Studia Scientiarum Mathematicarum Hungarica

AUXILIO
CONSILII INSTITUTI MATHEMATICI
ACADEMIAE SCIENTIARUM HUNGARICAE

REDIGIT
A. RÉNYI

ADIUVANTIBUS
M. ARATÓ, L. FEJES TÓTH, T. FREY,
G. FREUD, L. KALMÁR, A. PRÉKOPA,
K. TANDORI

TOMUS IV.
FASC. 1—4.
1969

AKADÉMIAI KIADÓ, BUDAPEST

Studia Scientiarum Mathematicarum Hungarica

A Magyar Tudományos Akadémia matematikai folyóirata

Szerkesztőség: Budapest V., Reáltanoda u. 13—15.

Technikai szerkesztő: Petruska Gy.

Kiadja az Akadémiai Kiadó, Budapest V., Alkotmány u. 21.

A *Studia Scientiarum Mathematicarum Hungarica* angol, német, francia vagy orosz nyelven közöl eredeti értekezéseket a matematika tárgyköréből. Félévenként jelenik meg, évi egy kötetben.

Előfizetési ára belföldre 120,— Ft, külföldre 165,— Ft. Megrendelhető a belföld számára az Akadémiai Kiadónál, a külföld számára pedig a Kultúra Könyv és Hírlap Külkereskedelmi Vállalatnál (Budapest II., Fő u. 32).

Cserekapcsolatok felvétele ügyében kérjük az MTA Matematikai Kutató Intézete Könyvtárához (Budapest V., Reáltanoda u. 13—15) fordulni.

Közlésre szánt dolgozatokat kérjük két példányban a szerkesztőség címére küldeni.

Studia Scientiarum Mathematicarum Hungarica is a journal of the Hungarian Academy of Sciences publishing original papers on mathematics, in English, German, French or Russian.

It is published semiannually, making up one volume per year.

Editorial Office: Budapest V., Reáltanoda u. 13—15, Hungary.
Technical Editor: Gy. Petruska

Subscription rate: Ft 165 per volume. Orders may be placed with *Kultúra* Trading Co. for Books and Newpapers, Budapest 62, P.O.B. 149 or with its representatives abroad.

For establishing exchange relations please write to the Library of the Mathematical Institute (Budapest V., Reáltanoda u. 13—15).

Papers intended for publication should be sent to Editor in 2 copies.

Studia Scientiarum Mathematicarum Hungarica

Auxilio
Consilii Instituti Mathematici
Academiae Scientiarum Hungaricae

Redigit

A. Rényi

Adiuvantibus

M. Arató, L. Fejes Tóth, T. Frey, G. Freud, L. Kalmár,
A. Prékopa, K. Tandori

Tomus IV

MAGYAR
TUDOMÁNYOS AKADÉMIA
KÖNYVTÁRA

Akadémiai Kiadó, Budapest
1969

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 3—8.

ÜBER DIE VERGLEICHBARKEIT DER MIT GEWICHTSFUNKTIONEN GEBILDETEN MITTELWERTE

von

M. BAJRAKTAREVIĆ

1. Definitionen, Bezeichnungen und Problemaufstellung

Es sei:

1. $I = (0, +\infty)$;

2. Ω die Menge der in I stetigen und streng monotonen Funktionen φ und Q die Menge der positiven und in jedem beliebigen endlichen Teilintervall von I beschränkten Funktionen f, die in I definiert sind;

3. $E = \bigcup_{n=2}^{\infty} I^n$ mit $\mathbf{a}' \neq \mathbf{a}''$, wenn

$$\mathbf{a}' = (a_1, \ldots, a_l), \qquad \mathbf{a}'' = (a_1, \ldots, a_{l+m}),$$

$$a_\mu \in I \quad (\mu = 1, \ldots, l), \qquad a_{l+i} = 0 \quad (i = 1, \ldots, m);$$

4.

$$M_\varphi[\mathbf{a}]_f = \varphi^{-1}\left\{\sum_{i=1}^{n} f(a_i)\varphi(a_i) \Big/ \sum_{i=1}^{n} f(a_i)\right\} \tag{1}$$

der mit Gewichtsfunktion $f \in Q$ gebildete Mittelwert von $\mathbf{a} \in I^n$ bezüglich der Abbildungsfunktion $\varphi \in \Omega$, wo φ^{-1} die zu φ inverse Funktion ist.

Der Verfasser [2] hat zuerst den Mittelwertstypus (1) eingeführt und das Gleichheitsproblem unter Differenzierbarkeitsbedingungen bei fixen $n \geqq 3$ gelöst. J. ACZÉL und Z. DARÓCZY [1] beschäftigten sich dann mit den grundlegenden Eigenschaften der Mittelwerte (1).

In der Theorie der quasiarithmetischen Mittelwerte spielt das Vergleichbarkeitsproblem verschiedener Mittelwerte eine große Rolle (S. [5]). Wir werden hier dieses Problem bezüglich der Mittelwerte (1) formulieren. Man sucht notwendige und hinreichende Bedingungen für die Funktionen $\varphi, \psi \in \Omega$ und $f, g \in Q$, damit die Ungleichheit

$$M_\varphi[\mathbf{a}]_f \leqq M_\psi[\mathbf{a}]_g \tag{2}$$

für alle $\mathbf{a} \in I^n$ ($n \in \{2, 3, \ldots\}$ beliebig aber fest) erfüllt wird. Die Lösung dieses Problems ist bis jetzt in dieser allgemeinen Form nicht bekannt. Z. DARÓCZY [4] hat sich mit speziellen Fällen (wo $\varphi = \psi$ bzw. $f = g$ vorausgesetzt ist) von (2) für alle $\mathbf{a} \in \bigcup_{n=2}^{\infty} I^n$ und mit einigen Folgerungen aus ihnen beschäftigt.

In der vorliegenden Arbeit wollen wir die Ungleichungen (2) sowohl für die eben erwähnten zwei Sonderfälle als auch für den allgemeinen Fall für alle $\mathbf{a} \in I^n$ ($n \in \{2, 3, \ldots\}$ beliebig und fest) behandeln, und wir werden dann eine Anwendung unserer Ergebnisse auf die Potenzmittelwerte von BECKENBACH [3] angeben.

2. Resultate und Beweise

SATZ 1. *Es sei* $\varphi, \psi \in \Omega$ *und* $f \in Q$. *Für eine beliebige aber feste natürliche Zahl* $n \geqq 2$ *und beliebiges* $\mathbf{a} \in I^n$ *gilt die Ungleichung*

$$M_\varphi[\mathbf{a}]_f \leqq M_\psi[\mathbf{a}]_f \tag{3}$$

(bzw. die Ungleichung

$$M_\varphi[\mathbf{a}]_f \geqq M_\psi[\mathbf{a}]_f \text{)} \tag{3a}$$

dann und nur dann, falls entweder $\psi\varphi^{-1}$ *konvex und* ψ *wachsend (bzw. abnehmend) oder* $\psi\varphi^{-1}$ *konkav und* ψ *abnehmend (bzw. wachsend) ist.*

BEWEIS. a) *Die Voraussetzung ist hinreichend.* Es sei z. B. $F=\psi\varphi^{-1}$ eine konvexe und ψ eine wachsende Funktion. Dann gilt die Ungleichung

$$F\left(\sum_{i=1}^{n} q_i b_i\right) \leqq \sum_{i=1}^{n} q_i F(b_i) \tag{4}$$

für beliebiges $\mathbf{b} \in J^n$ [1] ($n \geqq 2$ fest) und $\mathbf{q}=(q_1, \ldots, q_n)$ $\left(\sum_{i=1}^{n} q_i = 1,\ q_i > 0\right)$ (S. [5], Satz 86).

Setzen wir jetzt in (4) $q_i = f(a_i) \Big/ \sum_{i=1}^{n} f(a_i) > 0$ und $b_i = \varphi(a_i)$ $(i=1, \ldots, n)$, so erhalten wir

$$F\left(\sum_{i=1}^{n} f(a_i)\varphi(a_i) \Big/ \sum_{i=1}^{n} f(a_i)\right) \leqq \sum_{i=1}^{n} f(a_i) F[\varphi(a_i)] \Big/ \sum_{i=1}^{n} f(a_i).$$

Da ψ eine wachsende Funktion ist, so hat auch ψ^{-1} dieselbe Eigenschaft, d. h. aus der obigen Ungleichung folgt

$$\varphi^{-1}\left\{\sum_{i=1}^{n} f(a_i)\varphi(a_i) \Big/ \sum_{i=1}^{n} f(a_i)\right\} \leqq \psi^{-1}\left\{\sum_{i=1}^{n} f(a_i) F[\varphi(a_i)] \Big/ \sum_{i=1}^{n} f(a_i)\right\}.$$

Daraus folgt (3) wegen $F=\psi\varphi^{-1}$. Wenn F eine konkave Funktion und ψ eine abnehmende Funktion ist, so ist der Beweis ähnlich. Ähnlicherweise beweist man die Geltung der Ungleichung (3a), falls *F konvex und* ψ *abnehmend oder F konkav und* ψ *wachsend ist.*

b) *Die Voraussetzung ist notwendig.* Es sei jetzt $\mathbf{a}=(a_1, a_2, \ldots, a_2) \in I^{n+1}$ ($n \in \{1, 2, \ldots\}$ beliebig und fest) und ψ eine wachsende Funktion. Aus (3) folgt — mit der Bezeichnung $F=\psi\varphi^{-1}$ —

$$F\left\{\frac{f(a_1)\varphi(a_1)+nf(a_2)\varphi(a_2)}{f(a_1)+nf(a_2)}\right\} \leqq \frac{f(a_1)F[\varphi(a_1)]+nf(a_2)F[\varphi(a_2)]}{f(a_1)+nf(a_2)}.$$

Wir setzen $\varphi(a_i)=t_i$ und $f[\varphi^{-1}(t_i)]=g(t_i)=g_i>0$ $(i=1, 2)$, dann ist

$$F\left(\frac{g_1 t_1 + ng_2 t_2}{g_1+ng_2}\right) \leqq \frac{g_1 F(t_1)+ng_2 F(t_2)}{g_1+ng_2}. \tag{5}$$

[1] $J=(\varphi(0), \varphi(+\infty))$, falls φ eine wachsende Funktion ist; $J=(\varphi(+\infty), \varphi(0))$, falls φ eine abnehmende Funktion ist.

Die Kurve der Funktion $y=F(t)$ ist stetig und nach (5) gibt es auf einer beliebigen Sehne dieser Kurve, wegen der Ungleichung

$$t_1 < \frac{g_1 t_1 + n g_2 t_2}{g_1 + n g_2} < t_2,$$

einen von den Endpunkten verschiedenen Punkt der über oder auf der Kurve liegt. So ist die Funktion F laut der Behauptung des Satzes 88 von [5] konvex. Im Falle einer abnehmenden Funktion ψ ist der Beweis analog. Damit haben wir den Satz 1 bezüglich der Ungleichung (3) vollständig bewiesen. Im Falle der Ungleichung (3a) ist der Beweis ganz analog.

Aus dem Beweis des Satzes und aus dem Satz 89 von [5] ist leicht ersichtlich, daß die Gleichheit in (3) bzw. (3a) für beliebiges $\mathbf{a}\in I^n$ wo $n\in\{2, 3, \ldots\}$ beliebig aber fest ist, dann und nur dann gilt, falls $F=\psi\varphi^{-1}$ eine lineare Funktion ist, d. h. ψ die Gestalt $\psi = a\varphi+b$ $(a\neq 0)$ hat.

SATZ 2. *Es sei $\varphi\in\Omega$ und $f, g\in Q$. Für beliebige aber feste natürliche Zahl $n\in\{2, 3, \ldots\}$ und jedes $\mathbf{a}\in I^n$ gilt die Ungleichung*

$$M_\varphi[\mathbf{a}]_f \leqq M_\varphi[\mathbf{a}]_g \tag{6}$$

dann und nur dann, falls die Funktion $f(t)/g(t)$ abnehmend ist.

BEWEIS. a) *Die Voraussetzung ist notwendig.* Es sei auch jetzt $\mathbf{a}=(a_1, a_2, \ldots, a_2)\in$ $\in I^{n+1}$ $(n\in\{1, 2, \ldots\}$ beliebig und fest). Dann hat die Ungleichung (6) die folgende Form

$$\varphi^{-1}\left\{\frac{f(a_1)\varphi(a_1)+nf(a_2)\varphi(a_2)}{f(a_1)+nf(a_2)}\right\} \leqq \varphi^{-1}\left\{\frac{g(a_1)\varphi(a_1)+ng(a_2)\varphi(a_2)}{g(a_1)+ng(a_2)}\right\}. \tag{7}$$

Es sei z. B. φ eine wachsende Funktion, so erhalten wir aus (7)

$$f(a_1)g(a_2)[\varphi(a_1)-\varphi(a_2)] \leqq f(a_2)g(a_1)[\varphi(a_1)-\varphi(a_2)]. \tag{8}$$

Es sei $a_1 > a_2$. Dann ist $\varphi(a_1)-\varphi(a_2)$ wegen des Wachstums der Funktion φ positiv, so daß aus (8) $f(a_1)/g(a_1)\leqq f(a_2)/g(a_2)$ folgt, und dies ist gerade die Behauptung des Satzes. Wenn φ eine abnehmende Funktion ist, so kann man den Beweis ähnlich durchführen.

b) *Die Voraussetzung ist hinreichend.* Der Beweis des Satzes 2 ist wörtlich derjenige des entsprechenden Satzes 2 von DARÓCZY [4].

Man kann leicht sehen, daß in (6) die Gleichheit für alle $a\in I^n$ mit $n\in\{2, 3, \ldots\}$ beliebig und fest dann und nur dann gilt, wenn $g(t)=\alpha f(t)(\alpha\neq 0)$ ist.

SATZ 3. *Unter den Bedingungen:*

1. $\varphi, \psi\in\Omega$, $f, g\in Q$;
2. a) $\psi\varphi^{-1}$ *ist konvex,* ψ *wachsend bzw. abnehmend oder*
   b) $\psi\varphi^{-1}$ *ist konkav,* ψ *abnehmend bzw. wachsend,*
3. f/g *ist abnehmend bzw. wachsend,*

gilt die Ungleichung

$$M_\varphi[\mathbf{a}]_f \leqq M_\psi[\mathbf{a}]_g \quad \textit{bzw.} \quad M_\varphi[\mathbf{a}]_f \geqq M_\psi[\mathbf{a}]_g$$

für beliebiges aber festes $n\in\{2, 3, \ldots\}$ und für jedes $\mathbf{a}\in I^n$.

BEWEIS. Es gilt nach dem Satz 2 die Ungleichung

$$M_\psi[\mathbf{a}]_f \leqq M_\psi[\mathbf{a}]_g \quad \text{bzw.} \quad M_\psi[\mathbf{a}]_f \geqq M_\psi[\mathbf{a}]_g,$$

und nach dem Satze 1 die Ungleichung

$$M_\varphi[\mathbf{a}]_f \leqq M_\psi[\mathbf{a}]_f \quad \text{bzw.} \quad M_\varphi[\mathbf{a}]_f \geqq M_\psi[\mathbf{a}]_f.$$

Daraus folgt unmittelbar die Behauptung des Satzes 3.

BEMERKUNG. Die folgenden Beispiele zeigen, daß die Bedingungen dieses Satzes zu eng sind.

BEISPIEL 1. Wenn $\varphi(x)=x^4$, $\psi(x)=x^2$, $f(x)=1$, $g(x)=x^2$, $x>0$, dann ist $\psi\varphi^{-1}(x)=\sqrt{x}$ konkav, ψ wachsend, f/g abnehmend und $M_\varphi[\mathbf{a}]_f \leqq M_\psi[\mathbf{a}]_g$. Dies ist leicht ersichtlich aus der Gleichgiltigkeit der Ungleichungen

$$\left\{\sum_{i=1}^{n} x_i^4/n\right\}^{1/4} \leqq \left\{\sum_{i=1}^{n} x_i^4 \Big/ \sum_{i=1}^{n} x_i^2\right\}^{1/2},$$

$$2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} x_i^2 x_j^2 \leqq \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} (x_i^4 + x_j^4).$$

BEISPIEL 2. Wenn $\varphi=x^{1/4}$, $\psi=\sqrt{x}$, $f(x)=\sqrt[4]{x}$, $g(x)=1$, $x>0$, dann ist $\psi\varphi^{-1}$ konvex, ψ wachsend, f/g wachsend und $M_\varphi[\mathbf{a}]_f \geqq M_\psi[\mathbf{a}]_g$.

SATZ 4. *Setzen wir voraus, daß die gegebenen Funktionen* $f, g \in Q$, $\varphi, \psi \in \Omega$ *die folgenden Bedingungen erfüllen:*

1. $M_\varphi[\mathbf{a}]_f \leqq M_\psi[\mathbf{a}]_g$ *bzw.* $M_\varphi[\mathbf{a}]_f \geqq M_\psi[\mathbf{a}]_g$ *bei beliebigem aber festem* $n \in \{2, 3, \ldots\}$ *und bei jedem* $\mathbf{a} \in I^n$,

2. $\psi\varphi^{-1}$ *ist konvex und* ψ *abnehmend bzw.* $\psi\varphi^{-1}$ *konvex und* ψ *wachsend oder* $\psi\varphi^{-1}$ *ist konkav und* ψ *wachsend bzw.* $\psi\varphi^{-1}$ *konkav und* ψ *abnehmend, dann ist* f/g *abnehmend bzw. wachsend.*

BEWEIS. Es sei z. B. $F=\psi\varphi^{-1}$ eine konvexe und ψ eine abnehmende Funktion. Setzen wir voraus, f/g sei der Behauptung widersprechend nicht abnehmend. Dann gibt es zwei Punkte $x_1 < x_2$ so daß $f(x_1)/g(x_1) < f(x_2)/g(x_2)$ ist. Man kann immer zwei Funktionen f^* und g^*, $f^*, g^* \in Q$ derart finden, daß f^*/g^* wachsend und $f^*(x_i) = f(x_i)$, $g^*(x_i)=g(x_i)$ $(i=1, 2)$ ist. Ist nähmlich $G>0$ eine wachsende und $g^* \in Q$ eine beliebige Funktion und sind die Bedingungen $G(x_i)=f(x_i)/g(x_i)$, $g^*(x_i)=g(x_i)$ $(i=1, 2)$ erfüllt, dann ist $f^* \in Q$ durch $f^*=Gg^*$ definiert und es gilt $f^*(x_i)=f(x_i)$ $(i=1, 2)$. Dann gilt für $\mathbf{a}=(x_1, x_2, \ldots, x_2) \in I^{n+1}$ nach dem Satz 2 und nach der Beweisedurchführung dieses Satzes die Ungleichung

$$M_\varphi[\mathbf{a}]_f > M_\varphi[\mathbf{a}]_g.$$

Nach dem Satze 1 aus 2. folgt

$$M_\varphi[\mathbf{a}]_g \geqq M_\psi[\mathbf{a}]_g$$

und daraus

$$M_\varphi[\mathbf{a}]_f > M_\psi[\mathbf{a}]_g$$

im Widerspruch zu der Annahme 1.

So ist die Behauptung in diesem Falle ganz bewiesen. In anderen Fällen soll der Beweis ganz ähnlich durchgeführt werden. Damit ist der Beweis des Satzes 4 vollendet.

SATZ 5. *Falls die Funktionen* $\varphi, \psi \in \Omega$ $(\varphi' \neq 0, \psi' \neq 0)$ *und* $f, g \in Q$ *zweimal stetig differenzierbar sind, dann ist die Relation*

$$\frac{\varphi''}{\varphi'} - \frac{\psi''}{\psi'} + 2\left(\frac{f'}{f} - \frac{g'}{g}\right) \leqq 0 \tag{9}$$

eine notwendige Bedingung für das Bestehen der Ungleichung

$$M_\varphi[\mathbf{a}]_f \leqq M_\psi[\mathbf{a}]_g$$

für alle $\mathbf{a} \in I^n$ $(n \in \{2, 3, \ldots\}$ *beliebig aber fest).*

BEWEIS. Es sei $\mathbf{a} = (x, y, \ldots, y) \in I^{n+1}$ $(n \in \{1, 2, \ldots\})$ und

$$A = A(x, y) = \frac{f(x)\varphi(x) + nf(y)\varphi(y)}{f(x) + nf(y)}, \qquad B = B(x, y) = \frac{g(x)\psi(x) + ng(y)\psi(y)}{g(x) + ng(y)},$$

$$\Delta(x, y) = \varphi^{-1}(A) - \psi^{-1}(B) \leqq 0.$$

Dann folgt für $x = y$ aus der Relation

$$\frac{\partial \Delta(x, y)}{\partial x} = \varphi^{-1\prime}(A) \cdot A'_x - \psi^{-1\prime}(B) \cdot B'_x$$

die Relation

$$\frac{\partial \Delta(y, y)}{\partial x} = 0 \qquad (y \in I),$$

und aus der Relation

$$\frac{\partial^2 \Delta(x, y)}{\partial x^2} = \varphi^{-1\prime\prime}(A) \cdot (A'_x)^2 + \varphi^{-1\prime}(A) \cdot A''_{x^2} - \psi^{-1\prime\prime}(B) \cdot (B'_x)^2 - \psi^{-1\prime}(B) \cdot B''_{x^2}$$

die Relation

$$\frac{\partial^2 \Delta(y, y)}{\partial x^2} = \frac{n}{(n+1)^2}\left\{\frac{\varphi''(y)}{\varphi'(y)} - \frac{\psi''(y)}{\psi'(y)} + 2\left(\frac{f'(y)}{f(y)} - \frac{g'(y)}{g(y)}\right)\right\}.$$

Anderseits aus der Relation

$$\frac{\Delta(x, y)}{(x-y)^2} \leqq 0 \qquad (x \neq y)$$

bekommt man die Relation

$$\lim_{x \to y} \frac{\Delta(x, y)}{(x-y)^2} = \frac{1}{2} \lim_{x \to y} \frac{\partial^2 \Delta(x, y)}{\partial x^2} = \frac{1}{2} \frac{\partial^2 \Delta(y, y)}{\partial x^2} \leqq 0.$$

Somit ist alles bewiesen.

Als unmittelbare Folgerung und eine Anwendung auf die Potenzmittelwerte von Beckenbach (S. [3] und [4]) ist der folgende

SATZ 6. *Die Ungleichung* $p_1 - p_2 + 2(q_1 - q_2) \leqq 0$ *ist eine notwendige Bedingung für das Bestehen der Ungleichheit* $M_{p_1}[\mathbf{a}]_{q_1} \leqq M_{p_2}[\mathbf{a}]_{q_2}$ $(\mathbf{a} \in I^n,\ n \in \{2, 3, \ldots\}$ *beliebig und fest), wo*

$$M_p[\mathbf{a}]_q = \left\{ \frac{\sum_{i=1}^{n} a_i^{p+q}}{\sum_{i=1}^{n} a_i^q} \right\}^{\frac{1}{p}} \qquad (p \in (-\infty, 0) \cup (0, +\infty),\ q \in (-\infty, +\infty))$$

und

$$M_0[\mathbf{a}]_q = \exp\left\{ \frac{\sum_{i=1}^{n} a_i^q \log a_i}{\sum_{i=1}^{n} a_i^q} \right\} \qquad (q \in (-\infty, +\infty))$$

ist.

Die speziellen Potenzmittelwerte von Beckenbach $M_{\alpha-1}[\mathbf{a}]_1$ bzw. $M_0[\mathbf{a}]_1$ sind in enger Beziehung mit der Informationsmenge (Entropie) der Ordnung $\alpha > 0$, $\alpha \neq 1$ bzw. mit der Shannonschen Entropie ($\alpha = 1$) (s. [6] und [1]).

Am Ende sei noch bemerkt daß die Ungleichheit

$$M_2[\mathbf{a}]_0 \leqq M_1[\mathbf{a}]_q, \qquad \mathbf{a} = (x, y) \in I^2$$

dann und nur dann gilt, falls $q \geqq \frac{1}{2}$ ist. Wegen der wachsenden Monotonie der Mittelwerte $M_1[\mathbf{a}]_q$ mit wachsendem q (s. den Satz 5, [4]) ist es hinreichend nur die Ungleichheit

$$M_2[\mathbf{a}]_0 \leqq M_1[\mathbf{a}]_{\frac{1}{2}}, \qquad \mathbf{a} = (x, y) \in I^2$$

zu beweisen. Es ist aber leicht einzusehen, daß diese Ungleichheit gleichbedeutend mit der Ungleichheit $(x-y)^2 \geqq 0$ ist.

LITERATURVERZEICHNIS

[1] ACZÉL J. und DAROCZY Z.: Über verallgemeinerte quasilineare Mittelwerte, die mit Gewichtsfunktionen gebildet sind, *Publ. Math. Debrecen* **10** (1963) 171—190.
[2] BAJRAKTAREVIĆ, M.: Sur une équation fonctionnelle aux valeurs moyennes, *Glasnik Mat.-Fiz. Astr.* **13** (1958) 243—248.
[3] BECKENBACH, E. F.: A class of mean value functions, *Amer. Math. Monthly* **57** (1950) 1—6.
[4] DARÓCZY, Z.: Einige Ungleichungen über die mit Gewichtsfunktionen gebildeten Mittelwerte, *Monatsh. Math.* **68** 2. Heft (1964) 102—112.
[5] HARDY, J. E., LITTLEWOOD and PÓLYA, G.: *Inequalities*, Cambridge, 1952.
[6] RÉNYI, A.: On measures of entropy and information, *Proceedings of the Fourth Berkeley Symposium on Math. Statistics and Probability*, 1960, I, Berkeley, 1961, 547—561.

Naturwissenschaftlich-mathematische Fakultät, Sarajevo

(Eingegangen: 17. Februar, 1965.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 9—11.

A PROBLEM ON SEMI-GROUPS

by

D. A. BURGESS

It is well-known that if G is a finite group of order n and

$$x(1), x(2), \ldots, x(n) \tag{1}$$

are n (not necessarily distinct) elements of G, then there is a non-empty subset $x(i_1), \ldots, x(i_r)$ of (1) whose product

$$x(i_1)\ldots x(i_r)$$

is the identy element of G. The special case of this result in which G is the additive group of residue classes modulo the positive integer n appears in a paper [1] of P. ERDŐS. Dr. ERDŐS asked me if I could extend this result to semigroups, that is if I could prove:

(I) *If S is a semi-group of finite order n and if*

$$y(1), \ldots, y(n) \tag{2}$$

are n (not necessarily distinct) elements of S, then there is a non-empty subset $y(i_1), \ldots, y(i_r)$ of (2) whose product

$$y(i_1)\ldots y(i_r) \tag{3}$$

is an idempotent.

In this paper I give a partial answer to this question. The result presented is:

THEOREM. *Statement (I) holds if S contains only one idempotent.*

As a consequence I shall deduce:

COROLLARY. *Statement (I) holds if S is commutative.* Thus in particular (I) holds in the number theoretic case in which S is the set of residue classes modulo the positive integer n under multiplication.

PROOF OF THEOREM. Let S be a finite semigroup containing just one idempotent e. Then for each element y of S there exists a positive integer $m[y]$ so that

$$y^{m[y]} = e.$$

Note that consequently y and e commute.

Denote by N the subset consisting of elements y of S for which

$$ye = e,$$

and have a form of type (3). We suppose that (I) is false for S and the system (2) and show first that then N is non-empty. For consider the n products

$$y(1)y(2)\ldots y(i) \qquad (i=1, 2, \ldots, n).$$

On our assumption two of these must be equal, say

$$y(1)\ldots y(i)=y(1)\ldots y(j) \qquad (i<j).$$

From this it follows that

$$\begin{aligned} ey(i+1)\ldots y(j) &= \{y(1)\ldots y(i)\}^{m[y(1)\ldots y(i)]}\{y(i+1)\ldots y(j)\} \\ &= \{y(1)\ldots y(i)\}^{m[y(1)\ldots y(i)]-1}\{y(1)\ldots y(j)\} \\ &= \{y(1)\ldots y(i)\}^{m[y(1)\ldots y(i)]-1}\{y(1)\ldots y(i)\} \\ &= e \end{aligned}$$

and thus $y(i+1)\ldots y(j)$ is an element of N.

Now choose a maximal element $y(i_1)y(i_2)\ldots y(i_r)$ of N (maximal in the sense, that no product of type $y(i_1)y(i_2)\ldots y(i_r)y(i_{r+1})\ldots y(i_t)$ $(t>r;\ i_l\neq i_j$ if $j\neq l)$ is an element of N).

By renumbering the y's we may assume it to be

$$y(1)\ldots y(r) \qquad (r>0).$$

Applying the above argument we obtain again for some i, j

$$y(1)\ldots y(i)=y(1)\ldots y(j) \qquad (i<j),$$

and hence

$$ey(i+1)\ldots y(j) = e.$$

Now if $i\geqq r$ then

$$y(1)\ldots y(r)y(i+1)\ldots y(j)\in N$$

which contradicts the maximal property of $y(1)\ldots y(r)$. Thus $i<r$. But we also have

$$\begin{aligned} y(1)\ldots y(i) &= \{y(1)\ldots y(i)\}\{y(i+1)\ldots y(j)\} \\ &= \{y(1)\ldots y(i)\}\{y(i+1)\ldots y(j)\}^2 \\ &= \ldots \\ &= \{y(1)\ldots y(i)\}\{y(i+1)\ldots y(j)\}^{m[y(i+1)\ldots y(j)]} \\ &= y(1)\ldots y(i)e \end{aligned}$$

and so we see that

$$\begin{aligned} y(1)\ldots y(r) &= y(1)\ldots y(i)y(i+1)\ldots y(r) \\ &= y(1)\ldots y(i)ey(i+1)\ldots y(r) \\ &= y(1)\ldots y(i)y(i+1)\ldots y(r)e \\ &= e, \end{aligned}$$

since $y(1)\ldots y(r)$ belongs to N. This proves the theorem.

PROOF OF COROLLARY. A finite commutative semigroup can be partitioned into subsemigroups each containing a single idempotent (See e. g. [2]). For at least one of these subsemigroups the hypotheses of the theorem are satisfied.

REFERENCES

[1] ERDŐS, P.: On the representation of large integers as sums of distinct summands taken from a fixed set, *Acta Arith.* 7 (1962) 345—354, Lemma 3.
[2] CLIFFORD, A. H. and PRESTON, G. B.: *The algebraic theory of semigroups*, Vol. 1. § 1. 6, Exercise 4.

The University of Nottingham

(Received February 1, 1966)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 13—30.

SUR DEUX PROBLEMES D'AFFECTATION

par

HOÀNG TỤY et NGUYỄN QUANG THÁI

Introduction

Dans ce travail nous étudierons les deux problèmes suivants:

PROBLÈME A. Maximiser z, avec:

$$x_{ij} \geqq 0 \qquad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n) \tag{1}$$

$$\sum_{j=1}^{n} x_{ij} \leqq a_i \qquad (i = 1, 2, \ldots, m) \tag{2}$$

$$\sum_{i=1}^{m} b_{ij} x_{ij} = b_j z \qquad (j = 1, 2, \ldots, n). \tag{3}$$

PROBLÈME B. Minimiser $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$, avec:

$$x_{ij} \geqq 0 \qquad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n) \tag{4}$$

$$\sum_{j=1}^{n} x_{ij} \leqq a_i \qquad (i = 1, 2, \ldots, m) \tag{5}$$

$$\sum_{i=1}^{m} b_{ij} x_{ij} = b_j \qquad (j = 1, 2, \ldots, n), \tag{6}$$

où l'on suppose que: $a_i > 0$ $(i = 1, 2, \ldots, m)$, $b_j > 0$ $(j = 1, 2, \ldots, n)$, $b_{ij} \geqq 0$, $c_{ij} \geqq 0$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$, $\max\limits_i b_{ij} > 0$ $(j = 1, 2, \ldots, n)$.

Ces problèmes se rencontrent notamment dans la répartition des tâches entre plusieurs machines (ou entreprises): a_i est alors le fonds de temps disponible pour la machine i; b_{ij} — le volume de travail que peut effectuer par unité de temps la machine i quand elle travaille sur la tâche j; $b_j z$ (dans le problème A) ou b_j (dans le probléme B) — le volume total de travail à effectuer pour la tâche j, x_{ij} — le temps (à déterminer) que la machine i doit réserver à la tâche j, pour maximiser la production totale (problème A) ou minimiser le coût total (problème B).

Comme l'application de la méthode générale du simplexe à ces problèmes exigerait un volume important de calculs, à cause du grand nombre de variables, beaucoup d'auteurs ([2], [3], [4], [8], [9]) ont cherché des algorithmes plus appropriés, tenant suffisament compte de la structure spéciale des problèmes traités et de ce fait exigeant moins de calculs. Dans ce qui suit nous allons exposer une nouvelle méthode, plus simple, qui a l'avantage d'être exempte des considéra-

tions habituelles de dégénérescence et exige, semble-t-il, un volume de calcul sensiblement réduit par rapport aux méthodes actuellement disponibles.

Le point de vue général adopté dans ce travail sera celui de la théorie des graphes, dont les notions fondamentales sont supposées connues du lecteur [1].

§ 1. Problème A

1. Flot maximal dans un réseau. Formons le tableau T à m lignes $i=1, 2, \ldots, m$ et à $n+1$ colonnes: $j=0, 1, 2, \ldots, n$, et regardons ce tableau comme un graphe simple dont les sommets du premier groupe sont les lignes, ceux du second groupe les colonnes, chaque arête (i,j) étant représentée par la case située à l'intersection de la ligne i et de la colonne j. Soit G un sous-graphe quelconque de T. Désignons l'ensemble des sommets et des arêtes de G par la même lettre, ce qui permet d'écrire: $(i,j)\in G$, $i\in G$, $j\in G$, pour indiquer respectivement que la case (i,j), la ligne i, la colonne j, appartiennent au sous-graphe G. Nous disons qu'une matrice $[x_{ij}]$ est un *flot* dans le réseau G si l'on *a*:

$$x_{ij} \geqq 0 \qquad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n, 0) \tag{7}$$

$$x_{ij} = 0 \qquad (i,j) \notin G \tag{8}$$

$$\sum_{j=1}^{n} x_{ij} + x_{i0} = a_i \qquad (i = 1, 2, \ldots, m) \tag{9}$$

$$\sum_{i=1}^{m} b_{ij} x_{ij} = \theta b_j \qquad (j = 1, 2, \ldots, n). \tag{10}$$

Le nombre θ est appelé *valeur* du flot $x=[x_{ij}]$. S'il n'existe aucun flot $x'=[x'_{ij}]$ sur G de valeur $\theta'>\theta$, $x=[x_{ij}]$ est appelé un *flot maximal* dans le réseau G, et θ est la *capacité* de ce réseau. Le problème A revient évidemment à chercher la capacité du tableau T.

Le calcul de la capacité d'un réseau est particulièrement simple lorsque ce réseau est une forêt du tableau T, c'est à dire un graphe partiel G sans cycle de T. En effet, dans ce cas, la matrice des coefficients des $x_{ij}>0$ dans le système (8), (9), (10) est triangulaire, ce qui permet de résoudre ce système très facilement. On obient la solution de ce système sous la forme

$$x_{ij}(\theta) = \alpha_{ij} - \beta_{ij}\theta, \tag{11}$$

et la capacité du réseau G est

$$\bar{\theta} = \min\left\{\frac{\alpha_{ij}}{\beta_{ij}} : \beta_{ij} > 0\right\}, \tag{12}$$

à condition, bien entendu, qu'on ait

$$\max\left\{\frac{\alpha_{ij}}{\beta_{ij}} : \beta_{ij} < 0\right\} \leqq \min\left\{\frac{\alpha_{ij}}{\beta_{ij}} : \beta_{ij} > 0\right\}. \tag{13}$$

Le flot maximal est alors $\bar{x}=\{x_{ij}(\bar{\theta})\}$.

EXEMPLE. Pour chaque $j=1, 2, \ldots, n$ soit $i(j)$ l'indice tel que $b_{i(j)j}=\max_i b_{ij}$. L'ensemble des cases $(i(j), j)$ $(j=1, 2, \ldots, n)$ et des cases $(i, 0)$, $(i=1, 2, \ldots, m)$ constitue un arbre G, dont le flot associé est

$$x_{i(j)j} = \frac{\theta b_j}{b_{i(j)j}} \qquad (j = 1, 2, \ldots, n)$$

$$x_{i0} = a_i - \sum_{i(j)=i} x_{i(j)j} \qquad (i = 1, 2, \ldots, m)$$

et dont la capacité est

$$\bar{\theta} = \min_i \frac{a_i}{\sum_{i(j)=i} \frac{b_j}{b_{i(j)j}}} > 0.$$

2. Propriétés du flot maximal. Soit G un réseau quelcoque (non nécessairement une forêt) de capacité θ, $x=[x_{ij}]$ un flot maximal dans G. Comme toujours, nous supposons $b_{ij}>0$ pour chaque $(i, j)\in G$. Une case $(k, l)\in G$ est dite occupée si $x_{kl}>0$. Une colonne j est dite *puissante* si $j=0$ ou si elle est reliée à la colonne 0 par une chaîne Γ_j de cases de G, telle que les cases impaires de Γ_j soient occupées — les cases étant comptées dans le parcours de Γ_j, à partir de la colonne 0. Une colonne qui n'est pas puissante est *faible*.

THÉORÈME 1. *Il existe toujours au moins une colonne faible.*

DÉMONSTRATION. En vertu du théorème connu de dualité on peut trouver des nombres λ_i, μ_j tels que

$$\lambda_i - b_{ij}\mu_j \geqq 0 \qquad (i, j)\in G,\ j\neq 0 \tag{14}$$

$$\sum_{j=1}^{n} b_j \mu_j \geqq 1 \tag{15}$$

$$\lambda_i \geqq 0 \qquad (i, 0)\in G \tag{16}$$

$$\lambda_i - b_{ij}\mu_j = 0 \quad \text{si} \quad x_{ij} > 0 \tag{17}$$

$$\lambda_i = 0 \quad \text{si} \quad x_{i0} > 0. \tag{18}$$

Soit j une colonne puissante quelconque, $\Gamma_j=\{(i_0, 0), (i_0, j_1), (i_1, j_1), \ldots, (i_k, j_k), (i_k, j)\}$ la chaîne de cases de G telle que $x_{i_0, 0}>0$, $x_{i_1 j_1}>0, \ldots, x_{i_k j_k}>0$. D'après (18) on a $\lambda_{i_0}=0$, donc en vertu de (14) $-b_{i_0 j_1}\mu_{j_1}\geqq 0$, c'est-à-dire $\mu_{j_1}\leqq 0$. D'autre part, comme $x_{i_1 j_1}>0$ on a, d'après (17), $\lambda_{i_1}=b_{i_1 j_1}\mu_{j_1}\leqq 0$ d'où, en vertu de (1), $b_{i_1 j_2}\mu_{j_2}\leqq\lambda_{i_1}\leqq 0$ c'est-à-dire $\mu_{j_2}\leqq 0$ et ainsi de suite. Finalement, $\mu_j\leqq 0$. Donc si toutes les colonnes étaient puissantes on aurait $\mu_j\leqq 0$ pour tout $j=1, 2, \ldots, n$, contrairement à (15).

Une ligne i est dite *puissante* si elle coupe une colonne puissante suivant une case occupée; *faible*, si elle n'est pas puissante. Les propriétés suivantes sont faciles à vérifier:

P. 1) si la ligne i est puissante et $(i, j)\in G$, la colonne j est puissante,

P. 2) si la colonne $j\neq 0$ est puissante, toute ligne et colonne incidente à Γ_j est puissante.

P. 3) si la case (i, j) est à l'intersection d'une ligne faible et d'une colonne puissante, on a $x_{ij}=0$.

De ces propriétés on déduit une méthode très simple pour déterminer les lignes et colonnes puissantes: on marque la colonne 0; puis, si une colonne est déjà marquée, on marque toutes les lignes non marquées qui coupent cette colonne suivant une case occupée; si une ligne est déjà marquée, on marque toutes les colonnes non marquées qui coupent cette ligne suivant une case de G. Cela jusqu'à ce qu'on ne peut plus continuer. Alors les lignes et colonnes marquées constituent l'ensemble des lignes et colonnes puissantes.

Nous dirons qu'une case (i, j) est un *point mort*, si i est faible, j puissant; un *pont* si i est puissant, j faible. On a toujours, $x_{ij}=0$ pour un point mort, et $(i, j) \notin G$ pour un pont.

THÉORÈME 2. *Lorsqu'on ajoute à un réseau G un pont (r, s) et qu'on lui enlève un certain nombre de points morts, la capacité du réseau ou le nombre des colonnes puissantes doivent augmenter.*

DÉMONSTRATION. Supposons que la capacité du réseau n'augmente pas, c'est-à-dire que la capacité θ du nouveau réseau G' est égale à θ. En vertu de la propriété P, 3) ci-dessus, on a $x_{pq}=0$ pour chaque point mort (p, q), donc le flot x reste un flot maximal dans le réseau G'. Il est alors facile de voir que toute colonne puissante dans G reste puissante dans G'. En effet, pour toute colonne j puissante dans G, la chaîne Γ_j ne peut contenir un point mort (p, q), d'après la propriété P. 2). Donc la suppression des points morts n'influe pas sur Γ_j et la colonne j qui était puissante, le reste. D'autre part, puisque les cases occupées sont conservées, toute ligne puissante dans G le reste dans G'; en particulier la ligne r reste puissante dans G', et d'après la propriété P. 1), la colonne s qui était faible dans G, devient puissante dans G'. Par suite le nombre de colonnes puissantes dans G' est bien supérieur à celui dans G.

Comme on l'a vu, la détermination d'un flot maximal est particulièrement simple lorsque le réseau G est une forêt. Il y a donc intérêt à ce que les cycles ne se forment pas par l'adjonction des ponts. C'est ce que montre effectivement la proposition suivante;

THÉORÈME 3. — *Si G est une forêt, le nouveau réseau G' obtenu en ajoutant un pont (r, s) et en supprimant tous les points morts, sera encore une forêt.*

DÉMONSTRATION. — Le pont (r, s) forme avec les cases de la forêt G un cycle unique sur lequel doit se trouver au moins une case (p, q) située à l'intersection d'une ligne p et d'une colonne q de catégories différentes (sinon toutes les lignes et colonnes incidentes à ce cycle, en particulier r et s, seraient de même catégorie!). Or, en vertu de la propriete P. 1), puisque $(p, q) \in G$, cette case ne peut être un pont; c'est donc un point mort qui doit ainsi être supprimé. Il en résulte que G' ne peut avoir de cycles.

Les résultats précédents suggèrent une méthode pour augmenter pas à pas la capacité, en partant d'un réseau dont le flot maximal est trivial. Pour que l'absence des ponts assure que la capacité maximum est atteinte, nous opérons sur une classe particulière de réseaux qui va être définie.

3. Réseau normal. Un réseau G est dit *normal* s'il existe des nombres u_i, v_j tels que

$$u_i \geqq 1 \qquad i = 1, 2, \ldots, m \tag{19}$$

$$u_i \geqq b_{ij} v_j, \qquad i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n \tag{20}$$

$$u_i = 1 \qquad (i, 0) \in G \tag{21}$$

$$u_i = b_{ij} v_j \qquad (i, j) \in G,\ j \neq 0 \tag{22}$$

Le système $\{u_i, v_j\}$ est alors appelé un système de *potentiels* du réseau G.

Par exemple, le réseau construit à la fin du § 1.1. est normal. Son système de *potentiels* est

$$u_i = 1 \qquad (i = 1, 2, \ldots, m)$$

$$v_j = \frac{1}{b_{i(j)j}} \qquad (j = 1, 2, \ldots, n)$$

THÉORÈME 4. *Si pour un réseau normal G il n'existe aucun pont (i, j) avec $b_{ij} > 0$, la capacite θ de ce réseau est le maximum cherché.*

DÉMONSTRATION. Soit P l'ensemble des lignes et colonnes puissantes, F l'ensemble des lignes et colonnes faibles. Posons:

$$u_i' = \begin{cases} u_i & i \in P \\ \pi u_i & i \in F \end{cases}$$

$$v_j' = \begin{cases} v_j & j \in P \\ \pi v_j & j \in F \end{cases} \tag{23}$$

Il est facile de voir que, le cas $i \in P$, $j \in F$, $b_{ij} > 0$ étant ici exclu, le système (23) verifie pour tout $\pi \geqq 1$ les relations:

$$u_i' - b_{ij} v_j' \geqq 0, \qquad i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n \tag{24}$$

Supposons maintenant qu'il existe dans T un flot $x' = [x_{ij}]$ de valeur $\theta' > \theta$. En tenant compte de (24), et du fait que $u_i' \geqq 1$ $(i = 1, 2, \ldots, m)$ on peut alors écrire:

$$\sum_i a_i u_i' - \theta' \sum_j b_j v_j' = \sum_{i,j} (u_i' - b_{ij} v_j') x_{ij}' + \sum_i x_{i0}' u_i' \geqq \sum_i x_{i0}' \tag{25}$$

Or, on a aussi

$$\sum_i a_i u_i' - \theta' \sum_j b_j v_j' = \sum_{i \in P} a_i u_i - \theta' \sum_{j \in P} b_j v_j + \pi r(\theta') \tag{26}$$

où

$$r(\theta') = \sum_{i \in F} a_i u_i - \theta' \sum_{j \in F} b_j v_j \tag{27}$$

Comme $x_{ij} = 0$ pour $i \in F$, $j \notin F$, ou $i \notin F$, $j \in F$, et que $i \in F$ implique $x_{i0} = 0$, on voit aisément que:

$$\sum_{j \in F} x_{ij} = a_i \qquad (i \in F)$$

$$\sum_{i \in F} b_{ij} x_{ij} = \theta b_j \qquad (j \in F)$$

d'où il suit, en tenant compte de (22)

$$r(\theta) = \sum_{i\in F}\Big(\sum_{j\in F} x_{ij}\Big)u_i - \sum_{j\in F}\Big(\sum_{i\in F} b_{ij}x_{ij}\Big)v_j = \sum_{i\in F,\, j\in F}(u_i - b_{ij}v_j)x_{ij} = 0 \tag{28}$$

Si $\theta=0$, on aurait $x_{i0}>0$ pour tout i, donc toutes les lignes seraient puissantes, et la case (i_0, j_0) où $j_0\in F$, $b_{i_0 j_0}=\max\limits_i b_{ij_0}>0$ serait un pont. Donc $\theta>0$ et on doit avoir $v_j>0$ pour tout j, car chaque colonne j doit contenir alors au moins une case $(i,j)\in G$, pour laquelle on a, d'après (22)

$$v_j = \frac{u_i}{b_{ij}} \geqq \frac{1}{b_{ij}}$$

Il en résulte que

$$\sum_{j\in F} b_j v_j > 0$$

(car $F\neq\varnothing$) et par suite, en vertu de (27), (28), $r(\theta')<0$ pour tout $\theta'>\theta$. Dès lors, d'après (26), pour π suffisamment grand, on aura

$$\sum_i a_i u_i' - \theta' \sum_j b_j v_j' < \sum_i x_{i0}$$

contrairement à (25). L'existence dans T d'un flot de valeur $\theta'>\theta$ est ainsi impossible et le théorème est démontré.

Désignons par L l'ensemble des ponts (i,j) tels que $b_{ij}>0$:

$$L = \{(i,j): i\in P, j\in F, b_{ij} > 0\} \tag{29}$$

Supposons $L\neq\varnothing$ et cherchons la case $(r,s)\in L$ à ajouter au réseau normal G pour en déduire un réseau G' encore normal.

THÉORÈME 5. *Soit* (r,s) *un pont tel que*

$$\frac{u_r}{b_{rs}v_s} = \min\left\{+\frac{u_i}{b_{ij}v_j} : (i,j)\in L\right\} \tag{30}$$

Si on ajoute au réseau G ce pont, et qu'on lui enlève tous les points morts, le nouveau réseau G′ sera encore normal; *son système de potentiels sera déterminé par les égalités* (23), *où* π *est égal à la quantité* (30).

DÉMONSTRATION. En vertu de (20), on a $\pi\geqq 1$, et les relations $u_i'\geqq 1$ $(i=1, 2, \ldots, m$ $u_i' - b_{ij}v_j'\geqq 0$ $(i=1, 2, \ldots, m;\ j=1, 2, \ldots, n)$ sont vérifiées pour $i\in P$, $j\in P$, ou $i\in F$, $j\in F$, ou $i\in F$, $j\in P$, ou $i\in P$, $j\in F$, $b_{ij}=0$. Elles résultent immédiatement de (19) et (30) si $i\in P$, $j\in F$, $b_{ij}>0$.

D'autre part, pour la case (r,s) qui vient d'entrer dans le réseau, on a $u_i'=u_i$, $v_s'=\pi v_s$, d'où d'aprés la définition de π, $u_r - b_{rs}v_s' = 0$, tandis que pour les cases anciennes (i,j) qui sont conservées, on a soit $i\in P$, $j\in P$, soit $i\in F$, $j\in F$, de sorte que $u_i'=u_i$, $v_j'=v_j$ et les relations (22), (23) ne sont pas modifiées. Donc $\{u_i', v_j'\}$ est bien un système de potentiels pour le nouveau réseau G' qui est ainsi normal.

4 — Algorithme I — Ce qui précède conduit à l'algorithme suivant pour chercher la capacité du tableau T, c'est-à-dire pour résoudre le problème A.

On choisit au départ un réseau normal G_0, qui soit une forêt, de capacité $\theta_0 \geqq 0$. Ce réseau initial peut être pris égal au „réseau nul", formé par l'ensemble des cases de la colonne 0, auquel cas $\theta_0 = 0$, $u_i^0 = 1$ $(i = 1, 2, \ldots, m)$; $v_j^0 = 0$ $(j = 1, 2, \ldots, n)$. Mais il est mieux de prendre comme réseau initial celui construit suivant l'exemple donné à la fin du § 1. 1.

Dans chaque étape ultérieure, on dispose d'un réseau G, sans cycles, de capacité $\theta \geqq 0$, de flot maximal $x = [x_{ij}]$, et de potentiels $\{u_i, v_j\}$. On détermine, par le procédé de marquage, l'ensemble P des lignes et collonnes puissantes, et l'ensemble

$$L = \{(i,j): i \in P, j \in P, b_{ij} > 0\} \tag{31}$$

1. Si $L = \varnothing$, θ est le maximum cherché, et $x = x_{ij}$ est une solution optimale du problème.

2. Si $L \neq \varnothing$, on choisit un pont $(r, s) \in L$, satisfaisant au critère (30), qu'on ajoute au réseau G. On supprime de G tous les points morts pour former un nouveau réseau G', et on détermine, par la méthode indiquée au § 1. 1. le flot

$$[x_{ij}^*(\theta) = \alpha'_{ij} - \beta'_{ij}\theta]$$

la capacité

$$\theta' = \min\left\{\frac{\alpha'_{ij}}{\beta'_{ij}} : \beta'_{ij} > 0\right\}$$

puis de là le flot maximal dans G':

$$x' = [x'_{ij}] = [x_{ij}^*(\theta')]$$

On calcule ensuite le nouveau système de potentiels $\{u'_i, v'_j\}$ par les formules (23), où

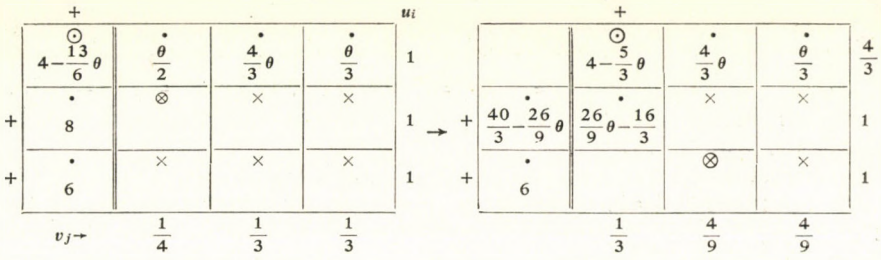
$$\pi = \frac{u_r}{b_{rs} v_s}$$

et on passe à l'étape suivante. Ainsi de suite.

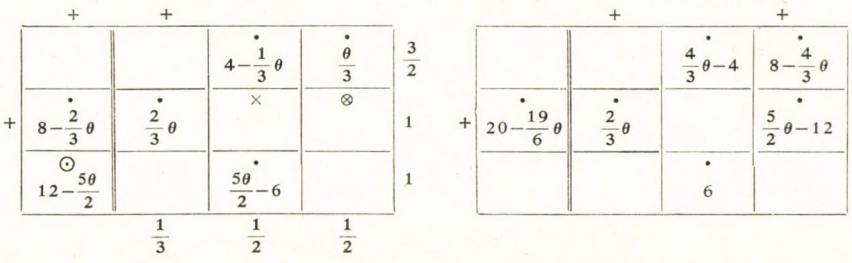
THÉORÈME 6. *L'algorithme décrit conduit à la solution optimale au bout d'un nombre fini d'étapes.* En effet, dans chaque étape, le réseau obtenu diffère de tous ceux rencontrés dans les étapes précédentes, soit par la capacité, soit par le nombre des colonnes puissantes (théorème 2). Comme chaque réseau est une forêt, et que le nombre total des forêts est fini, l'algorithme ne peut se poursuivre indéfiniment. Il est intéressant de remarquer que l'algorithme précédent est tout à fait analogue à l'algorithme de résolution du problème de transport proposé dans [5].

EXEMPLE NUMÉRIQUE. Résoudre le problème A, avec les données suivantes:

	2	4	1	(b_j)
4	4	3	3	
8	3	0	2	
6	2	2	1	
a_i				



$\theta = \frac{24}{13}$ $\pi = \frac{4}{3}$ $\theta = \frac{12}{5}$ $\pi = \frac{9}{8}$



$\theta = \frac{24}{5}$ $\pi = 1$ $\theta = 6$ $L = \emptyset$

0	4	0
4	0	3
0	6	0

Solution optimale

Les nombres $u_i(v_j)$ sont écrits à droite (en bas) de chaque tableau. Le signe · indique une case de G, ⊙ un point mort; × un pont; ⊗ le pont qui doit entrer dans le nouveau réseau. Les lignes et colonnes puissantes sont marquées du signe +.

§ 2. Problème B

1. Réseau simple. Dans cette partie, les termes „flot maximum", „capacité", etc ... désigneront des notions un peu différentes de celles introduites auparavant:

Considérons le graphe simple constitué par un tableau à $m+1$ lignes $i=0, 1, 2, \ldots, m$ et à $n+1$ colonnes $j=0, 1, 2, \ldots, n$, et un sous-graphe R de T. Nous dirons qu'une matrice $[x_{ij}]$ est un *flot* dans le réseau R, si l'on a

$$x_{ij} \geqq 0 \qquad (i = 1, 2, \ldots, m, 0;\ j = 1, 2, \ldots, n, 0) \tag{32}$$

$$x_{ij} = 0 \qquad (i, j) \notin R \tag{33}$$

$$\sum_{j=1}^{n} x_{ij} + x_{i0} = a_i \qquad (i = 1, 2, \ldots, m) \tag{34}$$

$$\sum_{i=1}^{m} b_{ij} x_{ij} + x_{0j} = b_j \qquad (j = 1, 2, \ldots, n) \tag{35}$$

Le nombre $\sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}x_{ij}$ est par définition la *valeur* du flot. S'il n'existe dans R aucun flot de valeur plus grande que le flot $\{x_{ij}\}$, celui-ci est appelé un *flot maximal* dans le réseau R et sa valeur est la *capacité* de ce réseau. Evidemment un flot maximal est aussi celui qui correspond au minimum de la quantité

$$\sum_{j=1}^{n} x_{0j}.$$

Nous supposerons toujours $b_{ij}>0$ pour chaque $(i,j)\in R$, $i\neq 0$, $j\neq 0$ et nous conviendrons que $b_{0j}=1$ quel que soit j. Alors pour chaque chaîne simple Γ formée de cases de R et ne contenant pas la colonne 0, on peut définir sa *caractéristique*

$$\varrho(\Gamma)=\Big(\prod_{\Gamma^+} b_{ij}\Big)\Big/\Big(\prod_{\Gamma^-} b_{ij}\Big),$$

où $\prod_{\Gamma^+}$, $\prod_{\Gamma^-}$ désignent les produits étendus respectivement aux cases paires et aux cases impaires de Γ, les cases de Γ étant toujours comptées dans le sens du parcours de la chaîne à partir de son origine. Lorsque Γ est un cycle, pour définir sa caractéristique, il faut choisir au préalable une de ses cases comme case impaire. Appelons *réserve* la ligne 0, ou la colonne 0, ou un cycle de caractéristique différente de 1 et ne passant ni par la ligne 0 ni par la colonne 0.

Un réseau R est dit *simple* si tout cycle qu'il contient est une réserve et si chacune de ses composantes connexes contient au plus une seule réserve.

L'intérêt des réseaux simples dans la question qui nous occupe réside dans le fait que la capacité d'un tel réseau est très facile à calculer. En effet, s'il existe un flot $x=[x_{ij}]$ dans un réseau simple R, il est déterminé univoquement de la façon suivante:

On prend une case (i_0,j_0) de R qui soit seule dans une ligne $i_0\neq 0$ ou seule dans une colonne $j_0\neq 0$ et on determine $x_{i_0j_0}$ par les conditions (33) et (34), ou (33) et (35); on efface la case utilisée et dans le réseau qui reste on prend une case (i_1,j_1), ayant les mêmes propriétés, pour laquelle on détermine la valeur $x_{i_1j_1}$ par les mêmes conditions en tenant compte de la valeur déjà calculée de $x_{i_0j_0}$, etc ... Lorsqu'il n'y a plus de case de R pendante dans une ligne $i\neq 0$ ou dans une colonne $j\neq 0$ il ne reste plus que des cycles isolés qui sont tous des réserves. On prend alors une case quelconque (i_k,j_k), située sur l'un de ces cycles et, en posant $x_{i_kj_k}=\theta$, on peut, par la méthode précédente, calculer de proche en proche, en fonction de θ, les valeurs x_{ij} correspondant aux autres cases du cycle considéré; il y aura alors exactement une inconnue x_{ij} exprimée de deux façons différentes en fonction de θ et en égalant ces deux valeurs, on obtient une équation linéaire qui permet de déterminer θ sans ambiguité, car il est facile de vérifier que le coefficient de θ dans cette équation est égal à $\varrho-1\neq 0$, ϱ désignant la caractéristique du cycle. En répétant les mêmes opérations pour les autres cycles, on obtient finalement un système de monbres x_{ij} verifiant les équations (33), (34), (35). Si alors la condition (32) est aussi satisfaite, la matrice $x=[x_{ij}]$ est le flot unique, et par suite le flot maximal, dans le reseau simple R. Si au contraire, cette condition n'est pas remplie, c'est que le réseau n'admet aucun flot.

2. — Propriétés du flot maximal. — Soit $x=[x_{ij}]$ un flot dans un réseau donné R, non necessairement simple. Comme auparavant, une case $(i,j)\in R$ est dite occupée, si $x_{ij}>0$. Une chaîne simple Γ issue de la ligne 0 et formée de cases de R est dite *insaturée*, si toutes ses cases impaires sont occupées. Un sous-graphe Γ de R est appelé un *tuyau* (de 1er, 2^{me}, 3^{me} type respectivement) s'il satisfait à l'une des trois conditions suivantes:

1. Γ est une chaîne élémantaire allant de la ligne 0 à la colonne 0;
2. Γ est un cycle élémentaire passant par la ligne 0;
3. Γ est formé d'un cycle élémentaire ne passant ni par la ligne 0 ni par la colonne 0 et d'une chaîne élémentaire allant de la ligne 0 à ce cycle. Il est clair qu'un réseau simple ne peut avoir aucun tuyau.

Un tuyau de ler type est dit *disponible*, si c'est une chaîne insaturée. Un tuyau de 2^{me} type est dit *disponible*, si on peut choisir les cases impaires sur ce tuyau de façon qu'elles soient toutes occupées, et que la caractéristique du tuyau soit supérieure à 1. Un tuyau de 3^{me} type est dit *disponible*, si la chaîne reliant la ligne 0 au cycle du tuyau est insaturée, et si on peut choisir les cases impaires sur ce cycle de façon qu'elles soient toutes occupées et que la caractéristique du cycle soit supérieure à 1.

Associons à chaque ligne $i\neq 0$ (colonne $j\neq 0$) un nombre λ_i (μ_j), appelé son *potentiel propre* dans le réseau R par rapport au flot donné x, et defini comme suit:

$$\lambda_i=\begin{cases}\max\limits_{\Gamma\in L_i}\varrho(\Gamma) & \text{si} \quad L_i\neq\emptyset\\ 0 & \text{si} \quad L_i=\emptyset\end{cases} \tag{36a}$$

$$\mu_j=\begin{cases}\max\limits_{\Gamma\in M_j}\varrho(\Gamma) & \text{si} \quad M_j\neq\emptyset\\ 0 & \text{si} \quad M_j=\emptyset\end{cases} \tag{36b}$$

où L_i (M_j) désigne l'ensemble des chaînes élémentaires insaturées allant de la ligne 0 à la ligne i (colonne j).

THÉORÈME 7 — *Les trois conditions suivantes sont équivalentes:*

1. *Le flot* $x=[x_{ij}]$ *est maximal dans* R;
2. *Il n'existe dans* R *aucun tuyau disponible*;
3. *Les potentiels propres* λ_i, μ_j *vérifient:*

$$\lambda_i\geqq 0 \qquad (i,0)\in R \tag{37}$$

$$\mu_j\leqq 1 \qquad (0,j)\in R \tag{38}$$

$$\lambda_i-b_{ij}\mu_j\geqq 0 \qquad (i,j)\in R \tag{39}$$

$$\lambda_i=0 \qquad x_{i0}>0 \tag{40}$$

$$\mu_j=1 \qquad x_{0j}>0 \tag{41}$$

$$\lambda_i-b_{ij}\mu_j=0 \qquad x_{ij}>0 \tag{42}$$

DÉMONSTRATION 1)⇒2). Il suffit de prouver que, si Γ est un tuyau disponible, on peut corriger le flot dans Γ, de manière à augmenter sa valeur.

Désignons par Γ^+ (Γ^-) l'ensemble des cases paires (impaires) de Γ, par ϱ_i la caractéristique de la portion de Γ allant de la ligne 0 à la ligne i ($\varrho_0=1$), par ϱ

la caractéristique du cycle de Γ (lorsque Γ en a). Si Γ est un tuyau disponible de 1er ou de 2me type, posons

$$x'_{ij} = \begin{cases} x_{ij} + \dfrac{\varepsilon}{\varrho_i} & (i,j) \in \Gamma^+ \\ x_{ij} - \dfrac{\varepsilon}{\varrho_i} & (i,j) \in \Gamma^- \\ x_{ij} & (i,j) \notin \Gamma \end{cases} \tag{43}$$

où

$$\varepsilon = \min \{\varrho_i x_{ij} : (i,j) \in \Gamma^-\} \tag{44}$$

Si Γ est un tuyau disponible de 3me type, posons

$$x'_{ij} = \begin{cases} x_{ij} + \dfrac{\varepsilon}{\varrho_i} & (i,j) \in \Gamma^+ \setminus \Delta \\ x_{ij} - \dfrac{\varepsilon}{\varrho_i} & (i,j) \in \Gamma^- \setminus \Delta \\ x_{ij} + \dfrac{\varepsilon}{\varrho_i} \dfrac{\varrho}{\varrho - 1} & (i,j) \in \Gamma^+ \cap \Delta \\ x_{ij} - \dfrac{\varepsilon}{\varrho_i} \dfrac{\varrho}{\varrho - 1} & (i,j) \in \Gamma^- \cap \Delta \\ x_{ij} & (i,j) \in \Gamma \end{cases}, \tag{45}$$

où Δ est le cycle du tuyau et

$$\varepsilon = \min \left\{ \varrho_i x_{ij} : (i,j) \in \Gamma^- \setminus \Delta ; \quad \varrho_i \frac{\varrho - 1}{\varrho} x_{ij} : (i,j) \in \Gamma^- \cap \Delta \right\} \tag{46}$$

On vérifie alors facilement que $x' = [x'_{ij}]$ est un flot dans R, de plus grande valeur que x. En effet, si Γ est de 1er ou de 3me type, en désignant par j_0 la colonne de la première case de Γ, on a

$$x'_{0j} = x_{0j} \qquad (j \neq j_0); \qquad x'_{0j_0} < x_{0j_0}$$

et si Γ est de 2me type en désignant respectivement par j_0 et j_1 les colonnes de la case impaire et de la case paire de Γ dans la ligne 0, on a

$$x'_{0j} = x_{0j} \qquad (j \neq j_0,\ j \neq j_1)$$

$$(x'_{0j_0} + x'_{0j_1}) - (x_{0j_0} + x_{0j_1}) = \varepsilon \left(\frac{1}{\varrho} - 1 \right) < 0,$$

d'où l'on déduit

$$\sum_{j=1}^{n} x'_{0j} < \sum_{j=1}^{n} x_{ij}$$

Donc si un tuyau disponible existe, le flot n'est pas maximal.

2)⇒3). Supposons qu'aucun tuyau disponible n'existe. Alors les relations (37) et (41) sont évidemment vérifiées, et les relations (40) et (38) résultent de l'absence

des tuyaux disponibles de 1er et de 2me types. Reste donc à établir les relations (39) et (42). Or à cause de l'absence des tuyaux disponibles de 3me type, si Γ est une chaîne insaturée, contenant un cycle Δ, alors $\varrho(\Delta) \leqq 1$, et la suppression de Δ ne fait pas diminuer la caractéristique de Γ; donc pour toute chaîne insaturée Γ il existe une chaîne insaturée élémentaire Γ' telle que $\varrho(\Gamma') \geqq \varrho(\Gamma)$. Il en résulte, d'après les égalités (36) qu'on a $\lambda_i \geqq \varrho(\Gamma)$ pour toute chaîne insaturée Γ élémentaire ou non, allant de la ligne 0 à la ligne i. Cela étant, prenons une case quelconque $(i,j) \in R$. On peut supposer $\mu_j > 0$, car les relations (39) sont évidentes lorsque $\mu_j = 0$. Soit Γ_j la chaîne insaturée telle que $\mu_j = \varrho(\Gamma_j)$. En ajoutant (i,j) à Γ_j si cette case ne lui appartient pas encore, ou en lui enlevant (i,j) si cette case lui appartient déjà, on obtient une chaîne insaturée Γ_i' allant à la ligne i, et par suite

$$\lambda_i \geqq \varrho(\Gamma_i') = b_{ij}\varrho(\Gamma_j) = b_{ij}\mu_j,$$

ce qui démontre l'inégalité (53). Si la case (i,j) est occupée, en désignant par Γ_i la chaîne insaturée telle que $\lambda_i = \varrho(\Gamma_i)$ et par Γ_j' celle obtenue de Γ_i en lui ajoutant (i,j) si $(i,j) \notin \Gamma_i$, ou en lui enlevant (i,j) si $(i,j) \in \Gamma_i$, on a $\mu_j \geqq \varrho(\Gamma_j') = \dfrac{\varrho(\Gamma_i)}{b_{ij}} = \dfrac{\lambda_i}{b_{ij}}$, d'où, grâce à (39), on tire (42). Donc toutes les relations (37)—(42) sont vérifies.

3)$\Rightarrow$1). Ceci résulte d'un théorème connu de dualité en programmation linéaire. Il est d'ailleurs facile de le démontrer directement.

Détermination des potentiels propres. Supposons que le flot $x = [x_{ij}]$ est maximal dans le réseau R et cherchons à déterminer les potentiels propres.

Considérons d'abord le cas particulier où R est un réseau simple. Alors, puisque le flot $x = [x_{ij}]$ est unique dans R, le système $\{\lambda_i, \mu_j\}$ est complètement determiné par le réseau R. Comme la composante connexe de R contennant la ligne 0 est un arbre, il existe, pour chaque ligne $i \neq 0$ donnée, au plus une seule chaîne de R reliant cette ligne à la ligne 0. Il en résulte immédiatement que les potentiels propres peuvent être calculés de proche en proche d'après la règle suivante:

On pose $\mu_j = 1$ pour toute colonne j telle que $x_{0j} > 0$ puis, si une colonne j a reçu un potentiel μ_j, on pose $\lambda_i = b_{ij}\mu_j$ pour toute ligne i coupant j suivant une case de R; si une ligne i a reçu un potentiel λ_i, on pose $\mu_j = \dfrac{\lambda_i}{b_{ij}}$ pour toute colonne j coupant i suivant une case occupée. Lorsqu'on ne peut plus continuer, on pose $\lambda_i = \mu_j = 0$ pour toute ligne i et toute colonne j, qui restent.

Dans le cas général, en s'appuyant sur le théorème 7, on pourrait démontrer que pour tous les flots maximaux d'un réseau donné, le système de potentiels propres $\{\lambda_i, \mu_j\}$ est le même, autrement dit, étant donné un réseau R, il existe un seul système $\{\lambda_i, \mu_j\}$ vérifiant les égalités (36), quel que soit le flot maximal $x = [x_{ij}]$ dans le réseau. Pour calculer ce système, on pose $\mu_j = 1$ pour toute colonne j telle que $x_{0j} > 0$, et $\lambda_i = \mu_j = 0$ pour les autres lignes et colonnes; puis, si $(i,j) \in R$ et $\lambda_i < b_{ij}\mu_j$, on augmente λ_i jusqu' ce qu'il soit égal à $b_{ij}\mu_j$; si $x_{ij} > 0$ et $\lambda_i > b_{ij}\mu_j$ on augmente μ_j jusqu'à ce qu'il soit égal à $\dfrac{\lambda_i}{b_{ij}}$, etc ... Comme il est facile de le montrer, cette procédure se termine au bout d'un nombre fini d'opérations et donne alors les potentiels cherchés.

3. Augmentation de la capacité. — Etant donnés un réseau R, un flot maximal $x=[x_{ij}]$ dans R et un système de potentiels propres λ_i, μ_j dans R, nous disons qu'une case (i,j) est un pont, si $\lambda_i-b_{ij}\mu_j<0$, un *point mort*, si $\lambda_i-b_{ij}\mu_j>0$. Pour que cette définition englobe aussi les cas $i=0$ ou $j=0$, convenons que $\lambda_0=1$, $\mu_0=0$, $b_{0j}=1$, pour tout j, $b_{i0}=0$ pour tout i. Les remarques suivantes sont évidentes:

1. On a toujours $(i,j)\notin R$, $b_{ij}>0$ pour un pont (i,j) et $x_{ij}=0$ pour un point mort (i,j).

2. Si aucun pont n'existe, le flot x est maximal dans tout le tableau T (théorème 7). Donc si aucun pont n'existe et que $x_{0j}>0$ pour un certain j, le problème B n'admet pas de solution réalisable.

3. En supprimant du réseau R un certain nombre de points morts, on ne modifie ni sa capacité ni ses potentiels propres.

En effet, soit λ_i le potentiel propre de la ligne i, Γ_i la chaîne insaturée dont la caractéristique donne λ_i. Si pour une case $(i_0,j_0)\in\Gamma_i$ on a $\lambda_{i_0}>b_{i_0j_0}\mu_{j_0}$, on pourra remplacer le tronçon de Γ_i allant de la ligne 0 à la ligne i_0 par la chaîne insaturée dont la caractéristique donne λ_{i_0}: on obtiendra alors une chaîne insaturée allant de la ligne 0 à la ligne i et de caractéristique supérieure à λ_i, ce qui est impossible. Donc Γ_i ne contient pas de points morts, et la suppression de ces points ne peut modifier λ_i. De même elle ne peut modifier les μ_j.

THÉORÈME 8. *Lorsqu'on ajoute à un réseau simple R un pont (r,s) et qu'on lui enlève tous les points morts, le réseau restera simple ou aura un tuyau unique. Dans ce dernier cas, on peut supprimer encore une case (p,q) convenable de ce tuyau pour obtenir un réseau simple R' satisfaisant à l'une des deux conditions suivantes:*

1. *La capacité de R' est supérieure à celle de R;*

2. *La capacité de R' reste égale à celle de R, mais les potentiels propres dans R' sont tels que $\lambda'_r>\lambda_r$, $\lambda'_i\geqq\lambda_i$ $(i\neq r)$, $\mu'_j\geqq\mu_j$ pour tout i,j.*

DÉMONSTRATION. — Désignons par R° le réseau obtenu de R après suppression des points morts; par $\bar{R}$ celui obtenu de R après suppression des points morts et adjonction du pont (r,s). Evidemment R° est un réseau simple.

Comme $\mu_s>\dfrac{r}{b_{rs}}\geqq 0$, la colonne s appartient à la composante connexe de R° contenant la ligne 0. Trois cas sont alors possibles: 1) la ligne r appartient à la même composante: puisque cette composante est un arbre, l'adjonction de (r,s) au réseau R° crée un cycle unique, et l'on obtient un tuyau unique (de 2$^{\text{me}}$ type si le cycle passe par la ligne 0, de 3$^{\text{me}}$ type sinon); 2) la ligne r appartient à la composante connexe de R° contenant la colonne 0: puisque cette composante est aussi un arbre, l'adjonction de (r,s) crée un tuyau unique de 1$^{\text{er}}$ type, constitué par une chaîne unique reliant la ligne 0 à la colonne 0; 3) la ligne r appartient à une composante de R° ne contennant ni la ligne 0 ni la colonne 0: si cette composante possède un cycle (qui est alors son cycle unique), l'adjonction de (r,s) crée un tuyau de 3$^{\text{me}}$ type; sinon, on n'a aucun tuyau. Donc dans tous les cas, si le réseau $\bar{R}$ obtenu de R par adjonction du pont (r,s) et la suppression des points morts n' est pas simple, il comporte un tuyau unique. La première partie du théorème est ainsi établie.

Supposons maintenant que $\bar{R}$ possède un tuyau Γ. Si ce tuyau est disponible, on peut corriger le flot sur ce tuyau d'après les formules (43), (44), (45), (46) et choisir comme (p,q) une case devenue non occupée après la correction (celle qui

correspond au minimum dans (44), (46): il est clair qu'en supprimant (p, q) le réseau obtenu R' satisfait à la condition 1). Comme le cycle d'un tuyau disponible a toujours une caractéristique différente de 1, en voit d'autre part que R' est simple.

Si le tuyau Γ n' est pas disponible, le flot $x=[x_{ij}]$ reste maximal dans le réseau $\bar{R}$, en vertu du théorème 7. En désignant par λ'_i, μ'_j les potentiels propres dans $\bar{R}$, on a évidemment $\lambda'_i \geqq \lambda_i$, $\mu'_j \geqq \mu_j$ pour tout i, j; d'autre part $\lambda'_r \geqq b_{rs}\mu_s > \lambda_r$ car (r, s) forme avec la chaîne insaturée dont la caractéristique donne $\mu_s > 0$, une chaîne insaturée conduisant à la ligne r, de caractéristique $b_{rs}\mu_s$. Montrons que Γ contient au moins un point mort dans $\bar{R}$. En effet, cela est clair, si Γ est de premier type car dans le cas contraire $\lambda'_i - b_{ij}\mu'_j = 0$ pour toute case $(i,j) \in \Gamma$ et puisque $\mu'_s \geqq \mu'_s > 0$, on en déduirait $\mu'_0 > 0$, ce qui est impossible. Donc si Γ ne contenait aucun point mort, il devrait être de 2me ou de 3me type. Alors, le cycle de Γ aurait pour caractéristique 1, comme on le voit aisément en multipliant membre à membre les égalités $\lambda'_{i_1} = b_{i_1 j_1}\mu'_{j_1}$, $b_{i_1 j_1}\mu'_{j_1} = \lambda'_{i_2}$, ..., $b_{i_k j_k}\mu'_{j_k} = \lambda'_{i_1}$ écrites successivement pour toutes les cases du cycle, et en remarquant que tous les λ'_i, μ'_j considérés sont positifs (car $\mu'_s \geqq \mu_s > 0$). Par suite ce cycle contiendrait la case (r, s), car tout cycle de R étant une réserve a une caractéristique différente de 1. Or, puisque Γ ne contient aucun point mort dans R (ces points morts ayant été supprimés), on a $\lambda_i - b_{ij}\mu_j = 0$ pour $(i,j) \in \Gamma \setminus \{(r, s)\}$, et comme $\lambda_r < b_{rs}\mu_s$ on en déduit comme précédemment que la caractéristique du cycle de Γ est différente de 1. Cette contradiction prouve que Γ doit contenir au moins un point mort dans $\bar{R}$. Il est clair qu'en prenant un tel point mort pour (p, q), le réseau R' satisfera à la condition 2). D'autre part, si Γ a un cycle, la case (p, q) ainsi choisie se trouve toujours sur se cycle, car toute case (i, j) de Γ extérieure à ce cycle est reliée à la ligne 0 par une chaîne insaturée unique, de sorte que $\lambda'_i = b_{ij}\mu'_j$. Cela étant, la suppression de (p, q) détruit toujours le cycle de Γ (si ce cycle existe) et l'on voit aisément que R' est simple. Le théorème est démontré.

Vérification de la disponibilité du tuyau. — Si le tuyau est de 1er type, on reconnaît facilement sa disponibilité. Si le tuyau est de 2me ou de 3me type, il faut, pour juger de sa disponibilité, savoir la caractéristique de son cycle. Pour cela, on pose $\lambda'_r = l_{rs}\mu_s$ puis on calcule de proche en proche les nombres λ'_i, μ'_j tels que $\lambda'_i = b_{ij}\mu'_j$ pour toute case (i, j) du tuyau (ces nombres sont les caractéristiques des portions successives du tuyau). Alors en désignant par Δ le cycle du tuyau et par (i_1, j_1) sa dernière case, on a pour un tuyau de 2me type: $\varrho(\Delta) = \mu_{j_1}$, et pour un tuyau de 3me type $\varrho(\Delta) = \dfrac{\lambda'_i}{b_{i_1 j_1}\mu'_{j_1}}$ ou $\dfrac{b_{i_1 j_1}\mu'_{j_1}}{\lambda'_i}$, selon que (i_1, j_1) est une case impaire ou paire du tuyau.

Lorsque le tuyau est disponible, les nombres λ'_i, μ'_j s'emploient au calcul du nouveau flot d'après les formules (43), (45). Dans le cas contraire, ces nombres servent à déterminer les nouveaux potentiels propres.

4. Réseau normal. — Un réseau est dit *normal* s'il existe des nombres u_i, v_j tels que

$$u_i \geqq 0 \qquad i = 1, 2, \ldots, m \tag{47}$$

$$u_i - b_{ij}v_j + c_{ij} \geqq 0 \qquad i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n \tag{48}$$

$$u_i = 0 \qquad (i, 0) \in R \tag{49}$$

$$u_i - b_{ij}v_j + c_{ij} = 0 \qquad (i, j) \in R \tag{50}$$

Les nombres u_i, v_j sont alors appelés *potentiels normaux* du réseau.

L'introduction des réseaux normaux est justifiée par le théorème suivant:

THÉORÈME 9. — *Si* $x=[x_{ij}]$ *est un flot maximal dans un réseau normal et sature toutes les colonnes* $j=1, 2, \ldots, n$ *c'est-à-dire*

$$x_{0j} = b_j - \sum_{i=1}^{m} b_{ij}x_{ij} = 0 \qquad (j = 1, 2, \ldots, n),$$

il est une solution optimale du problème B.

En effet, $x=[x_{ij}]$ constitue alors une solution réalisable du problème $\bar{B}$, et, comme, en vertu de (47)—(50) les conditions du théorème des écarts complémentaires sont satisfaites, c'est une solution optimale.

Soit maintenant $x=[x_{ij}]$ un flot maximal dans un réseau normal R, ne saturant pas encore toutes les colonnes. Nous pouvons supposer que:

$$(i, 0)\in R \Rightarrow x_{i0} > 0, \tag{51}$$

quitte à supprimer de R toutes les cases $(i, 0)$ telles que $x_{i0}=0$. Soit $\{\lambda_i, \mu_j\}$ le système correspondant de potentiels propres, L l'ensemble des ponts;

$$L = \{(i, j): \lambda_i - b_{ij}\mu_j < 0\} \tag{52}$$

Désignons par (r, s) un pont satisfaisant au critère

$$\min\left\{\frac{u_i - b_{ij}v_j + c_{ij}}{b_{ij}\mu_j - \lambda_i}: (i, j)\in L\right\} \tag{53}$$

THÉORÈME 10. — *Le réseau* R' *obtenu en ajoutant à* R *le pont* (r, s) *et en lui enlevant tous les points morts sera encore normal. Le nouveau système de potentiels normaux est défini par*

$$u'_i = u_i + \delta\lambda_i, \qquad v'_j = v_j + \delta\mu_j \tag{54}$$

où δ *désigne la quantité* (53).

DÉMONSTRATION. — On a évidemment $u'_i \geqq 0$, car u_i, δ, λ_i sont tous non négatifs. D'autre part

$$u'_i - b_{ij}v'_j + c_{ij} = u_i - b_{ij}v_j + c_{ij} + \delta(\lambda_i - b_{ij}\mu_j) \geqq 0$$

pour tout $i=1, 2, \ldots, m$; $j=1, 2, \ldots, n$, en vertu de la definition de δ.

Si $(i, 0)\in R'$ on a $(i, 0)\in R$ et d'après (51), $x_{i0}>0$, d'où d'après (54) et (40), $u'_i=0$. Si $(i, j)\in R'\setminus\{(r, s)\}$ on a $(i, j)\in R$ et $\lambda_i - b_{ij}\mu_j=0$, d'où d'après (50):

$$u'_i - b_{ij}v'_j + c_{ij} = 0$$

Enfin, pour la case (r, s) on a, d'après la définition de δ,

$$u'_r - b_{rs}v'_s + c_{rs} = 0$$

Donc le réseau R' est bien normal.

5. Algorithme II. — Ce qui précède conduit à l'algorithme suivant pour resoudre le problème B.

A l'étape initiale on prend un réseau simple normal R_0, de flot maximal x°, de potentiels normaux $\{u_i^0, v_j^0\}$. Par exemple, on peut prendre R_0 égal au „réseau nul" formé par l'ensemble des cases $(i, 0)$ $(i=1, 2, \ldots, m)$ et $(0, j)$ $(j=1, 2, \ldots, n)$: c'est bien un réseau simple normal, dont le flot maximal est: $x_{ij}^0=0$ $(i\neq 0, j\neq 0)$; $x_{i0}^0=a_i$, $x_{0j}^0=b_j$, et dont les potentiels normaux sont $u_i^0=0$ $(i=1, 2, \ldots, m)$, $v_j^0=0$ $(j=1, 2, \ldots, n)$.

Dans chaque étape ultérieure, on dispose d'un réseau simple normal R, de flot maximal $x=[x_{ij}]$, et de potentiels normaux $\{u_i, v_j\}$.

1. Si toutes les colonnes $j=1, 2, \ldots, n$ sont saturées, c'est-à-dire $x_{0j}=0$ pour tout $j=1, 2, \ldots, n$, x est la solution optimale.

2. S'il existe des colonnes insaturées, on détermine par le procédé indiqué au § 2.2 les potentiels propres $\{\lambda_i, \mu_j\}$ puis l'ensemble des ponts:

$$L = \{(i, j): \lambda_i - b_{ij}\mu_j < 0\}$$

a) Si $L=\emptyset$, le problème B n'admet pas de solution réalisable.

b) Si $L\neq\emptyset$, on choisit une case $(r, s)\in L$ d'après le critère (53). On ajoute à R cette case, on lui enlève tous les points morts, c'est-à-dire les cases $(i, j)\in R$ telles que $\lambda_i - b_{ij}\mu_j > 0$, et aussi les cases $(i, 0)\in R$ telles que $x_{i0}=0$.

On cherche par le procédé indiqué au paragraphe 3 le tuyau créé par (r, s).

α) Si le tuyau n'existe pas, ou n'est pas disponible, le flot $x=[x_{ij}]$ reste maximal dans le nouveau réseau. On corrige les potentiels propres en augmentant λ_r jusqu'à $b_{rs}\mu_s$ et en appliquant la règle indiquée au paragraphe 2, on determine les nouveaux potentiels normaux par les formules (54), puis on supprime un point mort sur le tuyau (si celui-ci existe) et on passe à l'étape suivante.

β) Si le tuyau existe et est disponible, on corrige le flot dans ce tuyau d'après les formules (43) ou (45) et on supprime du réseau une case devenue non occupée sur ce tuyau (celle réalisant le minimum (44) ou (46). On détermine par les procédés indiqués aux paragraphes 2 et 4 respectivement les potentiels propres et les potentiels normaux dans le nouveau réseau et on passe à l'étape suivante.

THÉORÈME 11. *L'algorithme précédent se termine au bout d'un nombre fini d'étapes, soit par le cas* 1), *soit par le cas* 2a).

En effet, à chaque étape, d'après le théorème 8, si on est dans le cas 2) b) *β)*, la capacité du réseau augenemte; si l'on est dans le cas 2) b) *α)*, le potentiel propre d'au moins une ligne augmente tandis qu'aucun autre potentiel ne diminue (la suppression des cases $(i, 0)\in R$ telles que $x_{i0}=0$ n'influe évidemment pas sur les potentiels propres, de sorte que le théorème 8 reste applicable). Donc le réseau simple obtenu à la fin de chaque étape diffère de tous ceux obtenus dans les étapes antérieures. Comme le nombre total de réseaux simples est fini, le cas 2) b) ne peut avoir lieu indéfiniment. Le théorème est démontré.

REMARQUE. — Lorsque $b_{ij}=1$ pour tout i, j, l'algorithme II coincide avec l'algorithme de résolution du problème des transports exposé dans [8] (voir aussi [5]).

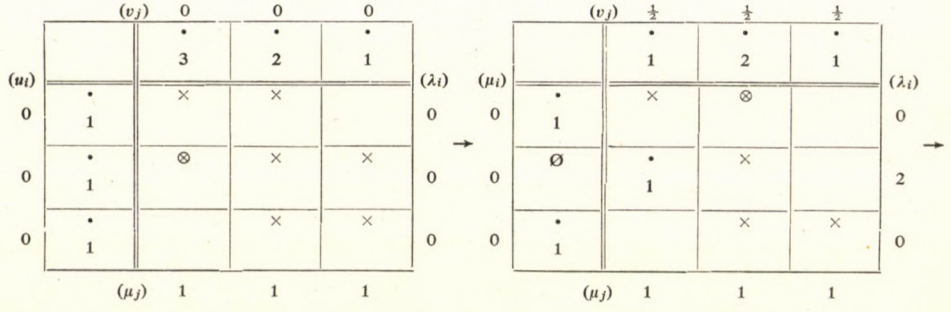
EXEMPLE NUMÉRIQUE. Résoudre le problème B avec les données suivantes:

$$a_i\colon \quad 1 \quad 1 \quad 1$$

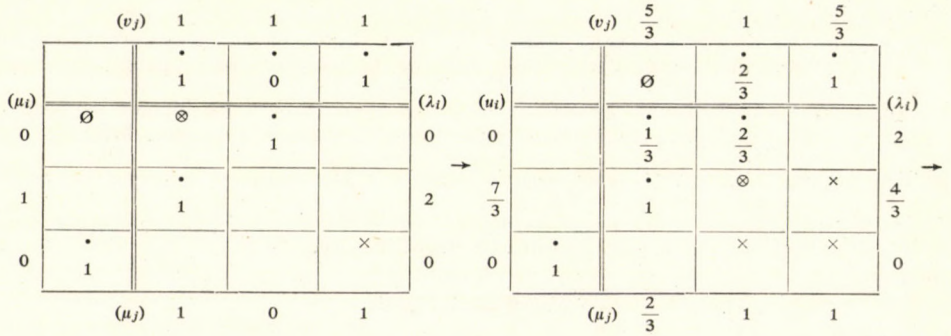
$$b_j\colon \quad 3 \quad 2 \quad 1$$

$$[b_{ij}] = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix} \qquad [c_{ij}] = \begin{bmatrix} 5 & 2 & 2 \\ 1 & 3 & 2 \\ 2 & 4 & 4 \end{bmatrix}$$

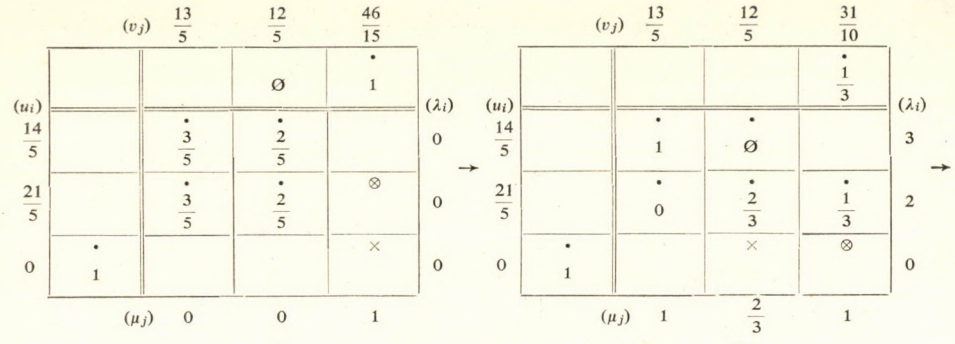
Comme auparavant, · indique ci-dessous une case de R; $\times$ un pont; $\otimes$ le pont qui doit entrer dans le nouveau réseau. On obtient la solution optimale après 7 étapes.



$$\delta = \frac{1}{2} \qquad \delta = \frac{1}{2}$$



$$\delta = \frac{2}{3} \qquad \delta = \frac{7}{5}$$



(solution optimale)

BIBLIOGRAPHIE

[1] BERGE, C.: *Théorie des graphes et ses applications*, Paris, 1958.
[2] DANTZIG, G.: *Linear programming and extension*, Princeton, University Press, 1963.
[3] JEWEL, W. S.: Optimal flow through networks with gains, *Operations Res.*, **10** (1962) 476—499.
[4] MARKOVITZ, H.: Concepts and computing procedures for certain x_{ij} programming problems, *Proceedings of the second symposium in linear programming* Vol. **2** (1955) 509—565.
[5] HOÀNG TỤY: Contribution à la théorie des réseaux, *Acta Sci. Vietnam.* **1** (1964) 11—67.
[6] NGUYỄN QUANG THÁI: Sur le problème de distribution, *Toán Lý* **1** (1965) 59—60.
[7] Гавурин, М. К., Рубинштейн, Г. Ш. и Сурин, С. С.: Об оптимальном использовании производственных средств при выполнении несколько видов работ (Обобщенная транспортная задача), *Сибирский Мат. Ж.* **4** (1962) 481—499.
[8] Брудно, А. Л.: Решение транспортной задачи методом вычеркивающей нумерации, *Сб. Применение цифровых вычислительных машин в экономике,* Москва, 1962, 17—38.
[9] Канторович, Л. В.: *Математические методы в организации и планировании производства,* Ленинград, 1939.
[10] Малков, У. Х.: Алгоритм решения распределительной задачи, *Журнал вычислительной математики и мат. физику,* **2** (1962) 358—366.

Université de Hanoi

(Reçu le 1er décembre 1966.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 31—35.

ON A CHARACTERISTIC PROPERTY OF POLYA'S URN

by

KRISHNA B. ATHREYA

Consider a POLYA's urn scheme with parameters W_0, B_0 and α_0. That is, we start with an urn containing W_0 and B_0 white and black balls, respectively. A *draw* consists of i) pick a ball at random from the urn, ii) note its color C and return it to the urn and iii) add α_0 balls of the color C. Let (W_n, B_n) be the composition of the urn after n draws. Then the stochastic process $\{(W_n, B_n);\ n=0, 1, 2, \ldots\}$ is called a POLYA's urn with parameters W_0, B_0 and α_0. We denote it by $P(W_0, B_0, \alpha_0)$. The classical POLYA's urn is $P(W_0, B_0, 1)$.

A well known result on $P(W_0, B_0, \alpha_0)$ is that (a simple proof appears later)

$$\lim_n \frac{W_n}{W_n+B_n} = Y \qquad \text{exists w.p. 1} \tag{1}$$

and Y has a $\beta(W_0/\alpha_0, B_0/\alpha_0)$ distribution, namely

$$P\{Y \leqq y\} = \begin{cases} 0 & \text{if } y<0 \\ \dfrac{\int_0^y x^{(W_0/\alpha_0)-1}(1-x)^{(B_0/\alpha_0)-1}\,dx}{\int_0^1 x^{(W_0/\alpha_0)-1}(1-x)^{(B_0/\alpha_0)-1}\,dx} & \text{if } 0 \leqq y \leqq 1 \\ 1 & \text{if } y>1. \end{cases} \tag{2}$$

Suppose we generalize the scheme $P(W_0, B_0, \alpha_0)$ a little bit by letting α_0 be a random variable with mean value α. It is the purpose of this paper to show (1) and (2) hold for this generalized scheme, if and only if α_0 is degenerate at α.

Actually we prove a more general result. We define a Generalized POLYA' Urn Scheme (GP) as follows. An urn has $(s_{01}, s_{02}, \ldots, s_{0k})$ balls $(s_{0i} \geqq 1)$ respectively of colors 1, 2, ..., k. A *draw* consists of the following operations:

i) Pick a ball at random from the urn, notice its color C and return it to the urn.

ii) If $C=$i, add N_i balls of color i where N_i is a random variable with probability generating function $f_i(z)$.

Let $s_n=(s_{n1}, s_{n2}, \ldots, s_{nk})$ denote the composition of the urn after n successive draws. We call the stochastic process $\{s_n;\ n=0, 1, 2, \ldots\}$ a GP and denote it by $GP\{k;\ s_0, f_1, \ldots, f_k\}$ to indicate the parameters involved. The classical POLYA's

urn is $GP\{2;\ (a, b),\ z,\ z\}$. Let $\mathbf{p}_n=(p_{n1}, p_{n2}, \ldots, p_{nk})$ where

$$p_{ni} = \frac{s_{ni}}{\sum_{i=1}^{k} s_{ni}} \tag{3}$$

Then we have the following

THEOREM 1. *Let for all* i, $f_i(0)=0$, $0<f_i'(1)=\lambda_i<\infty$, *and* $\sum_{j=1}^{\infty} q_{ij} j \log j < \infty$, *where* $f_i(z)=\sum_{j=1}^{\infty} q_{ij} z^j$. Let $\lambda_1=\lambda_2=\cdots=\lambda_r>\lambda_{r+1}\geqq\lambda_{r+2}\geqq\cdots\geqq\lambda_k$. *Then,*

$$\lim_n \mathbf{p}_n = \mathbf{p} = (p_1, \ldots, p_k) \quad \textit{exists} \quad \text{w.p. 1} \tag{4}$$

and for $i>r$, $p_i=0$ w.p. 1.

COROLLARY 1. Let $f_i(z)=z^{\lambda}$, $i=1, 2, \ldots, r$ and $f_i'(1)<\lambda$ for $i>r$. Then $(p_1, \ldots, p_r)$ as defined in (4) has a generalized

$$\beta\left(\frac{s_{01}}{\lambda}, \frac{s_{02}}{\lambda}, \ldots, \frac{s_{0r}}{\lambda}\right)$$

distribution over the simplex $\Delta=\{\mathbf{x}=(x_1, \ldots, x_r): x_i\geqq 0, \sum_{i=1}^{r} x_i=1\}$. That is,

$$P\{p_1\leqq y_1, p_2\leqq y_2, \ldots, p_{r-1}\leqq y_{r-1}\} =$$

$$=\int_0^{y_1}\int_0^{y_2}\cdots\int_0^{y_{r-1}} f(x_1, \ldots, x_{r-1})\,dx_1\ldots dx_{r-1} \quad \text{for} \quad 0\leqq y_i\leqq 1, \quad i=1, 2, \ldots, r-1,$$

where

$$f(x_1, \ldots, x_{r-1}) = \tag{5}$$

$$=\frac{\Gamma(s_{01}/\lambda)\cdots\Gamma(s_{0r}/\lambda)}{\Gamma((s_{01}+\cdots+s_{0r})/\lambda)} x_1^{s_{01}/\lambda} x_2^{s_{02}/\lambda}\cdots x_{r-1}^{s_{0,r-1}/\lambda}\{1-(x_1+\cdots+x_{r-1})\}^{s_{0,r}/\lambda}$$

and

$$\Gamma(p)=\int_0^{\infty} e^{-x}x^{p-1}\,dx \quad \text{for} \quad p>0.$$

THEOREM 2. *Consider a GP where* $f_i(z)$ *is independent of* i *and satisfies the hypothesis of Theorem* 1. *Then* $\mathbf{p}=(p_1, \ldots, p_k)$, *as defined by* (4), *has a* $\beta(s_{01}/\lambda, \ldots, s_{0k}/\lambda)$ *distribution for all initial set up* $s_0\neq 0$ *if and only if*

$$f(z)=z^{\lambda} \tag{6}$$

The result we stated earlier about POLYA's urn is a special case of the above theorem namely $GP\{2;\ (W_0, B_0),\ f(z), f(z)\}$.

Our approach to this problem is through a technique of imbedding urn schemes in continuous time multitype MARKOV branching processes. We introduced and exploited this a great deal in [1] and [6] to study FRIEDMAN's urn.

PROOF OF THEOREM 1. Consider a collection $\{X_{ij}(t);\ t \geqq 0\}$ of independent one dimensional continuous time branching processes where for $j=1, 2, \ldots, s_{0i}$ and $i=1, 2, \ldots, k$, $\{X_{ij}(t);\ t \geqq 0\}$ has the infinitesimal generating function (See [5].)

$$u_i(z) = [zf_i(z)-z] \tag{7}$$

Then for every i and j, $\{X_{ij}(t)e^{-\lambda_i t};\ t \geqq 0\}$ being a non-negative martingale,

$$\lim_t X_{ij}(t)e^{-\lambda_i t} = W_{ij} \quad \text{exists. w.p. 1} \tag{8}$$

Further, by our hypothesis about $f_i(z)$ we conclude that (see [2])

$$P\{W_{ij} > 0\} = 1 \quad \text{for all} \quad i \quad \text{and} \quad j \tag{9}$$

Thus

$$\lim_t \left\{ \frac{\sum_{j=1}^{s_{0i}} X_{ij}(t)}{\sum_{i=1}^{k} \sum_{j=1}^{s_{0i}} X_{ij}(t)} \right\} = \lim_t \left\{ \frac{\sum_{j=1}^{s_{0i}} X_{ij}(t)e^{-\lambda_1 t}}{\sum_{i=1}^{k} \sum_{j=1}^{s_{0i}} X_{ij}(t)e^{-\lambda_1 t}} \right\} \tag{10}$$

$$= p_i \quad \text{exists w.p. 1,} \tag{11}$$

where

$$p_i = \begin{cases} \dfrac{\sum_{j=1}^{s_{0i}} W_{ij}}{\sum_{i=1}^{r} \sum_{j=1}^{s_{0i}} W_{ij}} & \text{if} \quad i \leqq r \\ 0 & \text{if} \quad i > r \end{cases} \tag{12}$$

Now let τ_n^{ij} denote the time at which the nth split occurs for the branching process $\{X_{ij}(t);\ t \geqq 0\}$. Let $\tau_0 = 0$ and τ_n for $n=1, 2, \ldots$ be the sequence of time instants obtained by superposing all the $\{\tau_n^{ij}\}_{n=1}^{\infty}$, $j=1, 2, \ldots, s_{0i}$, $i=1, 2, \ldots, k$ and arranging them in increasing order.

The reader can convince himself (see [6] for a proof) that the stochastic processes $\{\mathbf{s}_n;\ n=0, 1, 2, \ldots\}$ and $\left\{\left(\sum_{j=1}^{s_{0i}} X_{ij}(\tau_n+0),\ i=1, 2, \ldots, k\right);\ n=0, 1, 2, \ldots\right\}$ are equivalent in the sense that all finite dimensional distributions are identical.

Also since $f_i'(1) = \lambda_i < \infty$ and $f_i(0) = 0$ for all i $P\{\lim_n \tau_n^{ij} = \infty\} = 1$ for every i and j. Thus

$$P\{\lim_n \tau_n = \infty\} = 1. \tag{13}$$

Now (11), (12), (13) and the above imbedding yield (4).

PROOF OF COROLLARY 1. This is immediate now from (12) and the fact that W_{ij} for $j=1, 2, \ldots, s_{0i}$, $i=1, 2, \ldots, r$ are independently and identically distributed random variables with a $\Gamma(1/\lambda, 1/\lambda)$ distribution (See [4].) that is

$$P\{W_{ij} \leqq x\} = \begin{cases} \dfrac{1}{(\lambda^{1/\lambda})} \left(\int_0^x e^{-u/\lambda} u^{(1/\lambda)-1}\, du\right) \Big/ \left(\int_0^\infty e^{-u} u^{(1/\lambda)-1}\, du\right) & \text{for} \quad x > 0 \\ = 0 & \text{for} \quad x \leqq 0. \end{cases} \tag{14}$$

REMARK 1. Recently BLACKWELL and KENDALL [3] computed the Martin boundary for POLYA's urn scheme and in the process proved a very special case of the Theorem 1 and the 'if' part of Corollary 1 using a lengthy argument. Theirs is the case where $\lambda=1$, $r=k$, $s_{0i}=1$ and $f_i(z)=z$ for all i.

REMARK 2. One of the important steps in the proof of Theorem 1 is to establish (11). We wish to point out that (11) is really a special case of a general phenomenon with multitype continuous time MARKOV branching processes. Let $X(t)=(X_1(t), \ldots, X_{1k_1}(t), X_{21}(t), \ldots, X_{2k_2}(t), \ldots, X_{r1}(t), \ldots, X_{rk_r}(t))$ be one such process such that we can decompose it into r independent positively regular branching processes (See [4] for definition.) which we denote by $(X_{j1}(t), \ldots, X_{jk_j}(t))$, $j=1, 2, \ldots, r$. For a positively regular branching process $\mathbf{X}(t)=(X_1(t), \ldots, X_k(t))$ there exists a real constant λ such that

$$\lim_{t\to\infty} \mathbf{X}(t)e^{-\lambda t} = uW \qquad \text{exists. w.p. 1,}$$

where u is a non-negative vector and W is a non-negative numerical random variable. (See [1].) Using obvious notation we then have w.p. 1

$$\lim_{t} (X_{j1}(t), \ldots, X_{jk_j}(t))e^{-\lambda_j t} = u_j W_j$$

for all $j=1, 2, \ldots, r$. This determines in a trivial way the almost sure behavior of many functionals on $X(t)$.

PROOF OF THEOREM 2. The 'if' part is just Corollary 1. The only 'if' part uses the following two lemmas.

