975)):

LEMMA 1. a) *Unter den Voraussetzungen von Satz* 1 *gilt für* $a \in (EX_1, a_1)$, $a_1 := \sup_{t<T} \varphi'(t)/\varphi(t)$, *die Beziehung*

$$\lim_{n\to\infty} [P(W_n \geqq na)]^{1/n} = \varrho(a).$$

b) *Unter den Voraussetzungen von Satz* 2 *gilt für* $a \in (0, a_1)$ *die Beziehung*

$$\lim_{n\to\infty} [P(W_n \geqq na)]^{1/n} = \varrho^*(a).$$

BEWEIS. Der Beweis ist in [3] und [7] für den Fall b) geführt worden. Der Beweis in [7] läßt sich jedoch sofort auf den Fall a) übertragen.

BEWEIS VON SATZ 1: a) Sei $\varepsilon > 0$ beliebig klein und $a' := a + \varepsilon$. Mit (2) gilt

$$W_{n+K} = \max(0,\ X_{n+K},\ X_{n+K} + X_{n+K-1},\ \ldots,\ X_{n+K} + \ldots + X_n,\ \ldots,\ X_{n+K} + \ldots + X_1),$$

$$W_n \quad = \max(0,\quad 0,\quad\quad 0,\ \ldots,\quad\quad\quad\quad X_n,\quad\quad\quad\quad \ldots,\ X_n + \ldots + X_1),$$

woraus folgt

$$W_{n+K} - W_n \leqq \max(0,\ X_{n+K},\ \ldots,\ X_{n+K} + \ldots + X_{n+1}).$$

Damit erhält man

$$P(D(N, K) \geqq a') \leqq \sum_{n=0}^{N-K} P(W_{n+K} - W_n \geqq Ka') \leqq$$

$$\leqq \sum_{n=0}^{N-K} P(\max(0,\ X_{n+K},\ \ldots,\ X_{n+K} + \ldots + X_{n+1}) \geqq Ka') =$$

$$= (N - K + 1) P(W_K \geqq Ka').$$

Für alle hinreichend großen K liefert Lemma 1 die Ungleichung

$$P(W_K \geqq Ka') \leqq \varrho'^K,$$

wobei $\varrho(a) > \varrho' > \varrho(a')$, ϱ wie in (5). Es existiert somit eine natürliche Zahl N' mit

$$P(D(N, [C \log N]) \geqq a') \leqq N\varrho'^{[C \log N]} \tag{7}$$

für $N \geqq N'$. Sei N_j die größte natürliche Zahl derart daß

$$[C \log N_j] = j,$$

$j \geqq j'$, j' fest. Sei o. B. d. A. $N_{j'} \geqq N'$, so folgt wegen

$$\exp(j/C) \leqq N_j < \exp((j+1)/C)$$

und der Ungleichung (7)

$$\sum_{j=j'}^{\infty} P\big(D(N_j, [C\log N_j]) \leqq a'\big) \leqq \sum_{j=j'}^{\infty} e^{(j+1)/C} \varrho'^{\,j},$$

wobei die letzte Reihe konvergiert wegen

$$e^{-1/C} = \varrho(a) > \varrho'.$$

Das Lemma von Borel—Cantelli liefert daher

$$\limsup_{j\to\infty} D(N_j, [C\log N_j]) \leqq a'$$

mit Wahrscheinlichkeit 1. Nach Definition der N_j gilt für $N_{j-1} < N \leqq N_j$

$$D(N, [C\log N]) \leqq D(N_j, [C\log N_j])$$

und folglich

$$\limsup_{N\to\infty} D(N, [C\log N]) \leqq a' \tag{8}$$

mit Wahrscheinlichkeit 1. Da $\varepsilon > 0$ beliebig klein gewählt war, kann a' in (8) durch a ersetzt werden.

b) Wegen der Rekursionsformel (1) gilt

$$W_{n+1} - W_n = [W_n + X_{n+1}]^+ - W_n \geqq X_{n+1}, \quad n = 0, 1, 2, \ldots$$

und folglich

$$W_{n+K} - W_n = \sum_{i=1}^{K} (W_{n+i} - W_{n+i-1}) \geqq \sum_{i=1}^{K} X_{n+i} = S_{n+K} - S_n.$$

Das Gesetzt der großen Zahlen von ERDŐS und RÉNYI (1970) liefert daher sofort

$$\liminf_{N\to\infty} D(N, [C\log N]) \geqq \lim_{N\to\infty} \max_{0\leqq n\leqq [C\log N]} \frac{S_{n+[C\log N]} - S_n}{[C\log N]} = a \tag{9}$$

mit Wahrscheinlichkeit 1.