LEMMA 1. Let $\{W_i\}_{i=1,2,\ldots}$ be a sequence of independently and identically distributed positive random variables with $EW_i=1$. Suppose there exists $\alpha>0$ such that for every pair n and m of integers, S_n/S_{n+m} has a $\beta(n/\alpha, m/\alpha)$ distribution where $S_n=\sum_{i=1}^{n} W_i$ for $n\geqq 1$. Then W_1 has a $\Gamma(1/\alpha, 1/\alpha)$ distribution.

PROOF. Let $Y_n = \dfrac{W_1}{W_1+(S_{n+1}-W_1)} = \dfrac{W_1}{W_1+\tilde{S}_n}$. By hypothesis, $Y_n \sim \beta(1/\alpha, n/\alpha)$ where $\sim$ stands for 'distributed as'. Now $W_1 = nY_n(\tilde{S}_n/n)/(1-Y_n)$. By strong law of large numbers, w.p. 1. $\tilde{S}_n/n \to 1$ and $Y_n \to 0$ as $n\to\infty$. Thus it suffices to show nY_n tends, as $n\to\infty$, in distribution to a $\Gamma(1/\alpha, 1/\alpha)$ random variable. But

$$P\{nY_n \leqq z\} = \frac{1}{\beta\left(\dfrac{1}{\alpha}, \dfrac{n}{\alpha}\right)} \int_0^{z/n} x^{(1/\alpha)-1}(1-x)^{(n/\alpha)-1}\,dx$$

and the right side tends to, as $n\to\infty$,

$$\frac{1}{\Gamma\left(\dfrac{1}{\alpha}\right)\alpha^{1/\alpha}} \int_0^{z} e^{-u/\alpha} u^{(1/\alpha)-1}\,du \qquad \text{q.e.d.}$$

LEMMA 2. Let $\{X(t),\ t \geqq 0\}$ be a one dimensional continuous time Markov branching process with associated infinitesimal generating function $u(z) = a[f(z) - z]$ satisfying $u'(1) = \lambda < \infty$. Let W be the almost sure limit of the non-negative martingale $\{X(t)e^{-\lambda t},\ t \geqq 0\}$. Let $P\{W = 0\} < 1$. Then W uniquely determines $f(z)$ provided a and λ are fixed.

PROOF is elementary and omitted.

Now we complete the proof of Theorem 2. From our hypothesis we have i) W_{ij} are identically distributed for all i and ii) the hypothesis of Lemma 1 is satisfied for W_{ij}. Now apply Lemma 2. q.e.d.

Acknowledgment. This note is a minor part of the author's doctoral thesis submitted to Stanford University. The author thanks his adviser Professor SAMUEL KARLIN for his help and encouragement.

REFERENCES

[1] ATHREYA, KRISHNA B.: Some Limit Theorems for Continuous Time Multitype Markov Branching Processes and B. Friedman's Urn, Unpublished doctoral thesis, Mathematics Department, Stanford University, 1967.

[2] ATHREYA, KRISHNA B. and KARLIN, S.: Limit Theorems for the Split Times of Branching Processes, *Journal Math. and Mech.* **17** (1969), 257-278.

[3] BLACKWELL, D. and. KENDALL, D. G.: The Martin Boundary for Polya's Urn and an Application to Stochastic Population Growth, *J. Appl. Probability* **1** No. 2 (1964).

[4] HARRIS, T.: *Theory of Branching Processes*, Springer-Verlag, Berlin, 1963.

[5] KARLIN, S.: *A First Course in Stochastic Processes*, Academic Press, New York, 1966.

[6] ATHREYA, KRISHNA B. and KARLIN, S.: Embedding of urn schemes into continuous time Markov branching processes and related limit theorems, *Ann. Math. Stat.* **39**, 6, 1801-1818.

The University of Wisconsin, Madison, Wisconsin, U.S.A.

(Received April 5, 1967.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 37–43.

RELATIONSHIP BETWEEN MIXED ZERO-ONE INTEGER LINEAR PROGRAMMING AND CERTAIN QUADRATIC PROGRAMMING PROBLEMS

by

F. FORGÓ

Introduction

Quadratic programming problems are generally defined as special constrained extremum problems of the following form:

$$Q(\mathbf{x}) = \mathbf{p}^*\mathbf{x} + \frac{1}{2}\mathbf{x}^*\mathbf{C}\mathbf{x} \to \max \tag{1}$$

subject to

$$\mathbf{A}\mathbf{x} \leqq \mathbf{b}$$

where $\mathbf{x}$ and $\mathbf{p}$ are points of the n dimensional Euclidean space, $\mathbf{b}$ is a vector of the m dimensional Euclidean space, $\mathbf{A}$ is an m by n matrix, $\mathbf{C}$ is a symmetric matrix of order n. Among the constraints $\mathbf{A}\mathbf{x} \leqq \mathbf{b}$ the nonnegativity restrictions $-\mathbf{x} \leqq \mathbf{0}$ are included. In addition let us suppose that $L = \{\mathbf{x} | \mathbf{A}\mathbf{x} \leqq \mathbf{b}\}$ has at least two extreme points and every basic solution of L is nondegenerate. This latter means that $\mathbf{x}_0 \in L$ is a basic solution of L if and only if $\mathbf{A}$ and $\mathbf{b}$ can be partitioned in the following manner:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} \quad \begin{matrix} \mathbf{A}_1\mathbf{x}_0 = \mathbf{b}_1 \\ \mathbf{A}_2\mathbf{x}_0 < \mathbf{b}_2, \end{matrix}$$

where $\mathbf{A}_1$ is nonsingular.

Furthermore let us suppose that $Q(\mathbf{x})$ attains its maximum on L.

In the special case when $\mathbf{C}$ is negative definite several methods are known for solving (1). (see: [1], [4]) In this case $Q(\mathbf{x})$ is concave, every local optimumpoint is also a global one and the set of optimal points is convex.

In this paper we want to deal with another special case when $\mathbf{C}$ is positive definite. We call

PROBLEM I.

$$Q(\mathbf{x}) = \mathbf{p}^*\mathbf{x} + \frac{1}{2}\mathbf{x}^*\mathbf{C}\mathbf{x} \to \max \tag{2}$$

subject to

$$\mathbf{x} \in L$$

$\mathbf{C}$ is positive definite,

PROBLEM II.

$$Q(\mathbf{x}) = \mathbf{p}^*\mathbf{x} + \frac{1}{2}\mathbf{x}^*\mathbf{C}\mathbf{x} \to \max \tag{3}$$

subject to

$$\mathbf{x} \in L$$

$\mathbf{C}$ is strictly positive definite.

The solution of Problem I. and II. is difficult because there can exist many local optimal points and they generally do not form a convex set. The aim of this paper is to show the relationship between Problem I—II. and the mixed zero-one integer linear programming problem.

I. The solution of problem II. can be reduced to the solution of a finite sequence of mixed zero-one integer linear programming problems.

First of all let us define what is a solution of Problem II. $\mathbf{x}_0 \in L$ is said to be a solution of Problem II. with precision ϱ ($\varrho > 0$) if $Q(\mathbf{y}) - Q(\mathbf{x}_0) \leqq \varrho$ where $\mathbf{y} \in L$ is one of the global maximumpoints of Problem II.

It is sufficient to search for the solution of Problem II. among the extreme points of L because it is well known that there is at least one extreme point of L among the optimal points of Problem II. So our aim can be formulated in the following manner: find $\mathbf{x}_0' \in L'$ for which $Q(\mathbf{y}) - Q(\mathbf{x}_0') \leqq \varrho$ where L' denotes the set of extreme points of L.

It is a consequence of the KUHN—TUCKER theorem [2] that if $\mathbf{x}^0$ is a local maximumpoint of (1), then the gradient vector $\mathbf{g}^*(\mathbf{x}^0) = \mathbf{p}^* + \mathbf{x}^{0*}\mathbf{C}$ can be expressed as a nonnegative linear combination of such row vectors of $\mathbf{A}$ which belong to active constraints.

The KUHN—TUCKER theorem says that if $\mathbf{x}^0$ is a local maximumpoint of (1), and the constraint qualification holds, then there exists a vector $\mathbf{u}^0$ so that

$$\left(\frac{\partial \Phi}{\partial x_j}\right)_{\mathbf{x}^0, \mathbf{u}^0} = 0 \qquad j = 1, 2, \ldots, n \tag{4}$$

$$\left.\begin{aligned} &\left(\frac{\partial \Phi}{\partial u_i}\right)_{\mathbf{x}^0, \mathbf{u}^0} \geqq 0 \qquad (5)\\ &\sum_{i=1}^{m} u_i^0 \left(\frac{\partial \Phi}{\partial u_i}\right)_{\mathbf{x}^0, \mathbf{u}^0} = 0 \qquad (6)\\ &u_i^0 \geqq 0, \qquad (7) \end{aligned}\right\} \quad i = 1, 2, \ldots, m$$

where

$$\Phi(\mathbf{x}, \mathbf{u}) = Q(\mathbf{x}) + \mathbf{u}^*(\mathbf{b} - \mathbf{A}\mathbf{x}) \tag{8}$$

Since our constraints are linear the constraint qualification always holds. Substituting (8) into (4) we obtain

$$\mathbf{p}^* + \mathbf{x}^{0*}\mathbf{C} - \mathbf{u}^{0*}\mathbf{A} = \mathbf{0}^*$$

Equality (6) means that such u_i^0 which corresponds to a nonactive constraint equals to 0, and this is what we have stated before.

Let us suppose that for an $\hat{\mathbf{x}} \in L$

$$\mathbf{g}^*(\hat{\mathbf{x}}) = \hat{\mathbf{z}}^* \mathbf{A}_1 \tag{9}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} \quad \begin{matrix} \mathbf{A}_1\hat{\mathbf{x}} = \mathbf{b}_1 \\ \mathbf{A}_2\hat{\mathbf{x}} < \mathbf{b}_2 \end{matrix}$$

and $\hat{\mathbf{z}} \geqq \mathbf{0}$.

Multiplying (9) by $\hat{\mathbf{x}}$ we get

$$\mathbf{g}^*(\hat{\mathbf{x}})\hat{\mathbf{x}} = \hat{\mathbf{z}}^*\mathbf{b}_1 \tag{10}$$

or with the substitution $\hat{\mathbf{u}}^* = [\hat{\mathbf{z}}^*, \mathbf{0}^*]$

$$\mathbf{g}^*(\mathbf{x})\hat{\mathbf{x}} = \hat{\mathbf{u}}^*\mathbf{b} \tag{11}$$

The following theorem will be helpful to show the desired relation between Problem II. and the mixed zero-one integer linear programming problem.

THEOREM 1. *There exist positive constants K, ε, M so that all local maximum-points of Problem I. satisfy the following system of inequalities:*

$$\mathbf{A}\mathbf{x} \leqq \mathbf{b} \tag{12}$$

$$\mathbf{u} \geqq \mathbf{0} \tag{13}$$

$$-\mathbf{C}\mathbf{x} + \mathbf{A}^*\mathbf{u} = \mathbf{p} \tag{14}$$

$$\left.\begin{matrix} b_i - \mathbf{a}_i^*\mathbf{x} \leqq K\delta_i & (15) \\ b_i - \mathbf{a}_i^*\mathbf{x} \geqq \varepsilon\delta_i & (16) \\ u_i + \delta_i M \leqq M & (17) \end{matrix}\right\} \quad i = 1, 2, \ldots, m,$$

$$\sum_{i=1}^{m} \delta_i = m - n \tag{18}$$

$$\left.\begin{matrix} 0 \leqq \delta_i \leqq 1 & (19) \\ \delta_i = \text{integer} & (20) \end{matrix}\right\} \quad i = 1, 2, \ldots, m$$

where $\mathbf{a}_i^*$ denotes the i-th row vector of $\mathbf{A}$, b_i is $\mathbf{b}$'s i-th component.

PROOF. Let

$$K \geqq K_{\max} = \max_i \left\{ \max_{\mathbf{x} \in L'} (b_i - \mathbf{a}_i^*\mathbf{x}) \right\}$$

$$\varepsilon \leqq \varepsilon_{\min} = \min_i \left\{ \min_{\substack{\mathbf{x} \in L' \\ b_i - \mathbf{a}_i^*\mathbf{x} > 0}} (b_i - \mathbf{a}_i^*\mathbf{x}) \right\} \qquad \varepsilon > 0,$$

$$M \geqq M_{\max} = \max_i \left\{ \max_{\mathbf{x} \in L'} u_i \right\}$$

$\max_i \left\{ \max_{\mathbf{x} \in L'} u_i \right\}$ means that at every $\mathbf{x} \in L'$ we consider the linear equation system

$$\mathbf{p} + \mathbf{C}\mathbf{x} - \mathbf{A}'^*\mathbf{u} = \mathbf{0}$$

where $\mathbf{A}'$ contains such row vectors of $\mathbf{A}$ which belong to active constraints, then by the nondegeneracy assumption $\mathbf{A}'$ is nonsingular, $\mathbf{u} = (\mathbf{A}'^*)^{-1}(\mathbf{p} + \mathbf{C}\mathbf{x})$ is uniquely determined and $\max_i \left\{ \max_{\mathbf{x} \in L'} u_i \right\}$ is the maximal component of $\mathbf{u}$ while $\mathbf{x}$ runs over all elements of L'.

Constraints (15), (16), (17), (19), (20) assure that (12—20) can be satisfied only by those $\mathbf{u}$ vectors which have 0 components corresponding to active constraints.

If for an $\mathbf{x}$ the equality $b_i - \mathbf{a}_i^* \mathbf{x} = 0$, holds then by (16) $\delta_i = 0$. (15) is also satisfied and (17) takes the form: $u_i \leqq M$.

If $b_i - \mathbf{a}_i^* \mathbf{x} > 0$, then (15) impiles $\delta_i = 1$, (16) is satisfied because of the definition of ε and (16) implies $u_i = 0$.

Constraint (18) says that $\delta_i = 0$ for exactly n indeces and it follows from the nondegeneracy assumption that (12—20) only by extreme points can be satisfied. Since the objective function of Problem II. is strictly convex all local maximumpoints are extreme points and satisfy (18).

By the above mentioned consequences of the KUHN—TUCKER conditions the inequality system (12—20) is satisfied by all local maximumpoints of L. Q. e. d.

Let us denote by H the set of those points $\mathbf{x} \in L$ for which $\mathbf{u}$ and $\delta_1, \delta_2, \ldots, \delta_m$ can be found such that (12—20) hold. According to Theorem 1. all the local maximumpoints of Problem II. belong to H. By (11) for all $\mathbf{x} \in H$ holds $\mathbf{u}^* \mathbf{b} = \mathbf{g}^*(\mathbf{x})\mathbf{x}$. Then we can re-form the objective function of Problem II. in the following manner:

$$Q(\mathbf{x}) = \mathbf{p}^* \mathbf{x} + \frac{1}{2} \mathbf{x}^* \mathbf{C} \mathbf{x} = \mathbf{p}^* \mathbf{x} + \mathbf{x}^* \mathbf{C} \mathbf{x} - \frac{1}{2} \mathbf{x}^* \mathbf{C} \mathbf{x} = \mathbf{g}^*(\mathbf{x}) \mathbf{x} - \frac{1}{2} \mathbf{x}^* \mathbf{C} \mathbf{x} =$$

$$= \mathbf{u}^* \mathbf{b} - \frac{1}{2} \mathbf{x}^* \mathbf{C} \mathbf{x} = R(\mathbf{u}, \mathbf{x})$$

Our aim is to find $\max\limits_{\mathbf{x} \in H} R(\mathbf{u}, \mathbf{x})$ or rather $\max\limits_{\mathbf{y} \in Y} R(\mathbf{y})$ if $Y = \{\mathbf{y}\}$ denotes the set of those vectors which satisfy (12—20) $(\mathbf{y}^* = [\mathbf{x}^*, \mathbf{u}^*, \mathbf{z}^*],\ \mathbf{z}^* = [\delta_1, \delta_2, \ldots, \delta_m])$.

Let us separate the nonnegativity conditions, write $\mathbf{A}'\mathbf{x} \leqq \mathbf{b}'$, $\mathbf{x} \geqq \mathbf{0}$ instead of $\mathbf{A}\mathbf{x} \leqq \mathbf{b}$ and introduce the following notations:

$$\begin{bmatrix} -\mathbf{A}' & \mathbf{0} & \mathbf{0} \\ \mathbf{A} & \mathbf{0} & K\mathbf{E} \\ \mathbf{A} & \mathbf{0} & \varepsilon\mathbf{E} \\ \mathbf{0} & \mathbf{E} & M\mathbf{E} \end{bmatrix} = \mathbf{B}_1$$

$$\begin{bmatrix} -\mathbf{C} & \mathbf{A}^* & \mathbf{0} \\ \mathbf{0}^* & \mathbf{0}^* & \mathbf{1}^* \end{bmatrix} = \mathbf{B}_2$$

$$[\mathbf{b}'^*, -\mathbf{b}^*, \mathbf{b}^*, M\mathbf{1}^*] = \mathbf{c}^*$$

$$[\mathbf{0}^*, \mathbf{b}^*, \mathbf{0}^*] = \mathbf{s}^*$$

$$\begin{bmatrix} \mathbf{C} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} = \mathbf{S}$$

$$\begin{bmatrix} \mathbf{p} \\ m-n \end{bmatrix} = \mathbf{q}$$

$$[1, 1, \ldots, 1] = \mathbf{1}^* \text{ (with } m \text{ components)}$$

So our original problem can be written in the following manner:

$$R(\mathbf{y}) = \mathbf{s}^*\mathbf{y} - \frac{1}{2}\mathbf{y}^*\mathbf{S}\mathbf{y} \to \max$$

subject to

$$\mathbf{y} \geqq \mathbf{0} \qquad \mathbf{y}^* = [y_1, y_2, \ldots, y_{n+2m}]$$

(21)
$$\mathbf{B}_1\mathbf{y} \leqq \mathbf{c}$$

$$\mathbf{B}_2\mathbf{y} = \mathbf{q}$$

$$\left.\begin{matrix} y_k \leqq 1 \\ y_k = \text{integer} \end{matrix}\right\} \quad k = n+m+1, n+m+2, \ldots, n+2m$$

$R(\mathbf{y})$ is a concave function of $\mathbf{y}$ disregarding the integrality constraints because $\mathbf{S}$ is positive definite. If the admissible error for Problem II. is ϱ, then it is admitted for (21), too. With other words this means that $\mathbf{y}_0$ is a solution of (21) if $R(\tilde{\mathbf{y}}) - R(\mathbf{y}_0) \leqq \varrho$, where $\tilde{\mathbf{y}}$ is one of the global maximumpoints of (21).

Let us consider the following problem:

$$\hat{R}(\mathbf{y}) = \mathbf{s}^*\mathbf{y} - \frac{1}{2}\mathbf{y}^*\hat{\mathbf{S}}\mathbf{y} \to \max$$

subject to

$$\mathbf{y} \geqq \mathbf{0}$$

(22)
$$\mathbf{B}_1\mathbf{y} \leqq \mathbf{c}$$

$$\mathbf{B}_2\mathbf{y} = \mathbf{q}$$

$$\left.\begin{matrix} y_k \leqq 1 \\ y_k = \text{integer} \end{matrix}\right\} \quad k = n+m+1, n+m+2, \ldots, n+2m,$$

where

$$\hat{\mathbf{S}} = \begin{bmatrix} \mathbf{C} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sigma\mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \sigma\mathbf{E} \end{bmatrix} \qquad \sigma > 0$$

Since $\mathbf{C}$ is strictly positive definite $\hat{\mathbf{S}}$ is strictly positive definite, too. Let $\hat{\mathbf{y}} = [\hat{\mathbf{x}}^*, \hat{\mathbf{u}}^*, \hat{\mathbf{z}}^*]$ be a global maximumpoint of (22). Obviously $R(\tilde{\mathbf{y}}) \geqq R(\hat{\mathbf{y}})$ and $\hat{R}(\hat{\mathbf{y}}) \geqq \hat{R}(\tilde{\mathbf{y}})$. From the latter expression we obtain

$$R(\tilde{\mathbf{y}}) - R(\hat{\mathbf{y}}) \leqq \frac{1}{2}\sigma\left[\sum_{i=1}^{m}(\tilde{u}_i^2 - \hat{u}_i^2) + \sum_{i=1}^{m}(\tilde{\delta}_i^2 - \hat{\delta}_i^2)\right].$$

If

(23)
$$\frac{1}{2}\sigma\left[\sum_{i=1}^{m}(\tilde{u}_i^2 - \hat{u}_i^2) + \sum_{i=1}^{m}(\tilde{\delta}_i^2 - \hat{\delta}_i^2)\right] \leqq \varrho$$

then $R(\tilde{\mathbf{y}}) - R(\hat{\mathbf{y}}) \leqq \varrho$. If we want (23) to hold for all $\mathbf{u}$, $\mathbf{z}$ then (23) must be satisfied even if the left side assumes its maximum. Since $0 \leqq u_i \leqq M$ and $0 \leqq \delta_i \leqq 1$ $i = 1, 2, \ldots m$ then

(24)
$$\sigma \leqq \frac{2\varrho}{m(M^2+1)}$$

So if σ is chosen according to (24) and instead of (21) we solve (22) and the optimal solution of (22) is $\mathbf{y}^{0*}=[\mathbf{x}^{0*}, \mathbf{u}^{0*}, \mathbf{z}^{0*}]$, then $\mathbf{x}^0$ is a solution to Problem II.

(22) can be solved because $\hat{\mathbf{S}}$ is strictly positive definite, the constraints are linear and the absolute maximumpoint is outside of the feasible region, if ϱ is chosen small enough.

A method has been given for the solution of problems of this type by KÜNZI and OETTLI [3]. They solve a sequence of mixed integer linear programming problems and if such a solution is obtained which can be found among the solutions of the previous programming problems, then this solution is an optimal one. In our case we can get to this final situation after finite number of steps, since every feasible solution of problem (22) is a local extreme point of Problem II. and L has only finite number of extreme points.

So the solution of Problem II. is reduced to the solution of a finite squence of mixed zero-one integer linear programming problems.

II. The solution of every mixed zero-one integer linear programming problem can be reduced to the solution of a finite sequence of programming problems of type Problem I.

Let us consider a mixed zero-one integer linear programming problem:

$$z(\mathbf{x}) = \sum_{j=1}^{n} c_j x_j \to \max$$

subject to

$$0 \leqq x_j \leqq 1 \quad x_j = \text{integer} \qquad j = 1, 2, \ldots, p$$

(25)
$$x_j \geqq 0 \qquad j = p+1, \ldots, n$$

$$\sum_{j=1}^{n} a_{ij} x_j \leqq b_i \qquad i = 1, 2, \ldots, m$$

Let us suppose that the feasible region is bounded. (25) is said to be solved with precision ε ($\varepsilon > 0$) if we find such feasible point $\mathbf{x}^0$ for which $z(\mathbf{y}) - z(\mathbf{x}^0) < \varepsilon$ where $\mathbf{y}$ is one of the global maximumpoints of (25).

An initial feasible solution can be obtained by solving the following quadratic programming problem:

$$Q(x_1, x_2, \ldots, x_p) = \sum_{j=1}^{p} x_j(1 - x_j) \to \min$$

subject to

$$0 \leqq x_j \leqq 1 \qquad j = 1, 2, \ldots, p$$

(26)
$$x_j \geqq 0 \qquad j = p+1, \ldots, n$$

$$\sum_{j=1}^{n} a_{ij} x_j \leqq b_i \qquad i = 1, 2, \ldots, m$$

If $Q(x_1^0, x_2^0, \ldots, x_p^0) > 0$ where $[x_1^0, \ldots, x_p^0, x_{p+1}^0, \ldots, x_n^0]$ is one of the global maximumpoints of (26), then (25) has no feasible solution. If $Q(x_1^0, x_2^0, \ldots, x_p^0) = 0$, then we proceed by solving the following linear programming problem:

$$\sum_{j=p+1}^{n} c_j x_j \to \max$$

subject to

$$x_j \geqq 0 \qquad j = p+1, \ldots, n \tag{27}$$

$$\sum_{j=p+1}^{n} a_{ij} x_j \leqq b_i - \sum_{j=1}^{p} a_{ij} x_j^0$$

Let the optimal solution be $x_{p+1}'^0, x_{p+2}'^0, \ldots, x_n'^0$. So in the first step we have just obtained the feasible solution $[x_1^0, x_2^0, \ldots, x_p^0, x_{p+1}'^0, \ldots, x_n'^0]$. Let the value of the objective function be z_1.

In the next step we add the following "cutting plane" to the constraint system of (26)

$$\sum_{j=1}^{n} c_j x_j \geqq z_1 + \varepsilon$$

In general in the k-th step the additional constraint is

$$\sum_{j=1}^{n} c_j x_j \geqq z_{k-1} + \varepsilon$$

The procedure is finished if after some steps $Q(x_1^t, x_2^t, \ldots, x_p^t) > 0$. The solution of problem (25) is the feasible solution obtained in the previous step. The number of steps is finite because the feasible set of (25) is bounded.

We can get the pure zero-one integer linear programming problem as a special case. In this case ε can be chosen 1 if all the c_j-s are integer.

Since problem (26) and (27) are of the type of Problem I., the solution of the mixed zero-one integer linear programming problem is reduced to the solution of a finite sequence of quadratic programming problems of type Problem I.

The author wishes to thank Professor A. Prékopa for helpful comments and discussions.

REFERENCES

[1] DANTZIG, G. B.: *Linear Programming and Extensions*, Princeton, 1963.
[2] KUHN, H. and TUCKER, W.: Nonlinear Programming, *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley, 1950.
[3] KÜNZI, H. and OETTLI, W.: *Quadratic Integer Programming* (in Recent Advances in Mathematical Programming), McGraw Hill, New York, 1963.
[4] KÜNZI, H., and KRELLE, W.: *Nichtlineare Programmierung*, Springer Verlag, Berlin, 1962.

Karl Marx University of Economics, Budapest

(Received January 27. 1967.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 45–53.

DERIVATION OF CERTAIN FINITE KONTOROVITCH TYPE OF INTEGRAL TRANSFORMS

by

A. CHAKRABARTI

Abstract. Integral transforms are often derived from GREEN's functions. Previous authors developed GREEN's functions corresponding to LAPLACE's equation having logarithmic singularity in the radial direction. Here the GREEN's function with logarithmic singularity in the angular direction gas been derived to find the finite integral transform of LEBEDEV—KONTOROVITCH type in an easier way.

LEBEDEV—KONTOROVITCH integral transform is applied in solution of problems of physics and engineering. Its form is known when the integral extends from zero to infinity. Its finite integral form has been recently developed by NAYLOR [1]. He used the GREEN's function corresponding to the equation

$$r^2\psi_{rr}+r\psi_r+\psi_{\theta\theta}+r^2\psi_{zz}=0$$

which has got the necessary logarithmic singularity at $r=r_0$, $\theta=\theta_0$, when approached in the radial direction. This approach lead him to long calculations, and introduction of additional functions. But the construction of GREEN's function having logarithmic singularity in the approach to $\theta=\theta_0$ in the θ-direction, solves the problem directly. This method was developed earlier (cf. SOMMERFELD [2]). It is easy to pass from one form of GREEN's function to another. In fact, the essential step of the method employed here to construct the desired integral transforms is to convert one form of a GREEN's function into another form by alternative kind of eigen-function expansion. In this paper, several forms of finite LEBEDEV—KONTOROVITCH transform are derived. In [1], the kernel of the integral generating the transform of the function $f(r)$, defined in the interval $0\leqq r\leqq a$, is composed of the modified Bessel-functions of the first and second kinds. It is to be noted that no special interest was given to generate an integral transform of a function $f(r)$ defined in the interval $\infty > r \geqq a$. In this case the kernel is a composition of the Hankel-functions. The transform as stated above of a function $f(r)$ adapted to the interval $[0, a]$ is derived in §. 1, and that adapted to the interval $[a, \infty)$ is derived in §. 2

§ 1. To generate an integral transform adapted to remove the operator

$$r^2\frac{\partial^2}{\partial r^2}+r\frac{\partial}{\partial r}-k^2r^2$$

in the finite interval $0\leqq r\leqq a$, let us consider the partial differential equation

$$r^2\psi_{rr}+r\psi_r-k^2r^2\psi+\psi_{\theta\theta}=0 \tag{1}$$

in the region $0 \leqq r \leqq a, 0 \leqq \theta \leqq \alpha$ with ψ determined by the conditions

$$\begin{aligned} &\psi = 0, \qquad r = a, \quad 0 \leqq \theta \leqq \alpha; \\ &\psi = 0, \qquad \theta = \alpha, \quad 0 < r < a; \\ &\psi = f(r), \quad \theta = 0, \quad 0 < r < a \end{aligned} \tag{2}$$

We assume that $\psi(r, \theta)$ is expansible in the form

$$\psi(r, \theta) = \sum_{n=1}^{\infty} c_n(\theta) I_{\nu_n}(kr), \tag{3}$$

where $c_n(\theta)$ is a function of θ alone and $I_{\nu_n}(kr)$ is the modified Bessel's function of the first kind and of order ν_n, where ν_n $(n = 1, 2, \ldots)$ are roots of the transcendental equation

$$I_\nu(ka) = 0 \qquad (\nu = \nu_1, \nu_2, \ldots)$$

The orthogonality relation for the modified Bessel's functions of first kind is given by

$$\int_0^a \frac{1}{r} I_{\nu_m}(kr) I_{\nu_n}(kr)\, dr = N_n \delta_{mn}, \tag{4}$$

where

$$N_n = \left[-\frac{1}{2\nu} \frac{\dfrac{\partial}{\partial \nu}\{I_\nu(ka)\}}{K_\nu(ka)} \right]_{\nu = \nu_n}, \tag{5}$$

$K_\nu(ka)$ being modified second-kind Bessel's function of order ν_n and δ_{mn} — Kronecker-deltas. This relation is directly derivable from the modified Bessel's equation. With the help of relation (4), the functions $c_n(\theta)[n = 1, 2, \ldots]$ can be determined from (3), by the equation:

$$c_n(\theta) = \frac{1}{N_n} \int_0^a \frac{1}{r} \psi(r, \theta) I_{\nu_n}(kr)\, dr \tag{6}$$

Equation (1) converts then to the ordinary differential equation

$$c_n''(\theta) + \nu_n^2 c_n(\theta) = 0, \tag{7}$$

where dashes denote differentiation with respect to θ. The solution of (7) can be put as

$$c_n(\theta) = A \sin \nu_n \theta + B \cos \nu_n \theta, \tag{8}$$

where A and B are constants. The constants A and B can be determined by the boundary conditions (2). From (2) and (3) we have for $\theta = \alpha$,

$$\sum_{n=1}^{\infty} c_n(\alpha) I_{\nu_n}(kr) = 0; \qquad 0 < r < a.$$

Therefore,

$$c_n(\alpha) = 0 \qquad (n = 1, 2, \ldots) \tag{9}$$

Again for $\theta = 0$,

$$f(r) = \sum_{n=1}^{\infty} c_n(0) I_{\nu_n}(kr), \qquad 0 < r < a.$$

Hence,

$$c_n(0) = \frac{1}{N_n} \int_0^a I_{\nu_n}(kr) \frac{f(r)}{r}\, dr. \tag{10}$$

From (8) with the help of (10),

$$B = \frac{1}{N_n} \int_0^a I_{\nu_n}(kr) \frac{f(r)}{r}\, dr \tag{11}$$

and from (8) with the help of (9),

$$A = -B \frac{\cos \nu_n \alpha}{\sin \nu_n \alpha}. \tag{12}$$

Finally the solution of (7) is given by with the help of eqs. (11), (12) and (8)

$$c_n(\theta) = \frac{\sin \nu_n(\alpha - \theta)}{\sin \nu_n \alpha} \cdot \frac{1}{N_n} \int_0^a \frac{f(r)}{r} I_{\nu_n}(kr)\, dr. \tag{13}$$

From (3),

$$\psi(r, \theta) = \sum_{n=1}^{\infty} \frac{1}{N_n} \frac{\sin \nu_n(\alpha - \theta)}{\sin \nu_n \alpha} \cdot I_{\nu_n}(kr) \int_0^a \frac{f(r)}{r} I_{\nu_n}(kr)\, dr$$

which by (5) is equivalent to

$$\psi(r, \theta) = -2 \sum_{\nu = \nu_1, \nu_2, \ldots} \frac{\sin \nu(\alpha - \theta)}{\sin \nu \alpha} \cdot \frac{\nu I_\nu(kr) K_\nu(ka)}{\frac{\partial}{\partial \nu}\{I_\nu(ka)\}} \cdot \int_0^a \frac{f(r)}{r} \cdot I_\nu(kr)\, dr. \tag{14}$$

We take the whole set of zeros of $I_\nu(ka)$ which lie on the imaginary axis of the ν-plane (cf. GRAY and MATHEWS [3], Chap. 3, p. 88). Let us consider the integral

$$\frac{1}{2\pi i} \int_\Gamma \left[\frac{\sin \nu(\alpha - \theta)}{\sin \nu \alpha} \cdot \frac{I_\nu(kr)}{I_\nu(ka)} \cdot \nu \int_0^a \frac{f(r)}{r} \{I_\nu(ka) K_\nu(Kr) - I_\nu(kr) K_\nu(ka)\}\, dr \right] d\nu, \tag{15}$$

where Γ is a positively oriented contour in the ν-plane containing the zeros of $I_\nu(ka)$ which lie on the imaginary axis of the ν-plane. The suitable path consists of two loops enclosing the zeros of $I_\nu(ka)$ on the half planes Im $\{\nu\} > 0$ and Im $\{\nu\} < 0$ respectively. This path obviously does not include the zeros of $\sin \nu\alpha = 0$.

MAGYAR TUDOMÁNYOS AKADÉMIA KÖNYVTÁ...

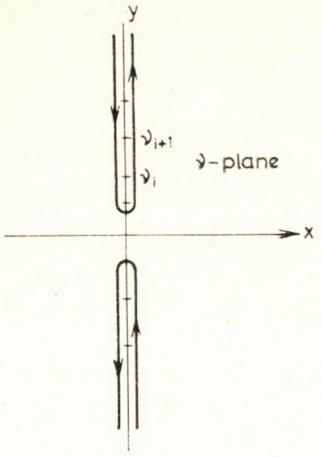


Fig. 1

The first term of the integrand in (15)

$$\frac{\sin \nu(\alpha-\theta)}{\sin \nu\alpha} \cdot \frac{I_\nu(kr)}{I_\nu(ka)} \cdot \nu \int_0^a \frac{f(r)}{r} \{I_\nu(ka) K_\nu(kr)\}\, dr =$$

$$= \frac{\sin \nu(\alpha-\theta)}{\sin \nu\alpha} \cdot I_\nu(kr) \cdot \nu \int_0^a \frac{f(r)}{r} K_\nu(kr)\, dr$$

is an analytic function inside Γ and hence it will have no residue, when necessary restrictions on $f(r)$ are satisfied. (See Appendix I). The second term of the integrand in (15)

$$-\frac{\sin \nu(\alpha-\theta]}{\sin \nu\alpha} \cdot \frac{I_\nu(kr)}{I_\nu(ka)} \cdot \nu \int_0^a \frac{f(r)}{r} I_\nu(kr)\, K_\nu(ka)\, dr =$$

(16)

$$= -\frac{\sin \nu(\alpha-\theta)}{\sin \nu\alpha} \cdot \nu K_\nu(ka) \cdot \frac{I_\nu(kr)}{I_\nu(ka)} \int_0^a \frac{f(r)}{r} I_\nu(kr)\, dr$$

has simple poles at $\nu = \nu_1, \nu_2, \ldots$ in Γ. Considering the pole at $\nu = \nu_i$, we have

$$I_\nu(ka) = I_{\nu_i}(ka) + (\nu - \nu_i) \left\{\frac{\partial}{\partial \nu} I_\nu(ka)\right\}_{\nu=\nu_i} + \cdots$$

Therefore, the residue of the function (16) at the pole $\nu = \nu_i$ is

$$-\frac{\sin \nu_i(\alpha-\theta)}{\sin \nu_i\alpha} \cdot \frac{I_{\nu_i}(kr)}{\left\{\frac{\partial}{\partial \nu} I_\nu(ka)\right\}_{\nu=\nu_i}} \cdot \nu_i K_{\nu_i}(ka) \int_0^a \frac{f(r)}{r} I_{\nu_i}(kr)\, dr.$$

Similarly taking into consideration all the poles inside Γ, the integral (15) can be written as the summation

$$-\sum_{\nu=\nu_1, \nu_2, \ldots} \frac{\sin \nu(\alpha-\theta)}{\sin \nu\alpha} \cdot \nu \cdot \frac{I_\nu(ka) K_\nu(ka)}{\frac{\partial}{\partial \nu}\{I_\nu(ka)\}} \int_0^a \frac{f(r)}{r} I_\nu(kr)\, dr$$

which is half the series (14) for $\psi(r, \theta)$. Hence the solution of the boundary value problem considered here is obtained in the form:

$$\psi(r, \theta) = \frac{1}{\pi i} \int_\Gamma \frac{\sin \nu(\alpha-\theta)}{\sin \nu\alpha} \cdot \frac{I_\nu(kr)}{I_\nu(ka)} \cdot \nu F(\nu)\, d\nu, \tag{17}$$

where

$$F(v) = \int_0^a \{I_v(ka)K_v(kr) - I_v(kr)K_v(ka)\}\frac{f(r)}{r}\,dr. \tag{18}$$

This $F(v)$ is the required finite LEBEDEV—KONTOROVITCH type transform of $f(r)$ adapted to the interval $(0, a)$, which is just the same as that obtained in [1].

Now when the boundary $\theta=0$ is approached, the value of ψ will approach the value $f(r)$. From (17) we arrive then at

$$f(r) = \frac{1}{\pi i}\int_\Gamma \frac{I_v(kr)}{I_v(ka)}\cdot vF(v)\,dv \tag{19}$$

which is the inversion integral. The contour however is not apparently identical with the contour considered in [1]. The contour can obviously be distorted to the line parallel to imaginary axis from $c-i\infty$ to $c+i\infty$, $\mathrm{Re}\{c\}>0$, as there is no root of $I_v(ka)=0$ in the right-half plane. The distortion of the contour is possible for the reasons that when v is large, and $\varkappa$ real and finite, (4, p. 134),

$$I_v(\varkappa)\sim\frac{1}{\Gamma(v+1)}\cdot\left(\frac{\varkappa}{2}\right)^v,$$

$$K_v(\varkappa)\sim\frac{2^{v-1}\Gamma(v)}{\varkappa^v}$$

so that

$$\frac{I_v(kr)}{I_v(ka)}\cdot v\cdot[I_v(ka)K_v(kr)-I_v(kr)K_v(ka)]$$

$$\sim\frac{\Gamma(v)\cdot v}{\Gamma(v+1)}\cdot\frac{1}{2}\left[1-\left(\frac{r}{a}\right)^{2v}\right]\to\frac{1}{2},\quad\text{if}\quad\mathrm{Re}\{v\}>0$$

and $v\to\infty$, as $0<r<a$.

§ 2. We consider now an integral transform of a type in which the kernel is composed of the Hankel functions and the interval under consideration is (a, ∞). Consider the differential equation

$$r^2\psi_{rr}+r\psi_r+k^2r^2\psi+\psi_{\theta\theta}=0\quad\text{in}\quad[0\leqq\theta\leqq\alpha,\ r\geqq a],$$

with ψ determined by the conditions:

$$\begin{aligned}&\psi=0, && r=a, && 0\leqq\theta\leqq\alpha;\\ &\psi=0, && \theta=\alpha, && r\geqq a;\\ &\psi=f(r), && \theta=0, && r>a,\end{aligned} \tag{20}$$

with the further condition that ψ vanishes as $r\to\infty$. Let

$$\psi(r,\theta)=\sum_{n=1}^{\infty}D_n(\theta)H_{\nu n}^{(1)}(kr), \tag{21}$$

where $D_n(\theta)$ is a function of θ alone and $H_{\nu_n}^{(1)}(kr)$ $(n=1, 2, \ldots)$ are the Hankel-functions of the first kind of order ν_n, $\nu_n's$ being the roots of the transcendental equation

$$H_\nu^{(1)}(ka) = 0. \qquad (\mathrm{Re}\{\nu\} > 0). \tag{22}$$

The orthogonality relation, in this case, is

$$\int_a^\infty \frac{1}{r} H_{\nu_m}^{(1)}(kr) H_{\nu_n}^{(1)}(kr)\, dr = M_n \delta_{mn}, \tag{23}$$

where

$$M_n = \left[\frac{2i}{\pi\nu} \cdot \frac{\dfrac{\partial}{\partial\nu}\{H_\nu^{(1)}(ka)\}}{H_\nu^{(2)}(ka)} \right]_{\nu=\nu_n}, \tag{24}$$

δ_{mn} being Kronecker-deltas.

From (21) and (22) it follows that

$$D_n(\theta) = \frac{1}{M_n} \int_a^\infty \frac{1}{r} \psi(r, \theta) H_{\nu_n}^{(1)}(kr)\, dr. \tag{25}$$

Then, by (25), the eqn. (1) transforms again to the ordinary differential equation for $D_n(\theta)$:

$$D_n''(\theta) + \nu_n^2 D_n(\theta) = 0 \tag{26}$$

From (26) and the first two boundary conditions of (20), proceeding as in the previous case, we get

$$D_n(\theta) = \frac{\sin \nu_n(\alpha - \theta)}{\sin \nu_n \alpha} \cdot \frac{1}{M_n} \int_a^\infty \frac{f(r)}{r} \cdot H_{\nu_n}^{(1)}(kr)\, dr. \tag{27}$$

From (21), therefore,

$$\psi(r, \theta) = \sum_{n=1}^{\infty} \frac{\sin \nu_n(\alpha - \theta)}{\sin \nu_n \alpha} \cdot \frac{H_{\nu_n}^{(1)}(kr)}{M_n} \int_a^\infty \frac{f(r)}{r} H_{\nu_n}^{(1)}(kr)\, dr.$$

Then, by (24),

$$\psi(r, \theta) = \sum_{\nu=\nu_1, \nu_2, \ldots} \frac{\pi}{2i} \cdot \frac{\nu \sin \nu(\alpha - \theta)}{\left\{ \dfrac{\partial}{\partial\nu} H_\nu^{(1)}(ka) \right\}} \cdot \int_a^\infty \frac{f(r)}{r} \cdot H_\nu^{(1)}(kr) H_\nu^{(2)}(ka)\, dr. \tag{28}$$

The summation is over the roots ν_n where $\mathrm{Re}\{\nu_n\} > 0$. In this connection it is to be mentioned that the roots of $H_\nu^{(1)}(ka) = 0$ lie in the first and the third quadrants of the ν-plane.

Writing the series (28) into a contour integral over a contour, we get

$$\psi(r,\theta)=\frac{1}{4}\int_{\Gamma}\frac{\nu\sin\nu(\alpha-\theta)}{\sin\nu\alpha}\cdot\frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(ka)}\cdot F_1(\nu)\,d\nu, \tag{29}$$

where

$$F_1(\nu)=\int_a^\infty [H_\nu^{(1)}(ka)H_\nu^{(2)}(kr)-H_\nu^{(2)}(ka)H_\nu^{(1)}(kr)]\cdot\frac{f(r)}{r}\,dr, \tag{30}$$

defining the transform of $f(r)$ adapted to the interval (a,∞) and Γ is the positively oriented loop in the ν-plane, enclosing the zeros of $H_\nu^{(1)}(ka)$ inside it, as shown in Fig. 2. None of the zeros of $\sin\nu\alpha$ is included in it.

Here also certain restrictions on $f(r)$ must be specified. (See Appendix II). Again $\psi(r,\theta)$ attains the value $f(r)$ on $\theta=0$ and from (29), therefore,

$$f(r)=\frac{1}{4}\int_{\Gamma}\frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(ka)}\cdot\nu F_1(\nu)\,d\nu \tag{31}$$

which is the inversion integral in this case.

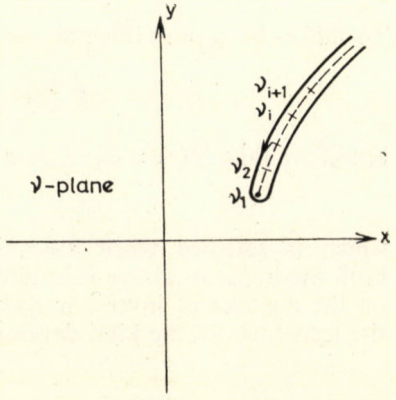


Fig. 2

The contour can be distorted to a line parallel to the imaginary axis from $a-i\infty$ to $a+i\infty$ where $0<\mathrm{Re}\,\{a\}<\delta$ and $\delta<|\,\mathrm{Re}\,\{\nu_1,\nu_2,\ldots$ lying in the upper half-plane$\}|$. This distortion is possible because of the asymptotic formulae for $H_\nu^{(p)}(\varkappa)[p=1,2]$ for large ν. (cf. [4]).

$$H_\nu^{(p)}(\varkappa)\approx i\left(\frac{2}{\varkappa}\right)^\nu\cdot\frac{\Gamma(\nu)}{\pi},\qquad \varkappa\to\text{finite},\quad \nu>0,$$

so that

$$\frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(ka)}\cdot\nu[H_\nu^{(1)}(ka)H_\nu^{(2)}(kr)-H_\nu^{(2)}(ka)H_\nu^{(1)}(kr)]\to 0,$$

if $\mathrm{Re}\,\{\nu\}>0$, since $a<r$.

Appendix-I.

Restrictions on $f(r)$ in the first case.

The necessary restrictions on $f(r)$ should be such that the integrals

$$\int_0^a\frac{f(r)}{r}I_\nu(kr)\,dr\quad\text{and}\quad\int_0^a\frac{f(r)}{r}K_\nu(kr)\,dr$$

may exist. To make the first integral convergent, let us take $|f(r)| > o(r^c)$, as $r \to 0$. We have (cf. [4]).

$$I_\nu(\varkappa) \approx \frac{\left(\frac{1}{2}\varkappa\right)^\nu}{\Gamma(\nu+1)}, \quad \text{as} \quad \varkappa \to 0.$$

The first integral therefore will converge, if

$$1 - c - \nu < 1,$$

i.e. if

$$c + \nu > 0.$$

To make the second integral convergent, we note (cf. [4])

$$K_\nu(\varkappa) \approx \frac{2^{\nu-1}\Gamma(\nu)}{\varkappa^\nu}, \quad \text{as} \quad \varkappa \to 0,$$

and if we take $|f(r)| > o(r^\alpha)$, as $r \to 0$ the second integral will converge, if

$$\nu + 1 - \alpha < 1,$$

which is satisfied when $\nu < \alpha$. Hence, if we take $|f(r)| > o(r^\alpha)$ where $\alpha > 0$, ν, both the integrals above exist and are convergent. The values of Re $\{\nu\}$ are positive on the contour of inverse transformation. Hence, if $f(r) \approx o(r^\alpha)$, $r \to 0$ where $\alpha > 0$, the transform of the kind developed here and its inverse is valid.

Appendix-II.

Restrictions on $f(r)$ in the second case.

The restrictions should be such that the integrals

$$\int_a^\infty \frac{f(r)}{r} H_\nu^{(1)}(kr)\,dr \quad \text{and} \quad \int_a^\infty \frac{f(r)}{r} H_\nu^{(2)}(kr)\,dr$$

both exist. We have, by the asymptotic formulae (cf. [4]) as $|\varkappa| \to \infty$ ($\varkappa$ real and positive)

$$H_\nu^{(p)}(\varkappa) \sim \left(\frac{2}{\pi\varkappa}\right)^{\frac{1}{2}} \exp\left\{\pm i\left(\varkappa - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)\right\}, \qquad (p = 1, 2).$$

If we take $|f(r)| < o(r^k)$ as $r \to \infty$, the above integrals will converge, if

$$1 + \frac{1}{2} - k > 1$$

or, if

$$k < \frac{1}{2}$$

I take this opportunity of thanking Dr. S. C. Das Gupta for suggesting this investigation and for guidance at various stages of the work.

REFERENCES

[1] NAYLOR, D.: On a finite Lebedev transform, *J. Math. Mech.* **12** (1963) 375.
[2] SOMMERFELD, A.: *Partial Differential Equations in Physics*, Academic Press, New York, 1949, 281.
[3] GRAY, A. and MATHEWS, G. B.: A treatise on Bessel functions, London, Macmillan, 1952, ch. 3, 88.
[4] LEBEDEV, N. N.: *Special functions and their applications*, Prentice Hall, New York, 1965, 134.

Bengal Engineering College, Howrah, West Bengal, India

(Received May 5, 1967.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 55—58.

UNE ÉVALUATION DU NOMBRE CHROMATIQUE D'UN GRAPHE FINI

par

I. TOMESCU

Dans cette note on présente une évaluation de la borne supérieure du nombre chromatique et on construit une arborescence qui permet de déterminer précisement le nombre chromatique d'un graphe fini sans boucles.

Notations. Soit $G=(X, \Gamma)$ un graphe fini symétrique et sans boucles avec n sommets. On désigne par $d(x_k)$ le degré du sommet $x_k \in X$ et on suppose que $d(x_1) \geqq d(x_2) \geqq \ldots \geqq d(x_n)$.

On définit une relation réflexive et symétrique $R_G \subset X \times X$ de façon que $(x, y) \in R_G$ si $y \notin \Gamma x$, et donc ni $x \notin \Gamma y$.

Le problème du coloriage des sommets du graphe G avec un nombre minimum de couleurs distinctes de façon que deux sommets adjacents soient de couleurs différentes devient le problème de la détermination d'une relation d'équivalence $R_G^* \subset R_G$ ainsi que R_G^* ait un nombre minimum de classes. Ce nombre minimum, nommé le nombre chromatique du graphe G, sera noté $K(G)$.

Par $A \setminus B$ on note la différence des ensembles A et B.

Définition. Un sous-ensemble $Sx_k \subset X$ est une classe maximale de l'élément $x_k \in X$ par rapport à la relation R_G si $x_k \in Sx_k$; pour tout $x_i, x_j \in Sx_k$, $(x_i, x_j) \in R_G$; pour tout $x_l \in X \setminus Sx_k$ il existe $x_m \in Sx_k$ tel que $(x_l, x_m) \notin R_G$.

Une classe maximale correspond à un sous-graphe complet maximal du graphe complémentaire $\bar{G}=(X, \bar{\Gamma})$ du graphe G, défini par $y \in \bar{\Gamma} x$ si $y \notin \Gamma x$.

Soit une partition $\Delta = \{X_k\}_{1 \leqq k \leqq q}$ de $X (q \leqq n)$, telle que pour tout k et $x_i, x_j \in X_k$ on ait $(x_i, x_j) \in R_G$.

On note $\alpha(G) = \max\limits_{k=1,2,\ldots,n} (\min(k, 1+d(x_k)))$ et

$$\beta_\Delta(G) = \max_{k=1,2,\ldots,q} (\min(k, 1+\max_{x_p \in X_k}(d(x_p)))) = \max_{k=2,\ldots,q} (\min(k, 1+\max_{x_p \in X_k}(d(x_p)))),$$

si

$$\sum_{p=1}^{n} d(x_p) \neq 0.$$

R. BROOKS dans [1] a établi l'inégalité $K(G) \leqq 1+\max\limits_{p=1,2,\ldots,n}(d(x_p))$ et D. WELSH et M. POWELL ont démontré dans [3] que $K(G) \leqq \alpha(G) \leqq 1+\max\limits_{p=1,2,\ldots,n}(d(x_p))$.

THÉORÈME 1. *$K(G) \leqq \beta_\Delta(G)$ pour toute partition Δ induite d'une relation d'équivalence $R_G^* \subset R_G$ et la condition*

$$\max_{x_p \in X_1}(d(x_p)) \geqq \max_{x_p \in X_2}(d(x_p)) \geqq \cdots \geqq \max_{x_p \in X_q}(d(x_p))$$

implique

$$K(G) \leqq \beta_{\Delta}(G) \leqq \alpha(G) \leqq 1 + \max_{p=1,2,\ldots,n} (d(x_p)).$$

Evidemment $\beta_{\Delta}(G) = \alpha(G)$ *si* $X_1 = \{x_1\}$; $X_2 = \{x_2\}$; ...; $X_n = \{x_n\}$ *et* $\beta_{\Delta}(G) = K(G)$, *si la partition* Δ *a un nombre minimum de classes parce que dans ce cas* $\beta_{\Delta}(G) \leqq \max_{k=1,\ldots,q} (k) = q = K(G)$ *et donc* $K(G) = \min_{\Delta} (\beta_{\Delta}(G))$.

DÉMONSTRATION. Si l'ensemble X_1 est une classe maximale par rapport à R_G, alors on vérifie si X_2 est une classe maximale par rapport à $R_G \cap (X \setminus X_1)^2$.

Si X_1 n'est pas une classe maximale par raport à R_G, alors on complète X_1 jusqu'à une classe maximale X_1', par l'extraction de quelques éléments des classes $X_2, \ldots, X_q$, qui deviennent $X_2^1, \ldots, X_q^1$ ($X_2^1 \subset X_2$; ...; $X_q^1 \subset X_q$).

On note avec $X_2^2, X_3^2, \ldots, X_{q_1}^2$ les ensembles non-vides de la suite $X_2^1, \ldots, X_q^1$ ainsi que $X_2^2 = X_{r_1}^1$; ...; $X_{q_1}^2 = X_{r_{q_1-1}}^1$ et $2 \leqq r_1 < r_2 < \ldots < r_{q_1-1} \leqq q$.

Si X_2^2 est une classe maximale par rapport à la relation $R_G \cap (X \setminus X_1')^2$, alors on note $X_2^2 = X_2'$ et on vérifie si X_3^2 est une classe maximale par rapport à $R_G \cap (X \setminus (X_1' \cup X_2'))^2$ etc.

Si X_2^2 n'est pas une classe maximale par rapport à $R_G \cap (X \setminus X_1')^2$, alors on complète X_2^2 jusqu'à une classe maximale X_2' par l'extraction de quelques éléments des classes $X_3^2, \ldots, X_{q_1}^2$.

Après un nombre fini de pas nous obtenons une nouvelle partition $\Delta' = \{X_k'\}_{1 \leqq k \leqq q'}$ de X, pour laquelle $q' \leqq q$ avec les propriétés suivantes:

La classe X_1' est maximale par rapport à la relation R_G, la classe X_k' est maximale par rapport à la relation $R_G \cap (X \setminus \bigcup_{i=1}^{k-1} X_i')^2$ pour tout $k = 2, 3, \ldots, q'$ et $X_k \subset \bigcup_{i=1}^{\min(k,q')} X_i'$ pour tout $k = 1, 2, \ldots, q$.

Si $x_m \in X \setminus \bigcup_{i=1}^{k} X_i'$ alors $d(x_m) \geqq k$, donc il résulte que $x_p \in \bigcup_{i=1}^{\varphi(p)} X_i'$ avec $\varphi(p) = \min(q', 1 + d(x_p))$ pour tout $x_p \in X$. Si l'on suppose que $x_p \in X_k$, alors on obtient $x_p \in \bigcup_{i=1}^{\min(k,q')} X_i'$. Mais $x_p \in \bigcup_{i=1}^{\varphi(p)} X_i'$ implique $x_p \in \bigcup_{i=1}^{\psi(k)} X_i'$ avec $\psi(k) = \min(q', \max_{x_p \in X_k} (\varphi(p))) = \min(q', \max_{x_p \in X_k} (1 + d(x_p))) = \min(q', 1 + \max_{x_p \in X_k} (d(x_p)))$.

Il résulte que $x_p \in \bigcup_{i=1}^{\eta(k)} X_i'$ pour tout $x_p \in X_k$, où $\eta(k) = \min(\min(k, q'), \psi(k)) = \min(q', \min(k, 1 + \max_{x_p \in X_k} (d(x_p))))$. Nous avons obtenu que $x_p \in \bigcup_{i=1}^{\min(q', \beta_{\Delta}(G))} X_i'$ pour tout $x_p \in X$ si l'on note $\beta_{\Delta}(G) = \max_{k=1,\ldots,q} (\min(k, 1 + \max_{x_p \in X_k} (d(x_p))))$, parce que $\min(q', \max_{k=1,\ldots,q} (\min(q', \min(k, 1 + \max_{x_p \in X_k} (d(x_p)))))) = \min(q', \beta_{\Delta}(G))$.

En conclusion $X = \bigcup_{i=1}^{\min(q', \beta_{\Delta}(G))} X_i' = \bigcup_{i=1}^{q'} X_i'$, donc $\min(q', \beta_{\Delta}(G)) = q'$, c'est-à-dire $\beta_{\Delta}(G) \geqq q' \geqq K(G)$.

Pour démontrer la deuxième partie du théorème nous considérons la relation $d(x_1) \geqq d(x_2) \geqq \ldots \geqq d(x_n)$ et du fait que $\max_{x_p \in X_1} (d(x_p)) \geqq \max_{x_p \in X_2} (d(x_p)) \geqq \ldots \geqq \max_{x_p \in X_q}$

$(d(x_p))$ on déduit: $d(x_1) = \max_{x_p \in X_1} (d(x_p)); d(x_2) \geqq \max_{x_p \in X_2} (d(x_p)); \ldots; d(x_q) \geqq \max_{x_p \in X_q} (d(x_p))$. Dans ces conditions $\min (k, 1 + \max_{x_p \in X_k} (d(x_p))) \leqq \min (k, 1 + d(x_k))$ pour tout $k = 1, 2, \ldots, q$, donc $\beta_A(G) \leqq \max_{k=1,\ldots,q} (\min (k, 1 + d(x_k))) \leqq \alpha(G)$ C. Q. F. D.

Il y a des cas où $\beta_A(G) = K(G)$, même si A n'a pas un nombre minimum de classes par rapport à R_G.

Pour la détermination exacte du nombre chromatique $K(G)$ on peut construire une arborescence $H(\tilde{X}, A)$ qui nous permettra de trouver au moins une partition minimale (avec un nombre minimum de classes) de X, partition qui est associée à une relation d'équivalence $R_G^* \subset R_G$.

Soit l'élément $x_i \in X$ dont les classes maximales par rapport à R_G sont $S_{x_i}^{(1)}$, $S_{x_i}^{(2)}, \ldots, S_{x_i}^{(r(i))}$.

On choisit un certain élément $y_{i,1} \in X \setminus Sx_i^{(1)}$ dont les classes maximales par rapport à la relation $R_G \cap (X \setminus Sx_i^{(1)})^2$ sont $Sy_{i,1}^{(1)}, Sy_{i,1}^{(2)}, \ldots, Sy_{i,1}^{(r_{i,1}(y_{i,1}))}$.

L'arborescence H a la racine x_i et l'application A est définie par récurrence comme suit: $A(x_i) = \{x_{i,k} | k = 1, 2, \ldots, r(i)\}$; $A(x_{i,1}) = \{x_{i,1,k} | k = 1, 2. \ldots, r_{i,1}(y_{i,1})\}$ etc.

En considérant le chemin $(x_i, x_{i,n_1}, x_{i,n_1,n_2}, \ldots, x_{i,n_1}, \ldots, n_s)$ de x_i à $x_{i,n_1}, \ldots, n_s$, si $X = S_{i,n_1,\ldots,n_s}$ où $S_{i,n_1,\ldots,n_s} = S_{x_i}^{(n_1)} \cup Sy_{i,n_1}^{(n_2)} \cup \ldots \cup Sy_{i,n_1,\ldots,n_{s-1}}^{(n_s)}$, alors on définit $A(x_{i,n_1,\ldots,n_s}) = \emptyset$. En ce cas, évidemment $n_s = 1$.

En cas contraire on considère un élément quelconque $y_{i,n_1,\ldots,n_s} \in X \setminus S_{i,n_1,\ldots,n_s}$ dont les classes maximales par rapport à la relation $R_G \cap (X \setminus S_{i,n_1,\ldots,n_s})^2$ sont $Sy_{i,n_1,\ldots,n_s}^{(1)}, \ldots, Sy_{i,n_1,\ldots,n_s}^{(r_{i,n_1,\ldots,n_s}(y_{i,n_1,\ldots,n_s}))}$ et on définit $A(x_{i,n_1,\ldots,n_s}) =$

$$= \{x_{i,n_1 \ldots,n_s,k} | k = 1, 2, \ldots, r_{i,n_1,\ldots,n_s}(y_{i,n_1,\ldots,n_s})\}.$$

La construction de l'arborescence H finit lorsqu'on obtient tous les sommets finaux.

THÉORÈME 2. *Le nombre chromatique $K(G)$ du graphe G est égal au minimum de la longueur des chemins qui réunissent la racine x_i avec les sommets finaux de l'arborescence H.*

DÉMONSTRATION. A chaque chemin $(x_i, x_{i,n_1}, \ldots, x_{i,n_1,n_2}, \ldots, n_s)$ de longueur s qui va de x_i à un sommet final $x_{i,n_1,\ldots,n_s}$ (avec $n_s = 1$) on fait correspondre une partition de X composée de s ensembles $Sx_i^{(n_1)}, Sy_{i,n_1}^{(n_2)}, \ldots, Sy_{i,n_1,\ldots,n_{s-1}}^{(n_s)}$, qui réalise une relation d'équivalence incluse dans R_G.

Il faut démontrer que l'ensemble des partitions de X associées à tous les chemins qui réunissent la racine x_i avec les sommets finaux de l'arborescence H contient au moins une partition minimale, induite dans l'ensemble X d'une relation d'équivalence $R_G^* \subset R_G$.

Soit une partition $\{X_j\}_{1 \leqq j \leqq r}$ de X avec un nombre minimum de classes telle que $X_j \times X_j \subset R_G$ pour tout $j = 1, 2, \ldots, r$.

L'élément x_i se trouve dans une classe X_q, mais on peut supposer que $x_i \in X_1$ et donc il y a un indice $1 \leqq m_1 \leqq r(i)$ pour lequel $X_1 \subset Sx_i^{(m_1)}$.

Si $X_1 \neq Sx_i^{(m_1)}$, alors on complète la classe X_1 jusqu'à $Sx_i^{(m_1)}$ par l'extraction des éléments de $Sx_i^{(m_1)}$ qui n'appartiennent pas à X_1 des classes restées.

On obtient une nouvelle partition $Sx_i^{(m_1)}, X_2', \ldots, X_r'$ de X ainsi que $X_j' \times X_j' \subset R_G$, $X_j' \subset X_j$ et $X_j' \neq \emptyset$ pour tout $j = 2, \ldots, r$.

On peut supposer que $y_{i,m_1} \in X_2'$, donc il y a un indice $1 \leqq m_2 \leqq r_{i,m_1}(y_{i,m_1})$ pour lequel $X_2' \subset Sy_{i,m_1}^{(m_2)}$ etc. On obtient une partition $Sx_i^{(m_1)}, Sy_{i,m_1}^{(m_2)}, \ldots, Sy_{i,m_1,\ldots,m_{r-1}}^{(m_r)}$ avec $m_r = 1$ de X, qui est une partition avec un nombre minimum de classes, associée au chemin $(x_i, x_{i,m_1}, \ldots, x_{i,m_1,m_2}, \ldots, m_r)$ de l'arborescence H. C. Q. F. D.

Si $Sx_q^{(1)}, Sx_q^{(2)}, \ldots, Sx_q^{(r(q))}$ sont toutes les classes maximales de $x_q \in X$ par rapport à la relation symétrique et réflexive $R_G \subset X \times X$ et si $X_1 \subset X$, alors les classes maximales de x_q par rapport à la relation $R_1 = R_G \cap X_1^2$ sont les ensembles de la suite $X_1 \cap Sx_q^{(1)}, X_1 \cap Sx_q^{(2)}, \ldots, X_1 \cap Sx_q^{(r(q))}$, maximaux par rapport à l'inclusion.

Ce fait permet de construire l'arborescence H en partant seulement des classes maximales des éléments de X par rapport à la relation R_G.

Pour la détermination de toutes les classes maximales des éléments de X par rapport à R_G on peut employer la procédure de calcul booléen pour la détermination des ensembles intérieurement stables maximaux d'un graphe fini, procédure proposée par MAGHOUT [5] et WEISSMAN [6], les méthodes combinatoires de PAULL et UNGER et de IOANIN, présentées dans [4] pour la détermination des compatibles maximaux qui interviennent dans le problème de la minimisation des automates séquentiels incomplètement spécifiés, ou l'algorithme de Bednarek et Taulbee [7].

BIBLIOGRAPHIE

[1] BROOKS, R. L.: On Colouring the Nodes of a Network, *Proc. Cambridge Philos. Soc.* **37** (1941), 194-197.
[2] ORE, O.: *Theory of Graphs*, Providence, 1962.
[3] WELSH, D. et POWELL, M.: An upper bound for the chromatic number of a graph and its application to timetabling problems, *Comput. J.* 3,1, May 1967, 85—86.
[4] CHINAL, J.: *Techniques booléennes et calculateurs arithmétiques*, Dunod, Paris, 1967.
[5] MAGHOUT, K.: Applications de l'algèbre de Boole à la théorie des graphes et aux programmes linéaires et quadratiques, *Cahiers du Centre d'études de Rech. Opérationnelle*, Bruxelles, **5**, 1—2 (1963).
[6] WEISSMAN, J.: Boolean algebra, map coloring and interconnections, *Amer. Math. Monthly*, **69** (1962) 608—613.
[7] BEDNAREK, A. et TAULBEE, O.: On maximal chains, *Revue Roum. de math. pures et appl.* **11** (1966), 1, 23-25.

Faculté de Mathématique et Mécanique, Bucarest

(Reçu le 25 juillet 1967.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 59—67.

SUR CERTAINS THÉORÈMES DE CARLEMAN ET DE HARDY

par

L. ALPÁR

§ 1. Introduction

1. 1. — Soient ζ $(0<|\zeta|<1)$ un nombre complexe donné, $h(z)=(z-\zeta)/(1-\bar{\zeta}z)$, et

$$f_1(z)=\sum_{\nu=0}^{\infty} a_\nu z^\nu$$

une fonction analytique dans le cercle $|z|<1$. La fonction

$$f_1[h(z)]=f_2(z)=\sum_{n=0}^{\infty} b_n z^n \qquad (b_n=b_n(\zeta)) \tag{1.1}$$

est également analytique pour $|z|<1$.

Nous avons obtenu plusieurs résultats concernant les transformations de genre (1.1). Nous en rappelons ceux dont nous aurons besoin par la suite.

THÉORÈME A. — *Il existe des fonctions $f_1(z)$ telles que*

$$\sum_{\nu=0}^{\infty} |a_\nu| < \infty \tag{1.2}$$

et, malgré cela,

$$\sum_{n=0}^{\infty} |b_n| = \infty \tag{1.3}$$

([1] et [2], Théorème 1, p. 287).

THÉORÈME B. — *Si la condition* (1.2) *est remplie, la série de puissances* (1.1) *est uniformément convergente sur la circonférence $|z|=1$ quel que soit ζ* ([2], Théorème 2, p. 288).

Les théorèmes A et B constituent une démonstration de la proposition suivante de G. H. HARDY [10]:

THÉORÈME C. — *Il existe des fonctions analytiques pour $|z|<1$, dont les séries de puissances développées autour de l'origine sont uniformément mais non absolument convergentes sur la circonférence $|z|=1$.*

Quand (1.2) est réalisé $f_1(z)$ et, par suite, $f_2(z)$ sont continues pour $|z|=1$, on a donc

$$\sum_{n=0}^{\infty} |b_n|^2 < \infty. \tag{1.4}$$

En comparant (1. 3) et (1. 4) nous avons soulevé, dans la note [1], la question suivante: Peut-on trouver un nombre ε $(0<\varepsilon<1)$ tel que l'on ait

$$\sum_{n=0}^{\infty} |b_n|^{2-\varepsilon} < \infty \tag{1. 5}$$

chaque fois que (1. 2) a lieu? Ce problème a été résolu par G. HALÁSZ [9] qui a démontré non seulement la non-existence d'un ε vérifiant (1. 5) pour toute série satisfaisant à (1. 2), mais il a obtenu le résultat ci-après plus fin.

THÉORÈME D. — *Étant donnés le nombre* ζ *et la suite* $\{\alpha_n\}_0^\infty$, *avec* $0<\alpha_n<2$, $\alpha_n \to 0$, $\alpha_n \log n \to \infty$, $n \to \infty$, *il existe des fonctions* $f_1(z)$ *telles que la condition* (1. 2) *soit remplie et, malgré cela,*

$$\sum_{n=0}^{\infty} |b_n|^{2-\alpha_n} = \infty.$$

1. 2. — Récemment nous avons étendu ces recherches aux séries de Fourier. Introduisons quelques notations pour pouvoir formuler d'une manière plus simple les résultats obtenus.

(a) t désigne une variable réelle, et soit $\mathscr{A}^\beta$ $(0<\beta)$ l'ensemble des fonctions

$$H(t) = \sum_{n=-\infty}^{\infty} C_n e^{int}, \quad \text{avec} \quad \sum_{n=-\infty}^{\infty} |C_n|^\beta < \infty.$$

$\mathscr{A}^1 = \mathscr{A}$ désigne alors l'ensemble des fonctions qui se représentent par des séries de Fourier absolument convergentes.

(b) Notons par $\mathscr{C}$ l'ensemble des fonctions continues,

(c) et par $\mathscr{U}$ l'ensemble des fonctions qui se développent en séries de Fourier uniformément convergentes.

(d) Soit enfin $\mathscr{I}$ l'ensemble des fonctions $f(t)$ à valeurs réelles qui, sur un segment $I=I(f)$ arbitrairement petit, sont deux fois dérivables avec $f''(t)>\varkappa>0$ (ou $f''(t)<-\varkappa<0$). L'ensemble des fonctions non linéaires et analytiques sur l'axe des t est un sous-ensemble de $\mathscr{I}$.

1. 3. — Soit

$$F(t) = \sum_{\nu=-\infty}^{\infty} A_\nu e^{i\nu t}, \quad \text{avec} \quad \sum_{\nu=-\infty}^{\infty} |A_\nu| < \infty, \tag{1. 6}$$

c'est que $F \in \mathscr{A}$. Appelons les transformées de F les fonctions $F(f(t))=G(t)$ périodiques de période 2π telles que l'on ait

$$F(f(t)) = \sum_{\nu=-\infty}^{\infty} A_\nu e^{i\nu t} = G(t) = \sum_{n=-\infty}^{\infty} B_n e^{int}. \tag{1. 7}$$

Z. L. LEIBENSON [12] et sous conditions plus générales A. BEURLING et H. HELSON [7] ont démontré la proposition suivante concernant les transformées du type (1. 7).

THÉORÈME E. — $F(f(t))=G(t)\in\mathscr{A}$ *pour tout* $F(t)\in\mathscr{A}$ *si, et seulement si* $f(t) = Nt+C$ *où* N $(-\infty<N<\infty)$ *est un entier et* C *est une constante réelle.*

Dans le même ordre d'idées nous avons trouvé les résultats suivants [3].

THÉORÈME F. — *Si* $f(t)$ *est analytique le long de l'axe des t, si* $f(t) \equiv f(t+2\pi) \pmod{2\pi}$, *et si* $F(t) \in \mathscr{A}$, *alors* $F(f(t)) = G(t) \in \mathscr{U}$.

Nous avons obtenu aussi de propositions de caractère négatif [4].

THÉORÈME G. — *Si* $f \in \mathscr{C} \cap \mathscr{I}$, *de plus si* $f(t) \equiv f(t+2\pi) \pmod{2\pi}$, *et* $\varepsilon > 0$ *est un nombre donné à l'avance aussi petit que l'on veut, il existe des fonctions* $F_\varepsilon(t) \in \mathscr{A}$ *telles que* $F_\varepsilon(f(t)) = G_\varepsilon(t) \in \mathscr{C}$, *mais* $G_\varepsilon(t) \notin \mathscr{A}^{2-\varepsilon}$.

THÉORÈME H. — *Si* $f(t)$ *est non linéaire, analytique sur l'axe des t, de plus si* $f(t) \equiv f(t+2\pi) \pmod{2\pi}$, *et* $\varepsilon > 0$ *est une nombre donné à l'avance aussi petit que l'on veut, il existe des fonctions* $F_\varepsilon(t) \in \mathscr{A}$ *telles que* $F_\varepsilon(f(t)) = G_\varepsilon(t) \in \mathscr{U}$, *mais* $G_\varepsilon(t) \notin \mathscr{A}^{2-\varepsilon}$.

Ces résultats évoquent la proposition suivante de T. CARLEMAN [8].

THÉORÈME I. — *Il existe une fonction* $g(t) \in \mathscr{C}$, *avec*

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

telle que

$$\sum_{n=-\infty}^{\infty} |c_n|^{2-\varepsilon} = \infty, \tag{1.8}$$

si petit que soit le nombre $\varepsilon > 0$.

Plus tard on a trouvé d'autres fonctions de cette nature, et l'on a construit aussi de fonctions analytiques pour $|z| < 1$, continues pour $|z| \leqq 1$ et dont les coefficients de Taylor possèdent la propriété (1. 8) (cf. p.e. P. TURÁN [13], cette note donne aussi une bibliographie détaillée de la question).

En comparant le théorème I aux précédents deux questions se posent. 1° Est-il possible d'établir le théorème de CARLEMAN ainsi que sa variante pour les séries de puissances à l'aide des propositions invoquées? 2° Peut-on étendre ces théorèmes, valables pour certaines fonctions appartenant à $\mathscr{C}$, à quelques fonctions éléments de $\mathscr{U}$? La réponse, comme nous allons voir, est affirmative à toutes les deux questions.

Dans le § 2 on va examiner le cas, à présent très simple, des séries de puissances. Dans le § 3 on fera connaître un lemme et l'on démontrera un autre. Le théorème de CARLEMAN sera établi et généralisé dans le § 4. Finalement dans le § 5 on appliquera le résultat obtenu sur la transformation conforme des séries de Faber.

§ 2. Cas des séries de puissances

THÉORÈME 1. — *Étant donnée une suite* $\{\alpha_n\}_0^\infty$, *avec* $0 < \alpha_n < 2$, $\alpha_n \to 0$, $\alpha_n \log n \to \infty$, $n \to \infty$, *il existe une fonction analytique pour* $|z| < 1$, *soit*

$$f_2(z) = \sum_{n=0}^{\infty} b_n z^n, \tag{2.1}$$

dont la série de puissances converge uniformément pour $|z| = 1$ *et, malgré cela,*

$$\sum_{n=0}^{\infty} |b_n|^{2-\alpha_n} = \infty. \tag{2.2}$$

COROLLAIRE 1. — *Si petit que soit le nombre* $\varepsilon>0$, *on a*

$$\sum_{n=0}^{\infty} |b_n|^{2-\varepsilon} = \infty. \tag{2.3}$$

DÉMONSTRATION. — Le théorème 1 est une conséquence directe des théorèmes B et D. D'après le théorème D, étant donnés ζ et la suite $\{\alpha_n\}_0^\infty$, il existe une fonction $f_1(z)$ telle que (1. 2) et, pour $f_1[h(z)]=f_2(z)$, (2. 2) soient réalisés. Selon le théorème B, la série de puissances (2.1) est alors toujours uniformément convergente sur la circonférence $|z|=1$.

(2. 3) est une conséquence immédiate de (2. 2).

Le théorème 1 et le corollaire 1 fournissent symultanément une généralisation directe du théorème C de HARDY et du théorème I de CARLEMAN.

§ 3. Deux lemmes

3. 1. — La démonstration des théorèmes principaux, que nous allons achever dans le § 4, se fonde sur deux lemmes. On doit à HALÁSZ [9] le premier et nous l'admettons sans preuve, tandis que nous allons démontrer le second.

Soit l l'ensemble des suites $x=\{x_\nu\}_{-\infty}^{\infty}$ telles que

$$\sum_{\nu=-\infty}^{\infty} |x_\nu| < \infty. \tag{3.1}$$

Notons par $\{\lambda_n\}_{-\infty}^{\infty}$ une suite avec $0<\lambda_n<\lambda$ où λ ne dépend pas de n, et par $l^{\{\lambda_n\}}$ l'ensemble des suites $y=\{y_n\}_{-\infty}^{\infty}$ telles que

$$\sum_{n=-\infty}^{\infty} |y_n|^{\lambda_n} < \infty. \tag{3.2}$$

Nous dirons qu'une fonction appartient à l'ensemble $\mathscr{A}^{\{\lambda_n\}}$ si ses coefficients de Fourier forment une suite appartenant à $l^{\{\lambda_n\}}$.

Désignons enfin par $[\gamma_{n\nu}]$ ($n=0, \pm1, \pm2, \ldots$; $\nu=0, \pm1, \pm2, \ldots$) une matrice infinie avec $|\gamma_{n\nu}|<M$ où M est une constante indépendant de n et de ν. Considérons les transformations

$$y_n = \sum_{n=-\infty}^{\infty} \gamma_{n\nu} x_\nu \qquad (n = 0, \pm1, \pm2, \ldots). \tag{3.3}$$

Le lemme 1 donne une condition *nécessaire* pour que (3.1) et (3. 3) entraîne (3. 2). (Les conditions *suffisantes* ne sont pas encore connues.)

LEMME 1. — *Pour que la matrice* $[\gamma_{n\nu}]$ ($n=0, \pm1, \pm2, \ldots$; $\nu=0, \pm1, \pm2, \ldots$) *transforme tout* $x\in l$ *en un* $y\in l^{\{\lambda_n\}}$ *il faut qu'il existe une constante* $K>0$ *telle que*

$$\sum_{n=-\infty}^{\infty} |\gamma_{n\nu}|^{\lambda_n} < K \qquad (\nu = 0, \pm1, \pm2, \ldots). \tag{3.4}$$

En d'autres termes, si (3. 4) n'a pas lieu, il existe une suite $x\in l$ pour laquelle $y\notin l^{\{\lambda_n\}}$.

3. 2. — Le lemme 2 est la généralisation d'un de notre résultat antérieur ([5], Théorème 1, p. 280) lequel, à son tour, généralise une proposition de J. P. KAHANE ([11], Théorème IV, p. 253).

LEMME 2. — *Soit* $f\in\mathscr{I}$ *et*

$$e^{ivf(t)}=\sum_{n=-\infty}^{\infty}a_{nv}e^{int}\qquad(v=0,\pm1,\pm2,\ldots). \tag{3.5}$$

$\{\alpha_n\}_{-\infty}^{\infty}$ *dénote une suite avec* $0<\alpha_n<2$, *et* $\alpha_n\to0$ *pour* $|n|\to\infty$.

(I) *Si* $\alpha_n\log|n|\to\infty$ $(|n|\to\infty)$, *on a*

$$\lim_{|v|\to\infty}\sum_{n=-\infty}^{\infty}|a_{vn}|^{2-\alpha_n}=\infty. \tag{3.6}$$

(II) *Si* $\alpha_n\log|n|=O(1)$ $(|n|\to\infty)$, *on a*

$$\sum_{n=-\infty}^{\infty}|a_{nv}|^{2-\alpha_n}=O(1)\qquad(|v|\to\infty). \tag{3.7}$$

Remarque. — Dans le cas particulier où

$$f(t)=-i\log\frac{e^{it}-\zeta}{1-\bar{\zeta}e^{it}}$$

(3. 6) a été déjà démontré par HALÁSZ [9].

Quant à (3. 7), $\sum_{n=-\infty}^{\infty}|u_n|^2=u<\infty$ entraîne $\sum_{n=-\infty}^{\infty}|u_n|^{2-\alpha_n}\leqq c_0u<\infty$, si la condition (II) a lieu (cf. [9] note 1). Les sommes représentées par la formule (3.7) sont donc uniformé ment bornées car, grâce à (3. 5), $\sum_{n=-\infty}^{\infty}|a_{nv}|^2=1$ pour tout v. Nous admettons donc (3. 7) sans démonstration. Nous en aurons besoin dans ce qui suit comme le complément de la relation (3. 6) faisant ressortir que les théorèmes que nous allons établir sont, dans certain sens, les meilleurs.

DÉMONSTRATION. — Nous allons faire usage d'une méthode due à LEIBENSON [12] (elle a été utilisée par KAHANE [11] et par nous même [5]). Cette fois-ci il faut modifier la méthode pour qu'elle soit applicable au problème envisagé. Comme $f\in\mathscr{I}$ (cf. 1. 2 (d)), on peut écrire

$$a_{nv}=b_{nv}+c_{nv},\quad b_{nv}=\frac{1}{2\pi}\int_I e^{ivf(t)-int}\,dt. \tag{3.8}$$

L'inégalité $|a_{nv}|>K|b_{nv}|$ $(K>1)$ entraîne $|a_{nv}|<\frac{K}{K-1}|c_{nv}|$ et, par suite,

$$\sum_{|a_{nv}|>K|b_{nv}|}|a_{nv}|^2\leqq\left(\frac{K}{K-1}\right)^2\sum_{n=-\infty}^{\infty}|c_{nv}|^2=\left(\frac{K}{K-1}\right)^2\left(1-\frac{|I|}{2\pi}\right)=1-\theta.$$

Pour un K convenablement choisi, on a $\theta > 0$. Il s'ensuit que

$$\sum_{|a_{nv}| \leqq K|b_{nv}|} |a_{nv}|^2 \geqq \theta. \tag{3.9}$$

D'autre part à chaque v donné on peut associer deux entiers n_0 et l_0, où $l_0 > 1$ est une constante et $n_0 = l_0|v|$, tels que l'on ait $|vn^{-1}f'(t) - 1| > c > 0$ pour $t \in I$ et $|n| \geqq n_0$. On a ainsi, d'après le premier lemme de VAN DER CORPUT ([14], I, p. 197), $|b_{nv}| \leqq c_1|n|^{-1}$ pour $|n| \geqq n_0$ et tout v. Il en découle

$$\sum_{|n| \geqq n_0} |b_{nv}|^2 = O(|v|^{-1}) \qquad (|v| \to \infty), \tag{3.10}$$

et l'on tire de (3.9) et (3.10)

$$\sum_{\substack{|a_{nv}| \leqq K|b_{nv}| \\ |n| < n_0}} |a_{nv}|^2 \geqq \theta(v) = \theta + o(1) \qquad (|v| \to \infty). \tag{3.11}$$

On déduit de plus à partir de (3.8), au moyen du second lemme de VAN DER CORPUT ([14], I, p. 197), que $|b_{nv}| < (2/\pi)(\varkappa|v|)^{-\frac{1}{2}} = \delta$ pour tout n (cf. 1.2 (d)). Posons ensuite $\alpha_{n(v)} = \min_{|n| < n_0} \alpha_n$ $(|n(v)| < n_0)$. Comme $\alpha_n \to 0$ $(|n| \to \infty)$ et $n_0 \to \infty$ $(|v| \to \infty)$, il vient $|n(v)| \to \infty$ et $\alpha_{n(v)} \log |n(v)| \to \infty$ $(|v| \to \infty)$. On en conclut, en vertu de (3.11),

$$\sum_{n=-\infty}^{\infty} |a_{nv}|^{2-\alpha_n} \geqq \sum_{\substack{|a_{nv}| \leqq K|b_{nv}| \\ |n| < n_0}} |a_{nv}|^{2-\alpha_n} \geqq \sum_{\substack{|a_{nv}| \leqq K|b_{nv}| \\ |n| < n_0}} |a_{nv}|^{2-\alpha_{n(v)}} \geqq \frac{\theta(v)}{(K\delta)^{\alpha_{n(v)}}} =$$

$$= c_2 \theta(v) e^{\frac{1}{2}\alpha_{n(v)} \log |v|}$$

Or, selon (3.11), $\theta(v) \to \theta$ $(|v| \to \infty)$, tandis que

$$\alpha_{n(v)} \log |v| > \frac{\log |v|}{\log n_0} \alpha_{n(v)} \log |n(v)| \to \infty \qquad (|v| \to \infty),$$

d'où le résultat.

§ 4. Théorème de CARLEMAN et ses généralisations

4.1. — Nous allons établir une proposition dont le théorème de CARLEMAN est un corollaire immédiat.

THÉORÈME 2. — *Étant données la fonction $f \in \mathscr{C} \cap \mathscr{I}$ et telle que $f(t) \equiv f(t + 2\pi)$ (mod 2π), ainsi que la suite $\{\alpha_n\}_{-\infty}^{\infty}$, avec $0 < \alpha_n < 2$, $\alpha_n \to 0$, $\alpha_n \log |n| \to \infty$, $|n| \to \infty$, il existe une fonction $F(t) \in \mathscr{A}$ telle que $F(f(t)) = G(t) \in \mathscr{C}$ et, malgré cela, $G(t) \notin \mathscr{A}^{\{2-\alpha_n\}}$, c'est que*

$$\sum_{n=-\infty}^{\infty} |B_n|^{2-\alpha_n} = \infty \tag{4.1}$$

où $\{B_n\}_{-\infty}^{\infty}$ est la suite des coefficients de Fourier de $G(t)$.

COROLLAIRE 2. — *Si petit que soit le nombre $\varepsilon > 0$, on a*

$$\sum_{n=-\infty}^{\infty} |B_n|^{2-\varepsilon} = \infty. \tag{4.2}$$

DÉMONSTRATION. — Employons les notations (1. 6) et (1. 7). Le fait que $f(t)\in\mathscr{C}$ et $F(t)\in\mathscr{A}$ assure que la série de fonctions

$$\sum_{\nu=-\infty}^{\infty} A_\nu e^{i\nu f(t)} = G(t)$$

soit uniformément convergente et, par conséquent, que $G(t)\in\mathscr{C}$.

D'autre part il est simple de voir que

$$B_n = \sum_{\nu=-\infty}^{\infty} a_{n\nu} A_\nu \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{4. 3}$$

On a, par hypothèse, $\{A_\nu\}_{-\infty}^{\infty} = A \in l$. Selon le lemme 1, la relation (4. 3) implique $\{B_n\}_{-\infty}^{\infty} = B \in l^{\{2-\alpha_n\}}$ pour tout $A\in l$, s'il existe une constante $K>0$ telle que

$$\sum_{n=-\infty}^{\infty} |a_{n\nu}|^{2-\alpha_n} < K \qquad (\nu = 0, \pm 1, \pm 2, \ldots).$$

Or, d'après le lemme 2, une telle K n'existe pas. Il y a donc de suites $A\in l$ qui ne se transforment pas par la matrice $[a_{n\nu}]$ en un élément de $l^{\{2-\alpha_n\}}$. Les termes d'une telle suite seront les coefficients de Fourier de $F(t)$ et $F(f(t)) = G(t)$ sera la fonction cherchée,

(4. 2) est la conséquence directe de (4.1) et exprime le théorème de CARLEMAN.

4. 2. — En imposant à $f(t)$ des conditions plus strictes, on peut établir (4.1) avec $G(t)\in\mathscr{U}$.

THÉORÈME 3. — *Étant données la fonction non linéaire $f(t)$ analytique sur l'axe des t, et telle que $f(t)\equiv f(t+2\pi)$ (mod 2π), ainsi que la suite $\{\alpha_n\}_{-\infty}^{\infty}$, avec $0<\alpha_n<2$, $\alpha_n\to 0$, $\alpha_n\log|n|\to\infty$, $|n|\to\infty$, il existe une fonction $F(t)\in\mathscr{A}$ telle que $F(f(t)) = G(t)\in\mathscr{U}$ et, malgré cela, $G(t)\notin\mathscr{A}^{\{2-\alpha_n\}}$, c'est que*

$$\sum_{n=-\infty}^{\infty} |B_n|^{2-\alpha_n} = \infty$$

où $\{B_n\}_{-\infty}^{\infty}$ est la suite des coefficients de Fourier de $G(t)$.

Un corollaire analogue au précédent a encore lieu.

DÉMONSTRATION. — Il résulte du théorème F que $G(t)\in\mathscr{U}$.

Si, tout en remplissant les autres conditions du théorème 3, $f(t)$ était linéaire, elle devrait avoir la forme $f(t) = Nt + C$ où N est un entier et C est une constante réelle, mais alors, en vertu du théorème E, on aurait $G(t)\in\mathscr{A}$.

Si $f(t)$ n'est pas linéaire, l'analyticité de $f(t)$ garantit que $f\in\mathscr{C}\cap\mathscr{S}$ et le résultat découle du théorème 2.

COROLLAIRE 3. — *Si la fonction f, remplissant les conditions du théorème* 2 *ou* 3, *est aussi monotone, et si $f(t+2\pi) = f(t)\pm 2\pi$, il existe de couples de fonctions $F(t)$ et $G(t) = F(f(t))$ qui, avec certain déplacement, admettent les mêmes valeurs dans la même succession et pourtant $F(t)\in\mathscr{A}$, tandis que $G(t)\notin\mathscr{A}^{\{2-\alpha_n\}}$, bien que $G(t)\in\mathscr{C}$ ou $G(t)\in\mathscr{U}$ selon que f est seulement continue ou analytique.*

Comme nous l'avons déjà fait observé, la formule (3. 7) met en évidence qu'on ne peut pas améliorer les résultats obtenus.

§ 5. Application à la transformation des séries de Faber

Soit C une courbe fermée simple dans le plan des z formée d'un seul arc analytique régulier, et dénote $\Phi_\nu(z)$ le ν-ième polynome de Faber associé à C. Toute fonction $F_1(z)$ holomorphe à l'intérieur de C s'y représente par sa série de Faber unique, soit

$$F_1(z) = \sum_{\nu=0}^{\infty} a_\nu^{(1)} \Phi_\nu(z). \tag{5.1}$$

C est la courbe de convergence de la série (5.1) si $F_1(z)$ a des singularités sur C. (En ce qui concerne la détermination des polynomes de Faber et des coefficients $a_\nu^{(1)}$ nous nous référons à notre travail [6] et à la bibliographie y indiquée.)

Il existe une fonction, et une seule $\varphi(z)$ qui, pour des $|z|$ assez élevés, s'écrit sous la forme

$$w = \varphi(z) = z + c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \dots \tag{5.2}$$

et applique d'une manière conforme et biunivoque l'extérieur de C sur celui de la circonférence K: $|w| = R$. Les conditions imposées à φ déterminent univoquement le rayon R et les coefficients c_n.

Soit $k(z)$ une transformation conforme biunivoque de l'intérieur de C sur lui-même. En raison de l'analyticité de C, $k(z)$ et $\varphi(z)$ sont définies même sur C. Soient $z_1 \in C$ et $z_2 \in C$ deux points homologues tels que $z_1 = k(z_2)$. Posons $w_1 = \varphi(z_1)$ et $w_2 = \varphi(z_2)$ où $w_1 \in K$, $w_2 \in K$ et $w_1/w_2 = e^{i\gamma}$. Si γ reste constante quand z_1 et z_2 parcourent C, nous disons que $k(z)$ est équivalente à une rotation. Lorsque γ varie avec z_1 et z_2, $k(z)$ sera dite non équivalente à une rotation.

La fonction $F_1[k(z)] = F_2(z)$ est aussi holomorphe à l'intérieur de C et peut être développée en série de Faber, soit

$$F_2(z) = \sum_{n=0}^{\infty} a_n^{(2)} \Phi_n(z). \tag{5.3}$$

THÉORÈME 4. — *Si $k(z)$ n'est pas équivalente à une rotation et $\{\alpha_n\}_0^\infty$ est une suite donnée, avec $0 < \alpha_n < 2$, $\alpha_n \to 0$, $\alpha_n \log n \to \infty$, $n \to \infty$, il existe des fonctions $F_1(z)$ holomorphes à l'intérieur de C, dont les séries de Faber sont absolument convergentes sur C, soit*

$$\sum_{\nu=0}^{\infty} |a_\nu^{(1)} \Phi_\nu(z)| < \infty, \qquad z \in C, \tag{5.4}$$

les séries de Faber (5.3) *de leurs transformées, $F_2(z)$ convergent uniformément sur C et, malgré cela,*

$$\sum_{n=0}^{\infty} |a_n^{(2)} \Phi_n(z)|^{2-\alpha_n} = \infty. \tag{5.5}$$

DÉMONSTRATION. — Nous avons prouvé dans la note [6] (§ 3, Théorème 2, p. 384) que (5.4) entraîne la convergence uniforme sur C de la série de Faber (5.3).

Nous avons démontré également dans la note [6] que l'étude de la série de Faber (5. 3) transformée de (5.1) se ramène à l'examen d'une certaine matrice $[\delta_{nv}]$. Plus exactement on peut montrer que si $[\delta_{nv}]$ ne remplit pas la condition (3. 4) (avec $\lambda_n = 2 - \alpha_n$), alors (5. 5) peut avoir lieu. Les éléments de la matrice $[\delta_{nv}]$ sont définis par la formule

$$\delta_{nv} = \frac{1}{2\pi} e^{i(nt_2 - vt_1)} \int_{-\pi}^{\pi} e^{iv\omega(t) - int}\, dt \qquad (n = 0, \pm 1, \pm 2, \ldots;\ v = 0, \pm 1, \pm 2, \ldots).$$

t_1 et t_2 sont des paramètres, $\omega(t) = \tau(t) + t$ où $\tau(t)$ est une fonction réelle, non constante, périodique de période 2π, et analytique sur l'axe des t. $\omega(t)$ satisfait donc aux conditions imposées à $f(t)$ dans le théorème 3. Il en résulte que $[\delta_{nv}]$ ne réalise pas la condition (3. 4), et le théorème est prouvé.

D'ailleurs $\omega(t)$ est défiinie de la façon suivante. Soit $z = \psi(w)$ la fonction inverse de $w = \varphi(z)$ (cf. (5. 2)) et soit $w = Re^{it}$, alors

$$\varphi(k[\psi(w)]) = Re^{i\omega(t)}, \quad \varphi(z_1) = Re^{it_1}, \quad \varphi(z_2) = Re^{it_2}.$$

BIBLIOGRAPHIE

[1] ALPÁR, L.: Egyes hatványsorok abszolút konvergenciája a konvergencia kör kerületén, *Mat. Lapok* **11** (1960) 312—322.

[2] ALPÁR, L.: Sur certaines transformées des séries de puissances absolument convergentes sur la frontière de leur cercle de convergence, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **7** (1962) 287—316.

[3] ALPÁR, L.: Abszolút konvergens Fourier-sorok transzformáltjairól, *Mat. Lapok* **18** (1967) 97—104.

[4] ALPÁR, L.: A független változó bizonyos transzformációinak hatása a Fourier-sorok abszolút konvergenciájára, *Mat. Lapok* **19** (1968) 49—54

[5] ALPÁR, L.: Sur une classe particulière de séries de Fourier à certaines puissances absolument convergentes. *Studia Sci. Math. Hungar.* **3** (1968) 279—286.

[6] ALPÁR, L.: Convergence absolue et représentation conforme, *Studia Sci. Math. Hungar.* **1** (1966) 379—388.

[7] BEURLING, A.—HELSON, H.: Fourier-Stieltjes transforms with bounded powers, *Math. Scand.* **1** (1953) 120—126.

[8] CARLEMAN, T.: Über die Fourierkoeffizienten einer stetigen Funktion, *Acta Math.* **41** (1918) 379—384.

[9] HALÁSZ, G.: On Taylor series absolutely convergent on the circumference of the circle of convergence I. *Publ. Math. Debrecen.* **14** (1967) 63—68.

[10] HARDY, G. H.: A theorem concerning Taylor's series, *The Quarterly Journal of Pure and Applied Mathematics* **44** (1913) 147—160.

[11] KAHANE, J. P.: Sur certaines classes de séries de Fourier absolument convergentes, *J. Math, Pures Appl.* **35** (1956) 249—259.

[12] Лейбенсон, З. Л.; О кольце функций с абсолутно сходящимися рядами фурье. *Успехи Мат. Наук* 9 (1954) 157—162.

[13] TURÁN, P.: On some examples in the theory of power series, *Bull. Amer. Math. Soc.* **54** (1948) 932—936.

[14] ZYGMUND, A.: *Trigonometric series, I—II.*, University Press, Cambridge, 1959.

Institut de Mathématique de l'Académie des Sciences de Hongrie, Budapest

(Reçu le 8 novembre 1967.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 69—91.

ON THE OPERATIONAL SOLUTION OF CERTAIN NON-LINEAR SINGULAR INTEGRAL EQUATIONS

by

T. FÉNYES

Introduction

In papers [1], [2] the integral equation

$$(1) \qquad (t+a)f(t)+\int_0^t f(\tau)g(t-\tau)\,d\tau = h(t)$$

has been solved by application of the operational calculus of MIKUSIŃSKI. The operational solutions of the algebraic differential equation

$$(2) \qquad Df-(a+\{g\})f = -\{h\}$$

in which D denotes the symbol of the so-called algebraic derivative were found. The existence problem of the locally integrable solutions of (2) was also considered in the homogeneous and inhomogeneous cases.

The present paper will deal with the integral equation

$$(3) \qquad (t-a)f(t)-\int_0^t f(\tau)g(t-\tau)\,d\tau = b\int_0^t f(\tau)f(t-\tau)\,d\tau. \qquad (0\leqq t<\infty)$$

Here $g(t)$ is given and defined on the interval $\langle 0, \infty)$ and locally integrable in the Lebesgue sense, $b\neq 0$ a real and a a non-negative constant. The function f is unknown.

Every equation of type (3) has the trivial locally integrable solution

$$f=0.$$

If an equation of type (3) has a locally integrable solution $f(t)$ which is different from zero on a set of positive measure, $f(t)$ is said to be a nontrivial locally integrable solution of (3).

Equation (3) is a nonlinear singular integral equation of convolution type. The operational calculus of MIKUSIŃSKI will be used in the investigation and practical solution of equation (3). Under some restriction to $g(t)$ we shall determine and give the so-called operational solutions and the necessary and sufficient conditions guaranteeing the existence of nontrivial locally integrable solutions. The operational notations and symbols of MIKUSIŃSKI [3] — the knowledge of which will be assumed — will be generally used.

If $a<0$ the equation is not singular. But in this case it is very easy to show that (3) cannot have a nontrivial locally integrable solution. This is the consequence

of the fact that every homogeneous integral equation of Volterra-type of the second kind

$$\varphi(t)+\int_0^t K(t,\tau)\varphi(\tau)\,d\tau = 0$$

has no nontrivial locally integrable solutions. So in the sequel it will be assumed that $a \not\equiv 0$. The problem is just then interesting, if (3) is singular.

Our investigation is joined with the concepts and properties of the so-called algebraic differentiation and integration of operators.

The next chapter summarizes the most important properties of the algebraic differentiation and integration of operators.

At the end of this paper some generalization of the theory will be treated which relates to the integral equations of the type

$$(t-a)f(t)-\int_0^t f(\tau)g(t-\tau)\,d\tau = b\int_0^t h(\tau)F(t-\tau)\,d\tau, \tag{4}$$

where $h(t)\neq 0$ is also a given locally integrable function and

$$F(t)=\int_0^t f(\tau)f(t-\tau)\,d\tau.$$

We also give some examples solving non-linear integral equations of type (3) and (4).

1. *The algebraic derivative and algebraic integral of Mikusiński operators*

Definition of the algebraic derivative (see [3]): If

$$\{a(t)\},\ \{b(t)\}\in C, \qquad \{b\}\neq 0,$$

then

$$D\{a(t)\} = \{-ta(t)\},$$

$$D\frac{\{a\}}{\{b\}} = \frac{\{b\}\{Da\}-\{a\}\{Db\}}{\{b\}^2}.$$

This definition involves that every operator is infinite times differentiable. An easy argument shows that the given definition can be extended to locally integrable functions $\{a\}$, $\{b\}$ the class of which let be denoted by the symbol L. The algebraic derivative has the following properties. (See [3]). For arbitrary operators p, q:

$$D(p+q) = Dp+Dq,$$

$$D(pq) = pDq+qDp,$$

$$D\left(\frac{p}{q}\right) = \frac{qDp-pDq}{q^2} \qquad (q\neq 0)$$

If α is a number, so

$$D\alpha = 0.$$

If

$$D\beta = 0,$$

then the operator β is an arbitrary complex number [4].

The algebraic differentiation of an arbitrary rational expression $R(s)$ of the operator s goes by the formal derivation rule according to s.

If the exponential function e^w exists then

$$De^w = e^w Dw.$$

If for a given operator w there exists an operator u that

$$Du = w \qquad \text{(see also later)}$$

and if u is a logarithm [4], then

$$x = Ce^u \qquad (C = \text{arbitrary number})$$

is the general solution of the so-called "algebraic" differential equation

$$Dx - wx = 0.$$

In the sequel a differential equation involving algebraic derivatives will be called an algebraic differential equation. In paper [8] there is a general operational theory of equations of this type called by the author differential equations "s".

The concept of the so-called algebraic integral introduced by E. GESZTELYI [5] is as follows:

If for an arbitrary operator p there exists an operator q that

$$Dq = p,$$

then q is called the algebraic integral of the operator p and denoted by

$$\int p\,ds = q.$$

The algebraic integral has the following properties:

Two algebraic integrals of an operator — if they exist — differ from each other in an arbitrary number.

Now we quote the most important theorem of the algebraic integrability of certain operators. This theorem was proved by E. GESZTELYI [5].

In [2] T. FÉNYES has used this theorem in another formulization. Our formulization here is quite equivalent to those, which have been communicated in [5] and [2], but it will be more convenient for our purposes.

The operator

$$x = s^k e^{-as}\{f\}$$

is algebraically integrable, where $a \geqq 0$, $-\infty < k < \infty$, $\{f\}$ is locally integrable on $\langle 0, \infty)$. The algebraic integral can be written as follows:

Let $a > 0$. Then

$$\int x\,ds = s^k e^{-as}\{u(t)\},$$

where the function

$$\{u(t)\} = \left\{-\frac{f(t)}{t+a} - k(t+a)^{k-1}\int_0^t \frac{f(\tau)\,d\tau}{(\tau+a)^{k+1}}\right\}$$

is locally integrable on $<0, \infty)$.

Now let $a=0$. Then

$$\int x\,ds = s^{k+1}\{u(t)\},$$

where

$$\{u(t)\} = \begin{cases} \left\{-t^k\int_0^t \frac{f(\tau)\,d\tau}{\tau^{k+1}}\right\} & \text{if} \quad k < -1 \\ \left\{-t^k\int_\varepsilon^t \frac{f(\tau)}{\tau^{k+1}}\right\} & \text{if} \quad k > -1 \end{cases}$$

$$(\varepsilon > 0 \text{ arbitrary})$$

and finally

$$\int \frac{1}{s}\{f\}\,ds = s\{u(t)\},$$

where

$$\{u(t)\} = \left\{-\int_\varepsilon^t \frac{F(\tau)\,d\tau}{\tau}\right\} \qquad (\varepsilon > 0 \text{ arbitrary}),$$

where $F(t) = \int_0^t f(\tau)\,d\tau$. (This is the case $k = -1$). It can be easily shown that the functions $\{u(t)\}$ occurring here are also locally integrable on $\langle 0, \infty)$. If for any fixed value $k \geqq -1$ the above integral remains convergent for $\varepsilon = 0$, we can take $\varepsilon = 0$. The operator x is also algebraic integrable in the case $a<0$ but the rule of algebraic integration of x when $a<0$ will not be needed. Thus it can be omitted.

Special cases:

1. If $\left\{\frac{f(t)}{t}\right\}$ is locally integrable on $\langle 0, \infty)$ then

$$\int \{f(t)\}\,ds = \left\{-\frac{f(t)}{t}\right\}.$$

2. $\int \{e^{at}\}\,ds = (\alpha - s)\{e^{at}\log t\}$ for every complex α, particularly for $\alpha = 0$ we get

$$\int \{1\}\,ds = -s\{\log t\}.$$

3. If $\lim_{t\to 0} f(t) = f(0)$ exists and the functions

$$\{f(t)\}, \quad \left\{\frac{f(t)-f(0)}{t}\right\}$$

are locally integrable then

$$\int \{f\}\,ds = \left\{\frac{f(t)-f(0)}{-t}\right\} - f(0)s\{\log t\}$$

which reduces to 1. when $f(0)=0$.

2. *The operational solutions of the algebraic differential equation of Bernoulli-type*

Let us consider the integral equation

$$(t-a)f(t) - \int_0^t f(\tau)g(t-\tau)\,d\tau = b\int_0^t f(\tau)f(t-\tau)\,d\tau, \tag{2. 1}$$

the operational form of which is

$$D\{f\} + (a+\{g\})f = -b\{f\}^2. \tag{2. 2}$$

At first we want to solve (2. 2) in the operator field, and do not take care of the fact whether the found solutions of (2. 2) are locally integrable functions or operators only. The curly brackets in (2. 2) will be omitted and so

$$Df + (a+g)f = -bf^2. \tag{2. 3}$$

Thus a *non-linear* algebraic differential equation will be obtained. Every locally integrable solution of (2. 1) is a solution of (2. 3), and conversely, but (2. 3) is more general than (2. 1). It can have generalized functions i.e. operators as solutions. If the solutions are restricted to locally integrable functions, then (2. 1) and (2. 3) are equivalent to each other.

The solution process of (2. 3) is very simple. Like the "classical" Bernoulli equation

$$\frac{dy(x)}{dx} + a(x)y(x) = y^2(x) \tag{2. 4}$$

(2. 3) can be regarded as an *algebraic differential equation of "Bernoulli type"*. But in the „classical" case the equation (2. 4) can always be reduced to a linear one by the substitution $u(x)=\dfrac{1}{y(x)}$. This gives the idea how to solve (2. 3).

Of course (2. 3) has the trivial solution $f=0$. Excluding it we obtain by substitution

$$u = \frac{1}{f}, \qquad Df = -\frac{Du}{u^2},$$

$$\frac{Du}{u^2} - (a+g)\frac{1}{u} = \frac{b}{u^2}$$

and finally

$$Du - (a+g)u = b \tag{2. 5}$$

which is a linear equation.

The general solution of the corresponding homogeneous equation

$$Du-(a+g)u = 0$$

is

$$u = Ce^{as}\exp\left[\int g\,ds\right] \qquad (C \text{ arbitrary constant}),$$

provided that the operator

$$\int g\,ds$$

is a logarithm in the sense of MIKUSIŃSKI [3]. We shall take the following restriction on the function $g(t)$, which guarantees the existence of the exponential function

$$e^{\int g\,ds}.$$

Let the value $\lim_{t\to 0} g(t)=g(0)$ *exist and let the function* $\dfrac{g(t)-g(0)}{t}$ *be locally integrable on* $\langle 0, \infty)$.

Then

$$\int \{g\}\,ds = \left\{\frac{g(t)-g(0)}{-t}\right\} - g(0)s\{\log t\}$$

and so

$$u = Ce^{as}\exp\left\{\frac{g(t)-g(0)}{-t}\right\}\exp\left[-g(0)s\{\log t\}\right]. \tag{2.6}$$

The operator (2. 6) certainly exists. Namely according to a rule of the operational calculus [3] the expansion

$$\exp\left\{\frac{g(t)-g(0)}{-t}\right\} = 1+\left\{\frac{g(t)-g(0)}{-t}\right\}+\frac{1}{2!}\left\{\frac{g(t)-g(0)}{-t}\right\}^2+\cdots$$

holds. The series converges in the operational sense and apart from its first term it represents a locally integrable function.

On th other hand Mikusiński has proved [7] that

$$\exp\left[\lambda(s\{\log t\}+\gamma)\right] = \frac{1}{s^{\lambda}} \qquad (\lambda \text{ real}, \gamma = \text{Euler constant}).$$

So we obtain

$$u = Ce^{as}\exp\left\{\frac{g(t)-g(0)}{-t}\right\}s^{g(0)} \qquad (C \text{ arbitrary constant}). \tag{2.7}$$

It will be determined by the method of variation of parameters a particular solution of the inhomogeneous differential equation (2. 5).

Let

$$u_{\text{part}} = C(s)\exp\left\{\frac{g(t)-g(0)}{-t}\right\}s^{g(0)};$$

substituting it in (2. 5) we get

$$C(s) = b\int e^{-as}s^{-g(0)}\exp\left\{\frac{g(t)-g(0)}{t}\right\}ds,$$

where

$$\exp\left\{\frac{g(t)-g(0)}{t}\right\} = 1+\sum_{k=1}^{\infty}\frac{1}{k!}\left\{\frac{g(t)-g(0)}{t}\right\}^k.$$

Here the power is understood in the operational sense.

Introducing the notation

$$\{G(t)\} = \sum_{k=1}^{\infty}\frac{1}{k!}\left\{\frac{g(t)-g(0)}{t}\right\}^k \tag{2.8}$$

by Gesztelyi's theorem we see that this algebraic integral exists and the general solution of (2. 5) is

$$u = \left[C+b\int e^{-as}s^{-g(0)}(1+\{G\})\,ds\right]e^{as}s^{g(0)}\exp\left\{\frac{g(t)-g(0)}{-t}\right\}. \tag{2.9}$$

The Bernoulli equation (2. 3) has the general operational solution

$$f=0$$

and

$$f=\frac{1}{u}=\frac{e^{-as}s^{-g(0)}(1+\{G\})}{C+b\int e^{-as}s^{-g(0)}(1+\{G\})\,ds} \qquad (C \text{ arbitrary}). \tag{2.10}$$

So we have proved the following

THEOREM 1. *Let* $g(0)$ *exist and* $\dfrac{g(t)-g(0)}{t}$ *be locally integrable* $a\geqq 0$. *The non-linear integral equation* (2. 1) *has the operational form* (2. 3) *which is an algebraic differential equation of Bernoulli type, having a family of solutions depending on an arbitrary constant.*

Now the problem arises to determine the so-called locally integrable solutions of the integral equation (3), namely to investigate the operational structure of (2. 10) and to obtain the conditions guaranteeing the existence of the locally integrable solu ions.

3. *The locally integrable solutions of the integral equation*

First let $a>0$. We can write the algebraic integral occurring in (2. 10) as

$$\int e^{-as}s^{-g(0)}(1+\{G\})\,ds = \int e^{-as}s^{-g(0)+1}\{1\}\,ds+\int e^{-as}s^{-g(0)}\{G\}\,ds. \tag{3.1}$$

On the one hand by GESZTELYI's theorem of algebraic integration we get

$$\int e^{-as}s^{-g(0)+1}\{1\}\,ds = s^{-g(0)+1}e^{-as}\left\{-\frac{1}{t+a}+(g(0)-1)(t+a)^{-g(0)}\int_0^t\frac{d\tau}{(\tau+a)^{2-g(0)}}\right\} =$$

$$= -s^{-g(0)+1}e^{-as}\{a^{g(0)-1}(t+a)^{-g(0)}\} =$$

$$= e^{-as}s^{-g(0)}\{g(0)a^{g(0)-1}(t+a)^{-g(0)-1}\}-\frac{e^{-as}s^{-g(0)}}{a}.$$

Here the well-known operational formula

$$s\{u\} = \{u'\} + u(0)$$

has been used.

On the other hand

$$\int e^{-as} s^{-g(0)} \{G\} ds = e^{-as} s^{-g(0)} \left\{ -\frac{G(t)}{t+a} + g(0)(t+a)^{-g(0)-1} \int_0^t \frac{G(\tau)\, d\tau}{(\tau+a)^{1-g(0)}} \right\}.$$

So we obtain the following expression for the algebraic integral (3. 1)

(3. 2)
$$e^{-as} s^{-g(0)} \cdot$$
$$\cdot \left[\left\{ g(0) a^{g(0)-1} (t+a)^{-g(0)-1} - \frac{G(t)}{t+a} + g(0)(t+a)^{-g(0)-1} \int_0^t \frac{G(\tau)\, d\tau}{(\tau+a)^{1-g(0)}} \right\} - \frac{1}{a} \right].$$

By substituting this in (2. 10) we get

(3. 3)
$$f = \frac{e^{-as} s^{-g(0)} (1+\{G\})}{C + be^{-as} s^{-g(0)} \left[\left\{ g(0) a^{g(0)-1} (t+a)^{-g(0)-1} - \frac{G(t)}{t+a} + g(0)(t+a)^{-g(0)-1} \int_0^t \frac{G(\tau)\, d\tau}{(\tau+a)^{1-g(0)}} \right\} - \frac{1}{a} \right]}.$$

If $C=0$, (3. 3) reduces to the form

(3. 4)
$$f = \frac{1+\{G\}}{b\left[\{g(0)\ldots\} - \frac{1}{a}\right]}.$$

It is easy to prove that (3. 4) is not a function. For (3. 4) can be written as

(3. 5)
$$f = \frac{1+\{G\}}{\alpha + \{H(t)\}},$$

where α is a number and $H(t)$ is a function. If the operator (3. 5) were a function, say $m(t)$, then it would be true that

$$\{\alpha m(t)\} + \left\{ \int_0^t m(\tau) H(t-\tau)\, d\tau \right\} = 1 + \{G\}$$

which is impossible because on the left side of the equation so obtained is a function and on the right is the sum of a function and a number.

Let now

$$C \neq 0.$$

If $g(0)=0$, (3. 3) reduces to the form

(3. 6)
$$\frac{e^{-as}(1+\{G\})}{C + be^{-as}\left[\left\{-\frac{G(t)}{t+a}\right\} - \frac{1}{a}\right]}.$$

It is easy to prove that (3. 6) is not a function. For (3. 6) can be written as

$$f = \frac{e^{-as} + \{M_1(t)\}}{C + \varrho e^{-as} + \{M_2(t)\}}, \tag{3.7}$$

where ϱ is a number and $M_1(t)$, $M_2(t)$ are functions.

If (3. 7) were a function, say $m(t)$, then it would hold that

$$\{Cm(t)\} + \varrho e^{-as}\{m(t)\} + \left\{\int_0^t m(\tau) M_2(t-\tau)\, d\tau\right\} = e^{-as} + \{M_1(t)\},$$

which is impossible because a function cannot agree with the sum of a function and the translation operator.

If $g(0) < 0$, we write (3. 3) in the form

$$f = \frac{e^{-as}(1 + \{G\})}{Cs^{g(0)} + be^{-as}[\cdots]}. \tag{3.8}$$

Apparently (3. 8) is not a function. Since

$$s^{g(0)} = \left\{\frac{t^{-g(0)-1}}{\Gamma(-g(0))}\right\}$$

(3. 8) can be written as

$$f = \frac{e^{-as} + \{M_1(t)\}}{\varrho e^{-as} + \{M_2(t)\}} \tag{3.9}$$

and the reasoning follows on the previous lines.

Finally it remains to settle the case $g(0) > 0$. Obviously (3. 3) is a function for every $C \neq 0$.

Namely (3. 3) can be written as

$$f = \frac{\{H_1(t)\}}{C + \{H_2(t)\}}, \tag{3.10}$$

where H_1 and H_2 are functions. But by a wellknown rule of the operational calculus (see [3])

$$\frac{1}{C + \{H_2(t)\}} = \frac{1}{C\left(1 + \left\{\frac{H_2}{C}\right\}\right)} = \frac{1}{C}\sum_{k=0}^{\infty}(-1)^k \left\{\frac{H_2}{C}\right\}^k, \tag{3.11}$$

where the power is understood in the convolution sense. The series (3.11) represents a function apart from its first term. Multiplying by $\{H_1(t)\}$ we see that (3. 10) is certainly a function as required. This proves the

THEOREM 2. (Existence-theorem of locally integrable solutions).

Let $a > 0$, and $g(0)$ exist and let $\frac{g(t) - g(0)}{t}$ be locally integrable on $\langle 0, \infty)$. The non-linear singular integral equation (3) *has nontrivial locally integrable solutions on $\langle 0, \infty)$, if and only if $g(0) > 0$. The solutions of* (3) *constitute a one-parameter family of locally integrable functions depending on a real parameter $C \gtrless 0$.*

Now let us discuss the case $a=0$. (2. 10) has the form

$$f=\frac{s^{-g(0)}(1+\{G\})}{C+b\int s^{-g(0)}(1+\{G\})\,ds}. \tag{3.12}$$

The occurring algebraic integral of (2. 10) can be written

$$\int s^{-g(0)}\,ds=\begin{cases}\dfrac{s^{-g(0)+1}}{1-g(0)}, & \text{if} \quad g(0)\neq 1\\ -s\{\log t\}, & \text{if} \quad g(0)=1\end{cases} \tag{3.13}$$

and by the use of GESZTELYI's rule of algebraic integration

$$\int s^{-g(0)}\{G\}\,ds=\begin{cases}\nearrow\\ \to\\ \searrow\end{cases}\begin{array}{l}-s^{-g(0)+1}\left\{t^{-g(0)}\displaystyle\int_0^t\frac{G(\tau)\,d\tau}{\tau^{1-g(0)}}\right\}, \quad \text{if} \quad g(0)>1\\ -s\left\{\displaystyle\int_\varepsilon^t\frac{\int_0^\tau G(u)\,du}{\tau}\,d\tau\right\}, \quad \text{if} \quad g(0)=1\\ -s^{-g(0)+1}\left\{t^{-g(0)}\displaystyle\int_\varepsilon^t\frac{G(\tau)}{\tau^{1-g(0)}}\,d\tau\right\}, \quad \text{if} \quad g(0)<1.\end{array} \tag{3.14}$$

First let $g(0)\neq 1$. By substituting (3. 13) and (3. 14) in (3. 12) we obtain

$$f=\frac{s^{-g(0)}(1+\{G\})}{C+bs^{-g(0)}\left[\dfrac{s}{1-g(0)}-s\left\{t^{-g(0)}\displaystyle\int_0^t\frac{G(\tau)\,d\tau}{\tau^{1-g(0)}}\right\}\right]}, \quad \text{if} \quad g(0)>1 \tag{3.15}$$

and

$$f=\frac{s^{-g(0)}(1+\{G\})}{C+bs^{-g(0)}\left[\dfrac{s}{1-g(0)}-s\left\{t^{-g(0)}\displaystyle\int_\varepsilon^t\frac{G(\tau)\,d\tau}{\tau^{1-g(0)}}\right\}\right]}, \quad \text{if} \quad g(0)<1.$$

The above operator (3. 15) is a function for every real value of C and $g(0)\gtrless 1$. For $C=0$, $g(0)>1$ we get

$$f=\frac{1+\{G\}}{s\left[\dfrac{b}{1-g(0)}-b\left\{t^{-g(0)}\displaystyle\int_0^t\frac{G(\tau)}{\tau^{1-g(0)}}\,d\tau\right\}\right]}. \tag{3.16}$$

But (3. 16) is the product of

$$f_1 = \frac{1+\{G\}}{s} = \left\{1+\int_0^t G(\tau)\,d\tau\right\}$$

and

$$f_2 = \frac{1}{\dfrac{b}{1-g(0)} - b\{\cdots\}}$$

which involves that (3. 16) is of type (3. 10. For $C=0$, $g(0)<1$ we proceed analogously.

The case $C\neq 0$, $g(0)>1$ may be treated similarly because $s^{-g(0)+1}$ itself is a function. For $C\neq 0$, $g(0)<1$ (3. 15) reads as

$$f = \frac{\dfrac{1}{s}[1+\{G\}]}{Cs^{g(0)-1} + b\left[\dfrac{1}{1-g(0)} - \left\{t^{-g(0)}\int_\varepsilon^t \cdots\right\}\right]}$$

and this is also of type (3. 10).

Now comes the discussion of the case $a=0$, $g(0)=1$. By (3. 12) and (3. 14) $(g(0)=1)$

(3. 17)
$$f = \frac{1+\{G\}}{Cs - bs^2\left\{\log t + \displaystyle\int_\varepsilon^t \frac{\int_0^\tau G(u)\,du}{\tau}\,d\tau\right\}}.$$

The decision of the structure of the operator (3. 17) is more difficult than in the preceding cases. Generally it is not easy to decide whether (3. 17) is a function or not. But with a simple restriction of $g(t)$ we can show that (3. 17) is certainly a function for every value of C.

Now in the case of $a=0$, $g(0)=1$, we take the following restriction with respect to $g(t)$:

Let $\dfrac{\displaystyle\int_0^t \frac{g(\tau)-1}{\tau}\,d\tau}{t}$ be locally integrable on $\langle 0, \infty)$. We make use of the following theorem (see [1]). If the functions $\{a(t)\}$, $\left\{\dfrac{a(t)}{t}\right\}$, $\{b(t)\}$ are locally integrable on $\langle 0, \infty)$, then

$$\frac{\displaystyle\int_0^t a(\tau)b(t-\tau)\,d\tau}{t}$$

is also locally integrable on $\langle 0, \infty)$.

Since

$$\left\{\int_0^t G(\tau)\,d\tau\right\} = \left\{\int_0^t \frac{g(\tau)-1}{\tau}\,d\tau\right\} + \tag{3.18}$$

$$+\left\{\frac{1}{2!}\int_0^t\int_0^\tau \frac{g(u)-1}{u}\,\frac{g(\tau-u)}{\tau-u}\,du\,d\tau\right\} + \cdots \frac{1}{k!}\int_0^t \left\{\frac{g(\tau)-1}{\tau}\right\}^k d\tau + \cdots$$

(the power is understood in the convolution sense).
With

$$b_k(t) = \frac{1}{k!}\left\{\frac{g(t)-1}{t}\right\}^{k-1} \qquad (k = 2, 3, 4 \ldots)$$

and

$$a(t) = \int_0^t \frac{g(\tau)-1}{\tau}\,d\tau$$

obviously every term of the series (3. 18) is the convolution of $a(t)$ and $b_k(t)$ and so the functions

$$\frac{\int_0^t a(\tau)\,b_k(t-\tau)\,d\tau}{t}$$

and

$$\frac{\int_0^t G(\tau)\,d\tau}{t} = \frac{a(t)+\int_0^t a(\tau)\sum_{k=2}^{\infty} b_k(t-\tau)\,d\tau}{t}$$

are also locally integrable.

If so, then ε can be chosen as zero and (3. 17) reads as

$$f = \frac{-(1+\{G\})}{s^2\{\log t\}\left[b - \dfrac{C}{s\{\log t\}} - b\,\dfrac{\left\{\int_0^t \dfrac{\int_0^\tau G(u)\,du}{\tau}\,d\tau\right\}}{\{\log t\}}\right]}. \tag{3.19}$$

We have only to show that all the following operators are functions:

$$\frac{1}{s^2\{\log t\}}, \quad \frac{1}{s\{\log t\}}, \quad \frac{\left\{\int_0^t \dfrac{\int_0^\tau G(u)\,du\,d\tau}{\tau}\right\}}{\{\log t\}}. \tag{3.20}$$

If so, then (3. 19) is of type (3. 10), consequently (3. 19) represents a function as required. Let us consider the following integral equation of the first kind

$$\int_0^t \varrho(\tau)\log(t-\tau)\,d\tau = H(t), \tag{3.21}$$

where $H(t)$ is any absolutely continuous function. This has been discussed by VOLTERRA—PÉRÈS [9]. The solution of (3. 21) is given by

$$\varrho(t) = -\frac{d^2}{dt^2}\int_0^t H(\tau)\mu(t-\tau)\,d\tau, \tag{3.22}$$

where

$$\mu(t) = \int_0^\infty \frac{t^v e^{\gamma v}}{\Gamma(v+1)}\,dv \qquad (\gamma = \text{Euler constant}).$$

So it is clear that the operators (3. 20) are certainly functions, namely they are the solutions of the corresponding integral equation of type (3. 21), with

$$H(t) = t; \quad \text{or} \quad 1; \quad \text{or} \quad \int_0^t \frac{\int_0^\tau G(u)\,du\,d\tau}{\tau}$$

respectively. This completes the proof of the following.

THEOREM 3. *(Existence theorem of locally integrable solutions). Let* $a=0$, $g(0)$ *exist and* $\dfrac{g(t)-g(0)}{t}$ *be locally integrable on* $\langle 0, \infty)$. *If* $g(0)=1$ *the integrability of the function*

$$\frac{\int_0^t \dfrac{g(\tau)-1}{\tau}\,d\tau}{t}$$

will be required too. The nonlinear singular integral equation (3) *has nontrivial solutions constituting a one-parameter family of locally integrable functions depending on a real parameter* $-\infty < C < \infty$,

REMARK 1. The case $a=0$, $g(0)=1$ seems to be too specialised, since it has been required the local integrability of

$$\frac{\int_0^t \dfrac{g(\tau)-1}{\tau}\,d\tau}{t}$$

which seems to be superfluous. It would be interesting to prove that (3. 17) agrees with a locally integrable function without the above restriction.

REMARK 2. By using the well-known rules of the Mikusiński's operational calculus the locally integrable solutions obtained in every case can be written in the „classical" form, i.e. as functions of the real variable t.

Sometimes in the solution process the complicated formulas obtained in the existence-theorems will not be applied. Occasionally it is more convenient to begin with the operational notations and express $g(t)$ as a function of the operator s. In this manner in many cases the solutions appear in a very simple closed form (see examples 1 and 4 in Chapter 5).

REMARK 3. The existence-theorems show that the operational solutions have a quite different behaviour, if $a>0$, or $a=0$, which is exhibited by the following

Table 1

	$a>0$	$a=0$
Existence of locally integrable solutions	$g(0)>0$, $C\neq 0$	$-\infty<g(0)<\infty$ $-\infty<C<\infty$
Non-existence of locally integrable solutions	$g(0)\leqq 0$	

4. *Further investigations*

Let us now consider the integral equation

$$(t-a)f(t)-\int_0^t f(\tau)g(t-\tau)\,d\tau=\int_0^t h(\tau)F(t-\tau)\,d\tau, \tag{4. 1}$$

where

$$F(t)=\int_0^t f(\tau)f(t-\tau)\,d\tau.$$

(4. 1) is of more complicated form then (1). The operational solution process reduces (4. 1) also to an algebraic differential equation of the Bernoulli type and so the solutions of (4. 1) can be obtained by using the method treated in the preceding chapters.

Equation (4. 1) can be brought in the following operational form:

$$Df+(a+g)f=-f^2h. \tag{4. 2}$$

Substituting

$$u=\frac{1}{f},\qquad Df=-\frac{Du}{u^2}$$

we get

$$Du-(a+g)u=h. \tag{4. 3}$$

Here a very interesting circumstance appears.

If the starting point is the integral equation

$$(4.4) \qquad (t+a)f(t)+\int_0^t f(\tau)g(t-\tau)\,d\tau = -h(t),$$

the following operational form will be obtained:

$$Df-(a+g)f = h$$

which *agrees* with (4. 3). But this is very convenient for us, because we may apply the results obtained in the discussion of equation (4. 4) (see [1], [2]).

Like [1], [2] in the case of $a=0$ we require the integrability of $\frac{h(t)}{t}$. Taking the above substitution and [1], [2] into account, the general operational solution of (4. 2) is

$$f=0$$

and

$$(4.4) \quad f = \frac{e^{-as}s^{-g(0)}(1+\{G\})}{C+e^{-as}s^{-g(0)}\left\{\frac{-h(t)-H(t)}{t+a}+g(0)\,(t+a)^{-g(0)-1}\int_0^t \frac{h(\tau)+H(\tau)}{(\tau+a)^{1-g(0)}}\,d\tau\right\}}$$

if $a>0$, or if $a=0$, $g(0)\geqq 0$

$$f = \frac{s^{-g(0)}(1+\{G\})}{C+s^{-g(0)}\left\{\frac{-h(t)-H(t)}{t}+g(0)\,t^{-g(0)-1}\int_\varepsilon^t \frac{h(\tau)+H(\tau)}{\tau^{1-g(0)}}\,d\tau\right\}}$$

if $a=0$, $g(0)<0$,

where C is an arbitrary real constant and $\varepsilon>0$, furthermore

$$H(t) = \int_0^t h(\tau)G(t-\tau)\,d\tau.$$

Quite analogously to chapter 3 we can obtain the conditions of the existence of the locally integrable solutions. The reasoning is similar to that in chapter 3, so it will be omitted.

This can be summarized in

THEOREM 4. *Let* $g(0)$ *exist,* $\frac{g(t)-g(0)}{t}$ *be locally integrable on* $\langle 0, \infty)$, $a\geqq 0$ *and in the case* $a=0$ *let* $\frac{h(t)}{t}$ *be also locally integrable on* $\langle 0, \infty)$. *Equation* (4. 1) *has the operational form* (4. 2), *which is an algebraic differential equation of Bernoulli-type, having a family of solutions depending on an arbitrary constant C, which are reciprocals of the solutions of the operational form of the linear integral equation*

$$(t+a)f(t)+\int_0^t f(\tau)g(t-\tau)\,d\tau = -h(t).$$

On the above conditions the obtained nontrivial solution (4. 4) *is a locally integrable function for every* $C \neq 0$, *if and only if* $g(0) > 0$.

REMARK 1. In the case $a = 0$ the condition of the local integrability of $\frac{h(t)}{t}$ is not necessary.

It is easy to show that if $g(0) > 1$ and $h(t)$ is integrable, then (4. 1) has nontrivial locally integrable solutions.

On the other hand let us consider

$$tf(t) = \int_0^t h(\tau) F(t-\tau)\, d\tau, \qquad F(t) = \int_0^t f(\tau) f(t-\tau)\, d\tau.$$

Here

$$Df = -f^2 h,$$

the general operational solution of which is

$$f = \frac{1}{C - s\left\{\int_\varepsilon^t \frac{h(\tau)}{\tau} d\tau\right\}}.$$

By choosing $h(t) = \{1\}$ we obtain locally integrable functions. But if $\frac{h(t)}{t}$ is integrable we can write

$$f = \frac{1}{C - \left\{\frac{h(t)}{t}\right\}}.$$

This cannot involve integrable solutions.

REMARK 2. of the preceding chapter holds here too. The only difference is that the operational notation must be introduced for the function $h(t)$ too.

5. *Examples*

EXAMPLE 1. Let us determine the nontrivial locally integrable solutions of the integral equation

$$tf(t) - \alpha \int_0^t f(t-\tau) e^{\beta\tau}\, d\tau = \int_0^t f(t) f(t-\tau)\, d\tau. \tag{5. 1}$$

By theorem 3 equation (5. 1) has nontrivial solutions for every real α. We show that it can be taken $\beta = 0$. Namely by the substitution

$$f(t) = e^{\beta t} F(t)$$

we obtain

$$te^{\beta t} F(t) - \alpha \int_0^t e^{\beta(t-\tau)} F(t-\tau) e^{\beta\tau}\, d\tau = \int_0^t e^{\beta(t-\tau)} F(t-\tau) e^{\beta\tau} F(\tau)\, d\tau$$

which involves

$$tF(t)-\alpha\int_0^t F(\tau)\,d\tau=\int_0^t F(t-\tau)F(\tau)\,d\tau \tag{5. 2}$$

and we have to solve (5. 2) only. Assume $\alpha\geqq 0$. If $\alpha<0$ the reasoning is similar. We exclude $\alpha=1$ which will be considered in the next example. By (3. 15)

$$F=\frac{1}{Cs^{\alpha}+\dfrac{s}{1-\alpha}}. \tag{5. 3}$$

If $C=0$ the solution reads as

$$F=\frac{1-\alpha}{s}=\{1-\alpha\}, \tag{5. 4}$$

if $\alpha=0$

$$F=\frac{1}{C+s}=\{e^{-ct}\}$$

(This is the special case $g(t)=0$).

If $C\neq 0$ and $\alpha>0$ the cases $\alpha=$integer, or not integer will be distinguished. If α is an integer $\geqq 2$ (5. 3) can be written in the form of partial fractions obtaining the wanted solutions in their „classical" forms, that is as functions of the variable t.

For example if $\alpha=2$

$$F=\frac{1}{Cs^2-s}=\frac{1}{Cs\left(s-\dfrac{1}{c}\right)}=\left\{\frac{1}{c}\int_0^t e^{\frac{1}{c}\tau}\,d\tau\right\}=\left\{e^{\frac{1}{c}t}-1\right\}, \tag{5. 5}$$

which is valid for every $C\neq 0$. (5. 5) has no meaning for $C=0$, but in this case (5. 4) gives

$$F=\{-1\}. \tag{5. 6}$$

It is obvious that (5. 6) is not a special case of (5. 5). They are quite „different" solutions.

Now let us discuss the case $\alpha=3$. We get from (5. 3)

$$F=\frac{1}{Cs^3-\dfrac{s}{2}}=\frac{1}{Cs\left[-\dfrac{1}{2C}+s^2\right]}. \tag{5. 7}$$

Since

$$\frac{1}{C\left(s^2-\dfrac{1}{2C}\right)}=\begin{cases}\sqrt{\dfrac{2}{C}}\,\mathrm{sh}\,\dfrac{t}{\sqrt{2C}}, & \text{if } C>0,\\[2ex] -\sqrt{-\dfrac{2}{C}}\,\sin\dfrac{t}{\sqrt{-2C}}, & \text{if } C<0,\end{cases}$$

$$\left\{2\left(\operatorname{ch}\frac{t}{\sqrt{2C}}-1\right)\right\}, \quad \text{if} \quad C>0$$

(5. 8) $$F(t)=\left\{2\left(\cos\frac{t}{\sqrt{-2C}}-1\right)\right\}, \quad \text{if} \quad C<0$$

$$\{-2\}, \quad \text{if} \quad C=0.$$

Now let us discuss the case where α is not an integer.
Let $\alpha = n+v$, where $n=0, 1, 2, \ldots$; $0<v<1$.
Then

$$F=\frac{1}{Cs^{n+v}+\dfrac{s}{1-n-v}}$$

For $n=0$

(5. 9) $$F=\frac{1}{s\left(Cs^{v-1}+\dfrac{1}{1-v}\right)}=\left\{(1-v)\sum_{k=0}^{\infty}(-1)^k C^k(1-v)^k s^{k(v-1)-1}\right\}=$$

$$=\left\{(1-v)\sum_{k=0}^{\infty}(-1)^k C^k\frac{t^{k(1-v)}}{\Gamma[k(1-v)+1]}\right\},$$

which contains also the solution (5. 4) ($C=0$).
If $n>0$

(5. 11) $$F=\frac{1}{Cs^{n+v}\left[1+\dfrac{s^{1-n-v}}{C(1-n-v)}\right]}=\frac{1}{Cs^{n+v}}\sum_{k=0}^{\infty}(-1)^k\frac{s^{k(1-n-v)}}{C^k(1-n-v)^k}=$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k(1-n-v)s^{(k+1)(1-n-v)-1}}{C^{k+1}}=$$

$$=\left\{(1-n-v)\sum_{k=0}^{\infty}\frac{(-1)^k t^{(k+1)(n+v-1)}}{C^{k+1}\Gamma[1+(k+1)(n+v-1)]}\right\},$$

which does *not* contain the solution (5. 4) ($C=0$). From (5. 4) we obtain the solution for $C=0$.

(5. 12) $$F=\{1-n-v\}.$$

REMARK. On the one hand the wanted solutions can be expressed as a power series of t also in the case where α is an integer, but it is more convenient to determine for fixed integer values of α the solutions in a closed form, as we have done it. On the other hand, if α is not an integer, sometimes we can obtain the solutions also in closed form. For example this can occur for $v=\dfrac{1}{2}$ and $n=1$, $0<v<1$.

It remains to discuss the case $\alpha = 1$.

EXAMPLE 2.
Let us solve

$$tf(t) - \int_0^t f(\tau)\,d\tau = \int_0^t f(\tau) f(t-\tau)\,d\tau .$$

From (3. 19)

$$f = \frac{1}{Cs - s^2\{\log t\}} = \frac{1}{s^2\left(\frac{C}{s} - \{\log t\}\right)} \tag{5.13}$$

(5. 13) can be written as an integral equation of the first kind

$$\int_0^t f(t-\tau)\,[\log t - C]\,d\tau = -t .$$

This is the VOLTERRA—PÉRÈS equation [9] investigated by FOCK [10] too, the solution of which is given by the formula

$$f(t) = \left\{ \int_0^\infty \frac{e^{(c-C)\gamma} t^\gamma}{\Gamma(\gamma+1)}\,d\gamma \right\} \quad (c = \text{Euler constant}) \tag{5.14}$$

(5. 14) is valid for every C.

EXAMPLE 3. Let us determine one solution of

$$tf(t) - \int_0^t f(\tau)\left(\sqrt{t-\tau} + \frac{3}{2}\right) d\tau = -\int_0^t f(\tau) f(t-\tau)\,d\tau . \tag{5.15}$$

The operational form of (5. 15) is

$$Df + f\left(\frac{\sqrt{\pi}}{2s^{3/2}} + \frac{3}{2s}\right) = -f^2 .$$

By substituting $u = \frac{1}{f}$

$$Du - \left(\frac{\sqrt{\pi}}{2s^{3/2}} + \frac{3}{2s}\right) u = -1 . \tag{5.16}$$

The general solution of the homogeneous equation is

$$u = C \exp\left[-\sqrt{\frac{\pi}{s}} - \frac{3}{2} s\{\log t\} \right]$$

and since

$$\exp\left[-\frac{3}{2}(s\{\log t\} + \lambda) \right] = s^{3/2} \qquad \text{(see [7])} \quad (\lambda = \text{Euler constant})$$

it holds

$$u = Cs^{3/2} \exp\left(-\sqrt{\frac{\pi}{s}} \right) . \tag{5.17}$$

We look for a particular solution of (5. 16) in the form

$$u_{\text{part}} = C(s)s^{3/2}\exp\left(-\sqrt{\frac{\pi}{s}}\right)$$

giving

$$C(s) = -\int s^{-3/2}\exp\left(\sqrt{\frac{\pi}{s}}\right)ds = \frac{2}{\sqrt{\pi}}\exp\left(\sqrt{\frac{\pi}{s}}\right)$$

and the general solution of (5. 16) is of the form

$$u = Cs^{3/2}\exp\left(-\sqrt{\frac{\pi}{s}}\right)+\frac{2}{\sqrt{\pi}}s^{3/2},$$

finally

$$f = \frac{1}{u} = \frac{1}{s^{3/2}\left[\frac{2}{\sqrt{\pi}}+C\exp\left(-\sqrt{\frac{\pi}{s}}\right)\right]}. \tag{5.18}$$

If $C=0$ we get the simple solution

$$f = \frac{\sqrt{\pi}}{2}s^{-3/2} = \{\sqrt{t}\}.$$

EXAMPLE 4.
Let us solve

$$(t-1)f(t)-\int_0^t f(\tau)\,d\tau = -\int_0^t\left(\frac{3\tau^2}{2}+\tau\right)F(t-\tau)\,d\tau, \tag{5.19}$$

where

$$F(t) = \int_0^t f(\tau)f(t-\tau)\,d\tau.$$

By Theorem 4 (5. 19) has nontrivial locally integrable solutions. (5. 19) has the operational form

$$Df+\left(1+\frac{1}{s}\right)f = \left(\frac{1}{s^2}+\frac{3}{s^3}\right)f^2.$$

Substituting μ by $\frac{1}{f}$ we obtain

$$Du-\left(1+\frac{1}{s}\right)u = -\left(\frac{1}{s^2}+\frac{3}{s^3}\right). \tag{5.20}$$

The homogeneous equation

$$Du-\left(1+\frac{1}{s}\right)u = 0$$

has the general solution

$$u = C\exp\int\left[1+\frac{1}{s}\right]ds = Cse^s.$$

Applying the method of variation of parameters we obtain a particular solution of (5. 20) as follows

$$u_{\text{part}} = C(s)se^s,$$

where

$$C(s) = -\int\left(\frac{1}{s^3}+\frac{3}{s^4}\right)e^{-s}ds. \tag{5.21}$$

The calculation of this algebraic integral is very simple. Since the integrand is

$$0, \quad \text{if} \quad t\leqq 1$$

and

$$\left\{\frac{(t-1)^2}{2}+\frac{(t-1)^3}{2}\right\} = \left\{\frac{(t-1)^2 t}{2}\right\} \quad \text{if} \quad t>1$$

therefore

$$C(s) = \{C(t)\} = \begin{cases} 0, & \text{if} \quad t\leqq 1 \\ \dfrac{(t-1)^2}{2} & \text{if} \quad t>1 \end{cases}$$

which implies

$$C(s) = e^{-s}\left\{\frac{t^2}{2}\right\} = \frac{e^{-s}}{s^3}$$

and

$$u_{\text{part}} = \frac{1}{s^2} = \{t\};$$

finally the general solution of (5. 20) is of the form

$$u = Cse^s+\frac{1}{s^2} \tag{5.22}$$

and

$$f = \frac{1}{Cse^s+\dfrac{1}{s^2}}. \tag{5.23}$$

If $C=0$ we obtain $f=s^2$, which is not a function in accordance with theorem 4. For every $C\neq 0$ (5. 23) is a function. Indeed if $C\neq 0$ (5. 23) can be written as

$$f = \frac{e^{-s}}{Cs}\left(\frac{1}{1+\dfrac{e^{-s}}{Cs^3}}\right) = \frac{e^{-s}}{Cs}\sum_{k=0}^{\infty}\left(\frac{-1}{C}\right)^k\frac{e^{-ks}}{s^{3k}} = \tag{5.24}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k e^{-(k+1)s}}{C^{k+1}s^{3k+1}} = \left\{\sum_{k=0}^{\infty}\frac{(-1)^k M_k(t)}{C^{k+1}(3k)!}\right\},$$

where

$$M_k(t) = \begin{matrix} 0 & \text{if} \quad t\leqq k+1 \\ \{t^{3k}\}, & \text{if} \quad t>k+1 \end{matrix}$$

So for every fixed t the series has only a finite number of terms.

REMARK. Let it be given an example of a $g(t)$ being not bounded in the neighbourhood of the origin. In general the method applied above is not applicable for any $g(t)$ unbounded in the neighbourhood of the origin. Namely we do not know in general whether the exponential function

$$e^{\int g\,ds}$$

exists or not, nevertheless in special cases the application of the operational calculus is very convenient.

Let us consider the integral equation

$$(5.25)\quad tf(t)-\lambda\int_0^t f(\tau)\,(t-\tau)^{-\frac{1}{n}}\,d\tau=\int_0^t f(\tau)f(t-\tau)\,d\tau \qquad (n=2,3,4,\ldots)$$

$$\lambda = \text{real number} \neq 0,$$

the operational form of which is

$$Df+\lambda\Gamma\left(1-\frac{1}{n}\right)s^{-\left(1-\frac{1}{n}\right)}f=-f^2.$$

Introducing $u=\dfrac{1}{f}$ and the denotation $v=\dfrac{1}{n}$ we obtain

$$(5.26)\qquad Du-\lambda\Gamma(1-v)s^{v-1}u=1.$$

The general solution of the homogeneous equation is

$$(5.27)\qquad u=C\exp\left[\lambda\frac{\Gamma(1-v)}{v}s^v\right]=C\exp\left[-\lambda\Gamma(-v)s^v\right].$$

The operator (5. 27) evidently exists. We look for a particular solution of (5. 26) of the form

$$u_{\text{part}}=C(s)\exp\left[-\lambda\Gamma(-v)s^v\right]$$

giving

$$(5.28)\qquad C(s)=\int\exp\left[\lambda\Gamma(-v)s^v\right]ds.$$

The algebraic integrand is for every real λ not integrable by GESZTELYI's method, i.e. it is not of type

$$s^k e^{-as}\{f(t)\} \qquad \text{(see Chapter 1).}$$

Nevertheless, (5. 28) exists and the following formula holds:

$$(5.29)\quad \int\exp\left[\lambda\Gamma(-v)s^v\right]ds=\exp\left[\lambda\Gamma(-v)s^v\right]\sum_{i=1}^{n}(-1)^{i+1}\frac{n!\,s^{1-iv}}{(n-i)!\,[\lambda\Gamma(-v)]^i}.$$

The proof of the validity of (5. 29) goes by simple algebraic differentiation of the right-hand side.

Hence the required particular solution is:

$$u_{\text{part}}=\sum_{i=1}^{n}(-1)^{i+1}\frac{n!\,s^{1-iv}}{(n-i)!\,[\lambda\Gamma(-v)]^i} \qquad \left(v=\frac{1}{n}\right)$$

and the general solution of the inhomogeneous equation (5. 26) is:

$$u = C\exp\left[-\lambda\Gamma(-v)s^{v}\right] + \sum_{i=1}^{n}(-1)^{i+1}\frac{n!\,s^{1-iv}}{(n-i)!\left[\lambda\Gamma(-v)\right]^{i}}$$

and finally

$$f = \frac{1}{u} = \frac{1}{C\exp\left[-\lambda\Gamma(-v)s^{v}\right] + \sum_{i=1}^{n}(-1)^{i+1}\frac{n!\,s^{1-iv}}{(n-i)!\left[\lambda\Gamma(-v)\right]^{i}}}. \tag{5.30}$$

It can be very easily shown that (5. 30) represents functions for every $-\infty < C < \infty$. We need the fact that

$$\exp(\beta s^{\alpha}) = \left\{\frac{1}{\pi}\int_{0}^{\infty}\exp(-xt - x^{\alpha}\beta\cos\alpha\pi)\sin(x^{\alpha}\beta\sin\alpha\pi)\,dx\right\},$$

$$\beta < 0, \quad 0 < \alpha < 1$$

(see [3]) is an infinitely many times differentiable function, every derivatives of which vanish in the neighbourhood of the origin. In all the three cases $C \gtreqless 0$ (5. 30) can be written in the form

$$f = \frac{\{M_1(t)\}}{1 + \{M_2(t)\}},$$

where M_1, M_2 are certain functions. So we have obtained an operator of type (3. 10) which is known to be a function. Here we let the detailed examinations to the reader.

REFERENCES

[1] FÉNYES, T.: Anwendung der Mikusińskischen Operatorenrechnung zur Lösung von Integralgleichungen dritter Art vom Faltungstypus, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **9** (1965) 365—399.

[2] FÉNYES, T.: A note on the solution of integral equations of convolution type of the third kind by application of the operational calculus of Mikusiński, *Studia Sci. Math.* Hungar. **2** (1967) 81—89.

[3] MIKUSIŃSKI, J.: *Operational Calculus*, Pergamon Press-Państwowe Wydawnistwo, 1959.

[4] MIKUSIŃSKI, J.: Remarks on the algebraic derivative in the Operational Calculus, *Studia Math.* **19** (1960) 187—192.

[5] GESZTELYI, E.: Anwendung der Operatorenrechnung auf lineare Differentialgleichungen mit Polynom-Koeffizienten, *Publ. Math. Debrecen.* **10** (1963) 215—243.

[6] ERDÉLYI, A.: *Operational Calculus and Generalized Functions*, California Institute of Technology Pasadena, 1959.

[7] MIKUSIŃSKI, J.: L'anneau algébrique et ses applications dans l'analyse fonctionelle (deuxième partie), *Annales Universitatis Mariae Curie Sklodowska, Sectio A.* **3** (1949) 25—33.

[8] WLOKA, J.: Über lineare *s*-Differentialgleichungen in der Operatorenrechnung, *Math. Ann.* **166** (1966) 233—256.

[9] VOLTERRA—PÉRÈS: *Théorie générale des fonctionelles*, Gauthier—Villars, Paris, 1936.

[10] FOCK, V.: Über eine Klasse von Integralgleichungen, *Math. Z.* **21** (1924) 161—173.

Mathematical Institute of the Hungarian Academy of Sciences, Budapest

(Received November 22, 1967.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 93—97.

ÜBER DIE NACHBARSCHAFT EINES KREISES IN EINER KREISPACKUNG

von

L. FEJES TÓTH

Meinem Freund, Professor J. Molnár zu seinem 50. Geburtstag gewidmet

§ 1. Einführung. Wir betrachten in der euklidischen Ebene eine Packung kongruenter Kreise, und fassen einen Kreis K ins Auge. Wir definieren die n-te *Nachbarschaft* oder Nachbarschaft n-ten Grades von K durch eine sukzessive Erweiterung in n Schritten. Im ersten Schritt betrachten wir diejenigen Kreise der Packung, die K berühren; im zweiten Schritt nehmen wir zu diesen Kreisen noch diejenigen, von K und von diesen Kreisen verschiedene Kreise hinzu, die die vorigen berühren, u.s.w. Die neuen Kreise, die im n-ten Schritt zu den vorigen hinzukommen, nennen wir n-te *Nachbarn* oder Nachbarn n-ten Grades von K.

Aus höchstens wievielen Kreisen kann die n-te Nachbarschaft eines Kreises bestehen? Die gesuchte Maximalzahl sei mit M_n bezeichnet.

Offensichtlich ist $M_1 = 6$. Es lässt sich ferner zeigen [1], daß $M_2 = 18$ ist. Für $n > 2$ ist der Wert von M_n nicht bekannt.

Im § 2 untersuchen wir das asymptotische Verhalten von M_n für große Werte von n. Es wird sich herausstellen, daß man die zahlreichste Nachbarschaft n-ten Grades für große Werte von n *nicht* so erhält, daß man in jedem Schritt soviel neue Kreise hinlegt wie möglich. Im § 3 werden wir zeigen, daß diese „gierige" Konstruktion für $n = 14$ nicht mehr erfolgreicht ist.

§ 2. Eine asymptotische Formel. In der dichtesten Gitterpackung hat jeder Kreis $6k$ Nachbarn k-ten Grades. Deshalb gilt

$$M_n \geqq 6(1 + \ldots + n) = 3n(n+1).$$

Es gibt aber auch andere Kreisanordnungen, in denen ein Kreis für jede natürliche Zahl k genau $6k$ Nachbarn k-ten Grades hat. Nimmt man nämlich 6 direkte Nachbarn, so kann man von den 12 zweiten Nachbarn einen willkürlich wählen, wodurch aber die übrigen 11 zweite Nachbarn schon eindeutig bestimmt sind (Abb. 1).

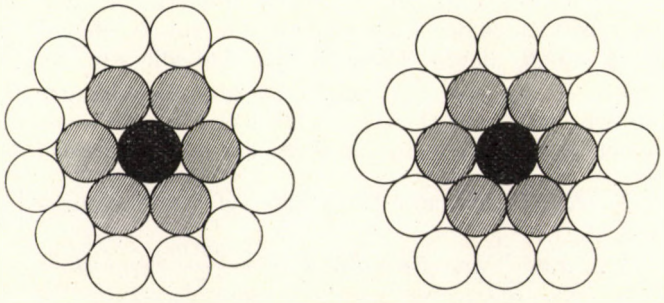


Abb. 1

Dann kann man von den 18 dritten Nachbarn wieder einen willkürlich wählen, u.s.w.

Man würde vermuten, daß die so konstruierte Nachbarschaften maximal sind, daß also $M_n = 3n(n+1)$ ist. Wir werden aber zeigen, daß für große Werte von n die asymptotische Formel $M_n \sim \frac{2\pi}{\sqrt{3}} n^2$ gilt, die die obige Vermutung widerlegt. Unser Hauptergebnis ist enthalten im folgenden

SATZ. *Bedeutet M_n das Maximum der Anzahl der Kreise, die bei verschiedenen Packungen kongruenter Kreise in der n-ten Nachbarschaft eines Kreises enthalten sind, so gilt*

$$\lim_{n\to\infty} M_n/n^2 = 2\pi/\sqrt{3}.$$

In einer Packung von Einheitskreisen sei K ein Kreis und O sein Mittelpunkt. Offensichtlich liegen die Nachbarn höchstens n-ten Grades von K ein einem um O geschlagenen Kreis vom Radius $2n+1$. Sind aber in ein konvexes Gebiet wenigstens zwei Kreise eingelagert, so ist nach einem bekannten Satz [2] die Lagerungsdichte $< \pi/\sqrt{12}$. Folglich haben wir

$$\frac{\pi(M_n+1)}{\pi(2n+1)^2} < \frac{\pi}{\sqrt{12}},$$

woraus sich

$$\overline{\lim_{n\to\infty}} M_n/n^2 \leqq 2\pi/\sqrt{3}$$

ergibt. Wir müssen noch die Ungleichung

$$\underline{\lim_{n\to\infty}} M_n/n^2 \geqq 2\pi/\sqrt{3}$$

beweisen.

Es sei w ein Winkelbereich mit dem Winkel $2\omega < 2\pi/3$. Es sei O_0 die Ecke von w, und $O_1, O_2, \ldots$ seien Punkte auf der winkelhalbierenden Halbgerade von w, für die $\overline{O_1O_0} = \overline{O_2O_1} = \ldots = 2$ gilt. Wir schlagen um $O_0, O_1, \ldots$ Einheitskreise und ergänzen diese Kreise zu einer dichtesten Gitterpackung G. Wir betrachten in G die Nachbarschaft k-ten Grades des um O_0 geschlagenen Kreises K_0 und greifen von den Kreisen die zu dieser Nachbarschaft gehören, diejenigen Kreise heraus, die ganz in w liegen. Wir wollen die Anzahl $a = a(k, \omega)$ dieser Kreise von unten abschätzen.

Wir betrachten die durch die Relationen $\sphericalangle A_iO_0O_i = \sphericalangle B_iO_0O_i = \omega$, $\sphericalangle A_iO_iO_0 = \sphericalangle B_iO_iO_0 = \pi/3$, $A_i \neq B_i$ definierte Punkte A_i und B_i, $i = 1, 2, \ldots$ (Abb. 2). Ferner betrachten wir diejenigen Kreise von G, deren Mittelpunkte auf dem Streckenzug $A_iO_iB_i$ liegen. Diese Kreise sind alle i-te Nachbarn von K_0. Ihre Zahl ist $> v_i - 1$, wo $v_i = \overline{A_iO_i} = \overline{O_iB_i}$. Da aber die Geraden O_0A_i und O_0B_i von diesen Kreisen, wegen $\sphericalangle O_0A_iO_i = \sphericalangle O_0B_iO_i = \pi - \pi/3 - \omega \geqq \pi/3 > \arcsin 1/2$, höchstens je einen Kreis treffen, ist die Zahl der in w liegenden Nachbarn i-ten Grades $> v_i - 3$. Deshalb gilt, mit Rücksicht auf $v_i = 2i \sin\omega / \sin\left(\frac{\pi}{3} + \omega\right)$,

$$a > \sum_{i=1}^{k} (v_i - 3) = k(k+1) \frac{\sin\omega}{\sin\left(\frac{\pi}{3} + \omega\right)} - 3k.$$

Um uns später bequemer ausdrücken zu können, nennen wir die jetzt betrachteten Kreise, zusammen mit denjenigen um $O_1, \ldots, O_k$ geschlagenen Einheitskreisen, die aus w herausragen, *Winkelbereichkreise*.

Wir betrachten wiederum eine dichteste Gitterpackung von Einheitskreisen. Wir fassen einen Kreis K mit dem Mittelpunkt O, sowie seine p-te Nachbarschaft N ins Auge. Die Mittelpunkte der p-ten Nachbarn von K liegen in den Ecken und

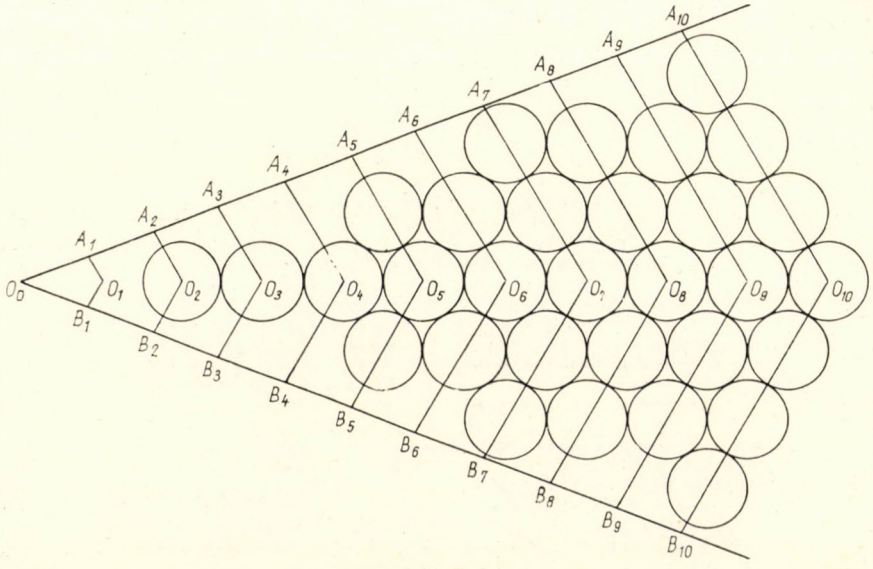


Abb. 2

auf den Seiten eines Sechsecks. Wir bezeichnen diese Punkte in ihrer zyklischen Reihenfolge mit $C_1, \ldots, C_{6p}$, so daß C_1 eine Ecke des Sechsecks sei.

Wir zerlegen die Ebene in $6p$ kongruente Winkelbereiche, die im Punkt O zusammentreffen, so daß die Halbgerade OC_1 einen Winkelbereich halbiert. Wir verschieben diesen Winkelbereich um den Vektor $\overrightarrow{OC_1}$, den zyklisch nachfolgenden Winkelbereich um $\overrightarrow{OC_2}$, u.s.w., und ergänzen N mit den entsprechenden Winkelbereichkreisen. Die so konstruierte Kreispackung zeigt, daß

$$M_{p+k} \geqq 3p(p+1)+6p\left\{k(k+1)\frac{\sin\dfrac{\pi}{6p}}{\sin\left(\dfrac{\pi}{3}+\dfrac{\pi}{6p}\right)}-3k\right\}.$$

Hieraus folgt

$$\underline{\lim_{n\to\infty}}\, M_n/n^2 = \underline{\lim_{k\to\infty}}\, M_{p+k}/k^2 \geqq 6p\sin\frac{\pi}{6p}\Big/\sin\left(\frac{\pi}{3}+\frac{\pi}{6p}\right).$$

Da die rechtsstehende Schranke mit $p\to\infty$ gegen $\pi/\sin\dfrac{\pi}{3}=2\pi/\sqrt{3}$ strebt, gilt tatsächlich $\underline{\lim_{n\to\infty}}\, M_n/n^2 \geqq 2\pi/\sqrt{3}$. Damit ist der Beweis des Satzes erbracht.

§ 3. Eine spezielle Konstruktion. Die obige Konstruktion würde erst für einen ziemlich großen Wert von n eine n-te Nachbarschaft mit mehr als $3n(n+1)$ Kreisen ergeben. Es sei hier gezeigt, daß $M_{14} \geqq 636$, was größer ist als $3 \cdot 14 \cdot 15 = 630$.

Es seien $\mathbf{a}$ und $\mathbf{b}$ zwei Vektoren, für die $|\mathbf{a}| = |\mathbf{b}| = \mathbf{ab} = 2$ ausfällt. Wir bezeichnen den Kreis, der aus einen vorgegebenen Einheitskreis durch Verschiebung um den Vektor $m\mathbf{a} + n\mathbf{b}$ entsteht, mit (m, n). Die Kreise (m, n) mit allen möglichen ganzzahligen Werten von m und n bilden eine dichteste Gitterpackung. Wir greifen von dieser Packung die 630 Nachbarn höchstens 14-ten Grades des Kreises (0, 0) heraus. Wir lassen nun die 12 Kreise (1, 2), (2, 1), (2, 2), ..., (4, 5), (5, 4), (5, 5) weg und tun dasselbe mit den 14 Kreisen (s, t) mit $s \geqq 5$, $t \geqq 5$, $10 < s + t \leqq 14$. Gleichzeitig fügen wir aber zu den gebliebenen Kreisen neue Kreise hinzu.

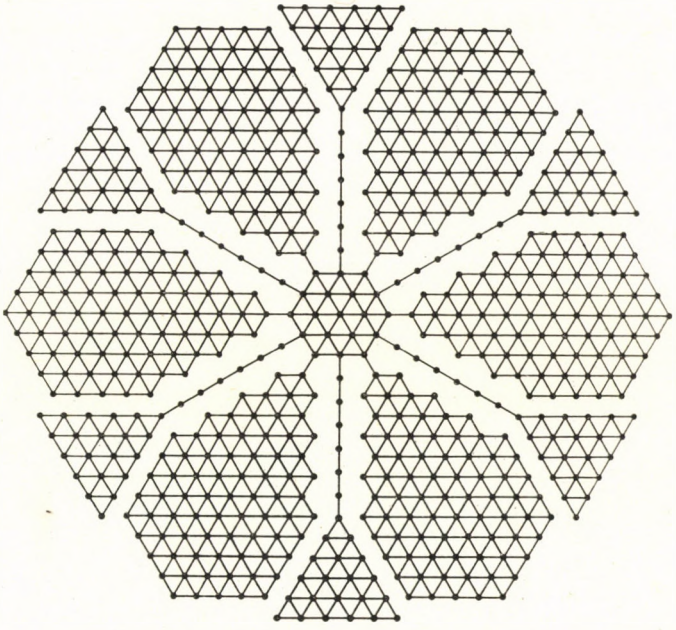


Abb. 3

Zunächst betrachten wir die 7 Kreise, die aus (1, 1) durch wiederholte Verschiebungen um den Vektor $2\,\frac{\mathbf{a}+\mathbf{b}}{|\mathbf{a}+\mathbf{b}|}$ entstehen, d.h. die Kreise $(1 + k/\sqrt{3},\ 1 + k/\sqrt{3})$ mit $k = 1, \ldots, 7$. Da $1 + 7/\sqrt{3} = 5{,}04\ldots > 5$ ist, liegt der Kreis $(1 + 7/\sqrt{3},\ 1 + 7/\sqrt{3})$ ein wenig weiter vom Kreis (0, 0) entfernt als der Kreis (5, 5). Während aber in der ursprünglichen Packung der Kreis (5, 5) bezüglich (0, 0) den Nachbarschaftsgrad 10 hatte, besitzt der Kreis $(1 + 7/\sqrt{3},\ 1 + 7/\sqrt{3})$ in der neuen Packung den Nachbarschaftsgrad 9. Deshalb können wir diese Packung durch die 20 Kreise $(s + 7/\sqrt{3},\ t + 7/\sqrt{3})$, $s \geqq 1$, $t \geqq 1$, $1 < s + t \leqq 6$ ergänzen ohne den Nachbarschaftsgrad 14 überzuschreiten.

Damit haben wir einen Kreis gewonnen. Führen wir diese Konstruktion symmetrisch in allen sechs Richtungen aus, so erhalten wir eine Kreispackung, die 636 Nachbarn höchstens 14-ten Grades des mittleren Kreises enthält. Abb. 3 stellt die Kreismittelpunkte in dieser Packung dar. Die Mittelpunkte der direckten Nachbarn sind mit Strecken der Länge 2 verbunden.

LITERATURVERZEICHNIS

[1] FEJES TÓTH, L. und HEPPES A.: A variant of the problem of the thirteen spheres, *Canad. J. Math.* **19** (1967) 1092—1100.

[2] FEJES TÓTH, L.: *Lagerungen in der Ebene, auf der Kugel und im Raum*, Berlin 1953.

Mathematisches Institut der Ungarischen Akademie der Wissenschaften, Budapest

(Eingegangen: 15. Dezember, 1967.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 99—108.

ON SOME DISTRIBUTIONS CONCERNING A RESTRICTED RANDOM WALK

by

B. R. HANDA and S. G. MOHANTY

1. Introduction and notations

Consider random sequences $\{\theta_1, \theta_2, \ldots, \theta_{(\mu+1)n}\}$ of μn $+1's$ and n $-\mu's$ (μ being a positive integer) each possible sequence being equally probable. Letting $s_0=0$, $s_i = \theta_1+\ldots+\theta_i$, $i=1, 2, \ldots, (\mu+1)n$, the following notations are defined.

λ: number of indices i for which $s_i=0$, $s_{i-1}s_{i+1} = -1$ or $-\mu^2$;

λ': number of indices i for which $s_{i-1}=0$, $s_i=1$;

π: number of positive $s_i's$ in $(s_0, s_1, \ldots, s_{(\mu+1)n})$.

In the definitions of λ, λ' and π, the sequecnes must satisfy $s_is_{i+1} \geqq 0$ for all i. This condition is refered to as "condition c" throughout in this paper.

$\varkappa$: $\max\limits_{0\leqq i\leqq(\mu+1)n} \{s_i\}$;

ϱ: $\min\{i: s_i=\varkappa\}$;

ψ: $\min\limits_{0\leqq i\leqq(\mu+1)n} \{s_i\}$;

φ: $\min\{i: s_i=\psi\}$ satisfying the conditions given below:

For all $0=i_0<i_1<\cdots<i_{-\psi-1}<i_{-\psi}=i$,

$$s_{i_{j-1}+1} > -j,\ s_{i_{j-1}+2} > -j,\ \ldots,\ s_{i_j-1} > -j,\ s_{i_j} = -j,$$
$$j = 1, 2, \ldots, -\psi.$$

In [1], CSÁKI and VINCZE have dealt the symmetric random walk which is a special case of the above by putting $\mu=1$. Their main results are

$$P(\chi=l) = \frac{1}{2}[P(\lambda=l-1)+P(\lambda=l)] \quad \text{for} \quad l=0, 1, \ldots, n, \tag{1}$$

and

$$P(\chi=l, \varrho=r) = P(\lambda'=l, \pi=r). \tag{2}$$

In this paper, besides deriving the distributions of λ, λ' and χ and the joint distributions of (λ', π), (χ, ϱ) and (ψ, φ), in the case of the generalised random walk (as stated at the beginning), we give the following equivalence relations:

$$P_c(\lambda'=l) = \frac{1}{2}[P_c(\lambda=l-1)+P_c(\lambda=l)], \qquad l=0, 1, \ldots, n; \tag{3}$$

$$P(\chi=l) = \begin{cases} P_c(\lambda'=l), & l=0; \\ \sum\limits_{i=1}^{l-1} \binom{l-1}{i} P_c(\lambda'=l-i), & 1\leqq l\leqq\mu; \end{cases} \tag{4}$$

and

$$P_c(\lambda' = l, \pi = r) = P(\psi = -l, \varphi = r), \tag{5}$$

where $P_c(\cdot)$ denotes the probability of $(\cdot)$ under the condition c.

Interesting enough to note is that (3), (4) and (5) reveal the exact nature of correspondence, whereas the symmetry induced by the case $\mu = 1$ leads to the results (1) and (2).

Mention might be made of another paper [4], in which one of the authors has obtained distribution of several characteristics associated with this restricted random walk.

An one to one correspondence between a set of lattice paths and the random walk is established by representing a $+1$ by a horizontal unit and a $-\mu$ by a vertical unit. Thus, any random sequence becomes a lattice path from (0,0) to (un, n), the total number of such paths being $\binom{(\mu+1)n}{n}$. This diagrametic representation helps us to visualize various aspects of the problem, more vividly than otherwise. For notational simplicity, we denote by $S_{\mu, n}$, any path from (0,0) to $(\mu n, n)$ and by $N(\cdot)$, the number of paths corresponding to $(\cdot)$. In determining the probability distribution $P(\cdot)$ of various characteristics as described earlier, it is readily seen that each such expression is equal to $N(\cdot)\Big/\binom{(\mu+1)n}{n}$. In subsequent sections, we therefore present results only in the form of $N(\cdot)$. $N_c(\cdot)$ denotes the paths corresponding to $(\cdot)$ under the condition c.

In what follows, a significant role is played by a result in GOULD's paper [2], which is stated below:

$$\sum_{k=0}^{n} \binom{x+\beta k}{k} \binom{y-\beta k}{n-k} \frac{p+qk}{(x+\beta k)(y-\beta k)} = \frac{p(x+y-\beta n)+nxq}{(x+y)x(y-\beta n)} \binom{x+y}{n}. \tag{6}$$

Furthermore, we state the expression for the number of paths from (0,0) to (m, n), $(m > \mu n)$, not crossing the line $x = \mu y$ as

$$\frac{m-\mu n+1}{m+n+1} \binom{m+n+1}{n}, \tag{7}$$

which is also equal to the number of paths from (0, 0) to $(m+1, n)$, never touching the line $x = \mu y$. Expressions (6) and (7) are used quite often in deriving and simplifying many results.

2. Distributions of λ, λ' and χ

THEOREM 1.

$$N_c(\lambda = l) = \begin{cases} \dfrac{2(l+1)}{n} \binom{(\mu+1)n}{n-l-1}, & l = 0, 1, \ldots, n-1; \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

PROOF. In the language of the lattice paths, expression (8) is the same as the number of $S_{\mu,n}$' s such that no path is allowed to cross the line $x=\mu y$ except at a lattice point and each path has l intersections with the line $x=\mu y$ (i.e. the number of times each path intersects the line $x=\mu y$ only at lattice points is l). Denoting by $N_c^+(\lambda=l)$, the number of paths $S_{\mu,n}$ with $\lambda=l$, the first step being horizontal one, it is easy to check that

$$N_c(\lambda=l) = 2N_c^+(\lambda=l). \tag{9}$$

Because of (9), (8) is equivalent to

$$N_c^+(\lambda=l) = \frac{l+1}{n}\binom{(\mu+1)n}{n-l-1}, \qquad l=0,1,\ldots,n-1. \tag{10}$$

For $l=0$, $N^+(\lambda=0)$ represents the number of $S_{\mu,n}$'s having initially a horizontal step and not crossing the line $x=\mu y$. Thus by (7)

$$N_c^+(\lambda=0) = \frac{1}{(\mu+1)n+1}\binom{(\mu+1)n+1}{n} = \frac{1}{n}\binom{(\mu+1)n}{n-1},$$

which confirms (10) for $l=0$. By induction we require to check (10) for l, given that it is true for all values upto $l-1$.

Evidently, any path counted in $N_C^+(\lambda=l)$ consists of two segments, where the first segment ends at the last point of intersection $(\mu y, y)$, $y=l, l+1, \ldots, n-1$, and contains exactly $l-1$ intersections. Thus

$$N_c^+(\lambda=l) = \sum_{y=l}^{n-1} \frac{l}{y}\binom{(\mu+1)y}{y-l}\frac{1}{(\mu+1)(n-y)+1}\binom{(\mu+1)(n-y)+1}{n-y} =$$

$$= \sum_{y=0}^{n-l-1} \frac{(\mu+1)l}{(\mu+1)(y+l)}\binom{(\mu+1)(y+l)}{y}\frac{1}{(\mu+1)(n-y-l)+1}\binom{(\mu+1)(n-y-l)+1}{n-y-l} =$$

$$= \frac{(\mu+1)l+1}{(\mu+1)n+1}\binom{(\mu+1)n+1}{n-l} - \frac{(\mu+1)l}{(\mu+1)n}\binom{(\mu+1)n}{n-l},$$

by (6), which simplifies to

$$\frac{l+1}{n}\binom{(\mu+1)n}{n-l-1}.$$

This completes the proof.

THEOREM 2.

$$N_c(\lambda'=l) = \begin{cases} \dfrac{(\mu+1)l+1}{(\mu+1)n+1}\dbinom{(\mu+1)n+1}{n-l}, & l=0,1,\ldots,n; \\ 0 & \textit{otherwise.} \end{cases} \tag{11}$$

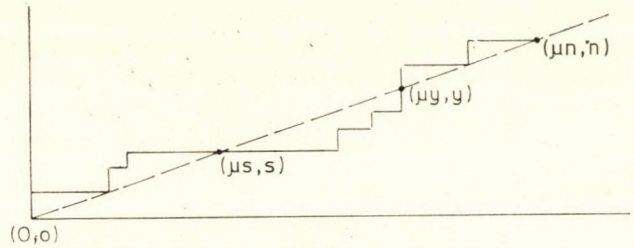


Fig. 1

PROOF. Trivially, it can be shown that $N_c(\lambda'=0)$ represents the total number of $S_{\mu,n}$'s, with the initial step to be vertical and not crossing $x=\mu y$ and this number by (7), is

$$\frac{1}{(\mu+1)n+1}\binom{(\mu+1)n+1}{n}.$$

Thus the result is true for $l=0$. For $l=1$, a typical path is given in figure 1. It is clear from the figure that

$$N_c(\lambda'=1)=\sum_{s=0}^{n-1}\frac{1}{(\mu+1)s+1}\binom{(\mu+1)s+1}{s}\sum_{y=sr'}^{n}\frac{\mu}{(\mu+1)(y-s)-1}\times$$

$$\times\binom{(\mu+1)(y-s)-1}{y-s-1}\frac{1}{(\mu+1)(n-y)+1}\binom{(\mu+1)(n-y)+1}{n-y},$$

which with the help of (6) simplifies to

$$\frac{\mu+2}{(\mu+1)n+1}\binom{(\mu+1)n+1}{n-1}.$$

Therefore, (11) is verified for $l=1$. Using induction, we can write

$$N_c(\lambda'=l+1)=\sum_{y=l}^{n-1}\frac{(\mu+1)l+1}{(\mu+1)y+1}\binom{(\mu+1)y+1}{y-l}\frac{\mu+1}{(\mu+1)(n-y)}\binom{(\mu+1)(n-y)}{n-y-1}=$$

$$=\frac{(\mu+1)(l+1)+1}{(\mu+1)n+1}\binom{(\mu+1)n+1}{n-l-1}.$$

This completes the proof.

As a corollary from Theorem 1 and Theorem 2, we obtain the following equivalence relation:

COROLLARY 1.

$$N_c(\lambda'=l)=\frac{1}{2}[N_c(\lambda=l-1)+N_c(\lambda=l)],\quad\text{for all } l.$$

THEOREM 3.

$$N(\chi=l)=\begin{cases}\dfrac{1}{(\mu+1)n+1}\binom{(\mu+1)n+1}{n} & \text{for } l=0;\\ \displaystyle\sum_{\alpha=0}^{n-s}{}'\frac{l}{(\mu+1)\alpha+l}\binom{(\mu+1)\alpha+l}{\alpha}\binom{(\mu+1)(n-\alpha)-l}{n-\alpha} & \\ -\displaystyle\sum_{k=s}^{n}\frac{l+1}{(\mu+1)(n-k)+l+1}\binom{(\mu+1)(n-k)+l+1}{n-k}\binom{(\mu+1)k-l-1}{k} & \\ \qquad\qquad \text{for } l=(s-1)\mu+\theta,\quad s=1,\ldots,n,\quad \theta=1,\ldots,\mu; & \\ 0 \qquad\qquad \textit{otherwise.} & \end{cases} \tag{12}$$

(It is observed that the second part of the theorem is independent of θ).

PROOF. Here the problem is to count the number of $S_{\mu,n}$'s not crossing the line $x=\mu y+l$ but touching it at least once. The first and the last part of the theorem is self-evident. Any such path in the second part consists of two segments viz. (a) from the origin to $(\mu\alpha+l,\alpha)$, $\alpha=0,1,\ldots,\left[n-\frac{l}{\mu}\right]$, not touching $x=\mu y+l$, except at the end, and (b) from $(\mu\alpha+l,\alpha)$ to $(\mu n,n)$ not crossing $x=\mu y+l$. Here $[\zeta]$ represents the greatest integer less than equal to ζ.

The number of paths in (a) according to (7) is

$$\frac{l}{(\mu+1)\alpha+l}\binom{(\mu+1)\alpha+l}{\alpha}, \tag{13}$$

whereas the number in (b), by some elementary transformation and with the help Theorem 3 in [3] becomes $N_{p,q}(1,\mu+1;c,1)$ in the notation of [3] where $p=n-\alpha-\left[\frac{l}{\mu}\right]$, $q=\left[\frac{l}{\mu}\right]$ and $c=\mu\left[\frac{l}{\mu}\right]+\mu+1-l$. The complete expression for $N_{p,q}(1,\mu+1;c,1)$ can be written as

$$\binom{(\mu+1)(n-\alpha)-l}{n-\alpha}- \tag{14}$$

$$-\sum_{k=\left[\frac{l}{\mu}\right]+1}^{n-\alpha}\frac{1}{(\mu+1)(n-\alpha-k)+1}\binom{(\mu+1)(n-\alpha-k)+1}{n-\alpha-k}\binom{(\mu+1)k-l-1}{k},$$

by using (6).

Next assume $(s-1)\mu<l<s\mu$. Then $\left[\frac{l}{\mu}\right]=s-1$ and $\left[n-\frac{l}{\mu}\right]=n-s$. We obtain

$$N(\chi=l)=\sum_{\alpha=0}^{n-s}{}'\frac{l}{(\mu+1)\alpha+l}\binom{(\mu+1)\alpha+l}{\alpha}\binom{(\mu+1)(n-\alpha)-l}{n-\alpha} \tag{15}$$

$$-\sum_{\alpha=0}^{n-s}{}'\sum_{k=s}^{n-\alpha}{}'\frac{l}{(\mu+1)\alpha+l}\binom{(\mu+1)\alpha+l}{\alpha}\frac{1}{(\mu+1)(n-\alpha-k)+1}\times$$

$$\times\binom{(\mu+1)(n-\alpha-k)+1}{n-\alpha-k}\binom{(\mu+1)k-l-1}{k}.$$

When the order of summation in the second term is interchanged it becomes

$$\sum_{k=s}^{n}\binom{(\mu+1)k-l-1}{k}\sum_{\alpha=0}^{n-k}\frac{l}{(\mu+1)\alpha+l}\binom{(\mu+1)\alpha+l}{\alpha}\times$$

$$\times\frac{1}{(\mu+1)(n-\alpha-k)+1}\binom{(\mu+1)(n-\alpha-k)+1}{n-\alpha-k},$$

which yields

$$\sum_{k=s}^{n}\frac{l+1}{(\mu+1)(n-k)+l+1}\binom{(\mu+1)(n-k)+l+1}{n-k}\binom{(\mu+1)k-l-1}{k}$$

by using (6). Thus the theorem is proved for $l\neq s\mu$. On the other hand, when $l=s\mu$, $\left[\frac{l}{\mu}\right]=s$, and $\left[n-\frac{l}{\mu}\right]=n-s$. This fact only introduces a change that the lower limit of k in (15) becomes $s+1$ in place of s. However, for $k=s$, the expiession in the second term of (15) takes the form

$$\frac{\mu s}{(\mu+1)\alpha+\mu s}\binom{(\mu+1)\alpha+\mu s}{\alpha}\times$$

$$\times\frac{1}{(\mu+1)(n-\alpha-s)+1}\binom{(\mu+1)(n-\alpha-s)+1}{n-\alpha-s}\binom{(\mu+1)s-\mu s-1}{s}$$

and this reduces to zero. Hence the theorem is also true for $l=s\mu$.

COROLLARY 2. When $\mu=1$

$$N(\chi=l)=N(\lambda'=l)=\frac{2l+1}{2n+1}\binom{2n+1}{n-l}. \tag{16}$$

The proof is very simple by using Theorems 2 and 3. We remark that because of the equivalence (3), relation (1) is satisfied for $\mu=1$, as derived in [1]. Note that condition c is trivial here and thus $N_c(\cdot)=N(\cdot)$

COROLLARY 3. If $l\leqq\mu$,

$$N(\chi=l)=\frac{\mu+l+1}{(\mu+1)n+l}\binom{(\mu+1)n+l}{n-1} \tag{17}$$

PROOF. If $l\leqq\mu$, then $s=1$. Application of (6) to Theorem 3 in this special case gives rise to the desired result.

Now we are in a position to state the second equivalence relation.

COROLLARY 4.

$$N(\chi=l)=\begin{cases}N_c(\lambda'=l), & \text{when} \quad l=0;\\ \sum_{i=0}^{l-1}\binom{l-1}{i}N_c(\lambda'=l-i), & \text{when} \quad 1\leqq l\leqq\mu.\end{cases}$$

PROOF. The proof of the first part directly follows from (11) and (12). Let

$$S_{l-i}=N_c(\lambda'=l-i)+N_c(\lambda'=l-i-1), \qquad l\geqq 1. \tag{18}$$

Then some elementary simplification shows that

$$S_l = \frac{(\mu+1)(l-1)+2}{(\mu+1)n+2}\binom{(\mu+1)n+2}{n-l+1}.$$

Considering the sum $\sum_{i=0}^{l-2}\binom{l-2}{i}S_{l-i}$, we assert that this sum equals to

$$\frac{\mu+l+1}{(\mu+1)n+l}\binom{(\mu+1)n+l}{n-1}, \tag{19}$$

which by Corollary 3 is the same as $N(\chi=l)$. To prove our assertion, we note that

$$\sum_{i=0}^{l-2}\binom{l-2}{i}S_{l-i} = \sum_{i=0}^{l-3}\left\{\binom{l-3}{i}S_{l-i}+\binom{l-3}{i}S_{l-i-1}\right\} \tag{20}$$

and that for $i=0$,

$$S_l + S_{l-1} = \frac{(\mu+1)(l-2)+3}{(\mu+1)n+3}\binom{(\mu+1)n+3}{n-l+2}. \tag{21}$$

Applying induction it is not difficult to get the second part of the corollary with the help of (20) and (21). Finally, changing S_{l-i} by (18) the desired relation is obtained.

Lasty we remark that several authors [5], [6], [7] have studied the distribution in Theorem 3.

3. Bivariate distributions

THEOREM 4.

(22)

$$N_c(\lambda'=l,\pi=r) = \begin{cases} \dfrac{1}{(\mu+1)n+1}\binom{(\mu+1)n+1}{n}, & \text{when} \quad l=r=0; \\ \dfrac{\mu l}{(\mu+1)(\alpha-l)+\mu l}\binom{(\mu+1)(\alpha-l)+\mu l}{\alpha-l}\dfrac{l+1}{(\mu+1)(n-\alpha)+l+1}\times \\ \times\binom{(\mu+1)(n-\alpha)+l+1}{n-\alpha} \\ \quad \text{when} \quad r=(\mu+1)\alpha-l,\ \alpha=l, l+1, \ldots, n,\ l=1, 2, \ldots, n; \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{otherwise.} \end{cases}$$

PROOF. $N_c(\lambda'=0,\ \pi=0)$ represents the number of $S_{\mu,n}$'s, which are above the line $x=\mu y$, but do not cross it and therefore equals to the first expression in (22). Moreover, it is readily verified that π cannot take any other value except those given in the second expression.

For $l=1$ and $r=(\mu+1)\alpha-1$, a typical path of this kind which is shown in Fig. 2 below, consists of two segments, (i) S_{μ,y_0} $(y_0=0, 1, \ldots, n-\alpha)$ that lies above $x=\mu y$ and never crosses it, and (ii) the segment starting from $(\mu y_0, y_0)$, initially

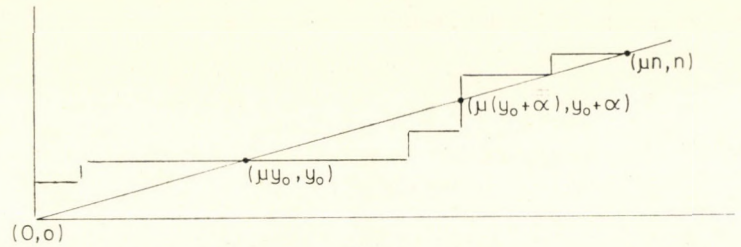


Fig. 2

contains a horizontal step, intersecting the line $x=\mu y$ at $(\mu(y_0+\alpha), y_0+\alpha)$, without touching it before and finally ends at $(\mu n, n)$, not crossing $x=\mu y$ after $(\mu(y_0+\alpha)$ $y_0+\alpha)$. Clearly, the number of first type of segments is

$$\frac{1}{(\mu+1)y_0+1}\binom{(\mu+1)y_0+1}{y_0}, \tag{23}$$

and that of the second is

$$\frac{\mu}{(\mu+1)\alpha-1}\binom{(\mu+1)\alpha-1}{\alpha-1}\frac{1}{(\mu+1)(n-y_0-\alpha)+1}\binom{(\mu+1)(n-y_0-\alpha)+1}{n-y_0-\alpha}. \tag{24}$$

Thus from (23) and (24)

$$N_c(\lambda'=1, \pi=(\mu+1)\alpha-1) = \frac{\mu}{(\mu+1)\alpha-1}\binom{(\mu+1)\alpha-1}{\alpha-1}\times$$

$$\times\sum_{y_0=0}^{n-\alpha}\frac{1}{(\mu+1)y_0+1}\binom{(\mu+1)y_0+1}{y_0}\frac{1}{(\mu+1)(n-\alpha-y_0)+1}\binom{(\mu+1)(n-\alpha-y_0)+1}{n-\alpha-y_0}$$

$$=\frac{\mu}{(\mu+1)\alpha-1}\binom{(\mu+1)\alpha-1}{\alpha-1}\frac{2}{(\mu+1)(n-\alpha)+2}\binom{(\mu+1)(n-\alpha)+2}{n-\alpha},$$

by using (6). This confirms the Theorem for $l=1$.

In general, any path to be counted in $N_c(\lambda'=l+1, \pi=(\mu+1)\alpha-l-1)$ contains two segments of which the first is $S_{\mu,y}$ with $\lambda'=l$ and $\pi=(\mu+1)(\alpha-\alpha_1)-l$ $(y=\alpha-\alpha_1, \ldots, n-\alpha;\ \alpha_1=1, \ldots, \alpha-l)$. Using induction we obtain,

$$N_c(\lambda'=l+1, \pi=(\mu+1)\alpha-l-1) = \sum_{\alpha_1=1}^{\alpha-l}\sum_{y=\alpha-\alpha_1}^{n-\alpha}\frac{l\mu}{(\mu+1)(\alpha-\alpha_1-l)+l\mu}\times$$

$$\times\binom{(\mu+1)(\alpha-\alpha_1-l)+l\mu}{\alpha-\alpha_1-l}\frac{l+1}{(\mu+1)(y-\alpha+\alpha_1)+l+1}\binom{(\mu+1)(y-\alpha+\alpha_1)+l+1}{y-\alpha+\alpha_1}\times$$

$$\times\frac{\mu}{(\mu+1)(\alpha_1-1)+\mu}\binom{(\mu+1)(\alpha_1-1)+\mu}{\alpha_1-1}\frac{1}{(\mu+1)(n-y-\alpha_1)+1}\times$$

$$\times\binom{(\mu+1)(n-y-\alpha_1)+1}{n-y-\alpha_1},$$

which with the help of (6) simplifies to

$$\frac{(l+1)\mu}{(\mu+1)(\alpha-l-1)+(l+1)\mu}\binom{(\mu+1)(\alpha-l-1)+(l+1)\mu}{\alpha-l-1}\times$$

$$\times\frac{l+2}{(\mu+1)(n-\alpha)+l+2}\binom{(\mu+1)(n-\alpha)+l+2}{n-\alpha}.$$

This completes the proof.

THEOREM 5.

$$N(\chi=l,\varrho=r)=\frac{l}{(\mu+1)\alpha+l}\binom{(\mu+1)\alpha+l}{\alpha}\left[\binom{(\mu+1)(n-\alpha)-l}{n-\alpha}-\right.$$

$$\left.-\sum_{k=\left[\frac{l}{\mu}\right]+1}^{n-\alpha}\frac{1}{(\mu+1)(n-\alpha-k)+1}\binom{(\mu+1)(n-\alpha-k)+1}{n-\alpha-k}\binom{(\mu+1)k-l-1}{k}\right], \tag{25}$$

for $r=(\mu+1)\alpha+l,\ l=1,2,\ldots,n\mu \quad \alpha=0,1,\ldots,\left[n-\frac{l}{\mu}\right]$.

PROOF. For proof, we can only say that expression (25) is the product of (13) and (14).

COROLLARY 5. When $l\leqq\mu$

$$N(\chi=l,\varrho=r)=\frac{l}{(\mu+1)\alpha+l}\binom{(\mu+1)\alpha+l}{\alpha}\frac{1}{(\mu+1)(n-\alpha)+1}\binom{(\mu+1)(n-\alpha)+1}{n-\alpha} \tag{26}$$

for $r=(\mu+1)\alpha+l,\ l=1,2,\ldots,n\mu,\ \alpha=0,1,\ldots,n-1$.

PROOF. In this case, the lower limit of k in (25) becomes 1. The rest involves only some elementary simplification.

COROLLARY 6. For $\mu=1$

$$N(\chi=l,\varrho=l+2\alpha)=\frac{l}{2\alpha+l}\binom{2\alpha+l}{\alpha}\frac{l+1}{2(n-\alpha)-l+1}\binom{2(n-\alpha)-l+1}{n-\alpha-l}, \tag{27}$$

for $l=1,2,\ldots,n,\ \alpha=0,1,\ldots,n-1$.

PROOF. It needs the application of (6).

A comparison of (22) for $\mu=1$ with (27) establishes (2).

Finally, we consider the joint distribution of ψ and φ and state the following equivalence relation as a theorem.

THEOREM 6.

$$N(\psi=-l,\varphi=r)=N_c(\lambda'=l,\pi=r). \tag{28}$$

PROOF. It suffices to show that the number represented by the left hand side is given by (22). Indeed it is trivially so except for the second expression in (22).

Any path counted in $N(\psi=-l,\ \varphi=(\mu+1)\alpha-l)\ \alpha=l,\ l+1,\ldots,n$, which meets the line $x=\mu y-l$ for the first time at $(\mu\alpha-l,\alpha)$, must also meet for the first time the line $x=\mu y-\beta$, $0<\beta<l$, at a lattice point before crossing it. Thus

such a path consists of two segments; (i) from the origin to $(\mu\alpha - l, \alpha)$ with the required restrictions, and (ii) from $(\mu\alpha - l, \alpha)$ to $(\mu n, n)$ not crossing the line $x = x = \mu y - l$. Obviously the number in (ii) is

$$\frac{l+1}{(\mu+1)(n-\alpha)+l+1}\binom{(\mu+1)(n-\alpha)+l+1}{n-\alpha}, \quad l = 1, \ldots, n. \tag{29}$$

In order to complete the proof, we require the number in (i) to be

$$\frac{\mu l}{(\mu+1)\alpha - l}\binom{(\mu+1)\alpha - l}{\alpha - l}, \quad l = 1, \ldots, n. \tag{30}$$

When $l=1$ the required number is equivalent to the number of paths from $(0, 0)$ to $(\mu\alpha - 1, \alpha - 1)$ never crossing $x = \mu y$, the expression for which is

$$\frac{\mu}{(\mu+1)\alpha - 1}\binom{(\mu+1)\alpha - 1}{\alpha - 1}.$$

This verifies (30) for $l=1$. Then apply induction on l. The restricted path from $(0, 0)$ to $(\mu\alpha - l - 1, \alpha)$ can be split into two parts of which the first one reaches the line $x = \mu y - l$ at $(\mu y_0 - l, y_0)$, $y_0 = l, l+1, \ldots, \alpha - 1$, with the desired restriction. Thus the total number of restricted paths from $(0, 0)$ to $(\mu\alpha - l - 1, \alpha)$ is obtained as

$$\sum_{y_0=l}^{\alpha-1} \frac{\mu l}{(\mu+1)y_0 - l}\binom{(\mu+1)y_0 - l}{y_0 - l} \frac{\mu}{(\mu+1)(\alpha - y_0) - 1}\binom{(\mu+1)(\alpha - y_0) - 1}{\alpha - y_0 - 1} =$$

$$= \frac{\mu(l+1)}{(\mu+1)\alpha - l - 1}\binom{(\mu+1)\alpha - l - 1}{\alpha - l - 1}.$$

We conclude with the following remark: When $\mu = 1$, there is an 1.1 correspondence between χ and ψ. In addition, the restrictions involved in the definition of φ are trivially satisfied by $S_{1,n}$, which touches the line $x = y - l$ at $(y - \alpha, \alpha)$ for the first time. Therefore

$$N(\psi = -l, \varphi = r) = N(\chi = l, \varrho = r) = N(\lambda' = l, \pi = r).$$

REFERENCES

[1] CSÁKI, E. and VINCZE, I.: On Some Combinatorial Relations Concerning the Symmetric Random Walk, *Acta Sci. Math.* **24** (1963) 231—235.

[2] GOULD, H. W.: Some Generalisations of Vandermonde's Convolution, *Amer. Math. Monthly* **63** (1956) 84—91.

[3] MOHANTY, S. G. and NARAYANA, T. V.: Some Properties of Compositions and their Applications to Probability and Statistics, I, *Biometrische Zeitschrift*, **3** (1961) 252.

[4] MOHANTY, S. G., On Some Generalization of a Restricted Random Walk, *Studia Sci. Math. Hungar.* 3 (1968) 225—241.

[5] KOROLJUK, V. S.: On the Discrepancy of Empirical Distribution Functions for the case of two independent Samples, *Izv. Akad. Nauk. USSR*, **1** (1955) 154—158.

[6] BLACKMAN, I.: An Extension of Kolmogorov Distribution, *Annal. Math. Stat.*, **27** (1956) 513.

[7] BLACKMAN, I.: Correction to an Extension of Kolmogorov Distribution, *Annal. Math. Stat.*, **29** (1958) 318.

Indian Institute of Technology, New Delhi, India

(Received August 9, 1967.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 109—125.

DIE LÖSUNG EINES WÄRMELEITUNGSPROBLEMS FÜR SYSTEME BESTEHEND AUS STÄBEN UND WÄRMEBEHÄLTERN

von

P. KOSIK

Einleitung

Die Lösung des Wärmeleitungsproblems im eindimensionalen Fall mit der ersten, zweiten und dritten Randbedingung ist bekannt. Bei der ersten Randbedingung ist die Temperatur, bei der zweiten ihre Ableitung in der Längsrichtung des Stabes, bei der dritten eine lineare Kombination dieser beiden Größen als Funktion der Zeit am Ende des Stabes vorgeschrieben. Man kommt zur dritten Randbedingung, wenn der Stab sich einem Körper mit bekannter Temperatur anschließt. Ist aber die Temperatur des anschließenden Körpers nicht bekannt, nur die diesem Körper zugeführte Wärmemenge $Q(t)$ cal/sec, so genügt zur Lösung des entsprechenden Wärmeleitungsproblems die Heranziehung der dritten Randbedingung nicht mehr.

In der Arbeit [1] werden die Bedingungen, die in diesem Falle zwischen dem Körper und dem Stab bestehen, „zusammengesetzte Randbedingungen" genannt. Mehrere Verfasser haben sich mit solchen Randbedingungen befaßt (siehe das Literaturverzeichnis).

FREUD untersucht in [1] das Problem, wenn einem Behälter von endlicher, nicht vernachlässigbarer Wärmekapazität sich ein Stab von endlicher oder unendlicher Länge anschließt, und dem Behälter die Wärmemenge $Q(t)$ cal/sec zugeführt wird. Im Falle eines endlichen Stabes wird am freien Ende entweder die Temperatur, oder ihre Ableitung in der Längsrichtung des Stabes als Funktion der Zeit vorgeschrieben. Die Anfangsbedingung muß nicht unbedingt homogen sein. Der Verfasser beweist in dieser Arbeit auch die Eindeutigkeit der Lösung.

ADLER untersucht im ersten Teil seiner Arbeit [2] das Wärmeleitungsproblem für den Fall, wenn einem Wärmebehälter zwei Stäbe von unendlicher Länge angeschlossen sind, und zu Beginn des Wärmeleitungsprozesses die Temperatur der Stäbe gleich null ist. Der zweite Teil der Arbeit beschäftigt sich mit dem Falle, wenn sich beide Enden eines Stabes von endlicher Länge je einem Wärmebehälter anschliessen, deren $Q_1(t)$ bzw. $Q_2(t)$ cal/sec Wärmemengen zugeführt werden. Auch hier beträgt die Anfangstemperatur des Stabes Null.

In der Arbeit [3] wird jenes Problem untersucht, wo einem punktförmigen Wärmebehälter mit nicht vernachlässigbarer Wärmekapazität beliebig viele — endliche oder unendliche — Stäbe angeschlossen sind. Unter den Stäben können solche vorkommen, an deren freien Enden die Temperaturwerte, und auch solche, an deren freien Enden die Ableitungen in der Längsrichtung des Stabes vorgeschrieben sind.

In der vorliegenden Arbeit werden solche Systeme von Stäben und Wärmebehältern untersucht, wo von jedem Wärmebehälter ausgehend jeder andere Wärmebehälter auf genau einem Wege erreicht werden kann, wobei diese Wege höchstens

einmal denselben Stab entlang führen und höchstens einmal denselben Wärmebehälter überqueren dürfen. Ferner können sich den Behältern auch solche Stabe anschließen, an deren anderen Enden die Temperatur oder ihre Ableitung in der Längsrichtung des Stabes vorgeschrieben ist.

Wir erwähnen auch den Fall, in dem das System geschlossen ist in dem Sinne, daß aus einem jeden Wärmebehälter genau zwei Stäbe zu anderen Wärmebehältern führen (Abbildung 2). Auch hier dürfen aus den Behältern Stäbe ausgehen, an deren anderem Ende die Temperatur oder das Temperaturgefälle gegeben ist. In beiden Fällen können sich unter den Stäben, die nur mit einem Ende einem Wärmebehälter angeschlossen sind, auch solche von unendlicher Länge befinden.

Wir erwähnen auch Systeme, in welchen zwei Wärmebehälter durch mehrere Stäbe verbunden sind. Hier können aus den Behältern ebenfalls beliebig viele Stäbe mit einem freien Ende ausgehen. An den freien Enden werden die oben angeführten Bedingungen festgelegt.

§ 1.

Betrachten wir eine aus n Elementen bestehende Menge von Wärmebehältern mit endlichen, nicht vernachläßigbaren Wärmekapazitäten und irgendeine gegebene Menge von Stäben. Man sagt von einem Stab, daß er vom Typus $\mathscr{A}$ ist, wenn er

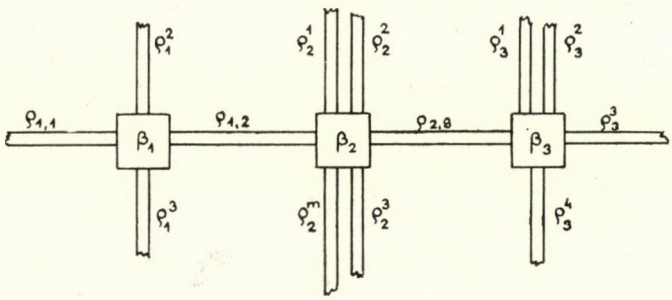


Abb. 1

an beiden Enden je einem Wärmebehälter angeschlossen ist, und man nennt ihn vom Typus $\mathscr{B}$, wenn er nur an einem Ende aus einem Wärmebehälter ausgeht.

Betrachten wir nun ein zusammenhängendes System von Stäben und Wärmebehältern, in dem einem Wärmebehälter höchstens zwei Stäbe vom Typus $\mathscr{A}$ angeschlossen sind, und es existiere in diesem System mindestens ein Behälter, welchem sich genau ein Stab vom Typus $\mathscr{A}$ anschließt. Wählen wir einen solchen Wärmebehälter aus, und bezeichnen wir diesen mit β_1. Von β_1 ausgehend, sind — die Stäbe vom Typus $\mathscr{A}$ entlang — alle Wärmebehälter erreichbar. Ihre Indizes sollen die Reihenfolge des Erreichens anzeigen. Denjenigen Stab, der die Behälter β_i und β_{i+1} verbindet, bezeichnen wir mit $\varrho_{i,i+1}$, seine Länge mit $l_{i,i+1}$. Die Stäbe vom Typus $\mathscr{B}$, die dem Wärmebehälter β_i angeschlossen sind, bezeichnen wir mit ϱ_i^j $(j=1, 2, \ldots, m_{i-1}, m_i)$, ihre Länge mit l_i^j (siehe Abbildung 1).

Nehmen wir an, daß den Behältern β_i die Wärmemengen $Q_i(t)$ cal/sec zugeführt werden $(1 \leqq i \leqq n)$. Gesucht wird die Temperaturverteilung in dem System unter der Annahme, daß die Temperatur $U_i(t)$ der Wärmebehälter β_i nur eine

Funktion der Zeit sei; ferner daß die Temperatur in einem Punkte irgendeines Stabes außer der Zeit nur von der Koordinate x an der Längs richtung des Stabes abhängt. Im Zeitpunkt $t=0$ ist die Temperatur der Behälter und der Stäbe nicht unbedingt gleich Null.

Die mathematische Formulierung der obigen Aufgabe

Man richte die Stäbe derart, daß die Anfangspunkte der Stäbe vom Typus $\mathscr{A}$ beim Wärmebehälter mit dem kleineren Index, und der Stäbe vom Typ $\mathscr{B}$ beim Wärmebehälter seien.

Bestimmen wir jenes Lösungssystem, welches bezüglich der Stäbe $\varrho_{i,i+1}$ die Gleichungen

$$V_{i,i+1_{xx}}(x,t) = a^2_{i,i+1} V_{i,i+1_t}(x,t) \qquad (1 \leqq i < n) \tag{1a}$$

$$V^j_{i_{xx}}(x,t) = (a^j_i)^2 V^j_{i_t}(x,t) \qquad (1 \leqq i \leqq n) \quad (1 \leqq j \leqq m_i) \tag{1b}$$

befriedigt. Durch $V_{i,i+1}(x,t)$, bzw. $V^j_i(x,t)$ wird die Temperatur im Stabe $\varrho_{i,i+1}$, bzw. ϱ^j_i in Punkte x zur Zeit t bezeichnet und $a_{\mu\nu}=\text{const.}$

Der Wärmeaustausch-Prozeß zwischen den Stäben und den Wärmebehältern läßt sich durch folgende Beziehungen beschreiben:

Für Stäbe vom Typus $\mathscr{A}$:

$$V_{i,i+1_x}(0,t) = -c_{i,i+1}[U_i(t) - V_{i,i+1}(0,t)] \qquad (1 \leqq i < n) \tag{2a}$$

$$V_{i,i+1_x}(l_{i,i+1},t) = c_{i+1,i}[U_{i+1}(t) - V_{i,i+1}(l_{i,i+1},t)] \qquad (1 \leqq i < n) \tag{2b}$$

$$C_{\mu,\nu} = \text{Const.}$$

Für Stäbe vom Typus $\mathscr{B}$:

$$V^j_{i_x}(0,t) = -c^j_i[U_i(t) - V^j_i(0,t)] \qquad (i = 1, 2, \ldots, n) \quad (j = 1, 2, \ldots, m_i) \tag{3}$$

$$c^\nu_\mu = \text{Const.}$$

Die Formeln (2) und (3) stellen das Newtonsche Wärmeübertragungsgesetz dar. Außerdem muß noch die sogenannte Wärmebilanz gültig sein:

$$Q_i(t) = A_i U_i'(t) + B_{i,i+1}[U_i(t) - V_{i,i+1}(0,t)] + B_{i,i-1}[U_i(t) - V_{i-1,i}(l_{i-1,i},t)] + \\ + \sum_{j=1}^{m_i} B^j_i[U_i(t) - V^j_i(0,t)] \qquad (1 < i < n)$$

und (4)

$$Q_i(t) = A_i U_i'(t) + \sum_{j=1}^{m_i} B^j_i[U_i(t) - V^j_i(0,t)] + \begin{cases} B_{i,i+1}[U_i(t) - V_{i,i+1}(0,t)] & (i=1) \\ B_{i,i-1}[U_i(t) - V_{i-1,i}(l_{i-1,i},t) & (i=n) \end{cases}$$

$$A_\nu, B_{\mu,\nu} \quad \text{und} \quad B^\nu_\mu = \text{Const.}$$

Durch die Beziehungen (4) wird ausgedrückt, daß aus der Wärmemenge $Q_i(t)$ die Menge $A_i[U_i'(t_1]$ zur Erwärmung des Wärmebehälters β_i verwendet wird, und die übrige Wärmemenge sich in den entsprechenden Stäben ausbreitet.

An den freien Enden der Stäbe vom Typus $\mathscr{B}$ haben wir:

$$V_{i_x}^j(l_i^j, t) = g_i^j(t) \qquad (1 \leqq j \leqq p_i), \tag{5}$$

$$V_i^j(l_i^j, t) = h_i^j(t) \qquad (p_i < j \leqq r_i), \tag{6}$$

wo $g_i^j(t)$ und $h_i^j(t)$ vorgegebene Funktionen von t sind.

Im Falle $l_i = \infty$ sei

$$\lim_{x \to \infty} V_i^j(x, t) e^{-\gamma x^2} = 0, \qquad (r_i < j \leqq m_i). \tag{7}$$

Im Anfangspunkt $t = 0$ gelten die folgenden Bedingungen:

$$\begin{aligned} &\text{für Stäbe vom Typus } \mathscr{A}\ V_{i,i+1}(x, 0) = f_{i,i+1}(x), \\ &\text{für Stäbe vom Typus } \mathscr{B}\text{: } V_i^j(x, 0) = f_i^j(x) \end{aligned} \tag{8}$$

(Die Funktionen $f_{i,i+1}(x)$ und $f_i^j(x)$ sind vorgegeben.)

Die Temperatur der Wärmebehälter β_i sei im Anfangspunkt $U_i(0)$.

Die Lösung der Aufgabe

Das Lösungssystem des obigen Problems wird als die Superposition von solchen Lösungssystemen hergestellt, die gewissen speziellen Bedingungen genügen. Im folgenden beschäftigen wir uns zunächst mit letzteren Lösungssystemen.

Betrachten wir das auf Abbildung 1. dargestellte — aus Stäben und Wärmebehältern bestehende — System, und benutzen wir die obigen Bezeichnungen. Es sei $k < n-1$, und es bezeichne P_k $(k = 1, 2, \ldots, n)$ diejenigen Lösungssysteme, die folgenden Bedingungen genügen:

Die Anfangsbedingungen sind im Falle von Stäben vom Typus $\mathscr{A}$:

$$V_{i,i+1}^{P_k}(x, 0) = \begin{cases} 0 & \text{für } i \neq k \\ f_{i,i+1}(x) & \text{für } i = k \end{cases} \tag{9a, 9b}$$

Im Falle von Stäben vom Typus $\mathscr{B}$:

$$V_i^{jP_k}(x, 0) = \begin{cases} 0 & \text{für } i \neq k;\ j = 1, 2, \ldots, m_i \\ f_i^j(x) & \text{für } i = k \end{cases} \tag{10}$$

Die Randbedingungen sind für Stäbe vom Typus $\mathscr{A}$

$$V_{i,i+1_x}^{P_k}(0, t) = 0, \qquad i \leqq k \tag{11a}$$

$$V_{i,i+1_x}^{P_k}(l_{i,i+1}, t) = 0, \qquad i \geqq k. \tag{11b}$$

Unter Einführung der Bezeichnung

$$U_i^{P_k}(t) = \begin{cases} V_{i,i+1}^{P_k}(0, t) & i \leqq k \\ V_{i-1,i}^{P_k}(l_{i-1,i}, t) & i > k \end{cases} \tag{12}$$

lassen sich die übriggebliebenen Randbedingungen bezüglich Stäben vom Typus $\mathscr{A}$ folgendermassen schreiben:

$$V_{i,i+1_x}^{P_k}(l_{i,i+1}, t) = c_{i+1,i}[U_{i+1}^{P_k}(t) - V_{i,i+1}^{P_k}(l_{i,i+1}, t)] \qquad i<k, \tag{13a}$$

$$V_{i,i+1_x}^{P_k}(0, t) = -c_{i,i+1}[U_i^{P_k}(t) - V_{i,i+1}^{P_k}(0, t)] \qquad i>k. \tag{13b}$$

Die Funktion $V_{k,k+1}^{P_k}$, wie wir es später sehen werden, erfüllt automatisch diese Bedingungen.