Mit (8) und (9) ist der Beweis von (6) erbracht.

BEWEIS VON SATZ 2: a) Der Nachweis von

$$\limsup_{N\to\infty} D(N, [C^*\log N]) \leqq a$$

mit Wahrscheinlichkeit 1 läßt sich analog zum Teil a) des Beweises von Satz 1 erbringen. Es ist lediglich $\varrho(a)$ durch $\varrho^*(a)$ und C durch C^* zu ersetzen.

b) Der Beweis von

$$\liminf_{N\to\infty} D(N, [C^*\log N]) \geqq a \tag{10}$$

mit Wahrscheinlichkeit 1 läßt sich für $a \in [a_0, a_1)$ wie in Satz 1 führen. Für $a \in (0, a_0)$ ist die dort benutzte Abschätzung jedoch zu grob. Hier führen wir den Beweis unter Benutzung geeigneter Wartezeiten in verschiedenen Arbeitsperioden. Dazu seien zunächst einige Bezeichnungen und vorbereitende Hilfssätze angegeben:

BEZEICHNUNG 1. Sei $\nu_0 := 0$ und $\nu_i > \nu_{i-1}$ die kleinste natürliche Zahl mit $W_{\nu_i} = 0$, $i = 1, 2, \ldots$. Ferner sei

$$\lambda_i := \nu_i - \nu_{i-1}, \quad i = 1, 2, \ldots .$$

λ_i heißt Länge der i-ten Arbeitsperiode des Bedienungssystems.

LEMMA 2. *Die Zufallsvariablen $\lambda_1, \lambda_2, \ldots$ sind unabhängig, identisch verteilt und besitzen endliche momenterzeugende Funktionen in einem Intervall $[0, T_1]$, etwa*

$$\varphi_\lambda(t) = Ee^{t\lambda_1} \leqq M_1 < \infty$$

für $t \in [0, T_1]$.

BEWEIS. Vgl. BÁRTFAI (1970), S. 319 f.

BEZEICHNUNG 2. Sei $\nu_{0,N} := 0$. Mit $K_N := [C^* \log N]$ sei ferner $\nu_{i,N} > \nu_{i-1,N} + K_N$ die kleinste natürliche Zahl mit $W_{\nu_{i,N}} = 0$, $i = 1, 2, \ldots$. Man setze

$$\lambda_{i,N} := \nu_{i,N} - \nu_{i-1,N} + K_N, \quad i = 1, 2, \ldots .$$

LEMMA 3. a) *Für festes N sind die Zufallsvariablen $\lambda_{1,N}, \lambda_{2,N}, \ldots$ unabhängig und identisch verteilt. Ferner gibt es eine Konstante M_2 (unabhängig von N) mit*

$$\varphi_{\lambda_{1,N}}(t) = Ee^{t\lambda_{1,N}} \leqq M_2 < \infty \tag{11}$$

für t aus einem Intervall $[0, T_2]$, T_2 unabhängig von N.

b) *Die Zufallsvariablen $W_{K_N}, W_{\nu_{1,N}+K_N}, W_{\nu_{2,N}+K_N}, \ldots$ sind unabhängig und identisch verteilt.*

BEWEIS. a) Setzt man $n_0 = 0$, $n_i = n_{i-1} + K_N + l_i$, $i = 1, 2, \ldots$, so ist

$$P(\lambda_{1,N} = l_1, \ldots, \lambda_{k,N} = l_k) = P\left(\bigcap_{i=1}^{k} A_{i,N}\right)$$

mit

$$A_{i,N} := \{W_{n_{i-1}+K_N+1} > 0, \ldots, W_{n_{i-1}+K_N+l_i-1} > 0, W_{n_{i-1}+K_N+l_i} = 0\},$$