Die Randbedingungen bezüglich Stäben vom Typus $\mathscr{B}$ lauten:

$$V_{i_x}^{jP_k}(0, t) = -c_i^j[U_i^{P_k}(t) - V_i^{jP_k}(0, t)] \qquad (1 \leqq i \leqq n \text{ und } 1 \leqq j \leqq m_i), \tag{14}$$

Es sei $i=k$, dann sollen an den Enden $x=l_i^j$ die Bedingungen (5), (6) und (7) erfüllt sein.

Falls $i \neq k$, dann gelten

$$V_{i_x}^{jP_k}(l_i^j, t) = 0, \qquad 1 \leqq j \leqq p_i \tag{15a}$$

und

$$V_i^{jP_k}(l_i^j, t) = 0 \qquad p_i < j \leqq r_i.$$

Falls $k = n-1$, so sollen die (9) Anfangs- und (11) Randbedingungen bezüglich Stäben vom Typus $\mathscr{A}$ erfüllt sein. Bezüglich $V_{i,i+1}^{P_k}(l_{i,i+1}, t)$ sei (13a) erfüllt, $1 \leqq i < n-1$. Diese Bedingung ist für $i = n-1$ automatisch erfüllt.

Für die Stäbe vom Typus $\mathscr{B}$ sei

$$V_i^{jP_k}(x, 0) = \begin{cases} 0, & \text{für } i<k \\ f_i^j(x) & \text{für } i \geqq k, \quad j = 1, 2, \ldots, m_i. \end{cases} \tag{16}$$

Für Stäbe vom Typus $\mathscr{B}$ schreiben wir an den Enden $x=l_i^j$ die folgenden Randbedingungen vor. Es sei zuerst $i<k$. Dann sei

$$\begin{aligned} V_{i_x}^{jP_k}(l_i^j, t) &= 0 \quad \text{für} \quad 1 \leqq j \leqq p_i \\ V_i^{jP_k}(l_i^j, t) &= 0 \quad \text{für} \quad p_i < j \leqq r_i, \end{aligned} \tag{17}$$

und es seien

$$\begin{aligned} V_{i_x}^{jP_k}(l_i^j, t) &= g_i^j(t) \qquad 1 \leqq j \leqq p_i \\ V_i^{jP_k}(l_i^j, t) &= h_i^j(t) \qquad p_i < j \leqq r_i, \end{aligned} \tag{18}$$

falls $i \geqq k$. Im Falle von unendlich langen Stäben sei die Bedingung (7) für $i \geqq k$ erfüllt.

Bezeichnet man durch $Q_i^{P_k}(t)$ jene Funktionen, die man erhält, wenn man in die linke Seite der Formel (4) die entsprechenden $V_{i,i+1}^{P_k}(x, t)$ bzw. $V_i^{jP_k}(x, t)$ und $U_i^{P_k}(t)$ einsetzt, so genügt die Summe der Lösungssysteme P_k den Anfangsbedingungen (8) und mit den $U_i^{P_k}(t)$ und $Q_i^{P_k}(t)$ den Randbedingungen (2) und (3), bzw. (5), (6) und (7).

Man erhält das Lösungssystem der ursprünglichen Aufgabe, wenn man der Summe der Lösungssysteme P_k jenes Lösungssystem P_n superponiert, welches bezüglich aller Stäbe homogenen Anfangsbedingungen genügt, an den Enden $x=l_i^j$

der Stäbe vom Typus $\mathscr{B}$ die entsprechenden homogenen Randbedingung erfüllt und

$$U_i^{P_n}(0) = U_i(0) - \sum_{k=1}^{n-1} U_i^{P_k}(0),$$

$$Q_i^{P_n}(t) = Q_i(t) - \sum_{k=1}^{n-1} Q_i^{P_k}(t) \tag{19}$$

und so

$$V_{i,i+1}(x, t) = V_{i,i+1}^{P_n}(x, t) + \sum_{k=1}^{n-1} V_{i,i+1}^{P_n}(x, t),$$

(20) und

$$U_i(t) = U_i^{P_n}(t) + \sum_{k=1}^{n-1} U_i^{P_k}(t).$$

Die Bestimmung des Systems P_n

Die Bestimmung des Systems P_n wird mit Hilfe der Operatorenrechnung auf die Lösung eines algebraischen Gleichungssystems zurückgeführt. Es wird $V_{i,i+1}^{P_n}(x, t)$ in der Form

$$V_{i,i+1}^{P_n}(x, t) = \int_0^t \frac{1}{\sqrt{(t-\tau)}} \left(e^{-\frac{a_{i,i+1}^2 x^2}{4(t-\tau)}} w_{i1}(\tau) + e^{-\frac{a_{i,i+1}^2 (l_{i,i+1}-x)^2}{4(t-\tau)}} w_{i2}(\tau) \right) d\tau \tag{21}$$

gesucht. Diese Funktion befriedigt die entsprechende Differentialgleichung der Wärmeleitung und für $t=0$ die vorgeschreibene homogene Anfangsbedingung. Die Funktionen $V_i^{jP_n}(x, t)$ werden in der Form

$$V_i^{jP_n}(x, t) = \int_0^t (U_i^{P_n})'(t-\tau)[v_i^{jP_n}(x, \tau)]\, d\tau \tag{22}$$

dargestellt, wo $v_i^{jP_n}(x, t)$ eine solche Lösung der Gleichung (1) bezeichnet, welche für $t=0$ gleich Null ist, am Ende $x=l_i^j$ der entsprechenden homogenen Randbedingung genügt und

$$v_{ix}^{jP_n}(0, t) - c_i^j v_i^{jP_n}(0, t) = 1. \tag{23}$$

Die Darstellung von $V_i^{jP_n}(x, t)$ in der Form (22) wird durch das Duhamelsche Prinzip ermöglicht [4]. Diese Methode ist deshalb nützlich, weil die Anzahl der Stäbe vom Typus $\mathscr{B}$ weder die Zahl der Gleichungen, noch der Unbekannten erhöht.

Die Funktionen $v_i^{jP_n}(x, t)$ können bestimmt werden, indem man für $1 \leqq j \leqq p_i$ solche Lösungen $v_i^{jP_n}(x, t)$ der Gleichung (1) sucht, welche den Bedingungen

$$v_i^{jP_n^*}(x, 0) = \frac{1}{c_i^j},$$

$$v_i^{jP_n^*}(l_i^j, t) = 0 \tag{24}$$

$$v_i^{jP_n^*}(0, t) - c_i^j v_i^{jP_n^*}(0, t) = 0$$

genügen. Für $p \leqq j \leqq r_i$, d. h. $V_i^{jP_n}(l_i^j, t) = 0$ muß die Funktion $v_i^{jP_n^*}(x, t)$ folgende Bedingungen erfüllen:

$$v_i^{jP_n^*}(x, 0) = -\frac{x}{c_i^j l_i^j + 1} + \frac{l_i^j}{c_i^j l_i^j + 1},$$

$$v_i^{jP_n^*}(l_i^j, t) = 0 \tag{25}$$

$$v_i^{jP_n^*}(0, t) - c_i^j v_i^{jP_n^*}(0, t) = 0,$$

Sind die Funktionen $v_i^{jP_n^*}(x, t)$ bekannt, so lassen sich die Funktionen $v_i^{jP_n}(x, t)$ leicht darstellen,

$$v_i^{jP_n}(x, t) = \begin{cases} -c_i^j \int_0^t (U_i^{P_n})'(t-\tau)\left[p_i^j(x, t) - \frac{1}{c_i^j}\right] & (1 \leqq j \leqq p_i) \\ -c_i^j \int_0^t (U_i^{P_n})'(t-\tau)\left[p_i^j(x, t) + \frac{x - l_i^j}{l_i^j c_i^j + 1}\right] & (p_i < j \leqq r_i) \end{cases} \tag{26}$$

und im Falle $1 \leqq j \leqq p_i$

$$p_i^j(x, t) = \sum_{k=1}^{\infty} A_{i,j}^k \left(\cos \mu_{i,j}^k x + \frac{c_i^j}{\mu_{i,j}^k} \sin \mu_{i,j}^k x\right) \exp\left(-\frac{(\mu_{i,j}^k)^2}{a_{i,j}^2} t\right),$$

wobei $\mu_{i,j}^k$ die k-te positive Wurzel der Gleichung

$$\operatorname{tg} \mu l_i^j = \frac{c_i^j}{\mu}$$

bezeichnet, und

$$A_{i,j}^k = \frac{\int_0^{l_i^j} \frac{1}{c_i^j}\left[\cos \mu_{i,j}^k x + \frac{c_i^j}{\mu_{i,j}^k} \sin \mu_{i,j}^k x\right] dx}{\left(\int_0^t \left[\cos \mu_{i,j}^k x \frac{c_i^j}{\mu_{i,j}^k} \sin \mu_{i,j}^k x\right]^2 dx\right)^{1/2}}.$$

Falls $p_i < j \leqq r_i$, so gilt

$$p_i^j(x, t) = \sum_{k=1}^{\infty} B_{i,j}^k \left(\cos \nu_{i,j}^k x + \frac{c_i^j}{\mu_{i,j}^k} \sin \nu_{i,j}^k x\right) \exp\left(-\frac{(\nu_{i,j}^k)^2}{x_i^2} t\right),$$

wobei $\nu_{i,j}^k$ die k-te positive Wurzel der Gleichung

$$\operatorname{tg} \nu l_i^j = -\frac{\nu}{c_i^j}$$

bezeichnet und

$$B_{i,j}^k = \frac{\int_0^{l_i^j} \left(\frac{1}{l_i^j c_i^j + 1}\right)\left(\cos \nu_{i,j}^k x + \frac{c_i^j}{\nu_{i,j}^k} \sin \nu_{i,j}^k x\right) dx}{\left[\int_0^{l_i^j} \left(\cos \nu_{i,j}^k x + \frac{c_i^j}{\nu_{i,j}^k} \sin \nu_{i,j}^k x\right)^2 dx\right]^{1/2}}.$$

Falls $r_i < j \leqq m_i$, so ist

$$V_i^{jP_n}(x,t) = \frac{1}{c_i^j} \exp\left(c_i^j x + \frac{c_i^{j2}}{a_i^{j2}} t\right) \operatorname{erfc}\left(\frac{a_i^j x}{2\sqrt{t}} + \frac{c_i^j}{a_i^j}\sqrt{t}\right) - \frac{1}{c_i^j} \operatorname{erfc}\left(\frac{a_i x}{2\sqrt{t}}\right)$$

(siehe [3]).

Falls $j > r_i$, d. h. $l_i^j = \infty$, so suchen wir $v_i^{jP_n}(x,t)$ in der Form

$$v_i^{jP_n}(x,t) = \int_0^t \frac{1}{\sqrt{(t-\tau)}} \exp - \left(\frac{a_i^j x^2}{4(t-\tau)}\right) w_{i,j}(\tau)\, d\tau. \tag{27}$$

Unter Benutzung der Bedingung (23) und für $v_i^{jP_n}(x,t)$ die Operatorschreibweise gebrauchend, erhält man für $w_{i,j}(t)$ eine algebraische Gleichung. In Kenntnis von $w_{i,j}(t)$ ergibt sich $v_i^{jP_n}(x,t)$ nach Durchführung der angezeigten Integrierung.

Durch Gebrauch der Bedingungen (2), (3) und (4) erhält man für die Bestimmung der unbekannten Funktionen $w_{i1}(t)$, $w_{i2}(t)$ und $U_i^{P_n}(t)$ ein System bestehend aus $3n-2$ Integrodifferentialgleichungen. Mit Hilfe der Operatorenrechnung läßt sich dieses Integrodifferentialgleichungssystem auf ein lineares algebraisches Gleichungssystem zurückführen.

Die Operatorenform von $V_{i,i+1}^{P_n}(x,t)$ ist

$$\sqrt{\frac{\pi}{s}}\, e^{-a_{i,i+1} x\sqrt{s}}\, w_{i1} + \sqrt{\frac{\pi}{s}}\, e^{-a_{i,i+1}(l_{i,i+1}-x)} w_{i2} \tag{28}$$

Nach Bestimmung der unbekannten Funktionen $w_{i1}(t)$, $w_{i2}(t)$ und $U_i^{P_n}(t)$ muß man zur Berechnung der Funktionen $V_{i,i+1}(x,t)$ bzw. $V_i^{jP_n}(x,t)$ noch die Integrationen (21) und (22) durchführen.

Nun wollen wir eine mögliche Bestimmung der Systeme P_k anführen. Dazu wird zuerst die Funktion $V_{k,k+1}^{P_k}(x,t)$ berechnet, z. B. mit der Fourierschen Methode.

Da

$$V_{k,k+1_x}^{P_k}(0,t) = V_{k,k+1_x}^{P_k}(l_{k,k+1}, t) = 0,$$

so sind die Eigenfunktionen $\cos \dfrac{n\pi}{l_{k,k+1}} x$, $n = 1, 2, \ldots$, und daher

$$V_{k,k+1}^{P_k}(x,t) = \frac{1}{l_{k,k+1}} \int_0^{l_{k,k+1}} f_{k,k+1}(x)\, dx + \sum_{n=1}^{\infty} \frac{2}{l_{k,k+1}} \int_0^{l_{k,k+1}} f_{k,k+1}(x) \cdot$$

$$\cdot \cos \frac{n\pi}{l_{k,k+1}} x\, dx \cdot \cos \frac{n\pi}{l_{k,k+1}} x \exp\left(-\frac{n^2\pi^2}{a_{k,k+1}^2 l_{k,k+1}^2} t\right).$$

Da $U_k^{P_k}(t) = V_{k,k+1}^{P_k}(0,t)$ und $U_{k+1}^{P_k}(t) = V_{k,k+1}^{P_k}(l_{k,k+1}, t)$, so sind auch die Funktionen $U_k^{P_k}(t)$ und $U_{k+1}^{P_k}(t)$ bekannt.

Ist $i \neq k$, so läßt sich $V_{i,i+1}^{P_k}(x,t)$ ebenfalls mit Hilfe des Duhamelschen Prinzips bestimmen. Betrachten wir zunächst den Fall $k > 1$ und $i = k-1$. In diesem Falle genügt $V_{i,i+1}^{P_k}(x,t)$ den Randbedingungen (11a) und (13a), mit homogenen Anfangsbedingungen. Es wird jene Funktion $v_{i,i+1}^{P_k}(x,t)$ gesucht, welche die Dif-

ferentialgleichung der Wärmeleitung befriedigt.

$$v_{i,i+1}^{P_k}(x,0) = \frac{-1}{c_{i+1,i}}$$

$$\left.\begin{matrix} v_{i,i+1_x}^{P_k}(0,t) \\ v_{i,i+1_x}^{P_k}(l_{i,i+1},t) + c_{i+1,i} v_{i,i+1}^{P_k}(l_{i,i+1},t) \end{matrix}\right\} = 0 \tag{29}$$

Die zu diesem Randwertproblem gehörenden Eigenfunktionen sind

$$\varphi_k = a_k \cos(\mu_k x),$$

wobei μ_k die k-te positive Wurzel der Gleichung

$$\frac{c_{i+1,i}}{\mu} - \operatorname{tg}(l_{i,i+1}\mu) = 0 \tag{30}$$

bezeichnet. Daher gelten

$$v_{i,i+1}^{P_k}(x,t) = \frac{2}{c_{i+1,i}} \sum_{k=1}^{\infty} \frac{\sin(\mu_k l_{i,i+1})}{2\mu_k l_{i,i+1} + \sin(\mu_k l_{i,i+1})} \cos(\mu_k x) e^{-\frac{\mu_k^2}{a_{i,i+1}^2} t} \tag{31}$$

$$V_{i,i+1}^{P_k}(x,t) = c_{i+1,i} \int_0^t U_k^{P_k}(t-\tau) \left[\frac{1}{c_{i+1,i}} + v_{i,i+1}^{P_k}(x,\tau) \right] d\tau. \tag{32}$$

Da $U_i^{P_k}(t) = V_{i,i+1}^{P_k}(0,t)$ $i \leqq k$, erhält man bei der Bestimmung von $V_{i,i+1}^{P_k}(x,t)$ zugleich auch $U_i^{P_k}(t)$. Durch Fortsetzung dieses Verfahrens lassen sich für $i<k$ alle Funktionen $V_{i,i+1}^{P_k}(x,t)$ und $U_i^{P_k}(x,t)$ berechnen.

Es seien nun $k < n-1$, und $i = k+1$. In diesem Falle muß $V_{i,i+1}^{P_k}(x,t)$ die entsprechenden Bedingungen (9) und (11b) bzw. (13b) erfüllen. Diese Aufgabe wurde — abgesehen von Konstanten bei der Lösung des Systems P_n für $1<j<p_i$ bereits besprochen.

Dem Fall $i<k$ ähnlich, lassen sich auch weitere $V_{i,i+1}(x,t)$-s und damit die Funktionen $U_{i+1}(t)$ bestimmen. Jene Funktionen $V_i^{jP_k}(x,t)$, bei denen im Falle $k < n-1$, $i \neq k$ oder im Falle $k = n-1$, $i<k$, lassen sich — ähnlich wie bei der Untersuchung des Systems P_n — mit Hilfe des Duhamelschen Prinzips bestimmen. Ist die Anfangsbedingung inhomogen ($k=i$, oder — falls $k = n-1$ und $i=k$ oder $i = k+1$), so wird jene Lösung $v_i^{jP_n^*}(x,t)$ der Differentialgleichung der Wärmeleitung gesucht, welche die inhomogene Anfangsbedingung erfüllt, d. h.

$$v_i^{jP_n^*}(x,0) = f_i^j(x), \tag{33}$$

am Rande $x=0$:

$$v_{i_x}^{jP_k^*}(0,t) - c_i^j v_i^{jP_n^*}(0,t) = 0 \tag{34}$$

$$v_{i_x}^{jP_k^*}(l_i^j,t) = g_i^j(t) \quad \text{für} \quad 1 \leqq j \leqq p_i$$

oder

$$v_i^{jP_k^*}(l_i^j,t) = h_i^j(t) \quad \text{für} \quad p_i < j \leqq r_i$$

Dieser Lösung muß eine solche Lösung $v_i^{jP_k}$ der Gleichung (1) superponiert werden, welche einer homogenen Anfangsbedingung genügt, und für $x=l_i^j$ die Gleichheiten

$$\left.\begin{matrix} v_{i_x}^{jP_k}(l_i^j,t) \\ v_i^{jP_k}(l_i^j,t) \end{matrix}\right\} = 0, \qquad \begin{matrix} 1 \leqq j \leqq p_i \\ p_i < j \leqq r_i \end{matrix} \quad \text{befriedigt,} \tag{35}$$

ferner am Ende $x=0$

$$v_{ix}^{jP_k}(0,t) = -c_i^j[U_i^{P_k}(t) - v_i^{jP_k}(0,t)] \quad \text{ist.} \tag{36}$$

Hiermit sind alle Systeme P_k bestimmt.

§ 2. Die Lösung im Falle eines geschlossenen Systems

Bilden die Stäbe vom Typus $\mathscr{A}$ ein geschlossenes System, in dem Sinne, daß sich einem jeden Wärmebehälter genau zwei Stäbe vom Typus $\mathscr{A}$ anschließen (siehe Abbildung 2), so ist die Lösung im wesentlichen der in § 1 besprochenen ähnlich.

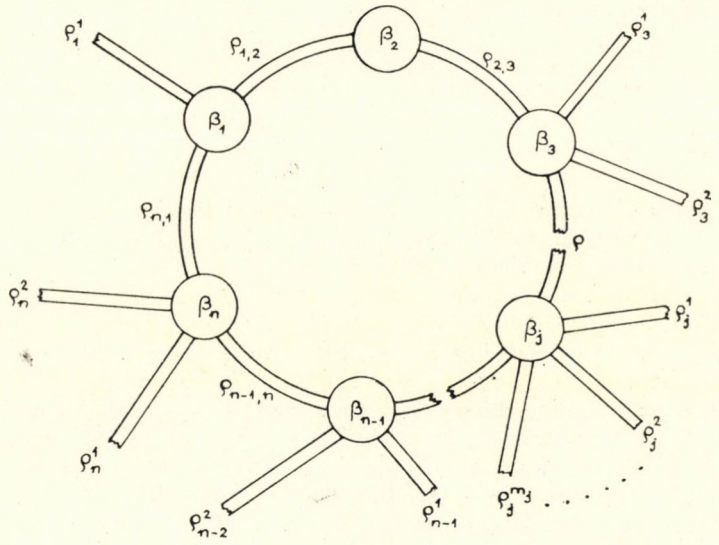


Abb. 2

Bezeichne man einen der Wärmebehälter mit β_1, und den über einem ihm angeschlossenen Stab vom Typus $\mathscr{A}$ erreichbaren anderen Behälter mit β_2. Die Bewegungsrichtung beibehaltend, werden die Behälter in der Reihenfolge ihres Erreichens mit Indizes versehen. Die Wärmebehälter β_1 und β_2 sind durch den Stab vom Typus $\mathscr{A}$ $\varrho_{1,2}$ verbunden, die Behälter β_2 und β_3 durch $\varrho_{2,3}$, ... die Behälter β_i, β_{i+1} durch $\varrho_{i,i+1}$. Hierbei wird mit $i+1$ die Indexziffer jenes Wärmebehälters bezeichnet, dem der i-te Behälter „vorangeht". So bedeutet z. B. im Falle $i=n$ $\varrho_{i,i+1}$ jenen Stab, der die Wärmebehälter β_n und β_1 verbindet. Die Temperatur des Stabes $\varrho_{i,i+1}$ vom Typus $\mathscr{A}$ sei mit $V_{i,i+1}(x,t)$ bezeichnet. Es sei die Temperatur des Behälters β_i gleich $U_i(t)$ $(i=1, 2, \ldots, n)$, und es sei die dem Wärmebehälter β_i zugeführte gesamte Wärmemenge $Q_i(t)$ cal/sec. Dem Behälter β_i können sich auch hier Stäbe vom Typus $\mathscr{B}$ anschließen, bezüglich denen die im vorigen Paragraphen angegebenen Bedingungen und Bezeichnungen gültig bleiben.

Es wird jenes Lösungssystem gesucht, in dem die Temperaturfunktionen der Stäbe vom Typus $\mathscr{A}$ die Differentialgleichung der Wärmeleitung (1a), vom Typus $\mathscr{B}$ (1b) befriedigen, unter folgenden Randbedingungen:

Im Falle von Stäben vom Typus $\mathscr{A}$ ist das Wärmeübertragungsgesetz:

$$V_{i,i+1_x}(0, t) = -c_{i,i+1}[U_i(t) - V_{i,i+1}(0, t)] \tag{2a}$$

$$V_{i,i+1_x}(l_{i,i+1}, t) = c_{i+1,i}[U_{i+1}(t) - V_{i,i+1}(l_{i,i+1}, t)]. \tag{2b}$$

Im Falle von Stäben vom Typus $\mathscr{B}$ wird das Wärmeübertragungsgesetz zwischen den Stäben und den Behältern β_i durch (3) angegeben.

Die Wärmebilanz kann in folgender Gestalt geschrieben werden:

$$Q_i(t) = A_i U_i'(t) + B_{i,i+1}[U_i(t) - V_{i,i+1}(0, t)] + B_{i,i-1}[U_i(t) - V_{i-1,i}(l_{i-1,i}, t)] + \tag{37}$$

$$+ \sum_{j=1}^{m_i} B_i^j [U_i(t) - V_i^j(0, t)].$$

Ähnlich wie im § 1, kann auch hier das vollständige Lösungssystem als die Summe von solchen Lösungssystemen dargestellt werden, welche gewissen speziellen Bedingungen genügen.

Bezeichne man mit P_k jenes Lösungssystem, in dem jeder Stab die Anfangstemperatur Null hat, mit Ausnahme von $\varrho_{k,k+1}$, dessen Anfangstemperatur

$$V_{k,k+1}^{P_k}(x, 0) = f_{k,k+1}(x)$$

sei und es sei

$$V_{k,k+1_x}^{P_k}(0, t) = V_{k,k+1_x}^{P_k}(l_{k,k+1}, t) = 0.$$

Bezeichne man mit $U_k^{P_k}(t)$, bzw. mit $U_{k+1}^{P_k}(t)$ die Funktionen $V_{k,k+1}^{P_k}(0, t)$ bzw. $V_{k,k+1}^{P_k}(l_{k\,k+1}, t)$, $(k = 1, 2, \ldots, n)$.

Ist $i = k+1$, so sei

$$V_{i,i+1_x}^{P_k}(0, t) = -c_{i,i+1}[U_i^{P_k}(t) - V_{i,i+1}^{P_k}(0, t)]$$

und

$$V_{i,i+1_x}^{P_k}(l_{i,i+1}, t) = 0.$$

Führen wir die Bezeichnung

$$V_{i,i+1}^{P_k}(l_{i,i+1}, t) = U_{i+1}^{P_k}(t)$$

ein; bezüglich $V_{i+1,i+2}^{P_k}(x, t)$ erhält man unter ähnlichen Bedingungen für $x = 0$, bzw. $x = l_{i+1,i+2}$ die Funktionen $V_{i+1,i+2}^{P_k}(x, t)$ bzw. $U_{i+2}^{P_k}(t)$. (Selbstverständlich zeigen hier die Indizes i, $i+1$, $i+2$ die Richtung der Fortbewegung an). Wenn i die Zahl $k-2$ erreicht, so hat man durch Bestimmung von $V_{i,i+1}^{P_k}(x, t)$ alle zu P_k gehörenden Funktionen $U_i^{P_k}(t)$ erhalten.

Ist $i = k-1$, so muss $V_{i,i+1}^{P_k}(x, t)$ folgenden Randbedingungen genügen:

$$V_{i,i+1_x}^{P_k}(0, t) = -c_{i,i+1}[U_i^{P_k}(t) - V_{i,i+1}^{P_k}(0, t) \tag{38}$$

und

$$V_{i,i+1_x}^{P_k}(l_{i,i+1}, t) = c_{i+1,i}[U_{i+1}^{P_k}(t) - V_{i,i+1}^{P_k}(l_{i,i+1}, t),$$

In diesem Falle kann die Funktion $V_{i,i+1}^{P_k}(x, t)$ in ähnlicher Form wie (21) gesucht werden, und auch ihre Bestimmung ist der dort beschriebenen identisch.

Bezüglich der Stäbe vom Typus $\mathscr{B}$ werden im System P_k die folgenden Bedingungen vorgeschrieben. Falls $i=k$, so seien

$$V_i^{jP_k}(x, 0) = f_i^j(x) \qquad (1 \leqq j \leqq m_i)$$

$$V_i^{jP_k}(l_i^j, t) = g_i^j(t) \qquad (1 \leqq j \leqq p_i)$$

$$V_i^{jP_k}(l_i^j, t) = h_i^j(t) \qquad (p_i < j \leqq r_i).$$

Falls $j > r_i$, so gilt für die Lösungsfunktion die für unendliche Stäbe festgelegte Bedingung.

Falls $i \neq k$, so sollen für die Stäbe vom Typus $\mathscr{B}$ die entsprechenden homogenen Anfangs- bzw. Randbedingungen bestehen. Die Funktionen $V_{i,i+1}^{P_k}(x, t)$ bzw. $V_i^{jP_k}(x, t)$ der Systeme P_k genügen den Differentialgleichungen der Wärmeleitung mit geeignetem Wärmeleitungskoeffizienten und erfüllen die Randbedingungen (2a), (2b) und (3) mit den Funktionen $U_i^{P_k}(t)$.

Bezeichnet man mit $Q_i^{P_k}(t)$ jene Funktion, die erhalten wird, wenn man in die linke Seite von (4) die Funktionen $V_{i,i+1}^{P_k}(0, t)$, $V_{i-1,i}^{P_k}(l_{i-1,i}, t)$ und $V_{i,j}^{P_k}(0, t)$, sowie $U_i^{P_k}(t)$ einsetzt, so genügt die Summe der entsprechenden Funktionen der Systeme P_k den vorgeschriebenen Anfangsbedingungen, und mit den Funktionen $U_i^{P_k}(t)$ bzw. $Q_i^{P_k}(t)$ den Bedingungen (2), (3), (4) und an den Enden $x = l_i$ der Stäbe vom Typus $\mathscr{B}$ den vorgeschriebenen Randbedingungen.

Das vollständige Lösungssystem wird erhalten, wenn man zur Summe der Funktionen der Systeme P_k die entsprechenden Funktionen jenes Lösungssystem P addiert, welches für alle Stäbe der homogenen Anfangsbedingung, und für die Stäbe vom Typus $\mathscr{B}$ an den Enden $x = l_i^j$ den entsprechenden homogenen Randbedingungen genügt:

$$U_i^P(0) = U_i(0) - \sum_{k=1}^{n} U_i^{P_k}(0) \tag{39}$$

und

$$Q_i^P(t) = Q_i(t) - \sum_{k=1}^{n} Q_i^{P_k}(t).$$

Die Bestimmung dieses Lösungssystems kann — ähnlich wie die des Lösungssystems P_n in § 1 — unter Benutzung des Duhamelschen Prinzips und der Operatorrechnung im Falle von n Wärmebehältern auf ein lineares algebraisches Gleichungssystem mit $3n$ Gleichungen und $3n$ Unbekannten zurückgeführt werden.

Die Funktionen $V_{i,i+1}^{P}(x, t)$ bzw. $V_i^{jP}(x, t)$ können in ähnlicher Gestalt gesucht werden, wie diejenigen vom System der P_n in § 1 (siehe (21), (22)).

§ 3

Schließlich betrachten wir den Fall, wenn zwei Wärmebehälter β_1 und β_2 durch mehrere Stäbe $\varrho_1, \varrho_2, \ldots, \varrho_n$ verbunden sind. Ähnlich wie bisher können den Wärmebehältern auch Stäbe vom Typus $\mathscr{B}$ angeschlossen werden. Die Länge des Stabes ϱ_I sei l_I, ihre Temperatur $V_I(x, t)$ $(I = 1, 2, \ldots, n)$, der Anfangspunkt

$x=0$ sei das sich an den Behälter β_1 ausschließende Ende. Die Funktion $V_I(x,t)$ sei für $t=0$ $f_I(x)$ $(I=1,2,\ldots,n)$, Die Bezeichnungen bezüglich der Stäbe vom Typus $\mathscr{B}$ seien wie vorhin, d. h. die Temperatur der zum Behälter β_1 führenden Stäbe der Reihe nach $V_{i,1}$, $V_{i,2}, \ldots, V_{i,p_i}, \ldots, V_{i,r_i}, \ldots, V_{i,m_i}$ $(i=1,2)$, mit entsprechender Randbedingung an der Stelle $x=l_{i,j}$. Die Temperatur des Behälters β_i wird mit $U_i(t)$ $(i=1,2)$ und die diesem Behälter zugeführte Wärmemenge mit $Q_i(t)$ cal/sec bezeichnet. Die Anfangstemperatur der Wärmebehälter sei $U_i(0)$.

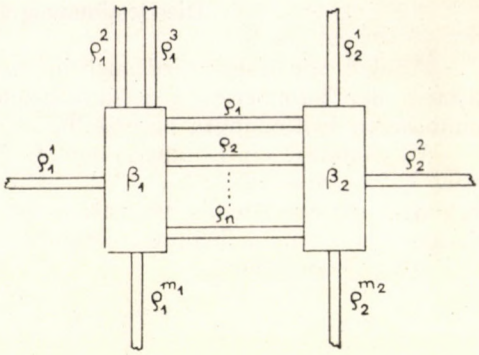


Abb. 3

Es wird jenes — aus dem Funktionen $V_1(x,t)$, $V_2(x,t), \ldots, V_n(x,t), V_{i,1}(x,t), \ldots, V_{i,m_i}(x,t)$ $(i=1,2)$ und $U_i(t)$ $(i=1,2)$ bestehendes — Lösungssystem gesucht, welches folgenden Bedingungen genügt: Jedes $V_I(x,t)$ und $V_i^j(x,t)$ genügt der Differentialgleichung

$$V_{I_{xx}}(x,t) = a_I^2 V_{I_t}(x,t) \qquad (I=1,2,\ldots,n)$$

bzw.

$$V_{i_{xx}}^j(x,t) = (a_i^j)^2 V_{i_t}^j(x,t) \qquad (i=1,2;\ j=1,2,\ldots,m_i)$$

mit folgenden Anfangs- und Randbedingungen:

$$\begin{aligned} &V_I(x,t) = f_I(x) && (I=1,2,\ldots,n)\\ &V_i^j(x,0) = f_i^j(x) && (i=1,2;\ j=1,2,\ldots,m_i)\\ &V_{i_x}^j(l_i^j,t) = g_i^j(t) && (i=1,2;\ j=1,2,\ldots,p_i)\\ &V_i^j(l_i^j,t) = h_i^j(t) && (i=1,2;\ j=p_{i+1},p_{i+2},\ldots,r_i) \end{aligned} \tag{40}$$

An der Stelle $x=0$:

$$\begin{aligned} &V_{i_x}^j(0,t) = -c_i^j[U_i(t)-V_i^j(0,t)]\\ &V_{I_x}(0,t) = -c_{1,I}[U_1(t)-V_I(0,t)]\\ &V_{I_x}(l_I,t) = c_{2,I}[U_2(t)-V_I(l_I,t)] \qquad (i=1,2;\ j=1,2,\ldots,m_i) \end{aligned} \tag{41}$$

$$Q_i(t) = A_i U_i'(t) + \sum_{j=1}^{m_i} B_i^j[U_i(t)-V_i(0,t)] + \tag{42}$$

$$+\begin{cases} \sum_{I=1}^{n} B_{i,I}[U_i(t)-V_I(0,t)] & (i=1)\\ \sum_{I=1}^{n} B_{i,I}[U_i(t)-V_I(l_I,t)] & (i=2). \end{cases}$$

Die Formeln (41) und (42) drücken das Newtonsche Wärmeübertragungsgesetz bzw. die Wärmebilanz aus.

Die Bestimmung des Lösungssystems

Ähnlich wie bisher wird auch in diesem Falle das gesuchte Lösungssystem als System der Summen von entsprechenden Funktionen in Lösungssystemen mit einfacheren Bedingungen hergestellt.

Es sei nun P_k ein Lösungssystem, in dem die Funktionen $V_I^{P_k}(x, t)$ $(I=1, 2, \ldots n)$ bzw. $V_i^{jP_k}(x, t)$ $(i=1, 2; j=1, 2, \ldots, m_i)$ der Differentialgleichung der Wärmeleitung mit dem entsprechenden Wärmeleitungskoeffizienten genügen, und folgende Anfangs- bzw. Randbedingungen erfüllen.

Anfangsbedingungen

$$V_I^{P_k}(x, 0) = \begin{cases} f_I(x) & (I=k) \\ 0, & (I \neq k) \end{cases}$$

(43)

$$V_i^{jP_k}(x, 0) = \begin{cases} f_i^j(x) & (k=1) \\ 0 & (k \neq 1; k = 1, 2, \ldots, n) \end{cases}$$

Randbedingungen

(44) $$V_{Ix}^{P_k}(x, 0) = V_{Ix}^{P_k}(l_I, t) = 0 \qquad (I=k;\ k = 1, 2, \ldots, n)$$

Führen wir für $I=k$ die Bezeichnungen

$$V_I^{P_k}(0, t) = U_1^{P_k}(t) \quad \text{bzw.} \quad V_I^{P_k}(l_I, t) = U_2^{P_k}(t)$$

ein.

Für $I \neq k$ sind die Randbedingungen bezüglich $V_I^{P_k}(x, t)$

(45) $$V_{Ix}^{P_k}(0, t) = -c_{1,I}[U_1^{P_k}(t) - V_I^{P_k}(0, t)]$$

und

$$V_{Ix}^{P_k}(l_I, t) = c_{2,I}[U_2^{P_k}(t) - V_I^{P_k}(l_I, t).$$

Am Rande $x=0$ des Definitionsbereiches der Funktionen $V_i^{jP_k}(x, t)$ sei

$$V_i^{jP_k}(0, t) = -c_i^j[U_i^{P_k}(t) - V_i^{jP_k}(0, t)] \quad (i = 1, 2;\ j = 1, 2, \ldots, m_i) \quad (k = 1, 2, \ldots, n)$$

An den Enden $x=l_i^j$ sind die vorgeschriebenen Randbedingungen für $k=1$ identisch mit den Bedingungen (40); im Falle $k>1$ gelten die entsprechenden homogenen Bedingungen.

Führen wir die Bezeichnung

(46) $$Q_i^{P_k}(t) = A_i U_i^{P_k}(t) + \sum_{j=1}^{m_i} B_i^j[U_i(t) - V_i^j(0, t)] +$$

$$+ \begin{cases} \sum_{I=1}^{n} B_I[U_i(t) - V_I(0, t)] & (i=1) \\ \sum_{I=1}^{n} B_I[U_i(t) - V_I(l_I, t)] & (i=2) \end{cases}$$

ein.

Die Summe der entsprechenden Funktionen der Systeme P_k genügt allen, in der ursprünglichen Aufgabe festgesetzten Anfangs- und Randbedingungen, und mit den Funktionen $U_i^{P_k}(t)$ und $Q_i^{P_k}(t)$ erfüllt sie die Gesetze der Wärmebilanz bzw. Newtonschen Wärmeübertragung.

Ähnlich wie in den vorausgehenden ist noch jenes Lösungssystem P zu bestimmen, in welchem die Funktionen $V_I^P(x, t)$ bzw. $V_i^{jP}(x, t)$ homogenen Anfangs-, bzw. Randbedingungen genügen, und

$$U_i^P(0) = U_i(0) - \sum_{k=1}^{n} U_i^{Pk}(0)$$

$$Q_i^P(t) = Q_i(t) - \sum_{k=1}^{n} Q_i^{Pk}(t) \qquad (i = 1, 2), \tag{47}$$

Die Bestimmung dieses Systems P erfolgt ähnlicherweise, wie die des Systems P_n, bzw. des zur Aufgabe 2. gehörenden Systems P. Wird $V_I(x, t)$ in der Form

$$V_I^P(x, t) = \int_0^{} \frac{1}{\sqrt{(t-\tau)}} \left[\exp\left(-\frac{a_I^2 x^2}{4(t-\tau)} \right) w_{1,I}(\tau) + \exp\left(\frac{a_I^2 (l_I - x)^2}{4(t-\tau)} \right) w_{2,I}(\tau) \right] d\tau \tag{48}$$

geschrieben, so erählt man unter Benutzung der Bedingungen (46) und (47) für die unbekannten Funktionen $w_{1,I}(t)$, $w_{2,I}(t)$ $(I = 1, 2, \ldots, n)$ bzw. $U_i^P(t)$ $(i = 1, 2)$ ein Integrodifferentialgleichungssystem vom Faltungstyp, bestehend aus $2n+2$ Gleichungen mit $2n+2$ Unbekannten.

Mit Hilfe der Operatorrechnung läßt sich dieses Integrodifferentialgleichungssystem auf ein lineares algebraisches Gleichungssystem, bestehend aus $2n+2$ Gleichungen mit $2n+2$ Unbekannten, zurückführen. Die Eindeutigkeit der Lösungen der bisher geschilderten Aufgaben läßt sich durch Anwendung des in der Arbeit [1] besprochenen Monotonitätsprinzips zeigen.

Betrachten wir nun zuerst ein im § 1 untersuchtes System von Stäben und Wärmebehältern. Durch einen oberen Index ϱ_α, $(\alpha = 1, 2)$ wollen wir zwei solche Lösungen der Aufgabe kenntlich machen, die denselben Anfangs-bzw. Randbedingungen genügen, insbesondere sind die Funktionen $V_{i,i+1}^{\alpha}(x, 0)$ $(i = 1, 2, \ldots, n-1)$; $V_i^{j\varrho_\alpha}(x, 0)$; $V_i^{j\varrho_\alpha}(l_i^j, t)$, $(i = 1, 2, \ldots, n$ und $j = 1, 2, \ldots, m_i)$ für $\alpha = 1$ und $\alpha = 2$ dieselben.

Falls

$$U_i^{\varrho_1}(0) > U_i^{\varrho_2}(0) \quad \text{und} \quad Q_i^{\varrho_1}(t) > Q_i^{\varrho_2}(t), \qquad (i = 1, 2, \ldots, n),$$

so gilt für jedes x und jedes $t > 0$

$$V_{i,i+1}^{\varrho_1}(x, t) > V_{i,i+1}^{\varrho_2}(x, t) \tag{49}$$

und

$$V_i^{j\varrho_1}(x, t) > V_i^{j\varrho_2}(x, t).$$

Nehmen wir an, dass es in dem durch ϱ_1 bezeichneten Lösungssystem eine Funktion $V_{i,i+1}(x, t)$ oder eine Funktion $V_i^j(x, t)$ gibt so, daß in irgendeinem Punkte (X, T) die Ungleichung

$$V_{i,i+1}^{\varrho_1}(X, T) < V_{i,i+1}^{\varrho_2}(X, T),$$

oder die Ungleichung

$$V_i^{j\varrho_1}(X, T) < V_i^{j\varrho_2}(X, T)$$

gilt. Des einfacheren Sprachgebrauchs wegen nennen wir „System ϱ" das System von Funktionen, welches wir erhalten, wenn wir die Differenzen der entsprechenden Funktionen der beiden durch ϱ_1 bzw. ϱ_2 bezeichneten Systeme bilden und lassen

wir für das Weitere die oberen Indizes ϱ_1 und ϱ_2 weg (statt ihrer werden wir nur 1 und 2 als untere Indizes schreiben). Das Funktionensystem ϱ genügt dann homogenen Rand-bzw. Anfangsbedingungen und für die Funktionen

$$U_i(t) = U_{i1}(t) - U_{i2}(t)$$
$$Q_i(t) = Q_{i1}(t) - Q_{i2}(t), \tag{51}$$

sind die Bedingungen (2), (3) und (4) erfüllt und so gibt es im System ϱ eine Funktion $V_{i,i+1}(x, t)$ oder eine Funktion $V_i^j(x, t)$ derart, daß in irgendeinem Punkte (X, T) die Ungleichung $V_{i,i+1}(X, T) < 0$ oder die Ungleichung $V_i^j(X, T) < 0$ gilt. Da die Konstanten $c_{i,i+1}$, $c_{i+1,i}$, c_i^j, $b_{i,i+1}$, $b_{i+1,i}$ und b_i^j positiv sind, $U_i(0) > 0$, $(i = 1, 2, \ldots, n)$, so folgt aus den Bedingungen (2) und (3), daß für genügend kleine Werte von $t (t > 0)$ die Größen $V_{i,i+1}(0, t)$, $V_{i,i+1}(l_{i,i+1}, t)$ und $V_i^j(0, t)$ positiv sind. Jedoch aus dem Maximum-Prinzip folgt, daß diejenigen Funktionen $V_{i,i+1}(x, t)$ bzw. $V_i^j(x, t)$, welche in irgendeinem Streifen $(0 < t \leqq T)$ negative Werte annehmen, auf dem Rande $x = 0$ oder $x = l_{i,i+1}$ des Streifens $(0, T)$ ihr Vorzeichen wechseln, im Falle der Funktion $V_i^j(x, t)$ erfolgt dieser Wechsel auf dem Rande $x = 0$.

Betrachten wir von diesen Funktionen diejenige, welche an einem dieser Ränder für einen kleinstmöglichen $t = t_1$ das Vorzeichen wechselt. Es sei dies die Funktion $V_{I,I+1}(x, t)$ und die Koordinaten des in Frage kommenden Punktes seien $l_{I,I+1}$ und t_1, also

$$V_{I,I+1}(l_{I,I+1}, t_1) = 0$$

und $V_{I,I+1}(x, t) \geqq 0$, wenn $0 \leqq x \leqq l_{I,I+1}$; $0 \leqq t \leqq t_1$. Hieraus folgt, daß auf dem Rande $x = l_{I,I+1}$ im Punkte $t = t_1$

$$V_{I,I+1_x}(l_{I,I+1}, t_1) \leqq 0$$

gilt. Doch durch Anwendung der Beziehung (2b) erhalten wir dann, daß

$$U_{I+1}(t_1) \leqq 0$$

ist. Weiterhin, da die Funktionswerte

$$V_{I+1,I+2}(0, t_1), \quad (i+1 < n), \quad V_{I+1}^j(0, t_1), \qquad (j = 1, 2, \ldots, m_i)$$

nicht negativ sind, $Q(t_1) > 0$, so folgt aus (4), daß

$$U'_{I+1}(t_1) > 0$$

ist. So gibt es für die Funktion $U_{I+1}(t)$ eine Zeichenwechselstelle t_2, $(0 < t_2 < t_1)$ und dort ist $U'_{I+1}(t_2) < 0$. In diesem Falle ist für das Bestehen der Bedingung (4) notwendig, daß mindestens einer der Funktionswerte

$$V_{I,I+1}(l_{I,I+1}, t_2); \quad V_{I+1,I+2}(0, t_2), \quad \text{wo} \quad I+1 < n; \quad V_{I+1}^j(0, t_2), \quad (j = 1, 2, \ldots, m_i)$$

negativ sei, was der Voraussetzung daß auf dem Rande $x = l_{I,I+1}$ der Punkt t_2 der kleinste t-Wert ist, für welchen ein Zeichenwechsel der Funktionen des Systems ϱ erfolgt.

Befindet sich diese kleinste Zeichenwechselstelle der Funktionen $V_{I,I+1}(x, t)$ oder $V_I^j(x, t)$ nicht auf dem Rande $x = l_{I,I+1}$ sondern auf dem Rande $x = 0$, so ist das Verfahren zum Herleiten des Widerspruchs dem Vorigen ähnlich.

Der Beweis der Gültigkeit des Monotonitätsprinzips für die Systeme der § 2 und 3 ist dem vorigen Beweis identisch.

LITERATURVERZEICHNIS

[1] FREUD, G.: Sur quelques problèmes de l'équation de la chaleur, *Comptes Rendus du Congrès* International „Les Mathématiques de l'Ingénieur". Mons et Bruxelles, 1958.
[2] ADLER, G.: Hővezetési és diffúziós feladatok összetett peremfeltételekkel, II, *Magyar Tud. Akad.* Mat. Kutató Int. Közl. **1** (1956) 167—183.
[3] FÉNYES, T. et KOSIK, P.: Sur les systèmes des barres conductrices de la chaleur, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **7** (1962) 181—189.

Mathematisches Institut der Ungarischen Akademie der Wissenschaften, Budapest

(Eingegangen: 8. September, 1967.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 127–136.

REMARK ON A PAPER BY I. BIHARI AND T. FÉNYES

by

Á. ELBERT

The paper in question [1] is related with the equation

$$x' + a(xy)' = -k_1 x$$

$$y' + a(xy)' = -k_2 y \qquad \left(' = \frac{d}{dt}\right) \quad (a, k_1, k_2 > 0)$$

which by a linear transformation can be turned into the simple form

$$\begin{aligned} x' + (xy)' &= -\nu x \\ y' + (xy)' &= -y, \end{aligned} \tag{1}$$

where $\nu = \frac{k_1}{k_2}$ (here we can assume $0 < \nu < 1$). The question was proposed: whether or not all the solutions of (1) with the initial conditions $y(0) > 0$, $x(0) > 0$ continue to be positive for all $t > 0$ and tend to 0 as t tends to ∞. The answer was affirmative and the authors gave asymptotic series for $x(t)$ and $y(t)$ depending on the algebraic property of ν:

I. if $\nu = 1$ then

$$x(t) \sim \sum_{n=1}^{\infty} a_n e^{-nt}, \qquad y(t) \sim \sum_{n=1}^{\infty} b_n e^{-nt}$$

II. if $\nu = \frac{p}{q}$ (rational) then

$$x(t) \sim \sum_{m=0}^{q-1} \sum_{n=0}^{\infty} a_{mn} e^{-\left(m\frac{1}{q}+n\right)t},$$

$$y(t) \sim \sum_{m=0}^{q-1} \sum_{n=0}^{\infty} b_{mn} e^{-\left(m\frac{1}{q}+n\right)t} \qquad (a_{00} = b_{00} = 0)$$

III. if ν is irrational then

$$x(t) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} e^{-(m\nu+n)t},$$

$$y(t) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{mn} e^{-(m\nu+n)t} \qquad (a_{00} = b_{00} = 0)$$

where the coefficients a_n, b_n, a_{mn}, b_{mn} can be calculated recursively and depend on two non-negative parameters λ, μ which are the coefficients of the terms $e^{-\nu t}$ and e^{-t} and the actual values of λ, μ are determined by the initial conditions $x(0)=x_0$ and $y(0)=y_0$.

In the paper [1] we can find a method for obtaining the actual values of λ, μ but it can be carried out only if the series of $x(t)$ and $y(t)$ are convergent in the common sense. In our paper we shall prove that the above series are convergent in the common sense for enough small λ and μ.

Suppose

$$\sqrt{1+\frac{1}{\nu}}\sqrt{\lambda}+\sqrt{1+\nu}\sqrt{\mu}<1 \qquad \lambda \geqq 0,\ \mu \geqq 0 \tag{2}$$

then by the identity

$$(1-a^2-b^2)^2-4a^2b^2 = (1+a+b)(1+a-b)(1-a+b)(1-a-b)$$

the inequality

$$\left[1-\left(1+\frac{1}{\nu}\right)\lambda-(1+\nu)\mu\right]^2-4(1+\nu)\left(1+\frac{1}{\nu}\right)\lambda\mu > 0 \tag{3}$$

holds.

THEOREM. *If the inequalities* (2) *hold for* λ, μ, *then the series given for* $x(t)$ *and* $y(t)$ *are absolute convergent.*

PROOF. I. $\nu=1$. Suppose that $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} \leqq 1$ and $\overline{\lim_{n\to\infty}} \sqrt[n]{|b_n|} \leqq 1$, then the series

$$x(t)=\sum_{n=1}^{\infty} a_n e^{-nt}, \qquad y(t)=\sum_{n=1}^{\infty} b_n e^{-nt} \tag{4}$$

are convergent for $t>0$ and the product series

$$\begin{aligned} x(t)y(t) &= \sum_{n=2}^{\infty} c_n e^{-nt} \\ c_n &= \sum_{i=1}^{n-1} a_i b_{n-i} \end{aligned} \tag{5}$$

is also convergent for $t>0$. Putting (4), (5) in (1) and equating the coefficients on both sides we have

$$\begin{aligned} -na_n-nc_n &= -a_n \\ -nb_n-nc_n &= -b_n \end{aligned} \qquad (n \geqq 2),$$

hence

$$a_n = b_n = -\frac{n}{n-1}c_n \qquad (n \geqq 2), \tag{6}$$

while the values of $a_1=\lambda$ and $b_1=\mu$ remain undetermined.

By (5) we get

$$c_2 = \lambda\mu, \qquad c_3 = -2(\lambda+\mu)\lambda\mu,$$

$$c_n = -\frac{n-1}{n-2}(\lambda+\mu)c_{n-1} + \sum_{i=2}^{n-2} \frac{i}{i-1}\,\frac{n-i}{n-i-1}\,c_i c_{n-i} \qquad (n \geqq 4),$$

hence

$$|c_n| \leqq 2(\lambda+\mu)|c_{n-1}| + 4\sum_{i=2}^{n-2} |c_i|\,|c_{n-i}| \qquad (n \geqq 4).$$

Let

$$d_0 = \lambda\mu, \quad d_1 = 2(\lambda+\mu)\lambda\mu,$$

(7)
$$d_{n+2} = 2(\lambda+\mu)d_{n+1} + 4\sum_{i=1}^{n} d_i d_{n-i},$$

and

(8)
$$f(z) = \sum_{n=0}^{\infty} d_n z^n.$$

It is obvious that

(9)
$$|c_{n+2}| \leqq d_n \qquad (n \geqq 0),$$

furthermore by the definitions (7), (8)

$$4z^2 f^2 = \sum_{n=0}^{\infty} z^{n+2} 4 \sum_{i=0}^{n} d_i d_{n-i} =$$

$$= \sum_{n=0}^{\infty} z^{n+2}[d_{n+2} - 2(\lambda+\mu)d_{n+1}] = [1-2(\lambda+\mu)z]f - \lambda\mu,$$

hence

$$4z^2 f^2 - [1-2(\lambda+\mu)z]f + \lambda\mu = 0,$$

therefore

$$f(z) = \frac{1-2(\lambda+\mu)z - \sqrt{[1-2(\lambda+\mu)z]^2 - 16\lambda\mu z^2}}{8z^2}.$$

The sign before the square root is $-$, because $f(z)$ takes on a finite value at $z=0$. The power series of the function $f(z)$ is convergent on the circle $|z| \leqq 1$ if the polynomial

$$[1-2(\lambda+\mu)z]^2 - 16\lambda\mu z^2$$

has no zero on the unit circle $|z| \leqq 1$. Consequently the inequality

$$[1-2(\lambda+\mu)]^2 - 16\lambda\mu > 0$$

must be valid which agrees with (3) when $v=1$. In this case the radius of the convergence of the power series of $f(z)$ is larger than 1, hence

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|d_n|} < 1$$

and by (6) and (9)

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = \overline{\lim_{n\to\infty}} \sqrt[n]{|b_n|} = \overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|} < 1,$$

as it was supposed.

MAGYAR TUDOMÁNYOS AKADÉMIA KÖNYVTÁRA

II. $\nu = \frac{p}{q}$, $(p, q) = 1$, p, q are integers.

Let us suppose again that $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} \leqq 1$, and $\overline{\lim_{n\to\infty}} \sqrt[n]{|b_n|} \leqq 1$, and we seek the solutions in the form

$$x(t) = \sum_{n=1}^{\infty} a_n e^{-\frac{n}{q}t},$$

(10)

$$y(t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n}{q}t}.$$

These series are a little more general as the series considered in the paper [1] for $\nu = \frac{p}{q}$, but for our purpose these forms are more convenient.

Let

$$x(t)y(t) = \sum_{n=1}^{\infty} c_n e^{-\frac{n}{q}t},$$

where

$$c_n = \sum_{i=2}^{n-1} a_i b_{n-i}. \tag{11}$$

By putting these series in (1) and comparing the coefficients of equal powers on both sides we obtain

$$-\frac{n}{q}a_n - \frac{n}{q}c_n = -\frac{p}{q}a_n$$

(12)

$$-\frac{n}{q}b_n - \frac{n}{q}c_n = -b_n.$$

Let $p = 1$. From (12) the coefficients $a_1 = \lambda$, $b_q = \mu$ remain undetermined and step by step

$$c_2 = \cdots = c_q = 0$$

$$a_2 = \cdots = a_q = 0$$

$$b_1 = \cdots = b_{q-1} = 0,$$

furthermore

$$c_{q+1} = \lambda\mu$$

$$c_{q+2} = -\frac{q+1}{1}\lambda^2\mu$$

$$\vdots$$

$$c_{2q} = (-1)^{q-1}\lambda^q\mu \prod_{i=1}^{q-1} \frac{q+i}{i}$$

$$c_{2q+1} = (-1)^q\lambda^{q+1}\mu \prod_{i=1}^{q} \frac{q+i}{i} - \frac{q+1}{q}\lambda\mu^2$$

and

$$a_n = -\frac{n}{n-1}\, c_n$$

(13) $$(n \geqq q+1)$$

$$b_n = -\frac{n}{n-q}\, c_n,$$

thus

$$c_n = -\lambda \frac{n-1}{n-1-q}\, c_{n-1} - \mu \frac{n-q}{n-1-q}\, c_{n-q} +$$

$$+ \sum_{i=q+1}^{n-1-q} \frac{i(n-i)}{(i-1)(n-i-q)}\, c_i c_{n-i} \qquad (n \geqq 2q+2),$$

hence

$$|c_n| \leqq \lambda(q+1)|c_{n-1}| + \mu \frac{q+1}{q}\, |c_{n-q}| + \frac{(q+1)^2}{q} \sum_{i=q+1}^{n-q-1} |c_i|\, |c_{n-i}| \qquad (n \geqq 2q+2).$$

Let

(14)
$$\begin{cases} d_n = (q+1)^n \lambda^{n+1} \mu \qquad (n = 0, 1, \ldots, q-1) \\ d_q = (q+1)^q \lambda^{q+1} \mu + \dfrac{q+1}{q}\, \lambda\mu^2 \\ d_{n+q+1} = (q+1)\lambda d_{n+q} + \dfrac{q+1}{q}\, \mu d_{n+1} + \dfrac{(q+1)^2}{q} \sum_{i=0}^{n} d_i d_{n-i} \qquad (n = 0, 1, \ldots) \end{cases}$$

and

$$f(z) = \sum_{n=0}^{\infty} d_n z^n,$$

then we have

(15) $$|c_n| \leqq d_{n-q-1} \qquad (n \geqq q+1)$$

and by (14)

$$\frac{(q+1)^2}{q}\, z^{q+1} f^2(z) = \sum_{n=0}^{\infty} z^{n+q+1} \frac{(q+1)^2}{q} \sum_{i=0}^{n} d_i d_{n-i} =$$

$$= \sum_{n=0}^{\infty} z^{n+q+1} \left[d_{n+q+1} - (q+1)\lambda d_{n+q} - \frac{q+1}{q}\, \mu d_{n+1} \right] =$$

$$= \left[1 - (q+1)\lambda z - \frac{q+1}{q}\, \mu z^q \right] f - \lambda\mu,$$

hence

$$f(z) = \frac{1 - (q+1)\lambda z - \dfrac{q+1}{q}\, \mu z^q - \sqrt{\left[1 - (q-1)\lambda z - \dfrac{q+1}{q}\, \mu z^q \right]^2 - 4\dfrac{(q+1)^2}{q}\, \lambda\mu z^{q+1}}}{2\dfrac{(q+1)^2}{q}\, z^{q+1}},$$

therefore the power-series of $f(z)$ is convergent on the circle $|z| \leqq 1$ if

$$\left(1-(q+1)\lambda-\frac{q+1}{q}\mu\right)^2-4\frac{(q+1)^2}{q}\lambda\mu>0.$$

which is equivalent with (3) being $\nu=\frac{1}{q}$ now. The radius of the convergence of the power series of $f(z)$ is larger than 1, hence $\overline{\lim_{n\to\infty}}\sqrt[n]{d_n}<1$, therefore by (13), (15)

$$\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|}=\overline{\lim_{n\to\infty}}\sqrt[n]{|b_n|}=\overline{\lim_{n\to\infty}}\sqrt[n]{|c_n|}<1,$$

as we supposed.

Now let $1<p<q$ be and k an integer defined by the inequalities $(k-1)p<q$, $kp>q$. From (11), (12) we have step by step

$$c_1=\cdots=c_{q+p-1}=0$$

$$a_1=\cdots=a_{p-1}=a_{p+1}=\cdots=a_{q+p-1}=0$$

$$b_1=\cdots=b_{q-1}=b_{q+1}=\cdots=b_{q+p-1}=0,$$

the coefficients $a_p=\lambda$, $b_q=\mu$ remain undetermined, furthermore

$$a_n=-\frac{n}{n-p}c_n \qquad (n\geqq p+q)$$

(16)

$$b_n=-\frac{n}{n-q}c_n.$$

A simple consideration shows that the coefficients c_n, different from 0, for $p+q\leqq n<2p+2q$ are

$$c_{a+p}=\lambda\mu$$

$$c_{q+2p}=-\frac{q+p}{p}\lambda^2\mu$$

$$\vdots$$

$$c_{q+kp}=(-1)^{k-1}\lambda^k\mu\prod_{i=1}^{k-1}\frac{q+ip}{ip}$$

$$c_{2q+p}=-\frac{p+q}{q}\lambda\mu^2,$$

By (11), (16) and the known values of the coefficients we have

$$c_n=-\frac{n-p}{n-p-q}\lambda c_{n-p}-\frac{n-q}{n-p-q}\mu c_{n-q}+\sum_{i=p+q}^{n-p-q}\frac{i(n-i)}{(i-p)(n-i-q)}c_ic_{n-i}$$

$$(n\geqq 2p+2q),$$

hence

$$|c_n| \leqq \frac{q+p}{p}\lambda|c_{n-p}| + \frac{q+p}{q}\mu|c_{n-q}| + \frac{(p+q)^2}{pq}\sum_{i=p+q}^{n-p-q}|c_i|\,|c_{n-i}| \qquad (n \geqq 2p+2q).$$

Now we let

$$d_{np} = \left(\frac{q+p}{p}\right)^n \lambda^{n+1}\mu \qquad (n = 0, 1, \ldots, k)$$

$$d_q = \frac{p+q}{q}\lambda\mu^2$$

$$d_{n+p+q} = \frac{q+p}{p}\lambda d_{n+q} + \frac{q+p}{q}\mu d_{n+p} + \frac{(p+q)^2}{pq}\sum_{i=0}^{n} d_i d_{n-i} \qquad (n = 0, 1, \ldots),$$

the remainder of the d_n are zero and $f(z) = \sum_{n=0}^{\infty} d_n z^n$. Hence as before

$$|c_n| \leqq d_{n-p-q} \qquad (n \geqq p+q) \tag{17}$$

and

$$\frac{(p+q)^2}{pq} z^{p+q} f^2(z) - \left[1 - \frac{p+q}{p}\lambda z^p - \frac{p+q}{q}\mu z^q\right] f(z) + \lambda\mu = 0,$$

therefore

$$f(z) = \frac{1 - \frac{p+q}{p}\lambda z^p - \frac{p+q}{q}\mu z^q - \sqrt{\left[1 - \frac{p+q}{p}\lambda z^p - \frac{p+q}{q}\mu z^q\right]^2 - 4\frac{(p+q)^2}{pq}\lambda\mu z^{p+q}}}{2\frac{(p+q)^2}{pq} z^{p+q}}.$$

The power series of $f(z)$ is convergent on the circle $|z| \leqq 1$ if

$$\left[1 - \frac{p+q}{p}\lambda - \frac{p+q}{q}\mu\right]^2 - 4\frac{(p+q)^2}{pq}\lambda\mu > 0$$

i.e. if the inequality (3) holds for $v = \frac{p}{q}$. This inequality and (16), (17) involve the convergence of the series (10) for all $t \geqq 0$ if v is rational.

III. v irrational. The solution is supposed in the form of double power series of e^{-vt} and e^{-t}:

$$x(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn} e^{-(mv+n)t} \qquad (a_{00} = 0)$$

$$y(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_{mn} e^{-(mv+n)t} \qquad (b_{00} = 0) \tag{18}$$

$$x(t)y(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{mn} e^{-(mv+n)t},$$

where

$$c_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} b_{m-i,n-j}. \tag{19}$$

We assume that the double power series obtained by replacing e^{-vt}, e^{-t} by u and v resp., are convergent on the region $|u| \leqq 1$, $|v| \leqq 1$. In this case we can derivate the series (18) term by term. Putting them in (1) and comparing the coefficients on both sides we have

$$\begin{aligned} -(mv+n)a_{mn} - (mv+n)c_{mn} &= -va_{mn} \\ -(mv+n)b_{mn} - (mv+n)c_{mn} &= -b_{mn}. \end{aligned} \tag{20}$$

By (19), (20) we get

$$\begin{aligned} c_{00} &= 0 \\ c_{01} &= a_{00}b_{01} + a_{01}b_{00} = 0 \\ c_{10} &= a_{00}b_{10} + a_{10}b_{00} = 0 \\ a_{10} &= \lambda \qquad \text{(undetermined)} \\ a_{01} &= 0 \\ b_{10} &= 0 \\ b_{01} &= \mu \qquad \text{(undetermined)} \end{aligned}$$

We shall prove the validity of the relations

$$\begin{aligned} &a_{10} = \lambda, \quad a_{01} = 0, \ldots, a_{0n} = 0, \ldots \\ &b_{01} = \mu, \quad b_{10} = 0, \ldots, b_{m0} = 0, \ldots \\ &c_{01} = \ldots = c_{0n} = \ldots = 0 \\ &c_{10} = \ldots = c_{m0} = \ldots = 0. \end{aligned} \tag{21}$$

The first relations of (21) are to be established now. If the relations hold until $n = k$ then

$$c_{0,k+1} = a_{01}b_{0k} + a_{02}b_{0,k-1} + \cdots + a_{0k}b_{01} = 0.$$

By (20) we have

$$a_{0,k+1} = 0.$$

The validity of the relations $b_{m0} = 0$ and $c_{m0} = 0$ can be proved similarly.

Taking the relations (21) into account we obtain from (20)

$$\begin{aligned} a_{m0} &= 0 \qquad (m > 1) \\ b_{0n} &= 0 \qquad (n > 1) \end{aligned}$$

From (19) and (20) we have

$$c_{1n} = \mu a_{1,n-1}$$

$$a_{1n} = -\frac{n+v}{n} c_{1n} = -\frac{n+v}{n} \mu a_{1,n-1},$$

hence

$$a_{1n} = (-1)^n \lambda \mu^n \prod_{i=1}^{n} \frac{i+\nu}{i} \qquad (n = 0, 1, 2, \ldots)$$

$$c_{1n} = (-1)^{n-1} \lambda \mu^n \prod_{i=1}^{n-1} \frac{i+\nu}{i} \qquad (n = 1, 2, \ldots)$$

$$b_{1n} = (-1)^n \lambda \mu^n \frac{n+\nu}{n-1+\nu} \prod_{i=1}^{n-1} \frac{i+\nu}{i} \qquad (n = 1, 2, \ldots).$$

Similarly we get

$$b_{m1} = (-1)^m \lambda^m \mu \prod_{i=1}^{m} \frac{i\nu+1}{i\nu} \qquad (m = 0, 1, 2, \ldots)$$

$$c_{m1} = (-1)^{m-1} \lambda^m \mu \prod_{i=1}^{m-1} \frac{i\nu+1}{i\nu} \qquad (m = 1, 2, \ldots)$$

$$a_{m1} = (-1)^m \lambda^m \mu \frac{m\nu+1}{(m-1)\nu+1} \prod_{i=1}^{m-1} \frac{i\nu+1}{i\nu} \qquad (m = 1, 2, \ldots).$$

The relations (20) give

$$a_{mn} = -\frac{m\nu+n}{(m-1)\nu+n} c_{mn} \quad \text{(except the case } m=1,\ n=0)$$

(22)

$$b_{mn} = -\frac{m\nu+n}{m\nu+n-1} c_{mn} \quad \text{(except the case } m=0,\ n=1)$$

and by (19)

$$c_{mn} = -\frac{(m-1)\nu+n}{(m-1)\nu+n-1} \lambda c_{m-1,n} - \frac{m\nu+n-1}{(m-1)\nu+n-1} \mu c_{m,n-1} +$$

$$+ \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \frac{i\nu+j}{(i-1)\nu+j} \frac{(m-i)\nu+n-j}{(m-i)\nu+n-j-1} c_{ij} c_{m-i,n-j} \qquad (m \geqq 2,\ n \geqq 2),$$

hence

$$|c_{mn}| < \frac{1+\nu}{\nu} \lambda |c_{m-1,n}| + (1+\nu)\mu |c_{m,n-1}| +$$

$$+ \frac{(1+\nu)^2}{\nu} \sum_{i=1}^{m-1} \sum_{j=1}^{n-1}{}' |c_{ij}|\, |c_{m-i,n-j}| \qquad (m \geqq 2,\ n \geqq 2).$$

Now we let

$$d_{00} = \lambda\mu$$

$$d_{0n} = (1+\nu)^n \lambda \mu^{n+1} \qquad (n = 1, 2, \ldots)$$

$$d_{m0} = \left(1+\frac{1}{\nu}\right)^m \lambda^{m+1} \mu \qquad (m = 1, 2, \ldots)$$

$$d_{m+1,n+1} = \left(1+\frac{1}{\nu}\right) \lambda d_{m,n+1} + (1+\nu)\mu d_{m+1,n} +$$

$$+ (1+\nu)\left(1+\frac{1}{\nu}\right) \sum_{i=0}^{m} \sum_{j=0}^{n} d_{ij} d_{m-i,n-j} \qquad (m \geqq 0,\ n \geqq 0)$$

and

$$f(u, v) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} d_{mn} u^m v^n.$$

We have now

$$|c_{mn}| \leqq |d_{m-1,n-1}| \qquad (m \geqq 1,\ n \geqq 1) \tag{23}$$

and

$$(1+v)\left(1+\frac{1}{v}\right) uvf^2 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u^{m+1} v^{n+1} (1+v)\left(1+\frac{1}{v}\right) \sum_{i=0}^{m} \sum_{j=0}^{n} d_{ij} d_{m-i,n-j} =$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u^{m+1} v^{n+1} \left[d_{m+1,n+1} - \left(1+\frac{1}{v}\right) \lambda d_{m,n+1} - (1+v)\mu d_{m+1,n}\right] =$$

$$= f - \sum_{n=1}^{\infty} d_{0n} v^n - \sum_{m=1}^{\infty} d_{m0} u^m - d_{00} - \left(1+\frac{1}{v}\right) \lambda u \left[f - \sum_{m=0}^{\infty} d_{m0} u^m\right] -$$

$$-(1+v)\mu v \left[f - \sum_{n=0}^{\infty} d_{0n} v^n\right] = \left[1 - \left(1+\frac{1}{v}\right) \lambda u + (1+v)\mu v\right] f - \lambda\mu,$$

hence

$$f(u, v) =$$

$$= \frac{1 - \left(1+\frac{1}{v}\right)\lambda u - (1+v)\mu v - \sqrt{\left[1 - \left(1+\frac{1}{v}\right)\lambda u - (1+v)\mu v\right]^2 - 4\lambda\mu\left(1+\frac{1}{v}\right)(1+v)uv}}{2(1+v)\left(1+\frac{1}{v}\right)}.$$

The double power series of $f(u, v)$ is convergent on the region $|u| \leqq 1$, $|v| \leqq 1$, if

$$\left[1 - \left(1+\frac{1}{v}\right)\lambda - (1+v)\mu\right]^2 - 4(1+v)\left(1+\frac{1}{v}\right)\lambda\mu > 0.$$

which agrees with (3) therefore by (22) and (23) the series of the solutions $x(t)$, $y(t)$ are also convergent. So we have completely proved our theorem for every v under the restriction (2) concerning λ and μ.

REFERENCE

[1] BIHARI, I. and FÉNYES, T.: On a first order nonlinear differential equation system, *Studia Sci. Math. Hungar.* **3** (1968) 257—277.

Mathematical Institute of the Hungarian Academy of Sciences, Budapest

(Received November 21, 1967.)

Studia Scientiarum Mathematicarum Hungarica 3 (1968) 137—144.

DISPROOF OF SOME CONJECTURES ON DIOPHANTINE APPROXIMATIONS

by

W. M. SCHMIDT

1. Introduction. Suppose $x_0 = 0, x_1, x_2, \ldots$ is a strictly increasing sequence of reals with $x_n \to \infty$ as $n \to \infty$ and with

$$\lim_{n\to\infty} x_{n+1}/x_n = 1. \tag{1}$$

For any λ in $0 < \lambda \leqq 1$ let $M(\lambda)$ be the set of numbers y which lie in one of the intervals

$$x_n \leqq y < x_n + \lambda(x_{n+1} - x_n) \qquad (n = 0, 1, 2, \ldots). \tag{2}$$

Let α be a positive number, and $F(N, \lambda)$ the number of positive integers $k \leqq N$ for which $k\alpha$ lies in $M(\lambda)$. If

$$\lim_{N\to\infty} F(N, \lambda)/N = \lambda \tag{3}$$

for every λ, we say that *the sequence*

$$\alpha, 2\alpha, 3\alpha, \ldots \tag{4}$$

is uniformly distributed relative to the sequence x_n. This concept is due to LEVEQUE [3]. DAVENPORT and ERDŐS [1] conjectured that the sequence (4) is uniformly distributed relative to the sequence x_n for almost all $\alpha > 0$.

By results of LEVEQUE and of DAVENPORT and LEVEQUE [2] the conjecture is true if $x_{n+1} - x_n$ is monotonic, and by a theorem of DAVENPORT and ERDŐS [1] it is true if $x_n \gg n^{\frac{1}{2}+\delta}$ for some $\delta > 0$. However, in general the conjecture is false.

Define $f(x)$ by

$$f(x) = \begin{cases} 1 \text{ if } x \in M\left(\frac{1}{2}\right) \\ -1 \text{ otherwise.} \end{cases} \tag{5}$$

If the conjecture were true, we would have that

$$\sum_{n=1}^{N} f(\alpha n) = o(N)$$

for almost every $\alpha > 0$.

THEOREM 1. *There is a function $f(x)$ of the type considered above such that*

$$\limsup_{N\to\infty} \left| N^{-1} \sum_{n=1}^{N} f(\alpha n) \right| = 1 \tag{6}$$

for almost every $\alpha > 0$.

The second conjecture to be discussed here is due to H. T. CROFT [1]. Let S be a set of positive reals with infinite Lebesgue measure. Then the conjecture is that for almost every $\alpha > 0$, infinitely many elements of the sequence (4) are in S. A weaker form of the conjecture is that there is *some* α with this property. Our next theorem will show that this conjecture is false.

Let $\mu_s(\varrho)$ be the measure of the intersection of S with the interval $0 < x < \varrho$.

THEOREM 2. *Let $\psi(\varrho)$ be a function satisfying $0 < \psi(\varrho) < 1$ which decreases to zero as ϱ tends to infinity. There is an open set S with*

$$\mu_S(\varrho) \geqq \varrho\psi(\varrho) \tag{7}$$

such that for almost all $\alpha > 0$, only finitely many of the numbers

$$\alpha, 2\alpha, 3\alpha, \ldots \tag{8}$$

lie in S. There is a measurable set S^* with (7) such that for all α, only finitely many of the numbers (8) are in S^*.

REMARKS: (1) The second assertion is an immediate consequence of the first. For if the open set S satisfies the conditions of the theorem, let S^* be the set obtained from S by removing the points α which have infinitely many integral multiples in S.

(2) The following can be proved: If S is open and unbounded, there is always some $\alpha > 0$ with infinitely many elements of (8) in S. See [4], Corollary 1 of Theorem 1.

(3) The sets S constructed in Theorem 2 have $\mu_S(\varrho) = o(\varrho)$. The following theorem shows that this is necessary if S has the properties of Theorem 2.

THEOREM 3. *Suppose that*

$$\limsup_{\varrho \to \infty} \mu_S(\varrho)/\varrho = c > 0 \tag{9}$$

(„*S has positive upper density*"). *Then for almost every α, infinitely many multiples $m\alpha$ lie in S.*

I could not decide the following question. Suppose S is a measurable set of positive reals such that the quotient of two distinct elements of S is never an integer. Is $\mu(S)$ necessarily finite?[2]

I am very indebted to P. ERDŐS who led my attention to the problems dealt with in this paper, and with whom I had valuable discussions.

2. A lemma.

LEMMA 1. *Let $N > 1$, $\varepsilon > 0$.* There is a *subdivision*

$$0 = x_0 < x_1 < \ldots < x_h = 1$$

of the unit interval with

$$x_{i+1} - x_i < \varepsilon \qquad (i = 0, 1, \ldots, h-1) \tag{10}$$

[1] Written communication to Professor ERDŐS. See also H. T. CROFT, Research problems, problem VII 8 page 27 (Mimeographed notes, Cambridge, 1967).

[2] *Added in proof* P. ERDŐS informed me that E. SZEMERÉDI can show that $\mu(S)$ is not necessarily finite.

and with the following property. Define $f(x)$ *in* $0 \leqq x < 1$ *by* (5). *There is a subset* σ_ε *of the unit interval of measure less than* ε *such that*

$$f(x) = f(mx) \tag{11}$$

if x, mx *are in the unit interval but* $x \notin \sigma_\varepsilon$, *and if* m *is an integer with* $1 \leqq m \leqq N$.

PROOF. By DIRICHLET's Theorem on simultaneous approximation, there are arbitrarily large integers q and integers $p_1, p_2, \ldots, p_N$ with

$$\left| \log m - \frac{p_m}{q} \right| < q^{-1-1/N} \qquad (1 \leqq m \leqq N). \tag{12}$$

Define $f^*(x)$ in $0 < x < 1$ by

$$f^*(x) = \begin{cases} 1 \text{ if } e^{t/q} \leqq x < \dfrac{1}{2}(e^{t/q} + e^{(t+1)/q}) \\ -1 \text{ if } \dfrac{1}{2}(e^{t/q} + e^{(t+1)/q}) \leqq x < e^{(t+1)/q} \end{cases}$$

where t runs through the negative integers. Write $b = b(q)$ for the number b between 0 and $1/q$ with

$$1 + e^{1/q} = 2e^b.$$

Then

$$\frac{1}{2}(e^{t/q} + e^{(t+1)/q}) = e^{(t/q)+b}.$$

Hence

$$f^*(x) = \begin{cases} 1 \text{ if } t/q \leqq \log x < (t/q) + b, \\ -1 \text{ if } (t/q) + b \leqq \log x < (t+1)/q. \end{cases} \tag{13}$$

Suppose that $1 \leqq m \leqq N$, $0 < mx < 1$. We infer from (12) that

$$\left| \log(mx) - \log x - \frac{p_m}{q} \right| < q^{-1-1/N} = c,$$

say. Hence in view of (13) we have:

$$f^*(x) = f^*(mx)$$

unless

$$(t/q) - c \leqq \log x \leqq (t/q) + c$$

or

$$(t/q) + b - c \leqq \log x \leqq (t/q) + b + c$$

for some negative integer t. These inequalities are equivalent with

$$e^{(t/q)-c} \leqq x \leqq e^{(t/q)+c}$$

and

$$e^{(t/q)+b-c} \leqq x \leqq e^{(t/q)+b+c}.$$

For fixed t these inequalities define intervals of length

$$\ll e^{t/q}(e^c - e^{-c}) \ll ce^{t/q}$$

and summation over t gives

$$\ll c \sum_{t=-1}^{-\infty} e^{t/q} \ll cq \ll q^{-1/N}$$

since $c = c(q) = q^{-1-1/N}$. Hence if x in $0 < x < 1$ is outside a set $\sigma(q)$ of measure $\ll q^{-1/N}$, then $f^*(x) = f^*(mx)$ for every integer m with $1 \leqq m \leqq N$ and $mx < 1$.

Now put

$$f(x) = \begin{cases} 1 \text{ if } 0 \leqq x < \frac{1}{2} e^{-q}, \\ -1 \text{ if } \frac{1}{2} e^{-q} \leqq x < e^{-q}, \\ f^*(x) \text{ if } e^{-q} \leqq x < 1. \end{cases}$$

The function $f(x)$ is of the type (5) with

$$x_1 = e^{-q}, \quad x_2 = e^{-q+1/q}, \quad x_2 = e^{-q+2/q}, \ldots, x_{q^2+1} = x_h = 1. \tag{14}$$

We have

$$x_{i+1} - x_i \ll e^{1/q} - 1 \ll 1/q \qquad (i = 0, 1, \ldots, h-1).$$

Moreover, if $x > e^{-q}$ and if x is not in $\sigma(q)$, then (11) holds for every m in $1 \leqq m \leqq N$ with $mx < 1$. When q is large, say if $q > q_0(\varepsilon)$, then the sequence (14) and $f(x)$ satisfy all the conditions of the lemma.

3. Proof of Theorem 1. Let $N_1 < N_2 < \ldots$ be a sequence of positive integers with

$$N_k \geqq 2k^2 N_{k-1} \qquad (k = 2, 3, \ldots). \tag{15}$$

Select a sequence $\varepsilon_1, \varepsilon_2, \ldots$ with

$$0 < \varepsilon_k < \min\left(\frac{1}{2k^2}, \frac{1}{kN_k}\right) \qquad (k = 1, 2, \ldots). \tag{16}$$

Let $f_k(x)$ be a function as described in Lemma 1 with $N = N_k$ and $\varepsilon = \varepsilon_k$, and denote the exceptional set σ_ε by $\sigma^{(k)}$. Its measure is less than ε_k.

We now define $f(x)$ as follows. In $0 \leqq x < N_1$ put

$$f(x) = f_1(x/N_1).$$

Suppose that $f(x)$ has already been defined in $0 \leqq x < N_{k-1}$. Let $0 = x_0^{(k)} < x_1^{(k)} < \ldots < x_{h_k}^{(k)} = 1$ be the subdivision belonging to f_k. Define j_k by

$$x_{j_k}^{(k)} < \frac{N_{k-1}}{N_k} \leqq x_{j_k+1}^{(k)}$$

and put

$$M_k = x_{j_k+1}^{(k)} N_k.$$

We then have

$$N_{k-1} \leqq M_k \leqq N_k.$$

In $N_{k-1} \leqq x < M_k$ define $f(x)$ by

$$f(x) = \begin{cases} 1 \text{ if } N_{k-1} \leqq x < \frac{1}{2}(N_{k-1}+M_k), \\ -1 \text{ if } \frac{1}{2}(N_{k-1}+M_k) \leqq x < M_k. \end{cases} \tag{17}$$

In $M_k \leqq x < N_k$ define $f(x)$ by

$$f(x) = f_k(x/N_k). \tag{18}$$

It is obvious that $f(x)$ is of the type (5) with respect to some sequence $0 = x_0 < x_1 < x_2 < \ldots$. The relation (1) holds: If $x_n = N_{k-1}$, $x_{n+1} = M_k$, then $x_{n+1} - x_n \leqq x^{(k)}_{j_k+1} N_k - x^{(k)}_{j_k} N_k$, and if x_n, x_{n+1} are both in $M_k \leqq x \leqq N_k$, then $x_{n+1} - x_n = (x^{(k)}_{j+1} - x^{(k)}_j) N_k$ for some j. In both cases we have $x_{n+1} - x_n \leqq \varepsilon_k N_k \leqq 1/k$ by (16).

As we have just noted, we have $M_k - N_{k-1} \leqq \varepsilon_k N_k$, and therefore by (15), (16),

$$M_k \leqq N_{k-1} + \varepsilon_k N_k < N_k/k^2.$$

Assume that $b > 1$ and I_b is the interval $b^{-1} \leqq \alpha \leqq b$. It will suffice to show that (6) holds for almost every α in I_b.

Let α be in I_b. Suppose $k > b$ and

$$N_k/k \leqq m < N_k/b. \tag{19}$$

Then

$$M_k < N_k/k^2 < (N_k/k)b^{-1} \leqq m\alpha < N_k.$$

This implies that

$$f(m\alpha) = f_k(m\alpha/N_k).$$

If α/N_k does not lie in $\sigma^{(k)}$, then the numbers

$$f_k(m\alpha/N_k)$$

with m satisfying (19) are all equal. Hence if α is in I_b but not in $I_b^{(k)} = I_b \cap N_k \sigma^{(k)}$, then the numbers

$$f(m\alpha)$$

with m in the interval (19) are equal. For such α we therefore have

$$\left| \sum_{m \leqq N_k/b} f(m\alpha) \right| \geqq (N_k/b) - 2(N_k/k) - 1 = (N_k/b)(1 + O(1/k)).$$

Since $I_b^{(k)}$ has measure at most $\varepsilon_k N_k$, and since this tends to zero by (16), we see that (6) holds for almost all α in I_b.

4. Proof of Theorem 2. Suppose $0 < N < M$, $0 < \varepsilon < 1$ and q is a positive integer. Write

$$[N, M; q, \varepsilon]$$

for the set of points x in $N < x < M$ which are in an interval

$$e^{t/q} < x < e^{(t+\varepsilon)/q}$$

with t an integer. Denote the measure of $[N, M; q, \varepsilon]$ by $\mu(N, M; q, \varepsilon)$.

LEMMA 2. *Let N, M, ε, q be as above, and assume that $M > 2N$ and $q > 32$. Then*

$$\frac{\varepsilon}{4}(M-N) < \mu(N, M; q, \varepsilon) < 3\varepsilon(M-N). \tag{20}$$

PROOF. Suppose first that both N, M are of the type $e^{t/q}$ where t is an integer. Then

$$\mu(N, M; q, \varepsilon) = (M-N)(e^{\varepsilon/q}-1)/(e^{1/q}-1).$$

Since

$$\varepsilon/2 < \varepsilon e^{-1/q} < (e^{\varepsilon/q}-1)/(e^{1/q}-1) < \varepsilon e^{1/q} < 2\varepsilon,$$

we have

$$(\varepsilon/2)(M-N) < \mu(N, m; q, \varepsilon) < (2\varepsilon)(M-N).$$

Therefore if M, N are arbitrary we have

$$(\varepsilon/2)(Me^{-1/q} - Ne^{1/q}) < \mu(N, M; q, \varepsilon) < (2\varepsilon)(Me^{1/q} - Ne^{-1/q}).$$

Note that

$$Me^{-1/q} - Ne^{1/q} < M - N < Me^{1/q} - Ne^{-1/q}.$$

The difference of the right hand side and the left hand of this inequality is

$$(M+N)(e^{1/q} - e^{-1/q}) \leqq (4/q)e^{1/q}M < (16/q)(M-N) < \frac{1}{2}(M-N).$$

This shows that

$$(\varepsilon/2)\left(1-\frac{1}{2}\right)(M-N) < \mu(N, M; q, \varepsilon) < (2\varepsilon)\left(1+\frac{1}{2}\right)(M-N).$$

We now are going to construct the set S of the theorem. Let $1 = \varepsilon_1 > \varepsilon_2 > \ldots$ be a decreasing sequence of positive reals with

$$\sum_{i=1}^{\infty} \varepsilon_i < \infty. \tag{21}$$

Let N_1 be a positive integer so large that $\psi(N_1) < \varepsilon_2/8$. When N_{k-1} is given, choose an integer N_k with

$$N_k > 2N_{k-1} \quad \text{and} \quad \psi(N_k) < \varepsilon_{k+1}/8. \tag{22}$$

By DIRICHLET's Theorem on simultaneous approximation there is an integer $q_k > \max(32, \varepsilon_k^{-N_k^2})$ and there are integers $p_k^{(1)}, \ldots, p_k^{(N_k^2)}$ with

$$\left| \log m - \frac{p_k^{(m)}}{q_k} \right| < q_k^{-1-1/N_k^2} < \frac{\varepsilon_k}{q_k} \qquad (1 \leqq m \leqq N_k^2). \tag{23}$$

Let S be the union of the interval

$$0 < x < N_1$$

and of the sets

$$[N_{k-1}, N_k; q_k, \varepsilon_k]$$

where $k = 2, 3, \ldots$.

Our next aim is to verify (7). This inequality is obvious if $0 < \varrho \leqq N_1$. Now

suppose that ϱ is in an interval $2N_{k-1} \leqq \varrho \leqq N_k$ where $k \geqq 2$. Then we have by Lemma 2 and by (22) that

$$\mu_S(\varrho) = N_1 + \mu(N_1, N_2; q_2, \varepsilon_2) + \ldots + \mu(N_{k-1}, N_k; q_k, \varepsilon_k) + \mu(N_k, \varrho; q_{k+1}, \varepsilon_{k+1}) \geqq$$
$$\geqq N\varepsilon_1 + (N_2 - N_1)\varepsilon_2/4 + \ldots + (N_k - N_{k-1})\varepsilon_k/4 + (\varrho - N_k)\varepsilon_{k+1}/4$$
$$\geqq \varrho\varepsilon_{k+1}/4 \geqq 2\varrho\psi(N_k) \geqq 2\varrho\psi(\varrho).$$

In particular we have $\mu_S(N_k) \geqq 2N_k\psi(N_k)$ for $k = 1, 2, \ldots$. Hence if ϱ is in an interval $N_k \leqq \varrho \leqq 2N_k$, then

$$\mu_S(\varrho) \geqq \mu_S(N_k) \geqq 2N_k\psi(N_k) \geqq \varrho\psi(\varrho).$$

Thus (7) is always true.

Suppose $b > 1$ and let I_b be the interval $b^{-1} \leqq \alpha \leqq b$. It will suffice to show that for almost no α in I_b, infinitely many of the numbers (8) are in S. Let S_k be the subset of I_b consisting of those reals α which have a multiple $m\alpha$ in $[N_{k-1}, N_k; q_k, \varepsilon_k]$. We shall show that

$$\mu(S_k) \ll \varepsilon_k$$

whence

$$\sum_{k=1}^{\infty} \mu(S_k) < \infty$$

by (21). This will imply that almost no α is in infinitely many of the sets S_k, and the theorem will follow.

If α is in I_b and $m\alpha$ in $[N_{k-1}, N_k; q_k, \varepsilon_k]$, then $m \leqq \alpha^{-1}N_k \leqq bN_k \leqq N_k^2$ if k is large. In particular (23) holds. Now $m\alpha$ is in an interval

$$e^{t/q_k} < m\alpha < e^{(t+\varepsilon_k)/q_k}$$

and therefore

$$e^{(t/q_k)-\log m} < \alpha < e^{((t+\varepsilon_k)/q_k)-\log m}.$$

Hence by (23) α lies in an interval

$$e^{(s-\varepsilon_k)/q_k} < \alpha < e^{(s+2\varepsilon_k)/q_k}$$

where s is an integer. Thus $\alpha e^{\varepsilon_k/q_k}$ lies in an interval $e^{s/q_k} \leqq x < e^{(s+3\varepsilon_k)/q_k}$ and therefore in $[b^{-1}e^{\varepsilon_k/q_k}, be^{\varepsilon_k/q_k}; q_k, 3\varepsilon_k]$. Using Lemma 2 again we obtain

$$\mu(S_k) \leqq e^{-\varepsilon_k/q_k}\mu(b^{-1}e^{\varepsilon_k/q_k}, be^{\varepsilon_k/q_k}; q_k, 3\varepsilon_k) \leqq 9\varepsilon_k(b - b^{-1}) = 9\varepsilon_k\mu(I_b).$$

By what we said above, the proof of Theorem 2 is now complete.

5. Proof of Theorem 3. Routine arguments show that the set T of numbers α with infinitely many multiples in S is measurable. We shall show that for any interval I of positive reals one has

$$\mu(I \cap T) \geqq c\mu(I). \tag{24}$$

This will suffice. For suppose the complement T^* of T in the set of positive reals had positive measure. There would for every $\varepsilon > 0$ be an interval I as above with $\mu(I \cap T^*) \geqq (1 - \varepsilon)\mu(I)$ and therefore $\mu(I \cap T) \leqq \varepsilon\mu(I)$, contradicting (24).

LEMMA 3. *Suppose $\varepsilon > 0$. There are infinitely many positive integers m with*

$$\mu(mI \cap S) \geqq (c - \varepsilon)\mu(mI). \tag{25}$$

The theorem follows from the lemma: Let $m_1 < m_2 < \ldots$ be a sequence of integers with (25). The set S_k of numbers α in I with $m_k\alpha$ in S has measure $\mu(S_k) \geqq \geqq (c-\varepsilon)\mu(I)$. We have

$$(T \cap I) \supset \bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} S_k \right)$$

and therefore $\mu(T \cap I) \geqq (c-\varepsilon)\mu(I)$.

It remains to prove the lemma. Suppose I has endpoints a, b and let K be a positive integer with

$$\frac{K(b-a)}{K(b-a)+b} > 1 - \frac{\varepsilon}{3}.$$

Suppose $\varrho \geqq Kb$. There is a largest integer n_1 such that $n_1 I$ lies in $0 \leqq x \leqq \varrho$. We have $n_1 \geqq K$. The ratio of the length of $n_1 I$ and of the interval $\varrho_1 = n_1 a \leqq x \leqq \varrho$ is

$$\frac{n_1(b-a)}{n_1(b-a)+\varrho-n_1 b} \geqq \frac{n_1(b-a)}{n_1(b-a)+b} \geqq \frac{K(b-a)}{K(b-a)+b} > 1 - \frac{\varepsilon}{3}.$$

If $\varrho_1 \geqq Kb$ we continue and find numbers n_2, ϱ_2, then n_3, ϱ_3, and so forth, until we reach n_k, ϱ_k with $\varrho_k < Kb$.

The intervals

$$n_k I, n_{k-1} I, \ldots, n_1 I$$

are disjoint and are contained in $0 \leqq x \leqq \varrho$ and

$$\mu(n_k I) + \ldots + \mu(n_1 I) \geqq \varrho \left(1 - \frac{\varepsilon}{3}\right) - Kb > \varrho \left(1 - \frac{\varepsilon}{2}\right)$$

if ϱ is large.

Now if ϱ is large and if $\mu_S(\varrho) \geqq \left(c - \frac{\varepsilon}{2}\right) \varrho$, then

$$\mu(n_k I \cap S) + \ldots + \mu(n_1 I \cap S) \geqq (c-\varepsilon)\varrho \geqq (c-\varepsilon)\left(\mu(n_k I) + \ldots + \mu(n_1 I)\right).$$

There is an n_i with

$$\mu(n_i I \cap S) \geqq (c-\varepsilon)\mu(n_i I).$$

Since $n_i \geqq K$ and since K was arbitrarily large, the lemma follows.

REFERENCES

[1] DAVENPORT, H. and ERDŐS, P.: A theorem on uniform distribution, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **8** (1963) 3—11.

[2] DAVENPORT, H. and LEVEQUE, W. J.: Uniform distribution relative to a fixed sequence, *Michigan Math. J.* **10** (1963) 315—319.

[3] LEVEQUE, W. J.: On uniform distribution modulo a subdivision, *Pacific J. Math.* **3** (1953) 757—771.

[4] KINGMAN, J. F. C.: Ergodic properties of continuous-time Markov Processes and their discrete skeletons, *Proc. London Math. Soc.* **13** (1963) 593—604.

University of Colorado, Boulder

(Received April 2, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 145–160.

ON SOME PROBLEMS OF ESTIMATION AFTER SELECTION IN THE RELIABILITY THEORY

by

A. BALOGH

Introduction

In the paper we deal with the problem of estimation after selection in connection with the design of a reliable equipment. In the design of an equipment it is essential to have a preliminary estimate of the equipment reliability on the basis of the reliaiblity of components to be built in the equipment.

In practice we often meet situations in which several types of selection are available for a component performing a given function in the equipment. For instance, if it is necessary to develop a new component type for performing the given function, several produced types of components are available in the course of the development.

The first step is to select the most suitable one of these component types or generations according to some predetermined decision rule. The second step is to give an estimator for one of the reliability characteristics of each component which can be used to estimate the reliability of the equipment. Both the selection and the estimation can be performed on the basis of results of reliability tests carried out for component types. Usually the same sample is available for both selection and estimation. The use of the same data for selection and subsequently for estimation causes bias. In practice this fact is often neglected and thus quite misleading estimates can result in case of high reliable equipment consisting of a large number of components.

The above mentioned problem can be solved by different procedures and methods depending on the nature of the practical task and on the adopted selection rule. In the literature this problem was discussed by SARKADI [4], who dealt with the possible solutions in a general case and gave estimators for the cases of the normal and the Poisson distributions. KITAGAWA [1] solved a similar problem by using the method of „sometimes pooling" in the case of the normal distribution. RUBINSTEIN [2] dealt with reliability estimates applied in the development programs of components in relation to equipment design. He discussed the problem related to the Poisson distributions. RUBINSTEIN gave an unbiased post-selection estimator of the reliability characteristic of the equipment. The unbiased estimator derived by RUBINSTEIN is based on the fact that for the variable

$$f_0 = \begin{cases} \lambda; & X \leqq a \\ 0; & X > a \end{cases}$$

where X has Poisson distribution, with parameter λ, the statistic

$$\hat{f}_0 = \begin{cases} X; & X \leqq a+1 \\ 0; & X > a+1 \end{cases}$$

is an unbiased estimator.

In Section 1. we deal with the modification of this problem when the original estimator has a binomial distribution. In this case, however, no unbiased estimator based on the original statistic, exists. Using some additional informations supposed to be available after selection an unbiased estimator is given for the variable

$$f_1 = \begin{cases} p; & X \leqq a \\ 0; & X > a \end{cases}$$

where X has a binomial distribution with parameters n and p. The estimator relating to the equipment is discussed, too. In Section 2. we deal with a similar problem in case of gamma distribution. An unbiased estimator is given for the variable where

$$f_2 = \begin{cases} \lambda; & T_r \geqq T_0 \\ 0; & T_r < T_0 \end{cases}$$

T_r has a gamma distribution with parameter r and λ. The estimator concerning the equipment is discussed in a similar way as in Section 1.

In Section 3. we prove a characterization theorem for the exponential distribution.

In Section 4. it is proved that the binomial and gamma distributions are characterized by the estimators given in Sections 1. and 2. In the case of the gamma distribution we use the characterization theorem proved in Section 3. We also proved that the Poisson distribution is characterized by an unbiased estimator given by RUBINSTEIN [2].

In Section 5. a selection procedure connected with a pooling method is discussed for the case of the normal distribution.

1. The case of the binomial distribution

Assume an equipment consists of N independent components. Each of these components is performing a given function in the equipment. Assume for each of these components a development program is carried out. In the course of this development program at most L different types for each of these components are produced. A produced type is named a „generation". If one of these generations is suitable and it is chosen, the development program is finished. Let us denote the jth generation of the ith component of the equipment by A_{ij} ($i = 1, 2, \ldots, N$; $j = 1, 2, \ldots, L$) and the proportion of defectives by p_{ij}, respectively. The samples of size n_{ij} drawn from the populations of the components A_{ij} are subjected to a reliability test and the number X_{ij} of components failed in the test are determined. Let X_{ij} ($i = 1, 2, \ldots, N$; $j = 1, 2, \ldots, L$) be a random variable having a binomial distribution with parameters n_{ij} and p_{ij}. Let the predetermined decision rule be as follows: for performing the ith function (the ith component) of the equipment that component A_{ij} will be chosen for which the inequality $X_{ij} \leqq a_{ij}$ holds for the first time, where the integers a_{ij} ($0 \leqq a_{ij} \leqq n_{ij} - 1$) are predetermined acceptance numbers. If the above inequality does not hold for any value of $j = 1, 2, \ldots, L-1$ the component A_{iL} is chosen. Having performed this selection procedure an estimator is given for the proportion of defectives of the ith component of the equipment. In this

case the proportion of defectives is a random variable and it is to be estimated. The selection and the estimation procedures should be performed for each value of i $(i=1, 2, \ldots, N)$.

Henceforth the notations will be simplified. Since in the equipment the components are independent, the estimators concerning the components of the equipment can be discussed separately. In the following, if we talk about the ith component of the equipment, we drop the index i, i.e. the first index of the above used pair of indices. A_j will denote the jth generation of the ith component (A_{ij}), similarly X_j will denote the random variable X_{ij}, the other notations are modified, similarly. We shall use the pair of indices only when discussing the reliability of the equipment.

Let us define the variable δ_j for the selection as follows:

$$\delta_j = \begin{cases} 1; & \text{if} \quad X_j \leqq a_j \\ 0; & \text{if} \quad X_j > a_j \end{cases}$$

The expectation of δ_j is

$$\mathsf{E}(\delta_j) = \mathsf{P}(X_j \leqq a_j) = \sum_{k=0}^{a_j} \binom{n_j}{k} p_j^k (1-p_j)^{n_j-k} = Q_j^{(n_j)}.$$

According to the above mentioned let us define the variable p^x as follows:

$$p^x = \sum_{j=1}^{L} p_j \delta_j \prod_{k=1}^{j-1} (1-\delta_k) = \sum_{j=1}^{L} f_j \prod_{k=1}^{j-1} (1-\delta_k), \tag{1.1}$$

where $f_j = p_j \delta_j \cdot p^x$ is a random variable which is equal to p_j, if the jth generation, i.e. the component A_j is selected.*

Let us suppose that $v_j^{(1)}$ is a random variable considered on the basis of additonal informations after selection; it has a binomial distribution with parameters 1 and p_j; $v_j^{(1)}$ and X_j are independent random variables. Taking into account these informations the following unbiased estimator can be given for p^x:

$$\hat{p} = \sum_{j=1}^{L} \hat{f}_j \prod_{k=1}^{j-1} (1-\delta_k), \tag{1.2}$$

where

$$\hat{f}_j = \begin{cases} \dfrac{X_j + v_j^{(1)}}{n_j + 1}; & \text{if} \quad X_j + v_j^{(1)} \leqq a_j + 1 \\ 0 \quad ; & \text{if} \quad X_j + v_j^{(1)} > a_j + 1 \end{cases}$$

Thus

$$\mathsf{E}(\hat{p} - p^x) = 0. \tag{1.3}$$

Since $(\hat{f}_j - f_j)$ is independent of δ_k for every $k < j$, $\mathsf{E}(\hat{p} - p^x)$ may be written in the following form:

$$\mathsf{E}(\hat{p} - p^x) = \sum_{j=1}^{L} \mathsf{E}(\hat{f}_j - f_j) \mathsf{E}\left(\prod_{k=1}^{j-1} (1-\delta_k) \right). \tag{1.4}$$

* $p_j \delta_j$ is a polynomial of degree (n_j+1), therefore an unbiased estimator based on the original statistic cannot be given (*cf.* [5]).

Since we have

$$\mathsf{E}(\hat{f}_j) = \sum_{k=0}^{a_j+1} \frac{1}{n_j+1} k \binom{n_j+1}{k} p_j^k (1-p_j)^{n_j-k+1} = p_j Q_j^{(n_j)} = \mathsf{E}(f_j).$$

(1. 3) follows from (1. 4) Let us calculate the expectation of $(\hat{p}-p^x)^2$.

$$\mathsf{E}[(\hat{p}-p^x)^2] = \mathsf{E}\left\{\left[\sum_{j=1}^{L}(\hat{f}_j - f_j)\prod_{k=1}^{j-1}(1-\delta_k)\right]^2\right\} = \sum_{j=1}^{L}\left\{\mathsf{E}[(\hat{f}_j-f_j)^2]\,\mathsf{E}\left[\prod_{k=1}^{j-1}(1-\delta_k)\right]\right\}.$$

(1. 5)

since the cross products vanish.

$$\mathsf{E}(\hat{f}_j^2) = \frac{n_j}{n_j+1}(Q_j^{(n_j-1)} - q_j^{(n_j-1)}) + \frac{p_j}{n_j+1} Q_j^{(n_j)}$$

$$\mathsf{E}(\hat{f}_j f_j) = p_j^2(Q_j^{(n_j)} - q_j^{(n_j)}) + \frac{p_j^2}{n_j+1}(a_j+1) q_j^{(n_j)}$$

$$\mathsf{E}(f_j^2) = p_j^2 Q_j^{(n_j)},$$

where

$$Q_j^{(n_j-1)} = \sum_{k=0}^{a_j} \binom{n_j-1}{k} p_j^k (1-p_j)^{n_j-1-k}$$

$$q_j^{(n_j-1)} = \binom{n_j-1}{a_j} p_j^{a_j}(1-p_j)^{n_j-1-a_j}$$

$$q_j^{(n_j)} = \binom{n_j}{a_j} p_j^{a_j}(1-p_j)^{n_j-a_j},$$

It follows from (1. 5) that the expectation of

(1. 6) $$s_{\hat{p}}^2 = (\hat{p}-p^x)^2$$

is

$$\mathsf{E}(s_{\hat{p}}^2) = \sum_{j=1}^{L}\left\{\frac{n_j}{n_j+1} p_j^2 (Q_j^{(n_j-1)} - q_j^{(n_j-1)}) + \frac{p_j}{n_j+1} Q_j^{(n_j)} - p_j^2 Q_j^{(n_j)} + \right.$$

$$\left. + \frac{2p_j(1-p_j)}{n_j+1}(a_j+1) q_j^{+(n_j)}\right\} \cdot \left\{\prod_{k=1}^{j-1}(1-Q_k^{(n_k)})\right\},$$

where

$$q_j^{+(n_j)} = \binom{n_j}{a_j+1} p_j^{a_j+1}(1-p_j)^{n_j-a_j-1}$$

It will be shown

(1. 7) $$\hat{s}_{\hat{p}}^2 = \sum_{j=1}^{L} \hat{g}_j \prod_{k=1}^{j-1}(1-\delta_k)$$

is an unbiased estimator of $s_{\hat{p}}^2 = (\hat{p}-p^x)^2$.

$$\hat{g}_j = \begin{cases} \hat{h}_j & ; \text{ if } X_j+v_j^{(1)}+v_j^{(2)} \leqq a_j+1 \\ \hat{h}_j + \dfrac{a_j(n_j-a_j)(X_j+v_j^{(1)}+v_j^{(2)})}{(n_j+1)^2(n_j+2)} & ; \text{ if } X_j+v_j^{(1)}+v_j^{(2)} = a_j+2 \\ 0 & ; \text{ if } X_j+v_j^{(1)}+v_j^{(2)} > a_j+2 \end{cases}$$

In the above mentioned expressions $v_j^{(1)}$ and $v_j^{(2)}$ have a binomial distribution with parameters 1 and p_j; they can be constructed on the basis of further informations after the selection. $v_j^{(1)}$, $v_j^{(2)}$ and X_j are independent random variables.

$$\hat{h}_j = \left(\frac{X_j+v_j^{(1)}}{n_j+1}\right)^2 - \frac{(X_j+v_j^{(1)}+v_j^{(2)})(X_j+v_j^{(1)}+v_j^{(2)}-1)}{(n_j+1)(n_j+2)}.$$

By a simple calculation we obtain

$$\mathsf{E}(\hat{g}_j) = \mathsf{E}[(\hat{f}_j-f_j)^2].$$

Thus we get

$$\mathsf{E}(\hat{s}_{\hat{p}}^2) = \mathsf{E}(s_{\hat{p}}^2).$$

We summarize the above discussed statements in the following theorem:

THEOREM 1. 1. *The estimator $\hat{p}$ in* (1. 2) *is an unbiased estimator of the random variable p^x in* (1. 1) *and the estimator $\hat{s}_{\hat{p}}^2$ in* (1. 7) *is an unbiased estimator of $s_{\hat{p}}^2$ in* (1. 6).

A biased estimator $\hat{\hat{p}}$ of p^x can be given, for which the inequality

$$\mathsf{E}[(\hat{\hat{p}}-p^x)^2] \leqq \mathsf{E}[(\hat{p}-p^x)^2] \tag{1.8}$$

holds.

Let be

$$\hat{\hat{p}} = \sum_{j=1}^{L} \frac{X_j+v_j^{(1)}}{n_j+1}\,\delta_j \prod_{k=1}^{j-1}(1-\delta_k). \tag{1.9}$$

Let us introduce the notation $\hat{\hat{f}}_j = \dfrac{X_j+v_j^{(1)}}{n_j+1}\,\delta_j$. The expectation of $\hat{\hat{f}}_j$ is

$$\mathsf{E}(\hat{\hat{f}}_j) = p_j Q_j^{(nj)} - \frac{(1-p_j)(a_j+1)}{n_j+1}\,q_j^{+(nj)}$$

and the bias is

$$\mathsf{E}(f_j-\hat{\hat{f}}_j) = \frac{(1-p_j)(a_j+1)}{n_j+1}\,q_j^{+(nj)}.$$

So we have

$$\mathsf{E}(p^x-\hat{\hat{p}}) = \sum_{j=1}^{L} \frac{(1-p_j)(a_j+1)}{n_j+1}\,q_j^{+(nj)} \prod_{k=1}^{j-1}(1-Q_k^{(nk)}).$$

On the other hand we obtain by calculation

$$\mathsf{E}[(\hat{\hat{f}}_j - f_j)^2] = \mathsf{E}[(\hat{f}_j - f_j)^2] - \frac{(1+p_j)(a_j+1)^2}{(n_j+1)^2} q_j^{+(nj)}.$$

So it follows

$$\mathsf{E}[(\hat{\hat{f}}_j - f_j)^2] \leqq \mathsf{E}[(\hat{f}_j - f_j)^2]$$

and hence we get the inequality in (1. 8).

Now we turn to the discussion of the proportion of defectives concerning to the equipment. The proportion of defectives of the equipment, if the components fail in the equipment independently, is

$$P^x = 1 - \prod_{i=1}^{N} (1 - p_i^*).$$

For the estimators of the proportion of defectives of equipment the following similar formulae may be written

$$\hat{P} = 1 - \prod_{i=1}^{N} (1 - \hat{p}_i)$$

$$\hat{\hat{P}} = 1 - \prod_{i=1}^{N} (1 - \hat{\hat{p}}_i).$$

One may ask: on what conditions does the inequality

$$\mathsf{E}[(\hat{P} - P^x)^2] \leqq \mathsf{E}[(\hat{\hat{P}} - P^x)^2]$$

hold? The calculations are tedious and lead to a complicated formula. Since in most of the practical cases a large number of components are operating in the equipment and the proportion of defectives may be assumed to be small, we may accept the „approximate proportion of defectives"

$$\tilde{P}^x = \sum_{i=1}^{N} p_i^*$$

as an approximation for the proportion of defectives. The estimators can be modified in a similar way:

$$\hat{\tilde{P}} = \sum_{i=1}^{N} \hat{p}_i$$

$$\hat{\hat{\tilde{P}}} = \sum_{i=1}^{N} \hat{\hat{p}}_i.$$

We shall discuss the following problem: on what conditions does the inequality

$$\mathsf{E}[(\hat{\tilde{P}} - \tilde{P}^x)^2] \leqq \mathsf{E}[(\hat{\hat{\tilde{P}}} - \tilde{P}^x)^2] \tag{1.10}$$

hold?

Theorem 1. 2 gives an answer to this question:

THEOREM 1. 2. *If the inequality in* (1. 10) *holds then the inequality*

$$\sum_{i=1}^{N}\left\{\sum_{j=1}^{L}(1-p_{ij})\left(\frac{a_{ij}+1}{n_{ij}+1}\right)^2 q_{ij}^{+(n_{ij})}\prod_{k=1}^{j-1}(1-Q_{ik}^{(n_{ik})})+\right.$$

$$+\left.\left[\sum_{j=1}^{L}(1-p_{ij})\left(\frac{a_{ij}+1}{n_{ij}+1}\right)q_{ij}^{+(n_{ij})}\prod_{k=1}^{j-1}(1-Q_{ik}^{(n_{ik})})\right]^2\right\}\leqq \tag{1.11}$$

$$\leqq\left[\sum_{i=1}^{N}\sum_{j=1}^{L}(1-p_{ij})\left(\frac{a_{ij}+1}{n_{ij}+1}\right)q_{ij}^{+(n_{ij})}\prod_{k=1}^{j-1}(1-Q_{ik}^{(n_{ik})})\right]^2$$

is fulfilled.

In order to prove this statement we consider the following expectations:

$$\mathsf{E}[(\hat{\tilde{P}}-\tilde{P}^x)^2]=\sum_{i=1}^{N}\left\{\sum_{j=1}^{L}\left[\mathsf{E}((\hat{f}_{ij}-f_{ij})^2)-(1-p_{ij})\left(\frac{a_{ij}+1}{n_{ij}+1}\right)^2 q_{ij}^{+(n_{ij})}\right]\cdot\right.$$

$$\left.\cdot\prod_{k=1}^{j-1}(1-Q_{ik}^{(n_{ik})})-\left[\sum_{j=1}^{L}(1-p_{ij})\left(\frac{a_{ij}+1}{n_{ij}+1}\right)q_{ij}^{+(n_{ij})}\prod_{k=1}^{j-1}(1-Q_{ik}^{(n_{ik})})\right]^2\right\}+$$

$$+\left[\sum_{i=1}^{N}\sum_{j=1}^{L}(1-p_{ij})\left(\frac{a_{ij}+1}{n_{ij}+1}\right)q_{ij}^{+(n_{ij})}\prod_{k=1}^{j-1}(1-Q_{ik}^{(n_{ik})})\right]^2 \tag{1.12}$$

$$\mathsf{E}[(\hat{\tilde{P}}-\tilde{P}^x)^2]=\sum_{i=1}^{N}\sum_{j=1}^{L}\mathsf{E}((\hat{f}_{ij}-f_{ij})^2)\prod_{k=1}^{j-1}(1-Q_{ik}^{(n_{ik})}). \tag{1.13}$$

Substituting (1. 12) and (1. 13) into (1. 10) after some calculations we obtain (1. 11). Another estimator $\alpha\hat{\tilde{P}}+(1-\alpha)\hat{\tilde{P}}(0\leqq\alpha\leqq 1)$ also can be given for the „approximate proportion of defectives" $\tilde{P}^x$. The bias and the mean square error are given by the following formulae:

$$\mathsf{E}(\tilde{P}^x-\alpha\hat{\tilde{P}}-(1-\alpha)\hat{\tilde{P}})=\sum_{i=1}^{N}\sum_{j=1}^{L}(1-\alpha)(1-p_{ij})\left(\frac{a_{ij}+1}{n_{ij}+1}\right)q_{ij}^{+(n_{ij})}\prod_{k=1}^{j-1}(1-Q_{ik}^{(n_{ik})});$$

$$\mathsf{E}[(\alpha\hat{\tilde{P}}+(1-\alpha)\hat{\tilde{P}}-\tilde{P}^x)^2]=\alpha^2\sum_{i=1}^{N}\sum_{j=1}^{L}(1-p_{ij})\left(\frac{a_{ij}+1}{n_{ij}+1}\right)^2 q_{ij}^{+(n_{ij})}\prod_{k=1}^{j-1}(1-Q_{ik}^{(n_{ik})})+$$

$$+\sum_{i=1}^{N}\left\{\mathsf{E}[(\hat{\tilde{p}}_i-p_i^x)^2]-(1-\alpha)^2\left[\sum_{j=1}^{L}(1-p_{ij})\left(\frac{a_{ij}+1}{n_{ij}+1}\right)q_{ij}^{+(n_{ij})}\prod_{k=1}^{j-1}(1-Q_{ik}^{(n_{ik})})\right]^2\right\}+$$

$$+(1-\alpha)^2\left[\sum_{i=1}^{N}\sum_{j=1}^{L}(1-p_{ij})\left(\frac{a_{ij}+1}{n_{ij}+1}\right)q_{ij}^{+(n_{ij})}\prod_{k=1}^{j-1}(1-Q_{ik}^{(n_{ik})})\right]^2.$$

In the following we deal with the estimate of the variable

$$(1-p)^x=\bar{p}^x=\sum_{j=1}^{L}(1-p_j)\delta_j\prod_{k=1}^{j-1}(1-\delta_k)=\sum_{j=1}^{L}\bar{f}_j\prod_{k=1}^{j-1}(1-\delta_k).$$

The expectation of the variable $\bar{p}^x$ is

$$\mathsf{E}(\bar{p}^x) = \sum_{j=1}^{L} (1-p_j) Q_j^{(nj)} \prod_{k=1}^{j-1} (1-Q_k^{(nk)}),$$

where

$$Q_j^{(nj)} = \sum_{k=0}^{a_j} \binom{n_j}{k} p_j^k (1-p_j)^{n_j-k}.$$

The following estimator $\bar{p}$ is an unbiased estimator of $\bar{p}^x$:

$$\hat{\bar{p}} = \sum_{j=1}^{L} \hat{\bar{f}}_j \prod_{k=1}^{j-1} (1-\delta_k),$$

where

$$\hat{\bar{f}}_j = \begin{cases} 1 - \dfrac{X_j + v_j^{(1)}}{n_j+1}; & \text{if} \quad X_j + v_j \leqq a_j \\ 0 \qquad\qquad\quad ; & \text{if} \quad X_j + v_j > a_j \end{cases}$$

The expectation of $(\hat{\bar{p}} - \bar{p}^x)^2$ is

$$\mathsf{E}(\hat{\bar{p}} - \bar{p}^x)^2 = \sum_{j=1}^{L} \left[(1-p_j)^2 \frac{n_j}{n_j+1} Q_j^{(nj-1)} + \frac{(1-p_j)}{n_j+1} Q_j^{(nj)} - (1-p_j)^2 Q_j^{(nj)} \right] \prod_{k=1}^{j-1} (1-Q_k^{(nk)}).$$

Let us define the following biased estimator $\hat{\hat{\bar{p}}}$ of $\bar{p}^x$ as follows

$$\hat{\hat{\bar{p}}} = \sum_{j=1}^{L} \hat{\hat{\bar{f}}}_j \prod_{k=1}^{j-1} (1-\delta_k),$$

where

$$\hat{\hat{\bar{f}}}_j = \left(1 - \frac{X_j + v_j^{(1)}}{n_j+1}\right) \delta_j.$$

Calculating the bias we get

$$\mathsf{E}(\hat{\hat{\bar{p}}} - p) = \sum_{j=1}^{L} \frac{n_j - a_j}{n_j+1} q_j^{(nj)} p_j \prod_{k=1}^{j-1} (1-Q_k^{(nk)}),$$

where

$$q_j^{(nj)} = \binom{n_j}{a_j} p_j^{a_j} (1-p_j)^{n_j-a_j}.$$

The expectation of $(\hat{\hat{\bar{p}}} - \bar{p}^x)^2$ is

$$\mathsf{E}\left[(\hat{\hat{\bar{p}}} - \bar{p}^x)^2\right] = \mathsf{E}\left[(\hat{\bar{p}} - \bar{p}^x)^2\right] + \sum_{j=1}^{L} \left[\left(\frac{n_j - a_j}{n_j+1}\right)^2 - \frac{2(1-p_j)(n_j-a_j)}{n_j+1} \right] q_j^{(nj)} p_j \prod_{k=1}^{j-1} (1-Q_k^{(nk)})$$

Hence it can be seen that $\mathsf{E}\left[(\hat{\bar{p}} - \bar{p}^x)^2\right] \leqq \mathsf{E}\left[(\hat{\hat{\bar{p}}} - \bar{p}^x)^2\right]$ if for every $j (1 \leqq j \leqq L)$ $1 - p_j \leqq \dfrac{n_j - a_j}{2(n_j+1)}$. By calculation we get that

$$\mathsf{E}\left[(\varepsilon \hat{\bar{p}} + (1-\varepsilon)\hat{\hat{\bar{p}}} - \bar{p}^x)^2\right] \leqq \mathsf{E}\left[(\hat{\hat{\bar{p}}} - \bar{p}^x)^2\right]; \qquad (0 \leqq \varepsilon \leqq 1)$$

if for every $j (1 \leqq j \leqq L)$

$$1 - p_j \leqq \frac{n_j - a_j}{n_j+1} \left(1 - \frac{\varepsilon}{2}\right).$$

2. The case of the gamma distribution

Suppose the equipment consists of N independent components. Each of these components is performing a given function in the equipment. At most L generations are carried out for each of these components. If one of the generations is accepted the development program is finished. The jth generation of the ith component is denoted by A_{ij} $(i=1, 2, ..., N)$ $(j=1, 2, ..., L)$ and the failure rate by λ_{ij}. Let us suppose that the life time of components has an exponential distribution with parameter λ_{ij} in the case of the component A_{ij}. Hereinafter, unless otherwise stated we deal with the jth component and therefore we drop the index „i" for sake of simplification.

The samples of size n_j drawn from generations $A_1, A_2, ..., A_j, ..., A_L$ are subjected to life test, the times of the first r failures — $0=t_{j,0}<t_{j,1}<t_{j,2}<...<t_{j,r}$ — are observed, where r is a predetermined number. That component A_j will be selected for which the inequality $T_{j,r}=\sum_{s=1}^{r} t_{j,s}+(n_j-r)t_{j,r} \geqq T_{j,0}$ holds for the first time. If the above inequality does not hold for any value of $j=1, 2, ..., L-1$, the component A_L is chosen. $T_{j,0}>0$ is a given number. Since the variables $\tau_{j,s-1} = (n_j-s)(t_{j,s+1}-t_{j,s})$ are independent and have the same exponential distribution with parameter λ_j, $T_{j,r}$ has a gamma distribution with parameters λ_j and r.

Let us define the random variable for the selection as follows:

$$\delta_j = \begin{cases} 1; & \text{if} \quad T_{j,r} \geqq T_{j,0} \\ 0; & \text{if} \quad T_{j,r} < T_0 \end{cases}$$

Let us suppose the simple case $T_{j,0}=T_0$ for every j. The expectation of the random variable δ_j is

$$\mathsf{E}(\delta_j) = \mathsf{P}(T_{j,r} \geqq T_0) = \int_{T_0}^{\infty} \frac{\lambda_j^r t^{r-1}}{(r-1)!} e^{-\lambda_j t}\, dt = P_j^{(r)}.$$

Let us define the variable λ^x as follows

$$\lambda^x = \sum_{j=1}^{L} \lambda_j \delta_j \prod_{k=1}^{j-1} (1-\delta_k) = \sum_{j=1}^{L} f_j \prod_{k=1}^{j-1} (1-\delta_k), \tag{2.1}$$

where $f_j=\lambda_j\delta_j$.

λ^x is a random variable which is equal to λ_j if the component A_j is selected. The following estimator $\hat{\lambda}$ is an unbiased one for λ^x:

$$\hat{\lambda} = \sum_{j=1}^{L} \hat{f}_j \prod_{k=1}^{j-1} (1-\delta_k);$$

$$\hat{f}_j = \begin{cases} \dfrac{r}{T_{j,r+1}}; & \text{if} \quad T_{j,r+1} \geqq T_0 \\ 0 \quad ; & \text{if} \quad T_{j,r+1} < T_0 \end{cases} \tag{2.2}$$

$$T_{j,r+1} = \sum_{s=1}^{r+1} t_{j,s} + (n_j-r-1)_{j,r+1}.$$

Since $(\hat{f}_j - f_j)$ is independent of δ_k for $k<j$ and $E(\hat{f}_j)=E(f_j)=\lambda_j P_j^{(r)}$ we get $E(\hat{\lambda}-\lambda^x)=0$. Let us calculate the expectation of

$$s_{\hat{\lambda}}^2 = (\hat{\lambda}-\lambda^x)^2. \tag{2.3}$$

It is equal to

$$\mathsf{E}[(\hat{\lambda}-\lambda^x)^2] = \sum_{j=1}^{L}\left[\frac{r}{r-1}\lambda_j^2 P_j^{(r-1)} + 2\lambda_j^2 S_j^{(r)} - \lambda_j^2 P_j^{(r)}\right]\cdot \prod_{k=1}^{j-1}(1-P_k^{(r)}),$$

where

$$P_j^{(r-1)} = \int_{T_0}^{\infty} \frac{\lambda_j^{r-1} t^{r-2}}{(r-2)!} e^{-\lambda_j t}\, dt,$$

$$S_j^{(r)} = \int_{T_0}^{\infty} \frac{(\lambda_j T_0)^r}{(r-1)!}\frac{e^{-\lambda_j t}}{t}\, dt.$$

The considerations described in Section 1. and the following expectations are used in calculation

$$\mathsf{E}(\hat{f}_j^2) = \frac{r}{r-1}\lambda_j^2 P_j^{(r-1)}$$

$$\mathsf{E}(\hat{f}_j f_j) = \lambda_j^2 P_j^{(r)} - \lambda_j^2 S_j^{(r)}$$

$$\mathsf{E}(f_j^2) = \lambda_j^2 P_j^{(r)}.$$

The following estimator $\hat{s}_{\hat{\lambda}}^2$ is an unbiased one for $s_{\hat{\lambda}}^2$

$$\hat{s}_{\hat{\lambda}}^2 = \sum_{j=1}^{L} \hat{g}_j \prod_{k=1}^{j-1}(1-\delta_k), \tag{2.4}$$

where

$$\hat{g}_j = \begin{cases} \left(\dfrac{r}{T_{j,r+1}}\right)^2 - \dfrac{r^2}{T_{j,r+1}, T_{j,r+2}}; & \text{if} \quad T_{j,r+1} \geqq T_0 \\ \dfrac{r^2}{T_{j,r+1}\, T_{j,r+2}} \quad ; & \text{if} \quad T_{j,r+1} < T_0, T_{j,r+2} \geqq T_0 \\ 0 \quad ; & \text{if} \quad T_{j,r+1} < T_0, T_{j,r+2} < T_0 \end{cases}$$

$$T_{j,r+1} = \sum_{s=1}^{r+1} t_{j,s} + (n_j - r - 1)\, t_{j,r+1}$$

$$T_{j,r+2} = \sum_{s=1}^{r+2} t_{j,s} + (n_j - r - 2)\, t_{j,r+2}.$$

After some calculations we obtain

$$\mathsf{E}(\hat{g}_j) = \mathsf{E}[(\hat{f}_j - f_j)^2]$$

and therefore

$$\mathsf{E}(\hat{s}_{\hat{\lambda}}^2) = \mathsf{E}(s_{\hat{\lambda}}^2).$$

We summarize the above discussed statements in the following theorem.

THEOREM 2. 1. *The statistic* $\hat{\lambda}$ *given in* (2. 2) *is an unbiased estimator for the random variable* λ^x *in* (2. 1), *the statistic* $\hat{s}_{\hat{\lambda}}^2$ *in* (2. 4) *is an unbiased estimator for* $s_{\hat{\lambda}}^2$ *in* (2. 3). We give a biased estimator $\hat{\hat{\lambda}}$ of λ^x, but it holds for $\hat{\hat{\lambda}}$

$$\mathsf{E}[(\hat{\hat{\lambda}} - \lambda^x)^2] \leqq \mathsf{E}[(\hat{\lambda} - \lambda^x)^2]. \tag{2. 5}$$

Let be

$$\hat{\hat{\lambda}} = \sum_{j=1}^{L} \frac{r}{T_{j,r+1}} \delta_j \prod_{k=1}^{j-1} (1 - \delta_k). \tag{2. 6}$$

The bias of $\hat{\hat{\lambda}}$ is

$$\mathsf{E}(\hat{\hat{\lambda}} - \lambda^x) = \sum_{j=1}^{L} \lambda_j S_j^{(r)} \prod_{k=1}^{j-1} (1 - P_k^{(r)}).$$

The mean square error of $\hat{\hat{\lambda}}$ is

$$\mathsf{E}[(\hat{\hat{\lambda}} - \lambda^x)^2] = \mathsf{E}[(\hat{\lambda} - \lambda^x)^2] - \sum_{j=1}^{L} \lambda_j Z_j^{(r)} \prod_{k=1}^{j-1} (1 - P_k^{(r)}), \tag{2. 7}$$

where

$$Z_j^{(r)} = \int_{T_0}^{\infty} \frac{(\lambda_j T_0)^r}{(r-1)!} \frac{e^{-\lambda_j t}}{t^2} dt.$$

The inequality (2. 5) follows from (2. 7).

Considering the selection we obtain for the failure rate Λ^x of the equipment

$$\Lambda^x = \sum_{i=1}^{N} \lambda_i^x$$

since the operations of the components are independent in the equipment. We get for the estimators in a similar way

$$\hat{\Lambda} = \sum_{i=1}^{N} \hat{\lambda}_i,$$

$$\hat{\hat{\Lambda}} = \sum_{i=1}^{N} \hat{\hat{\lambda}}_i.$$

Similarly to the Theorem 1. 2 described in Section 1. we have

THEOREM 2. 2. *If fhe inequality*

$$\mathsf{E}[(\hat{\Lambda} - \Lambda^x)^2] \leqq \mathsf{E}[(\hat{\hat{\Lambda}} - \Lambda^x)^2] \tag{2. 8}$$

holds, then

$$\sum_{i=1}^{N}\left\{\sum_{j=1}^{L} r\lambda_{ij}Z_{ij}^{(r)}\prod_{k=1}^{j-1}(1-P_{ik}^{(r)})+\left[\sum_{j=1}^{L} S_{ij}^{(r)}\lambda_{ij}\prod_{k=1}^{j-1}(1-P_{ik}^{(r)})\right]^2\right\}\leqq$$

(2. 9)

$$\leqq\left[\sum_{i=1}^{N}\sum_{j=1}^{L}\lambda_{ij}S_{ij}^{(r)}\prod_{k=1}^{j-1}(1-P_{ik}^{(r)})\right]^2.$$

The proof of the Theorem 2. 2. is similar to the proof of the Theorem 1. 2. therefore we don't enter into the details of it.

3. A characterization theorem of the exponential distribution

In the literature several theorems are known for the characterization of the exponential distribution e.g. [6]. In the course of our investigations a new characterization theorem has been found.

THEOREM 3. 1. Let $f(x)$ be a continuous density function in the interval $(0, \infty)$ If the convolution of $f(x)$ with itself is equal to $xf(x)$, then $f(x)=e^{-x}$.

PROOF. If follows from the assumption that

$$xf(x)=\int_0^x f(x-y)f(y)dy. \tag{3. 1}$$

Let us denote the Laplace transformation of $f(x)$ by $\varphi(z)$. Taking the Laplace transformation of both sides of the equation (3. 1), we get the following differential equation.

$$-\varphi'(z)=\varphi^2(z). \tag{3. 2}$$

Solving (3. 2) we obtain

$$\varphi(z)=\frac{1}{\varphi(o)+z}.$$

Taking into consideration that

$$\varphi(o)=\int_0^\infty f(x)\,dx=1,$$

we get

$$\varphi(z)=\frac{1}{1+z}.$$

Since $\varphi(z)=\dfrac{1}{1+z}$ is the Laplace transformation of $f(x)=e^{-x}$, we proved our statement.

4. The characterization of the Poisson, the binomial and the gamma distributions by unbiased estimators of a given form

In this Section it will be shown that the Poisson, the binomial and the gamma distributions are characterized by unbiased estimators of a given form. RUBINSTEIN [2], using a selection procedure as described in our paper for the binomial distribution, gave an unbiased estimator for the parameter (λ) of the Poisson distribution. The unbiased estimator given by Rubinstein is based on the fact that the estimator

$$\hat{f} = \begin{cases} cX; & \text{if} \quad X \leqq a+1 \\ 0\,; & \text{if} \quad X > a \end{cases}$$

is an unbiased estimator for the random variable

$$f = \begin{cases} \lambda; & \text{if} \quad X \leqq a \\ 0; & \text{if} \quad X > a \end{cases}$$

where X has a Poisson distribution with parameter $\frac{\lambda}{c}$. Similar considerations are true in the cases of the binomial and the gamma distributions discussed in our paper, too. It will be shown that the above mentioned distributions are characterized by estimators of a given form.

THEOREM 4. 1. *Let X be a random variable taking on the values $0, 1, \ldots, n, \ldots$ with probabilities $p_0(\lambda), p_1(\lambda) \ldots p_n(\lambda) \ldots$, where $\lambda > 0$ is the parameter of the distribution. If the estimator*

$$\hat{f} = \begin{cases} cX; & \text{if} \quad X \leqq a+1 \\ 0\,; & \text{if} \quad X > a+1 \end{cases}$$

is an unbiased estimator of the random variable

$$f = \begin{cases} \lambda; & \text{if} \quad X \leqq a \\ 0; & \text{if} \quad X > a \end{cases}$$

for any positive real number c and for every integer $a \geqq 0$, then X has a Poisson distribution with parameter $\frac{\lambda}{c}$.

PROOF. It follows from the assumption that

$$c \sum_{k=0}^{a+1} k p_k(\lambda) = \lambda \sum_{k=0}^{a} p_k(\lambda). \tag{4. 1}$$

Replacing in (4. 1) $(a+1)$ by a and subtracting the equation obtained in this way from the original one, we get

$$p_{a+1}(\lambda) = \frac{\lambda}{c} \frac{p_a(\lambda)}{a+1}. \tag{4. 2}$$

Replacing in (4. 2) a successively by $0, 1, \ldots, a$ and multiplying all the equations thus formed, we obtain

$$p_{a+1}(\lambda) = \left(\frac{\lambda}{c}\right)^{a+1} \frac{p_0(\lambda)}{(a+1)!}; \qquad (a = 0, 1\ldots) \tag{4. 3}$$

Hence for every integer $k>0$ we get

$$p_k(\lambda) = \left(\frac{\lambda}{c}\right)^k \frac{p_0(\lambda)}{k!}.$$

Taking into account $\sum_{k=0}^{\infty} p_k(\lambda)=1$, we obtain

$$p_0(\lambda) = e^{-\frac{\lambda}{c}}.$$

Therefore we have

$$p_k(\lambda) = \left(\frac{\lambda}{c}\right)^k \frac{1}{k!} e^{-\frac{\lambda}{c}},$$

i.e. the random variable X has a Poisson distribution with parameter $\frac{\lambda}{c}$.

THEOREM 4. 2. *Let X be a discrete random variable taking on the values $0, 1, \ldots, n$ with probabilities $p_0(\alpha)$, $p_1(\alpha)\ldots, p_n(\alpha)$ where α $(0<\alpha<1)$ is the parameter of the ditsribution, and let Y be a random variable being independent of X and taking on the values 0, 1 with probabilities $1-\alpha$, α, respectively. If the statistic*

$$\hat{f} = \begin{cases} \dfrac{X+Y}{n+1}; & \text{if} \quad X+Y \leqq a+1 \\ 0 \quad ; & \text{if} \quad X+Y > a+1 \end{cases}$$

is an unbiased estimator for the random variable

$$f = \begin{cases} \alpha; & \text{if} \quad X \leqq a \\ 0; & \text{if} \quad X > a \end{cases}$$

for every integer a $(0 \leqq a \leqq n-1)$, then X has a binomial distribution with parameters n, and α.

The PROOF is omitted, because it is similar to the proof of Theorem 4. 1.

As a corollary of Theorem 3. 1. the following theorem holds:

THEOREM 4. 3. *Let $t_1, t_2, \ldots, t_r$ be identically distributed continuous, independent and non-negativ random variables having the same density function $f(x; \lambda)$, where $\lambda>0$ is the parameter of the distribution. Let us denote the random variable $t_1+t_2+ +\ldots t_r$ by T_r. If the estimator*

$$\hat{f}_2 = \begin{cases} \dfrac{1}{T_2}; & \text{if} \quad T_2 \geqq T_0 \\ 0 \; ; & \text{if} \quad T_2 < T_0 \end{cases}$$

is an unbiased estimator of the variable

$$f_1 = \begin{cases} \lambda; & \text{if} \quad T_1 \geqq T_0 \\ 0; & \text{if} \quad T_1 < T_0 \end{cases}$$

for every real number $T_0 \geqq 0$, then the variables $t_1, t_2, \ldots, t_r$ are exponentially distributed with parameter λ and consequently T_r has a gamma distribution with parameters λ and r.

5. The application of a pooling method for the estimation after selection

KITAGAWA [1] discussed the following problem: Consider two samples drawn from two normal populations with parameters (μ_1, σ_1) and (μ_2, σ_2), respectively. The sample sizes are n_1 and n_2, respectively. Supposing σ_1 and σ_2 to be unknown KITAGAWA gave the following estimator for μ_1

$$\bar{\bar{X}} = \begin{cases} \dfrac{n_1\bar{X}_1 + n_2\bar{X}_2}{n_1+n_2}; & \text{if} \quad |t| < t_\alpha(n_1+n_2-2) \\ \bar{X}_1 \qquad ; & \text{if} \quad |t| \geqq t_\alpha(n_1+n_2-2) \end{cases}$$

$$|t| = \frac{|\bar{X}_1 - \bar{X}_2|}{\left(\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1+n_2-2}\right)^{\frac{1}{2}} \left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)^{\frac{1}{2}}}$$

$$\bar{X}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij}; \qquad (i = 1, 2)$$

where

$$s_i^2 = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2; \qquad (i = 1, 2)$$

$t_\alpha(n_1+n_2-2)$ is the α percentage point of the non-central t distribution with (n_1+n_2-2) degrees of freedom.

In our paper the following problem will be discussed:
Let us suppose we have two types of components, A_1 and A_2, the theoretical values of their reliability characteristics μ_1 and μ_2 are unknown, but we know the values of their estimators X_1 and X_2; let us suppose these estimators to be unbiased.

$$\mathsf{E}(X_i) = \mu_i; \qquad (i = 1, 2)$$

Let us suppose that the component having the less observed value X_i will be chosen. In addition let us suppose that X_1 and X_2 are normally distributed random variables with unknown expectations μ_1 and μ_2, and with known variances σ_1^2 and σ_2^2. If A_i is the component chosen, let us introduce the notation

$$m = \mu_i,$$

i.e.

$$m = \begin{cases} \mu_1; & \text{if} \quad X_1 \leqq X_2 \\ \mu_2; & \text{if} \quad X_2 < X_1 \end{cases} \tag{5.1}$$

SARKADI [4] gave a biased estimator of the random variable in (5. 1), but the bias can be made as small as desired. Another estimator of m is given by using a pooling

method (see [4. p. 344])

$$\hat{m} = \begin{cases} X_1 & ; \text{ if } X_2 - X_1 > K \\ \dfrac{\sigma_2^2 X_1 + \sigma_1^2 X_2}{\sigma_1^2 + \sigma_2^2}; & \text{if } |X_2 - X_1| < K \\ X_2 & ; \text{ if } X_1 - X_2 > K \end{cases}$$

where K is *a* predetermined percentage point of the random variable $|X_2 - X_1|$. The expectation of m (see [4]):

$$\mathsf{E}(\hat{m}) = \mu_1 + (\mu_2 - \mu_1)\,\Phi\left(\frac{\mu_1 - \mu_2}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right),$$

where $\Phi(X)$ denotes the standardized normal distribution function.

After some calculations we obtain the expectation of $\hat{m}$

$$\mathsf{E}(\hat{m}) = \mu_1 + \frac{\mu_2 - \mu_1}{\sigma_1^2 + \sigma_2^2}\left[\sigma_1^2 \Phi\left(\frac{K - (\mu_2 - \mu_1)}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right) + \sigma_2^2 \Phi\left(\frac{-K - (\mu_2 - \mu_1)}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right) - \right.$$

$$\left. - \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}}\left[\sigma_1^2 \varphi\left(\frac{K - (\mu_2 - \mu_1)}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right) + \sigma_2^2 \varphi\left(\frac{K + (\mu_2 - \mu_1)}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right)\right]\right],$$

where

$$\varphi(x) = \Phi'(x).$$

It can be seen $\hat{m}$ is a biased estimator of m; the bias cannot be made as small as desired and the estimator is biased in the case $\mu_1 = \mu_2$, too.

A similar estimator can be given in the case of unknown variances, σ_1^2 and σ_2^2, too. The properties of the estimator are the same as in the above discussed case. Here we do not enter into the details of calculations leading to complicated formulae.

REFERENCES

[1] KITAGAWA, T.: Estimation after Preliminary Test of Significance, *University of California Publications in Statistics*. **3** (1963) 147—186.
[2] RUBINSTEIN, D.: On the Estimation of Failure Rates of Systems in Development. (In print).
[3] RUBINSTEIN, D.: On the Estimation of Failure Rates of Systems in Development, *Ann. Math. Statist.* **32** (1964) 924.
[4] SARKADI, K.: Estimation after Selection, *Studia Sci. Math. Hungar.* **2** (1967) 341—350.
[5] КОЛМОГОРОВ, А. Н.: Несмещенные оценки, *Изв. Акад. Наук СССР Сер. Мат.* **14** (1950) 303—326.
[6] LUKÁCS, E.: *Characteristic Function*, New York, Hafner, 1960.

Research Institute for Industrial Electronics, Budapest

(Received November 28, 1967.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 161–170.

CORRELATION PROPERTIES OF THE INTERVALS IN THE SUPERPOSITIONS OF INDEPENDENT STATIONARY RECURRENT POINT PROCESSES*

by

R. V. AMBARTZUMIAN

1. Let Π_i, $i=1, \ldots, n$ be independent, stationary, recurrent point processes and let Π be their superposition.

We denote by

(I) $$X_0, X_1, X_2, \ldots$$

the lengths of consecutive intervals between the events (points) in Π, starting from an arbitrary event in Π. In the present paper the sequence ϱ_K, $K=1, 2, \ldots$ of correlation coefficients between X_0 and X_k is investigated, while the following assumptions are imposed upon the distribution functions $1-F_i(x)$ of renewal times in Π_i, $i=1, 2, \ldots, n$.

ASSUMPTION A. *There exists $a>0$ such that the integrals representing the Laplace transforms*

$$l_i(s) = -\int e^{-sx} dF_i(x), \qquad i = 2, 3, \ldots, n$$

converge in the stripe $-a < \text{real } s < 0$.

ASSUMPTION B. *Each of the d. f's $1-F_i(x)$ for $i=2, 3, \ldots, n$ possesses non-vanishing absolute continuous component.*

As to regard of the d. f. $1-F_1(x)$, we always assume the existence of

$$\sigma_1^{-1} = -\int x\, dF_1(x)$$

and in the sequel stronger assumptions about the behaviour of the Laplace transform

$$l_1(s) = -\int e^{-sx} dF_1(x)$$

at $s=0$ are being stated when needed.

By assumptions A and B it is shown (for the case $n=2$) that the generating function

$$\varphi(z) = \sum_{n=1}^{\infty} \varrho_n z^n$$

is analytical everywhere on the periphery of the unit circle except may be of the

* This work was carried out during author's stay in Budapest on the exchange program of the Hungarian Academy of Sciences, Sept. 1967.

MAGYAR TUDOMÁNYOS AKADÉMIA KÖNYVTÁRA

point $z=1$. The $\varphi(z)$ is shown to possess K derivatives at $z=1$, if and only if the moment

$$-\int x^{K+1}\,dF_1(x)$$

is finite. $\varphi(z)$ is shown to be analytical at $z=1$, if and only if $l_1(s)$ is analytical at $s=0$.

The relationship

$$\frac{c_0}{2}+(c_0-1)\varphi(1)=\sum_{K=1}^{n}\frac{\mu_K\sigma_K^3}{2}$$

or equivalently

$$\frac{c_0}{2}+(c_0-1)\varphi(1)-1=\lim_{t\to\infty}[M(t)-t],$$

where

$$c_0=\mathsf{E}X_0^2,\qquad \mu_K=-\int x^2\,dF_K(x),\qquad \sigma_K^{-1}=-\int x\,dF_K(x),$$

$M(t)$ is the „renewal function" for Π, is derived.

In the special case, when $n=2$ and Π_2 is assumed poissonian it is shown, that if $\varphi(z)$ is reduced to a polynom, we have $\varphi(z)\equiv 0$ and Π_1 is a Poisson process.

2. Without loss of generality we can (and do) assume, that

$$\sigma_1+\ldots+\sigma_n=1.$$

We introduce the distributions

$$\Phi^{\langle m\rangle}(x)=\mathsf{P}\{X_0+X_1+\cdots+X_m>x\}$$

as well as the distributions $F_i^{(m)}(x)$, so that $1-F_i^{(m)}(x)$ is the $m+1$. fold convolution of $1-F_i(x)$, $i=1,\ldots,n$. It is well known, that $\Phi^{\langle 0\rangle}(x)$ is expressed in terms of $F_i^{(0)}(x)$ by the following relation (see [1]).

$$\int_x^\infty \Phi^{\langle 0\rangle}(u)\,du=\sigma_1\int_x^\infty F_1(u)\,du\ldots\sigma_n\int_x^\infty F_n(u)\,du. \tag{2}$$

One can obtain similar expressions for $\Phi^{\langle m\rangle}(x)$ by the following reasoning.

For any stationary point process Γ, we can introduce a new one $\overset{p}{\Gamma}$ by cancelling some of its events, the cancelling being carried out in such a way, that each event may be cancelled with probability p (and retained in its place with probability $1-p$), independent of cancelling of any other event and the position of events on the line. It is easily seen, that $\overset{p}{\Pi}$ remains to be a superposition of independent stationary recurrent point processes, namely $\overset{p}{\Pi}_i$, $i=1,\ldots,n$. So (2) remains valid, when we substitute in (2) instead of each quantity another one, which has analogous meaning for the corresponding process $\overset{p}{\Pi}$ or $\overset{p}{\Pi}_i$, $i=1,\ldots,n$. In this way we come to the identity

$$(1-p)\int_0^\infty\sum_{k=0}^{\infty}(1-p)p^k\Phi^{\langle k\rangle}(u)\,du=\prod_{i=1}^{n}(1-p)\sigma_i\int_x^\infty\sum_{k=0}^{\infty}(1-p)p^kF_i^{(k)}(u)\,du.$$

This identity was used to obtain the expressions for $\Phi^{\langle m\rangle}(x)$ in [2]. Restricting ourselves to double integral

$$(1-p)\int_0^\infty dx \int_x^\infty \sum_{k=0}^\infty (1-p)p^k \Phi^{\langle k\rangle}(u)\,du$$

we shall show, that it is closely related with the function $\varphi(z)$. Indeed, changing the order of summation and integration, we get

$$(3)\qquad (1-p)^2\int_0^\infty dx\int_x^\infty \sum_{k=0}^\infty p^k \Phi^{\langle k\rangle}(u)\,du = (1-p)^2 \sum_{k=0}^\infty P^k \int_0^\infty dx \int_x^\infty \Phi^{\langle k\rangle}(u)\,du =$$

$$= (1-p)^2 \sum_{k=0}^\infty P^k \frac{m_k}{2},$$

where

$$m_k = \mathsf{E}(X_0+X_1+\cdots+X_k)^2,$$

Put

$$c_k = \mathsf{E}X_0X_k, \qquad c_0 = m_0$$

Then making use of the stationarity of the sequence (1), we find

$$m_i = (i+1)c_0 + 2ic_1 + 2(i-1)c_2 + \ldots + 2c_i, \qquad i \geqq 1$$

the equation (3) is transformed to

$$(1-p)^2\int_0^\infty dx\int_x^\infty \sum_{k=0}^\infty p^k \Phi^{\langle k\rangle}(u)\,du =$$

$$= \frac{(1-p)^2}{2}\sum_{i=0}^\infty p^i\left[(i+1)c_0 + \sum_{k=1}^i (i+1-k)c_k\right] = \frac{c_0}{2} + \sum_{k=1}^\infty c_k p^k$$

and finally substituting instead of c_k their expressions

$$c_k = \varrho_k(c_0-1)+1,$$

we find

$$(1-p)^2\int_0^\infty dx\int_x^\infty \sum_{k=0}^\infty p^k \Phi^{\langle k\rangle}(u)\,du = \frac{c_0}{2} + (c_0-1)\varphi(p) + \frac{p}{1-p}.$$

Thus the generating function $\varphi(z)$ is to be found by calculating the right-hand side integral in the equation

$$(4)\qquad \frac{c_0}{2} + (c_0-1)\varphi(p) + \frac{p}{1-p} = \int_0^\infty dx \prod_{k=1}^\infty (1-p)^2\sigma_k \int_x^\infty \sum_{i=0}^\infty p^i F_k^{(i)}(u)\,du.$$

For this purpose we use the theorem on complex convolutions.

Denote by $L_k(s)$ the Laplace transforms

$$L_k(s) = \int_0^\infty e^{-sx}\,dx \prod_{i=1}^k (1-p)^2\sigma_i \int_x^\infty \sum_{l=0}^\infty p^l F_i^{(l)}(u)\,du, \qquad k = 1, 2, \ldots, n$$

then, on the basis of the mentioned theorem, we find

$$L_n(s) = \frac{1}{2\pi j} \int_{\alpha - j\infty}^{\alpha + j\infty} L_{n-1}(z) \hat{l}_n(s-z, p)\, dz, \tag{5}$$

where

$$\hat{l}_n(s, p) = (1-p)^2 \sigma_n \int_0^\infty e^{-sx}\, dx \int_x^\infty \sum_{k=0}^{\infty} p^k F_n^{(k)}(u)\, du$$

and integration is performed along the vertical line of the complex plane, connecting the points $\alpha - j\infty$ and $\alpha + j\infty$ with α and s satisfying

$$\text{real } s > a, \quad a > 0.$$

Using the relationships

$$\int_0^\infty e^{-sx}\, dx \int_x^\infty F_n^{(k)}(u)\, du = \frac{k+1}{s\sigma_n} - \frac{1}{s^2} + \frac{l_n^{k+1}(s)}{s^2}, \qquad k = 0, 1, 2, \ldots$$

where

$$l_n(s) = -\int e^{-sx}\, dF_n(x), \tag{6}$$

rewrite the equation (4) as follows

$$L_n(s) = \frac{1}{2\pi j} \int_{\alpha - j\infty}^{\alpha + j\infty} L_{n-1}(z) \left[\frac{1}{s-z} - \frac{(1-p)\sigma_n}{(s-z)^2} + \frac{(1-p)^2 \sigma_n}{(s-z)^2} \frac{l_n(s-z)}{1 - p l_n(s-z)} \right] dz. \tag{7}$$

To evaluate the right-hand side of the equation (4) we need only to find the limit $\lim_{s \to 0} L_n(s)$.

Now to simplify the calculations, assume that $n = 2$. Making use of the assumption A, we find that the function

$$\hat{l}_2(z, p) = \frac{1}{z} - \frac{(1-p)\sigma_2}{z^2} + \frac{(1-p)^2 \sigma_2}{z^2} \frac{l_2(z)}{1 - p l_2(z)}$$

is analytical in a certain stripe

$$-a(p) < \text{real } z < 0,$$

whose width now depends on p. This permits to put $s = 0$ in (5), choosing α to satisfy $0 < \alpha < a(p)$. Notice, that

$$L_1(z) = \frac{1}{z} - \frac{(1-p)\sigma_1}{z^2} + \frac{(1-p)^2 \sigma_1}{z^2} \frac{l_1(z)}{1 - p l_1(z)}.$$

So we get

$$L_2(0) = \frac{1}{2\pi j} \int_{\alpha - j\infty}^{\alpha + j\infty} \left[\frac{1}{z} - \frac{(1-p)\sigma_1}{z^2} + \frac{(1-p)^2 \sigma_1 l_1(z)}{z^2[1-pl_1(z)} \right] \cdot \tag{8}$$

$$\cdot \left[-\frac{1}{z} - \frac{(1-p)\sigma_2}{z^2} + \frac{(1-p)^2 \sigma_2 l_2(-z)}{z^2[1-pl_2(-z)]} \right] dz .$$

After some straightforward calculations (6) is transformed to

$$\frac{c_0}{2} + (c_0 - 1)\,\varphi(p) + \frac{p}{1-p} = L_2(0) = \frac{\sigma_2}{2} \frac{l_2''(0)(1-p) + 2pl_2'^2(0)}{1-p} + \tag{9}$$

$$+ \frac{\sigma_1 \sigma_2}{6} \frac{(1-p)^2 l_2'''(0) + 6p(1-p) l_2'(0) l_2''(0) + 6p^2 l_2'^3(0)}{1-p} +$$

$$+ \frac{(1-p)^4 \sigma_1 \sigma_2}{2\pi j} \int_{\alpha - j\infty}^{\alpha + j\infty} \frac{l_1(z) l_2(-z)\, dz}{z^4 [1-pl_1(z)][1-pl_2(-z)]} .$$

This representation is convenient for the investigation of the behaviour of $\varphi(z)$, which is analytical inside the unit circle, in the points of its periphery. Note that the singular points of the function under integral in (9)

$$\frac{1}{z^4} \frac{l_1(z)}{1-pl_1(z)} \frac{l_2(-z)}{1-pl_2(-z)} \tag{10}$$

which lie in the stripe $0 < \text{real}\, z < a$ coincide with the solutions of the equation

$$1 - pl_2(-z) = 0. \tag{11}$$

From the assumption B it follows that

$$\text{supremum}\ |l_2(jt)| < 1, \text{ when } t \text{ is real}$$

$$|t| > \varepsilon, \quad \varepsilon > 0.$$

So the assumption A and B imply the existence of a small positive quantity $\delta(\varphi)$, $0 < \varphi < 2\pi$ which is independent of p, such that the equation (11) has no solutions in the stripe real $z < \delta(\varphi)$ when p changes in the sufficiently small neighbourhood of $e^{i\varphi}$, $\varphi \neq 0$.

Thus, when $p \to e^{i\varphi}$, $\varphi \neq 0$, the path of integration in (9) may be left unchanged, with α, say, being taken equal to $\delta(\varphi)$.

At the same time the uniform convergent in some neighbourhood of $p = e^{i\varphi}$, $\varphi \neq 0$ the integral

$$\int_{\delta(\varphi) - j\infty}^{\delta(\varphi) + j\infty} \frac{1}{z^4} \frac{l_1(z)}{1-pl_1(z)} \frac{l_2(-z)}{1-pl_2(-z)} dz$$

represents an analytical function of p in that neighbourhood. We come to the result:

By the assumptions A) and B) $\varphi(z)$ is analytical in every point of the periphery of the unit circle, with exception, may be, of the point $z=1$.

Consider the behaviour of $\varphi(z)$ near the point $z=1$. Denote by $w(u)$ that branch of the solution of the equation

$$1-(1-u)l_2(-w) = 0$$

which vanishes when $u=0$. The specification of such a branch is possible because $l_2'(0) = -\sigma_2^{-1} \neq 0$. Assumptions A) and B) ensure the existence of such $c>0$ that $w(u)$ is the only singular point of the function (10) in the stripe $0<$ real $z<c$ when u is small enough. Rewrite (9) changing the path of integration to real $z=c$ and taking account of the residue at the point $z=w(u)$ as follows

$$\frac{c_0}{2}+(c_0-1)\varphi(1-u)+\frac{1-u}{u} = \frac{\sigma_2}{2}\frac{l_2''(0)u+2(1-u)l_2'^2(0)}{u}+ \tag{12}$$

$$+\frac{\sigma_1\sigma_2}{6}\frac{u^2 l_2'''(0)+6u(1-u)l_2'(0)l_2''(0)+6(1-u)^2 l_2'^3(0)}{u}+$$

$$+u^4\sigma_1\sigma_2\frac{1}{w^4(u)}\frac{l_1(w(u))w'(u)}{1-(1-u)l_1(w(u))}+$$

$$+u^4\sigma_1\sigma_2\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}\frac{1}{z^4}\frac{l_1(z)}{1-(1-u)l_1(z)}\frac{l_2(-z)}{1-(1-u)l_2(-z)}dz.$$

Note that the integral

$$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}\frac{1}{z^4}\frac{l_1(z)}{1-(1-u)l_1(z)}\frac{l_2(-z)}{1-(1-u)l_2(-z)}dz$$

is analytical at $u=0$; so is $w(u)$. A straightforward calculation shows, that the term, proportional to u^{-1} in the expansion of $\varphi(1-u)$, read from (12), vanishes. By the assumption that the first one of the moments

$$\mu_K = -\int x^2\, dF_k(x), \qquad k=1,2,$$

is finite, the calculation in (10) of the term, proportional to u^0 gives

$$\frac{c_0}{2}+(c_0-1)\varphi(1) = \frac{\mu_1\sigma_1^3}{2}+\frac{\mu_2\sigma_2^3}{2}. \tag{13}$$

In general, the function $\varphi(1-u)$ is seen from (12) to possess k derivatives at $u=0$, if and only if $l_2(z)$ possesses $k+1$ derivatives at $z=0$. Again, the function $\varphi(1-u)$ is seen to be analytical at $u=0$, if and only if $l_1(z)$ is analytical at $z=0$.

To extend (13) to the case of general, return to (7). First change the path of integration in (7) to become

$$\text{real } z=\beta, \quad \text{real } s<\beta<\text{ real }(s+w_n(u)),$$

where $w_i(u)$ are those branches of the solutions of the equations

$$1-(1-u)l_i(-w(u)) = 0, \qquad i=1, \ldots, n$$

which vanish at $u=0$. Since the residue due to the point $z=s$ vanishes, we get

$$L_n(s, u) = \frac{\sigma_n u^2}{2\pi j} \int_{\beta-j\infty}^{\beta+j\infty} L_{n-1}(z, u) \frac{dz}{(s-z)^2} \frac{l_n(s-z)}{1-(1-u)l_n(s-z)}. \tag{14}$$

Choose γ in such a way, that the point $s+w_n(u)$ is the only singular point of the function under integral in (14) in the stripe

$$\text{real } s < \text{real } z < \gamma + \text{reals } s,$$

when u is taken small enough. Next, choosing s to satisfy $\text{real } s < \gamma - \text{real } w(u)$, we may shift the path of integration in (14) to become

$$\text{real } z = \gamma,$$

which implies

$$L_n(s, u) = \frac{\sigma_n u^2 L_{n-1}(s+w_n(u), u)}{w_n^2(u)} w_n'(u) + \tag{15}$$

$$+ \frac{\sigma_n u^2}{2\pi j} \int_{\gamma-j\infty}^{\gamma+j\infty} \frac{L_{n-1}(z, u) l_n(s-z)\, dz}{(s-z)^2[1-(1-u)l_n(s-z)]}.$$

The integral

$$\int_{\gamma-j\infty}^{\gamma+j\infty} L_{n-1}(z, u) \frac{1}{(s-z)^2} \frac{l_n(s-z)}{1-(1-u)l_n(s-z)} dz$$

is an analytical function of u at the point $u=0$ and remains such when we pass to the limit when $s \to 0$. For that reason (15) may be written as

$$L_n(s, u) = \frac{\sigma_n u^2 L_{n-1}(s+w_n(u), u)}{w_n^2(u)} w_n'(u) + O(u^2), \tag{16}$$

the above equation being valid when $s=0$, also. From (16) it is easy to derive, that

$$\frac{c_0}{2} + (c_0-1)\varphi(1-u) + \frac{1-u}{u} = \prod_{k=2}^{n} \frac{\sigma_k u^2 w_k'(u)}{w_k^2(u)} L_1(w_2 + \ldots + w_n, u) + o(u^2) \tag{17}$$

with

$$L_1(z, u) = \frac{1}{z} + \frac{\sigma_1 u}{z^2} + \frac{\sigma_1 u^2}{z^2} \frac{l_1(z)}{1-(1-u)l_1(z)}.$$

The desired generalisation of (13)

$$\frac{c_0}{2} + (c_0-1)\varphi(1) = \sum_{K=1}^{n}{}' \frac{\mu_k \sigma_k^3}{2} \tag{18}$$

now may be derived by a straightforward calculation of the term, proportional to u^0 in the expansion of (17) on the assumption, that μ is finite. One may check (18) by assuming that the components are poissonian. Indeed, putting $\mu_k = \frac{2}{\sigma_k^2}$, $c_0 = 2$ and $\varphi(1) = 0$ reduces (18) to an identity.

The following observation may be of interest: Consider the expected number $M(t)$ of the events in Π, happening during the period $(0, t)$ on the condition that at $t = 0$ an arbitrary event of Π has taken place. Making use of the assumption of stationarity of the components and their independence, we find

$$M(t) = \sum_{i=1}^{n} \sigma_i [(1-\sigma_i)t + m_i(t)],$$

where $m_i(t)$ is the renewal function for Π_i, $i = 1 \ldots n$.

The well-known result of the renewal theory states

$$m_i(t) = \sigma_i t + \frac{\mu_i}{2}\sigma_i^2 - 1 + o(1) \tag{19}$$

and thus

$$M(t) = t + \sum_{i=1}^{n} \frac{\mu_i \sigma_i^3}{2} - 1 + o(1).$$

So, we come to the result

$$\frac{c_0}{2} + (c_0 - 1)\varphi(1) - 1 = \lim_{t \to \infty} [M(t) - t]. \tag{20}$$

The corresponding relationship stated in terms of the Laplace transform $\int e^{-st} dM(t)$ was first proposed by J. A. MCFADDEN in [3], where pure analytical conditions for its validity were outlined. In fact, in [3] these conditions were justified only for recurrent point processes, when (20) is reduced to (19).

3. In this section we consider the case, when $n = 2$ and Π_2 is Poissonian, with positive parameter λ. The assumptions A and B are in this case satisfied, and we can begin directly from (12), where we have to put

$$\sigma_1 = 1 - \lambda, \quad \sigma_2 = \lambda, \quad l_2(s) = \frac{\lambda}{\lambda + s}, \quad l_2'(o) = -\frac{1}{\lambda},$$

$$l_2''(o) = \frac{2}{\lambda^2}, \quad l_2'''(o) = -\frac{6}{\lambda^3}, \quad w(u) = \lambda u.$$

The integral in the right-hand side of (12) vanishes and we establish

$$\frac{c_0}{2} + (c_0 - 1)\varphi(1-u) + \frac{1-u}{u} = \frac{1}{\lambda u} - \frac{1-\lambda}{\lambda^2 u} + \frac{1-\lambda}{\lambda^2} \frac{l_1(\lambda u)}{1 - (1-u) l_1(\lambda u)}. \tag{21}$$

From (21), $l_1(z)$ is found to be

(22)
$$l_1(z) = \frac{\lambda z\left(\frac{c_0}{2}-1\right)+\lambda z(c_0-1)\varphi\left(1-\frac{z}{\lambda}\right)+(1-\lambda)^2}{\left(1-\frac{c_0}{2}\right)z^2+\lambda\left(\frac{c_0}{2}-1\right)z+(1-\lambda)z+z(\lambda-z)(c_0-1)\varphi\left(1-\frac{z}{\lambda}\right)+(1-\lambda)^2}.$$

Thus the quantities λ, c_0 and $\varphi(z)$ uniquely determine the distribution function $1-F_1(x)$. Note that, as it was shown in [2], λ may be expressed in terms of Π as the solution of the equation

$$(3m_0^2-2v)\lambda^2+(6m_1-24m_0+2v)\lambda-6m_1+12m_0+12=0,$$

where

$$m_0=\mathsf{E}X_0^2,\quad m_1=\mathsf{E}(X_0+X_1)^2,\quad v=\mathsf{E}X_0^3$$

which lies in the interval (0, 1). Assume now, that $\varphi(z)\equiv 0$; that means, then the intervals between the events of Π are assumed to be noncorrelated. The expression (22) is reduced to

$$l_1(z)=\frac{\lambda z\left(\frac{c_0}{2}-1\right)+(1-\lambda)^2}{\left(1-\frac{c_0}{2}\right)z^2+\lambda\left(\frac{c_0}{2}-1\right)z+(1-\lambda)z+(1-\lambda)^2}.$$

If $c_0\neq 2$, then for large positive values of z the function (12) has to become negative, and this contradicts to its definition (6). Thus, the only possibility for $\varphi(z)$ to vanish identically arises when $c_0=2$, or in other words, when

$$l_1(z)=\frac{1-\lambda}{z+1-\lambda}.$$

This corresponds to the case of poissonian Π_1.

The same result is valid by a milder assumption of $\varphi(z)$ being reduced to a polynom. Indeed, let

$$\varphi(z)=\varrho_n z^n+\varrho_{n-1}z^{n-1}+\ldots+\varrho_1 z,\quad \varrho_n\neq 0.$$

When z tends to infinity, remaining positive, (17) is seen from (22) again to become negative, this time due to the fact, that the main terms in the numerator and denumerator of (22) are $(-1)^n\frac{z^{n+1}}{\lambda^{n-1}}(c_0-1)$ and $(-1)^{n+1}\frac{z^{n+2}}{\lambda^n}(c_0-1)$, correspondingly, the whole expression being equivalent to

$$l_2(z)=-\frac{\lambda}{z}+o\left(\frac{1}{z}\right).$$

This ensures, that $\varphi(z)\equiv 0$, and thus Π_1 is poissonian. By a similar argument one can also show, that if $\varrho_n=\varrho_1^n$, then $\varrho_1=0$, and Π_1 is again Poissonian.

REFERENCES

[1] COX, D. R. and SMITH, WALTER L.: On the Superposition of renewal processes, *Biometrika*, **41** (1954) 91—99.
[2] AMBARTZUMIAN, R. V.: Two inverse problems concerning the superposition of recurrent point processes, *J. Appl. Probability* **2** (1965) 449—454.
[3] MCFADDEN, J. A.: On the lengths of intervals in a stationary point processes, *J. Roy. Statist. Soc. Ser. A.* **13** (24) 364—382.

Institute of Mathematics and Mechanics Erevan, Armenia, USSR

(Received December 27, 1967.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 171—178.

ÜBER DIE KONVERGENZ MULTIPLIKATIVER ZAHLENTHEORETISCHER FUNKTIONEN

von

G. HALÁSZ

Zur Beleuchtung des Problems erinnern wir an das Verhalten der Reihenentwicklung

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (s = \sigma + it)$$

der Riemannschen ζ-Funktion auf der Grenzlinie $\sigma = \mathrm{Re}\, s = 1$ der Konvergenzhalbebene $\sigma > 1$. Es ist eine elementare Tatsache, daß die Reihe überall auf dieser Geraden divergiert, ihre Partialsummen jedoch mit Ausnahme von $s = 1$ beschränkt sind. In der vorliegenden Arbeit beweisen wir, daß dies auch im allgemeinen der Fall ist, wenn die Eins im Zähler durch eine beliebige streng multiplikative zahlentheoretische Funktion $g(n)$ (d.h. $g(nm) = g(n)g(m)$ für ganze $n, m \geqq 1$) mit $|g(n)| \leqq 1$ ersetzt wird; die Ausnahmemenge besteht dann natürlich nicht aus einem einzigen Punkt, bildet aber eine Menge vom Maß 0. Mit anderen Worten:

Ist $g(n)$ streng multiplikativ, $|g(n)| \leqq 1$, so sind die partiellen Summen der Reihe

$$G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

fast überall auf der Gerade $\sigma = 1$ beschränkt. Ferner konvergieren sie sogar fast überall, falls sie es in einem einzigen Punkt tun.

Was die Multiplikativität anbelangt, sei zuerst erwähnt, dass sie nicht einfach weggelassen werden kann: die Partialsummen können in jedem Punkt unbeschränkt sein, wenn man nur $|g(n)| \leqq 1$ fordert. Zweitens bemerken wir, daß die Annahme, $g(n)$ sei streng multiplikativ, keine wesentliche Einschränkung gegenüber der üblichen Multiplikativität (d. h. wenn $g(nm) = g(n)g(m)$ nur für $(n, m) = 1$ verlangt wird) bedeutet: die erste Aussage gilt unverändert, die zweite — abgesehen von einem wenig interessanten Fall — auch für schwach multiplikative Funktionen. Durch diese Einschränkung wollten wir den bekannten elementaren Übergang von schwach zu streng multiplikativem $g(n)$ ersparen. Überigens wird diese zahlentheoretische Eigenschaft natürlich durch die Produktendarstellung

$$G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s} = \prod_p \left(1 + \frac{g(p)}{p^s} + \frac{g(p^2)}{p^{2s}} + \ldots\right) =$$

$$= \prod_p \left(1 + \frac{g(p)}{p^s} + \left(\frac{g(p)}{p^s}\right)^2 + \ldots\right) = \prod_p \frac{1}{1 - \frac{g(p)}{p^s}} \qquad (\sigma > 1,\ p \text{ prim})$$

ausgenützt.

Betreffend der Ausnahmemenge bemerken wir, daß man eine beliebige abzählbare Punktfolge aus $\sigma=1$ zu Unbeschränktheitspunkten machen kann. Wählt man sie überall dicht, so wird die Unbeschränktheitsmenge, nach allgemeinen mengentheoretischen Sätzen, die Mächtigkeit des Kontinuums haben.

Da die Beweise unserer zwei Aussagen sehr ähnlich sind, wird die erstere ausführlich begründet, und an manchen Stellen, wo der Beweis der letzteren etwas abweicht, wird auf die Abweichung in Fußnoten kurz hingewiesen.

Der Beweis wird jetzt aus zwei Teilen bestehen. Zuerst werden wir eine (auch notwendige) Bedingung geben, die die Beschränktheit in einem Punkt nach sich zieht (Satz 1). Als zweiter Schritt wird diese Bedingung für fast alle Punkte verifiziert (Satz 2).

Im ersten Teil verletzt man die Allgemeinheit nicht, wenn $s=1$ als der zuuntersuchende Punkt gewählt wird. Eine notwendige Bedingung kann man leicht finden. Es sei $s(x)$ die in Frage kommende Partialsumme

$$s(x)=\sum_{n\leq x}\frac{g(n)}{n}.$$

Durch partiellen Integration ergibt sich

$$G(s)=\sum_{n=1}^{\infty}\frac{g(n)}{n^s}=\int_1^{\infty}\frac{\mathrm{d}s(x)}{x^{s-1}}=(s-1)\int_1^{\infty}\frac{s(x)}{x^s}dx=(\text{falls } s(x)=O(1) \text{ wäre})=$$

$$=(s-1)\int_1^{\infty}\frac{O(1)}{x^s}dx=O\left(\frac{|s-1|}{\sigma-1}\right)^1$$

gleichmäßig für $\sigma>1$. (Zur Orientierung sei bemerkt, daß $G(s)$ in jedem Winkelraum mit Spitzenpunkt in $s=1$, aber übrigens ganz auf $\sigma>1$, beschränkt ist.) Wir zeigen, daß diese Bedingung schon hinreichend ist.

SATZ 1. *Ist $g(n)$ streng multiplikativ, $|g(n)|\leq 1$, und erfüllt die zugehörige Dirichletsche Reihe*

$$G(s)=O\left(\frac{|s-1|}{\sigma-1}\right)$$

gleichmäßig in der Halbebene $\sigma>1$, dann ist

$$s(x)=\sum_{n\leq x}\frac{g(n)}{n}=O(1).$$

BEWEIS. Dieser Satz wird auf eine elementare Weise auf eine frühere Arbeit [1] von uns zurückgeführt. Dort haben wir das entsprechende Taubersche Problem

[1] bzw. $=(\text{falls } s(x)=C+o(1) \text{ ist})=(s-1)\int_1^{\infty}\frac{C+o(1)}{x^s}dx=C+o\left(\frac{|s-1|}{\sigma-1}\right)$ für die zweite Aussage (im folgenden kurz z. A.).

für den Mittelwert

$$\frac{1}{x}\sum_{n\leqq x} f(n)$$

untersucht und haben gefunden, daß die entsprechende notwendige Bedingung ebenfalls auch hinreichend ist, nicht nur im Falle $|f(n)|\leqq 1$ (wo die Beschränktheit des Mittelwertes trivial ist), sondern unter den folgenden allgemeineren Voraussetzungen:

SATZ.

a) *$f(n)$ ist multiplikativ,*

b) $f(p)=O(1)$,

c) $\sum_{p}\sum_{k\geqq 2}\frac{|f(p^k)|}{p^k}<+\infty$,

d) *keiner der Faktoren*

$$1+\frac{f(p)}{p^s}+\frac{f(p^2)}{p^{2s}}+\ldots$$

der entsprechenden Dirichletschen Reihe

$$F(s)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s}=\prod_{p}\left(1+\frac{f(p)}{p^s}+\frac{f(p^2)}{p^{2s}}+\ldots\right)$$

verschwindet auf der Geraden $\sigma=1$,

e) $F(s)=O\left(\frac{|s|}{\sigma-1}\right)$ (*die notwendige Bedingung*).

Dann ist

$$\sum_{n\leqq x} f(n)=O(x).^{2}$$

Das Wesentlichste für uns ist die Tatsache, daß in b nicht $|f(p)|\leqq 1$, sondern nur $f(p)=O(1)$ gefordert ist (was einfach bloß $\sum_{n\leqq x} f(n)=O(x\log^A x)$ implizieren würde). Wir wenden nämlich diesen Satz für die Funktion

$$f(n)=\sum_{d|n} g(d)$$

an. Sie ist auch multiplikativ,

$$|f(p^k)|=|1+g(p)+g(p^2)+\ldots|\leqq k+1,$$

[2] Für die z. A. statt e) $F(s)=\frac{C}{s-1}+o\left(\frac{|s|}{\sigma-1}\right)$ mit der Behauptung $\sum_{n\leqq x} f(n)=Cx+o(x)$. Der Satz 3 in [1] wurde tatsächlich für diesen Fall ausgesprochen, der Beweis gibt ihn aber auch in der oben zitierten Form.

womit wir zugleich die Voraussetzungen a, b und c verifiziert haben, die letzte wegen

$$\sum_p \sum_{k\geqq 2} \frac{k+1}{p^k} = \sum_p \frac{1}{p^2} \frac{3-\frac{2}{p}}{\left(1-\frac{1}{p}\right)^2} \leqq 12 \sum_p \frac{1}{p^2} < +\infty.$$

Für die zugehörige Dirichletsche Reihe hat man

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{n=1}^{\infty} \frac{g(n)}{n^s} = \zeta(s)G(s).$$

Was also d betrifft, verschwinden weder die Faktoren $\dfrac{1}{1-\dfrac{1}{p^s}}$ von $\zeta(s)$, noch $\dfrac{1}{1-\dfrac{g(p)}{p^s}}$ von $G(s)$ auf $\sigma=1$. Endlich ist für $|s|\leqq 2$

$$\zeta(s) = O\left(\frac{1}{|s-1|}\right)$$

auf Grund des Pols in $s=1$ und nach Annahme

$$G(s) = O\left(\frac{|s-1|}{\sigma-1}\right),$$

d.h.

$$F(s) = \zeta(s)G(s) = O\left(\frac{1}{\sigma-1}\right) = O\left(\frac{|s|}{\sigma-1}\right),$$

während für $|s|\geqq 2$

$$\zeta(s) = O(\log|s|)$$

gleichmässig für $\sigma\geqq 1$ nach der bekannten Abschätzung und

$$|G(s)| = \left|\sum_{n=1}^{\infty} \frac{g(n)}{n^s}\right| \leqq \sum_{n=1}^{\infty} \frac{1}{n^\sigma} = \zeta(\sigma) \leqq \frac{2}{\sigma-1} \qquad (1<\sigma\leqq 2),$$

d.h.

$$F(s) = \zeta(s)G(s) = O\left(\frac{\log|s|}{\sigma-1}\right) = O\left(\frac{|s|}{\sigma-1}\right).^3$$

Der angeführte Satz behauptet jetzt

$$\sum_{n\leqq x} f(n) = \sum_{n\leqq x} \sum_{d|n} g(d) = \sum_{d\leqq x} g(d)\left[\frac{x}{d}\right] = O(x).$$

[3] Für die z. A. benützt man die entsprechenden Relationen mit $o(\ )$ für $|s|\leqq K$ statt $|s|\leqq 2$, während man für $|s|\geqq K$ auf obige Weise verfährt. Dann läßt man zuerst σ zu 1, nachher K zu ∞ streben.

Das Zeichen [ ] weggelassen, ergibt sich ebenfalls ein Fehler $O(x)$:

$$x \sum_{d \leqq x} \frac{g(d)}{d} = O(x),$$

$$\sum_{d \leqq x} \frac{g(d)}{d} = O(1)^4$$

und Satz 1 ist bewiesen.

SATZ 2. *Wie oben, sei $g(n)$ streng multiplikativ, $|g(n)| \leqq 1$. Für fast jedes feste a $(-\infty < a < +\infty)$ gilt dann*

$$G(s) = O\left(\frac{|s-(1+ia)|}{\sigma-1}\right)$$

gleichmäßig für $\sigma > 1$.[5]

BEWEIS. Wir beginnen mit einer allgemeinen Bemerkung. Wenn eine Dirichletsche Reihe

$$A(s) = \sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s}$$

für $\sigma > 1$ absolut konvergiert, so ist durch partielle Integration mit der Bezeichnung $\sum_{n \leqq x} \alpha(n) = M(x)$

$$A(s) = \int_1^{\infty} \frac{dM(x)}{x^s} = s \int_1^{\infty} \frac{M(x)}{x^{s+1}}\, dx,$$

$$\frac{A(s)}{s} = \int_1^{\infty} \frac{M(x)}{x^{s+1}}\, dx = \int_0^{\infty} \frac{M(e^u)}{e^u}\, e^{-(s-1)u}\, du.$$

Nach der Parsevalschen Gleichung gilt also

$$\int_{(\sigma)} \left|\frac{A(s)}{s}\right|^2 |ds| = 2\pi \int_0^{\infty} \frac{|M(e^u)|^2}{e^{2u}}\, e^{-2(\sigma-1)u}\, du,$$

[4] Für die z. A., um den Fehler durch Weglassung von [ ] kleiner zu machen, zerlegt man die Summe $\sum_{n \leqq x} f(n) = \sum_{md \leqq x} g(d)$ in zwei Teile: $d \leqq \frac{x}{K}$, $m \leqq \frac{x}{d}$ und $m < K$, $\frac{x}{K} < d \leqq \frac{x}{m}$. Im zweiten Teil kommt $\sum_{n \leqq y} g(n)$ vor, welches $o(y)$ ist, wie der zitierte Satz es mit $f(n) = g(n)$ gibt. Man läßt x und dann K zu ∞ streben.

[5] Für die z. A. muß man noch $G(s) = o\left(\frac{|s|}{\sigma-1}\right)$ annehmen und $G(s) = C + o\left(\frac{|s-(1+ia)|}{\sigma-1}\right)$ kann behauptet werden. Die z. A. folgt in der Tat aus Satz 2 und 1, da die Konvergenz in einem Punkt diese zusätzliche Bedingung $G(s) = o\left(\frac{|s|}{\sigma-1}\right)$ zur Folge hat (s. Fußnote 3, wobei $|s-1| \leqq |s|$ ist).

wobei das Integral auf der linken Seite über die Gerade Re $s=\sigma(>1)$ zu erstrecken ist. Je größer daher $|M(x)|$ ist, desto größer wird das Quadratintegral von $\left|\frac{A(s)}{s}\right|$ auf (σ).

Jetzt betrachten wir die Produktendarstellung

$$G(s)=\prod_p \frac{1}{1-\frac{g(p)}{p^s}}=e^{\sum_p \log \frac{1}{1-\frac{g(p)}{p^s}}}=e^{\sum_p \sum_{k=1}^{\infty} \frac{g^k(p)}{kp^{ks}}} \underset{\text{def}}{=} e^{\sum_{n=1}^{\infty} \lambda(n) \frac{g(n)}{n^s}},$$

wo

$$\lambda(n)=\begin{cases}\frac{1}{k}, & \text{wenn} \quad n=p^k \text{ ist,}\\ 0 & \text{sonst.}\end{cases}$$

Man kann $G(s)$ zur $^1/_3$-ten Potenz erheben:

$$G^{\frac{1}{3}}(s)=e^{\frac{1}{3}\sum_{n=1}^{\infty} \lambda(n) \frac{g(n)}{n^s}}$$

und e^z in seine Taylorsche Reihe entwickelnd, erhält man wieder eine für $\sigma>1$ absolut konvergente Dirichletsche Reihe. Dies sei das obige $A(s)$. Die Koeffizienten, folglich auch die summatorische Funktion $M(x)$, werden wegen $\lambda(n)\geqq O$ und $|g(n)|\leqq 1$ die größten sein, wenn $g(n)\equiv 1$, d.h. $G(s)$ die ζ-Funktion ist. Die vorausgeschickte Bemerkung gibt daher

$$\int_{(\sigma)} \left|\frac{G^{1/3}(s)}{s}\right|^2 |ds| \leqq \int_{(\sigma)} \left|\frac{\zeta^{1/3}(s)}{s}\right|^2 |ds|.$$

Das rechtsseitige Integral bleibt aber für $\sigma \to 1$ beschränkt, da wegen des Poles in $s=1$

$$\int_{\substack{(\sigma)\\ |s|\leqq 2}} \left|\frac{\zeta^{1/3}(s)}{s}\right|^2 |ds| \leqq \text{const} \int_{\substack{(\sigma)\\ |s|\leqq 2}} \frac{|ds|}{|s-1|^{2/3}} \leqq 2\,\text{const} \int_0^2 \frac{dt}{t^{2/3}} < +\infty,$$

und wegen $\zeta(s)=O(\log|s|)$ $(|s|\geqq 2)$

$$\int_{\substack{(\sigma)\\ |s|\geqq 2}} \left|\frac{\zeta^{1/3}(s)}{s}\right|^2 |ds| \leqq \text{const} \int_{1,5}^{\infty} \frac{\log^{2/3} t}{t^2}\, dt < +\infty$$

sind. Wir können also den bekannten Satz von FATOU heranziehen: Ist das Quadratintegral von $\frac{G^{1/3}(s)}{s}$ beschränkt, so strebt die Funktion $\frac{G^{1/3}(s)}{s}$ für fast alle a zu einem Grenzwert, wenn s zu $1+ia$ strebt, nicht nur auf der horizontalen Geraden,

sondern auch in jedem Winkelraum $\frac{|t-a|}{\sigma-1} \leqq \text{const}$ ($t = \text{Im}\, s$). Dasselbe trifft daher für $G^{1/3}(s)$, also auch für $G(s)$ selbst zu. Dies bedeutet speziell, daß die Behauptung des Satzes

$$G(s) = O\left(\frac{|s-(1+ia)|}{\sigma-1}\right)$$

für ein solches a mindestens im Winkelraum $\frac{|t-a|}{\sigma-1} \leqq 1$ besteht. Es gilt noch zu zeigen, daß daraus auch das Bestehen in der Allgemeinheit folgt. Dazu werden wir wieder von der Multiplikativität und der Annahme $|g(n)| \leqq 1$ einen wesentlichen Gebrauch machen. Ohne Beschränkung der Allgemeinheit sei $a = 0$.

Es liege also $s = \sigma + it$ außerhalb des Winkelraumes $\frac{|t|}{\sigma-1} \leqq 1$. Wir können etwa $|t| \geqq 1$ annehmen, andernfalls wäre die Abschätzung

$$|G(s)| = \left|\sum_{n=1}^{\infty} \frac{g(n)}{n^s}\right| \leqq \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \leqq \frac{\sigma}{\sigma-1} = O\left(\frac{|s-1|}{\sigma-1}\right) \qquad (|t| \geqq 1)$$

trivial. Wir müssen $|G(s)|$ mit $\frac{1}{\sigma-1}$ oder — auf Grund der mehrmals gebrauchten Asymptotik — mit $\zeta(\sigma)$ vergleichen. Beide werden in der exponentiellen Form der Produktendarstellung geschrieben:

$$\frac{|G(s)|}{\zeta(\sigma)} = \frac{\left|e^{\sum_{n=1}^{\infty} \lambda(n) \frac{g(n)}{n^s}}\right|}{e^{\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^\sigma}}} = e^{\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^\sigma} \text{Re}\frac{g(n)}{n^{it}} - \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^\sigma}} = e^{-\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^\sigma}\left(1-\text{Re}\frac{g(n)}{n^{it}}\right)}.$$

Wegen $|g(n)| \leqq 1$ hat die Summe im Exponent nichtnegative Glieder. Für festes t ist also der ganze Ausdruck eine monoton zunehmende Funktion von σ. Faßt man außer s auch den in derselben Höhe liegenden Punkt $s_1 = 1 + |t| + it$ des Randes unseres Winkelraumes ins Auge, so kann man schreiben

$$\frac{|G(s)|}{\zeta(\sigma)} \leqq \frac{|G(s_1)|}{\zeta(1+|t|)}.$$

Im Winkelraum ist aber $G(s_1)$ beschränkt, so daß hieraus

$$|G(s)| \leqq |G(s_1)| \frac{\zeta(\sigma)}{\zeta(1+|t|)} = O\left(\frac{|t|}{\sigma-1}\right) = O\left(\frac{|s-1|}{\sigma-1}\right)$$

folgt, wie behauptet wurde.[6]

[6] Für die z. A. benützt man für $|t| \geqq \frac{1}{K}$ die zusätzliche Bedingung $G(s) = o\left(\frac{|s|}{\sigma-1}\right)$. Für $|t| \leqq \frac{1}{K}$ betrachtet man den größeren Winkelraum $\frac{|t|}{\sigma-1} \leqq M$ und nützt die von Fatouschen Satz gesicherte Konvergenz von $G(s)$ im Mittelwerte nicht nur durch die Beschränktheit aus, übrigens gilt aber dieselbe Überlegung. Darauf führt man die Grenzübergänge $\sigma \to 1$, $K \to \infty$, $M \to \infty$ nacheinander durch.

LITERATURVERZEICHNIS

[1] HALÁSZ, G.: Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen, *Acta Math. Acad. Sci. Hung.* **19** (1968) 365—403.

Mathematisches Institut der Ungarischen Akademie der Wissenschaften, Budapest

(Eingegangen: 15. Januar, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1968) 179—190.

LAGRANGESCHE INTERPOLATION ÜBER DIE NULLSTELLEN DER HERMITESCHEN ORTHOGONALPOLYNOME

von

G. FREUD

A. Einleitung

Es seien

$$x_{1n} > x_{2n} > \ldots > x_{nn} \tag{1}$$

die Nullstellen des Hermiteschen Orthogonalpolynoms

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

und es seien

$$l_{kn}(x) = \frac{H_n(x)}{H_n'(x_{kn})(x - x_{kn})}$$

die Grundpolynome der Lagrangeschen Interpolation mit den Stellen (1) als Knotenpunkte.

Es sei $f(x)$ eine in $(-\infty, +\infty)$ gleichmäßig stetige beschränkte Funktion

$$\|f\| = \sup_{-\infty < x < +\infty} |f(x)| \tag{2}$$

und der Stetigkeitsmodul r-ter Ordnung von $f(x)$ sei

$$\omega_r(f; \delta) = \sup_{\substack{|h| \leqq \delta \\ -\infty < x < +\infty}} \left| \sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^\nu f(x + \nu h) \right|. \tag{3}$$

Wir betrachten die Folge der Interpolationspolynome

$$L_n(f; x) = \sum_{k=1}^{n} l_{kn}(x) f(x_{kn})$$

welche an den Stellen x_{kn} $(k = 1, 2, \ldots, n)$ den gleichen Wert wie $f(x)$ annehmen. Ziel vorliegender Arbeit ist der Beweis nachstehenden Satzes (Satz 2)

Es gilt für jedes reelle x mit absuluten Konstanten $K_r^{(1)}$ und $K_r^{(2)}$ $(r = 1, 2, \ldots)$

$$|f(x) - L_n(f; x)| \leqq [K_r^{(1)} \log n + K_r^{(2)} e^{x^2/2}] \omega_r(f; n^{-1/2}) + 2\|f\|(2e^{-1})^{n/2} e^{x^2/2}.$$

In dieser Abschätzung können die Konstanten $K_r^{(1)}$ und $K_r^{(2)}$ nicht durch die gleiche feste Nullfolge $\{\varepsilon_n\}$ ersetzt werden. (vgl. Satz 4 und Satz 5.)

Es ist recht merkwürdig, daß der Faktor von $\log n$ von x unabhängig ist.

B. Einige Abschätzungen

Die orthonormierten Polynome mit der Gewichtsfunktion e^{-x^2} sind

$$h_n(x) = \pi^{-1/4}(2^n n!)^{-1/2} H_n(x) = \gamma_n x^n + \dots \qquad (n=0, 1, \dots) \tag{4}$$

so daß

$$\gamma_n = \pi^{-1/4}\sqrt{\frac{2^n}{n!}} \tag{5}$$

ist. Es gilt dann (vgl. G. FREUD [4])

$$l_{kn}(x) = \frac{\gamma_{n-1}}{\gamma_n}\lambda_{kn}\frac{h_n(x)h_{n-1}(x_{kn})}{x-x_{kn}} = \sqrt{\frac{n}{2}}\lambda_{kn}\frac{h_{n-1}(x_{kn})}{x-x_{kn}}h_n(x), \tag{6}$$

wo $\lambda_{kn}>0 (k=1, 2, \dots n)$ die Christoffelschen Zahlen der Gauss—Jacobischen Quadraturformel

$$\int_{-\infty}^{+\infty} \pi_{2n-1}(x)e^{-x^2}\,dx = \sum_{k=1}^{n} \lambda_{kn}\pi_{2n-1}(x_{kn})$$

sind. Diese Formel ist für jedes Polynom $\pi_{2n-1}(x)$ höchstens $2n-1$-ten Grades gültig.

Es wird uns die Ungleichung von J. BALÁZS—P. TURÁN [1]

$$\sum_{k=1}^{n} \lambda_{kn} e^{(1/2)x_{kn}^2} \leqq \int_{-\infty}^{+\infty} e^{-(1/2)x^2}\,dx = c_0 \tag{7}$$

von Nutzen sein. Mit $c_0, c_1, \dots$ bezeichnen wir positive absolute Konstanten.

Die Funktion $R_n(x)=e^{-x^2/2}H_n(x)$ befriedigt die Differentialgleichung

$$R_n'' + q_n(x)R_n = 0, \quad q_n(x) = 2n+1-x^2 \tag{8}$$

und es ist

$$x_{1n} = |x_{nn}| < \sqrt{2n+1} \tag{9}$$

(vgl. G. SZEGŐ [8] Formeln (5. 5. 2) bzw. (6. 3. 2. 6)).

Es ist also $q_n(x) \leqq 2n+1$ und mit Hilfe des Sturmschen Satzes folgt

$$x_{in} - x_{i+1,n} \geqq \frac{\pi}{\sqrt{2n+1}} \qquad (i = 1, 2, \dots, n-1). \tag{10}$$

In dem Intervall $[\sqrt{-n}, \sqrt{n}]$ ist $q_n(x) \geqq n+1$ und aus dem Sturmschen Satz erhalten wir

$$x_{in} - x_{i+1,n} \leqq \frac{\pi}{\sqrt{n+1}} \qquad (|x_{in}|,\ |x_{i+1,n}| \leqq \sqrt{n}). \tag{11}$$

Es sei $|x_{kn}| \leqq \sqrt{n}-4$; dann fallen wegen (11) die zu x_{kn} banachbarten Nullstellen

$x_{k-1,n}$ und $x_{k+1,n}$ in $[-\sqrt{n}, \sqrt{n}]$. Mit Hilfe der Markoff—Stieltjesschen Ungleichung erhalten wir unter Beachtung von (11)

$$0 < \lambda_{kn} = \sum_{i=k}^{n} \lambda_{in} - \sum_{i=k+1}^{n} \lambda_{in} \leqq \int_{-\infty}^{x_{k-1,n}} e^{-x^2}\,dx -$$

$$(12) \qquad - \int_{-\infty}^{x_{k+1,n}} e^{-x^2}\,dx < (x_{k-1,n} - x_{k+1,n}) e^{-\left(|x_{kn}| - \frac{\pi}{\sqrt{n+1}}\right)^2} \leqq$$

$$\leqq \frac{2\pi}{\sqrt{n+1}} e^{\frac{2\pi}{\sqrt{n+1}}|x_{kn}|} e^{-x_{kn}^2} < c_1 n^{-1/2} e^{-x_{kn}^2}$$

$$(|x_{kn}| \leqq \sqrt{n-4}).$$

Es ist infolge der asymptotischen Formel von M. PLANCHEREL—W. ROTACH [7] (vgl. auch G. SZEGŐ [8] (8. 22. 12))

$$(13a) \qquad |h_n(x)| \leqq c_2 n^{-1/4} e^{x^2/2} \qquad (|x| \leqq \sqrt{n})$$

und nach einem Resultate von H. CRAMÉR (A. ERDÉLYI [4], Formel (19) S. 208 und C. V. L. CHARLIER [2] S. 49—52)

$$(13b) \qquad |h_n(x)| \leqq c_3 e^{x^2/2} \qquad (-\infty < x < +\infty).$$

Endlich sei bemerkt, daß die Beziehung

$$(14) \qquad H_n'(x) = 2nH_{n-1}(x) \quad \text{d. h.} \quad h_n'(x) = \sqrt{2n}\, h_{n-1}(x)$$

gültig ist.

C. Hilfssätze

HILFSSATZ 1. *Es gilt*

$$(15) \qquad K_n(x) = \sum_{k=1}^{n} \frac{l_{kn}^2(x)}{\lambda_{kn}} \leqq c_4 n^{1/2} e^{x^2}.$$

BEWEIS. (Vgl. G. FREUD [6]). Es gilt die Identität*

$$K_n(x) = \sum_{\nu=0}^{n-1} h_\nu^2(x)$$

Wir setzen in die Mehlersche Formel

$$\Phi(w, x) = \sum_{\nu=0}^{\infty} h_\nu^2(x) w^\nu = (1-w^2)^{-1/2} e^{2x^2 \frac{w}{1+w}}$$

* Vgl. etwa das Buch G. FREUD: Orthogonalpolynome, *Birkhäuser Verlag*, Basel, 1968, Formel I. (4. 7).

(vgl. A. ERDÉLYI [3], ferner A. ERDÉLYI [4], Formel (22) auf S. 194) $w=1-\frac{1}{n}$:

$$\left(1-\frac{1}{n}\right)^n K_n(x) \leqq \Phi\left(1-\frac{1}{n}, x\right) \leqq c_5 n^{1/2} e^{x^2},$$

und wegen $\left(1-\frac{1}{n}\right)^n \to e^{-1}$ folgt (15), w.z.b.w.

HILFSSATZ 2. *Es gilt für jedes n und* $1 \leqq k \leqq n$

$$|h_{n-1}(x_{kn})| \leqq c_6 n^{-1/4} e^{x_{kn}^2/2}. \tag{16}$$

BEWEIS. Aus der Christoffel—Darbouxschen Summenformel ergibt sich unter Beachtung von (5) und (14)

$$\sum_{\nu=0}^{n-1} h_\nu^2(x_{kn}) = \frac{\gamma_{n-1}}{\gamma_n} h_n'(x_{kn}) h_{n-1}(x_{kn}) = n h_{n-1}^2(x_{kn}), \tag{16a}$$

und wir erhalten (16) mit $c_6 = \sqrt{c_4}$ aus Hilfssatz 1, w.z.b.w.

HILFSSATZ 3. *Für* $|x| \leqq \sqrt{n}$ *gilt*

$$\sum_{|x-x_{kn}| \geqq 1} |l_{kn}(x)| \leqq c_7 e^{x^2/2}. \tag{17}$$

BEWEIS. Aus (6), (13a), (16) und (7) erhalten wir

$$\sum_{|x-x_{kn}| \geqq 1} |l_{kn}(x)| \leqq \sqrt{\frac{n}{2}}\, c_2 n^{-1/4} e^{x^2/2} \sum_{k=1}^{n} c_6 n^{-1/4} e^{x_{kn}^2/2} \lambda_{kn} \leqq c_2 c_6 c_0 e^{x^2/2}$$

w. z. b. w.

HILFSSATZ 4. *Es gilt*

$$\sum_{|x_{kn}| \geqq 2\sqrt{\log n}} |l_{kn}(x)| \leqq c_8 e^{x^2/2} \tag{18}$$

BEWEIS. Aus (7) und Hilfssatz 1 folgt

$$\sum_{k=1}^{n} |l_{kn}(x)| e^{(1/4) x_{kn}^2} \leqq \left[\sum_{k=1}^{n} \lambda_{kn}^{-1} l_{kn}^2(x)\right]^{1/2} \left(\sum_{k=1}^{n} \lambda_{kn} e^{(1/2) x_{kn}^2}\right)^{1/2} \leqq c_9 n^{1/4} e^{x^2/2}.$$

Für $|x_{kn}| \geqq 2\sqrt{\log n}$ ist $e^{(1/4) x_{kn}^2} \geqq n$,

so daß aus der letzten Formel (18) folgt, w. z. b. w.

HILFSSATZ 5. *Für* $|x| \geqq n^{\frac{1}{4}}$ *gilt*

$$\sum_{k=1}^{n} |l_{kn}(x)| \leqq c_{10} e^{x^2/2}. \tag{19}$$

BEWEIS. Infolge Hilfssatz 4 reicht es den Teil $|x_{kn}| \leqq 2\sqrt{\log n}$ der Summe (19) für $n^{1/4} > 2\sqrt{\log n}$ abzuschätzen. Es ist dann wegen (6), (13b), (16) und (7)

$$\sum_{|x_{kn}|<2\sqrt{\log n}} |l_{kn}(x)| \leqq \sqrt{\frac{n}{2}}\, c_3 e^{x^2/2} \frac{1}{n^{1/4}-2\sqrt{\log n}} c_6 n^{-1/4} \sum_{k=1}^{n} \lambda_{kn} e^{x_{kn}^2/2} \leqq c_{10} e^{x^2/2}$$

w. z. b. w.

HILFSSATZ 6. *Unter den Voraussetzungen* $|x| \leqq n^{1/4}$, $|x - x_{jn}| \leqq 10n^{-1/2}$ *gilt*

$$|l_{jn}(x)| \leqq c_{11}. \tag{20}$$

BEWEIS. Ohne Einschränkung der Allgemeinheit kann vorausgesetzt werden, daß n so groß ist, daß $n^{1/4} + 10n^{-1/2} \leqq n^{1/2} - 4$ erfüllt ist. Aus (12) und (15) ergibt sich wegen $|x_{jn}| \leqq |x| + |x - x_{jn}| \leqq n^{1/4} + 10n^{-1/2} \leqq n^{1/2} - 4$

$$|l_{jn}(x)| \leqq \left(\lambda_{jn} \sum_{k=1}^{n} \lambda_{kn}^{-1} l_{kn}^2(x)\right)^{1/2} \leqq e^{(1/2)x^2} e^{-(1/2)x_{jn}^2}.$$

Für $|x_{jn}| \geqq |x|$ folgt (20) aus dieser Ungleichung unmittelbar. Ist $|x_{jn}| \leqq |x|$, dann ist

$$e^{-(1/2)x_{jn}^2} = e^{-(1/2)[|x|-(|x|-|x_{jn}|)]^2} \leqq e^{-(1/2)x^2} e^{|x|(|x|-|x_{jn}|)} \leqq$$
$$\leqq e^{-(1/2)x^2} e^{n^{1/2}(On^{-1/2})} \leqq c_{11} e^{-(1/2)x^2},$$

und das zeigt die Gültigkeit von (20), w.z.b.w.

HILFSSATZ 7. Für $|x| \leqq n^{1/2} - 5$ gilt

$$\sum_{10n^{-1/2} \leqq |x - x_{kn}| \leqq 1} |l_{kn}(x)| \leqq c_{12}(\log n + e^{|x|}). \tag{21}$$

BEWEIS. (Aus (6), (12), (13a) und $|x - x_{kn}| \geqq 10n^{-1/2}$ folgt

$$|l_{kn}(x)| \leqq c_{13} e^{x^2/2} n^{-1/2} \frac{e^{-(1/2)x_{kn}^2}}{|x - x_{kn}|} \leqq c_{14} e^{x^2/2} \int_{x_{kn}-n^{-1/2}}^{x_{kn}+n^{-1/2}} \frac{e^{-(1/2)t^2}}{|x-t|}\, dt.$$

Aus (10) folgt, daß die Intervalle $[x_{kn} - n^{-1/2}, x_{kn} + n^{-1/2}]$ nicht übereinander greifen. Es ist also

$$\sum_{10n^{-1/2} \leqq |x - x_{kn}| \leqq 1} |l_{kn}(x)| \leqq c_{14} e^{x^2/2} \left\{ \int_{x-1}^{x-9n^{-1/2}} \frac{e^{-(1/2)t^2}}{|x-t|}\, dt + \int_{x+9n^{-1/2}}^{x+1} \frac{e^{-(1/2)t^2}}{|x-t|}\, dt \right\}$$
$$\leqq 2c_{14} e^{x^2/2} \int_{|x|-1}^{|x|-9n^{-1/2}} \frac{e^{-(1/2)t^2}}{|x|-t}\, dt. \tag{22}$$

Durch Einführung der neuen Variablen $u=|x|-t$ ergibt sich

$$e^{x^2/2}\int_{|x|-1}^{|x|-9n^{-1/2}}\frac{e^{-(1/2)t^2}}{|x|-t}\,dt = e^{x^2/2}\int_{9n^{-1/2}}^{1}\frac{e^{-(1/2)(|x|-u)^2}}{u}\,du \leqq$$

$$\leqq \int_{9n^{-1/2}}^{1}\frac{e^{(|x|+1)u}}{u}\,du = \int_{9n^{-1/2}(|x|+1)}^{|x|+1}\frac{e^t}{t}\,dt \leqq$$

$$\leqq e\int_{9n^{-1/2}(|x|+1)}^{1}\frac{dt}{t}+\int_{1}^{|x|+1}e^t\,dt \leqq e(\log n+e^{|x|}).$$

Wir setzen diese Ungleichung in (22) und erhalten (21), w.z.b.w.

D. Die Hauptsätze

SATZ 1. *Es gilt für jedes reelle x und natürliches n*

$$\sum_{k=1}^{n}|l_{kn}(x)| \leqq c_{15}\log n + c_{16}e^{x^2/2}. \tag{23}$$

BEWEIS. Für $n\leqq 64$ läßt sich (23) durch passende Wahl von c_{16} befriedigen. Des weiteren sei $n>64$, also $n^{1/2}-5>n^{1/4}$.

Für $|x|\geqq\sqrt{n}-5>n^{1/4}$ folgt (23) aus Hilfssatz 5. Es sei nun $|x|<\sqrt{n}-5$. Wegen Hilfssatz 3 und Hilfssatz 7 ist

$$\sum_{k=1}^{n}|l_{kn}(x)| \leqq \sum_{|x-x_{kn}|\leqq 10n^{-1/2}}|l_{\kappa n}(x)|+c_{12}(e^{|x|}+\log n)+c_7 e^{x^2/2}. \tag{24}$$

Die Anzahl der Glieder in der Summe rechts ist infolge (10) weniger als $2\dfrac{10n^{-1/2}}{\pi(2n+1)^{-1/2}}<20$ und nach Hilfssatz 6 ist jedes einzelne Glied kleiner als c_{11}, es ist also

$$\sum_{|x-x_{kn}|\leqq 10n^{-1/2}}|l_{kn}(x)| \leqq 20c_{11}. \tag{25}$$

Aus (24) und (25) erhalten wir (23), w.z.b.w.

SATZ 2. *Für jede beschränkte und an der ganzen reellen Achse gleichmäßig stetige Funktion $f(x)$ gilt für jedes natürliche r*

$$|f(x)-L_n(f;x)| \leqq \begin{cases}[K_r^{(1)}\log n+K_r^{(2)}e^{x^2/2}]\omega_r(f;n^{-1/2}) & \text{für} \quad |x|\leqq\sqrt{2n+1}\\ 2\|f\|(2e^{-1})^{n/2}e^{x^2/2} & \text{für} \quad |x|\geqq\sqrt{2n+1}\end{cases} \tag{26}$$

wo $K_r^{(1)}$ und $K_r^{(2)}$ nur von r abhängen.

BEWEIS. Für die Funktion $f_n(x)=f(\sqrt{2n+1}\,x)$ gilt

$$\omega_r(f_n;\delta)=\omega_r(f;\sqrt{2n+1}\,\delta)\leqq c_{17}\omega_r(f;n^{1/2}\delta).$$

Es gibt also einen Polynom höchstens $n-1$-ten Grades $\pi_{n-1}(x)$, so daß für $|x|\leqq 1$ $|\pi_{n-1}(x)|\leqq 2\|f\|$ ist und

$$|f_n(x)-\pi_{n-1}(x)|\leqq C_r\omega_r(f_n;n^{-1})\leqq c_{17}C_r\omega_r(f;n^{-1/2})$$

gültig ist**, d.h. es ist

$$\left|f(x)-\pi_{n-1}\left(\frac{x}{\sqrt{2n+1}}\right)\right|\leqq c_{17}C_r\omega_r(f;n^{-1/2})\qquad(|x|\leqq\sqrt{2n+1}). \tag{27}$$

Unter Beachtung von (27), (23) und der Identität

$$\pi_{n-1}\left(\frac{x}{\sqrt{2n+1}}\right)=\sum_{k=1}^{n}l_{kn}(x)\pi_{n-1}\left(\frac{x_{kn}}{\sqrt{2n+1}}\right)$$

folgt, daß für $|x|\leqq\sqrt{2n+1}$

$$\left|f(x)-\sum_{k=1}^{n}l_{kn}(x)f(x_{kn})\right|\leqq\left|f(x)-\pi_{n-1}\left(\frac{x}{\sqrt{2n+1}}\right)\right|+$$

$$+\sum_{k=1}^{n}|l_{kn}(x)|\left|f(x_{kn})-\pi_{n-1}\left(\frac{x_{kn}}{\sqrt{2n+1}}\right)\right|\leqq$$

$$\leqq[1+c_{15}\log n+c_{16}e^{x^2/2}]c_{17}C_r\omega_r(f;n^{-1/2})$$

gültig ist. Das zeigt die Gültigkeit der oberen Hälfte von (26).

Es sei jetzt $|x|\geqq\sqrt{2n+1}$, also

$$|f(x)|\leqq e^{-\frac{2n+1}{2}}e^{x^2/2}\|f\|\leqq e^{-n}e^{x^2/2}\|f\|. \tag{28}$$

Aus $|\pi_{n-1}(x)|\leqq 2\|f\|$ $(|x|\leqq 1)$ folgt mit Hilfe einer klassischen Ungleichung von P. L. TSCHEBYSCHEFF, daß für $|x|>1$

$$|\pi_{n-1}(x)|\leqq 2\|f\|T_n(|x|)\leqq\|f\|(2|x|)^n$$

gültig ist. Dabei ist $T_n(x)$ das Tschebyscheffsche Polynom n-ten Grades. Es folgt weiter

$$e^{-x^2/2}\left|\pi_{n-1}\left(\frac{x}{\sqrt{2n+1}}\right)\right|\leqq\|f\|\left(\frac{2}{n}\right)^{n/2}\operatorname{Max}(|x|^ne^{-x^2/2})=$$

$$=\|f\|\left(\frac{2}{n}\right)^{n/2}n^{n/2}e^{-n/2}=(2e^{-1})^{n/2}\|f\|. \tag{29}$$

Die Formeln (28) und (29) zeigen die Gültigkeit der zweiten Hälfte von (26), w.z.b.w.