$i = 1, \ldots, k$. Wegen $W_{n_{i-1}} = 0$ und der Rekursion $W_n = [W_{n-1} + X_n]^+$, $n = 1, 2, \ldots$, hängt das Ereignis $A_{i,N}$ nur von $X_{n_{i-1}+1}, \ldots, X_{n_{i-1}+K_N+l_i}$ ab, $i = 1, \ldots, k$. Die Unabhängigkeit und identische Verteilung der X_i liefert daher

$$P\left(\bigcap_{i=1}^{k} A_{i,N}\right) = \prod_{i=1}^{k} P(A_{i,N}) = \prod_{i=1}^{k} P(\lambda_{1,N} = l_i),$$

d. h. die Unabhängigkeit und identische Verteilung der Zufallsvariablen $\lambda_{1,N}$, $\lambda_{2,N}, \ldots$. Für $\lambda_{1,N}$ gelten die Abschätzungen

$$P(\lambda_{1,N} > l) = P(W_{K_N+1} > 0, \ldots, W_{K_N+1} > 0) =$$

$$= P(W_{K_N} + X_{K_N+1} > 0, \ldots, W_{K_N} + X_{K_N+1} + \ldots + X_{K_N+l} > 0) =$$

$$= P(W_{K_N} + \min(X_{K_N+1}, X_{K_N+1} + X_{K_N+2}, \ldots, X_{K_N+1} + \ldots + X_{K_N+l}) > 0).$$

Sei $M_{l,N}:=\min(X_{K_N+1}, \ldots, X_{K_N+1}+\ldots+X_{K_N+l})$ und F_l die Verteilungsfunktion von $M_{l,N}$, so erhält man wegen der Unabhängigkeit von W_{K_N} und $M_{l,N}$

$$P(W_{K_N}+M_{l,N}>0)=\int_{-\infty}^{\infty} P(W_{K_N}>-x)\,dF_l(x)=$$

$$=P(M_{l,N}>0)+\int_{-\infty}^{0} P(W_{K_N}>-x)\,dF_l(x).$$

Mit Bezeichnung 1 und Lemma 2 gilt

$$P(M_{l,N}>0)=P(\lambda_1>l)\leqq M_1 e^{-T_1 l}=:M_1\varrho_1^l$$

Ferner ist $P(W_{K_N}>-x)\leqq e^{t_0 x}$ $(x\leqq 0)$ (vgl. BÁRTFAI (1970), S. 323) und folglich für $\delta>0$

$$\int_{-\infty}^{0} P(W_{K_N}>-x)\,dF_l(x)\leqq \int_{-\infty}^{0} e^{t_0 x}\,dF_l(x)\leqq$$

$$\leqq \int_{-\infty}^{-l\delta} e^{t_0 x}\,dF_l(x)+\int_{-l\delta}^{0} e^{t_0 x}\,dF_l(x)\leqq e^{-t_0 l\delta}+P(X_{K_N+1}+\ldots+X_{K_N+l}\geqq -l\delta).$$

Für $EX_1<-\delta<0$ gilt nach CHERNOFF (1952)

$$P(X_{K_N+1}+\ldots+X_{K_N+l}\geqq -l\delta)=P(X_1+\ldots+X_l\geqq -l\delta)\leqq \varrho_2^l$$

mit $\varrho_2=\varrho(-\delta)=\inf_t \varphi(t)e^{t\delta}<1$. Setzt man noch $\varrho_3:=e^{-t_0\delta}$ so folgt insgesamt wegen $M_1\geqq 1$

$$P(\lambda_{1,N}>l)\leqq M_1\max\{\varrho_1^l, \varrho_2^l, \varrho_3^l\},$$

was wegen $\max\{\varrho_1, \varrho_2, \varrho_3\}<1$ die Beziehung (11) impliziert.

b) Nach Definition der Zufallsvariablen $v_{i,N}$ liegen W_{K_N}, $W_{v_{1,N}+K_N}$, $W_{v_{2,N}+K_N}$, ... in verschiedenen Beschäftigungsperioden, was deren Unabhängigkeit und mit Teil a) auch deren identische Verteilung zur Folge hat.