Vgl. etwa А. Ф. Тиман, : Теория приближении функции, Москва 1959, s. 274, § 5. 1. 32.

E. Abschätzungen nach unten. Gegenbeispiele

SATZ 3. *Ist ξ eine Nullstelle von $H_{n+1}(x)$, welche $2 \leqq |\xi| \leqq \sqrt{n}-4$ befriedigt, dann gilt für jedes n*

$$\sum_{k=1}^{n} |l_{kn}(\xi)| \geqq c_{18} \log n + c_{19} |\xi|^{-1} e^{\xi^2/2}. \tag{30}$$

BEWEIS. Aus (16a), (15) und (12) ergibt sich unter der Voraussetzung $|x_{kn}| \leqq \sqrt{n}-4$

$$|h_{n-1}(x_{kn})| = n^{-1/2} K_n^{1/2}(x_{kn}) = n^{-1/2} \lambda_n^{-1/2} \geqq c_{20} n^{-1/4} e^{x_{kn}^2/2}. \tag{31}$$

Setzen wir in diese Formel $n+1$ an Stelle von n, dann ergibt sich

$$|h_n(\xi)| \geqq c_{20}(n+1)^{-1/4} e^{\xi^2/2}. \tag{32}$$

Aus (15) erhalten wir

$$\lambda_{kn} = K_n^{-1}(x_{kn}) \geqq c_4^{-1} n^{-1/2} e^{-x_{kn}^2}. \tag{33}$$

Aus den letzten drei Ungleichungen ergibt sich mit Hilfe von (6)

$$|l_{kn}(\xi)| \geqq c_{21} e^{-x_{kn}^2/2} e^{\xi^2/2} n^{-1/2} |\xi - x_{kn}|^{-1} \quad (|x_{kn}| \leqq \sqrt{n}-4). \tag{34}$$

Da der Ausdruck in (30) eine gerade Funktion von ξ ist, reicht es Werte $2 < \xi < \sqrt{n}-4$ zu betrachten. Wir können auch (unter Beachtung von $\sum |l_{kn}(x)| \geqq \sum l_{kn}(x) = 1$) ohne Einschränkung der Allgemeinheit $n \geqq 100$ voraussetzen. Wegen (11) ist die Anzahl der x_{kn}, welche in $(\xi-1, \xi)$ fallen und auch die Anzahl der x_{kn}, welche in $(0, 1)$ fallen, für $n \geqq 100$ größer als $c_{22} n^{1/2} + 1$. Es ist also

$$\sum_{0 < x_{kn} < 1} |l_{kn}(\xi)| \geqq c_{21} e^{-1/2} e^{\xi^2/2} n^{-1/2} |\xi|^{-1} c_{22} n^{1/2} = c_{23} |\xi|^{-1} e^{\xi^2/2} \tag{35}$$

und unter nochmaliger Beachtung von (11)

$$\begin{aligned} \sum_{\xi-1 < x_{kn} < \xi} |l_{kn}(\xi)| &\geqq c_{21} n^{-1/2} \sum_{\xi-1 < x_{kn} < \xi} |\xi - x_{kn}|^{-1} \geqq \\ &\geqq c_{22} n^{-1/2} \sum_{r=1}^{c_{22} n^{1/2}+1} \frac{1}{r n^{1/2}} \geqq c_{23} \log n. \end{aligned} \tag{36}$$

Die Intervalle $(0, 1)$ und $(\xi-1, \xi)$ haben wegen $\xi \geqq 2$ keine gemeinsame Punkte, so daß aus (35) und (36) die zu beweisende Ungleichung (30) folgt.

SATZ 4. *Es gibt eine Folge $\{f_n\}$ von Funktionen, welche auf der reellen Achse gleichmäßig stetig und beschränkt sind und eine reelle Zahlenfolge $\{\xi_n\}$, so daß für $n \geqq c_{24}$*

$$|f_n(\xi_n) - L_n(f_n; \xi_n)| \geqq c_{25} 2^{-r} \omega_r(f; n^{-1/2})(\log n + |\xi_n|^{-1} e^{\xi_n^2/2}) \tag{37}$$

befriedigt ist. Dabei kann als Folge $\{\xi_n\}$ jede Zahlenfolge dienen, deren Glieder

$$H_{n+1}(\xi_n) = 0 \quad \textit{und} \quad 2 \leqq |\xi_n| \leqq \sqrt{n}-4 \tag{38}$$

befriedigen.

BEWEIS. Wir wählen eine Zahlenfolge $\{\xi_n\}$, deren Glieder (38) erfüllen. Die Funktion $f_n(x)$ definieren wir wie folgt. Es sei $f_n(x_{kn}) = \operatorname{sign} l_{kn}(\xi_n)$ und für die anderen Werte von x setzen wir es so fort, daß $f_n(x)$ an der reellen Achse gleichmäßig stetig wird und $\|f\| \leqq 1$ befriedigt. Eine der möglichen Verfahren ist z.b., daß wir zwischen $x_{k+1,n}$ und $x_{k,n}$ linear interpolieren und $f(x) = f(x_{1n})$ für $x > x_{1n}$ bzw. $f(x) = f(x_{nn})$ für $x < x_{nn}$ setzen. Es ist dann offenbar

$$\omega_r(f; \delta) \leqq 2^r \|f\| \leqq 2^r \tag{39}$$

und

$$L_n(f_n; \xi_n) = \sum_{k=1}^{n} |l_{kn}(\xi_n)|. \tag{40}$$

Aus (38), (39), (40) und Satz 3 folgt (37) für hinreichend große Werte von n, w.z.b.w.

SATZ 5. *Es sei $\omega_0(\delta)$ eine solche für $\delta \geqq 0$ nichtabnehmende Funktion, daß $\omega_0(+0) = 0$, $\omega_0(\delta) > 0$ für $\delta > 0$ und*

$$\omega_0(2\delta) \leqq 2\omega_0(\delta) \tag{41}$$

gültig ist. Für jedes solche $\omega_0(\delta)$ gibt es eine auf der reellen Zahlengeraden gleichmäßig stetige beschränkte Funktion $F(x)$ mit

$$\omega(F; \delta) \leqq \omega_0(\delta) \tag{42}$$

ferner eine wachsende Folge natürlicher Zahlen $\{n_r\}$ und eine Zahlenfolge $\{\xi_r\}$, so daß

$$|F(\xi_r) - L_{n_r}(F; \xi_r)| \geqq c_{26}\left[\log n_r + e^{\xi_r^2/2}\right]\omega_0(n_r^{-1/2}) \tag{43}$$

befriedigt ist.

BEWEIS. Die Idee der nachfolgenden Konstruktion ist die gleiche, durch welche P. DU BOIS RAIMOND das erste Beispiel einer stetigen Funktion mit divergenter Fourierschen Reihe gegeben hat.

a) *Konstruktion von $\{n_r\}$, $\{\xi_r\}$ und $F(x)$.*

Wir wählen eine beliebige natürliche Zahlenfolge $\{n_r\}$, welche so verdünnt ist, daß $\log\log n_1 > 100$ ist und

$$\sqrt{\log\log n_{r+1}} > n_r + 4 \qquad (r = 1, 2, \ldots) \tag{44}$$

gilt. Dann werden die Intervalle $I_r = [\log\log n_r - 2, \log\log n_r + 2]$ $(r = 1, 2, \ldots)$ sicher nicht übereinandergreifen. Es sei ferner ξ_r eine Nullstelle von $H_{n_r+1}(x)$ mit

$$\sqrt{\log\log n_r} - \frac{\pi}{\sqrt{n_r}} < \xi_r \leqq \sqrt{\log\log n_r}. \tag{45}$$

Die Existenz einer solchen Zahl ξ_r ist durch (11) gesichert. Die Intervalle

$[\xi_r - 1, \xi_r] \subset I_r$ greifen nicht übereinander. Ist x_{k,n_r} eine in $[\xi_r - 1, \xi_r]$ liegende Nullstelle von $H_{n_r}(x)$, dann sei

$$F(x_{k,n_r}) = \frac{1}{2}\,\omega_0(n_r^{-1/2}) \operatorname{sign} l_{k,n_r}(\xi_r) \tag{46}$$

$$(\xi_r - 1 \leqq x_{k,n_r} \leqq \xi_r;\ r = 1, 2, \ldots).$$

Es sei ferner ξ_r^* die rechts von ξ_r liegende erste Nullstelle von $H_{n_r}(x)$ und ξ_r^{**} die links von $\xi_r - 1$ gelegene erste Nullstelle von $H_{n_r}(x)$. Es sei dann

$$F(\xi_r^*) = F(\xi_r^{**}) = 0. \tag{47}$$

In den Intervallen $I_r^* = [\xi_r^*, \xi_r^{**}] \subset I_r$ definieren wir $F(x)$ so, daß wir zwischen den benachbarten Stellen (46), bzw. (46) und (47) linear interpolieren. Fällt x in keine der Intervalle I_r^*, dann setzen wir $F(x) = 0$. $F(x)$ ist offenbar beschränkt, und $\|F\| = \frac{1}{2}\,\omega_0(n^{-1/2})$.

b) *Abschätzung des Stetigkeitsmoduls von* $F(x)$.

Aus (41) folgt leicht, daß

$$\frac{\omega_0(\delta_2)}{\delta_2} \leqq 2\,\frac{\omega_0(\delta_1)}{\delta_1} \quad \text{für} \quad 0 < \delta_1 < \delta_2 \tag{48}$$

gilt. Für $h \geqq 1$ gilt

$$|F(x+h) - F(x)| \leqq 2\|F\| \leqq \omega_0(n_1^{-1/2}) \leqq \omega_0(h).$$

Es sei jetzt $h < 1$. Fällt x und $x+h$ in keine der Intervalle I_r^* $(r = 1, 2, \ldots)$, dann ist $F(x+h) - F(x) = 0$. Fällt wenigstens einer der Punkte x, $x+h$ in I_r^*, dann kann der andere Punkt nicht in ein I_s^* mit $s \neq r$ fallen. Es ist also wegen (46)

$$|F(x+h) - F(x)| \leqq \omega_0(n_r^{-1/2}) \leqq \omega_0(h) \quad \text{für} \quad h \geqq n_r^{-1/2}.$$

Es sei endlich $[x, x+h] \cap I_r^* \neq 0$ und $0 < h < n_r^{-1/2}$. Die Funktion $F(x)$ ist zwischen x und $x+h$ stückweise linear, und der Betrag der Steigung $F'(x)$ ist infolge (46) und (10) kleiner als

$$\pi^{-1}\sqrt{2n_r+1} \cdot \frac{1}{2}\,\omega_0(n_r^{-1/2}) < \pi^{-1}\omega_0(n_r^{-1/2}).$$

Es folgt unter Beachtung von (48)

$$|F(x+h) - F(x)| \leqq \pi^{-1} n_r^{1/2}\,\omega_0(n_r^{-1/2})h \leqq \pi^{-1} 2\omega_0(h) < \omega_0(h).$$

Es ist also für jedes reelle x und $h > 0$

$$F(x+h) - F(x) \leqq \omega_0(h),$$

so daß (42) befriedigt ist.

c) *Abschätzung nach unten von* $|F(\xi_r)-L_{n_r}(F;\xi_r)|$.

Es sei

$$F_r(x)=\begin{cases}F(x) & \text{für } x\in I_r^*\\ 0 & \text{für } x\notin I_r^*.\end{cases} \tag{49}$$

Wegen (46), (36) und (45) ist dann

$$L_{n_r}(F_r;\xi_r)=\frac{1}{2}\omega_0(n_r^{-1/2})\sum_{x_{k,n_r}\in I_r^*}|l_{k,n_r}(\xi_r)|\geqq c_{27}\omega_0(n_r^{-1/2})\log n_r. \tag{50}$$

Für die Funktion $F_r^*(x)=F(x)-F_r(x)$, welche wegen (49) in I_r^* verschwindet und ausserhalb I_r^* gleich $F(x)$ ist, gilt wegen der schon bewiesenen Formel (42)

$$\omega(F_r^*;\delta)\leqq\omega_0(\delta).$$

Nach dem Muster von (27) konstruieren ein Polynom $\Phi_r(x)$, dessen Grad niedriger als $n_r/2$ ist, für welches

$$|F_r^*(x)-\Phi_r(x)|\leqq c_{28}\omega_0(n_r^{-1/2})\qquad(|x|\leqq\sqrt{2n_r+1}) \tag{51}$$

gültig ist.

Es sei $f_r(x)=0$ für $|x-\xi_r|\leqq 1$, $f_r(x)=1$ für $|x-\xi_r|\geqq 2$ und in $[\xi_r-2,\ \xi_r-1]$ bzw. $[\xi_r+1,\ \xi_r+2]$ sei $f_r(x)$ linear. Dann ist $|f_r(x+h)-f_r(x)|\leqq|h|$, so daß es wegen (27) ein Polynom $\psi_r(x)$ mit niedrigerem Grad als $n_r/2$ gibt, so daß

$$|f_r(x)-\Psi_r(x)|\leqq c_{29}n_r^{-1/2}\quad\text{für}\quad|x|\leqq\sqrt{2n_r+1}$$

gültig ist. Der Grad des Polynoms $\chi_r(x)=\Phi_r(x)\Psi_r(x)$ is dann kleiner als n_r und es folgt aus einer leichten Rechnung

$$|F_r^*(x)-\chi_r(x)|=|\chi_r(x)|\leqq c_{30}n_r^{-1/2}\omega_0(n_r^{-1/2})\quad\text{für}\quad|x-\xi_r|\leqq 1 \tag{52}$$

und

$$|F_r^*(x)-\chi_r(x)|\leqq c_{31}\omega_0(n_r^{-1/2})\quad\text{für}\quad|x|\leqq\sqrt{2n_r+1}. \tag{53}$$

Es ist dann

$$|F_r^*(\xi_r)-L_{n_r}(F_r^*;\xi_r)|\leqq|F_r^*(\xi_r)-\chi_r(\xi_r)|+|L_{n_r}(F_r^*-\chi_r;\xi_r)|\leqq c_{31}\omega_0(n_r^{-1/2})+$$

$$+\sum_{|x_{k,n_r}-\xi_r|\leqq 1}|l_{k,n_r}(\xi_r)|\cdot c_{30}n_r^{-1/2}\omega_0(n_r^{-1/2})+\sum_{|x_{k,n_r}-\xi_r|>1}|l_{k,n_r}(\xi_r)|\cdot c_{31}\omega_0(n_r^{-1/2}),$$

und unter Anwendung von (23) und (17)

$$\begin{aligned}|F_r^*(\xi_r)-L_{n_r}(F_r^*;\xi_r)|&\leqq c_{31}\omega_0(n_r^{-1/2})+c_{32}(\log n_r+e^{\xi_r^2/2})n_r^{-1/2}\omega_0(n_r^{-1/2})+\\&+c_{33}e^{\xi_r^2/2}\omega_0(n_r^{-1/2})\prec c_{34}e^{\xi_r^2/2}\omega_0(n_r^{-1/2}).\end{aligned} \tag{54}$$

Infolge (45) ist $e^{\xi_r^2/2} < \sqrt{\log n_r}$, so daß aus (50) und (54) unter Beachtung von $F(x) = F_r(x) + F_r^*(x)$

$$|F(\xi_r) - L_{n_r}(F; \xi_r)| \geqq |L_{n_r}(F_r; \xi_r)| - |F_r(\xi_r)| - |F_r^*(\xi_r) - L_{n_r}(F_r^*; \xi_r)| \geqq$$

$$\geqq (c_{27} \log n_r - 1 - c_{34} e^{\xi_r^2/2}) \omega_0(n_r^{-1/2}) \geqq c_{35}(\log n_r + e^{\xi_r^2/2}) \omega_0(n_r^{-1/2})$$

folgt, w. z. b. w.

LITERATURVERZEICHNIS

[1] BALÁZS, J. and TURÁN, P.: Notes on interpolation, VIII, *Acta Math. Acad Sci. Hungar.* **12** (1961) 469—474.

[2] CHARLIER, C. V. P.: *Application de la théorie des probabilités a l'astronomie,* Gauthier-Villars, Paris 1931.

[3] ERDÉLYI, A.: Über eine erzeugende Funktion von Produkten Hermitescher Polynome, *Math. Z.* **44** (1938) 201—211.

[4] ERDÉLYI, A.: (Unter Mitwirkung von W. Magnus, F. Oberhettinger und F. F. Tricomi) *Higher Transcendental Functions, Bd. II.* Mc Graw Hill, New York, 1953.

[5] FREUD, G.: Über die Lebesgueschen Funktionen der Lagrangeschen Interpolation, *Acta Math. Acad. Sci. Hungar.* **4** (1953) 137—142.

[6] FREUD, G.: On the convergence of Lagrange interpolation processes on infinite intervalls (ungarisch mit englischen und russischen Auszug), *Mat. Lapok* **18** (1967) 289—292.

[7] PLANCHEREL, M. et ROTACH, W.: Sur les valeurs asymptotiques des polynomes d'Hermite, *Comment. Math. Helv.* **1** (1929) 227—254.

[8] SZEGŐ, G.: *Orthogonal Polynomials,* 2. Aufl. Amer. Math. Soc., New York, 1959.

Mathematisches Institut der Ungarischen Akademie der Wissenschaften, Budapest

(Eingegangen: 4. März, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 191–194.

ÜBER FUNKTIONEN, DIE Δ^2-STETIG BZW. STETIG VON 2. ORDNUNG SIND

von

M. TASCHE

K. BÖGEL betrachtete in [1], [2] reellwertige Funktionen $f: R^2 \to R$ zweier reeller Variabler. Eine Funktion f nennt er *Δ^2-stetig* im Punkt (x_0, y_0) (bezüglich des Cartesischen (x, y)-Koordinatensystems der Euklidischen Ebene R^2), wenn es zu jedem $\varepsilon > 0$ ein $\delta > 0$ gibt, so daß für beliebige $h, k \in R$ mit $|h| < \delta$, $|k| < \delta$

$$|f(x_0, y_0) - f(x_0 + h, y_0) - f(x_0, y_0 + k) + f(x_0 + h, y_0 + k)| < \varepsilon$$

gilt.

Wie K. BÖGEL zeigte, ist die Funktion f genau dann Δ^2-stetig in (x_0, y_0), wenn f sich in der Form

$$f(x, y) = g(x, y) + g_1(x) + g_2(y)$$

darstellen läßt, wobei $g(x, y)$ in (x_0, y_0) stetig ist und $g_1(x), g_2(y)$ gewisse Funktionen einer Variablen sind.

Der Begriff der Δ^2-Stetigkeit ist, im Gegensatz zur Stetigkeit, nicht drehungsinvariant. Um Strukturaussagen für Δ^2-stetige Funktionen zu gewinnen, stellte mir G. FREUD* folgendes

PROBLEM. *Wodurch kann man die Funktionen $f: R^2 \to R$ charakterisieren, die bezüglich aller (Cartesischen) Koordinatensysteme in einem Punkt Δ^2-stetig sind.*

Die vorliegende Arbeit gibt eine teilweise Antwort auf diese Frage. Im Teil 1. wird die Menge der von 2. Ordnung stetigen Funktionen angegeben, die bezüglich aller Koordinatensysteme Δ^2-stetig sind. Ein Beispiel zeigt, daß diese Funktionen i.a. nicht stetig sind. Wir bemerken aber, daß es umgekehrt Funktionen gibt, die bezüglich aller Cartesischen Koordinatensysteme Δ^2-stetig, jedoch nicht stetig von 2. Ordnung sind. In mehreren Schritten beweisen wir im zweiten Teil, daß eine meßbare, von 2. Ordnung stetige Funktion stetig ist.

1. Es seien X und Y normierte Räume. Die Funktion $f: X \to Y$ bilde X in Y ab.

M. NICOLESCU (siehe [4], S. 456) nennt eine Funktion $f: X \to Y$ *stetig von 2. Ordnung* im Punkt $a \in X$, wenn für jedes $\varepsilon > 0$ ein $\delta > 0$ derart existiert, daß für beliebige $h, k \in X$ mit $\|h\| < \delta$, $\|k\| < \delta$

$$\|f(a) - f(a + h) - f(a + k) + f(a + h + k)\| < \varepsilon$$

gilt.

Ist X der Produktraum zweier normierter Räume X_1 und X_2, so gilt der folgende

* Herrn G. FREUD möchte ich für seine Unterstützung sehr danken.

SATZ 1 (siehe [4]): *Die Funktion $f: X_1 \times X_2 \to Y$ ist genau dann in $a=(a_1, a_2)$ stetig von 2. Ordnung, wenn*

1. *$f(x_1, a_2)$ und $f(a_1, x_2)$ in a_1 bzw. a_2 stetig von 2. Ordnung bezüglich x_1 bzw. x_2 sind und*
2. *$f(x_1, x_2)$ in (a_1, a_2) Δ^2-stetig ist.***

Es sei nun $X=R^2$ und $Y=R$. Der Begriff der Stetigkeit von 2. Ordnung ist, wie man sofort aus seiner Definition ersieht, drehungsinvariant. Ist $f: R^2 \to R$ in einem Punkt $a \in R^2$ stetig von 2. Ordnung, so ist dann f wegen 2. von Satz 1 bezüglich aller Koordinatensysteme Δ^2-stetig.

Umgekehrt braucht aber eine Funktion, die bezüglich aller Cartesischen Koordinatensysteme Δ^2-stetig ist, nicht stetig von 2. Ordnung zu sein. Ist z. B. $g(z)$ eine unstetige Lösung der Cauchyschen Funktionalgleichung $g(z)+g(z') = g(z+z')$, so ist $f(x, y) = g(x^2+y^2)$ im Punkt (0, 0) bezüglich aller Cartesischen Koordinatensysteme Δ^2-stetig, aber nicht stetig von 2. Ordnung.

Wir betrachten jetzt nur noch Funktionen, die stetig von 2. Ordnung sind. Wenn die Funktion $f: R^2 \to R$ im Punkte $a=(0, 0)$ stetig von 2. Ordnung ist, so braucht f noch nicht stetig in diesem Punkt zu sein. G. HAMEL zeigte in [3], daß es unstetige Lösungen der Cauchyschen Funktionalgleichung gibt. Da sich sein Konstruktionsverfahren einer unstetigen Lösung ohne weiteres auf die mehrdimensionale Funktionalgleichung

$$f(x, x')+f(y, y') = f(x+y, x'+y')$$

übertragen läßt und $f(0, 0)=0$ ist, haben wir eine Funktion gefunden, die in (0, 0) stetig von 2. Ordnung, aber unstetig ist. Betrachten wir die Funktionalgleichung

$$f(x, x')+f(x+y+z, x'+y'+z') = f(x+y, x'+y')+f(x+z, x'+z'),$$

so können wir wie oben einsehen, daß eine Funktion $f: R^2 \to R$ in allen Punkten von R^2 stetig von 2. Ordnung (also auch Δ^2-stetig) sein kann, ohne irgendwo stetig zu sein. Falls f aber Stetigkeitspunkte hat und diese in R^2 dicht liegen, so ist f in allen Punkten stetig.

2. Es seien X und Y normierte Räume und $f: X \to Y$ eine Funktion, die X in Y abbildet. Wenn f stetig von 2. Ordnung ist, unter welchen Bedingungen ist f dann stetig?

SATZ 2. *Die Funktion $f: X \to Y$ sei in $a \in X$ stetig von 2. Ordnung. Es existiere eine Umgebung $U(a)$ von a derart, daß die Menge $\{\|f(x)\|\,;\, x \in U(a)\}$ beschränkt ist. Dann ist f im Punkt a stetig.*

BEWEIS. O. B. d. A. können wir annehmen, daß $a=\theta_X$ und $f(a)=\theta_Y$ sind, wenn θ das Nullelement des betreffenden Raumes ist. Angenommen, daß f in θ unstetig ist. Sei $\varphi(x)=\|f(x)\|$. Nach Voraussetzung ist $\bar{\varphi}(\theta)=\overline{\lim}\ \varphi(x)$ für $x \to \theta$ endlich. Dann gibt es eine zu θ konvergente Folge $\{x_n\}$ derart, daß $\varphi(x_n) \to \bar{\varphi}(\theta)$. ε_0 sei eine beliebig kleine, positive Zahl mit $\bar{\varphi}(\theta)-3\varepsilon_0 > 0$. Weiterhin gibt es zu ε_0 ein $\delta_0' > 0$, so daß für $\|x\| < \delta_0'$

$$\bar{\varphi}(\theta)+\varepsilon_0 > \varphi(x) \tag{1}$$

** Die Definition der Δ^2-Stetigkeit wollen wir auf Funktionen $f: X_1 \times X_2 \to Y$ übertragen, indem wir die Absolutbeträge durch die entsprechenden Normen ersetzen.

ist, und ein $\delta_0'' > 0$, so daß für alle x_n mit $\|x_n\| < \delta_0''$

$$|\bar{\varphi}(\theta) - \varphi(x_n)| < \varepsilon_0 \tag{2}$$

ist. Da $x_n \to \theta$, gibt es zu jedem positiven $\delta \leqq \min\left\{\frac{\delta_0'}{2}, \delta_0''\right\}$ ein x_m mit $\|x_m\| < \delta$. Nach (1) und (2) gelten die Ungleichungen

$$\bar{\varphi}(\theta) + \varepsilon_0 > \varphi(2x_m),$$

$$\bar{\varphi}(\theta) + \varepsilon_0 > \varphi(x_m) > \bar{\varphi}(\theta) - \varepsilon_0.$$

Folglich ist

$$2\varphi(x_m) - \varphi(2x_m) > \bar{\varphi}(\theta) - 3\varepsilon_0 > 0,$$

d.h. φ ist in θ nicht stetig von 2. Ordnung. Da aber

$$\|2f(x_m) - f(2x_m)\| \geqq 2\varphi(x_m) - \varphi(2x_m) > \bar{\varphi}(\theta) - 3\varepsilon_0 > 0$$

ist, kann f in θ im Widerspruch zur Voraussetzung nicht stetig von 2. Ordnung sein. Damit ist der Satz bewiesen.

Ist $Y = R$, so läßt sich die Voraussetzung in Satz 2 abschwächen.

SATZ 2'. *Wenn die Funktion $f: X \to R$ in $a \in X$ stetig von 2. Ordnung und in einer gewissen Umgebung von a nach oben (unten) beschränkt ist, dann ist f in a stetig.*

Der BEWEIS verläuft wie der von Satz 2.

Es seien nun X der n-dimensionale Euklidische Raum R^n und Y ein normierter Raum.

SATZ 3. *Die Funktion $f: R^n \to Y$ sei in $a \in R^n$ stetig von 2. Ordnung. In einer hinreichend kleinen Umgebung $U(a)$ von a existiere eine Menge M mit positivem Lebesgue-Maß, auf der f beschränkt ist. Dann ist f im Punkt a stetig.*

BEWEIS. Es sei $a = \theta$ und $f(\theta) = \theta_Y$. Da f nach Voraussetzung in θ stetig von 2. Ordnung ist, gibt es eine Kugelumgebung V von θ derart, daß für $h', k' \in V$

$$\|f(h') + f(k') - f(h' + k')\| \leqq 1$$

ist. Sind $h = h' + k'$, $k = k'$, so gilt für beliebige h, $k \in \frac{1}{2}V$

$$\|f(h - k) - f(h) + f(k)\| \leqq 1.$$

Nach Voraussetzung sei nun $U = U(a) \subset \frac{1}{2}V$. Aufgrund eines Satzes von H. STEINHAUS (siehe [6], [5]) existiert ein $\delta > 0$ derart, daß jedes $x \in R^n$ mit $\|x\| < \delta$ in der Form $x = m - n$, wobei m, $n \in M$ sind, dargestellt werden kann. Seien $W = \frac{1}{2}V \cap \{x; \|x\| < \delta\}$ und k eine obere Schranke von $\|f\|$ auf M. Dann gilt für beliebiges $x = m - n \in W$, wobei m, $n \in M$ sind,

$$\|f(x)\| = \|f(m - n)\| \leqq \|f(m) - f(n)\| + 1 \leqq 2k + 1.$$

Folglich ist f auf der Umgebung W von θ beschränkt. Nach Satz 2 ist f stetig in $a = \theta$.

SATZ 4. *Die Funktion $f: R^n \to Y$ sei überall in R^n stetig von 2. Ordnung. $\|f\|$ sei fast überall durch eine meßbare, endliche Funktion $g: R^n \to R$ beschränkt. Dann ist f in R^n stetig.*

BEWEIS. Es sei $a \in R^n$ ein beliebiger Punkt. Wir zeigen, daß f in a stetig ist. $U(a) = \{x; \|x-a\| < \delta\}$ sei eine beliebige Umgebung von a. Da g eine meßbare Funktion ist, sind die Mengen $G_n = \{x \in U(a);\ g(x) \leqq n\}$ $(n = 1, 2, \ldots)$ sämtlich meßbar. Es ist $G_n \subseteqq G_{n+1}$ $(n = 1, 2, \ldots)$ und $\bigcup_{n=1}^{\infty} G_n = U(a)$. Ist μ das n-dimensionale Lebesgue-Maß, so gilt

$$\lim_{n \to \infty} \mu(G_n) = \mu(U(a)) > 0,$$

d.h. es gibt einen Index i mit $\mu(G_i) > 0$. Da fast überall $\|f\| \leqq g$ ist, gibt es in jeder Umgebung von a eine Menge von positivem Mass, auf der f beschränkt ist. Aus Satz 3 ergibt sich, daß f in a stetig ist.

FOLGERUNG 5. *Jede meßbare und von* 2. *Ordnung stetige Funktion* $f: R^n \to R$ *ist stetig.*

LITERATURVERZEICHNIS

[1] BÖGEL, K.: Mehrdimensionale Differentation von Funktionen mehrerer Veränderlicher, *J. reine angew. Math.* **170** (1934) 197—217.
[2] BÖGEL, K.: Über die mehrdimensionale Differentation, Jahresbericht d. DMV **65** (1962) 45—71.
[3] HAMEL, G.: Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung $f(x+y) = f(x)+f(y)$, *Math. Ann.* **60** (1905) 459—462.
[4] NICOLESCU, M.: Analiză matematică II, ed. tehnică Bucureşti, 1958.
[5] RUZIEWICZ, S.: Contribution à l'étude des ensembles de distances de points, *Fund. Math.* **7** (1925) 141—143.
[6] STEINHAUS, H.: Sur les distances des points des ensembles de mesure positive, *Fund. Math.* **1** (1920) 93—104.

Universität Rostock, DDR

(Eingegangen: 25. Januar 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 195–200.

ON A PROBLEM OF L. FUCHS

by

A. R. RHEMTULLA

1. Introduction

1.1 *The Problem:* In a finite group G of order n, every product of the n distinct elements of G lies in the same coset of the derived subgroup G'. Can every element of this coset be represented as a product of the n distinct elements? The question, put forward by L. FUCHS, was discussed by J. DÉNES in [2] and in section 3 of [1]. Here we give an affirmative answer to this problem for the class of all finite soluble groups.

In [2] J. DÉNES obtained an affirmative answer to the problem for a group G of order n in which (i) the number of elements of order two does not exceed $n/2$ and (ii) all the elements of G' are commuators. His result can be used in the case of many finite simple groups like the simple alternating groups and the simple projective special linear groups where condition (ii) are known to hold. (See [3] and [4]).

1. 2. Notations. For a subset $X=\{x_1, \ldots, x_m\}$ of a group G we write $p(X)$ to denote the subset of G consisting of all the elements of G that can be expressed as the product of the m elements of X. Thus, for example, if X consists of just one element x, then $p(x)=x$; If $X=\{x_1, x_2\}$ then $p(x)=x_1x_2$ if $x_2x_1=x_1x_2$ and $p(x)=\{x_1x_2, x_2x_1\}$ otherwise.

We denote by F the class of all finite groups G in which $|p(G)|=|G'|$. Thus F-groups are precisely those groups for which Fuchs' problem has an affirmative answer.

Let X, Y be subsets of a group G. If $X \leqq Y$, then $(Y \setminus X)$ denotes the set $\{y; y \in Y,\ y \notin X\}$. If X and Y are disjoint, then $(X+Y)$ denotes $X \cup Y$. We shall write $\langle X \rangle$ for the subgroup generated by X.

For any elements h, k of G, $[h, k]=h^{-1}k^{-1}hk$ and if H, K are subgroups of G, then $[H, K]=\langle [h, k],\ h \in H,\ k \in K \rangle$. We shall write $\overline{H}$ for $(H \setminus 1)$ and $H \lhd G$ if H is normal in G.

As usual we shall write $G^{(r)}$ for the r^{th} derived subgroup of G and $\xi_s(G)$ for the s^{th} centre of G.

1. 3. Statement of Results: The main result of the paper is

THEOREM 1. *Every finite soluble group is an F-group.*

This is proved in Section 2 where the preliminary results necessary for the proof of the theorem are also stated and proved. In Section 3 we prove the following.

THEOREM 2. *For a finite group G, $p(G)$ and G' are disjoint, if and only if the following three conditions hold.*

(i) $|G|$ *is even,*

(ii) $|G'|$ *is odd,*

(iii) *Sylow 2-subgroups of* G *are cyclic.*

Groups satisfying these three conditions are soluble by the celebrated result of FEIT—THOMPSON concerning odd order groups. Hence by Theorem 1, such groups are always F-groups.

2. Proofs

LEMMA 2. 1. *If* X *is a subset of a finite group* G *and* $|p(X)| = |G'|$, *then* $G \in F$.

PROOF. Let $Y = (G \setminus X)$. Then for some $g \in G$, $g \in p(Y)$, and $p(G) \cong p(Y) \cdot p(X) \cong g \cdot p(X)$. Since $|g \cdot p(X)| = |p(X)| = |G'|$, the result follows.

If $G = A \times B$ is the direct product of two non-trivial F-groups, then $G \in F$. This follows as a corollary to the above Lemma. Since $A \in F$, $|p(A)| = |A'|$. Similarly $|p(B)| = |B'|$. Now $G = (A + \bar{B} + X)$ where $X = \{ab; a \in \bar{A}, b \in \bar{B}\}$. $|p(A+B)| \cong \cong |p(A) \cdot p(B)| = |A'||B'| = |G'|$. Hence the result.

LEMMA 2. 2. *If* $G = \langle A, x_1, \ldots, x_n \rangle$ *where* $A \lhd G$ *and abelian, then the set*

$$S = \{[a_1, x_1] \ldots [a_n, x_n];\ a_i' s \in A\}$$

is precisely the subgroup $[A, G]$.

PROOF. For any fixed $g \in G$, the mapping

$$a \to [a, g],\ a \in A$$

is homomorphic since A is abelian and normal in G. Thus the set $Sg = \{[a, g] : a \in A\}$ is a subgroup of A. Also g normalizes Sg since $g^{-1}[a, g]g = [g^{-1}ag, g] \in Sg$. Evidently A/Sg is a central factor of $\langle A, g \rangle$ and thus $Sg \cong [A, \langle g \rangle]$. But trivially $Sg \leqq [A, \langle g \rangle]$ so that $Sg = [A, \langle g \rangle]$.

Now write S_i for Sx_i so that $S = S_1 S_2 \ldots S_n$. As shown above, x_i centralizes A/S_i. Since $S_i \leqq S \leqq A$, x_i centralizes A/S. This holds for all $i = 1, 2, \ldots, n$, so that $G = \langle A, x_1, \ldots, x_n \rangle$ centralizes A/S, whence $[A, G] \leqq S$. But again trivially $S \leqq [A, G]$ so that $S = [A, G]$.

LEMMA 2. 3. *If* $A \lhd G$ *and abelian, and* G *finite, then for any* $\xi \varepsilon A$, $[\xi, g] \in p(\bar{A}g + \bar{A}g^{-1})$ *whenever* $Ag \neq Ag^{-1}$.

PROOF. Since $A \lhd G$, $\bar{A}g^{-1} = g^{-1}\bar{A}$. Let ξ be any element in $\bar{A}$. The map

$$g^{-1}a\xi \to a^{-1}g, \quad a \in A$$

is one-one from $g^{-1}A$ onto Ag. Under this map, $g^{-1} \to \xi g$ and $g^{-1}\xi \to g$. Hence for all $a \neq 1$, ξ^{-1} in A, $g^{-1}a\ \xi \in g^{-1}\bar{A}$ and its image $a^{-1}g \in \bar{A}g$. If $|A| = n > 2$, then for each of the $n-2$ such values of a, the product of $g^{-1}a\xi$ and its image $a^{-1}g$ gives ξ or $g^{-1}\xi g$ depending on the order in which we take the product. Hence ξ^{n-2} and $\xi^{n-3}g^{-1}\xi g$ both belong to $p(\bar{A}g + g^{-1}\bar{A} \setminus g^{-1}\xi \setminus \xi g)$. But $\xi g \cdot g^{-1}\xi = \xi^2$ so

that ξ^n and $\xi^{n-1}g^{-1}\xi g$ are both in $p(\bar{A}g+\bar{A}g^{-1})$. Since $\xi^n=1$ for all ξ in A, we conclude that 1 and $[\xi, g]$, $\xi\in\bar{A}$ all lie in $p(\bar{A}g+\bar{A}g^{-1})$.

If $A=\{1, a\}$ is of order two, then $p(\bar{A}g+g^{-1}\bar{A})$ consists of $ag\cdot g^{-1}a=1$. Since a normal subgroup of order two is always in the centre of G, $[a, g]=1$, for all $g\in G$.

LEMMA 2. 4. *Let $A=\langle a\rangle$ be a cyclic group of order p for some odd prime p, and let t be an element of order two such that $t^{-1}at=a^{-1}$. Then $A=p(\bar{A}+\bar{A}t)$.*

PROOF. Since p is odd, it is enough to show that $a^{2r}\in p(\bar{A}+\bar{A}t)$ for all $r=0, 1, \ldots, p-1$. Establish a one-one mapping $a^{-i}\to a^it$ $i=1, 2, \ldots, p-1$, from $\bar{A}$ onto $\bar{A}t$. The product of a^{-i} and a^it is t or $a^{2i}t$ depending on the order in which we take the product. For $i\neq r$, take the product $a^{-i}\cdot a^it$ to obtain t and for $i=r$, take $a^rt\cdot a^{-r}=$ $=a^{2r}t$. Thus if $r\neq 0$ then

$$a^{2r}t\cdot t^{p-2}=a^{2r}\in p(\bar{A}+\bar{A}t)$$

and if $r=0$, then

$$t^{p-1}=1\in p(\bar{A}+\bar{A}t).$$

This completes the proof.

LEMMA 2. 5. If $1\neq c=[x, y]\in\xi_1(G)$, $C=\langle c\rangle$ is of prime order and $G/C\in F$, then $G\in F$.

PROOF. When c is of odd order p, Cx, Cx^{-1}, ..., $Cx^{p-1/2}$, $Cx^{1-p/2}$. Cy, Cy^{-1} are all distinct cosets of C. Let S be a set of coset representatives of C in G such that $\{x, x^{-1}, \ldots, x^{p-1/2}, x^{1-p/2}, y, y^{-1}\}\leqq S$. Since $G/C\in F$, $p(S)\equiv G'z$, mod C, for some suitable $z\in G$. By Lemma 2. 1 it is enough to show that

$$C\leqq p\{cx, c^{-1}x^{-1}, \ldots, cx^{p-1/2}; c^{-1}x^{1-p/2}, cy, c^{-1}y^{-1}\}.$$

For any positive integer $\alpha\leqq p-1/2$,

$$c^\alpha=[x^\alpha, y]=c^{-1}x^{-\alpha}\cdot c^{-1}y^{-1}\cdot cx^\alpha\cdot cy$$

$$c^{-\alpha}=[y, x^\alpha]=c^{-1}y^{-1}\cdot c^{-1}x^{-\alpha}\cdot cy\cdot cx^\alpha$$

$$1=c^{-1}x^{-\alpha}\cdot cx^\alpha\cdot c^{-1}y^{-1}\cdot cy$$

and

$$p\{cx^r, c^{-1}x^{-r}; r=1, 2, \ldots, \alpha-1, \alpha+1, \ldots, p-1/2\}=1.$$

Hence

$$C\leqq p\{cx, c^{-1}x^{-1}, \ldots, cx^{p-1/2}, c^{-1}x^{1-p/2}, cy, c^{-1}y^{-1}\}.$$

When C is of order two, Cx^{-1}, Cy^{-1} and Cxy are all distinct, and using argument similar to above, it would follow that $G\in F$ if $C\leqq p(cx^{-1}, cy^{-1}, cxy)$. Since $c^2=1$, $cx^{-1}\cdot xy^{-1}\cdot cxy=c^4=1$ and $xy^{-1}\cdot cx^{-1}\cdot cxy=c^3=c$. This completes the proof.

PROOF OF THEOREM 1. Suppose that the theorem is false. Let G be a soluble group of smallest order such that $G\notin F$. Since every finite abelian group is trivially an F-group, we can assume that $G^{(r)}\neq 1$, $G^{(r+1)}=1$ for some integer $r\geqq 1$.

Corresponding to every abelian subgroup $A\lhd G$, we define a subgroup H_A by

$$H_A=\langle x, x\in G, x^2\notin A\rangle.$$

Note that for a general element h of H_A, h^2 may well lie in A, but we can choose a suitable set $\{x_1, \ldots, x_m\}$ of H_A such that $H_A = \langle x_1, \ldots, x_m \rangle$ and $x_i^2 \notin A$, $i = 1, 2, \ldots, m$. Also note that $H_A \lhd G$ and

$$B \lhd G,\ B \leqq A \text{ implies } H_B \geqq H_A. \tag{1}$$

If $[A, H_A] = A$ for some non-trivial abelian $A \lhd G$, then it is easy to show that $G \in F$. For let $AH_A = \langle A, x_1, \ldots, x_m \rangle$ where m is minimal and $x_i^2 \notin A$, $i = 1, 2, \ldots, m$. Choose a set S of coset representatives of A in G such that

$$\{x_1, x_1^{-1}, \ldots, x_m, x_m^{-1}\} \leqq S.$$

This is possible since $Ax_i = Ax_j^{\pm 1}$ implies $x_i x_j^{\mp 1} \in A$. If $j \neq i$, then $AH_A = \langle A, x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m \rangle$ contradicting the minimality of m. If $j = i$ and $Ax_i = Ax_i^{-1}$ then $x_i^2 \in A$ contradicting the choice of x_i. Now $G/A \in F$ so that

$$p(S) \equiv G'z (\mathrm{mod}\ A)$$

for some suitable coset $G'z$ of G', and it is sufficient, by Lemma 2. 1, to show that

$$A \leqq p(\bar{A}x_1 + \bar{A}x_1^{-1} + \ldots + \bar{A}x_m + \bar{A}x_m^{-1}).$$

By Lemma 2. 2,

$$[A, H_A] = \{[a_1, x_1] \ldots [a_m, x_m];\ a_i's \in A\}$$

and by Lemma 2. 3,

$$[a_i, x_i] \in p(\bar{A}x_i + \bar{A}x_i^{-1}),$$

so that

$$A = [A, H_A] \leqq p(\bar{A}x_1 + \bar{A}x_1^{-1} + \ldots + \bar{A}x_m + \bar{A}x_m^{-1}).$$

We shall hence-forth assume that for every non-trivial abelian $A \lhd G$,

$$[A, H_A] < A. \tag{2}$$

If $\xi_2(G) > \xi_1(G)$, then choose an element $x \in \xi_2(G)$, $x \notin \xi_1(G)$ of smallest order, and let y be an element of G such that $1 \neq c = [x, y]$. Then for any integer α, $c^\alpha = [x^\alpha, y]$ so that by our choice of x, c is of prime order. Lemma 2. 5 now applies and $G \in F$. We can therefore assume that

$$\xi_2(G) = \xi_1(G). \tag{3}$$

If A is any normal abelian subgroup of G, then G/AH_A is an elementary abelian 2-group since for any $x \in G$, $x^2 \in A$ or $x \in H_A$. By virtue of (1) and (2), $D = A_\cap \xi_1(AH_A) \neq 1$. Choose a cyclic subgroup $A_1 = \langle a \rangle$ of prime order in D such that the centralizer C of A_1 is maximal. We now show that $A_1 \lhd G$ and for any $x \in G$, $x^{-1}ax$ is either a or a^{-1}. Note that $G' \leqq AH_A \leqq C$ since G/AH_A is abelian and for any x in G, $x^2 \in C$. If $x^{-1}ax = a^x \neq a$, a^{-1}, then $\langle C, x \rangle$ centralizes aa^x for $(aa^x)^x = a^x a = aa^x$ and for any $c \in C$,

$$(aa^x)^c = a^c a^{xc} = aa^{c[c, x^{-1}]x} = aa^x.$$

Now $1 \neq aa^x \in D$ since $D \lhd G$, and aa^x is of prime order so that the maximality of C is contradicted.

Hence every abelian subgroup $A \lhd G$ contains a cyclic subgroup $A_1 = \langle a \rangle$ of prime order such that $A_1 \leqq \xi_1(AH_A)$ and $A_1 \lhd G$. Either $A_1 \leqq \xi_1(G)$ or for some

$x \in G$, $[x, a] = a^2$. If $[x, a] = a^2$, then by Lemma 2. 4, $A_1 \leqq p(\bar{A}_1 + \bar{A}_1 x)$ so that for any set S, containing x, of coset representatives of A_1 in G, $|p(S + \bar{A}_1 + \bar{A}_1 x)| = |G'|$ and hence $G \in F$.

We can thus conclude that for every non-trivial abelian subgroup $A \lhd G$,

$$A \cap \xi_1(G) \neq 1. \tag{4}$$

If $G^{(r)} \leqq \xi_1(G)$, then choose an element x of smallest order in $G^{(r-1)}$ such that $[x, G^{(r-1)}] \neq 1$. Hence for some $y \in G^{(r-1)}$, $1 \neq c = [x, y] \in \xi_1(G)$ and c of prime order by our choice of x. Lemma 2. 5 applies and we conclude that $G \in F$.

Thus not every normal abelian subgroup of G lies in $\xi_1(G)$. Let A be the smallest abelian subgroup such that $A \not\leqq \xi_1(G)$. By (2), $A_1 = [A, H_A] < A$ so that $A_1 \in \xi_1(G)$. If $A_1 \neq 1$, then we can find an element $x \in A$, $y \in H_A$ such that $1 \neq [x, y] \in \xi_1(G)$ is prime order, and then conclude by Lemma 2. 5, that $G \in F$.

If $A_1 = 1$, then for any $x \in G$,

$$[A, \langle x \rangle] = \{[a, x],\ a \in A\}$$

by Lemma 2. 2. Since $A \cap \xi_1(G) > 1$, the set $[\{a, x\},\ a \in A\}$ has fewer elements than A. Also

$$[a, x]^g = [a^g, x[x, g]] = [a^g, x] \in \{[a, x];\ a \in A\}$$

for any $a \in A$, $g \in G$ so that $[A, \langle x \rangle] \lhd G$. Thus $[A, \langle x \rangle] \leqq \xi_1(G)$ by our choice of A. Since x was arbitrary, we conclude that $A \leqq \xi_2(G)$. But $\xi_2(G) = \xi_1(G)$ by (4) so that $A \leqq \xi_1(G)$. This contradiction completes the proof.

3. Finite groups G with G' and $p(G)$ disjoint

LEMMA 3. 1. *$p(G) \leqq G'$ if G is an odd order group.*

PROOF. It is clear that either $p(G) \leqq G'$ or $p(G)$ is disjoint from G'. Since $|G|$ is odd, for every $x \neq 1$ in G, $x \neq x^{-1}$. Thus we can pair every element with its inverse to get $1 \in p(G)$. Hence $p(G) \leqq G'$.

LEMMA 3. 2. *If G is an abelian group of even order, then $p(G) = 1$ if the Sylow 2-subgroup of G is non-cyclic and $p(G)$ consists of the unique element of order two in G in the other case.*

PROOF. Consider first the case when the Sylow 2-subgroup S of G is cyclic. We can express G as the direct product $H \times S$ where H is of odd order m for some $m \geqq 1$ and $S = \langle a \rangle$ is cyclic of order 2^r for some $r \geqq 1$. Now $G = H + Ha + \ldots + Ha^{2^r - 1}$ and $p(H) = 1$ by Lemma 3. 1. Hence $p(Ha^i) = a^{mi}$ and $p(G) = a^{m(1 + 2 + \ldots + 2^r - 1)} = a^{2^r - 1} \neq 1$.

Assume induction hypothesis in the case where the Sylow 2-subgroup is non-cyclic. We can write $G = B \times C$, where $C = \langle c \rangle$ is cyclic of order 2^s for some $s \geqq 1$ and B is of even order. Since $|B| < |G|$, we may assume that $p(B) = 1$ or $p(B) = b$ with $b^2 = 1$. Now $G = B + Bc + \ldots + Bc^{2^s - 1}$. Let $|B| = 2n$ for some $n \geqq 1$, then $p(Bc^i) = p(B) \cdot c^{2in}$ so that $p(G) = (p(B))^{2^s} \cdot c^{2n(1 + 2 + \ldots + 2^s - 1)} = 1$.

PROOF OF THEOREM 2. $p(G) \leqq G'$ for odd order groups by Lemma 3. 1. Hence we may confine our attention to the case when $|G| = n$ is even. Let $|G'| = m$ and let

T be a set of coset representatives of G' in G. Then for any $g \in p(G)$, $g \equiv (p(T))^m \pmod{G'}$. Also $(p(T))^2 = 1, \pmod{G'}$ by Lemma 3. 2 so that if m is even, then $p(G) \leqq G'$. If the Sylow 2-subgroup of G/G' is cyclic, then $p(T) \not\equiv 1 \pmod{G'}$ by Lemma 3. 2, so that if m is odd and the Sylow 2-subgroup of G is cyclic, then $(p(T))^m$ $(p(T))^m \not\equiv 1 \pmod{G'}$, and hence $p(G) \not\leqq G'$. If the Sylow 2-subgroup of G/G' is non-cyclic, then $p(T) \equiv 1 \pmod{G'}$ by Lemma 3. 2 so that $(p(T))^m \in G'$ and hence $p(G) \leqq$ $\leqq G'$. Finally, the Sylow 2-subgroup of G and that of G/G' are isomorphic when $|G'|$ is odd. This completes the proof.

REFERENCES

[1] DÉNES, J.: On some properties of commutator subgroups, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* 7 (1964) 123—127.

[2] DÉNES, J.: On commutator subgroups of finite groups, *Comment. Math. Univ. St. Paul. (Tokyo)* 1967.

[3] ITO, N.: A theorem on alternating group A_n $(n \geqq 5)$, *Math. Japon.* **2** (1951) 59—60.

[4] THOMPSON, R. C.: Commutators in the special and general linear groups, *Trans. Amer. Math. Soc.* **101** (1961) 16—33.

University of Alberta

(Received February 5, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 201—205.

О ПРИБЛИЖЕНИИ ПЕРИОДИЧЕСКИХ ФУНКЦИЙ ЛИНЕЙНЫМИ ПРОЦЕССАМИ

В. В. ЖУК

1°. В работе рассматриваются вопросы, связанные с порядком приближения периодических функций линейными методами аппроксимации.

2°. Примем следующие обозначения и предположения. $\tilde{C}$ — пространство непрерывных 2π-периодических функций с обычной нормировкой. Функция $f\in\tilde{C}$. $E_n(f)$ — наилучшее приближение функции f тригонометрическими полиномами порядка не выше n, $\omega_k(\delta,f)$ — её модуль непрерывности порядка k. $\tilde{f}$ — функция тригонометрически сопряжённая с функцией f. Запись $f\in K$ означает, что f бесконечно дифференцируемая 2π-периодическая функция. A — множество операторов U, удовлетворяющих условиям: 1. U — линейный оператор из $\tilde{C}$ в $\tilde{C}$, 2. для любой $f\in K$ $U'(f)\in\tilde{C}$ и $U'(f)=U(f')$. B — множество операторов U, принадлежащих A и удовлетворяющих для любой $f\in K$ условию: $U(\tilde{f})=\tilde{U}(f)$. $\tilde{F}(x)$ — функция тригонометрически сопряжённая к первообразной для $f(x)-\frac{a_0}{2}$, где $a_0=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}f(x)dx$. C — неотрицательная постоянная, зависящая только от тех аргументов, которые будут выписаны.

3°. В дальнейшем нам понадобятся следующие результаты.

Теорема А. *Пусть U — линейный оператор из $\tilde{C}$ в $\tilde{C}$, r и n — натуральные числа. Если для любого тригонометрического полинома порядка не выше n $T_n(x)$*

$$\|U(T_n)\|\leqq C_1\|T_n^{(r)}(x)\|,$$

то для любой $f\in\tilde{C}$

$$\|U(f)\|\leqq C_2(r)\,(\|U\|+C_1 n^r)\,\omega_r\left(\frac{1}{n},f\right).$$

Теорема В. *Пусть U линейный оператор из $\tilde{C}$ в $\tilde{C}$, r и n — натуральные числа. Если для любого тригонометрического полинома порядка не выше n $T_n(x)$*

$$\|U(T_n)\|\leqq C_3\|\tilde{T}_n^{(r)}(x)\|,$$

то для любой $f\in\tilde{C}$

$$\|U(f)\|\leqq C_4(r)\left[\|U\|E_n(f)+C_3 n^{r+1}\,\omega_{r+1}\left(\frac{1}{n},\,\tilde{F}\right)\right].$$

Теоремы A и B суть простые следствия теоремы 1 работы [1] и теоремы 2 работы [2] и обобщают в некоторых направлениях более ранние результаты Г. Фройда [3], С. Б. Стечкина [4], стр. 60 и автора [5], стр. 93—96.

4⁰. Теорема 1. *Пусть операторы* $U_k \in B(k=1, 2, \dots i, \dots l)$, *целые числа* $p_k > 0$ *и* $m_k \geqq 0$ *таковы, что* $r = \sum_{k=1}^{l} p_k m_k > 0$. *Положим* $g = \sum_{k=1}^{i} p_k m_k$. *Тогда, если для любой* $f \in K$

$$\|U_k(f)\| \leqq C_k \|\tilde{f}_{(x)}^{(p_k)}\|, \qquad (k = 1, 2 \dots i)$$
$$\|U_k(f)\| \leqq C_k \|f_{(x)}^{(p_k)}\|, \qquad (k = i+1, \dots l)$$

то для любой $f \in \tilde{C}$ *и при любом натуральном* n*)

$$\left\| \left(\prod_{k=1}^{l} U_k^{m_k}\right)(f) \right\| \leqq C_6(r) \left[\prod_{k=1}^{l} \|U_k\|^{m_k} + n^r \prod_{k=1}^{l} C_k^{m_k} \right] \omega_r\left(\frac{1}{n}, f\right),$$

если g — *чётное число и*

$$\left\| \left(\prod_{k=1}^{l} U_k^{m_k}\right)(f) \right\| \leqq C_7(r) \left[\left(\prod_{k=1}^{l} \|U_k\|^{m_k}\right) E_n(f) + (n^{r+1} \prod_{k=1}^{l} C_k^{m_k} \omega_{r+1}) \left(\frac{1}{n}, \tilde{F}\right) \right],$$

если g — *нечётное число.*

Доказательство. Замечая, что если $f \in K$ и оператор $U \in A$, то $U(f) \in K$, нетрудно проверить, что для $f \in K$ при любых неотрицательных целых числах γ_k и s

$$\left[\left(\prod_{k=h}^{l} U_k^{\gamma_k}\right)(f) \right]^{(s)} = \left(\prod_{k=h}^{l} U_k^{\gamma_k}\right)(f^{(s)}) \qquad (h = 1, 2 \dots l)$$

и

$$\widetilde{\left(\prod_{k=h}^{l} U_k^{\gamma_k}\right)(f)} = \left(\prod_{k=h}^{l} U_k^{\gamma_k}\right)(\tilde{f}), \qquad (h = 1, 2 \dots l)$$

где знак ~ означает тригонометрическое сопряжение.

Пусть $g > 0$ — чётное число, $m_1 > 0$. Тогда для любой $f \in K$

$$\left\| \left(\prod_{k=1}^{l} U_k^{m_k}\right)(f) \right\| = \left\| U_1 \left(U_1^{m_1 - 1} \left(\left(\prod_{k=2}^{l} U_k^{m_k}\right)(f) \right) \right) \right\| \leqq$$

$$\leqq C_1 \left\| \left(U_1^{m_1 - 1} \left(\left(\prod_{k=2}^{l} U_k^{m_k}\right)(f) \right) \right)^{(p_2)} \right\| = C_1 \left\| U_1^{m_1 - 1} \left(\left(\prod_{k=2}^{l} U_k^{m_k}\right)(f^{(p_1)}) \right) \right\|.$$

Поступая аналогичным образом нетрудно убедиться, что неравенство

$$\text{(1)} \qquad \left\| \left(\prod_{k=1}^{l} U_k^{m_k}\right)(f \right\| \leqq \left(\prod_{k=1}^{l} C_5^{m_k} p_k\right) \|f_{(x)}^{(r)}\|$$

*) по определениию полагаем $\left(\prod_{k=1}^{3} U_k^{m_k}\right)(f) = U_1^{m_1}(U_2^{m_2}(U_3^{m_3}(f)))$. Выражение $\left(\prod_{k=1}^{l} U_k^{m_k}\right)(f)$ опрегелятся аналогично.

справедливо для любой $f \in K$ при единственном условии g — чётное число, если же g — нечётное число, то для $f \in K$ имеет место соотношение

$$\left\| \left(\prod_{k=1}^{l} U_k^{m_k} \right)(f) \right\| \leqq \left(\prod_{k=1}^{l} C_k^{m_k} \right) \| f_{(x)}^{(r)} \|. \tag{2}$$

Остальное следует из сопоставления неравенств (1) и (2) с теоремами A и B.

Замечание. Если $g=0$, то теорема справедлива и когда вместо условия $U_k \in B$ предполагается $U_k \in A$ $(k=1, 2 \ldots l)$. Справедливость замечания сразу следует из доказательства теоремы.

5⁰. Г. Фройдом (см. [6], стр. 10) установлена следующая теорема.

Теорема (Г. Фройд). Пусть $A_n = A_n(f)$ — последовательность линейных операторов, преобразующих $\tilde{C}$ в $\tilde{C}$ и подмножество неотрицательных функций в себя и пусть для целого $m > 0$ имеем

$$A_n(f_0) = f_0, \quad A_n(f_k) = f_k(x) + O(\lambda_n^{2m}), \qquad k = 1, 2 \ldots 2m,$$

где $$f_0 = 1, \qquad f_{2k-1}(x) = \sin kx, \qquad f_{2k}(x) = \cos kx.$$

Тогда для любой $f \in \tilde{C}$

$$\| f - A_n(f) \| \leqq O(1) \omega_{2m}(\lambda_n, f).$$

Эта теорема обобщает некоторые более ранние результаты Х. Бохмана, П. П. Коровкина и самого Г. Фройда.

Из теоремы Г. Фройда сразу следует, что если $f \in K$, то для последовательности $A_n(f)$ имеет место соотношение

$$\| f - A_n(f) \| \leqq O(1) \lambda_n^{2m} \| f_{(x)}^{(2m)} \|,$$

так как хорошо известно, что (см., например, [7], стр. 116)

$$\omega_{2m}(\delta, f) \leqq \delta^{2m} \| f_{(x)}^{(2m)} \|, \quad \text{если} \quad f_{(x)}^{(2m)} \in \tilde{C}.$$

Сопоставление последнего утверждения с теоремой 1 позволяет получить для операторов A_n, удовлетворяющих дополнительному условию $A_n \in A$ следующее обобщение приведённой выше теоремы Г. Фройда.

Теорема 2. *Пусть $A_n \in A$ — линейные операторы, удовлетворяющие условиям теоремы Г. Фройда, E — оператор тождественного преобразования $\tilde{C}$ в $\tilde{C}$, r — натуральное число, $\lim_{n \to \infty} \lambda_n = 0$. Тогда для любой $f \in \tilde{C}$*

$$\| (E - A_n)^r (f) \| \leqq O(1) \omega_{2mr}(\lambda_n, f).$$

Следствие. *Пусть*

$$A_n(f) = \int_{-\pi}^{\pi} f(x+t) \Phi_n(t) \, dt,$$

где функция $\Phi_n(t) \geqq 0$ удовлетворяет условиям:

1. $\int_{-\pi}^{\pi} \Phi_n(t)dt = 1$, 2. $\Phi_n(t)$ — *чётная функция, и пусть*

$$\varrho_1^{(n)} = \int_{-\pi}^{\pi} \cos t\, \Phi_n(t)\, dt \underset{n\to\infty}{\to} 1.$$

Тогда для любой $f \in \tilde{C}$

$$\|(E-A_n)^r(f)\| \leqq C_8(r)\,\omega_{2r}\left(\sqrt{1-\varrho_1^{(n)}}, f\right).$$

Доказательство. Очевидно, что

$$1 - A_n(1) = 0,$$

$$\|\sin x - A_n(\sin x)\| \leqq 1 - \varrho_1^{(n)},$$

$$\|\cos x - A_n(\cos x)\| \leqq 1 - \varrho_1^{(n)}.$$

Остальное следует из теоремы 2.

Отметим, что процессы, рассматриваемые в следствии, впервые изучались И. П. Натансоном [8], а в случае, когда $\Phi_n(t)$-ядро Джексона также, независимо, и И. Ю. Харрик [9].

6⁰. Проведём некоторые приложения изложенных выше результатов к конкретным методам аппроксимации.

1. Пусть

$$\sigma_n(f) = \frac{1}{n\pi} \int_{-\pi}^{\pi} f(x+t) \left(\frac{\sin \frac{nt}{2}}{\sin \frac{t}{2}} \right)^2 dt$$

интеграл Фейера,

$$I_n(f) = \frac{3}{2\pi n(2n^2+1)} \int_{-\pi}^{\pi} f(x+t) \left(\frac{\sin \frac{nt}{2}}{\sin \frac{t}{2}} \right)^4 dt$$

интеграл Джексона, числа $r \geqq 0$ и $k \geqq 0$ целые и таковы, что $m = r + 2k > 0$. Тогда для любой $f \in \tilde{C}$

$$\|((E-\sigma_n)^r(E-I_n)^k)(f)\| \leqq C_9(r,k)\,\omega_m\left(\frac{1}{n}, f\right),$$

если r — чётное число и

$$\|((E-\sigma_n)^r(E-I_n)^k)(f)\| \leqq C_{10}(r,k)\left[E_n(f) + n\omega_{m+1}\left(\frac{1}{n}, \tilde{F}\right)\right],$$

если r — нечётное число.

Доказательство. Г. Алексичем [10] показано, что

$$\|\sigma_n(f) - f(x)\| \leqq C_{11}\, n^{-1}\, \|\tilde{f}'(x)\|, \tag{3}$$

ели $\tilde{f}^{r} \in \tilde{C}$. Легко проверить справедливость неравенств

$$\|I_n(f)-f(x)\| \leqq C_{12} n^{-2} \|f''(x)\|,$$

если $f'' \in \tilde{C}$ и

$$\|(E-\sigma_n)^r (E-I_n)^k\| \leqq \|E-\sigma_n\|^r \|E-I_n\|^k \leqq 2^{r+k}. \tag{5}$$

Остальное получается сопоставлением оценок (3)—(5) с теоремой 1.

2. Пусть

$$V_n(f) = \frac{(2n)!!}{(2n-1)!!\,2\pi} \int_{-\pi}^{\pi} f(x+t) \cos^{2n} \frac{t}{2}\, dt$$

синтеграл Валле-Пуссена, r — натуральное число. Тогда для любой $f \in \tilde{C}$

$$\|(E-V_n)^r (f)\| \leqq C_{13}(r)\, \omega_{2r}\left(\frac{1}{\sqrt{n}}, f\right).$$

Доказательство этого утверждения легко получить, опираясь на следствие теоремы 2.

ЛИТЕРАТУРА

[1] Жук, В. В.: О приближении периодической функции ограниченным полуаддитивным оператором, *Докл. Акад. Наук СССР* **169** (1966) 515—518.

[2] Жук, В. В.: О приближении периодических функций линейными методами суммирования рядов Фурье, *Известия ЛЭТИ им. В. И. Ульянова (Ленина). Труды I конферен. проф.-препод. состава Новгородск. филиала,* Новгород, 1967, 202—210.

[3] Freud, G.: Sui procedimenti lineari d'approssimazione, *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat.* **26** (1959) 641—643.

[4] Стечкин, С. Б.: О приближении периодических функций суммами Фейера, *Труды Мат. Инст. Стеклов.* **72** (1961) 48—60.

[5] Жук, В. В.: О приближении 2π-периодической функции линейным оператором, *Сб. научн. тр. Ленингр. механ. ин-та* № **50** (1965) 93—115.

[6] Freud, G.: On approximation by positive linear methods, *Тезисы кратких научных сообщений. У. С. М.* (1966) 10.

[7] Тиман, А. Ф.: *Теория приближения функций действительного переменного,* Физматгиз, Москва, 1960.

[8] Натансон, И. П.: О приближении к многократно дифференцируемым периодическим функциям при помощи сингулярных интегралов, *Докл. Акад. Наук СССР* **82** (1952) 337—339.

[9] Харрик, И. Ю.: К проблеме аппроксимации функций, связанной с исследованием сходимости вариационных процессов, *Докл. Акад. Наук СССР* **81** (1951) 157—160.

[10] Alexits, G.: Sur l'ordre de grandeur de l'approximation d'une fonction par les moyennes de sa série de Fourier, *Mat. Fiz. Lapok* **48** (1941) 410—422.

Новгородский филиала Ленинградсково электротехнического инсмитута ии. В. И. Ульянова (Ленина)

(Поступила 5-ого мая 1967 г.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 207—211.

ON BASIS OPTIMALITY IN LINEAR PROGRAMMING

by

A. MAJTHAY

1. Introduction

The so-called optimality condition plays an important role both in the theory of linear programming and in the simplex method. It is a sufficient condition for the optimality of a feasible basis. Using the lexicographic simplex method we can prove that in case the problem has a solution, there always exists a feasible basis which satisfies the optimality condition [6]. Nevertheless if we have a feasible basis which does not satisfy the optimality condition, the question arises whether it is optimal or not.

In this remark we give a necessary and sufficient condition for the optimality of a feasible basis. This condition imples the well-known optimality condition.

2. Definitions and notations

Let $\mathbf{a}_p$ $p=0, 1, \ldots, n$ denote m-component column vectors. Let furthermore $A=(\mathbf{a}_1, \ldots, \mathbf{a}_n)$.

Among the columns of A we choose r linearly independent vectors and denote them by $\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_r}$, where r is the rank of A, and construct the matrix $B=(\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_r})$. The columns of B form a basis of the subspace generated by the columns of A.

Let us suppose that the system of equations

$$A\,\mathbf{x}=\mathbf{a}_0 \tag{2. 1}$$

is solvable. In this case $\mathbf{a}_0$ is a uniquely determined linear combination of the columns of B. If in this linear combination all coefficients are nonnegative, then B is called a feasible basis. Let us introduce the following sets of subscripts

$$I=\{i_1, \ldots, i_r\},$$

$$\bar{I}=\{1, \ldots, n\}-I.$$

In view of the definition of the basis, the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ can uniquely be expressed as linear combinations of the basis vectors:

$$\sum_{i\in I}\mathbf{a}_i d_{ip}=\mathbf{a}_p \qquad p=0, 1, \ldots, n. \tag{2. 2}$$

For a given p we shall construct the vector $\mathbf{d}_p$ from the components d_{ip} $i\in I$. Then we can write the system of equations (2. 1) in the following equivalent form:

$$\sum_{p=1}^{n}\mathbf{d}_p x_p=\mathbf{d}_0. \tag{2. 3}$$

The following solution of (2. 1)

$$x_p = \begin{cases} d_{p0} & \text{for} \quad p \in I \\ 0 & \text{for} \quad p \in \bar{I} \end{cases}$$

is called a basic solution defined by the basis B.

Consider now the linear programming problem:

Maximize

(2. 4) $$\sum_{p=1}^{n} c_p x_p$$

subject to the linear constraints

(2. 5) $$\sum_{p=1}^{n} \mathbf{a}_p x_p = \mathbf{a}_0$$

and nonnegativity restrictions

(2. 6) $$x_p \geqq 0 \qquad p = 1, \ldots, n.$$

A basis B is said to be optimal if the basic solution defined by B is optimal, i.e. if (2. 5) and (2. 6) implies

(2. 7) $$\sum_{p=1}^{n} c_p x_p \leqq \sum_{i \in I} c_i d_{i0}$$

for all n-tuples $x_1, \ldots, x_n$ satisfying (2. 5) and (2. 6).

In what follows we shall make use of the following

THEOREM OF HAAR [3], [4]: *Let us consider a real $m \times n$ matrix G, an m-component column vector* $\mathbf{g}$, *an n-component column vector* $\mathbf{h}$, *and a constant k.*

The consistent system of linear inequalities

(2. 9) $$G\mathbf{y} \leqq \mathbf{g}$$

implies the linear inequality

(2. 10) $$\mathbf{h}^T \mathbf{y} \leqq k$$

if and only if there exists an m-component vector $\mathbf{u}$ *which satisfies the following constraints:*

$$\mathbf{u} \geqq \mathbf{0}$$

$$\mathbf{u}^T G = \mathbf{h}^T$$

$$\mathbf{u}^T \mathbf{g} \leqq k$$

where the superscript T refers to the transpose.

For the sake of completeness we give the proof of this theorem, more exactly we show that this theorem is a consequence of the well-known theorem of FARKAS.

SUFFICIENCY. If $\mathbf{y}$ is a solution to (2. 9) then

$$\mathbf{h}^T \mathbf{y} = \mathbf{u}^T G \mathbf{y} \leqq \mathbf{u}^T \mathbf{g} \leqq k.$$

NECESSITY. First we prove that the homogeneous linear inequality

$$\mathbf{h}^T\mathbf{y} - kz \leqq 0 \tag{2.11}$$

is a consequence of the system of homogeneous linear inequalites

$$\begin{aligned} G\mathbf{y} - \mathbf{g}z &\leqq \mathbf{0} \\ -z &\leqq 0. \end{aligned} \tag{2.12}$$

For, if $(\mathbf{y}, z)$ is a solution to (2.12) and $z > 0$, then $\frac{1}{z}\mathbf{y}$ is a solution to (2.9) and so it is a solution to (2.10) according to our hypothesis. But this means that $(\mathbf{y}, z)$ is a solution to (2.11).

If $(\mathbf{y}, z)$ is a solution to (2.12) and $z = 0$, let us choose a vector $\mathbf{y}_1$, which solves the system of inequalities (2.9). Such a $\mathbf{y}_1$ exists, because the system (2.9) is consistent as we assumed. The vectors $\mathbf{y}_1 + \lambda\mathbf{y}$, $\lambda \geqq 0$ are solutions to (2.9) for arbitrary λ and so they solve (2.10). But this is possible only if $\mathbf{h}^T\mathbf{y} \leqq 0$ or in other words if (2.11) is a consequence of (2.12).

By the theorem of FARKAS [2] we know that there exists a non-negative m-component vector $\mathbf{u}$ and a non-negative constant u_0 such that

$$\begin{aligned} \mathbf{h}^T &= \mathbf{u}^T G \\ -k &= \mathbf{u}^T(-\mathbf{g}) - u_0 . \end{aligned}$$

This proves the theorem of HAAR.

3. The optimality of a basis

THEOREM. *A basis B is an optimal basis of problem* (2.4)—(2.6) *if and only if there exists an r-component vector* $\mathbf{v}$ *such that*

$$\begin{aligned} \mathbf{v} &\geqq \mathbf{0} \\ \mathbf{v}^T\mathbf{d}_0 &= 0 \\ \mathbf{v}^T\mathbf{d}_p &\geqq c_p - z_p \qquad p = 1, \ldots, n. \end{aligned} \tag{3.1}$$

PROOF. Let us write the constraints (2.5), (2.6) in the following equivalent form:

$$\begin{aligned} \sum_{i \in I} \mathbf{d}_i x_i &= \mathbf{d}_0 - \sum_{p \in I} \mathbf{d}_p x_p \\ x_p &\geqq 0 \qquad p = 1, \ldots, n. \end{aligned} \tag{3.2}$$

Looking at (2.2) we see that the vector $\mathbf{d}_i$ $i \in I$ is the i-th unit vector, having all components zero except the i-th one which is 1. Consequently (3.2) can be written as follows

$$\begin{aligned} x_i &= d_{i0} - \sum_{p \in I} d_{ip} x_p \quad i \in I \\ x_p &\geqq 0 \qquad p = 1, \ldots, n. \end{aligned} \tag{3.3}$$

Hence we obtain the system of inequalities:

$$\sum_{p\in I} \mathbf{d}_p x_p \leqq \mathbf{d}_0 \tag{3. 4}$$

$$x_p \geqq 0 \quad p\in \bar{I}.$$

By means of (3. 3) we can write the objective function in the following form:

$$\sum_{p=1}^{n} c_p x_p = \sum_{i\in I} c_i d_{i0} + \sum_{p\in I} (c_p - \sum_{i\in I} c_i d_{ip}) x_p.$$

Thus in accordance with (2. 8)

$$\sum_{p=1}^{n} c_p x_p = z_0 + \sum_{p\in I} (c_p - z_p) x_p. \tag{3. 5}$$

We see that the linear programming problem (2. 4)—(2. 6) is completely equivalent to the following problem: maximize (3. 5) subject to the constraints (3. 4). For there is a one to one correspondence between the feasible solutions of the two programs and the values of the two objective functions on corresponding solutions are equal.

Thus the basis B is optimal if and only if the linear inequality

$$\sum_{p\in I} (c_p - z_p) x_p \leqq 0 \tag{3. 6}$$

is a consequence of the system of linear inequalities (3. 4).

The theorem of HAAR implies that (3. 6) is a consequence of (3. 4) if and only if there exists an r-vector $\mathbf{v}\geqq\mathbf{0}$ and constants $w_p \geqq 0$ $p\in\bar{I}$ such that

$$c_p - z_p = \mathbf{v}^T \mathbf{d}_p - w_p \quad p\in\bar{I} \tag{3. 7}$$

$$0 \geqq \mathbf{v}^T \mathbf{d}_0$$

or equivalently

$$\mathbf{v}^T \mathbf{d}_p \geqq c_p - z_p \quad p\in\bar{I} \tag{3. 8}$$

$$\mathbf{v}^T \mathbf{d}_0 = 0$$

where we took into account that $\mathbf{v}\geqq\mathbf{0}$ and $\mathbf{d}_0\geqq\mathbf{0}$. For $p\in I$ the inequalities (3. 1) are trivially fulfilled and thus the theorem is proved.

REMARK 1. If a basis B satisfies the optimality condition

$$c_p - z_p \leqq 0 \qquad p = 1, \ldots, n$$

then $\mathbf{v}=\mathbf{0}$ satisfies the conditions (3. 1). Hence B is an optimal basis. Thus the classical basis-optimality theorem of linear programming is a special case of our theorem.

REMARK 2. In the general case the optimality condition is not necessary for the optimality as we can see from the following trivial example. In the linear program

$$\max \{x_1 + 2x_2 \mid x_1 + x_2 \leqq 0,\ x_1 \geqq 0,\ x_2 \geqq 0\}$$

the only — and so trivially optimal — solution is

$$x_1 = x_2 = 0.$$

We can consider this solution as a basic feasible solution corresponding to the basis $B = \mathbf{a}_1 = (1)$. It is easy to see that $z_2 - c_2 = -1 < 0$, so this optimal basis does not satisfy the optimality criterion. Our theorem makes it easy to construct such examples.

REMARK 3. In the case of nondegeneracy, i.e. if $\mathbf{d}_0 > \mathbf{0}$, the orthogonality relation in the theorem implies $\mathbf{v} = \mathbf{0}$ so the optimality condition is also necessary that the basis is optimal.

REFERENCES

[1] DANTZIG, G. B.: *Linear Programming and Extensions*, Princeton Univ. Press, Princeton, New Jersey, 1963.
[2] FARKAS, J.: Über die Theorie der einfachen Ungleichungen, *J. Reine Angew. Math.* **124** (1902) 1—24.
[3] HAAR, A.: A lineáris egyenlőtlenségekről, *Matematikai és Természettudományi Értesítő* **36** (1918) 279—296.
[4] HAAR, A.: Über lineare Ungleichungen. *Acta Sci. Math. (Szeged)* **2** (1924) 1—24.
[5] *Linear Inequalities and Related Systems*, H. W. Kuhn and A. W. Tucker eds., Princeton Univ. Press, 1956.
[6] PRÉKOPA, A.: *Lineáris programozás*, Bolyai János Matematikai Társulat, 1968.

Mathematical Institute of the Hungarian Academy of Sciences, Budapest

(Received March 19, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 213—224.

ON COMPLEMENTARY PIVOT THEORY

by

A. MAJTHAY

1. Introduction

As it is well known [8], the convex quadratic minimumproblem can be reduced to the solution of the system

$$\mathbf{w} = G\mathbf{z} + \mathbf{g}, \tag{1.1}$$

$$\mathbf{w} \geqq \mathbf{0}, \quad \mathbf{z} \geqq \mathbf{0}, \tag{1.2}$$

$$\mathbf{z}^T\mathbf{w} = \mathbf{0}. \tag{1.3}$$

Here G is an m by m real matrix, $\mathbf{g}$ is a fixed while $\mathbf{w}$ and $\mathbf{z}$ are variable vectors.

This reduction can be especially advantageous if the quadratic programming problem has a special structure. As an example let us consider the problem of finding the point of a convex polyhedron nearest a given point. This problem can be reduced in a trivial manner to the following.

Minimize

$$\frac{1}{2}\mathbf{x}^T\mathbf{x} \tag{1.4}$$

subject to the constraints

$$A\mathbf{x} \leqq \mathbf{b} \tag{1.5}$$

where A is an $m \times n$ matrix.

In the case of $\mathbf{b} \geqq \mathbf{0}$, the vector $\mathbf{x} = \mathbf{0}$ is (the only one) optimal solution. If $\mathbf{b}$ has at least one negative component, then the problem is not trivial. The constraints are linear and the objective function is convex, so the KUHN—TUCKER optimality conditions are both necessary and sufficient.

According to the KUHN—TUCKER theorem $\mathbf{x}$ is an optimal solution of problem (1.4)—(1.5) if and only if a vector $\mathbf{y} \geqq \mathbf{0}$ can be found such that

$$\mathbf{x} + A^T\mathbf{y} = \mathbf{0} \tag{1.7}$$

$$\mathbf{b} - A\mathbf{x} \geqq \mathbf{0} \tag{1.8}$$

$$\mathbf{y}^T(\mathbf{b} - A\mathbf{x}) = 0. \tag{1.9}$$

From (1.7)

$$\mathbf{x} = -A^T\mathbf{y}$$

so we can find the Lagrange multipliers by solving the following system:

$$AA^T\mathbf{y}+\mathbf{b} \geqq \mathbf{0}, \tag{1.12}$$

$$\mathbf{y}^T(AA^T\mathbf{y}+\mathbf{b}) = 0, \tag{1.13}$$

$$\mathbf{y} \geqq \mathbf{0}. \tag{1.14}$$

Introducing the notation

$$G = AA^T$$

$$\mathbf{g} = \mathbf{b}$$

$$\mathbf{z} = \mathbf{y}$$

$$\mathbf{w} = AA^T\mathbf{y}+\mathbf{b}$$

we see that our problem is reduced to (1. 1)—(1. 3). This result seems to be very useful because the size of the matrix AA^T is m by m, while the general theory of quadratic programming leads to an $m+n$ by $m+n$ matrix.

For the solution of (1. 1)—(1. 3) Wolfe proposed the simplex method with some modifications [8]. Recently DANTZIG and COTTLE [1], LEMKE [5] and GRAVES [3] proposed new methods which use deeply the special structure of the problem.

Some authors regard problem (1. 1)—(1. 3) as a programming problem: minimize $\mathbf{z}^T\mathbf{w}$, subject to the constraints $\mathbf{w} = G\mathbf{z}+\mathbf{g}$, $\mathbf{w} \geqq \mathbf{0}$, $\mathbf{z} \geqq \mathbf{0}$, while in this paper we prefer to regard it as a combinatorial one.

2. Definitions and notations

Let $\mathbf{g}_1, \ldots, \mathbf{g}_m$ denote the columns of the matrix G i.e.

$$G = \begin{bmatrix} g_{11} \cdots g_{1m} \\ \cdots\cdots\cdots \\ g_{m1} \cdots g_{mm} \end{bmatrix} = [\mathbf{g}_1, \ldots, \mathbf{g}_m].$$

Let furthermore

$$\mathbf{g} = \begin{bmatrix} g_1 \\ \vdots \\ g_m \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}.$$

If the pair $\mathbf{z}$, $\mathbf{g}$ satisfy the conditions (1. 1)—(1. 3) then

$$z_i w_i = 0 \qquad i = 1, \ldots, m,$$

so in some sense the variables w_i and z_i are complements of each other. The vectors $\mathbf{w}$, $\mathbf{z}$ satisfying (1. 1) are said solutions, the solutions satisfying (1. 2) are said feasible solutions and the solutions satisfying (1. 3) are said complementary solutions.

Let us denote by $\mathbf{e}_i$ $i = 1, \ldots, m$ the i-th unit vector i.e. the m-vector, the components of which are all zeros except the i-th one which is one. Let

$$\mathbf{e} = \sum_{i=1}^{m} \mathbf{e}_i.$$

We consider now the following auxiliary problem:

$$\mathbf{w} = G\mathbf{z} + \mathbf{e}t + \mathbf{g} \tag{2. 1}$$

$$\mathbf{w} \geqq \mathbf{0}, \quad \mathbf{z} \geqq 0, \quad t \geqq 0 \tag{2. 2}$$

$$\mathbf{z}^T\mathbf{w} = 0. \tag{2. 3}$$

Here t is a real variable. We speak about solutions, feasible solutions and complementary solutions of problem (2. 1)—(2. 3) analogously as above.

It is immediate that problem (2. 1)—(2. 3) has a feasible solution e.g.

$$\mathbf{z} = \mathbf{0}$$

$$t = \max(-\min g_i, 0)$$

$$\mathbf{w} = \mathbf{e}t + \mathbf{g}.$$

We give another form to the above equations

$$[E, -\mathbf{e}; -G]\begin{bmatrix}\mathbf{w}\\ t\\ \mathbf{z}\end{bmatrix} = \mathbf{g}$$

here E is a unit matrix of size m. The variables w_i and z_i $i = 1, \ldots, m$ are called complementary variables — each one is a complement of the other one — and in accordance with this the vectors $\mathbf{e}_i$ and $-\mathbf{g}_i$ $i = 1, \ldots, m$ in the matrix

$$A = [E, -\mathbf{e}, -G]$$

are called complementary vectors. It is convenient to give a unified notation for the columns of A:

$$A = [\mathbf{a}_1, \ldots, \mathbf{a}_{2m+1}].$$

Thus

$$\mathbf{a}_p = \begin{cases} \mathbf{e}_p & p = 1, \ldots, m, \\ -\mathbf{e} & \text{for } p = m+1, \\ -\mathbf{g}_{p-m-1} & p = m+2, \ldots, 2m+1. \end{cases} \tag{2. 4}$$

Let

$$\mathbf{x} = \begin{bmatrix}\mathbf{w}\\ t\\ \mathbf{z}\end{bmatrix},$$

and

$$\mathbf{a}_0 = \mathbf{g}.$$

Using these notations we can write (2. 1), (2. 2) in the following form:

$$A\mathbf{x} = \mathbf{a}_0$$

$$\mathbf{x} \geqq \mathbf{0}. \tag{2. 5}$$

The matrix A contains a unit matrix E, so the rank of A is m. Consequently an arbitrary basis of the linear vectorspace generated by the columns of A has m ele-

ments. We denote by B (sometimes with a superscript) a basis of this vectorspace consisting of columns of A. If

$$B = [\mathbf{a}_{i1}, \ldots, \mathbf{a}_{im}],$$

then let

$$I = \{i_1, \ldots, i_m\}.$$

Here B denotes a matrix. We shall, however, use the notation B for the set of vectors $\{\mathbf{a}_{i1}, \ldots, \mathbf{a}_{im}\}$ as well.

To every basis B there corresponds a matrix

$$D = [\mathbf{d}_0, \mathbf{d}_1, \ldots, \mathbf{d}_{2m+1}] = \begin{bmatrix} \boldsymbol{\delta}_1 \\ \vdots \\ \boldsymbol{\delta}_m \end{bmatrix} = [d_{i,p}],$$

as a unique solution of the equation system

$$BD = (\mathbf{a}_0, A).$$

We see that $\mathbf{d}_p$ contains the components of $\mathbf{a}_p$ $p = 0, 1, \ldots, 2m+1$ relative to B. In the following if B has a superscript then all the symbols depending on B will have the same superscript.

The solution

$$x_p = \begin{cases} d_{p,0} & p \in I \\ 0 & p \notin I \end{cases}$$

of the system of equations (2. 5) is called a basic solution defined by the basis B. The basis B is feasible if $\mathbf{d}_0 \geqq \mathbf{0}$ and it is called a complementary basis if it contains no complementary pairs of vectors.

The row vector $\boldsymbol{\delta} \neq \mathbf{0}$ is said to be lexicographically positive $\boldsymbol{\delta} \succ \mathbf{0}$ if its first nonzero component from the left is positive. We say that the vector $\boldsymbol{\delta}_1$ is lexicographically greater than the vector $\boldsymbol{\delta}_2$ and denote it by $\boldsymbol{\delta}_1 \succ \boldsymbol{\delta}_2$, if $\boldsymbol{\delta}_1 - \boldsymbol{\delta}_2 \succ \mathbf{0}$.

If we have a finite number of n-component vectors, there always exists a lexicographically minimal and a lexicographically maximal among them, because the relation $\succ$ is a total ordering. We denote these extrema by the symbols l-min and l-max, respectively.

The matrix D is said to be lexicographically positive $D \succ 0$ if each of its rows is lexicographically positive. It is clear that in such a case $\mathbf{d}_0 \geqq \mathbf{0}$ i. e. B is feasible and this is the reason for the following definition:

The basis B is called l-feasible if the matrix D belonging to B is lexicographically positive.

3. A complementary pivot algorithm

We want to find a complementary feasible solution to (1. 1)—(1. 3) or to show that such a solution does not exist.

If $\mathbf{g} \geqq \mathbf{0}$ then the problem can be solved trivially. In fact in this case $\mathbf{w} = \mathbf{g}$, $\mathbf{z} = \mathbf{0}$ is a complementary feasible solution. So we shall suppose that

$$\mathbf{g} \ngeqq \mathbf{0}.$$

It is clear that our problem is equivalent to the following: find a complementary feasible solution to (2. 1)—(2. 3), in which $t=0$, or show that it has no solution. In the following we shall deal with this modified problem.

Preparing the algorithm suppose we have an l-feasible basis B. Let us choose one of the vectors $\mathbf{a}_p$ $p\in\{1, 2, \ldots, 2m+1\}-I$, say $\mathbf{a}_k$. Assume we inserted $\mathbf{a}_k$ into B and removed from B an $\mathbf{a}_j$ $j\in I$. The question arises whether the new basis $B^{(1)}$ is l-feasible, too. If so, we will call it the neighbour of B.

THEOREM 1. *The l-feasible basis B has a neighbour, obtained from B by inserting $\mathbf{a}_k$ if and only if $\mathbf{d}_k$ has a positive component. In this case the neighbour is uniquely determined.*

PROOF. Let us insert $\mathbf{a}_k$ into B and remove an $\mathbf{a}_j$ $j\in I$. The subscript j is undefined yet. $B^{(1)}$ is a basis therefore necessarily $d_{j,k}\neq 0$. Assuming this to be the case, let us examine whether the new basis $B^{(1)}$ is l-feasible or not.

As it is well-known [6], the transformation formulas in terms of the rows of D resp. $D^{(1)}$ are the following:

$$\boldsymbol{\delta}_k^{(1)} = \frac{1}{d_{j,k}}\boldsymbol{\delta}_j \tag{3. 1}$$

$$\boldsymbol{\delta}_i^{(1)} = \boldsymbol{\delta}_i - \frac{d_{i,k}}{d_{j,k}}\boldsymbol{\delta}_j \quad i\in I^{(1)}-\{k\}. \tag{3. 2}$$

The vector $\boldsymbol{\delta}_j\succ\mathbf{0}$, so we see from (3. 1) that necessarily $d_{j,k}>0$. Supposing this, let us examine the equations (3. 2). $\boldsymbol{\delta}_i\succ\mathbf{0}$ and $\boldsymbol{\delta}_j\succ\mathbf{0}$, so $d_{i,k}\leqq 0$ impies $\boldsymbol{\delta}_i^{(1)}\succ\mathbf{0}$, therefore we have to examine those subscripts only for which $d_{i,k}>0$. Such a $d_{i,k}$ always exists, e.g. $d_{j,k}>0$ in accordance with our hypothesis. For such subscripts i, the vector $\delta_i^{(1)}$ is lexicographically positive if and only if

$$\frac{1}{d_{i,k}}\boldsymbol{\delta}_i \succ \frac{1}{d_{j,k}}\boldsymbol{\delta}_j.$$

Thus we see that the subscript j is defined by means of the following relation

$$\frac{1}{d_{j,k}}\boldsymbol{\delta}_j = l - \min_{\{i\,|\,i\in I,\, d_{i,k}>0\}} \frac{1}{d_{i,k}}\boldsymbol{\delta}_i.$$

The subscript j is uniquely determined by this relation, because in the opposite case two rows of the matrix D were linearly dependent. We know, however, that the rows of D are linearly independent, because D contains an identity matrix. Thus the theorem is proved.

After this preparation we can define the algorithm as follows:

(i) Let $B^{(0)}=[\mathbf{a}_1, \ldots, \mathbf{a}_m]$. According to the definition of the vectors $\mathbf{a}_p$ we see that $D^{(0)}=(\mathbf{a}_0, A)$ therefore $B^{(0)}$ is not feasible.

(ii) We define now the basis $B^{(1)}$. Let $k=m+1$ and let us insert $\mathbf{a}_k$ into the basis $B^{(0)}$. The subscript h of the vector to be removed from $B^{(0)}$ will be defined

by the following relation:

$$\frac{1}{d_{h,k}^{(0)}}\,\delta_h^{(0)} = l - \max_{i \in I^{(0)}} \frac{1}{d_{i,k}^{(0)}}\,\delta_i. \tag{3. 4}$$

We can see as above that h is uniquely determined.

Remembering to the transformation formulas (3. 1), (3. 2), to the definition of $\mathbf{a}_{m+1}$, furthermore to our hypothesis concerning $\mathbf{a}_0 = \mathbf{g}$ (that it has at least one negative component) we see that $B^{(1)}$ is l-feasible.

It is easy to see that $\mathbf{d}_h^{(1)} < \mathbf{0}$ i.e. all components are negative. It is therefore an immediate consequence of Theorem 1 that we cannot construct a neighbour to $B^{(1)}$, by inserting $\mathbf{a}_h$ into $B^{(0)}$. It is clear, too, that $\mathbf{a}_{m+1}$ is on a positive level in $B^{(1)}$.

(iii) After having constructed $B^{(0)}$ and $B^{(1)}$ we construct the series of neighbouring bases

$$B^{(0)};\ B^{(1)},\ B^{(2)},\ B^{(3)},\ \ldots \tag{3. 5}$$

in the following manner:

if in the construction of $B^{(q)}$ $q = 1, 2, 3, \ldots$ $\mathbf{a}_j$ is removed from $B^{(q-1)}$ then denoting by $\mathbf{a}_k$ the complementary pair of $\mathbf{a}_j$, in the following step $\mathbf{a}_k$ will be inserted into $B^{(q)}$. For $q = 1$ we have $j = h$ which was well defined in (3. 4), so according to Theorem 1 the series is uniquely defined.

It is clear from the definition of the series (3. 5) that each basis in it is l-feasible. Also, each basis is of a complementary type by the construction. The sequence (3. 5) terminates if one of the two possibilities arises.

(iv′) we remove $\mathbf{a}_{m+1}$ from the basis, or it remains in the basis on a zero level.

(iv″) $\mathbf{a}_{m+1}$ is in the basis on a positive level, but we cannot insert $\mathbf{a}_k$ into the basis because $\mathbf{d}_k \leqq 0$.

4. The finiteness of the algorithm

THEOREM 2. *The series of bases* (3. 5) *is finite, i.e. after a finite number of steps we reach either case* (iv′) *or* (iv″).

PROOF. The number of bases selected from the columns of A is finite, so it is enough to prove that in the series there is no basis appearing twice.

If we had a basis appearing twice in the series then we would have a first one, say $B^{(q)}$. Here $q \geqq 1$, because all the bases are l-feasible except for $B^{(0)}$.

We defined the bases of the series in such a manner that for each basis exactly one pair of complementary vector is outside the basis and the neighbours of the basis are constructed by inserting one of them into the basis. Thus remembering to Theorem 1 we see that each basis has at most two neighbours in the series.

The first basis appearing twice in the series must have at least three neighbours, because at the first appearing it has a different preceding basis than at the second appearing. In the opposite case it would not be the first basis appearing twice.

THEOREM 3. *In case* (iv′) *the last basic solution is a feasible complementary solution to problem* (1. 1)—(1. 3).

DEFINITION. *We call an m by m real matrix G positive semidefinite, if for all n-vectors* $\mathbf{z}$

$$\mathbf{z}^T G\mathbf{z} \geqq 0.$$

As we see this definition is a little more general than the usual.

THEOREM 4. *If G is a positive semidefinite matrix, then in case* (iv″) *the problem* (1. 1)—(1. 3) *has no feasible solution, so it has a fortiori no feasible complementary solution.*

As a preparation to the proof of this theorem, we prove two lemmas

LEMMA 1. *Let G be a positive semidefinite matrix, and let*

$$\mathbf{z}^T G\mathbf{z} = 0. \tag{4. 1}$$

Then

$$(G+G^T)\mathbf{z} = \mathbf{0}. \tag{4. 2}$$

PROOF. Let $\mathbf{y}$ be an arbitrary vector and λ an arbitrary real number. The positive semidefiniteness of G implies

$$(\mathbf{y}+\lambda\mathbf{z})^T G(\mathbf{y}+\lambda\mathbf{z}) \geqq 0$$

or as a consequence of (4. 1)

$$\mathbf{y}^T G\mathbf{y} + \lambda\mathbf{y}^T(G+G^T)\mathbf{z} \geqq 0.$$

This inequality is true for arbitrary λ values hence necessarily

$$\mathbf{y}^T(G+G^T)\mathbf{z} = 0$$

which is again true for an arbitrary $\mathbf{y}$ thus

$$(G+G^T)\mathbf{z} = \mathbf{0}.$$

LEMMA 2. *The system of inequalities*

$$G\mathbf{z}+\mathbf{g} \geqq \mathbf{0} \tag{4. 3}$$

$$\mathbf{z} \geqq \mathbf{0}$$

has a solution $\mathbf{z}$ *if and only if the system*

$$\mathbf{d} \geqq \mathbf{0}$$

$$G^T\mathbf{d} \leqq \mathbf{0} \tag{4. 4}$$

$$\mathbf{g}^T\mathbf{d} < 0$$

has no solution $\mathbf{d}$.

PROOF. If there exists a vector $\mathbf{d}$ satisfying (4. 4) than for an arbitrary vector $\mathbf{z} \geqq \mathbf{0}$ we have

$$\mathbf{d}^T G\mathbf{z} \leqq 0,$$

thus

$$\mathbf{d}^T(G\mathbf{z}+\mathbf{g}) < 0$$

therefore the second factor of the product has at least one negative component.

On the other hand if there is no vector $\mathbf{d}$ satisfying (4. 4) then the inequality

$$\mathbf{g}^T\mathbf{d} \geqq 0$$

is a consequence of the system of inequalities

$$\mathbf{d} \geqq \mathbf{0}$$

$$(-G^T)\mathbf{d} \geqq \mathbf{0}.$$

Then by the theorem of Farkas we know that there exists a pair of vectors $\mathbf{z} \geqq \mathbf{0}$, $\mathbf{u} \geqq \mathbf{0}$ satisfying

$$\mathbf{g} = -G\mathbf{z} + \mathbf{u},$$

i.e. the system (4. 3) is solvable.

5. The proof of Theorem 4

Let us suppose that G is positive semidefinite and the algorithm stopped at case (iv″). Denote the last basis of the series by B. We have to investigate two possibilities: 1) $B = B^{(1)}$ and 2) $B \neq B^{(1)}$.

1) If

$$B = B^{(1)} = [\mathbf{e}_1, \ldots, \mathbf{e}_{h-1}, -\mathbf{e}, \mathbf{e}_{h+1}, \ldots, \mathbf{e}_m] \tag{5. 1}$$

then $I = I^{(1)} = \{1, \ldots, h-1, m+1, h+1, \ldots, m\}$, because in the first step $\mathbf{a}_{m+1} = -\mathbf{e}$ was inserted into $B^{(0)} = [\mathbf{e}_1, \ldots, \mathbf{e}_m]$ and $\mathbf{e}_n$ was removed from it. The complementary pair of $\mathbf{e}_h$ is $\mathbf{a}_k = -\mathbf{g}_h$, so we have

$$B\mathbf{d}_k = -\mathbf{g}_h. \tag{5. 2}$$

Here, in accordance with our hypothesis,

$$\mathbf{d}_k \leqq \mathbf{0}. \tag{5. 3}$$

From (5. 1) and (5. 2) we see that

$$\begin{aligned} \mathrm{d}_{m+1,k} &= g_{h,h} \\ d_{i,k} &= g_{h,h} - g_{i,h} \qquad i \in I - \{m+1\} \end{aligned} \tag{5. 4}$$

G is positive semidefinite thus

$$g_{hh} = \mathbf{e}_h^T G \mathbf{e}_h \geqq 0. \tag{5. 5}$$

From (5. 3), (5. 4) and (5. 5) we conclude that

$$g_{hh} = \mathbf{e}_h^T G \mathbf{e}_h = 0. \tag{5. 6}$$

We know from Lemma 1 that

$$(G + G^T)\mathbf{e}_h = \mathbf{0}$$

which says

$$g_{ih} + g_{hi} = 0 \qquad i \in I. \tag{5. 7}$$

From (5. 3), (5. 4) and (5. 6)

$$g_{ih} \geqq 0 \qquad i \in I.$$

So from (5. 7) we see that

$$g_{hi} \leqq 0 \qquad i \in I$$

i.e.

(5. 8) $$G^T \mathbf{e}_h \leqq \mathbf{0}.$$

The vector $\mathbf{e}_h$ was removed from the basis $B^{(0)}$, therefore $g_h < 0$ hence

(5. 9) $$\mathbf{g}^T \mathbf{e}_h < 0.$$

If we define

(5. 10) $$\mathbf{d} = \mathbf{e}_h \geqq \mathbf{0},$$

then from Lemma 2 and from (5. 8), (5. 9) and (5. 10) we see that our Theorem is true.

2) In the second case, i.e. if $B \neq B^{(1)}$, B contains at least one vector of $-G$, it is a feasible complementary basis to the auxiliary problem and it contains the vector $\mathbf{a}_{m+1}$ on a positive level which is

(5. 11) $$d_{m+1,0} > 0.$$

As we supposed

(5. 12) $$\mathbf{d}_k \leqq \mathbf{0},$$

here $\mathbf{a}_k$ is the vector to be inserted into the basis B.

For the sake of simplifying the notation we can suppose that

(5. 13) $$B = (\mathbf{a}_1, \ldots, \mathbf{a}_r, \mathbf{a}_{m+1}, \mathbf{a}_{m+r+3}, \ldots, \mathbf{a}_{2m+1}).$$

We know that

(5. 14) $$0 \leqq r < m-1.$$

From this it is immediate that

(5. 15) $$\mathbf{a}_k = \begin{cases} \mathbf{e}_{r+1} & \text{if} \quad k < m+1 \\ -\mathbf{g}_{r+1} = \mathbf{a}_{m+r+2} & \text{if} \quad k > m+1. \end{cases}$$

We can express the vectors $\mathbf{a}_0$ and $\mathbf{a}_k$ in terms of B, namely

$$\sum_{i=1}^{r} \mathbf{a}_i d_{i,0} + \mathbf{a}_{m+1} d_{m+1,0} + \sum_{i=m+r+3}^{2m+1} \mathbf{a}_i d_{i,0} = \mathbf{a}_0,$$

$$\sum_{i=1}^{r} \mathbf{a}_i d_{i,k} + \mathbf{a}_{m+1} d_{m+1,k} + \sum_{i=m+r+3}^{2m+1} \mathbf{a}_i d_{i,k} = \mathbf{a}_k.$$

By means of (5. 13)—(5. 15) we can write

$$\sum_{i=1}^{r} \mathbf{e}_i(d_{i,0}-d_{m+1,0}) - \sum_{i=r+1}^{m} \mathbf{e}_i d_{m+1,0} - \sum_{i=r+2}^{m} \mathbf{g}_i d_{i+m+1,0} = \mathbf{g}, \tag{5. 16}$$

$$\sum_{i=1}^{r} \mathbf{e}_i(d_{i,k}-d_{m+1,k}) - \sum_{i=r+1}^{m} \mathbf{e}_i d_{m+1,k} - \sum_{i=r+2}^{m} \mathbf{g}_i d_{i+m+1,k} = \tag{5. 17}$$

$$= \begin{cases} \mathbf{e}_{r+1} & \text{if} \quad k < m+1 \\ -\mathbf{g}_{r+1} & \text{if} \quad k > m+1. \end{cases}$$

Let us define

$$\mathbf{d} = \begin{cases} -\sum_{i=r+2}^{m} \mathbf{e}_i d_{i+m+1,k} & \text{if} \quad k<m+1 \\ \mathbf{e}_{r+1} - \sum_{i=r+2}^{m} \mathbf{e}_i d_{i+m+1,k} & \text{if} \quad k>m+1. \end{cases} \tag{5. 18}$$

Clearly

$$\mathbf{d} \geqq \mathbf{0}. \tag{5. 19}$$

We multiply the system of equations (5. 17) by $\mathbf{d}^T$:

$$\begin{aligned} -d_{m+1,k}\left(\sum_{i=m+r+3}^{2m+1}(-d_{i,k})\right) + \mathbf{d}^T G\mathbf{d} = 0 \quad \text{if} \quad k<m+1, \\ -d_{m+1,k}\left(1+\sum_{i=m+r+3}^{2m+1}(-d_{i,k})\right) + \mathbf{d}^T G\mathbf{d} = 0 \quad \text{if} \quad k>m+1. \end{aligned} \tag{5. 20}$$

The positive semidefiniteness of G implies that the first term of these sums is non-positive. On the other hand $\mathbf{d}_k \geqq \mathbf{0}$ so it cannot be negative, hence it is zero. In the second equation we see that the second factor of the first term is positive. It is positive in the first equation, too, because in the opposite case we had $d_{i,k}=0$ $i = m+$ $+r+3, \ldots, 2m+1$ and so the system of equations (5. 17) would be in a contradiction to (5. 14). The results of our considerations are

$$d_{m+1,k} = 0, \tag{5. 21}$$

and

$$\mathbf{d}^T G\mathbf{d} = 0. \tag{5. 22}$$

Taking these into account (5. 17) can be rewritten as

$$G\mathbf{d} = \begin{cases} \sum_{i=1}^{r} \mathbf{e}_i(-d_{i,k}) + \mathbf{e}_{r+1} & \text{if} \quad k<m+1 \\ \sum_{i=1}^{r} \mathbf{e}_i(-d_{i,k}) & \text{if} \quad k>m+1. \end{cases} \tag{5. 23}$$

In view of (5. 19) and (5. 22) Lemma 1 implies

$$G^T\mathbf{d} = -G\mathbf{d} = \begin{cases} \sum_{i=1}^{r} \mathbf{e}_i d_{i,k} - \mathbf{e}_{r+1} & \text{if} \quad k < m+1 \\ \sum_{i=1}^{r} \mathbf{e}_i d_{i,k} & \text{if} \quad k > m+1. \end{cases} \tag{5. 24}$$

A special case of this

$$(G^T\mathbf{d})_i = 0 \quad \text{if} \quad i > r+1 \tag{5. 25}$$

and in general

$$G^T\mathbf{d} \leqq \mathbf{0}. \tag{5. 26}$$

Multiply (5. 16) by $\mathbf{d}^T$. Then by means of (5. 25) we get

$$\mathbf{d}^T\mathbf{g} = \begin{cases} -d_{m+1,0}\left(-\sum_{i=r+2}^{m} d_{i+m+1,k}\right) & \text{if} \quad k < m+1 \\ -d_{m+1,0}\left(1-\sum_{i=r+2}^{m} d_{i+m+1,k}\right) & \text{if} \quad k > m+1. \end{cases}$$

Here, according to (5. 11) the first factor is negative and the second one is positive as we saw it above, thus

$$\mathbf{d}^T\mathbf{g} < 0. \tag{5. 27}$$

The relations (5. 19), (5. 26), (5. 27) and Lemma 2 prove our theorem.

An immediate consequence of our Theorem 4 is the following

THEOREM 5. *Let G be an m by m positive semidefinite real matrix and $\mathbf{g}$ an m component real vector. If the system of inequalities*

$$G\mathbf{z}+\mathbf{g} \geqq \mathbf{0}$$

$$\mathbf{z} \geqq \mathbf{0}$$

is consistent, then among its solutions we have at least one for which

$$\mathbf{z}^T(G\mathbf{z}+\mathbf{g}) = 0.$$

The author would like to express his appreciation to Prof. A. PRÉKOPA for many stimulating discussions and helpful suggestions toward the development of this paper.

REFERENCES

[1] DANTZIG, G. B. and COTTLE, R. W.: Positive (semi) Definite Programming, in *Nonlinear Programming*, ed. J. Abadie, North-Holland, 1967.
[2] FARKAS, J.: Über die Theorie der einfachen Ungleichungen, *J. Reine Angew. Math.* **124** (1902) 1—24.
[3] GRAVES, R. L.: A Principal Pivoting Simplex Algorithm for Linear and Quadratic Programming, *Operations Res.* **15** (1967) 482—494.
[4] LEMKE, C. E.: Bimatrix Equilibrium Points and Mathematical Programming, *Managament Sci.* **11/2** (1965) 681—689.
[5] LEMKE, C. E. and F. T. HOWSON,: Equilibrium Points of Bimatrix Games, *J. Soc. Indust. Appl. Math.* **12** (1964) 413—423.
[6] PRÉKOPA, A.: *Lineáris programozás, Bolyai János Mat. Társ.*, Budapest, 1968.
[7] TUCKER, A. W.: Pivotal Algebra, Seminar Notes by T. D. Parsons, Princeton, 1967.
[8] WOLFE, PH.: The Simplex Method for Quadratic Programming, *Econometrica* **27** (1959) 382—398.

Mathematical Institute of the Hungarian Academy of Sciences, Budapest

(Received March 19, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 225–247.

KREISÜBERDECKUNGEN DER SPHÄRE

von

G. FEJES TÓTH

§ 1. Einleitung. Unter einer *Kreisunterdeckung* der Sphäre verstehen wir eine Menge offener Kugelkappen, in der jeder Punkt der Sphäre zu höchstens einer Kugelkappe gehört. In ähnlicher Weise definieren wir eine *Kreisüberdeckung* der Sphäre als eine Menge abgeschlossener Kugelkappen, wobei jeder Punkt der Sphäre zu mindestens einer Kugelkappe gehört. Eine Kugelkappe wollen wir eine Kreisscheibe, oder schlechthin einen Kreis nennen. Die *Dichte* einer Kreismenge wird als der Quotient aus der Kreisinhaltssumme und dem Flächeninhalt der Sphäre definiert.

Wir richten unsere Aufmerksamkeit auf zwei wohlbekannte, einander dual gegenüberstehende Probleme, nämlich auf die Probleme der dichtesten Kreisunterdeckung und der dünnsten Kreisüberdeckung der Sphäre:

Gesucht werden diejenigen Unterdeckungen der Sphäre durch n kongruente Kreise, bei denen die Kreisdichte den grösstmöglichen Wert erreicht.

Gesucht werden diejenigen Überdeckungen der Sphäre durch n kongruente Kreise, bei denen die Kreisdichte den kleinstmöglichen Wert erreicht.

Das erste Problem lässt sich nach MESCHKOWSKI [9] folgendermassen interpretieren:

Auf einem Planeten herrschen n einander feindliche Diktatoren. Wie sind die Residenzen dieser Herren zu verteilen, damit der Minimalabstand zwischen zwei beliebigen von ihnen möglichst gross wird?

Um die Dualität der beiden Probleme zu betonen, geben wir dem zweiten Problem folgende Interpretation:

Auf einem Planeten herrschen n miteinander befreundete Diktatoren. Wie müssen diese Herren ihre Residenzen verteilen, damit sie den Planeten am besten kontrollieren können in dem Sinne, dass der Maximalabstand zwischen einem Planetenpunkt und der nächsten Residenz möglichst klein wird?

Das Unterdeckungsproblem scheint zuerst der Biologe TAMMES [14] aufgeworfen zu haben, der die eigenartige Verteilung der Öffnungen an den Pollenkörnern gewisser Blumen durch dieses Extremalprinzip zu erklären versuchte. Das Problem scheint auch in der Struktur des haploiden Zellkerns [11] und im Aufbau gewisser Viren [6] eine Rolle zu spielen. Aber auch das Überdeckungsproblem scheint von biologischem Interesse zu sein [3]. Bemerkenswert ist auch der Zusammenhang der analogen höherdimensionalen Probleme mit der Informationstheorie [8, 15].

Die erste mathematische Behandlung des Unterdeckungsproblems rührt von L. FEJES TÓTH [1] her. Er gab für die Unterdeckungsdichte eine obere Schranke an, die für $n = 3, 4, 6$ und 12 die Lösung des Problems ergibt. Für $n = 5, 7, 8$ und 9 wurde das Problem von SCHÜTTE und VAN DER WAERDEN [13] und für $n = 24$ von ROBINSON

[10] gelöst. Ausserdem wurden günstige Unterdeckungen für $n=10$, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 25, 26, 28, 30, 31, 32, 33, 42, 44, 48, 52, 60, 80, 110, 120 und 122 angegeben (s. z. B. [5]).

Trotz der vollkommenen Dualität der beiden Probleme erregte das zweite Problem weniger Interesse. Eine von L. FEJES TÓTH [2] stammende untere Abschätzung der Überdeckungsdichte enthält die Lösung für $n=3$, 4, 6 und 12. Für $n=5$ und 7 hat das Problem SCHÜTTE [12] gelöst, der zugleich eine günstige Überdeckung mit $n=8$ Kreisen konstruierte. Das ist aber alles, was über das Überdeckungsproblem bekannt war.

§ 2. Ergebnisse. In diesem Aufsatz lösen wir das Überdeckungsproblem für $n=10$ und 14. Weiterhin konstruieren wir einige lokal extremale Überdeckungen, von denen eine, die wir ikosidodekaedrisch nennen werden, vermutlich die dünnste Überdeckung der Sphäre durch 32 kongruente Kreise darstellt.

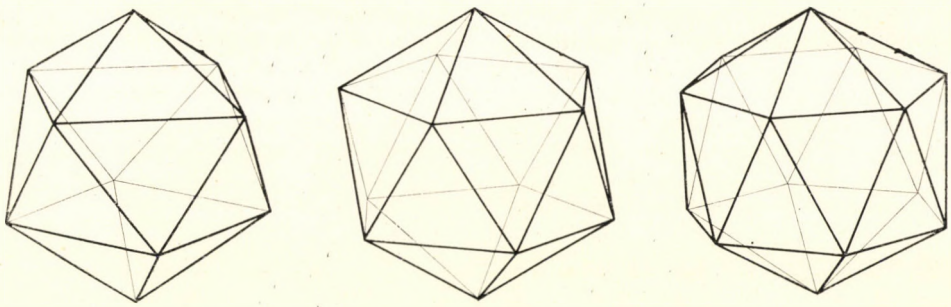


Abb. 1

Das oben erwähnte Resultat von Schütte bezüglich der Überdeckung mit 5 und 7 Kreisen besagt, dass in der extremalen Lage je ein Kreismittelpunkt im Nord- und Südpol liegt, während die übrigen Kreismittelpunkte auf dem Äquator gleichmässig verteilt sind. Da dasselbe auch bei der dünnsten Überdeckung mit 6 Kreisen zutrifft, gilt folgender

Satz von Schütte. In der dünnsten Überdeckung der Sphäre durch 5, 6 oder 7 kongruente Kreise sind die Kreismittelpunkte Ecken einer regulären Doppelpyramide.

Um unser Hauptresultat in einem analogen Satz aussprechen zu können, nennen wir den Körper, der durch Aufsetzen zweier kongruenter gerader Pyramiden auf die Grund- und Deckfläche eines Antiprismas entsteht, eine *antiprismatische Doppelpyramide*. Ist ein solcher Körper einer Kugel einbeschrieben, und sind alle seiner Flächen kongruent, so nennen wir ihn regulär (Abb. 1).

Nun gilt folgender

SATZ. *In der dünnsten Überdeckung der Sphäre durch* 10, 12 *oder* 14 *kongruente Kreise sind die Kreismittelpunkte Ecken einer regulären antiprismatischen Doppelpyramide.*

§ 3. Gedankengang des Beweises. Da die Lösung des Überdeckungsproblems für 12 Kreise bekannt ist, können wir uns auf den Fall von 10 und 14 Kreisen beschränken.

Es sei S ein System von n Punkten auf der Einheitssphäre, die nicht alle auf einer Halbsphäre liegen, K die konvexe Hülle von S, und M das Mosaik, das durch

Zentralprojektion der Flächen von K auf die Sphäre entsteht. Wir stellen uns M als ein Dreiecksmosaik vor, indem wir die eventuell auftretenden mehr als dreiseitigen Flächen durch einander nicht kreuzende Diagonalen irgendwie in Dreiecke zerlegen.

Es sei r der kleinste Raduiswert mit der Eigenschaft, dass die um die Punkte von S mit dem Radius r geschlagenen Kreise die Sphäre überdecken. Wir betrachten eine Fläche f von M sowie den Umkreis k von f, d. h. den Kreis, der die Ecken von f auf seinem Rand enthält. Da kein Punkt von S ausserhalb K liegt, liegt kein Punkt von S innerhalb k. Deshalb ist der Radius von k nicht grösser als r. Sonst könnten nämlich die um die Punkte von S mit dem Radius r geschlagenen Kreise den Mittelpunkt von k nicht überdecken.

Ist der Umkreisradius eines Dreiecks nicht grösser als ein gewisser Wert a, so nennen wir das Dreieck ein *a-Dreieck*. Ein Dreiecksmosaik mit lauter a-Dreiecken sei ein *a-Mosaik* genannt. Das Mosaik M ist also ein r-Mosaik.

Wir setzen $n_1 = 10$ und $n_2 = 14$. Wir bezeichnen das Mosaik einer regulären antiprismatischen Doppelpyramide mit n_i Ecken mit M_i und die entsprechenden Werte von r mit r_i (1, 2). Nach einer elementaren Rechnung gilt

$$r_1 = \operatorname{arc\,tg} \sqrt{2(\sqrt{2}-1)} \approx 42^\circ 18' 28''$$

$$r_2 = \operatorname{arc\,tg} \sqrt{\frac{2}{3}(\sqrt{3}-1)} \approx 34^\circ 56' 17''.$$

Wir werden zeigen, dass *ein r_i-Mosaik mit n_i Ecken mit dem Mosaik M_i identisch ist.*

Der Beweis beruht auf folgendem

HILFSSATZ 1. *Wir betrachten k r-Dreiecke, die eine gemeinsame Ecke E haben und eine Umgebung von E schlicht und lückenlos bedecken. An die E gegenüberliegenden Seiten der Dreiecke sollen sich weitere k r-Dreiecke anschliessen. Ist $r < \pi/4$ und $k > 5$ oder $r = r_1$ und $k > 3$, so erreicht die Inhaltssumme dieser 2k Dreiecke ihren grösstmöglichen Wert für kongruente gleichschenklige Dreiecke vom Umkreisradius r, die so angeordnet sind, dass in E lauter kongruente Dreiecksseiten zusammentreffen.*

Zuerst werden wir zeigen, dass ein r_i-Mosaik mit n_i Ecken für $i = 1$ nur 4- und 5-kantige Ecken und für $i = 2$ nur 5- und 6-kantige Ecken haben kann. Es wird sich dann herausstellen, dass dadurch die topologischen Typen der beiden fraglichen Mosaike eindeutig bestimmt sind, dass also diese Mosaike zu M_1 und M_2 isomorph sind.

Wir greifen nun die beiden 4- bzw. 6-kantigen Ecken E_1 und E_2 eines r_i-Mosaiks mit n_i Ecken heraus. Die Inhaltssumme der in E_i zusammentreffenden Dreiecke und der weiteren Dreiecke, die sich an die vorigen entlang ganzer Seiten anschliessen, sei s_j ($j = 1, 2$). Nach Hilfssatz 1 gilt $s_j \leqq 2\pi$. Andererseits haben wir $s_1 + s_2 = 4\pi$, woraus $s_1 = s_2 = 2\pi$ folgt. Dies ist aber nach Hilfssatz 1 nur so möglich, dass das Mosaik mit M_i identisch ist.

§ 4. Hilfssätze. In diesem Paragraphen beweisen wir einige Hilfssätze, die wir einerseits beim Beweis des Hilfssatzes 1, andererseits bei der Ausschliessung der Ecken mit unzulässigen Kantenzahlen brauchen werden. Unsere Betrachtungen

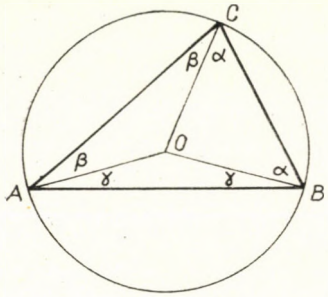


Abb. 2

beziehen sich auf die sphärische Geometrie. Dies wird im folgenden stets stillschweigend vorausgesetzt.

HILFSSATZ 2. *Wir betrachten in einem festen Kreis einen Peripheriewinkel vorgegebener Grösse. Dann erreicht die Länge des zugehörigen Kreisbogens ihr Minimum im symmetrischen Fall.*

Offensichtlich ist Hilfssatz 2 äquivalent mit folgender Behauptung: In einem Kreis K erreicht der zu dem festen Bogen AB gehörige Peripheriewinkel $\omega = \sphericalangle ACB$ sein Maximum im Falle $AC = CB$. Um dies einzusehen, bezeichnen wir den Mittelpunkt von K mit 0 und schreiben $\gamma = \pm \sphericalangle OAB$, $\alpha = \pm \sphericalangle OBC$, $\beta = \pm \sphericalangle OCA$, je nachdem ob der Bogen AB, BC bzw. CA kleiner oder grösser ist als ein Halbkreis (Abb. 2). Dann ist der Inhalt t des Dreiecks ABC

$$t = 2\alpha + 2\beta + 2\gamma - \pi = 2\omega + 2\gamma - \pi.$$

Da aber γ konstant ist, erreicht der Winkel ω sein Maximum in derselben Lage wie t. Ist nun $AC \neq CB$, so bringen wir C auf dem Lexellschen Kreis in die symmetrische Lage, wodurch t unverändert bleibt und C in das Innere von K rückt. Bringen wir jetzt C unter Behaltung der Symmetrie wieder auf den Rand von K, so nimmt t zu. Deshalb ist t, ebenso wie ω in der symmetrischen Lage grösser als in einer asymmetrischen. Diesen Beweis, der einfacher ist als mein ursprünglicher analytischer Beweis, verdanke ich Herrn Professor G. HAJÓS.

HILFSSATZ 3. *Ist das Dreieck ABC einem festen Kreis einbeschreiben und ist die Grösse des Winkels $\sphericalangle ACB$ vorgegeben, so erreicht der Inhalt von ABC sein Maximum, wenn $AC = CB$ ist.*

Es sei d der Durchschnitt des Winkelbereichs ACB und des Kreises. Wir setzen voraus, dass $AC < CB$ ist und drehen den Winkelbereich um C solange, bis $AC = CB$ wird. Inzwischen überstreichen die Sehnen CA und CB je einen Kreissektor. Da aber während der Drehung stets $AC < CB$ ist, ist der Inhalt des ersten Sektors kleiner als der Inhalt des zweiten. Deshalb ist der Inhalt von d in der symmetrischen Lage grösser als in einer asymmetrischen.

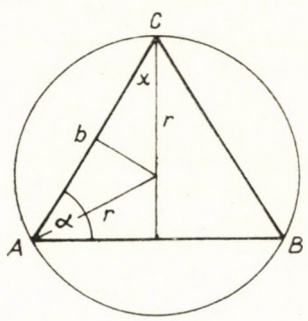


Abb. 3

Nun entsteht das Dreieck ABC so, dass man von d das zum Bogen AB gehörige Kreissegment abschneidet. Da aber der Inhalt dieses Segments sein Minimum nach Hilfssatz 2 im symmetrischen Fall erreicht, nimmt der Inhalt des Dreiecks sein Maximum ebenfalls im symmetrischen Fall an.

HILFSSATZ 4. *Haben vier r_2-Dreiecke einen gemeinsamen Eckpunkt E, und bedecken sie eine Umgebung von E schlicht und lückenlos, so erreicht die Inhaltssume der Dreiecke ihr Maximum für kongruente gleichschenklige Dreiecke vom Umkreisradius r_2, deren Schenkel in E zusammentreffen.*

Es sei ABC ein Dreieck vom Umkreisradius $r<\pi/2$ mit $AC=CB$. Wir setzen $\sphericalangle ACB=2x$, $\sphericalangle BAC=\alpha$ und $AC=b$ (Abb. 3). Wir betrachten die Funktion

$$T(x)=2x+2\operatorname{arc\,tg}\frac{\operatorname{ctg} x(1+\operatorname{tg}^2 r\cos^2 x)}{1-\operatorname{tg}^2 r\cos^2 x}-\pi,\quad 0<x<\pi/2,$$

die mit Rücksicht auf die Relationen $\operatorname{tg} b/2=\operatorname{tg} r\cos x$ und $\cos b=\operatorname{ctg} x\operatorname{ctg}\alpha$ den Inhalt des Dreiecks ABC angibt. Wir wollen zeigen, dass für $r=r_2$

$$S\leqq 4T(\pi/4)$$

ist, wo S die Inhaltssumme der Dreiecke bedeutet.

Aus Hilfssatz 3 folgt, dass

$$S\leqq T(x_1)+\ldots+T(x_4),$$

wobei $2x_1,\ldots,2x_4$ die bei E liegenden Winkel sind. Es genügt also zu zeigen, dass

$$T(x_1)+\ldots+T(x_4)\leqq 4T(\pi/4)$$

ist. Setzen wir

$$p=\operatorname{tg}^2 r,$$

so ergibt eine elementare Rechnung

$$\frac{1}{4}T'=1-\frac{p\cos^2 x+1}{p^2\cos^4 x+2p\cos^2 x\cos 2x+1},$$

$$\frac{1}{4}T''=-p\sin 2x\,\frac{(p^2+4p)\cos^4 x+(2p+8)\cos^2 x-3}{(p^2\cos^4 x+2p\cos^2 x\cos 2x+1)^2}.$$

Da

$$(p^2+4p)\cos^4 x+(2p+8)\cos^2 x-3$$

eine abnehmende Funktion von x ist, die für

$$\tilde{x}=\operatorname{arc\,cos}\sqrt{\frac{1}{p}\left(2\sqrt{\frac{1+p}{4+p}}-1\right)}$$

verschwindet, ist $T(x)$ im Intervall $(0,\tilde{x})$ eine konkave und im Intervall $(\tilde{x},\pi/2)$ eine konvexe Funktion von x.

Es sei $2\bar{x}$ ein Winkel und Δ der Inhalt des gleichseitigen Dreiecks vom Umkreisradius r. Offensichtlich ist $\Delta=T(\bar{x})=6\bar{x}-\pi$ das Maximum von $T(x)$. Bezeichnen wir die zu $r=r_i$ gehörigen Werte von $\tilde{x}$, $\bar{x}$ und Δ mit dem Index i, so gilt

$$\tilde{x}_1\approx 58^\circ 8'46'',\quad \bar{x}_1\approx 37^\circ 58'43'',\quad \Delta_1\approx 47^\circ 52'18''$$

$$\tilde{x}_2\approx 56^\circ 7'30'',\quad \bar{x}_2\approx 35^\circ\ 9'23'',\quad \Delta_2\approx 30^\circ 56'18''.$$

Wir können voraussetzen, dass die Winkel $x_1,\ldots,x_4$ alle kleiner sind als $\pi/3$. Sonst hätten wir

$$T(x_1)+\ldots+T(x_4)\leqq T(\pi/3)+3\Delta_2\approx 104^\circ 39'18'',$$

was erheblich kleiner ist als

$$4T(\pi/4)\approx 109^\circ 42'32''.$$

Wir betrachten die lineare Funktion $1(x)$, die durch $1(\pi/4)=T(\pi/4)$ und $1'(\pi/4)=T'(\pi/4)$ definiert ist. Wegen

$$16°58'53'' \approx 1(60°) > T(60°) \approx 12°50'24''$$

haben wir für $\pi/4 < x \leqq \pi/3$

$$1(x) > T(x).$$

Wir definieren jetzt die Funktion $G(x)$ durch

$$G(x) = \begin{cases} T(x) & \text{für} \quad 0 < x \leqq \pi/4 \\ 1(x) & \text{für} \quad \pi/4 < x \leqq \pi/3. \end{cases}$$

Da $G(x)$ in $(0, \pi/3)$ konkav ist, gilt nach der Jensenschen Ungleichung

$$S \leqq T(x_1)+\ldots+T(x_4) \leqq G(x_1)+\ldots+G(x_4) \leqq 4G(\pi/4) = 4T(\pi/4).$$

Der Fall der Gleichheit leuchtet ein.

HILFSSATZ 5. *Es sei ABC ein beliebiges r-Dreieck mit $r<\pi/4$. Ist die Seitenlänge AB sowie die Grösse des Winkels $\sphericalangle ACB$ vorgegeben, so erreicht der Inhalt von ABC sein Maximum entweder, wenn $AC=CB$ oder wenn der Umkreisradius von ABC gleich r ist.*

Wir setzen $\sphericalangle CAB=\alpha$, $\sphericalangle ABC=\beta$, $\sphericalangle BCA=\gamma$ und setzen voraus, dass $\alpha<\beta$ ist. Wegen $r<\pi/4$ gilt dann $BC<CA<\pi/2$. Deshalb ist der Inhalt des Dreiecks ABC kleiner als der halbe Inhalt des durch die „Halbgeraden" CA und CB begrenzten Zweiecks: $\alpha+\beta+\gamma-\pi < \gamma$.

Wir bewegen die Strecke AB so, dass A und B auf den Halbgeraden CA und CB bleiben und dass CB zunimmt. Ist $\beta \geqq \pi/2$, so ist, wegen $\alpha+\beta < <\pi$, $\alpha<\pi/2$, weshalb auch CA und zugleich der Inhalt von ABC offensichtlich zunimmt. Ist dagegen $\alpha<\beta<\pi/2$, so nimmt CA ab. Es sei D der Schnittpunkt der von A und B ausgehenden inneren Normalen des Zweiecks und r' der Fusspunkt des von D auf die Strecke AB gefällten Lotes (Abb. 4). Da D das momentane Drehzentrum der Bewegung ist, dreht sich die Gerade AB um F. Wegen $\sphericalangle DAF = 90°- -\alpha > 90°-\beta = \sphericalangle DBF$ gilt aber $AF<FB$, weshalb der Inhalt von ABC auch in diesem Fall monoton zunimmt, bis entweder $AC=CB$ oder der Umkreisradius von ABC gleich r wird.

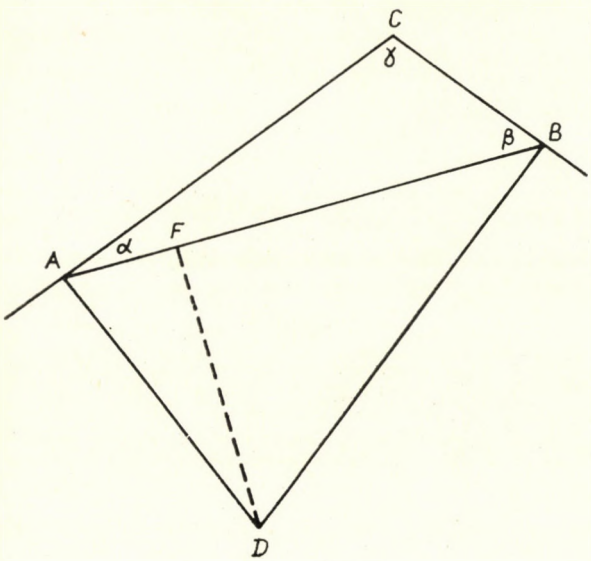


Abb. 4

Es sei bemerkt, dass Hilfssatz

5 auch ohne die Beschränkung $r<\pi/4$ gilt, doch erfordert der Beweis einige Fallunterscheidungen.

HILFSSATZ 6. *Es seien ABC und ABD zwei r-Dreiecke mit $r<\pi/4$, die bei einem konstanten Wert von r und des Winkels $\sphericalangle ACB$ frei variieren. Dann kann die Inhaltssumme der Dreiecke ihr Maximum nur in einer solchen Lage erreichen, in der $AC=CB$, $AD=DB$ und der Umkreisradius von ABD gleich r ist.*

Wir halten die Seite AB fest und bringen D durch eine Bewegung auf dem Lexellschen Kreis in eine solche Lage, dass $AD=DB$ sei. Dadurch bleibt der Inhalt von ABD unverändert. Lassen wir jetzt die gemeinsame Länge der Seiten AD und DB zunehmen, bis der Umkreisradius von ABD den Wert r erreicht und ABD seinen Umkreismittelpunkt enthält, so nimmt der Inhalt von ABD zu. Nach Hilfssatz 5 müssen wir uns also nur mit demjenigen Fall beschäftigen, in dem der Umkreisradius beider Dreiecke gleich r und $AD=DB$ ist und ABD seinen Umkreismittelpunkt enthält.

Wir variieren das Dreieck ABC so, dass sein Umkreisradius r und der Winkel $\omega=\sphericalangle ACB$ unverändert bleiben, und fügen an die Seite AB das oben betrachtete gleichschenklige Dreieck ABD hinzu. Bei dieser Operation ändert sich auch die Länge der Seite AB, und es kann vorkommen, dass, während der Inhalt des einen Dreiecks zunimmt, der Inhalt des anderen abnimmt. Wir wollen zeigen, dass die Inhaltssumme S der Dreiecke ihr Maximum im Falle $AC=CB$ erreicht.

Im Beweis wird die Funktion $g(x)$ eine Rolle spielen, die wir in $(0, \pi)$ durch die Formel

$$g(x)=\operatorname{arc\,tg}(h\operatorname{tg}x), \quad h=\cos r$$

und durch folgende Vereinbarung definieren: 1. für $x<\pi/2$ ist der Hauptwert der arc tg-Funktion zu nehmen, 2. es sei $g(\pi/2)=\pi/2$ und 3. für $x>\pi/2$ ist der zwischen $\pi/2$ und $3\pi/2$ liegende Wert der arc tg-Funktion zu nehmen. Wir haben dann

$$g(x)+g(\pi-x) = \pi$$

und

$$g'(x) = \frac{h}{1-k\sin^2 x}, \quad k = \sin^2 r.$$

Es sei 0 der Umkreismittelpunkt von ABC. Wir drehen die Halbgerade OA um 0, so dass sie zuerst mit OB, dann mit OC koinzidiert, und nach einer Drehung um 2π in ihre ursprüngliche Lage OA zurückkehrt. Bezeichnen wir die entsprechenden Drehwinkel mit 2γ, 2α, 2β, so haben wir

$$\alpha+\beta+\gamma = \pi.$$

Ferner sei φ der mit einem positiven oder negativen Vorzeichen versehene Winkel $\sphericalangle OCB$, je nachdem ob $\alpha<\pi/2$ oder $\alpha>\pi/2$ ist. Dann gilt

$$\cos r = \operatorname{ctg}\alpha \operatorname{ctg}\varphi,$$

woraus sich

$$\frac{\pi}{2}-\varphi = g(\alpha)$$

ergibt. Ist $\alpha<\pi/2$, so ist der Inhalt u des Dreiecks OBC

$$u = 2\alpha+2\varphi-\pi = 2\alpha-2g(\alpha).$$

Ist dagegen $\alpha>\pi/2$, so haben wir

$$u = 2\pi-2\alpha-2\varphi-\pi = -2\alpha-2g(\alpha).$$

Deshalb ergibt die Funktion

$$t(\alpha) = \alpha-g(\alpha)$$

den halben Inhalt von OBC mit einem positiven oder negativen Vorzeichen, je nachdem ob $\alpha<\pi/2$ oder $\alpha>\pi/2$ ist. Folglich haben wir

$$\frac{1}{2}S = t(\alpha)+t(\beta)+2t\left(\frac{\pi-\alpha-\beta}{2}\right) = \pi-g(\alpha)-g(\beta)-2g\left(\frac{\pi-\alpha-\beta}{2}\right),$$

bzw.

$$\frac{1}{2}S = t(\alpha)+t(\beta)+t(\gamma)+2t\left(\frac{\alpha+\beta}{2}\right)+t(\gamma) =$$

$$= 2\pi-g(\alpha)-g(\beta)-2g(\gamma)-2g\left(\frac{\alpha+\beta}{2}\right) = -g(\alpha)-g(\beta)+2g(\alpha+\beta)-2g\left(\frac{\alpha+\beta}{2}\right),$$

je nachdem ob $\alpha+\beta \leqq \pi/2$ oder $\alpha+\beta > \pi/2$ ausfällt (Abb. 5).

Da andererseits

$$\pi-g(\alpha)-g(\beta) = \omega \tag{1}$$

ist, müssen wir das Maximum der Funktion

$$\frac{1}{4}S = F(\alpha,\beta) = \begin{cases} \dfrac{\omega}{2}-g\left(\dfrac{\pi-\alpha-\beta}{2}\right), & 0<\alpha+\beta\leqq\pi/2 \\ \dfrac{\omega-\pi}{2}+g(\alpha+\beta)-g\left(\dfrac{\alpha+\beta}{2}\right), & \pi/2<\alpha+\beta<\pi \end{cases}$$

unter der Bedingung (1) bestimmen.

Wir fassen $\beta=\beta(\alpha)$ als Funktion von α auf und zeigen, dass für $\alpha<\beta$

$$\frac{d}{d\alpha}F(\alpha,\beta(\alpha)) > 0$$

ausfällt.

Aus (1) ergibt sich

$$g'(\alpha)+\frac{d\beta}{d\alpha}g'(\beta) = 0.$$

Deshalb gilt wegen $F_\alpha=F_\beta$

$$\frac{d}{d\alpha}F(\alpha,\beta(\alpha)) = F_\alpha+\frac{d\beta}{d\alpha}F_\beta = F_\alpha\left(1-\frac{g'(\alpha)}{g'(\beta)}\right).$$

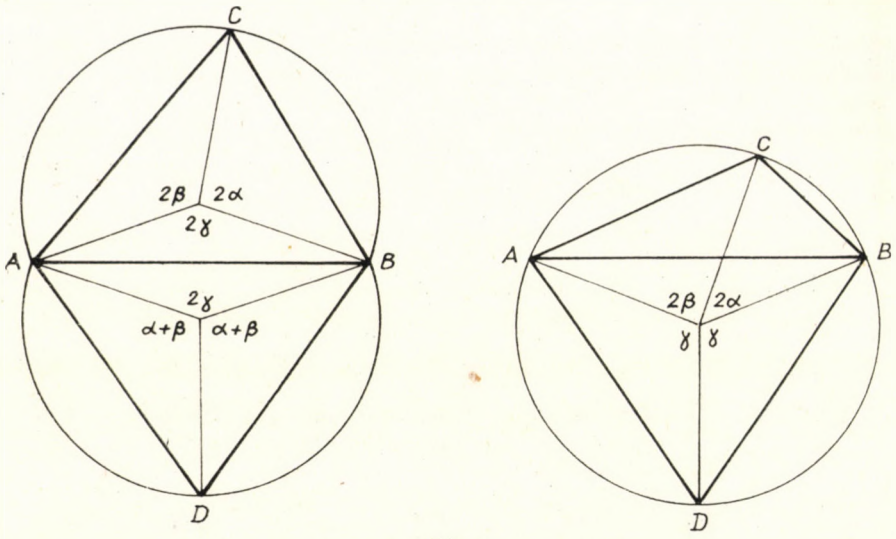


Abb. 5

Da $g'(x)$ in $(0, \pi/2)$ eine positive zunehmende Funktion und $g'(x)=g'(\pi-x)$ ist, gilt wegen $\alpha+\beta<\pi$ und der Voraussetzung $\alpha<\beta$

$$1-\frac{g'(\alpha)}{g'(\beta)}>0.$$

Wir müssen deshalb noch zeigen, dass auch F_α positiv ist.

Für $\alpha+\beta\leqq\pi/2$ ist dies wegen $F_\alpha=\frac{1}{2}g'\left(\frac{\gamma}{2}\right)$ trivial. Für $\alpha+\beta>\pi/2$ ist dagegen die zu beweisende Ungleichung

$$F_\alpha=g'(2x)-\frac{1}{2}g'(x)=\frac{h}{1-k\sin^2 2x}-$$

$$\frac{1}{2}\frac{h}{1-k\sin^2 x}>0, \quad \frac{\alpha+\beta}{2}=x$$

äquivalent mit

$$1+2k\sin^2 x\cos 2x>0,$$

was wegen $k=\sin^2 r<\sin^2\pi/4=1/2$ tatsächlich richtig ist.

§ 5. Beweis des Hilfssatzes 1. Wir beginnen mit dem Fall, dass $k>5$ ist.

Wir betrachten einen fest vorgegebenen Wert $r<\pi/4$ sowie ein konvexes Viereck $ACBD$ mit folgenden Eigenschaften: 1. $AC=CB$, 2. $AD=DB$, 3. ABC ist ein r-Dreieck, 4. der Umkreisradius des Dreiecks ABD ist gleich r und 5. ABD enthält seinen Umkreismittelpunkt. Wir fassen den Inhalt V des Vierecks als Funktion der Winkel $x=\frac{1}{2}\sphericalangle ACB$ und $y=\frac{1}{2}\sphericalangle ADB$ auf: $V=V(x,y)$. Offensichtlich gilt $x\geqq y$.

Es sei $2\bar{x}$, wie früher, ein Winkel eines gleichseitigen Dreiecks vom Umkreisradius r. Wir behaupten, dass $V(x,y)$ bei einem festen Wert von x für $0<y\leqq\bar{x}$

eine zunehmende Funktion von y ist. Ist nämlich $y<\bar{x}$ und lassen wir y zunehmen, so nimmt sowohl der Inhalt von ABD, wie die Länge der Seite AB zu. Nimmt aber die Länge von AB zu, so nimmt wegen der Voraussetzung $r<\pi/4$ auch die gemeinsame Länge der Seiten AC und CB zu. Hieraus folgt, dass auch der Inhalt von ABC zunimmt.

Damit haben wir zugleich bewiesen, dass für $x\leqq\bar{x}$

$$V(x, y)\leqq V(x, x)=2T(x)$$

gilt.

Wir definieren die Funktion $H(x)$ durch

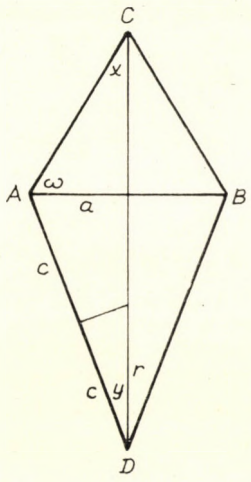


Abb. 6

$$H(x)=\begin{cases}2T(x), & 0<x\leqq\bar{x}\\ 2T(\bar{x}), & \bar{x}<x<\pi/2.\end{cases}$$

Da $T(x)$ in $(0, \bar{x})$ eine konkave, zunehmende Funktion ist, ist $H(x)$ im ganzen Intervall $(0, \pi/2)$ konkav. Da ferner für $\bar{x}<x<\pi/2$

$$V(x, y)<2\Delta=2T(\bar{x})=H(x)$$

gilt, haben wir überall

$$V(x, y)\leqq H(x).$$

Es sei nun W die Inhaltssumme der im Hilfssatz 1 betrachteten k Dreieckspaare. Ist $r<\pi/4$, so ist der Inhalt eines Dreieckspaares nach Hilfssatz 6 und den obigen Überlegungen $\leqq H(x_i)$, wo $2x_i$ der bei E liegende Winkel des betreffenden Dreiecks ist $(i=1, \ldots, k)$. Mit Rücksicht auf die Konkavität von $H(x)$ gilt also

$$W\leqq H(x_1)+\ldots+H(x_k)\leqq kH(\pi/k).$$

Ist aber $k>5$, so haben wir, wegen $\bar{x}>\pi/6$, $H(\pi/k)=2T(\pi/k)$ und folglich

$$W\leqq 2kT(\pi/k).$$

Gleichheit gilt dabei offensichtlich nur in dem im Hilfssatz 1 angedeuteten Fall.

Wir wenden uns jetzt dem erheblich schwierigeren Fall zu, dass $r=r_1$ und $k=4$ oder 5 ist.

Wir zeigen zunächst, dass für $r=r_1$ und $0<x\leqq\gamma=47°$ $V(x, y)$ eine zunehmende Funktion von y ist. Da dies für $0<y\leqq\bar{x}=\bar{x}_1$ bereits gezeigt wurde, können wir uns auf den Fall $\bar{x}_1<y\leqq x\leqq\gamma$ beschränken.

Führen wir die Bezeichnungen $AD=2c$, $AB=2a$, $\sphericalangle BAC=\omega$, $\operatorname{tg}^2 r_1=p$ ein (Abb. 6), so erhalten wir im Hinblick auf die Relationen $\operatorname{tg} c=\cos y \operatorname{tg} r$, $\sin a=\sin 2c \sin y$, $\cos x=\sin\omega\cos a$ durch etliche Rechnungen den Inhalt des Dreiecks ABC:

$$U(x, y)=2x+2\omega-\pi=2x-\pi+2\arcsin\frac{\cos x(1+p\cos^2 y)}{\sqrt{(1+p\cos^2 y)^2-p\sin^2 2y}}.$$

Es gilt also

$$V(x, y)=T(y)+U(x, y).$$

Wir wollen zeigen, dass $V_y(x, y)$ für $\bar{x}_1\leqq y\leqq x\leqq\gamma$ positiv ist. Bemerken wir

hierzu, dass $V_y(x, y) = T'(y) + U_y(x, y)$ eine abnehmende Funktion von x ist, wie dies aus der Darstellung

$$\frac{1}{4} U_y(x, y) = \frac{p \cos x \sin 2y (\cos 2y + p \cos^2 y)}{\sqrt{\sin^2 x (1 + p \cos^2 y)^2 - p \sin^2 2y}\,[(1 + p \cos^2 y)^2 - p \sin^2 2y]}$$

klar ersichtlich ist. Deshalb haben wir nur die Positivität der Funktion

$$z(y) = T'(y) + U_y(\gamma, y), \quad \bar{x}_1 \leqq y \leqq \gamma$$

nachzuweisen. Führen wir die Bezeichnungen $\alpha = 37°30'$, $\beta = 45°$ ein, so haben wir für $\alpha \leqq y \leqq \beta$

$$z(y) \geqq T'(\beta) + \frac{4p \cos \beta \sin 2\alpha (\cos 2\beta + p \cos^2 \beta)}{\sqrt{\sin^2 \beta (1 + p \cos^2 \alpha)^2 - p \sin^2 2\alpha}\,[(1 + p \cos^2 \alpha)^2 - p \sin^2 2\alpha]} \approx 0{,}0314$$

und für $\beta \leqq y \leqq \gamma$

$$z(y) \geqq T'(\gamma) + \frac{4p \cos \gamma \sin 2\beta (\cos 2\gamma + p \cos^2 \gamma)}{\sqrt{\sin^2 \gamma (1 + p \cos^2 \beta)^2 - p \sin^2 2\gamma}\,[(1 + p \cos^2 \beta)^2 - p \sin^2 2\gamma]} \approx 0{,}1593,$$

womit die Positivität von $z(y)$ und damit die Ungleichung

$$V(x, y) \leqq V(x, x) = 2T(x), \quad 0 < y \leqq x \leqq \gamma$$

dargetan ist.

Lassen wir x von dem Wert γ ausgehend zunehmen, so hört die Gültigkeit der Ungleichung $V(x, y) \leqq V(x, x)$ rasch auf. Etwa für $x = 48°$ gilt sie sicher nicht mehr. Beispielweise sei erwähnt, dass $V(x, y)$ für $x = 50°$ sein Maximum für $y \approx 47°$ erreicht. Eine explizite Darstellung der Funktion $\max_{0 < y \leqq x} V(x, y)$ scheint für $x > \gamma$ recht kompliziert zu sein. Statt dessen werden wir diese Funktion durch eine, unseren Zwecken entsprechende Funktion $w(x)$ von oben abschätzen, die wir in $(0, \pi/4)$ durch

$$w(x) = V(x, x)$$

und in $(\pi/4, \pi/2)$ durch

$$w(x) = 2\pi - 3w\left(\frac{\pi - x}{3}\right).$$

definieren. Es wird genügen, die Ungleichung $V(x, y) < w(x)$ für $\pi/4 < x \leqq \pi/3$ zu beweisen.

Im Falle $\pi/4 < x < \gamma$ haben wir uns nur auf die oben bewiesene Ungleichung $V(x, y) \leqq V(x, x)$ und auf die Tatsachen zu berufen, dass in $(\pi/4, \gamma)$ $V(x, x)$ konkav, $w(x)$ dagegen konvex ist, und dass diese Funktionen an der Stelle $x = \pi/4$ eine gemeinsame Derivierte haben. Deshalb müssen wir die Positivität der Differenz

$$D(x, y) = w(x) - V(x, y) =$$

$$= w(x) - T(y) - 2x + \pi - 2 \arcsin \frac{\cos x (1 + p \cos^2 y)}{\sqrt{(1 + p \cos^2 y)^2 - 4p \sin^2 y \cos^2 y}}$$

nur in dem durch die Ungleichungen

$$\gamma < x < \pi/3, \quad \bar{x}_1 \leqq y \leqq x$$

definierten Trapez nachweisen.

Wir zerlegen die (x, y)-Ebene durch die Geraden $x = 3'i$, $y = 3'j$ $(i, j = 0, \pm 1,$

$\pm 2, \ldots$) in Quadrate, und betrachten diejenigen Quadrate, die mit dem Trapez gemeinsame innere Punkte haben. Ist etwa

$$a \leqq x \leqq b, \quad c \leqq y \leqq d$$

ein solches Quadrat, so gilt hier

$$D(x, y) > w(b) - T(c) - 2b + \pi - 2 \arcsin \frac{\cos a(1 + p\cos^2 c)}{\sqrt{(1 + p\cos^2 d)^2 - 4p \sin^2 d \cos^2 c}}.$$

Diese unteren Schranken wurden für sämtliche Quadrate mit einem ICT-Computer (Zentrales Physikalisches Institut der Ungarischen Akademie der Wissenschaften) berechnet. Es stellte sich heraus, dass die Argumente der arc sin-Funktion alle <1 ausfielen, womit die benützte Abschätzung nachträglich gerechtfertigt wurde. Im Quadrat $47° \leqq x, y \leqq 47° 3'$ ergab sich für diese Schranke der Wert $0{,}0015 \pm \frac{1}{2} \cdot 10^{-4}$ und in den übrigen Quadraten ein grösserer Wert.

Auf Grund dieser Rechnungen und der obigen Überlegungen haben wir also im ganzen Dreieck $0 < y \leqq x \leqq \pi/3$

$$V(x, y) \leqq w(x).$$

Nach dieser Vorbereitung wird der Beweis der Ungleichung

$$W \leqq 2kT(\pi/k), \quad k > 3$$

ziemlich einfach sein. Wir beginnen mit dem Fall $k = 4$.

Ist ein Dreieckswinkel bei E grösser als oder gleich $2\pi/3$, so ist

$$W \leqq T(\pi/3) + 7\Delta_1 < 358° < 8T(\pi/4) = 2\pi.$$

Wir können also voraussetzen, dass $x_i < \pi/3$ gilt ($i = 1, \ldots, 4$). Dann haben wir

$$W \leqq w(x_1) + \ldots + w(x_4).$$

Es sei $x_1 \leqq \ldots \leqq x_4$. Ist $x_3 > \pi/4$, so lassen wir x_3 ab- und x_4 um ebensoviel zunehmen, bis x_3 den Wert $\pi/4$ erreicht. Da $w(x)$ in $(\pi/4, \pi/2)$ konvex ist, nimmt bei dieser Operation $w(x_3) + w(x_4)$ zu. Ist auch $x_2 > \pi/4$, so wiederholen wir die obige Operation mit x_2 und x_4. Da ursprünglich $x_4 < \pi/3$ war, gilt in der neuen Lage $x_4 < \pi/2$. Jedenfalls können wir also voraussetzen, dass von den Werten x_i drei im abgeschlossenen Intervall $(0, \pi/4)$ liegen, während einer, etwa x_4, in das abgeschlossene Intervall $(\pi/4, \pi/2)$ fällt. Da aber $w(x)$ in $(0, \pi/4)$ konkav ist, haben wir

$$W \leqq 3w\left(\frac{x_1 + x_2 + x_3}{3}\right) + w(x_4) = 3w\left(\frac{\pi - x_4}{3}\right) + w(x_4) = 2\pi.$$

Gleichheit gilt nur, wenn $x_1 = \ldots = x_4 = \pi/4$ ausfällt. Auch die übrigen geometrischen Bedingungen für den Fall der Gleichheit sind klar.

Ist $r = r_1$ und $k = 5$, so können wir wegen $\pi/5 < \bar{x}_1$ die im Falle $k > 5$ benützte Methode anwenden, indem wir $V(x, y)$ mit Hilfe derjenigen Funktion von oben abschätzen, die sich aus der oben definierten Funktion $H(x)$ im Spezialfall $r = r_1$, $\bar{x} = \bar{x}_1$ ergibt.

§ 6. Zulässige Ecken. In diesem Paragraphen zeigen wir, dass ein r_1-Mosaik mit 10 Ecken nur 4- und 5-kantige Ecken und ein r_2-Mosaik mit 14 Ecken nur 5- und 6-kantige Ecken haben kann.

Die 3-kantigen Ecken lassen sich mit einer von SCHÜTTE [12] herrührenden

Methode leicht ausschliessen. Stossen in einer Ecke eines r-Mosaiks genau drei Dreiecke zusammen, so bilden diese ein neues Dreieck, dessen Seiten offenbar kleiner sind als $2r$. Deshalb ist der Inhalt dieses Dreiecks kleiner als der Inhalt eines gleichseitigen Dreiecks der Seitenlänge $2r$, dh. als

$$Q(r) = 6 \arcsin \frac{1}{2\cos r} - \pi.$$

Nun hat aber ein Dreiecksmosaik mit n Ecken $2n$-4 Flächen. Hätte deshalb ein r_1-Mosaik mit 10 Ecken eine 3-kantige Ecke, so wäre die Inhaltssumme der Dreiecke höchstens

$$Q(r_1)+13\Delta_1 < 700^\circ < 720^\circ,$$

was unmöglich ist.

In ähnlicher Weise folgt aus

$$Q(r_2)+21\Delta_2 < 696^\circ < 720^\circ,$$

dass in einem r_2-Mosaik mit 14 Ecken keine 3-kantige Ecke auftreten kann.

Um uns darauf später berufen zu können, formulieren wir diese Ergebnisse ein wenig allgemeiner. Wir betrachten in einem Dreiecksmosaik ein *Kantenpolygon*, d. h. ein einfach zusammenhängendes Polygon, dessen sämtliche Seiten Mosaikkanten sind, das aber selbst keine Mosaikfläche ist. Dann gilt die

BEMERKUNG 1. In einem r_i-Mosaik mit n_i Ecken ($i=1, 2$) kann kein Kantendreieck vorkommen.

Ein Kantendreieck würde nämlich wenigstens drei Flächen enthalten, weshalb der Gesamtinhalt der übrigen Flächen zur Bedeckung des komplementären Kantendreiecks nicht ausreichen würde.

In einem r-Mosaik mit n Ecken sei E eine k-kantige Ecke und E' der zu E antipodische Punkt. Die um E und um die mit E benachbarten Ecken mit dem Radius r geschlagenen Kreise können nicht in den um E' mit dem Radius $\pi-3r$ geschlagenen Kreis hineingreifen. Deshalb müssen die um die übrigen $n-k-1$ Ecken mit dem Radius r geschlagenen Kreise diesen Kreis überdecken.

Wäre also in einem r_1-Mosaik mit 10 Ecken eine 6- oder mehrkantige Ecke vorhanden, so müssten 3 oder weniger Kreise einen Kreis vom Radius $\pi-3r_1 > 53^\circ$ überdecken. Der Radius eines Kreises, den drei Kreise vom Radius r überdecken können, kann aber nicht den Umkreisradius eines gleichseitigen Dreiecks der Seitenlänge $2r$, d. h.

$$R(r) = \arcsin \frac{2}{\sqrt{3}} \sin r$$

übertreffen. Da aber $R(r_1)<51^\circ<53^\circ$ ist, hat unser Mosaik höchstens 5-kantige Ecken.

Wäre in einem r_2-Mosaik mit 14 Ecken eine 7-kantige Ecke vorhanden, so wäre die Inhaltssumme der 24 Dreiecke nach Hilfssatz 1 höchstens

$$14T(\pi/7)+10\Delta_2 < 700^\circ < 720^\circ,$$

was unmöglich ist. In ähnlicher Weise lässt sich der Fall von mehr als 7-kantigen Ecken ausschliessen.

Schwieriger ist die Ausschliessung 4-kantiger Ecken.

Wir setzen zunächst voraus, dass in einem r_2-Mosaik mit 14 Ecken eine ein-

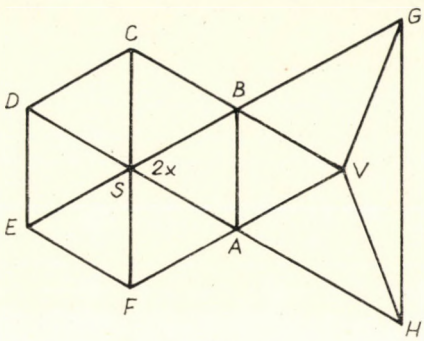


Abb. 7

zige 4-kantige Ecke V auftritt. Wir werden im § 7 zeigen, dass es dann eine 6-kantige Ecke S geben muss, die kein Nachbar von V ist. Die in S bzw. V zusammenkommenden Dreiecke bilden ein Sechseck s und ein Viereck v. Wir setzen voraus, dass s und v eine gemeinsame Seite AB haben.

Wir behaupten, dass $s = ABCDEF$ und $v = ABGH$ ausser AB keine weitere gemeinsame Seite haben (Abb. 7). Die Seiten AF und AH, oder die Seiten BC und BG können nämlich nicht zusammenfallen, da sonst A bzw. B eine zweite 4-kantige Ecke wäre. Wäre aber etwa G mit D identisch, so würden die Kanten GB, BS, SD zwei Kantendreiecke begrenzen, was nach Bemerkung 1 ausgeschlossen ist.

Hieraus folgt, dass keines der fünf Dreiecke, die sich an s entlang den von AB verschiedenen Seiten anschliessen, mit einem in v liegenden Dreieck identisch ist. Setzen wir also $2x = \sphericalangle ASB$, so ist die Inhaltssumme der Mosaikflächen nach Hilfssatz 5 und den beim Beweis von Hilfssatz 1 benützten Überlegungen höchstens

$$S(x) = \frac{1}{2} H(x) + 5H\left(\frac{\pi - x}{5}\right) + 4T\left(\frac{\pi}{4}\right) + 9\Delta_2 .$$

Da aber $H(x)$ konkav ist, gilt dasselbe für $S(x)$. Nun zeigen die Funktionswerte

$$S(24^\circ) \approx 718^\circ\, 43'\, 44'', \quad S(26^\circ) \approx 718^\circ\, 52'\, 30'', \quad S(28^\circ) \approx 718^\circ\, 41'\, 22'',$$

dass

$$0 < S(26^\circ) - S(24^\circ) < S(26^\circ) - S(28^\circ) < 12'$$

ist. Hieraus folgt, dass $S(x)$ sein Maximum im Intervall $(24^\circ, 28^\circ)$ erreicht und dass hier, also zugleich im ganzen Intervall $(0^\circ, 90^\circ)$,

$$S(x) < S(26^\circ) + S(26^\circ) - S(28^\circ) < 719^\circ\, 5' < 720^\circ$$

gilt, womit wir zu einem Widerspruch gekommen sind.

Diese Abschätzung gilt auch dann, wenn s und v keine gemeinsame Seite haben. Wir haben also nur noch den Fall zu untersuchen, dass wenigstens zwei 4-kantige Ecken V_1 und V_2 auftreten.

Sind V_1 und V_2 keine Nachbarn, so ist die Inhaltssumme der Dreiecke nach Hilfssatz 4 höchstens

$$8T(\pi/4) + 16\Delta_2 < 715^\circ < 720^\circ.$$

Sind dagegen V_1 und V_2 Nachbarn, so bilden die um die Kante V_1 und V_2 liegenden sechs Dreiecke ein Viereck, dessen Inhalt kleiner ist als der Inhalt eines regelmässigen Vierecks mit der Seitenlänge $2r$. Deshalb ist der totale Inhalt der Mosaik-

flächen kleiner als

$$8 \arcsin \frac{\cos \pi/4}{\cos r_2} + 18\Delta_2 < 673^\circ < 720^\circ,$$

womit auch die letzten beiden Möglichkeiten ausgeschlossen sind.

§ 7. Zulässige Mosaiktypen. Wir beginnen mit einer Bemerkung, die zusammen mit der Bemerkung 1 im § 6 die Untersuchung der möglichen Mosaiktypen erleichtern wird.

BEMERKUNG 2. In einem Dreiecksmosaik sei K ein Kantenpolygon, aus dessen Ecken entweder genau eine Kante oder genau zwei Kanten in K hineingehen. Dann ist keine Kante eine Diagonale von K.

Sonst gäbe es eine Kante AB, die K so in zwei Teilpolygone K_1 und K_2 zerlegt, dass etwa K_1 keine Kantendiagonale mehr enthält. Wir betrachten die in K_1 und K_2 liegenden Dreiecke ABC_1 und ABC_2, die AB als gemeinsame Seite haben. Da nach Voraussetzung von jeder von A und B verschiedenen Ecke von K_i wenigstens eine Kante in K_i hineinweist, kann ABC_i nicht mit K_i identisch sein ($i=1, 2$). Da ferner weder AC_1 noch BC_1 eine Diagonale von K_1 ist, ist weder AC_1 noch BC_1 eine Seite von K_1. Was nun die Seiten AC_2 und BC_2 betrifft, können wir offensichtlich behaupten, dass wenigstens eine von ihnen, etwa AC_2, keine Seite von K_2 ist. Deshalb gehen von A drei Kanten, nämlich AC_1, AB und AC_2 nach K aus, was unserer Voraussetzung widerspricht (Abb. 8).

Im § 6 haben wir bei der Ausschliessung einer 4-kantigen Ecke in einem r_2-Mosaik mit 14 Ecken einen Fall nicht untersucht: Es gibt eine einzige 4-kantige Ecke, keine 3- oder mehr als 6-kantige Ecke und keine 6-kantige Ecke, die mit der 4-kantigen nicht benachbart ist. Wir zeigen vor allem, dass dieser Fall nicht zustande kommen kann.

In einem Dreiecksmosaik mit n Ecken sei A_i die Anzahl der i-kantigen Ecken. Dann haben wir

$$A_3 + A_4 + \ldots = n.$$

Da andererseits die Flächenzahl $2n-4$ und deshalb die zweifache Kantenzahl $3(2n-4)$ beträgt, gilt

$$3A_3 + 4A_4 + \ldots = 6n - 12.$$

Ist $n=14$, $A_3 = A_7 = A_8 = \ldots = 0$ und $A_4 = 1$, so haben wir

$$A_5 + A_6 = 13$$

$$5A_5 + 6A_6 = 68,$$

und folglich $A_5 = 10$ und $A_6 = 3$.

Es sei A die einzige 4-kantige Ecke und $BCDE$ das Viereck, das aus den in A zusammentreffenden Dreiecken besteht. Wir setzen voraus, dass B, C

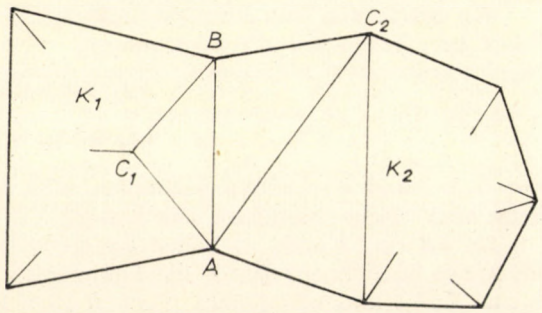


Abb. 8

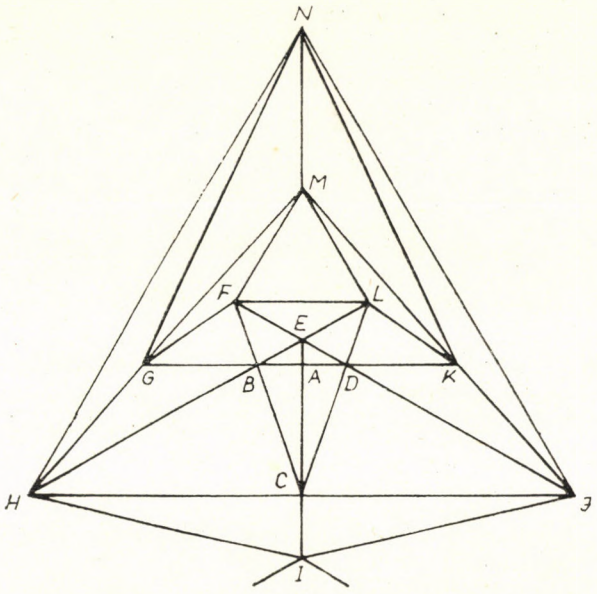


Abb. 9

und D 6-kantige Ecken sind und zeigen, dass es unmöglich ist, das Mosaik unter der Bedingung zu konstuieren, dass die übrigen zehn Ecken alle 5-kantig sind.

Es seien $AEFGHC$, $ABHIJD$ und $ACJKLE$ die Sechsecke, die aus den in B, C bzw. D zusammenkommenden Dreiecken bestehen (Abb. 9). Wir behaupten, dass die 12 Punkte $A, \ldots, L$ alle voneinander verschieden sind. Da nämlich E eine 5-kantige Ecke ist, sind die Kanten EF und EL verschieden, und EFL ein neues Dreieck. Wäre nun etwa $K \equiv G$, so würde der Streckenzug KLF zwei Kantendreiecke begrenzen, was nach Bemerkung 1 unmöglich ist. Deshalb sind die Punkte F, G, H, I, J, K, L Ecken eines Kantensiebenecks S, wobei immer dasjenige Kantenpolygon gemeint wird, in das noch keine Dreiecke eingezeichnet sind.

In S gehen von G, I und K je zwei Kanten, und von F, H, J und L je eine Kante hinein. Deshalb führt die etwa von F ausgehende Kante nach Bemerkung 2 zu einer neuen Ecke M, wodurch die neuen Dreiecke MFL, MFG und MLK entstehen. Wir können jetzt diese Konstruktion mit dem Kantensechseck $S' = MGHIJK$ fortsetzen. Die von M in S' hineinweisende Kante führt zu einer neuen Ecke N. Dann haben wir zugleich die neuen Dreiecke NMG, NMK, NGH, NKJ und das Kantenviereck $S'' = NHIJ$ mit der unmöglichen Eigenschaft, dass in S'' von N, H und J keine Kante, von I aber zwei Kanten hineingehen.

Damit ist der Beweis der Tatsache, dass in einem r_2-Mosaik mit 14 Ecken keine 4-kantige Ecke sein kann, beendet.

Wir betrachten jetzt ein r_1-Mosaik mit 10 Ecken. Da es nur 4- und 5-kantige Ecken hat, haben wir die Gleichungen

$$A_4 + A_5 = 10$$

$$4A_4 + 5A_5 = 48,$$

also $A_4 = 2$ und $A_5 = 8$. Wir behaupten, dass die 4-kantigen Ecken A und B weder erste, noch zweite Nachbarn sein können.

Ist AB eine Kante, so bilden die sechs Dreiecke um AB ein Viereck $CDEF$, das wir so bezeichnen, dass in das komplementäre Kantenviereck von F und D aus je eine Kante und von C und E aus je zwei Kanten hineinweisen (Abb. 10). Wir können dann zu diesen Dreiecken nach Bemerkung 2 sukzessiv neue Kanten und

Dreiecke adjungieren, etwa in folgender Reihenfolge: *FG*, *GFC*, *GFE*, *DH*, *HDC*, *HDE*, *GEH*, *GCH*. Da aber jetzt die Dreiecke die Sphäre lückenlos bedecken, endet die Konstruktion mit einem Mosaik mit 8 Ecken.

Ist *AB* keine Kante und *C* eine Ecke, die sowohl mit *A* wie mit *B* benachbart ist, so gibt es eine Kante *CD*, so dass auch die Ecke *D* mit *A* und *B* benachbart ist (Abb. 11). Die in *A* bzw. *B* zusammenkommenden Dreiecke bilden zwei Vierecke *CDEF* und *CDHG*. Dann ist aber *EFGH* ein Kantenviereck, aus dessen jeder Ecke genau eine Kante in das Viereck eintritt. Deshalb treffen sich diese Kanten in einem Punkt, womit wir zu einem Mosaik mit 9 Ecken gekommen sind.

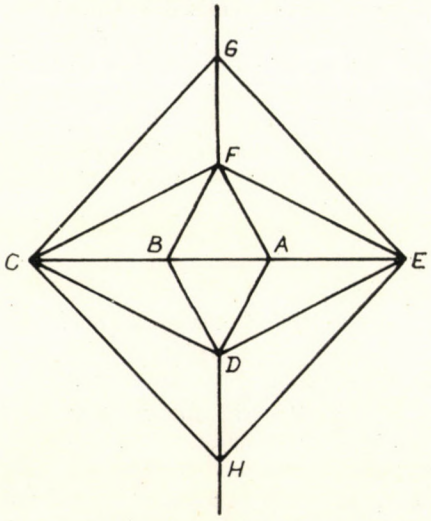


Abb. 10

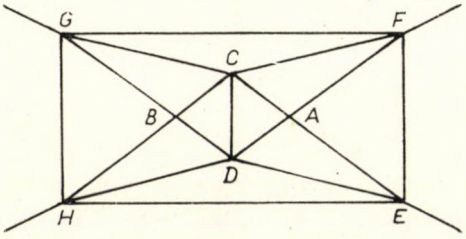


Abb. 11

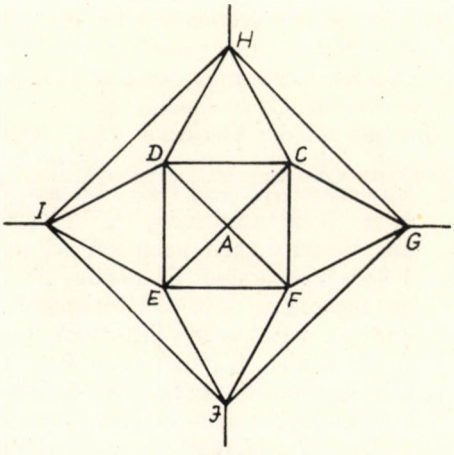


Abb. 12

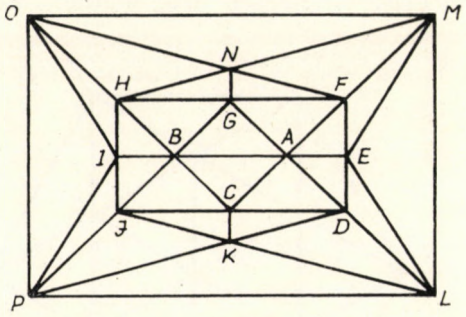


Abb. 13

Es sei jetzt *CDEF* das von den in *A* zusammentreffenden Dreiecken gebildete Viereck. Wir haben nur noch den Fall zu untersuchen, dass *C*, *D*, *E*, *F* und die von *A* verschiedenen Nachbarn dieser Ecken alle 5-kantig sind (Abb. 12). Da die von *C* aus in das komplementäre Viereck weisenden Kanten in zwei verschiedenen

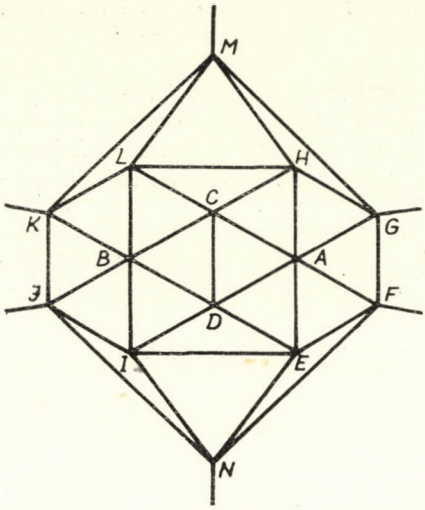


Abb. 14

Punkten G und H enden, haben wir die neuen Dreiecke FGC, GCH, CHD, zu denen dann sukzessiv die weiteren Dreiecke HDI, DIE, IEJ, EJF, JFG hinzukommen. Nun geht von jeder Ecke des Vierecks $GHIJ$ noch genau eine weitere Kante aus. Deshalb treffen sich diese Kanten im Punkt B, womit wir ein zu M_1 isomorphes Mosaik vor uns haben.

Die Untersuchung der topologischen Struktur eines r_2-Mosaiks mit 14 Ecken geschieht in ähnlicher Weise. Aus den Gleichungen

$$A_5 + A_6 = 14$$

$$5A_5 + 6A_6 = 72$$

folgt $A_5 = 12$ und $A_6 = 2$. Wir werden zeigen, dass die 6-kantigen Ecken A und B weder erste, noch zweite Nachbarn sind, wodurch sich das Mosaik automatisch und eindeutig konstruieren lässt.

Es sei AB eine Kante und $BCDEFG$ und $AGHIJC$ die beiden Sechsecke, die aus den in A bzw. B zusammentreffenden Dreiecken bestehen (Abb. 13). Wir adjungieren die neuen Kanten und Dreiecke CK, KCD, KCJ, DL, LDK, LDE, EM, MEL, MEF, GN, NGF, NFM, NGH, HO, OHN, OHI, ONM, IP, PIJ, PJK, PKL, PIO, wodurch ein Mosaik mit 16 Ecken und der Viereckfläche $LMOP$ entsteht.

Sind A und B zweite Nachbarn, so haben die beiden Sechsecke, die aus den in A bzw. B zusammenkommenden Dreiecken bestehen, eine gemeinsame Seite, so dass diese Sechsecke mit $CDEFGH$ und $CDIJKL$ bezeichnet werden können (Abb. 14). Um uns zu überzeugen, dass die verschieden bezeichneten Ecken tatsächlich verschieden sind, streichen wir vorübergehend die Dreiecke BLK, BKJ, BJI, zeichnen aber gleichzeitig die Dreiecke CHL und DEI, und gehen vom Kantensiebeneck $EFGHLBI$ aus, dessen Ecken nach Bemerkung 1 verschieden sind. Es genügt dann zu bemerken, dass die Endpunkte der von B aus in das Siebeneck weisende beiden Kanten nach Bemerkung 2 von den Ecken des Siebenecks und offensichtlich auch voneinander verschieden sind. Deshalb können wir die gestrichenen Dreiecke wieder aufnehmen, und vom Kantenachteck $EFGHLKJI$ ausgehend die neuen Dreiecke LHM, MLK, MHG, EIN, NEF, NIJ zeichnen. Es entsteht das Kantensechseck $FGMKJN$, von dessen jeder Ecke

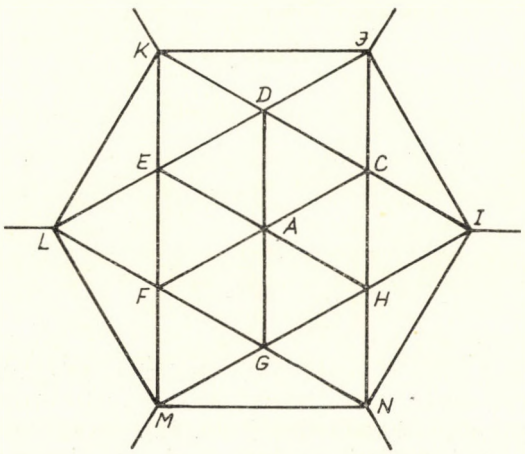


Abb. 15

noch genau eine weitere Kante ausgeht, die in einer neuen Ecke zusammenstossen. Unsere Konstruktion führt also zu einem Mosaik mit 15 Ecken.

Es sei nun $CDEFGH$ das Sechseck, das aus den in A zusammentreffenden Dreiecken besteht (Abb. 15). Sind die Ecken dieses Sechsecks alle 5-kantig, so liegen um C die weiteren Dreiecke HIC, ICJ, CJD, an die sich sukzessiv die Dreiecke JDK, DKE, KEL, ELF, LFM, FMG, MGN, GNH, NHI anschliessen. Sind auch die Ecken des neuen Kantensechsecks $IJKLMN$ alle 5-kantig, so sind sie noch mit der zweiten 6-kantigen Ecke B durch je eine Kante verbunden. Das erhaltene Mosaik ist mit M_2 isomorph.

Damit sind alle Behauptungen, auf die wir uns im § 3 ohne Beweis berufen haben, bestätigt, und der Beweis unseres Satzes dargetan.

§ 8. Bemerkungen. Eine verhältnismässig günstige Überdeckung mit 9 Kreisen entsteht, wenn die Kreismittelpunkte folgendermassen angeordnet sind. Es seien ABC, $A'B'C'$ und $A''B''C''$ drei gleichseitige Dreiecke, die dem Äquator und zwei

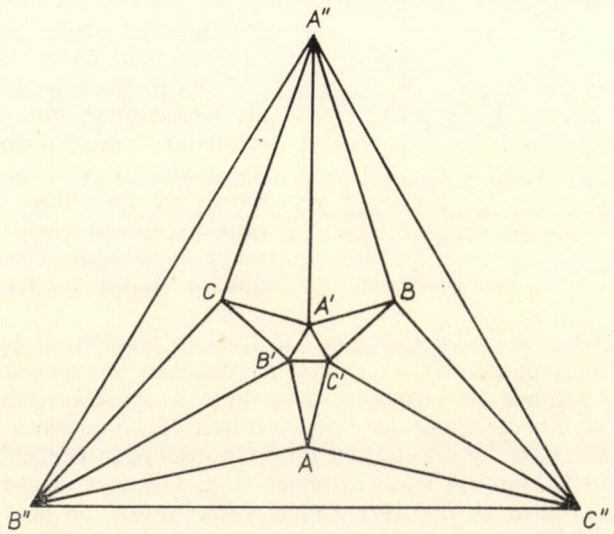


Abb. 16

kongruenten Breitenkreisen so einbeschrieben sind, dass $AC'BC''$, $BA'CA''$ und $CB'AB''$ kongruente Rhomben sein sollen (Abb. 16). Die Breitenkreise seien so gewählt, dass die Umkreisradien von $A'B'C'$ und $AB'B''$ einen gleichen Wert r haben. Leider ist der Umkreisradius von $AB'C'$ kleiner als r, so dass die um die Ecken der Dreiecke mit dem Radius r geschlagenen Kreise gewisse Flächenteile dreifach überdecken. Deshalb ist die Dichte dieser Überdeckung grösser als die Dichte der von Schütte konstruierten Überdeckung mit 8 Kreisen.

Wir richten jetzt unsere Aufmerksamkeit auf das Problem der dünnsten Überdeckung der elliptischen Ebene durch m kongruente Kreise, oder — was auf dasselbe hinauskommt — auf das Problem der dünnsten zentralsymmetrischen Überdeckung der Sphäre durch $2m$ Kreise. Das analoge Unterdeckungsproblem haben

für $m=4$ und 5 L. FEJES TÓTH [4] und für $m=7$ K. BÖRÖCZKY gelöst. Da die dichtesten Unterdeckungen der Sphäre durch 6 und 12 Kreise automatisch zentralsymmetrisch sind, ist damit das elliptische Unterdeckungsproblem für $m \leqq 7$ gelöst.

In der dünnsten zentralsymmetrischen Überdeckung der Sphäre durch 8 Kreise liegen die Kreismittelpunkte — wie A. HEPPES [7] gezeigt hat — in den Ecken einer regulären sechsseitigen Doppelpyramide.

Bei 10 Kreisen scheint es günstig zu sein, die Kreismittelpunkte in die Ecken einer „prismatischen Doppelpyramide" zu legen d.h. in die Ecken eines Mosaiks, das aus 8 kongruenten gleichschenkligen Dreiecken und 4 kongruenten gleichwinkligen Vierecken besteht, wobei die Umkreisradien der Dreiecke und Vierecke übereinstimmen.

Eine günstige Anordnung von 14 Kreisen ergibt sich, wenn man die Kreismittelpunkte in die Ecken und Flächenmittelpunkte eines oktaedrischen oder, was dasselbe ist, eines hexaedrischen Mosaiks legt. In bemerkenswerter Weise ergibt diese Punktanordnung die Lösung des entsprechenden Unterdeckungsproblems. Wir wollen diese Anordnung *kuboktaedrisch* nennen, weil hier die Punkte die Flächenmittelpunkte eines kuboktaedrischen Mosaiks sind (Abb. 17).

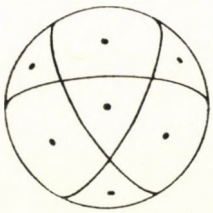


Abb. 17

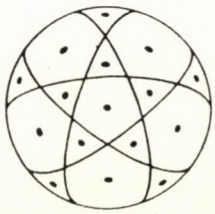


Abb. 18

Bei 16 Kreisen ist es zweckmässig, als Kreismittelpunkte die Ecken einer regulären antiprismatischen Doppelpyramide zu wählen. Diese Anordnung scheint nicht nur unter den zentralsymmetrischen, sondern auch unter beliebigen Überdeckungen der Sphäre durch 16 Kreise günstig zu sein.

Eine sehr sparsame Überdeckung mit 32 Kreisen entsteht, wenn man die Kreise um die Flächenmittelpunkte eines ikosidodekaedrischen Mosaiks schlägt (Abb. 18). Wir haben gute Gründe zu vermuten, dass diese *ikosidodekaedrische* Kreisanordnung die dünnste Überdeckung der Sphäre durch 32 kongruente Kreise darstellt.

Bemerken wir, dass die drei letzten Punktanordnungen bezüglich des Problems der befreundeten Diktatoren lokal extremal sind. Genauer ist jede dieser Punktverteilungen unter allen zu ihr isomorphen Verteilungen die bestmögliche, wobei unter *isomorphen* Punktverteilungen Anordnungen mit topologisch isomorphen Mosaiken verstanden werden.

Für die Punktanzahl 16 folgt dies unmittelbar aus Hilfssatz 1. Allgemeiner bilden die Ecken einer mehr als dreiseitigen regulären antiprismatischen Doppelpyramide eine beste Punktanordnung unter sämtlichen isomorphen Anordnungen. Da bei grosser Seitenzahl diese Anordnung sicher nicht die absolut beste ist, haben wir unendlich viele Beispiele lokal extremaler Punktsysteme vor uns, die nicht absolut extremal sind.

Weitere Beispiele ergeben sich aus folgender Bemerkung. Wir betrachten ein Mosaik, das aus kongruenten regelmässigen k-Ecken besteht. Ist $k>3$, so ist die Punktverteilung, die aus den Ecken und Flächenmittelpunkten des Mosaiks besteht, unter allen isomorphen Punktanordnungen die beste.

Gehen wir vom hexaedrischen und dodekaedrischen Mosaik aus, so erhalten wir die kuboktaedrische und ikosidodekaedrische Anordnung, von denen die erste

nur lokal, die zweite aber wahrscheinlich auch absolut extremal ist. Gehen wir dagegen vom diedrischen Mosaik aus, so erhalten wir die Ecken einer regulären Doppelpyramide, die für die Seitenzahlen 3, 4 und 5 eine absolut, sonst aber nur eine lokal beste Anordnung liefert.

Um die Richtigkeit unserer Bemerkung einzusehen, erwähnen wir, dass Hilfssatz 4 seine Gültigkeit unter viel allgemeineren Bedingungen behält. Da nämlich die Tangente, die die Kurve der Funktion $T(x)$ im Punkt $x=\pi/5$ berührt, für jeden Wert von $r<\pi/2$ im ganzen Intervall $(\pi/5, \pi/2)$ oberhalb der Kurve liegt, gilt der Hilfssatz auch dann, wenn r beliebig und die Anzahl k der Dreiecke >4 ist. Es ist nicht zu bezweifeln, dass Hilfssatz 4 auch für $k=4$ bei beliebigen Werten von $r<\pi/2$ gilt, doch scheint der Beweis noch einige weitere Überlegungen zu erfordern. Jedenfalls gilt der Hilfssatz bei $k=4$ für denjenigen Wert von $r\approx 36°12'$, der der kuboktaedrischen Kreisüberdeckung entspricht. Dies lässt sich in gleicher Weise zeigen wie im Falle $r=r_2$. Die Richtigkeit der obigen Bemerkung folgt nun, wenn wir in den Mosaiken der fraglichen Punktsysteme die Inhaltssumme der in den k-kantigen Ecken zusammentreffenden Dreiecke mit Hilfe dieses verallgemeinerten Hilfssatzes von oben abschätzen.

Unsere Tabelle enthält für einige Kreisanzahlen n die Näherungswerte des extremalen Radius- und Dichtenwertes r und $d=\frac{n}{2}(1-\cos r)$, bzw. Schranken derselben, die sich einerseits aus der Abschätzungsformel [2]

$$r \geqq \arccos \frac{1}{\sqrt{3}} \operatorname{ctg} \frac{n}{n-2} \frac{\pi}{6},$$

andererseits aus den betrachteten günstigen Überdeckungen ergeben. Ist bei geraden Werten von n die beste bzw. die konstruierte günstige Überdeckung nicht zentralsymmetrisch, so enthält die Tabelle weitere Werte, die sich auf das Problem der dünnsten zentralsymmetrischen Überdeckung beziehen.

n	r			d	
2	90°			1	
3	90°			1,5	
4	70° 32′			1,333	
5	61° 1′			1,382	
6	54° 44′			1,268	
7	51° 2′			1,299	
8	46° 31′	48° 9′	1,246		1,331
	49° 6′			1,381	
9	43° 36′	45° 53′	1,232		1,368
10	42° 18′			1,302	
		47° 4′			1,596
12	37° 23′			1,232	
14	34° 56′			1,261	
		36° 12′			1,351
16	32° 8′	33° 32′	1,226		1,332
32	22° 29′	22° 41′	1,221		1,238

Wir bemerken noch, dass sich die obige Abschätzung von r noch ein wenig verschärfen lässt. Es seien nämlich in einer Überdeckung der Sphäre durch n Kreise vom Radius r $D_1, \ldots, D_n$ die Flächeninhalte und $k_1, \ldots, k_n$ die Seitenzahlen der Dirichletschen Zellen. Dann gilt $D_i \leqq P(k_i)$, wo

$$P(k) = 2\pi - 2k \operatorname{arc\,tg}\left(\cos r \operatorname{tg} \frac{\pi}{k}\right)$$

den Inhalt eines regulären k-Ecks vom Umkreisradius r bedeutet. Da aber einerseits $P(3), P(4), \ldots$ eine zunehmende konkave Folge, andererseits $k_1 + \ldots + k_h \leqq \leqq 6n - 12$ ist, haben wir

$$4\pi = D_1 + \ldots + D_h \leqq P(k_1) + \ldots + P(k_h) \leqq n_1 P(p_1) + n_2 P(p_2),$$

wo n_1 und n_2 nicht negative und p_1 und p_2 positive ganze Zahlen sind, für die

$$n_1 + n_2 = n, \quad p_2 - p_1 = 1, \quad n_1 p_1 + n_2 p_2 = 6n - 12$$

ausfällt.

Für $n = 32$ haben wir

$$12P(5) + 20P(6) \geqq 4\pi,$$

woraus sich $r \geqq 22° 32'$ ergibt. Diese Schranke ist nur um weniger als 0,7% kleiner als der vermutete Minimalwert von r.

Schliesslich spreche ich meinen aufrichtigen Dank meinem Vater L. FEJES TÓTH aus, der meine Aufmerksamkeit auf diese Probleme gerichtet und meine Untersuchungen durch wertvolle Bemerkungen unterstützt hat.

LITERATURVERZEICHNIS

[1] FEJES TÓTH, L.: Über die Abschätzung des kürzesten Abstandes zweier Punkte eines auf einer Kugelfläche liegenden Punktsystems, *Jber. Deutsch. Math.-Ver.* **53** (1943) 66—68.
[2] FEJES TÓTH, L.: Egy gömbfelület befedése egybevágó gömbsüvegekkel, *Mat. Fiz. Lapok* **50** (1943) 40—46.
[3] FEJES TÓTH, L.: *Reguläre Figuren*, Budapest, 1965, S. 221.
[4] FEJES TÓTH, L.: Distribution of points in the elliptic plane, *Acta Math. Acad. Sci. Hungar.* **16** (1965) 437—440.
[5] GOLDBERG, M.: Axially symmetric packing of equal circles on a sphere, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **10** (1967) 37—48.
[6] GOLDBERG, M.: Viruses and a mathematical problem, *J. Mol. Biol.* **24** (1967) 337—338.
[7] HEPPES, A.: Az elliptikus sík legritkább fedése négy körrel, *Mat. Lapok* (im Druck).
[8] LANDAU, H. J. und SLEPIAN, D.: On the optimality of the regular simplex code, Bell Sistem Tech. J. 45 (1966) 1247 — 1372.
[9] MESCHKOWSKI, H.: *Ungelöste und unlösbare Probleme der Geometrie*, Braunschweig, 1960.
[10] ROBINSON, R. M.: Arrangement of 24 points on a sphere, *Math. Ann.* **144** (1961) 17—48.
[11] ŞERBAN M. und STROILĂ, C.: Un modele mathématique du noyau cellulaire, *Rev. Roumaine. Math. Pures Appl.* **11** (1966) 287—316.

[12] SCHÜTTE, K.: Überdeckung der Kugel mit höchstens acht Kreisen, *Math. Ann.* **129** (1955) 181—186.
[13] SCHÜTTE, K. und VAN DER WAERDEN, B. L.: Auf welcher Kugel haben 5, 6, 7, 8 oder 9 Punkte mit Mindestabstand Eins Platz? *Math. Ann.* **123** (1951) 96—124.
[14] TAMMES, P. M. L.: On the origin of number and arrangement of the places of exit on the surface of pollen grains, *Rec. Trav. Bot. Neerl.* **27** (1930) 1—84.
[15] VAN DER WAERDEN, B. L.: Pollenkörner, Punkverteilungen auf der Kugel und Informationstheorie, *Die Naturwissenschaften* **48** (1961) 189—192.

Eötvös L. University, Budapest

(Eingegangen: 22. März, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 249—255.

ON AN ITERATIVE METHOD WITH DIFFERENCE QUOTIENTS OF THE SECOND ORDER

by

M. BALÁZS and G. GOLDNER

In this paper an iterative method is given with difference quotients by analogy with ČEBYŠEV's method [1], for solving equation

$$P(x)=0, \tag{1}$$

where $P(x)$ is a continuous operator defined on a Banach space X, with values in a Banach space Y.

S. ULM [2] has given such a method, but his recursive relation for finding succesive approximations contains more terms than the corresponding formula in this paper. In [2], also the author supposes the existence of the solution of the equation (1), proving only the convergence of the method to the solution assumed as existing. In our paper we prove the existence and uniqueness of the solution.

In the paper we use some properties of the difference quotients used in [3] and [4], but we also need some other properties wich will be given in § 1.

§ 1. Some properties of the difference quotients

In this paragraph we adopt SCHRÖDER's definition [5] for difference quotients of the first order. The difference quotients of the higher orders can be succesively defined. We denote by

$$D(x_1, x_2;\ T(x)) \quad \text{respectively} \quad D(x_1, x_2, x_3;\ T(x))$$

the difference quotients of the first, respectively second order of operator $T(x)$.

LEMMA 1. *If $B[u, v]$ is a bilinear operator $(Y\times Y \xrightarrow{B} Z)$ and $u(x)$, $v(x)$ are operators $(X \xrightarrow{u} Y)$, $(X \xrightarrow{v} Y)$ with difference quotients of the first order in the points x_1, x_2, then*

$$D(x_1, x_2;\ B[u(x),\ v(x)])\Delta x = \\ = B[D(x_1, x_2;\ u(x))\Delta x, v(x_1)] + B[u(x_2), D(x_1, x_2;\ v(x))\Delta x]$$

PROOF.

$$D(x_1, x_2;\ B[u(x), v(x)])\,(x_1 - x_2) = \\ = B[u(x_1), v(x_1)] - B[u(x_2), v(x_1)] + B[u(x_2), v(x_1)] - B[u(x_2), v(x_2)] = \\ = B[u(x_1), v(x_1)] - B[u(x_2), v(x_2)].$$

We assume that

$$D(x_1, x_2, x_3;\ B[u(x), v(x)]) = D(x_1, x_3;\ D(x, x_2;\ B[u(x), v(x)])).$$

LEMMA 2. *If the conditions of Lemma* 1 *take place and* $u(x)$, $v(x)$ *have the difference quotients of the second order in the points* x_1, x_2, x_3, *then*

$$D(x_1, x_2, x_3;\ B[u(x), v(x)])\Delta_1 x\Delta_2 x = \\ = B[D(x_1, x_2, x_3; u(x))\Delta_1 x\Delta_2 x, v(x_1)] + B[D(x_2, x_3; u(x))\Delta_1 x, \\ D(x_1, x_3;\ v(x))\Delta_2 x] + B[u(x_2), D(x_1, x_2, x_3;\ v(x))\Delta_1 x\Delta_2 x]$$

The PROOF is analogous with the proof of Lemma 1. Thus we assume that

$$D(x_1, x_2, x_3, x_4;\ B[u(x), v(x)]) = D(x_1, x_4;\ D(x, x_2, x_3;\ B[u(x), v(x)]))$$

We obtain immediately

LEMMA 3. *Under conditions of Lemma* 2, *and if* $u(x), v(x)$ *have difference quotients of the third order in the points* x_1, x_2, x_3, x_4, *then*

$$D(x_1, x_2, x_3, x_4;\ B[u(x), v(x)]) = B[D(x_1, x_2, x_3, x_4; u(x))\Delta_1 x\Delta_2 x\Delta_3 x, v(x_1)] + \\ + B[D(x_1, x_3, x_4;\ u(x))\Delta_1 x\Delta_2 x,\ D(x_1, x_4;\ v(x))\Delta_3 x] + \\ + B[D(x_2, x_3;\ u(x))\Delta_1 x,\ D(x_1, x_3, x_4;\ v(x))\Delta_2 x\Delta_3 x)] + \\ + B[u(x_2), D(x_1, x_2, x_3, x_4;\ v(x))\Delta_1 x\Delta_2 x\Delta_3 x],$$

LEMMA 4. *The following Newton's formula takes place for any operator* $P(x)$, *which has the difference quotients of the first, second and third order:*

$$P(x) = P(x_1) + D(x_1, x_2;\ P(x))(x - x_1) + D(x_1, x_2, x_3;\ P(x))(x - x_2)(x - x_1) + \\ + D(x_1, x_2, x_3, x;\ P(x))(x - x_3)(x - x_2)(x - x_1),$$

In the following paragraph we shall denote:

$$P(x_1, x_2) = D(x_1, x_2;\ P(x)),\ P(x_1, x_2, x_3) = D(x_1, x_2, x_3;\ P(x)) \quad \text{and}$$
$$P(x_1, x_2, x_3, x_4) = D(x_1, x_2, x_3, x_4;\ P(x))$$

§ 2. Convergence of the method, existence and uniqueness of the solution

Let equation (1) hold, where $P(x)$ is a continuous operator with symmetrical difference quotients[1] [3] of the first, second and third order in the space X. The method studied is given by the algorithm

$$(2) \qquad x_{n+1} = x_n - \Gamma_n P(x_n) - \Gamma_n P(x_n, x_{n-1}, x_{n-2})\Gamma_{n-1} P(x_{n-1})\bar{\Gamma}_n P(x_n), \qquad (n = 0, 1, 2, \ldots)$$

where

$$\Gamma_i = [P(x_i, x_{i-1})]^{-1}, \quad \bar{\Gamma}_{i+1} = [P(x_{i+1}, x_{i-1})]^{-1} \qquad (i = -1, 0, 1, 2, \ldots).$$

[1] Evidently, in this case the lemmas of § 1 apply too.

THEOREM 1. *If in the domain of definition of the operator* $P(x)$, *there exist points* x_{-2}, x_{-1}, x_0 *for which the following conditions hold:*

1^0 $\Gamma_{-1}, \bar{\Gamma}_0, \Gamma_0$ *exist, in the sphere* $S_0(x_0, R_0)$ *and* $\Gamma = [P(x', x'')]^{-1}$ *exists too, for any point* $(x', x'') \in S_0 \times S_0$, *and*

$$\max\{\|\Gamma_{-1}\|, \|\bar{\Gamma}_0\|, \|\Gamma_0\|, \|\Gamma\|\} = B;$$

2^0 $\|P(x_{-2})\| \leqq \eta_{-2}, \quad \|P(x_{-1})\| \leqq \eta_{-1}, \quad \|P(x_0)\| \leqq \eta_0, \qquad (\eta_0 \leqq \eta_{-1} \leqq \eta_{-2});$

3^0 *For any points* x', x'', x''', x^{IV} *from* S_0

$$\|P(x', x'', x''')\| \leqq M, \quad \|P(x', x'', x''', x^{IV})\| \leqq N;$$

4^0 $h_{-2}E_{-2} < 1$, *where* $h_{-2} = B^2 M \eta_{-2}$,

$$E_{-2}^2 = \frac{1}{H_{-2}}\left\{3 + \left(\frac{N}{BM^2} + 1\right)(1 + h_{-2})^3\right\}$$

and

$$H_{-2} = 1 - h_{-2}(1 + h_{-2}) > 0$$

then equation (1) *has a solution* $x^* \in S_0(x_0, R_0)$, *where* $R_0 = \dfrac{2B\eta_{-2}}{1-(h_{-2}E_{-2})^2}$, x^* *being the limit of the sequence* (2), *and the rapidity of convergence is given by the inequality*

$$(3) \qquad \|x^* - x_n\| \leqq \frac{2B\eta_{-2}}{1-(h_{-2}E_{-2})^6}(h_{-2}E_{-2})^{2^{\sum_{i=2}^{n-1} t_i}} \leqq R_0(h_{-2}E_{-2})^{2^{\sum_{i=2}^{n-1} t_i}},$$

where t_i *is the general term of the generalized Fibonacci's sequence with* $t_i = t_{i-3} + t_{i-2} + t_{i-1}$; $(i = 2, 3, \ldots)$ and $t_{-1} = -1, \ t_0 = 1, \ t_1 = 1$.

REMARK 1. From the condition 4^0 results that $h_{-2} < 1$.

REMARK 2. The condition $E_{-2}h_{-2} < 1$ holds, if, for example, $h_{-2} < \dfrac{1}{4}$ and $\dfrac{N}{BM^2} \leqq \dfrac{387}{125}$.

REMARK 3. In the conditions of the theorem $x_{-1}, x_{-2} \in S_0$. In fact

$$\|x_{-2} - x_0\| = \|\bar{\Gamma}_0 P(x_0, x_{-2})(x_{-2}, x_0)\| \leqq \|\bar{\Gamma}_0\|(\|P(x_{-2})\| + \|P(x_0)\|) \leqq$$
$$\leqq 2B\eta_{-2} < R_0.$$

Analogically can be proved that $x_{-1} \in S_0$.

PROOF OF THE THEOREM. From the conditions $1^0 - 3^0$, formula (2) and Remark 3, results the possibility of the construction of the point x_1 and

$$\|x_1 - x_0\| \leqq B\eta_0(1 + h_{-2}) \leqq R_0$$

whence

$$\|P(x_1, x_0, x_{-1}, x_{-2})\| \leqq N.$$

We prove that the points x_{-1}, x_0, x_1 satisfy the conditions of the theorem

1^0 $\Gamma_0, \bar{\Gamma}_1, \Gamma_1$ exist and

$$\max\{\|\Gamma_0\|, \|\bar{\Gamma}_1\|, \|\Gamma_1\|\} \leqq B.$$

2^0 Let be the operator:

$$F_n(x) = x - \lambda_n P(x) - \mu_n [P(x)]^2, \qquad (n = 0, 1, \ldots), \text{ where}$$

$$(4) \qquad \begin{cases} \lambda_n \Delta x = \Gamma_n \Delta x + \Gamma_n P(x_n, x_{n-1}, x_{n-2}) \Gamma_{n-1} \Delta x \bar{\Gamma}_n P(x_n) + \\ \qquad + \Gamma_n P(x_n, x_{n-1}, x_{n-2}) \Gamma_{n-1} P(x_{n-1}) \bar{\Gamma}_n \Delta x, \\ \mu_n \Delta_1 x \Delta_2 x = -\Gamma_n P(x_n, x_{n-1}, x_{n-2}) \Gamma_{n-1} \Delta_1 x \bar{\Gamma}_n \Delta_2 x. \end{cases}$$

We obtain immediately the following properties:

$$F_n(x_{n-2}) = F_n(x_{n-1}) = F_n(x_n) = x_{n+1}, \quad F_n(x_n, x_{n-1}) = 0, \quad F_n(x_n, x_{n-1}, x_{n-2}) = 0,$$

$$F_n(x_{n+1}, x_n, x_{n-1}, x_{n-2}) \Delta_1 x \Delta_2 x \Delta_3 x = \lambda_n P(x_{n+1}, x_n, x_{n-1}, x_{n-2}) \Delta_1 x \Delta_2 x \Delta_3 x -$$
$$- \mu_n P(x_{n+1}, x_n, x_{n-1}, x_{n-2}) \Delta_1 x \Delta_2 x \Delta_3 x P(x_n) -$$
$$- \mu_n P(x_{n+1}, x_{n-1}, x_{n-2}) \Delta_1 x \Delta_2 x P(x_{n+1}, x_n) \Delta_3 x -$$
$$- \mu_n P(x_{n-1}, x_{n-2}) \Delta_1 x P(x_{n+1}, x_n, x_{n-2}) \Delta_2 x \Delta_3 x -$$
$$- \mu_n P(x_{n-1}) P(x_{n+1}, x_n, x_{n-1}, x_{n-2}) \Delta_1 x \Delta_2 x \Delta_3 x.$$

Applying Lemma 4 to the operator $F_0(x)$ in the point $x = x_1$ and using the conditions of the theorem we obtain

$$\|P(x_1)\|[1 - h_{-2}(1 + h_{-2})] \leqq (N + M^2 B)\|x - x_{-2}\| \cdot \|x - x_{-1}\| \cdot \|x - x_0\| + 3h_{-2}^2 \eta_0.$$

Using the property

$$F_0(x_{-2}) = F_0(x_{-1}) = F_0(x_0) = x_1$$

we obtain

$$(5) \qquad \|P(x_1)\| \leqq (h_{-2} E_{-2})^2 \eta_0 = \eta_1 < \eta_0.$$

3^0 The conditions are evidently satisfied.

4^0 $h_{-1} = B^2 M \eta_{-1} \leqq h_{-2}$, $H_{-1} = 1 - h_{-1}(1 + h_{-1}) \geqq H_{-2}$

whence

$$E_{-1} \leqq E_{-2}, \quad \text{hence} \quad h_{-1} E_{-1} < 1.$$

From the proved properties results the possibility of the construction of x_2 and we have

$$\|x_2 - x_0\| \leqq B\eta_0[1 + (E_{-2} h_{-2}) h_{-1}] \leqq 2B\eta_{-2} < R_0.$$

We show that the condition $1^0 - 4^0$ are valid for the points x_0, x_1, x_2.

1^0 This condition is evidently satisfied.

2^0 Putting in the formula (4) $n = 1$ and applying Lemma 4 to the operator, it results

$$\|P(x_2)\| \leqq (E_{-1} h_{-1})^2 \eta_1 = \eta_2 < \eta_1.$$

3^0 The conditions are evidently satisfied.

4^0 $h_0 = B^2 M\eta_0 \leqq h_{-1}$, $H_0 \geqq H_{-1}$, $E_0 \leqq E_{-1}$ and $E_0 h_0 < 1$.
The conditions can be verified succesively by induction, and the following relations can be proved:

$$\eta_{n+3} \leqq (h_n E_n)^2 \eta_{n+2} \tag{6}$$

$$h_{n+3} \leqq \frac{(h_n E_n)^3}{E_n} \tag{7}$$

$$h_n E_n < 1, \tag{8}$$

where $\|P(x_n)\| \leqq \eta_n$, $h_n = B^2 M\eta_n$,

$$E_n^2 = \frac{1}{H_n}\left[3 + \left(\frac{N}{BM^2} + 1\right)(1 + h_n)^3\right]$$

and $H_n = 1 - h_n(1 + h_n)$.

From (6) it results that $x_n \in S_0(x_0, R_0)$, $x_n \in S_0'(x_{-1}, R_0)$ and $x_n \in S_0''(x_{-2}, R_0)$. In fact, we have

$$\|x_n - x_0\| \leqq 2B(\eta_{n-1} + \ldots + \eta_1 + \eta_0) \leqq \frac{2B\eta_0}{1 - (E_{-2}h_{-2})^2} \leqq R_0.$$

Analogically we can prove the other relations.

From the relations (6) and (7) we obtain

$$\eta_{n+1} \leqq (E_{-2}h_{-2})^{2\sum_{i=2}^{n} t_i} \cdot \eta_0 \tag{9}$$

whence it follows

$$\|x_{n+p} - x_n\| \leqq \frac{2B\eta_{-2}}{1 - (h_{-2}E_{-2})^6}(E_{-2}h_{-2})^{2\sum_{i=2}^{n-1} t_i} < R_0(E_{-2}h_{-2})^{2\sum_{i=2}^{n-1} t_i}. \tag{10}$$

From (8) and (10) it results that the sequence (2) is fundamental, hence it is convergent to an element $x^* \in X$. Also

$$x^* \in S(x_0, R_0), \quad x^* \in S_0'(x_{-1}, R_0), \quad x^* \in S_0''(x_{-2}, R_0).$$

If in (9), $p \to \infty$, results the formula (3).

We have to prove that

$$P(x^*) = 0.$$

From (9) it follows

$$\|P(x_n)\| \leqq (h_{-2}E_{-2})^{2\sum_{i=2}^{n-1} t_i} \cdot \eta_0.$$

For $n \to \infty$, using (9) and the continuity of $P(x)$, we obtain

$$P(x^*) = 0.$$

THEOEM 2. *If conditions of the Theorem* 1 *take place, then equation* (1) *has a unique solution in the sphere* $S_0(x_0, R_0)$.

MAGYAR TUDOMÁNYOS AKADÉMIA KÖNYVTÁRA

PROOF. Let $\tilde{x} \in S_0(x_0, R_0)$ be a solution of equation (1), $\tilde{x} \neq x^*$ and (x_n) the sequence obtained by (2). According to the condition 1^0 of Theorem 1 $[P(\tilde{x}, x_n)]^{-1}$ exists, and

$$\|[P(\tilde{x}, x_n)]^{-1}\| \leqq B.$$

We have

$$\tilde{x} - x_n = [P(\tilde{x}, x_n)]^{-1} P(\tilde{x}, x_n)(\tilde{x} - x_n) = -[P(\tilde{x}, x_n)]^{-1} P(x_n),$$

whence $\|\tilde{x} - x_n\| \leqq B\eta_n$. Using the relation (9) we obtain

$$\|\tilde{x} - x_n\| \leqq B(E_{-2}h_{-2})^{2^{\sum_{i=2}^{n-1} t_i}} \eta_0.$$

Using the relation (8) it results

$$\lim_{n \to \infty} x_n = \tilde{x}$$

and from the uniqueness of the limit

$$\tilde{x} = x^*.$$

REMARK 4. In case of algebraical or transcendental equations the formula (2) has the following form:

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n) + f(x_{n-1}) f(x_n) \cdot \tag{11}$$

$$\left[\frac{x_n}{[f(x_n) - f(x_{n-1})][f(x_n) - f(x_{n-2})]} - \frac{x_{n-1}}{[f(x_n) - f(x_{n-1})][f(x_{n-1}) - f(x_{n-2})]} + \right.$$

$$\left. + \frac{x_{n-2}}{[f(x_n) - f(x_{n-2})][f(x_{n-1}) - f(x_{n-2})]} \right].$$

NUMERICAL EXAMPLE. Let the algebrical equation be $x^3 + 30x - 12 = 0$ and $f(x) = \frac{1}{6}x^3 + 5x - 2$. We let $x_{-2} = 0$; $x_{-1} = 0{,}6$; $x_0 = 0{,}3$.