Zum Beweis von Satz 2, Teil b) legen wir noch fest:

BEZEICHNUNG 3. $L_N:=$ Anzahl der zufälligen Indizes $v_{i,N}$ mit $v_{i,N}\leqq N-K_N$, $i=0, 1, 2, \ldots$.

BEMERKUNG 4. $L_N\leqq[N/K_N]$.

LEMMA 4. *Sei* $\delta>0$ *und* $N_\delta:=[(1-\delta)[N/K_N]]$. *Dann gilt:*

$$\sum_{N=1}^{\infty} P(L_N\leqq N_\delta)<\infty. \tag{12}$$

BEWEIS. Es ist

$$P(L_N\leqq N_\delta)=P(v_{N_\delta,N}+K_N>N)=$$

$$=P\left(\sum_{i=1}^{N_\delta}\lambda_{i,N}+(N_\delta+1)K_N>N\right)\leqq P\left(\sum_{i=1}^{N_\delta}\lambda_{i,N}>N_\delta a+N\delta'\right),$$

wobei $0<\delta'<\delta$, $a>0$ beliebig und N hinreichend groß ist, etwa $N\geqq N'$. Mit Lemma 3 folgt

$$P\left(\sum_{i=1}^{N_\delta}\lambda_{i,N} > N_\delta a + N\delta'\right) \leqq \{M_2 e^{-T_2 a}\}^{N_\delta} e^{-NT_2\delta'}.$$

Die positive Konstante a kann so gewählt werden, daß $M_2 e^{-T_2 a}\leqq 1$ gilt. Damit erhält man insgesamt

$$P(L_N \leqq N_\delta) \leqq e^{-NT_2\delta'}, \quad N \geqq N',$$

was (12) zur Folge hat.

Mit Hilfe von Lemma 4 vervollständigen wir nun den Beweis von Satz 2.

BEWEIS VON SATZ 2. b) Sei $\varepsilon>0$ beliebig klein und $a'':=a-\varepsilon$. Es gilt

$$D(N, K_N) = \max_{0\leqq n\leqq N-K_N}\frac{W_{n+K_N}-W_n}{K_N} \geqq \max_{v_{i,N}\leqq N-K_N}\frac{W_{v_{i,N}+K_N}-W_{v_{i,N}}}{K_N} =$$

$$= \max_{v_{i,N}\leqq N-K_N}\frac{W_{v_{i,N}+K_N}}{K_N} =: D^*(N, K_N).$$

Damit erhält man

$$P(D(N, K_N) < a'') \leqq P(D^*(N, K_N) < a'') \leqq \tag{13}$$

$$\leqq P(D^*(N, K_N) < a'', L_N > N_\delta) + P(L_N \leqq N_\delta).$$

Aus Lemma 3 folgt

$$P(D^*(N, K_N) < a'', L_N > N_\delta) \leqq \{P(W_{K_N} < K_N a'')\}^{N_\delta}.$$

Aus Lemma 1 erhält man für alle hinreichend großen N

$$P(W_{K_N} < K_N a'') \leqq 1-\varrho''^{K_N} \leqq \exp(-\varrho''^{K_N}),$$

wobei $\varrho^*(a'')>\varrho''>\varrho^*(a)$, etwa $\varrho''=\exp(-1/C'')$ mit $C^*(a'')>C''>C^*(a)$. Setzt man $C^*(a)/C''=1-2\delta^*$ für ein $\delta^*>0$, so ergibt sich

$$\varrho''^{K_N} = \varrho''^{[C^*\log N]} \geqq \varrho''^{C^*\log N} = N^{-(1-2\delta^*)}$$

und für N hinreichend groß, etwa $N\geqq N^*$,

$$N_\delta = [(1-\delta)[N/K_N]] \geqq N^{1-\delta^*}.$$

Damit gilt

$$\sum_{N=N^*}^{\infty} P(D^*(N, K_N) < a'', L_N > N_\delta) \leqq \sum_{N=N^*}^{\infty} e^{-N^{\delta^*}} < \infty,$$

und mit Lemma 4 unter Beachtung von (13)

$$\sum_{N=1}^{\infty} P(D(N, K_N) < a'') < \infty.$$

Das Lemma von Borel—Cantelli liefert daher

$$\liminf_{N\to\infty} D(N, K_N) \geqq a''$$

mit Wahrscheinlichkeit 1. Da $a''=a-\varepsilon$ mit beliebig kleinem ε gilt, ist (10) bewiesen.

BEMERKUNG 5. Die Existenz der momenterzeugenden Funktion $\varphi(t)=E\exp(tX_1)$ in einem Intervall $(0, T)$ ist notwendig für die Gültigkeit der Sätze 1 und 2. Andernfalls erhält man nach [8] unter Beachtung von $W_{n+K}-W_n \geqq S_{n+K}-S_n$ die Beziehung

$$\limsup_{N\to\infty} \max_{0\leqq n\leqq N-[C\log N]} \frac{W_{n+[C\log N]}-W_n}{[C\log N]} = \infty$$

mit Wahrscheinlichkeit 1 für jede positive Konstante C.

BEMERKUNG 6. BÁRTFAI (1966) hat gezeigt, daß durch eine einzige Realisation der Irrfahrt, d. h. der Folge $S_1=X_1$, $S_2=X_1+X_2$, ... die Verteilungsfunktion von X_1 mit Wahrscheinlichkeit 1 bestimmt ist, falls die momenterzeugende Funktion von X_1 in einem Intervall existiert. Dabei sind für S_n noch Meßfehler in der Größenordnung $o(\log n)$ zulässig. ERDŐS und RÉNYI (1970) haben als Anwendung ihres Gesetzes der großen Zahlen einen neuen Beweis des Ergebnisses von Bártfai angegeben, indem sie zeigten, daß die Gültigkeit der Aussage (4) für Werte a aus einem Intervall die zugrundeliegende momenterzeugende Funktion in einem Intervall — und somit auch die Verteilung — mit Wahrscheinlichkeit 1 bestimmt. Aus den Sätzen 1 und 2 folgt, daß Irrfahrt und Warteprozeß in einem bestimmten Intervall von a-Werten ein identisches Konvergenzverhalten (im Sinne des Gesetzes der großen Zahlen von Erdős—Rényi) besitzen, falls im Existenzbereich $[0, T)$ der momenterzeugenden Funktion $\varphi(t)=E\exp(tX_1)$ eine Stelle $t_0\geqq 0$ existiert mit $\varphi(t_0)=1$ und $\varphi(t)>1$ für $t\in(t_0, T)$. Hieraus ergibt sich, daß auch eine einzige (möglicherweise mit Fehlern behaftete) Realisation der Folge $W_1, W_2, \ldots$ von Wartezeiten die Verteilungsfunktion von X_1 mit Wahrscheinlichkeit 1 bestimmt.

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FIXED POINTS, COMMON FIXED POINTS AND CONSTANT MAPPINGS

by

B. FISHER and M. S. KHAN

We first of all prove the following theorem:

THEOREM 1. *If S and T are mappings of the complete metric space X into itself such that*

$$d(Sx, Ty) \leqq \frac{bd(x, Sx)d(x, Ty)+cd(y, Ty)d(y, Sx)}{d(x, Ty)+d(y, Sx)} \tag{1}$$

for all x, y in X with $d(x, Ty)+d(y, Sx)\neq 0$, where $b, c\geqq 0$ and $bc<1$, and $d(Sx, Ty)=0$ otherwise, then S and T have a unique common fixed point z.

PROOF. Let x be an arbitrary point in X. Suppose first that there exists an integer n such that $d((ST)^n x, T(ST)^n x)=0$. Then $z=(ST)^n x$ is a fixed point of T. Assuming $d(Sz, z)\neq 0$, relation (1) yields $d(Sz, Tz)\leqq 0$ i.e. $d(Sz, z)\leqq 0$, a contradiction. It follows that z is a common fixed point of S and T.

Similarly, if there exists an integer n such that

$$d((ST)^{n+1}x, T(ST)^n x) = 0$$

it follows that $z=T(ST)^n x$ is a common fixed point of S and T.

We will now suppose that

$$d((ST)^n x, T(ST)^n x) \neq 0, \quad d((ST)^{n+1}x, T(ST)^n x) \neq 0$$

for $n=0, 1, 2, \ldots$. Then

$$d((ST)^n x, T(ST)^n x) \leqq bd(T(ST)^{n-1}x, (ST)^n x) \leqq$$

$$\leqq bcd((ST)^{n-1}x, T(ST)^{n-1}x) \leqq \ldots \leqq (bc)^n d(x, Tx).$$

Since $bc<1$, it follows that the sequence

$$\{x, Tx, STx, TSTx, \ldots, (ST)^n x, T(ST)^n x, \ldots\}$$

is a Cauchy sequence in the complete metric space X and so has a limit z in X.

We have either

$$d((ST)^n x, Tz) \leqq \frac{bd(T(ST)^{n-1}x, (ST)^n x)d(T(ST)^{n-1}x, Tz)+cd(z, Tz)d(z, (ST)^n x)}{d(T(ST)^{n-1}x, Tz)+d(z, (ST)^n x)}$$

or $d((ST)\ x, Tz)=0$ and on letting n tend to infinity in either case, we see that $d(z, Tz)=0$. Thus z is a fixed point of T and it follows as above that z is a common fixed point of S and T.

We will now suppose that w is a second common fixed point of S and T. Then either $d(z, w)=d(Sz, Tw)\leqq 0$ or $d(z, w)=d(Sz, Tw)=0$ and each possibility implies that $z=w$. The common fixed point must therefore be unique. This completes the proof of the theorem.

Note that if we stipulate in Theorem 1, that the mappings S and T only satisfy the inequality

$$\{d(x, Ty)+d(y, Sx)\}d(Sx, Ty) \leqq bd(x, Sx)d(x, Ty)+cd(y, Ty)d(y, Sx) \tag{2}$$

for all x, y in X, where $b, c\geqq 0$ and $bc<1$, then S and T do not necessarily have any fixed points. This is very easily seen by letting $X=\{0, 1\}$ and defining identical mappings S and T on X by $S(0)=T(0)=1$, $S(1)=T(1)=0$. Then

$$\{d(x, Ty)+d(y, Sx)\}d(Sx, Ty) = 0$$

for all x, y in X, whatever metric is used, and so inequality (2) is satisfied with $b=c=\frac{1}{2}$. S and T, however, have no fixed points.

On putting $S=T$ in Theorem 1, we have the following

COROLLARY. *If T is a mapping of the complete metric space X into itself such that*

$$d(Tx, Ty) \leqq \frac{bd(x, Tx)d(x, Ty)+cd(y, Ty)d(y, Tx)}{d(x, Ty)+d(y, Tx)}$$

for all x, y in X with $d(x, Ty)+d(y, Tx)\neq 0$, where $b, c\geqq 0$ and $bc<1$, and $d(Tx, Ty)=0$ otherwise, then T has a unique fixed point z.

If $b=0$ in Theorem 1, we have the following stronger result:

THEOREM 2. *If S and T are mappings of the metric space X into itself such that*

$$d(Sx, Ty) \leqq \frac{cd(y, Ty)d(y, Sx)}{d(x, Ty)+d(y, Sx)}$$

for all x, y in X with $d(x, Ty)+d(y, Sx)\neq 0$, where $c\geqq 0$, and $d(Sx, Ty)=0$ otherwise, then S and T have a unique common fixed point z. Further, $Sx=z$ for all x in X.

PROOF. Let x be an arbitrary point in X. Then

$$d(Sx, TSx) \leqq \frac{cd(Sx, TSx)d(Sx, Sx)}{d(x, TSx)+d(Sx, Sx)} = 0$$

if $d(x, TSx)\neq 0$, and $d(Sx, TSx)=0$ if $d(x, TSx)=0$. Both of these possibilities imply that $TSx=Sx$ and so $Sx=z$ is a fixed point of T. As in the proof of Theorem 1, it follows that z is a unique common fixed point of S and T. Since z is unique, it now follows that $Sx=z$ for all x in X and so S is a constant mapping. This completes the proof of the theorem.

On putting $S=T$ in Theorem 2, we have the following

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COROLLARY. *If T is a mapping of the metric space X into itself such that*

$$d(Tx, Ty) \leqq \frac{cd(y, Ty)d(y, Tx)}{d(x, Ty)+d(y, Tx)}$$

for all x, y in X with $d(x, Ty)+d(y, Tx)\neq 0$, where $c\geqq 0$, and $d(Tx, Ty)=0$ otherwise, then T is a constant mapping.

We will now show that both of S and T in Theorem 1 may be nonconstant mappings. To see this, let X be the set of real numbers x, where $0\leqq x\leqq 1$, with metric $d(x, y)=|x-y|$ for all x, y in X. Define mappings S and T on X, identical to each other by $Sx=Tx=x/4$ for all x in X. Then, with $b=c=2/3$, inequality (1) is satisfied for all distinct x, y in X, but $S=T$ is not a constant mapping.

We finally prove a theorem for compact metric spaces:

THEOREM 3. *If S and T are continuous mappings of the compact metric space X into itself such that*

$$d(Sx, Ty) < \frac{c^{-1}d(x, Sx)d(x, Ty)+cd(y, Ty)d(y, Sx)}{d(x, Ty)+d(y, Sx)}$$

for all x, y in X with $d(x, Ty)+d(y, Sx)\neq 0$, where $c>0$, and $d(Sx, Ty)=0$ otherwise, then S and T have a unique common fixed point z.

PROOF. If there exists b, with $b<c^{-1}$, such that

$$d(Sx, Ty) \leqq \frac{bd(x, Sx)d(x, Ty)+cd(y, Ty)d(y, Sx)}{d(x, Ty)+d(y, Sx)}$$

for all x, y in X with $d(x, Ty)+d(y, Sx)\neq 0$, then the result follows from Theorem 1.

If no such b exists, let $\{b_n\}$ be a monotonically increasing sequence of positive numbers such that $\lim_{n\to\infty} b_n=c^{-1}$. Then we can find sequences $\{x_n\}$ and $\{y_n\}$ in X such that $d(x_n, Ty_n)+d(y_n, Sx_n)\neq 0$ and

$$d(Sx_n, Ty_n) > \frac{b_n d(x_n, Sx_n)d(x_n, Ty_n)+cd(y_n, Ty_n)d(y_n, Sx_n)}{d(x_n, Ty_n)+d(y_n, Sx_n)}$$

for $n=1, 2, \ldots$. Since X is compact, we can choose convergent subsequences $\{x_{n(k)}\}$ and $\{y_{n(k)}\}$ of $\{x_n\}$ and $\{y_n\}$ converging to x and y respectively. Putting $x_{n(k)}=x'_k$, $y_{n(k)}=y'_k$, $b_{n(k)}=b'_k$ we have

$$d(Sx'_k, Ty'_k) > \frac{b'_k d(x'_k, Sx'_k)d(x'_k, Ty'_k)+cd(y'_k, Ty'_k)d(y'_k, Sx'_k)}{d(x'_k, Ty'_k)+d(y'_k, Sx'_k)}$$

for $k=1, 2, \ldots$. Letting k tend to infinity we have, since S and T are continuous,

$$d(Sx, Ty) \geqq \frac{c^{-1}d(x, Sx)d(x, Ty)+cd(y, Ty)d(y, Sx)}{d(x, Ty)+d(y, Sx)}$$

giving a contradiction unless

$$d(Sx, Ty) = 0 = d(x, Ty)+d(y, Sx).$$

This implies that $x=Ty=Sx=y$ and so $x=y=z$ is a common fixed point of S and T.

Now suppose that S and T have another common fixed point w. Then

$$d(z, w) = d(Sz, Tw) < \frac{c^{-1}d(z, Sz)d(z, Tw)+cd(w, Tw)d(w, Sz)}{d(z, Tw)+d(w, Sz)} = 0,$$

a contradiction. The common fixed point must therefore be unique. This completes the proof of the theorem.

On putting $S=T$ in Theorem 3, we have the following

COROLLARY. *If T is a continuous mapping of the compact metric space X into itself such that*

$$d(Tx, Ty) < \frac{c^{-1}d(x, Tx)d(x, Ty)+cd(y, Ty)d(y, Tx)}{d(x, Ty)+d(y, Tx)}$$

for all x, y in X with $d(x, Ty)+d(y, Tx)\neq 0$, where $c>0$, and $d(Tx, Ty)=0$ otherwise, then T has a unique fixed point z.

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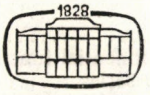

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