It is easy to verify, that these points satisfy the conditions of Theorem 1, and $0{,}83 < R_0 < 0{,}854$.

Using the formula (11) we obtain

$$x_1 = 0{,}39766 \quad \text{and} \quad x_2 = 0{,}3979.$$

Using the method of chord in [0,3; 0,6] we have

$$\bar{x}_1 = 0{,}39771 \quad \text{and} \quad \bar{x}_2 = 0{,}3978977.$$

We can easily prove that x_2 is nearer to the solution than $\bar{x}_2$.

REFERENCES

[1] Нечепуренко, М. И.: О методе Чебышева для функциональных уравнений, *Успехи Мат. Наук* **9** (2) (60) (1954) 163—170.
[2] Ульм, С. Ю.: Итерационные методы с разделенными разностями второго порядка, *Докл. Акад. Наук СССР* **158** (1964) 56—58.
[3] Сергеев, А. С.: О методе хорды, *Сибирск. Мат. Ж.* **2** (1961).
[4] Goldner, G. and Balázs, M.: On method of chord and an its modification for the solving of the nonlinear operational equations *Stud. Cer. Mat. (Bucarest)* **7**, 20 (1968) 981—990.
[5] Schröder, J.: Nichtlineare Majoranten beim Verfahren der schrittenweisen Näherung, *Arch. Math. (Basel)* **7** (1956) 471—484.

Babeş—Bolyai University, Cluj, Roumania

(Received April 16, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 257–266.

ON THE SOLUTIONS OF THE DIFFERENTIAL EQUATION $y''+q(x)y=0$, WHERE $[q(x)]^v$ IS CONCAVE, II.

by

Á. ELBERT

Let $y=y(x)$ be the solution of the differential equation

$$y''+q(x)y=0 \qquad \left(' = \frac{d}{dt}\right), \tag{1}$$

where $q(x)$ is continuous, $q(x)\geqq 0$. Let us assume that the solution $y(x)$ is oscillatory and $x_0(q), x_1(q), \ldots, x_n(q), \ldots$ denote the consecutive roots of the equation $y(x)=0$ and $x_0'(q), x_1'(q), \ldots, x_n'(q), \ldots$ the roots of $y'(x)=0$ with the validity of the inequalities $x_0(q)<x_0'(q)<x_1(q)<x_1'(q)<\ldots<x_n(q)<x_n'(q)<\ldots$.

We make the following

DEFINITION. The function $q(x)$ belongs to $C_v[a(q), b(q)]$ $(0<v<\infty)$ if $q(x)$ is a nonnegative continuous function on $[a(q), b(q)]$ and $[q(x)]^v$ is concave (no point of an arc lies below the corresponding chord).

This class of functions $q(x)$ was considered in [1] and our main tool now is based upon the method expounded there.

If $q(x)$ is twice continuously differentiable then the relation $q(x)\in C_v[a(q), b(q)]$ is equivalent to

$$(1-v)q'^2-qq''\geqq 0 \qquad (a(q)\leqq x\leqq b(q)). \tag{2}$$

Let $C_v[x_0, x_n]$, $C_v[x_0', x_n]$ be defined by

$$C_v[x_0, x_n]=\{q;\, q(x)\geqq 0,\ \exists x_0(q), x_1(q), \ldots, x_n(q),\ q(x)\in C_v[x_0(q), x_n(q)]\}$$

$$C_v[x_0', x_n]=\{q;\, q(x)\geqq 0,\ \exists x_0'(q), x_1(q), \ldots, x_n(q),\ q(x)\in C_v[x_0'(q), x_n(q)]\}$$

and similarly $C_v[x_0, x_n']$, $C_v[x_0', x_n']$, too.

We are interested in the quantities

$$\int_{x_0(q)}^{x_n(q)} \sqrt{q}\,dx, \quad \int_{x_0'(q)}^{x_n(q)} \sqrt{q}\,dx, \quad \int_{x_0(q)}^{x_n'(q)} \sqrt{q}\,dx, \quad \int_{x_0'(q)}^{x_n'(q)} \sqrt{q}\,dx$$

if $q(x)$ belongs to $C_v[x_0, x_n]$, $C_v[x_0', x_n]$, $C_v[x_0, x_n']$, $C_v[x_0', x_n']$. Concerning the first of these quantities, E. MAKAI proved [2] that

$$\int_{x_0(q)}^{x_n(q)} \sqrt{q}\,dx \leqq n\pi \qquad (n=1, 2, \ldots) \tag{3}$$

if $\frac{5}{4}q'^2 - qq'' \geqq 0$. In [3] it is proved for a nondecreasing $q(x)$ satisfying (2) with $0 < \nu < \infty$ that

$$\int_{x_0(q)}^{x_n(q)} \sqrt{q}\,dx > n\pi - \frac{\pi}{2} + \mu\pi, \qquad \int_{x_0(q)}^{x'_n(q)} \sqrt{q}\,dx > n\pi + \mu\pi \qquad (n = 0, 1, 2, \ldots),$$

where $\mu = \dfrac{\nu}{1+2\nu}$.

In this paper we shall give sharp upper and lower bounds for these quantities. These bounds depend on the property of $q(x)$ that $q(x)$ is nondecreasing or nonincreasing or increasing-decreasing.

We can restrict ourselves to twice continuously differentiable functions $q(x)$ as it was done in [1] because the integrals $\int_{x_0(q)}^{x_n(q)} \sqrt{q}\,dx$, ... depend continuously on q and the function $q(x)$ can be approximated by

$$q_\varepsilon(x) = \left[\frac{1}{\varepsilon^2}\int_x^{x+\varepsilon}\left(\int_u^{u+\varepsilon} q^\nu(v)\,dv\right)du\right]^{1/\nu} \qquad (\varepsilon > 0),$$

where q_ε has the same monotonity property as q, $q_\varepsilon^\nu(x)$ is concave if $q^\nu(x)$ is concave and $\lim_{\varepsilon \to +0} q_\varepsilon(x) = q(x)$.

We introduce the function $\varphi(x)$ by

$$\cos\varphi = \frac{y'}{\sqrt{qy^2 + y'^2}}, \qquad \sin\varphi = \frac{y\sqrt{q}}{\sqrt{qy^2 + y'^2}}, \qquad \varphi(x_0(q)) = 0. \tag{4}$$

By (1), (4) we have

$$\varphi' = \sqrt{q} + \frac{1}{4}\frac{q'}{q}\sin 2\varphi \tag{5}$$

and

$$\varphi(x_n(q)) = n\pi, \quad \varphi(x'_n(q)) = (n+\tfrac{1}{2})\pi \qquad (n = 0, 1, 2, \ldots). \tag{6}$$

It was proved in [3] that $\varphi'(x) > 0$, therefore the inverse function $x(\varphi)$ of the increasing function $\varphi(x)$ exists.

The following lemmas are needed.

LEMMA 1. *If $q(x)$ is nondecreasing and belongs to $C_\nu[x'_0, x_1]$ then*

$$\int_{x'_0(q)}^{x_1(q)} \sqrt{q}\,dx \geqq \frac{\pi}{2}.$$

PROOF. Since by (6) $\pi/2 \leqq \varphi(x) \leqq \pi$ for $x'_0(q) \leqq x \leqq x_1(q)$ we obtain by integrating (5) over $[x'_0(q), x_1(q)]$

$$\frac{\pi}{2} - \int_{x'_0(q)}^{x_1(q)} \sqrt{q}\,dx = \int_{x'_0(q)}^{x_1(q)} \frac{1}{4}\frac{q'}{q}\sin 2\varphi\,dx \leqq 0,$$

which proves Lemma 1.

LEMMA 2. *If $q(x)$ is nondecreasing and belongs to $C_v[x_0', x_1']$ then*

$$\int_{x_0'(q)}^{x_1'(q)} \sqrt{q}\, dx \geqq \pi.$$

PROOF. Let $s(x)=\frac{1}{4}\, q'q^{-3/2}$. Then by (2) $s'(x)\leqq 0$, therefore $s(x)$ is decreasing. Integrating (5) over $[x_0'(q), x_1'(q)]$ and taking into account (6) we have

$$\pi - \int_{x_0'(q)}^{x_1'(q)} \sqrt{q}\, dx = \int_{x_0'(q)}^{x_1'(q)} \frac{1}{4}\frac{q'}{q} \sin 2\varphi\, dx = \int_{x_0'(q)}^{x_1'(q)} \frac{s(x)\sin 2\varphi(x)}{1+s(x)\sin 2\varphi}\, \varphi'(x)\, dx =$$

$$= \int_{\frac{\pi}{2}}^{3\frac{\pi}{2}} \frac{s(x(\varphi))\sin 2\varphi}{1+s(x(\varphi))\sin 2\varphi}\, d\varphi =$$

$$= -\int_0^{\frac{\pi}{2}} \left[\frac{s(x(\pi-\psi))\sin 2\psi}{1-s(x(\pi-\psi))\sin 2\psi} - \frac{s(x(\pi+\psi))\sin 2\psi}{1+s(x(\pi+\psi))\sin 2\psi}\right] d\psi \leqq 0$$

which proves Lemma 2.

LEMMA 3. *If $q(x)$ is nondecreasing and belongs to $C_v[x_0, x_0']$ then*

$$\int_{x_0(q)}^{x_0'(q)} \sqrt{q}\, dx \leqq \frac{\pi}{2}.$$

PROOF. Similar to the proof of Lemma 1.

LEMMA 4. *If $q(x)$ is nondecreasing and belongs to $C_v[x_0, x_1]$ then*

$$\int_{x_0(q)}^{x_1(q)} \sqrt{q}\, dx \leqq \pi.$$

PROOF. This lemma is a special case of (3) proved by E. MAKAI. Alternatively, one can prove it on the lines of the proof of Lemma 2.

LEMMA 5. *If $q(x)$ is nondecreasing and belongs to $C_v[x_0, x_0']$ then*

$$\int_{x_0(q)}^{x_0'(q)} \sqrt{q}\, dx \geqq j_{\mu-1}$$

where $j_{\mu-1}$ is the first positive root of the Bessel function of order $\mu-1$ and $0<j_{\mu-1}<\frac{\pi}{2}$ and μ is defined above.

PROOF. We can assume that $x_0(q)=0$. (If it were not so we would make use of the substitution $\bar{x} = x-x_0(q)$.) If $q'(x)\equiv 0$, i.e. $q(x)\equiv$ const. we have by (5), (6) $\int_0^{x_0'(q)} \sqrt{q}\, dx = \frac{\pi}{2} > j_{\mu-1}$, therefore the inequality holds for $q(x)\equiv$ const.. If $q(x)\not\equiv$

const., then let $x^* = \sup\{x; q'(x) > 0,\ 0 \leqq x \leqq x_0'(q)\}$. We define the function $\tau(x)$ by

$$\tau(x) = 2\left(\frac{q(x)}{q'(x)} - vx\right) \quad 0 < x < x^*. \tag{7}$$

By (2) $\tau(x)$ is nondecreasing in $(0, x^*)$. Let $q_\xi(x) = s_\xi^2(x)$ be defined for $0 < \xi < x^*$ by

$$s_\xi(x) = \begin{cases} \sqrt{q} & 0 \leqq x \leqq \xi \\ C(x+\gamma)^{1/2v} & x \geqq \xi, \end{cases} \tag{8}$$

where $C = C(\xi)$ and $\gamma = \gamma(\xi)$ are determined by $s_\xi(\xi) = \sqrt{q(\xi)}$, $s_\xi'(\xi) = \left(\sqrt{q(x)}\right)'_{x=\xi}$ $(C > 0,\ \gamma \geqq 0)$. It is clear that $q_\xi = s_\xi^2 \in C_v[x_0, x_0']$ and $\lim_{\xi \to x^* - 0} s_\xi(x) = \sqrt{q(x)}$ and $s_\xi(x)$ is twice continuously differentiable except at $x = \xi$ where $s''(x)$ may be discontinuous.

Let $\psi_\xi(x)$ be defined for $x \geqq 0$ by

$$\psi_\xi' = s_\xi + \frac{1}{2}\frac{s_\xi'}{s_\xi}\sin 2\psi_\xi, \quad \psi_\xi(0) = 0. \tag{9}$$

We introduce the notations $x_0'(q_\xi) = a(\xi)$ and

$$J(\xi) = \int_0^{a(\xi)} s_\xi(x)\,dx. \tag{10}$$

Let ξ be any place where $\tau'(\xi) > 0$; differentiating (10) we have

$$\frac{dJ(\xi)}{d\tau(\xi)} = \int_0^{a(\xi)} \frac{ds_\xi(x)}{d\tau(\xi)}\,dx + s_\xi(a(\xi)) \cdot \frac{da(\xi)}{d\tau(\xi)}. \tag{11}$$

By (6) we have $\psi_\xi(a(\xi)) = \frac{\pi}{2}$ and by (9) $\psi_\xi'(a(\xi)) = s_\xi(a(\xi))$. From these we get

$$s_\xi(a(\xi))\frac{da(\xi)}{d\tau(\xi)} + \left.\frac{d\psi_\xi(x)}{d\tau(\xi)}\right|_{x=\xi} = 0. \tag{12}$$

From the differential equation (9) it follows

$$\left(\frac{d\psi_\xi(x)}{d\tau(\xi)}\right)' = \tag{13}$$

$$= \frac{ds_\xi(x)}{d\tau(\xi)} + \frac{1}{2}\left(\frac{d}{d\tau(\xi)}\frac{s_\xi'(x)}{s_\xi(x)}\right)\sin 2\psi_\xi(x) + \frac{s_\xi'(x)}{s_\xi(x)}\cos 2\psi_\xi(x) \cdot \frac{d\psi_\xi(x)}{d\tau(\xi)} \quad (x \geqq \xi),$$

which is a first order linear differential equation for $d\psi_\xi(x)/d\tau(\xi)$ with the initial value $d\psi_\xi(x)/d\tau(\xi)|_{x=\xi} = 0$. To solve this differential equation and to determine

the sign of $dJ(\xi)/d\tau(\xi)$ we need some relations proved already in [1]:

$$\frac{ds_\xi(x)}{d\tau(\xi)} = \begin{cases} 0 & 0 \leqq x < \xi \\ -s_\xi(x)\int_\xi^x \left[\frac{s'_\xi(u)}{s_\xi(u)}\right]^2 du & x \geqq \xi \end{cases}$$

$$\frac{d}{d\tau(\xi)}\frac{s'_\xi(x)}{s_\xi(x)} = -\left[\frac{s'_\xi(x)}{s_\xi(x)}\right]^2 \quad x \geqq \xi.$$

Let $\varrho(x)$, $t(x)$ be defined by

$$\varrho = Y'^2+s^2Y^2, \quad t = \varrho\frac{s}{s'}+YY' \quad (x \geqq 0), \tag{14}$$

where $s=s_\xi(x)$ and $Y=Y_\xi(x)$ is the solution of the differential equation $Y''+s^2Y=0$ with the initial conditions $Y_\xi(0)=0$, $Y'_\xi(0)=1$. Using these functions the other relations needed are

$$\varrho' = 2ss'Y^2 \quad (x \geqq 0)$$

$$\sin\psi_\xi = \frac{sY}{\sqrt{\varrho}}, \quad \cos\psi_\xi = \frac{Y'}{\sqrt{\varrho}} \quad (x \geqq 0)$$

$$t' = (2v+1)\varrho \quad (x \geqq \xi)$$

$$\frac{s'}{s}\cos 2\psi_\xi = \frac{\varrho}{s}\left(\frac{s'}{\varrho}\right)' \quad (x \geqq 0).$$

Applying these relations we have by (13)

$$\left(\frac{d\psi_\xi(x)}{d\tau(\xi)}\right)' = -s\int_\xi^x\left(\frac{s'}{s}\right)^2 du - \left(\frac{s'}{s}\right)^2\frac{s}{\varrho}YY' + \frac{\varrho}{s}\left(\frac{s}{\varrho}\right)'\frac{d\psi_\xi(x)}{d\tau(\xi)} \quad (x \geqq \xi),$$

hence

$$\left(\frac{\varrho}{s}\frac{d\psi_\xi(x)}{d\tau(\xi)}\right)' = -\varrho\int_\xi^x\left(\frac{s'}{s}\right)^2 du - \left(\frac{s'}{s}\right)^2 YY' \quad (x \geqq \xi),$$

therefore

$$\frac{d\psi_\xi(x)}{d\tau(\xi)} = -\frac{s(x)}{\varrho(x)}\left[\int_\xi^x \varrho(v)\int_\xi^v\left(\frac{s'}{s}\right)^2 du\,dv + \int_\xi^x\left(\frac{s'}{s}\right)^2 YY'\,du\right] \quad (x \geqq \xi).$$

Taking into account that $s(x) = C(x+\gamma)^{\frac{1}{2v}}$ for $x \geqq \xi$ it follows

$$\left.\frac{d\psi_\xi(x)}{d\tau(\xi)}\right|_{x=a(\xi)} = -\frac{1}{2v}\frac{s(a(\xi))}{\varrho(a(\xi))}\left\{\frac{1}{1+2v}\frac{s'(\xi)}{s(\xi)}[t(a(\xi))-t(\xi)]+\right.$$

$$\left.+\frac{s'(\xi)}{s(\xi)}Y(\xi)Y'(\xi)-[\varrho(a(\xi))-\varrho(\xi)]\right\}$$

and similarly

$$\int_0^{a(\xi)} \frac{ds_\xi(x)}{d\tau(\xi)}\,dx = -\frac{1}{2\nu}\left\{\frac{1}{2\nu+1}\,\frac{s'(\xi)}{s(\xi)}\left[\frac{s^2(a(\xi))}{s'(a(\xi))}-\frac{s^2(\xi)}{s'(\xi)}\right]-[s(a(\xi))-s(\xi)]\right\}.$$

We get also by (11), (12)

$$\frac{dJ(\xi)}{d\tau(\xi)} = \frac{1}{2\nu+1}\,\frac{s(\xi)s(a(\xi))}{\varrho(a(\xi))}\,[z(\xi)-z(a)], \tag{15}$$

where $z(x) = \dfrac{s'(x)}{s^2(x)}\,t(x)$ and by (8) $z'(x) = -\dfrac{1+2\nu}{2\nu}\,\dfrac{s'}{s^2}\,YY' < 0$

for $\xi \leqq x < a(\xi)$, therefore $dJ(\xi)/d\tau(\xi) > 0$. The derivative $dJ(\xi)/d\tau(\xi)$ can be defined by (15) for all $\xi \in (0, x^*)$. We obtain in this way

$$\int_0^{x_0'(q)} \sqrt{q}\,dx = J(x^*) = J(\xi) + \int_\xi^{x^*} \frac{dJ(u)}{d\tau(u)}\,\tau'(u)\,du \quad (0 < \xi < x^*). \tag{16}$$

Let $\sigma_\xi(x)$ be defined for $0 < \xi < x^*$ by

$$\sigma_\xi(x) = C(x+\gamma)^{1/2\nu} \quad (x \geqq 0), \tag{17}$$

where $C = C(\xi)$ and $\gamma = \gamma(\xi)$ are the same as in (8). Let $\chi_\xi(x)$ be the solution of the differential equation

$$\chi' = \sigma_\xi + \frac{1}{2}\,\frac{\sigma_\xi'}{\sigma_\xi}\sin 2\chi_\xi \quad (x \geqq 0) \tag{18}$$

with the initial condition $\chi_\xi(\xi) = \psi_\xi(\xi)$ and let $\alpha(\xi)$ be the value of x where $\chi_\xi(\alpha(\xi)) = 0$ Like $\varphi(x)$ and $\psi_\xi(x)$, the function $\chi_\xi(x)$, too, is an increasing function of x, therefore $\alpha(\xi) < \xi$. The inequality $\sigma_\xi \geqq s_\xi$ valid by (2) on $[0, \xi]$ gives $\chi_\xi \leqq \psi_\xi$ for $0 \leqq x \leqq \xi$, therefore $\alpha(\xi) \geqq 0$. Taking into account the relation

$$\lim_{\xi \to +0}\left\{J(\xi) - \int_{\alpha(\xi)}^{a(\xi)} \sigma_\xi(x)\,dx\right\} = \lim_{\xi \to +0} \int_0^\xi \sqrt{q}\,dx - \lim_{\xi \to +0} \int_{\alpha(\xi)}^{\xi} \sigma_\xi(x)\,dx = 0,$$

assuming $\tau'(x) \not\equiv 0$ (i.e. $(1-\nu)q'^2 - qq'' \not\equiv 0$) and using (16) there is $\xi_0 \in (0, x^*)$ with

$$\int_0^{x_0'(q)} \sqrt{q}\,dx > \int_{\alpha(\xi_0)}^{a(\xi_0)} \sigma_{\xi_0}(x)\,dx. \tag{19}$$

We remark here that for $q = \sigma_{\xi_0}^2$ the sign of equality holds in (2) and conversely, if equality holds in (2) then q has the form $\sigma_\xi^2(x)$ where $\sigma_\xi(x)$ is defined by (17). The proof for such a q goes as for $q = \sigma_{\xi_0}^2$ here.

If $\gamma_0 = \gamma(\xi_0) + \alpha(\xi_0) > 0$ then the domain of the definition of the differential equation (18) can be extended from $[0, \infty)$ to $[-\gamma_0, \infty)$. Let $\chi^{(0)}(x)$ be the solution of this extended differential equation with the initial condition $\chi^{(0)}(-\gamma_0) = 0$ and

let a_0 be the value of x where $\chi^{(0)}(a_0)=\pi/2$. The uniqueness of the solutions gives $\chi^{(0)}(x)>\chi_{\xi_0}(x)$ for $x\geqq\alpha(\xi_0)$, therefore for their inverse functions $x^{(0)}(\chi)$ and $x_{\xi_0}(\chi)$ the inequality

$$x^{(0)}(\chi)<x_{\xi_0}(\chi) \quad (\chi\geqq 0) \tag{20}$$

holds. Putting into (18) the function $\chi_{\xi_0}(x)$ and integrating it in $[\alpha(\xi_0), a(\xi_0)]$ with $\sigma_\xi=\sigma_{\xi_0}$ we have

$$\frac{\pi}{2}-\int_{\alpha(\xi_0)}^{a(\xi_0)}\sigma_{\xi_0}(x)\,dx=\int_{\alpha(\xi_0)}^{a(\xi_0)}\frac{1}{2}\frac{\sigma'_{\xi_0}}{\sigma_{\xi_0}}\sin 2\chi_{\xi_0}\,dx= \tag{21}$$

$$=\int_{\alpha(\xi_0)}^{a(\xi_0)}\frac{\sin 2\chi_{\xi_0}}{4\nu C(x+\gamma)^{1+1/2\nu}+\sin 2\chi_{\xi_0}}\chi'_{\xi_0}\,dx=\int_0^{\pi/2}\frac{\sin 2\chi}{4\nu C[x_{\xi_0}(\chi)+\gamma]^{1+1/2\nu}+\sin 2\chi}\,d\chi$$

and similarly for $\chi^{(0)}(x)$

$$\frac{\pi}{2}-\int_{-\gamma_0}^{a_0}\sigma_{\xi_0}\,dx=\int_0^{\pi/2}\frac{\sin 2\chi}{4\nu C[x^{(0)}(\chi)+\gamma]^{1+1/2\nu}+\sin 2\chi}\,d\chi, \tag{22}$$

hence by (20)

$$\int_{\alpha(\xi_0)}^{a(\xi_0)}\sigma_{\xi_0}(x)\,dx>\int_{-\gamma_0}^{a_0}\sigma_{\xi_0}(x)\,dx=\int_0^{\gamma_0+a_0}Cx^{1/2\nu}\,dx= \tag{23}$$

$$=\int_0^{x'_0(C^2x^{1/\nu})}Cx^{1/2\nu}\,dx=\int_0^{x'_0(x^{1/\nu})}x^{1/2\nu}\,dx,$$

where the last equality can be proved by the substitution

$$x=\lambda u, \quad \lambda=C^{-\frac{2\nu}{1+2\nu}}=C^{-2\mu}.$$

The solutions of the differential equation $y''+x^{1/\nu}y=0$ with the initial conditions $y(0)=0$, $y'(0)\neq 0$ are $y(x)=c\sqrt{x}J_\mu(2\mu x^{1/2\mu})$, where $\int_0^x u^{1/2\nu}\,du=2\mu x^{1/2\mu}$ and c is an arbitrary constant, hence $y'(x)=cJ_{\mu-1}(2\mu x^{1/2\mu})/(2\mu\sqrt{x})$, therefore for $x'_0(x^{1/\nu})$ the relation

$$\int_0^{x'_0(x^{1/\nu})}x^{1/2\nu}\,dx=j_{\mu-1}$$

holds. This and (19), (23) complete the proof of LEMMA 5.

LEMMA 6. *If $q(x)$ is nondecreasing and belongs to $C_\nu[x'_0, x_1]$ then*

$$\int_{x'_0(q)}^{x_1(q)}\sqrt{q}\,dx\leqq j_{-\mu},$$

where $j_{-\mu}$ is the first positive root of the Bessel function of order $-\mu$ and $j_{-\mu}>\frac{\pi}{2}$.

PROOF. The method of the proof of Lemma 5 can be used now, too, only the following changes are needed: $\varphi(0)=\pi/2$, $\varphi(x_1(q))=\pi$, $\psi_\xi(0)=\pi/2$, $\psi_\xi(a(\xi))=\pi, \ldots,$ $\ldots,$ $Y_\xi(0)=1$, $Y'_\xi(0)=0$ and $z'(x)=-(1+2\nu)s'YY'/(2\nu s^2)>0$ for $\xi\leqq x<a(\zeta)$ because $Y>0$, $Y'<0$ for these x-s and therefore $dJ(\xi)/d\tau(\xi)<0$ by (15) and the relation $<$ holds in (19). In (21) and (22) we integrate over $[\pi/2, \pi]$ instead of $[0, \pi/2]$, therefore the relation $<$ is needed in (23). The solutions of the differential equation $y''+x^{1/\nu}y=$ with the initial conditions $y(0)\neq 0$, $y'(0)=0$ are $y(x)=c\sqrt{x}J_{-\mu}(2\mu x^{1/2\mu})$, hereby the Lemma 6 is completely proved.

REMARK. It is clear that the inequalities in these Lemmas are sharp.

THEOREM 1. *If $q(x)$ is nondecreasing and belongs to a) $C_\nu[x_0, x_n]$, b) $C_\nu[x'_0, x_n]$, c) $C_\nu[x_0, x'_n]$, d) $C_\nu[x'_0, x'_n]$ then*

$$a) \qquad n\pi-\frac{\pi}{2}+j_{\mu-1}\leqq\int_{x_0(q)}^{x_n(q)}\sqrt{q}\,dx\leqq n\pi \qquad (n=1,2,\ldots)$$

$$b) \qquad n\pi-\frac{\pi}{2}\leqq\int_{x'_0(q)}^{x_n(q)}\sqrt{q}\,dx\leqq(n-1)\pi+j_{-\mu} \qquad (n=1,2,\ldots)$$

$$c) \qquad n\pi+j_{\mu-1}\leqq\int_{x_0(q)}^{x'_n(q)}\sqrt{q}\,dx\leqq(n+\tfrac{1}{2})\pi \qquad (n=0,1,2,\ldots)$$

$$d) \qquad n\pi\leqq\int_{x'_0(q)}^{x'_n(q)}\sqrt{q}\,dx\leqq(n-\tfrac{1}{2})\pi+j_{-\mu} \qquad (n=1,2,\ldots).$$

PROOF. *a)* We have the left inequality by the partition $[x_0(q), x_n(q)]=[x_0(q), x'_0(q)]\cup\bigcup_{i=1}^{n-1}[x'_{i-1}(q), x'_i(q)]\cup[x'_{n-1}(q), x_n(q)]$ and using Lemmas 5, 2, 1. The right inequality is the same as in (3).

b) For the left inequality we need the partition $[x'_0(q), x_n(q)]=\bigcup_{i=1}^{n-1}[x'_{i-1}(q), x'_i(q)]\cup[x'_{n-1}(q), x_n(q)]$ and Lemma 2, 1. For the right inequality $[x'_0(q), x_n(q)]=$ $=[x'_0(q), x_1(q)]\cup\bigcup_{i=1}^{n-1}[x_i(q), x_{i+1}(q)]$ and Lemmas 6, 4.

The proof of *c)* and *d)* is similar.

A similar theorem holds for nonincreasing $q(x)$-s, too (see Tables I, II).

LEMMA 7. *If $q(x)$ is nondecreasing on $[x_0(q), \infty)$ and belongs to $C_\nu[x_0(q), \infty]$, then*

$$0\leqq\varphi(x)-\int_{x_0(q)}^{x}\sqrt{q}\,du\leqq\frac{\pi}{2}-j_{\mu-1} \quad \text{for} \quad x\geqq x_0(q).$$

PROOF. By (5), (6) the function $\varphi(x)-\int_{x_0(q)}^{x}\sqrt{q}\,du$ has local minima at $x_n(q)$ $(n=0, 1, 2, \ldots)$ and local maxima at $x'_n(q)$ $(n=0, 1, 2, \ldots)$. Applying Theorem 1 (cases *a)*, *c)*) we have the desired inequalities.

LEMMA 8. *If $q(x)$ is nondecreasing on $[x_0'(q), \infty)$ and belongs to $C_\nu[x_0'(q), \infty]$ then*

$$\frac{\pi}{2}-j_{-\mu} \leqq \varphi(x)-\frac{\pi}{2}-\int_{x_0'(q)}^{x}\sqrt{q}\,du \leqq 0.$$

PROOF. Similar to the proof of Lemma 7.

THEOREM 2. *If $q(x)$ belongs to a) $C_\nu[x_0, x_n]$, b) $C_\nu[x_0', x_n]$, c) $C_\nu[x_0, x_n']$, d) $C_\nu[x_0', x_n']$ then*

a) $$(n-1)\pi+2j_{\mu-1} \leqq \int_{x_0(q)}^{x_n(q)}\sqrt{q}\,dx \leqq n\pi \qquad (n=1,2,\ldots)$$

b) $$(n-1)\pi+j_{\mu-1} \leqq \int_{x_0'(q)}^{x_n(q)}\sqrt{q}\,dx \leqq (n-1)\pi+j_{-\mu} \qquad (n=1,2,\ldots)$$

c) $$n\pi+j_{\mu-1} \leqq \int_{x_0(q)}^{x_n'(q)}\sqrt{q}\,dx \leqq n\pi+j_{-\mu} \qquad (n=0,1,2,\ldots)$$

d) $$n\pi \leqq \int_{x_0'(q)}^{x_n'(q)}\sqrt{q}\,dx \leqq (n-1)\pi+2j_{-\mu} \qquad (n=1,2,\ldots).$$

PROOF. *a)* If $q(x)$ is monotonic then there is no news in this theorem. If $q(x)$ is increasing-decreasing then we consider the intervals where $q(x)$ is monotonic: $[x_0(q), x^{**}]$ and $[x^{**}, x_n(q)]$. Let $\bar{q}(x)$ be defined by

$$\bar{q}(x)=\begin{cases} q(x) & x_0(q) \leqq x \leqq x^{**} \\ q(x^{**}) & x \geqq x^{**} \end{cases}$$

then $\bar{q} \in C_\nu[x_0(q), \infty]$, nondecreasing, applying Lemma 7 we have

$$0 \leqq \varphi(x^{**})-\int_{x_0(q)}^{x^{**}}\sqrt{q}\,dx \leqq \frac{\pi}{2}-j_{\mu-1},$$

similarly on $[x^{**}, x_n(q)]$

$$0 \leqq \varphi(x_n(q))-\varphi(x^{**})-\int_{x^{**}}^{x_n(q)}\sqrt{q}\,dx \leqq \frac{\pi}{2}-j_{\mu-1},$$

hence

$$0 \leqq n\pi-\int_{x_0(q)}^{x_n(q)}\sqrt{q}\,dx \leqq \pi-2j_{\mu-1},$$

which was to be proved. Since the proof of the other statements *b), c), d)* involves no new ideas, we omit the proof.

Finally let the results of this paper be collected in two Tables as follows.

TABLE I. $\inf_{q\in C_v[a,b]} \int_{a(q)}^{b(q)} \sqrt{q}\,dx$

a	b	$q(x)$		
		nondecreasing	nonincreasing	general
x_0	x_n	$(n-\frac{1}{2})\pi-j_{\mu-1}$	$(n-\frac{1}{2})\pi+j_{\mu-1}$	$(n-1)\pi+2j_{\mu-1}$
x_0'	x_n	$(n-\frac{1}{2})\pi$	$(n-1)\pi+j_{\mu-1}$	$(n-1)\pi+j_{\mu-1}$
x_0	x_n'	$n\pi+j_{\mu-1}$	$(n+\frac{1}{2})\pi$	$n\pi+j_{\mu-1}$
x_0'	x_n'	$n\pi$	$n\pi$	$n\pi$

TABLE II. $\sup_{q\in C_v[a,b]} \int_{a(q)}^{b(q)} \sqrt{q}\,dx$

a	b	$q(x)$		
		nondecreasing	nonincreasing	general
x_0	x_n	$n\pi$	$n\pi$	$n\pi$
x_0'	x_n	$(n-1)\pi+j_{-\mu}$	$(n-\frac{1}{2})\pi$	$(n-1)\pi+j_{-\mu}$
x_0	x_n'	$(n+\frac{1}{2})\pi$	$n\pi+j_{-\mu}$	$n\pi+j_{-\mu}$
x_0'	x_n'	$(n-\frac{1}{2})\pi+j_{-\mu}$	$(n-\frac{1}{2})\pi+j_{-\mu}$	$(n-1)\pi+2j_{-\mu}$

REFERENCES

[1] ELBERT, Á.: On the solutions of the differential equation $y''+q(x)y=0$, where $[q(x)]^v$ is concave, I, *Acta Math. Hungar.* **20** (1969),1 — 11.

[2] MAKAI, E.: Über die Nullstellen von Funktionen, die Lösungen Sturm-Liouville'scher Differentialgleichungen sind, *Comment. Math. Helv.* **16** (1943—44) 153—199.

[3] ELBERT, Á.: On the solutions of the differential equation of second order, *Publ. Math. Debrecen* **15** (1968), 13 — 17

Mathematical Institute of the Hungarian Academy of Sciences, Budapest

(Received April 17, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 267–278.

SYNTACTIC ANALYSIS AND UNILATERAL CONTEXT-SENSITIVE GRAMMARS

by

GY. RÉVÉSZ

Introduction

The formal syntax of a programming language is usually given in the form of a context-free grammar [1] and consequently, the syntactic analysis performed by the corresponding compiler is based on that grammar. It is a natural requirement that the analyzing algorithm should be in a direct connection with the formal grammar. In addition it should be efficient from a practical point of view.

There have been developed general parsing algorithms for context-free languages [2, 3, 4] though, these are not as simple as it might be desired, since the possibility of processing strictly from left to right cannot be ensured in general. In other words, every general parsing algorithm for context-free languages must adopt trials. This means that the application of some syntactic rules are subject to reaffirmation or rejection in the course of the analysis later on. In order to avoid such trials, or at least minimize their number, we must either restrict ourselves to some subclass of context-free languages [5] or the original rules of the grammar must somehow be transformed for the actual analysis [6].

It is a usual feature of these transformed grammars that some elements of the context will also be taken into account during the analysis of the input string [7, 8]. This has directed my attention to context-sensitive grammars that have already been introduced and investigated at some length as an extension of context-free grammars. In a previous paper I introduced a special subclass of phrase-structure grammars which I called „operator-operand" grammars [9]. This type of grammars has been used for constructing a parsing algorithm that proceeds from left to right and can be utilized for syntactic analysis on certain conditions. In [4] DÖMÖLKI investigated also the problem of the syntactic analysis from left to right in a class of languages that is more general than context-sensitive languages.

It is essential to apply the same rules (only in the opposite sense) in the course of the analysis that serve for the definition of the language in question since the semantic contents of the language must also be related to these rules. Furthermore, any changes in the syntax must obviously affect the analysis in a possibliy direct way. Of course this requirement influences the class of grammars that can be made use of.

The above mentioned operator-operand grammars belong to a subclass of context-sensitive grammars where the restrictive context in the rules consists of at most one symbol which appears to the right of the symbol to be replaced.

In this connection DÖMÖLKI has raised the question as to whether unilateral context-sensitive grammars are more general in generative power than context-free grammars. The answer to this question is in the affirmative since E. FARKAS

has constructed such a grammar that generates a language that cannot be generated by a context-free grammar [10].

But the parsing algorithm developed for operator-operand languages in [9] can be generalized for these unilateral context-sensitive languages, so can the conditions ensuring the possibility to carry out a deterministic analysis (i.e. without trials). Moreover, these conditions can be met also by languages that are not context-free, that can be shown by a suitable modification of FARKAS' example.

In the present paper we describe a parsing algorithm developed for these special context-sensitive languages. Then we shall specify a few conditions and prove that they are sufficient for the applicability of this algorithm which is deterministic and proceeds from left to right in some sense.

The context-free languages form a subclass of the languages dealt with in this paper, hence the algorithm and the corresponding conditions also hold for context-free languages. On the other hand in the light of our results, it would seem that the „monopolistic" role of context-free grammars in the definition of programming languages can be questioned if the efficiency of the analysis is taken into account.

§ 1. The class of right-sensitive grammars

Let us recall the formal definition of context-sensitive phrase-structure grammar. The set of strings over a set of symbols V will be denoted by $\Sigma(V)$. This set is equivalent to the free semigroup generated by the symbol set V including the empty string Λ as its unit element.

DEFINITION 1. A context-sensitive phrase-structure grammar G is a quadruple $G=(T, I, s, P)$ where T and I are finite sets of terminal and non-terminal (auxiliary) symbols, respectively, $T\cap I=\emptyset$, $s\in I$ and P is a finite set of ordered pairs (called rules) of the form (XqY, XQY) where $q\in I$ and $X, Y, Q\in\Sigma(T\cup I)$ and $Q\neq\Lambda$ (i.e., Q is nonempty). Remark: In this paper the term grammar will stand for context-sensitive phrase-structure grammar.

DEFINITION 2. Given a grammar G and two strings A, $B\in\Sigma(T\cup I)$, B is an immediate consequence of A (in symbols $A\vdash B$), iff there exists a rule $(XqY, XQY)\in P$ such that $A=ZXqYW$ and $B=ZXQYW$ with some $Z, W\in\Sigma(T\cup I)$.

DEFINITION 3. A finite sequence of strings $X_1, X_2, \ldots, X_n$ is called a derivation with respect to a given grammar G, iff $X_i\vdash X_{i+1}$ for $1\leqq i<n$.

DEFINITION 4. Given a grammar G and two strings $U, V\in\Sigma(T\cup I)$, V is a consequence of U (in symbols $U\models V$) iff there exists a derivation $X_1, X_2, \ldots, X_n$ with respect to G, where $U=X_1$ and $V=X_n$.

DEFINITION 5. Given a grammar G, the set of strings

$$L_G=\{W|s\models W\}\cap\Sigma(T)$$

is called the language generated by G.

DEFINITION 6 (RSG). A grammar is called right-sensitive, if the string X is empty in each rule of P, i.e., all the rules of P are of the form (qY, QY), where $q \in I$, Y and $Q \in \Sigma(T \cup I)$, and $Q \neq \Lambda$.

The right-sensitive grammars can be classified according to the maximum length of Y in their rules:

DEFINITION 7. A right-sensitive grammar is of order n if the length of each Y occurring in its rules is not greater than n.

REMARK. The length of a string Y (i. e. the number of its symbols) is denoted sometimes by $|Y|$.

The class of right-sensitive grammars of order n will be denoted by RSG_n and the class of all languages L_G with $G \in \mathrm{RSG}_n$ will be denoted by RSL_n. According to the above definitions $\mathrm{RSG}_n \subset \mathrm{RSG}_{n+1}$, hence $\mathrm{RSL}_n \subseteq \mathrm{RSL}_{n+1}$ for $n = 0, 1, 2, \ldots$ but it is a question, whether RSL_n forms a proper subset of RSL_{n+1}. As I have mentioned in the introduction, for $n = 0$ the answer is in the affirmative, i.e., the context-free languages form a proper subset of RSL_1.

This was shown by E. FARKAS with the aid of the language $a^k b^n c^n$ where $k \leqq n$ [10]. Here we shall make use of this language in the modified form $a^{2k} p b^{2n} q c^{2n}$ with $k \leqq n$. This particular language cannot be generated by a context-free grammar just because of the restriction $k \leqq n$. This can indirectly be shown in the following way.

With the assumption that there exists a context-free grammar G generating this language, we may argue similary to the proof of Theorem 4.1 in [11]. Namely, let us consider the generation trees of the terminal strings $a^{2k} p b^{2n} q c^{2n}$. By a generation tree we mean a directed graph, where to each node a symbol of $T \cup I$ will be assigned that occurs in the derivation of the given terminal string, while the arrows drawn from a node correspond to the applicatin of a rule replacing the symbol assigned to the initial node by those assigned to the points of these arrows.

The tree below (see Figure 1), for instance, corresponds to the generation of the terminal string $a^2 p b^2 q c^2$, by means of the set of rules P^*.

Now, if the maximum length of the right-hand sides of the rules is m and the length of the longest path in a tree is l, then the string corresponding to the terminal nodes of the tree is not longer than m^l. So, if $2k > m^{i+1}$ where i denotes the number of the elements of I, then there is a path in the generation tree of the string $a^{2k} p b^{2n} q c^{2n}$ that leads from s to some a and is longer than $i+1$. Thus, some noterminal symbol q_0 must occur twice along that path.

This means that $q_0 \models Z q_0 W$ for some Z, $W \in \Sigma(T)$ and $q_0 \models AaB$ for some A, $B \in \Sigma(T)$, which implies that $q_0 \models \models Z^h AaB W^h$ and, thus, the string $UZ^h AaB W^h V \in L_G$ with some U, $V \in \Sigma(T)$ for $h = 1, 2, \ldots$. Now, it follows that the string Z must be built up from the terminal symbol a only, while the string W cannot contain b and c both. If the string W contains a only, the restriction $k \leqq n$ will be violated for some h, if it contains

P^* (s, axy)
(x, apx)
(x, bb)
(x, qy)
(z, cc)

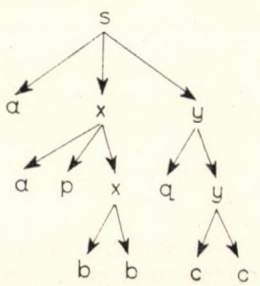


Fig. 1

b or c the number of these terminal symbols will be different in some string of the language for some h. Qu. e. d.

On the other hand, the language $a^{2k}pb^{2n}qc^{2n}$ with $k \leqq n$ can be generated by the following right-sensitive grammar:

EXAMPLE 1. $G=(T, I, s, P)$, where $T=\{a, b, c, p, q\}$, $I=\{s, x_1, x_2, y_1, y_2, b_1, b_2, u_1, u_2, z_1, z_2\}$, and the rules of P are as follows

$$(s, x_1y_1)$$

$$(y_1, b_2z_2) \quad (x_1b_2, u_2b_2) \quad (u_2, ax_2)$$

$$(z_2, y_2c) \quad (b_1b_2, b_2b_2) \quad (u_1, ax_1)$$

$$(y_2, b_1z_1) \quad (x_2b_1, u_1b_1) \quad (x_1, p)$$

$$(z_1, y_1c) \quad (b_2b_1, b_1b_1) \quad (y_1, q)$$

$$(b_1, b)$$

E.g. the derivation of the string $a^2pb^4qc^4$ can be given in the following way:

$$s \vdash x_1y_1 \vdash x_1b_2z_2 \vdash u_2b_2z_2 \vdash ax_2b_2z_2 \vdash ax_2b_2y_2c \vdash ax_2b_2b_1z_1c \vdash$$

$$ax_2b_1b_1z_1c \vdash au_1b_1b_1z_1c \vdash a^2x_1b_1b_1z_1c \models a^2pb^2z_1c \vdash a^2pb^2y_1c^2 \vdash$$

$$a^2pb^2b_2z_2c^2 \vdash a^2pb^2b_2y_2c^3 \vdash a^2pb^2b_2b_1z_1c^3 \vdash a^2pb^2b_2b_1y_1c^4 \models a^2pb^4qc^4.$$

In general the string $a^{2k}pb^{2n}qc^{2n}$ can be derived if we apply the rule (y_1, b_2z_2) n times and the rule (x_1b_2, u_2b_2) k times.

§ 2. The algorithm of the syntactic analysis

All grammars $G \in \mathrm{RSG}_1$ can be reduced such that the right-hand sides of the rules will contain no more than two symbols.

DEFINITION 8. A grammar $G \in \mathrm{RSG}_n$ is said to be in reduced form if the length of the right-hand side of each rule of G is not greater than $n+1$.

THEOREM 1. *To each grammar* $G \in \mathrm{RSG}_n$ *for* $n \geqq 1$ *there exists a* G^* *such that* $L_G = L_{G^*}$ and G^* *is in reduced form.*

PROOF. Each rule whose right-hand side is longer than $n+1$ is of the form $(qY, r_1r_2 \ldots r_mY)$ where $m > 1$. We may replace this rule by m new rules of the form (qY, u_mY), $(u_m, u_{m-1}r_m)$, ..., (u_3, u_2r_3), (u_2, r_1r_2), where the symbols $u_2, \ldots, u_m$ are newly introduced auxiliary symbols ($u_i \notin I$, but $u_i \in I^*$). Thus, we can construct another grammar G^* that generates the same language and is in reduced form since $|Y| \leqq n$.

REMARK. It may be observed that the maximum length of the right-hand sides of the rules of a context-free grammar can be reduced to 2 only, though $|Y|=0$ in that case [6].

The algorithm developed for the syntactic analysis in the class RSL_1 makes use of the reduced form of the grammar, which means that the grammar G must be transformed first into some G^* according to Theorem 1. Therefore, we may assume that for every $L_G \in \mathrm{RSL}_1$ each rule of G has one of the following three forms

1. (x, y)
2. (x, yz)
3. (xt, yt).

According to these schemata the rules can be divided into three groups P_1, P_2 and P_3, and they can be represented by three functions F_1, F_2 and F_3, where $x_i = F_1(y_i)$ iff $(x_i, y_i) \in P_1$, $x_j = F_2(y_j, z_j)$ iff $(x_j, y_j z_j) \in P_2$, and $x_k = F_3(y_k, t_k)$ iff $(x_k t_k, y_k t_k) \in P_3$.

The right-hand sides of the rules of P_1, P_2 and P_3 will specify the sets R_1, R_2 and R_3, respectively and the functions F_1, F_2 and F_3 are defined on them.

Let us denote the string to be analysed by $x_1 x_2 \ldots x_n$, where $x_i \in T$ for $1 \leqq i \leqq n$.

The algorithm $\mathfrak{A}$ of the analysis makes use of a push-down-like store, where the processed part of the input string will be stored. The symbols of the input string will be read into this store one by one strictly from left to right, but not only the top of this store will be operated on. The contents of this store will be denoted by $v_1 v_2 \ldots v_m$, where $v_i \in T \cup I$ for $1 \leqq i \leqq m$.

The algorithm $\mathfrak{A}$ of the syntactic analysis is represented by the flow-chart in Figure 2. The subscripts here i, j and k have nothing to do with those occurring in the above definitions of the functions F_1, F_2 and F_3, which are considered here as already given parameters of the algorithm.

The algorithm works in the following way. It performs always the leftmost replacement (or at least one of them). This is achieved by a reverse scan whenever a new symbol is placed into the push-down store, since this makes it possible that another replacement could take place below the top by applying a rule whose right-hand side terminates just in the new symbol etc. New symbols are added to the contents of the push-down store only if the string contained there so far is no more reducible.

If the functions F_1, F_2 and F_3 are unique then this algorithm is deterministic. But in any case it cannot handle all the possibilities that may result from the different choices of the rules applicable to some string. Therefore, our next aim is to select a class of grammars such that for the languages generated by them the algorithm $\mathfrak{A}$ should represent an effective decision procedure. Ambiguous sentences will be analysed, of course, in one way only. To each input string only one syntactic structure (if any) will be assigned.

The algorithm $\mathfrak{A}$ has a very important property with respect to the RSG_1 grammars in reduced form.

LEMMA 1: *For a grammar $G \in \mathrm{RSG}_1$ in reduced form the algorithm $\mathfrak{A}$ performs always the „leftmost" reduction (replacement) in the course of the processing of an arbitrary input string, i.e., the initial part $v_1 v_2 \ldots v_{i-1}$ of the string stored in the push-down store just before the application of a function F is irreducible with respect to G. This means that there is no substring of $v_1 v_2 \ldots v_{i-1}$ that belongs to $R_1 \cup R_2 \cup R_3$.*

PROOF. As the first symbol of the input string is placed into the push-down store, the index i takes the value 1 for which the assertion is trivial. (The string in

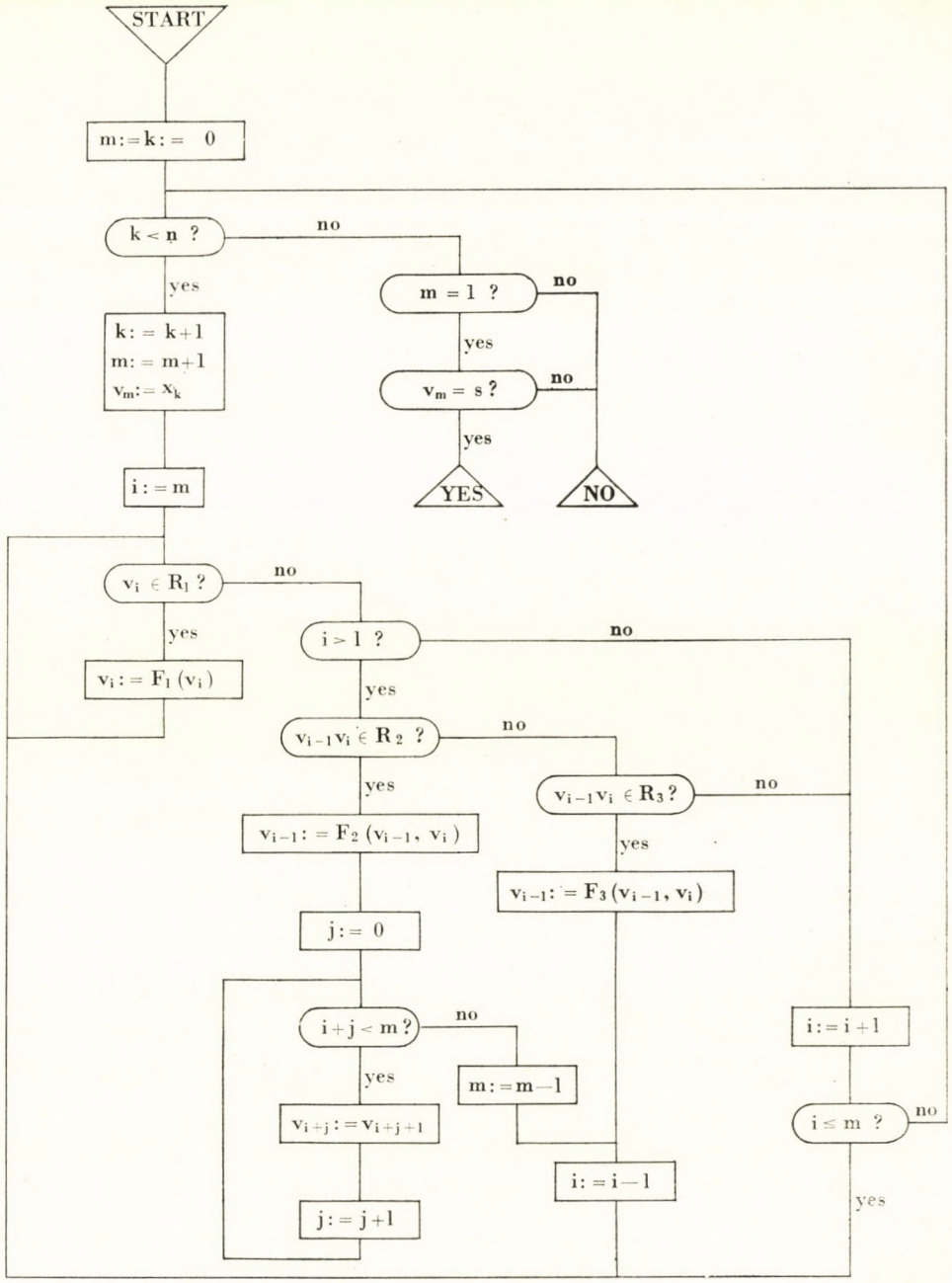

START
m:=k:= 0
k < n ?
no
yes
m = 1 ?
no
yes
k: = k+1
m: = m+1
v_m:= x_k
v_m = s ?
no
yes
YES
NO
i: = m
v_i ∈ R_1 ?
no
yes
i > 1 ?
no
yes
v_i: = F_1 (v_i)
v_{i-1}v_i ∈ R_2 ?
no
yes
v_{i-1}v_i ∈ R_3 ?
no
yes
v_{i-1}: = F_2 (v_{i-1}, v_i)
v_{i-1}: = F_3 (v_{i-1}, v_i)
j:= 0
i+j < m ?
no
yes
i:=i+1
m:=m—1
v_{i+j}:=v_{i+j+1}
i ≤ m ?
no
yes
i:=i—1
j:=j+1

Fig. 2

question is empty.) Then it will reach the value 2, if and only if the single symbol contained in the push-down store cannot be replaced by another one, i.e., if v_i is irreducible.

Now let us suppose that the irreducibility of the string $v_1 v_2 \ldots v_{i-1}$ has been preserved throughout a number of steps performed by the algorithm $\mathfrak{A}$ in the course of processing some input string. We can show that this property will be preserved also in the next step of $\mathfrak{A}$.

Namely, the value of i is decreased by 1 whenever the symbol v_{i-1} is changed. (The symbols $v_1, v_2, \ldots, v_{i-2}$ are left unchanged.) The value of i is increased by 1 only if the whole string $v_1 v_2 \ldots v_{i-1} v_i$ is irreducible. Qu. e. d.

COROLLARY 1. When the algorithm $\mathfrak{A}$ stops then $k=n$ and the whole string $v_1 v_2 \ldots v_m$ stored in the push-down store is irreducible.

Remark: If for an input, string the algorithm $\mathfrak{A}$ stops with $m>1$ or with $v_1 \neq s$ then the input string may however, be derivable in the grammar. Thus, the algorithm $\mathfrak{A}$ may be blocked on a dead-end in spite of the fact that the input string belongs to the language. It may also happen that $\mathfrak{A}$ does not stop at all.

§ 3. The class of left-deterministic languages

DEFINITION 9. Any language generated by some $G \in \mathrm{RSG}_1$ in reduced form is called left-deterministic if the algorithm $\mathfrak{A}$ above represents an effective decision procedure for it, i.e., if the algorithm $\mathfrak{A}$ will stop for any string $U \in \Sigma(T)$ after performing a finite number of replacements and will reduce the string U to the single initial symbol s iff $U \in L_G$.

This definition is of no real value if we do not know how to decide whether a given grammar G generates a left-deterministic language or not. For that purpose we shall give a sufficient condition in Theorem 2. In order to do that we have to define some sets of symbols. So to each symbol $x \in T \cup I$ a set $\Phi(x)$ will be assigned as follows:

DEFINITION 10. Given a grammar $G=(T, I, s, P) \in \mathrm{RSG}_1$ in reduced form and a symbol $x \in T \cup I$ then the set $\Phi(x)$ is defined recursively as follows:

(1) $u \in \Phi(x)$, if $(u, x) \in P$

or $(u, xy) \in P$ for some $y \in T \cup I$

or $(ut, xt) \in P$ for some $t \in T \cup I$.

(2) $v \in \Phi(x)$, if for some $u \in \Phi(x)$

either $(v, u) \in P$

or $(v, uy) \in P$ for some $y \in T \cup I$

or $(vt, ut) \in P$ for some $t \in T \cup I$.

As can be seen from the definition the symbol x need not belong to $\Phi(x)$.

LEMMA 2. *If* $xY \models xZ$ *and* $x \notin \Phi(x)$ *then* $Y \models Z$.

PROOF. This can be proved by induction according to the length of the derivation $xY \models xZ$.

Basis: $xY \vdash xZ$ and $x \notin \Phi(x)$. The rule that is applied here cannot replace the first symbol of the string xY, because in that case both of its sides would have to begin with x, which is excluded by $x \notin \Phi(x)$. Thus, only some symbol of Y can be replaced by that rule, which implies that $Y \vdash Z$.

Induction step: If the derivation $xY \models xZ$ is of length $n+1$ then there exist some x' and Z' for which $xY \models x'Z'$, and $x'Z' \vdash xZ$ where the derivation $xY \models x'Z'$ is of length n. If $x' \neq x$ then $x \in \Phi(x')$ and $x' \in \Phi(x)$ and therefore, $x \in \Phi(x)$ which is again excluded so that $x' = x$ must be the case. Then $Y \models Z'$ by the hypothesis and $Z' \vdash Z$ follows similarly to the basis of the induction which implies that $Y \vdash Z$ qu. e. d.

DEFINITION 11. A sequence of rules $\pi_1, \pi_2, \ldots, \pi_n$ is called a loop, if for some $t \in T \cup I$ and for $i = 1, 2, \ldots, n-1$, $\pi_i = (x_i, x_{i+1})$ or $\pi_i = (x_i t, x_{i+1} t)$ and $x_1 = x_n$. If the rules are all of type 1 (type 3) then the loop is called type 1 (type 3) otherwise it is mixed.

Remark: For a given grammar $G \in \mathrm{RSG}_1$ in reduced form it can be decided (obviously in finite steps) whether the set of rules P contains any loop. (The length of a loop cannot be greater than the total number of the rules in $P_1 \cup P_3$.)

THEOREM 2. *A language is left-deterministic if it can be generated by a grammar* $G = (T, I, s, P) \in \mathrm{RSG}_1$ *in reduced form which satisfies the following five conditions:*

(1) *The right-hand sides of the rules are different from each other.*

(2) *If the right-hand side of a rule consists of only one symbol then this symbol must not occur on the right-hand side of any other rule.*

(3) *If* $(z, xy) \in P$ *and* $y^* \notin \Phi(y)$ *then*
   a) $xy^* \notin R_3$ *and*
   b) *if* $(u, xy^*) \in P$ *then* $z = y$ *and* $u = y^*$.

(4) *If* $(zt, xt) \in P$ *and* $t^* \in \Phi(t)$ *then*
   a) *if* $(u, xt^*) \in P$ *then* $(u, zt^*) \in P$ *and*
   b) *if* $(vt^*, xt^*) \in P$ *and* $v \neq z$ *then* $(vt^*, zt^*) \in P$.

(5) *There is no loop in* P

PROOF. Let the grammar G satisfy these conditions. We have to show first that the algorithm $\mathfrak{A}$ will stop for any $U \in \Sigma(T)$ after a finite number of steps.

This can be shown indirectly. Assuming that $\mathfrak{A}$ does not stop for some $U \in \Sigma(T)$, i.e., it performs infinitely many replacements by applying the rules of G, then it will produce an infinite sequence of strings $U = U_0, U_1, U_2, \ldots$ where $U_{i+1} \vdash U_i$. Since the application of a rule of type 2 (i.e., of the form (z, xy)) decreases the length of the string, these rules can be applied only finite many times. Thus, if the subscript M is great enough then the strings $U_M, U_{M+1}, \ldots$ will be all of the same length. But the set of symbols $T \cup I$ is finite, hence $U_{M+j} = U_M$ must hold for some j, which implies that $U_M \models U_M$.

Let x be the rightmost symbol of the string U_M that is replaced by some other symbol in the course of this derivation. If this is the last symbol of U_M then it can be replaced only at the application of a rule of type 1, say (x, x_1). But at the end

of the derivation $U_M \models U_M$ we must get back x as the last symbol of the string, which implies that a loop of type 1 must exist in P.

If x is not the last symbol of U_M then there is some symbol t next to it which is left unchanged in the course of the derivation $U_M \models U_M$. Now the symbol x can be changed only by the application of a rule of type 1 or else of type 3 with t as the second symbol on its both sides, say (xt, x_1t). But again at the end of the derivation we must get back x in front of t which means that there must be a loop in P.

Thus, if P does not contain any loop then the algorithm $\mathfrak{A}$ will stop for every $U \in \Sigma(T)$. It can be observed that this has been shown independently from Conditions (2)—(4).

Now we can prove that the algorithm $\mathfrak{A}$ will reduce the input string U to the single initial symbol s whenever $U \in L_G$.

Assuming hat $U \in L_G$, there is a derivation $s \models U$ of finite length, say N. This means that there is a sequence of strings $X_0, X_1, \ldots, X_N$ where $X_0 = U$, $X_N = s$ and $X_{i+1} \vdash X_i$ for $0 \leqq i < N$. The algorithm $\mathfrak{A}$ will produce, on the other hand, a sequence of strings $U = U_0, U_1, \ldots, U_M$ where $U_{i+1} \vdash U_i$. Let J denote the highest subscript for which all the relations $X_0 = U_0$, $X_1 = U_1, \ldots, X_J = U_J$ hold. Here we have obviously $0 \leqq J \leqq \min(M, N)$. If $J = N$ then $M = N$ and we have nothing to prove. If $J < N$ then also $J < M$, since the algorithm $\mathfrak{A}$ must not stop at U_J in this case because of Corollary 1 and $X_{J+1} \models X_J = U_J$. Thus, the algorithm $\mathfrak{A}$ performs at least one more replacement producing U_{J+1}, for which $X_{J+1} \neq U_{J+1}$.

Now the hypothetical derivation $s = X_N \vdash \ldots \vdash X_J \vdash \ldots \vdash X_0 = U$ can be modified in such a way that this first deviation of the two sequences will be eliminated, i.e., $X_i = U_i$ will hold for $0 \leqq i \leqq J+1$. The length of the hypothetical derivation will be either left unchanged or increased by one through this modification. By repeating this procedure every deviation can be eliminated throughout the whole derivation $s \models U$.

Let us consider the above reasoning in details. Let the string U_J be denoted by $AxyB$ where xy corresponds to the substring $v_{i-1}v_i$ (see Figure 2) just before $\mathfrak{A}$ performs the replacement that yields U_{J+1}. The following cases must be considered:

Case 1: $\mathfrak{A}$ *applies a rule of type* 1 *to* Axy.

Let this rule be (z, y), hence $U_{J+1} = AxzB$ and $X_{J+1} = Axy'B'$, according to Condition (2) and Lemma 1. Here we have $y'B' \vdash yB$ and $y' \in \Phi(y)$ unless $y' = y$ and $B' \vdash B$. In the former case y' should be equal to z according to Conditions (1) and (2) which implies that $B' = B$, but this is excluded since $X_{J+1} \neq U_{J+1}$. In the latter case let us follow the sequence $X_{J+1}, X_{J+2}, \ldots$ up to the point where the indicated occurrence of y will be replaced by another symbol. Because of Condition (1) this may happen only by applying the same rule that is applied by $\mathfrak{A}$ to Axy. This means that the hypothetical derivation must be of the form

$$s \models AxzB^* \vdash AxyB^* \models AxyB \models U$$

where $B^* \models B$. But this can be modified simply by changing the order of the replacements such that we obtain.

$$s \models AxzB^* \models AxzB \vdash AxyB \models U$$

which eliminates the deviation in question.

Case 2: $\mathfrak{A}$ *applies a rule of type* 2 *to* Axy.

This rule will be of the form (z, xy) and $U_{J+1} = AzB$ while $X_{J+1} = Axy'B'$ according to Condition (1). Now let us follow the sequence $X_{J+1}, X_{J+2}, \ldots$ up to the point where the indicated occurrence of x will be involved in a replacement. This may happen only by applying a rule of type 2 or type 3, for type 1 rules are excluded by Condition (2). The string $AxyB$ is reduced in the meantime to some Axy^*B^* where $y^*B^* \models yB$.

If $y^* \notin \Phi(y)$ then $y^* = y$ must be the case and $B^* \models B$ according to Lemma 2. In this case the rule in question must be the same as the one applied by $\mathfrak{A}$ to Axy, so that the hypothetical derivation is of the form

$$s \models AzB^* \vdash AxyB^* \models AxyB \models U$$

which can be modified into

$$s \models AzB^* \models AzB \vdash AxyB \models U$$

that eliminates the deviation in question.

If $y^* \in \Phi(y)$ then, according to Condition (3), only type 2 rules are to be considered. Moreover, the rule applied in the hypothetical derivation must be of the form (y^*, xy^*) and the one applied by $\mathfrak{A}$ to Axy must be of the form (y, xy). Thus, the hypothetical derivation

$$s \models Ay^*B^* \vdash Axy^*B^* \models AxyB \models U$$

can be modified such that

$$s \models Ay^*B^* \models AyB \vdash AxyB \models U$$

Case 3: $\mathfrak{A}$ *applies a rule of type* 3 *to* Axy.

Let the rule in question be (zy, xy) hence $U_{J+1} = AzyB$ while $X_{J+1} = Axy'B'$. Let the sequence $X_{J+1}, X_{J+2}, \ldots$ be followed again until the indicated occurrence of x will be involved in a replacement. The string $AxyB$ is meanwhile reduced to some Axy^*B^* where $y^*B^* \models yB$.

If $y^* \notin \Phi(y)$, then by a similar argument as given in Case 2 the hypothetical derivation must be of the form

$$s \models AzyB^* \vdash AxyB^* \models AxyB \models U$$

which can be changed into

$$s \models AzyB^* \models AzyB \vdash AxyB \models U$$

that eliminates the deviation under discussion.

If $y^* \in \Phi(y)$ then according to Condition (4) two subcases arise.

3/a. In the first subcase a rule of the form (u, xy^*) is applied in the hypothetical derivation. According to Condition (4) there exists also a rule of the form (u, zy^*). Thus, the hypothetical derivation is of the form

$$s \models AuB^* \vdash Axy^*B^* \models AxyB \models U$$

that can be replaced by

$$s \models AuB^* \vdash Azy^*B^* \models AzyB \vdash AxyB \models U$$

which removes the deviation in question but the length of the hypothetical derivation is increased by one.

3/b. In the second subcase a rule of the form (vy^*, xy^*) is applied in the hypothetical derivation. According to Condition (4) there exists now a rule of the form (vy^*, xy^*) unless $v=z$. Thus, the hypothetical derivation is of the form

$$s \models Avy^* B^* \vdash Axy^* B^* \models AxyB \models U$$

that can be replaced, by

$$s \models Azy^* B^* \models AzyB \vdash AxyB \models U$$

if $v=z$, otherwise it can be replaced by

$$s \models Avy^* B^* \vdash Azy^* B^* \models AzyB \vdash AxyB \models U.$$

In both cases the deviation under discussion is removed, only the length of the derivation is increased by one in the latter case.

Thus, every deviation can be successively eliminated until $J=M=N$ holds. Hence the proof of Theorem 2 is complete.

EXAMPLE 2. Let the *unsigned numbers*, *variables* and *function designators* be all denoted by v. The following set of rules may serve as the definition of the *primary* expression in the sense of ALGOL 60 [12].

$$\begin{array}{llll} v \to (e & a \to v+ & b \to v\times & p \to v\uparrow \\ e \to v) & a \to v- & b \to v/ & \\ & & b \to v\div & \\ e \to ae & a \to aa & & \\ e \to be & a \to ba & b \to bb & \\ e \to pe & a \to pa & b \to pb & p \to pp \end{array}$$

Here the rules are described with the aid of the arrow $\to$ since the parantheses are symbols of the language. It can be easily shown that the conditions of Theorem 2 are satisfied by these rules. This is obvious for the first two conditions and also for the last two ones because all the rules are of type 2. Condition (3) is concerned with the sets Φ assigned to the symbols that occur as the last symbol on the right-hand side of a rule of type 2. It can immediately be seen that all these sets are empty except for three, i.e., only $\Phi(a)$, $\Phi(b)$ and $\Phi(p)$ are non-empty. Namely

$$\Phi(a) = \{e, a\}$$

$$\Phi(b) = \{e, a, b\}$$

$$\Phi(p) = \{e, a, b, p\}.$$

Now, for instance, the rules $a \to aa$ and $e \to ae$ where $e \in \Phi(a)$ satisfy Condition 3/b and the same holds for all other pairs of rules in question.

This example shows the application of Theorem 2 to a context-free language, but it can be applied also to non-context-free languages.

THEOREM 3. *There are left-deterministic languages that cannot be generated by a context-free grammar.*

PROOF. The language $a^{2k}pb^{2n}qc^{2n}$ where $k \leqq n$ is left-deterministic. In order to prove this two more rules, (y_1, b_1z_2) and (y_2, b_2z_1), must be added to those given in Example 1. It can be easily shown that the introduction of these two redundant rules has no influence on the language generated by the grammar, though, the grammar becomes ambiguous. The fulfilment of the conditions of Theorem 2 for that augmented grammar can be verified without difficulty by the reader.

In virtue of Theorem 3 the class of left-deterministic languages is not contained in the class of context-free languages, but it is hoped that most programming languages are left-deterministic and thus can be analysed by the algorithm $\mathfrak{A}$. It is, however, an open question whether the grammar to be constructed for the application of $\mathfrak{A}$ could be unambiguous. It is ambiguous in our both examples, though, the corresponding languages can be generated also by unambiguous grammars.

It should be mentioned that if we restrict ourselves only to context-free grammars then the algorithm $\mathfrak{A}$ might be much simpler, but it is quite possible that a context-free language requires a right sensitive grammar for the algorithm $\mathfrak{A}$.

REFERENCES

[1] CHOMSKY, N.: On certain formal properties of grammars, *Information and Control* **2** (1959) 137—167.
[2] CHOMSKY, N.: Formal properties of grammars, *Handbook of Mathematical Psychology*, Vol. 2. (1963) 323—418.
[3] DÖMÖLKI, B.: An algorithm for syntactic analysis, *Computational Linguistics* **3** (1964) 29—46.
[4] DÖMÖLKI, B.: Voprosy syntaxicheskovo analyza dlja formalnyh jazykov, *Computational Linguistics* **5** (1967) 41—93.
[5] FLOYD, R.: Syntactic analysis and operator precedence, *J. Assoc. Comput. Mach.* **10** (1963) 316—333.
[6] PAUL, M.: A general processor for certain formal languages, *Symbolic Languages in Data Processing, ICC Symposium Rome*, 1962, 65—74.
[7] EICKEL, J., PAUL, M., BAUER, F. and SAMELSON, K.: A Syntax controlled generator of formal language processors, *Comm. ACM* **6** (1963) 451—455.
[8] KNUTH, D.: On the translation of Languages from left to right, *Information and Control* **8** (1965) 607—639.
[9] RÉVÉSZ, GY.: An efficient syntactic analyser of certain formal languages, (to appear in *Math. Systems Theory*).
[10] FARKAS, E.: Matematikai grammatikák (in Hungarian), M. Thesis, Eötvös L. University, Budapest, 1968.
[11] BAR-HILLEL, Y., PERLES, M., and SHAMIR, E.: On formal properties of simple phrase structure grammars, *Zeitschrift für Phonetik, Sprachwissenschaft und Kommunikationsforschung* **14** (1961) 143—172.
[12] NAUR, P. (editor): Revised Report on the Algorithmic Language ALGOL 60, *Comm. ACM* **6** (1963) 1—17.

Ministry of Finance, Budapest

(Received April 18, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 279—289.

ЗАМЕЧАНИЯ ОБ ОЦЕНКЕ ПОГРЕШНОСТИ ИНТЕРПОЛИРОВАНИЯ

O. KIS

§ 1. Введение

Обозначим через $g(x)$ любую 2π-периодическую непрерывную функцию и через $\omega(g, t)$ ее модуль непрерывности. Пусть тригонометрический многочлен n-ого порядка $T_n(g, x)$ совпадает с $g(x)$ в узлах

$$x_k = \frac{2k+1}{2n+1}\pi \qquad (k = 0, 1, \ldots, 2n). \tag{1.1}$$

Обозначим через λ_n число Лебега тригонометрического интерполирования по узлам x_k. Основная цель настоящей заметки доказать, что

$$|T_n(g, x) - g(x)| \leqq \left(\frac{1}{2}\lambda_n + \frac{3}{2} + \frac{4}{\pi}\right)\omega\left(g, \frac{2\pi}{2n+1}\right) \qquad (n = 0, 1, 2, \ldots). \tag{1.2}$$

Это неравенство мы докажем в § 2.*

Заметим, что более грубая оценка

$$|T_n(g, x) - g(x)| \leqq (\lambda_n + 1)\,\omega\left(g, \frac{\pi}{n+1}\right) \qquad (n = 0, 1, 2, \ldots) \tag{1.3}$$

является следствием хорошо известного неравенства

$$|T_n(g, x) - g(x)| \leqq (\lambda_n + 1)\,E_n(g) \qquad (n = 0, 1, 2, \ldots), \tag{1.4}$$

где $E_n(g)$ — наилучшее приближение функции $g(x)$ тригонометрическими многочленами n-ого порядка, и соотношения

$$E_n(g) \leqq \omega\left(g, \frac{\pi}{n+1}\right) \qquad (n = 0, 1, 2, \ldots), \tag{1.5}$$

которое Н. П. Корнейчук доказал в работе [4].

Заметим также, что для тех функций $g(x)$, модуль непрерывности которых есть выпуклая кверху функция, выполняется более точное неравенство

$$|T_n(g, x) - g(x)| \leqq \left(\frac{1}{2}\lambda_n + \frac{1}{2}\right)\omega\left(g, \frac{\pi}{n+1}\right) \qquad (n = 0, 1, 2, \ldots). \tag{1.6}$$

* Известно, что $\lambda_n \leqq \frac{2}{\pi}\ln n + c$, где c — положительное число.

Этот факт является следствием неравенства (1. 4) и оценки

$$E_n(g) \leqq \frac{1}{2}\,\omega\left(g, \frac{\pi}{n+1}\right) \qquad (n = 0, 1, 2, \ldots), \tag{1. 7}$$

которую Н. П. Корнейчук доказал в статье [3].

Возможно, что справедливо более точное чем (1. 2) неравенство

$$|T_n(g, x) - g(x)| \leqq \left(\frac{1}{2}\,\lambda_n + \frac{1}{2}\right)\omega\left(g, \frac{2\pi}{2n+1}\right) \qquad (n = 0, 1, 2, \ldots), \tag{1. 8}$$

однако эту оценку уже нельзя улучшить, так как существует зависящая от n функция $g_n(x)$, для которой имеет место равенство

$$T_n(g_n, 0) - g_n(0) = \left(\frac{1}{2}\,\lambda_n + \frac{1}{2}\right)\omega\left(g_n, \frac{2\pi}{2n+1}\right), \tag{1. 9}$$

где n — любое фиксированное натуральное число. Такую функцию мы построим в §3.

Равенство (1. 9) выполняется лишь для одного значения n. Однако можно построить такую функцию $g(x)$, для которой

$$T_n(g, 0) - g(0) > (1-\varepsilon)\left(\frac{1}{2}\,\lambda_n + \frac{1}{2}\right)\omega\left(g, \frac{2\pi}{2n+1}\right) \qquad (n = n_1, n_2, n_3, \ldots), \tag{1. 10}$$

где ε сколь угодно малое положительное число и $n_1, n_2, n_3, \ldots$ возрастающая последовательность натуральных чисел. При этом функцию $g(x)$ можно выбрать так, чтобы выполнялось условие

$$\omega\left(g, \frac{2\pi}{2n+1}\right) = \omega\left(\frac{2\pi}{2n+1}\right) \qquad (n = n_1, n_2, n_3, \ldots), \tag{1. 11}$$

где $\omega(t)$ любая такая функция, которая удовлетворяет условиям

$$\omega(t) > 0 \qquad (t > 0), \tag{1. 12}$$

$$\lim_{t=+0} \omega(t) = 0, \tag{1. 13}$$

$$\lim_{t=+0} \frac{t}{\omega(t)} = 0. \tag{1. 14}$$

Этот факт будет доказан в § 4.

Для тригонометрических интерполяционных многочленов n-ого порядка, не содержащих члена $a_n \cos nx$ или $b_n \sin nx$ и совпадающих с функцией $g(x)$ в узлах

$$x_k^* = \frac{2k-1}{2n}\,\pi \qquad (k = 1, 2, \ldots, 2n) \tag{1. 15}$$

или

$$x_k^{**} = \frac{k\pi}{n} \qquad (k = 1, 2, \ldots, 2n), \tag{1. 16}$$

можно доказать результаты, аналогичные тем, которые мы сформулировали для тригонометрических интерполяционных многочленов $T_n(g, x)$. Эти результаты мы приведем в § 5.

Из теорем о тригонометрическом интерполировании следуют аналогичные теоремы о четном тригонометрическом интерполировании. Мы сформулируем их в § 6.

Следствиями этих результатов являются аналогичные теоремы о Лагранжевом интерполировании. Мы приведем их в § 7.

§ 2. Доказательство неравенства (1. 2)

Так как

$$|T_n(g,x)-g(x)| \leqq \left|\frac{1}{2}T_n(g,x)+\frac{1}{2}T_n\left(g,x-\frac{2\pi}{2n+1}\right)-g(x)\right| + \left|\frac{1}{2}T_n(g,x)-\frac{1}{2}T_n\left(g,x-\frac{2\pi}{2n+1}\right)\right|, \tag{2.1}$$

то неравенство (1. 2) будет доказано, если мы убедимся в том, что

$$\left|\frac{1}{2}T_n(g,x)+\frac{1}{2}T_n\left(g,x-\frac{2\pi}{2n+1}\right)-g(x)\right| \leqq \left(\frac{3}{2}+\frac{4}{\pi}\right)\omega\left(g,\frac{2\pi}{2n+1}\right) \tag{2.2}$$

и

$$\left|\frac{1}{2}T_n(g,x)-\frac{1}{2}T_n\left(g,x-\frac{2\pi}{2n+1}\right)\right| \leqq \frac{1}{2}\lambda_n\omega\left(g,\frac{2\pi}{2n+1}\right). \tag{2.3}$$

Первое из этих утверждений по существу доказал С. Н. Бернштейн в работе [2]. Точнее говоря, в [2] доказано, что

$$\left|\frac{1}{2}T_n(g,x)+\frac{1}{2}T_n\left(g,x-\frac{2\pi}{2n+1}\right)\right| \leqq \frac{4}{\pi}\max|g(x)|. \tag{2.4}$$

Однако, дословно повторяя рассуждения, с помощью которых С. Н. Бернштейн в работе [1] оценил разность

$$\left|\frac{1}{2}S_n(g,x)+\frac{1}{2}S_n\left(g,x-\frac{2\pi}{2n+1}\right)-g(x)\right|,$$

где $S_n(x)$ отрезок ряда Фурье функции $g(x)$, и обозначая через $U(x)$ тригонометрический многочлен n-ого порядка, наименее уклоняющийся от функции $g(x)$, получаем:

$$\left|\frac{1}{2}T_n(g,x)+\frac{1}{2}T_n\left(g,x-\frac{2\pi}{2n+1}\right)-g(x)\right| \leqq \left|\frac{1}{2}T_n(g-U,x)+\frac{1}{2}T_n\left(g-U,x-\frac{2\pi}{2n+1}\right)\right| + \left|\frac{1}{2}U(x)-\frac{1}{2}g(x)\right| + \left|\frac{1}{2}U_n\left(x-\frac{2\pi}{2n+1}\right)-\frac{1}{2}g\left(x-\frac{2\pi}{2n+1}\right)\right| + \left|\frac{1}{2}g\left(x-\frac{2\pi}{2n+1}\right)-\frac{1}{2}g(x)\right| \leqq \left(1+\frac{4}{\pi}\right)E_n(g)+\frac{1}{2}\omega\left(g,\frac{2\pi}{2n+1}\right). \tag{2.5}$$

Чтобы получить неравенство (2. 2), нам остается сослаться на формулу (1. 5), и очевидное соотношение

$$\omega\left(g, \frac{\pi}{n+1}\right) \leqq \omega\left(g, \frac{2\pi}{2n+1}\right). \tag{2. 6}$$

Чтобы доказать (2. 3), воспользуемся тождеством

$$T_n(g, x) = \sum_{k=0}^{2n} g(x_k)\, t_n(x - x_k), \tag{2. 7}$$

где

$$t_n(x) = \frac{\sin \frac{2n+1}{2} x}{(2n+1) \sin \frac{x}{2}}. \tag{2. 8}$$

Очевидно

$$T_n\left(g, x - \frac{2\pi}{2n+1}\right) = \sum_{k=0}^{2n} g(x_k)\, t_n(x - x_{k+1}) = \sum_{k=1}^{2n+1} g(x_{k-1})\, t_n(x - x_k) = \tag{2. 9}$$

$$= \sum_{k=0}^{2n} g(x_{k-1})\, t_n(x - x_k).$$

Поэтому

$$\left| \frac{1}{2} T(g, x) - \frac{1}{2} T_n\left(g, x - \frac{2\pi}{2n+1}\right) \right| \leqq \frac{1}{2} \sum_{k=0}^{2n} |g(x_k) - g(x_{k-1})| \cdot |t_n(x - x_k)| \leqq \tag{2. 10}$$

$$\leqq \frac{1}{2} \omega\left(g, \frac{2\pi}{2n+1}\right) \sum_{k=0}^{2n} |t_n(x - x_k)| \leqq \frac{1}{2} \omega\left(g, \frac{2\pi}{2n+1}\right) \lambda_n,$$

что и требовалось доказать.

§ 3. Доказательство равенства (1.9)

Определим 2π-периодическую непрерывную функцию $g_n(x)$ следующим образом: пусть

$$g_n(0) = -1, \quad g_n(x_k) = (-1)^k \qquad (k = 0, 1, \ldots, 2n) \tag{3. 1}$$

и пусть $g_n(x)$ линейна на отрезках

$$[0, x_0], [x_0, x_1], [x_1, x_2], \ldots, [x_{2n-1}, x_{2n}], [x_{2n}, 2\pi].$$

Ввиду (2. 7) и (3. 1)

$$T_n(g_n, 0) - g_n(0) = \sum_{k=0}^{2n} (-1)^k t_n(-x_k) + 1. \tag{3. 2}$$

Из (2. 8) и (1. 1) видно, что

$$(3.3)\qquad \sum_{k=0}^{2n}{}'(-1)^k t_n(-x_k) = \frac{1}{2n+1}\sum_{k=0}^{2n}{}'\frac{1}{\sin\frac{2k+1}{2n+1}\frac{\pi}{2}}.$$

В работе [5] доказано, что

$$(3.4)\qquad \lambda_n = \frac{1}{2n+1}\sum_{k=0}^{2n}{}'\frac{1}{\sin\frac{2k+1}{2n+1}\frac{\pi}{2}}.$$

Поэтому

$$(3.5)\qquad T_n(g_n, 0) - g(0) = \lambda_n + 1.$$

Очевидно

$$(3.6)\qquad \omega\left(g_n, \frac{2\pi}{2n+1}\right) = 2.$$

Из двух последних неравенств следует (1. 9), что и требовалось доказать

§ 4. Доказательство соотношений (1.11) и (1.10)

Пусть $0<\varepsilon<1$ и натуральные числа n_i удовлетворяют условиям

$$(4.1)\qquad 2n_{i+1}+1 \geqq 2(2n_i+1) \qquad (i = 1, 2, 3, \ldots),$$

$$(4.2)\qquad \omega_{i+1} \leqq \frac{\varepsilon}{7}\omega_i \qquad (i = 1, 2, 3, \ldots),$$

$$(4.3)\qquad (2n_{i+1}+1)\omega_{i+1} \geqq \frac{14}{\varepsilon}(2n_i+1)\omega_i \qquad (i = 1, 2, 3, \ldots),$$

где

$$(4.4)\qquad \omega_i = \omega\left(\frac{2\pi}{2n_i+1}\right) \qquad (i = 1, 2, 3, \ldots).$$

Это возможно, так как мы предположили, что выполняются условия (1. 13) и (1. 14).

Определим числа c_i как решение бесконечной системы линейных уравнений

$$(4.5)\qquad \sum_{k=1}^{i-1}\frac{2\omega_k}{\omega_i}\frac{2n_k+1}{2n_i+1}c_k + c_i + \sum_{k=i+1}^{\infty}\frac{\omega_k}{\omega_i}c_k = 1 \qquad (i = 1, 2, 3, \ldots).$$

Эта система имеет единственное решение, так как ввиду (4. 3) и (4. 2)

$$(4.6)\qquad \sum_{k=1}^{i-1}\frac{2\omega_k}{\omega_i}\frac{2n_k+1}{2n_i+1} + \sum_{k=i+1}^{\infty}\frac{\omega_k}{\omega_i} \leqq \sum_{k=1}^{i-1}\left(\frac{\varepsilon}{7}\right)^{i-k} + \sum_{k=i+1}^{\infty}\left(\frac{\varepsilon}{7}\right)^{k-i} < 2\sum_{k=1}^{\infty}\left(\frac{\varepsilon}{7}\right)^k =$$

$$= \frac{2\varepsilon}{7-\varepsilon} < \frac{\varepsilon}{3} < 1.$$

Заметим, что

$$1-\frac{\varepsilon}{3} < c_i < 1 \qquad (i = 1, 2, 3, \ldots). \tag{4.7}$$

Положим, наконец,

$$g(x) = \frac{1}{2}\sum_{k=1}^{\infty} c_k \omega_k g_{n_k}(x). \tag{4.8}$$

Ряд справа равномерно сходится ввиду (4. 7), (4. 2) и неравенства

$$|g_n(x)| \leqq 1 \qquad (n = 0, 1, 2, \ldots), \tag{4.9}$$

которое является очевидным следствием определения функций $g_n(x)$, приведенного в предыдущем параграфе. Так как эти функции 2π-периодичны и непрерывны, то и их сумма $g(x)$ также 2π-периодична и непрерывна.

Покажем, что для нее имеет место равенство (1. 11). Очевидно

$$\omega(g_n, t) = \begin{cases} 2\dfrac{2n+1}{\pi}\, t, & \text{если} \quad 0 \leqq t \leqq \dfrac{\pi}{2n+1}, \\ 2, & \text{если} \quad t \geqq \dfrac{\pi}{2n+1}. \end{cases} \tag{4.10}$$

Ввиду (4. 1)

$$\omega\left(g_{n_k}, \frac{2\pi}{2n_i+1}\right) = \begin{cases} 4\dfrac{2n_k+1}{2n_i+1}, & \text{если} \quad i > k, \\ 2, & \text{если} \quad i \leqq k. \end{cases} \tag{4.11}$$

Отсюда, из (4. 8) и из (4. 5) получаем:

$$\omega\left(g, \frac{2\pi}{2n_i+1}\right) = \sum_{k=1}^{i-1} 2\frac{2n_k+1}{2n_i+1}\omega_k c_k + \sum_{k=i}^{\infty} \omega_k c_k = \omega_i, \tag{4.12}$$

что и требовалось доказать.

Докажем неравенство (1. 10). Очевидно

$$T_{n_i}(g, 0) - g(0) = \frac{1}{2}\sum_{k=1}^{\infty} c_k \omega_k [T_{n_i}(g_{n_k}, 0) - g_{n_k}(0)] \geqq \tag{4.13}$$

$$\geqq \frac{1}{2} c_i \omega_i [T_{n_i}(g_{n_i}, 0) - g_{n_i}(0)] - \frac{1}{2}\sum_{\substack{k=1 \\ k \neq i}}^{\infty} \omega_k c_k |T_{n_i}(g_{n_k}, 0) - g_{n_k}(0)|.$$

Ввиду (1. 3) и (4. 11)

$$|T_{n_i}(g_{n_k}, 0) - g_{n_k}(0)| \leqq \begin{cases} 4(\lambda_{n_i}+1)\dfrac{2n_k+1}{2n_i+1}, & \text{если} \quad i > k, \\ 2(\lambda_{n_i}+1), & \text{если} \quad i < k. \end{cases} \tag{4.14}$$

Поэтому и в силу (1. 9)

$$(4.15)\qquad T_{n_i}(g,0)-g(0)\leqq\frac{1}{2}(\lambda_{n_i}+1)\left(c_i\omega_i-\sum_{k=1}^{i-1}4c_k\omega_k\frac{2n_k+1}{2n_i+1}-\sum_{k=i+1}^{\infty}2c_k\omega_k\right).$$

На основании (4. 7) пишем здесь $\left(1-\frac{\varepsilon}{3}\right)\omega_i$ вместо $c_i\omega_i$. Кроме того, воспользовавшись неравенствами (4. 7) и (4. 6), получаем:

$$(4.16)\qquad \sum_{k=1}^{i-1}4c_k\omega_k\frac{2n_k+1}{2n_i+1}+\sum_{k=i+1}^{\infty}2c_k\omega_k\leqq 2\omega_i\left(\sum_{k=1}^{i-1}2\frac{\omega_k}{\omega_i}\frac{2n_k+1}{2n_i+1}+\sum_{k=i+1}^{\infty}\frac{\omega_k}{\omega_i}\right)\leqq\frac{2}{3}\varepsilon\omega_i.$$

Поэтому

$$(4.17)\qquad T_{n_i}(g,0)-g(0)\leqq\frac{1-\varepsilon}{2}(\lambda_{n_i}+1)\omega_i,$$

что и требовалось доказать.

§ 5. Оценка погрешности тригонометрического интерполирования с четным числом узлов

Обозначим через $T_n^*(g,x)$ тригонометрический интерполяционный многочлен вида

$$a_0+\sum_{k=1}^{n-1}(a_k\cos kx+b_k\sin kx)+b_n\sin nx,$$

совпадающий с 2π-периодической функцией $g(x)$ в узлах (1. 15), и пусть λ_n^* соответствующие числа Лебега. Также, как было доказано неравенство (1. 2), можно показать, что

$$(5.1)\qquad |T_n^*(g,x)-g(x)|\leqq\left(\frac{1}{2}\lambda_n^*+\frac{3}{2}+\frac{4}{\pi}\right)\omega\left(g,\frac{\pi}{n}\right)\qquad(n=1,2,3,\ldots).$$

Как и в § 3, можно построить функцию $g_n(x)$, для которой

$$(5.2)\qquad T_n^*(g_n,0)-g_n(0)=\left(\frac{1}{2}\lambda_n^*+\frac{1}{2}\right)\omega\left(g_n,\frac{\pi}{n}\right).$$

Рассуждая также, как в § 4, можно найти функцию $g(x)$, для которой

$$(5.3)\qquad \omega\left(g,\frac{\pi}{n}\right)=\omega\left(\frac{\pi}{n}\right)\qquad(n=n_1,n_2,n_3,\ldots)$$

и

$$(5.4)\qquad T_n^*(g,0)-g(0)\leqq(1-\varepsilon)\left(\frac{1}{2}\lambda_n^*+\frac{1}{2}\right)\omega\left(g,\frac{\pi}{n}\right)\qquad(n=n_1,n_2,n_3,\ldots).$$

Аналогичные утверждения справедливы и для тригонометрического многочлена $T_n^{**}(g, x)$ вида

$$a_0 + \sum_{k=1}^{n-1} (a_k \cos kx + b_k \sin kx) + a_n \cos nx,$$

совпадающего с $g(x)$ в узлах (1. 16), только в формулах, соответствующих равенству (5. 2) и неравенству (5. 4), следует писать π вместо 0. (Числа Лебега для узлов (1. 15) и (1. 16) очевидно совпадают.)

§ 6. Оценка погрешности четного тригонометрического интерполирования

Обозначим через $C_{n-1}^*(g, x)$ четный тригонометрический интерполяционный многочлен $n-1$-ого порядка, совпадающий в узлах

$$x_k^* = \frac{2k-1}{2n}\pi \qquad (k = 1, 2, \ldots, n) \tag{6. 1}$$

с четной 2π-периодической функцией $g(x)$, а через Λ_{n-1}^* соответствующие числа Лебега. Из (5. 1) следует неравенство

$$|C_{n-1}^*(g, x) - g(x)| \leqq \left(\frac{1}{2}\Lambda_{n-1}^* + \frac{3}{2} + \frac{4}{\pi}\right)\omega\left(g, \frac{\pi}{n}\right) \qquad (n = 1, 2, 3, \ldots), \tag{6. 2}$$

так как очевидно

$$C_{n-1}^*(g, x) = T_n^*(g, x) \qquad (n = 1, 2, 3, \ldots) \tag{6. 3}$$

и

$$\Lambda_{n-1}^* = \lambda_n^* \qquad (n = 1, 2, 3, \ldots) \tag{6. 4}$$

(это по существу доказано в [5].) Следствием формулы (5. 2) является равенство

$$C_{n-1}^*(g_n, 0) - g_n(0) = \left(\frac{1}{2}\Lambda_{n-1}^* + \frac{1}{2}\right)\omega\left(g_n, \frac{\pi}{n}\right), \tag{6. 5}$$

а из неравенства (5. 4) получаем:

$$C_{n-1}^*(g, 0) - g(0) \geqq (1-\varepsilon)\left(\frac{1}{2}\Lambda_{n-1}^* + \frac{1}{2}\right)\omega\left(g, \frac{\pi}{n}\right) \qquad (n = n_1, n_2, n_3, \ldots). \tag{6. 6}$$

Аналогичным образом для четного тригонометрического многочлена n-ого порядка $C_n^{**}(g, x)$, совпадающего с $g(x)$ в узлах

$$x_k^{**} = \frac{k\pi}{n} \qquad (k = 0, 1, \ldots, n), \tag{6. 7}$$

выполняется неравенство

$$|C_n^{**}(g, x) - g(x)| \leqq \left(\frac{1}{2}\Lambda_n^{**} + \frac{3}{2} + \frac{4}{\pi}\right)\omega\left(g, \frac{\pi}{n}\right) \qquad (n = 1, 3, 5, \ldots), \tag{6. 8}$$

так как

$$C_n^{**}(g, x) = T_n^{**}(g, x) \qquad (n = 1, 2, 3, \ldots) \tag{6.9}$$

и, как доказано в [5],

$$\Lambda_n^{**} = \lambda_n^{*} \qquad (n = 1, 3, 5, \ldots). \tag{6.10}$$

При этом для некоторой функции $g_n(x)$

$$C_n^{**}\left(g_n, \frac{\pi}{2}\right) - g_n\left(\frac{\pi}{2}\right) = \left(\frac{1}{2}\Lambda_n^{**} + \frac{1}{2}\right)\omega\left(g_n, \frac{\pi}{n}\right) \tag{6.11}$$

и для некоторой функции $g(x)$ выполняется условие (5. 3) и

$$C_n^{**}\left(g, \frac{\pi}{2}\right) - g_n\left(\frac{\pi}{2}\right) \geqq (1-\varepsilon)\left(\frac{1}{2}\Lambda_n^{**} + \frac{1}{2}\right)\omega\left(g, \frac{\pi}{n}\right) \quad (n = n_1, n_2, n_3, \ldots). \tag{6.12}$$

Наконец, для четного тригонометрического многочлена n-ого порядка $C_n(g, x)$, совпадающего с любой четной 2π-периодической непрерывной функцией $g(x)$ в узлах

$$x_k = \frac{2k+1}{2n+1}\pi \qquad (k = 0, 1, \ldots, n), \tag{6.13}$$

выполняется неравенство

$$|C_n(g, x) - g(x)| \leqq \left(\frac{1}{2}\Lambda_n + \frac{3}{2} + \frac{4}{\pi}\right)\omega\left(g, \frac{2\pi}{2n+1}\right) \qquad (n = 0, 1, 2, \ldots), \tag{6.14}$$

так как

$$C_n(g, x) = T_n(g, x) \qquad (n = 0, 1, 2, \ldots) \tag{6.15}$$

и

$$\Lambda_n = \lambda_n \qquad (n = 0, 1, 2, \ldots). \tag{6.16}$$

При этом для некоторой $g_n(x)$

$$C_n(g_n, 0) - g_n(0) = \left(\frac{1}{2}\Lambda_n + \frac{1}{2}\right)\omega\left(g_n, \frac{2\pi}{2n+1}\right) \tag{6.17}$$

и для некоторой функции $g(x)$ имеет место (5. 3) и

$$C_n(g, 0) - g(0) \geqq (1-\varepsilon)\left(\frac{1}{2}\Lambda_n + \frac{1}{2}\right)\omega\left(g, \frac{2\pi}{2n+1}\right) \qquad (n = n_1, n_2, n_3, \ldots). \tag{6.18}$$

§ 7. Оценка погрешности Лагранжева интерполирования

Обозначим через $P_{n-1}^{*}(f, y)$ интерполяционный многочлен Лагранжа $n-1$-ой степени, совпадающий с непрерывной на отрезке $[-1, +]$ функцией $f(y)$ в узлах

$$y_k^{*} = \cos\frac{2k-1}{2n}\pi \qquad (k = 1, 2, \ldots, n), \tag{7.1}$$

а через L^*_{n-1} соответствующие числа Лебега. Из (6. 2) следует неравенство

$$(7.2)\qquad |P^*_{n-1}(f,y)-f(y)| \leqq \left(\frac{1}{2}L^*_{n-1}+\frac{3}{2}+\frac{4}{\pi}\right)\omega\left(f,\frac{\pi}{n}\right) \qquad (n=1,2,3,\ldots),$$

так как, полагая

$$(7.3)\qquad g(x)=f(\cos x),$$

получаем:

$$(7.4)\qquad P^*_{n-1}(f,\cos x)=C^*_{n-1}(g,x) \qquad (n=1,2,3,\ldots)$$

и очевидно

$$(7.5)\qquad L^*_{n-1}=\Lambda^*_{n-1} \qquad (n=1,2,3,\ldots),$$

$$(7.6)\qquad \omega(g,t)\leqq\omega(f,t) \qquad (0\leqq t\leqq 2).$$

Рассуждая как и в §3—§4, можно построить непрерывную функцию $f_n(y)$, для которой

$$(7.7)\qquad P^*_{n-1}(f_{n-1},1)-f_{n-1}(1)=\left(\frac{1}{2}L^*_{n-1}+\frac{1}{2}\right)\omega\left(f_{n-1},\frac{\pi}{n}\right)$$

и непрерывную функцию $f(y)$, для которой

$$(7.8)\qquad \omega\left(f,\frac{\pi}{n}\right)=\omega\left(\frac{\pi}{n}\right) \qquad (n=n_1,n_2,n_3,\ldots)$$

и

$$(7.9)\qquad P^*_{n-1}(f,1)-f(1)\geqq(1-\varepsilon)\left(\frac{1}{2}L^*_{n-1}+\frac{1}{2}\right)\omega\left(f,\frac{\pi}{n}\right) \qquad (n=n_1,n_2,n_3,\ldots).$$

Аналогичные результаты можно получить для многочленов n-ой степени $P^{**}_n(f,y)$, совпадающих с $f(y)$ в узлах

$$(7.10)\qquad y^{**}_k=\cos\frac{k\pi}{n} \qquad (k=0,1,\ldots,n).$$

В этом случае вместо (7. 2), (7. 7) и (7. 9) имеем:

$$(7.11)\qquad |P^{**}_n(f,y)-f(y)| \leqq \left(\frac{1}{2}L^{**}_n+\frac{3}{2}+\frac{4}{\pi}\right)\omega\left(f,\frac{\pi}{n}\right) \qquad (n=1,3,5,\ldots),$$

$$(7.12)\qquad P^{**}_n(f_n,0)-f_n(0)=\left(\frac{1}{2}L^{**}_n+\frac{1}{2}\right)\omega\left(f_n,\frac{\pi}{n}\right) \qquad (n=n_1,n_2,n_3,\ldots),$$

$$(7.13)\qquad P^{**}_n(f,0)-f(0)\geqq(1-\varepsilon)\left(\frac{1}{2}L^{**}_n+\frac{1}{2}\right)\omega\left(f,\frac{\pi}{n}\right) \qquad (n=n_1,n_2,n_3,\ldots).$$

Наконец, в случае многочленов n-ой степени $P_n(f,y)$, интерполирующих $f(y)$ в узлах

$$(7.14)\qquad y_k=\cos\frac{2k+1}{2n+1}\pi \qquad (k=0,1,\ldots,n),$$

получаем: для любой непрерывной функции $f(y)$

$$(7.15)\quad |P_n(f, y)-f(y)| \leqq \left(\frac{1}{2}L_n+\frac{3}{2}+\frac{4}{\pi}\right)\omega\left(f, \frac{2\pi}{2n+1}\right) \qquad (n = 0, 1, 2, \ldots),$$

для некоторой функции $f_n(y)$

$$(7.16)\quad P_n(f_n, 1)-f_n(1) = \left(\frac{1}{2}L_n+\frac{1}{2}\right)\omega\left(f_n, \frac{2\pi}{2n+1}\right) \qquad (n = n_1, n_2, n_3, \ldots),$$

для некоторой функции $f(y)$ имеет место

$$(7.17)\qquad \omega\left(f, \frac{2\pi}{2n+1}\right) = \omega\left(\frac{2\pi}{2n+1}\right) \qquad (n = n_1, n_2, n_3, \ldots)$$

и

$$(7.18)\quad P_n(f, 1)-f(1) \geqq (1-\varepsilon)\left(\frac{1}{2}L_n+\frac{1}{2}\right)\omega\left(f, \frac{2\pi}{2n+1}\right) \qquad (n = n_1, n_2, n_3, \ldots).$$

ЛИТЕРАТУРА

[1] Бернштейн, С. Н.: Об одном методе суммирования тригонометрических рядов, *Собрание сочинений*, том 1, 523—525.

[2] Бернштейн, С. Н.: О тригонометрическом интерполировании по способу наименьших квадратов, *Собрание сочинений*, том 2, 161—165.

[3] Корнейчук, Н. П.: О наилучшем равномерном приближении на некоторых классах непрерывных функций, *Докл. Акад. Наук* **140** (1961) 748—751.

[4] Корнейчук, Н. П.: Точная константа в теореме Д. Джексона о наилучшем равномерном приближении непрерывных периодических функций, *Докл. Акад. Наук* **145** (1962) 514—515.

[5] Ehlich, H. und Zeller, K.: Auswertung der Normen von Interpolationsoperatoren, *Math. Ann.* **164** (1966) 105—112.

Политехнический Институт, Будапешт

(Поступила 26-ого апреля 1968 г.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 291—300.

A CLASS OF METHODS FOR FUNCTION MINIMIZATION

by

L. VARGA

1. Introduction

The solution to many different types of practical problems can be obtained by minimizing a function of several variables. Such problems are e.g. the fitting of a formula to a set of measurements, the optimization of the solution to economic or technical problems, etc.

In [1], A. CAUCHY proposed the steepest descent method for finding the minimum of certain functions of more than one variable. The steepest descent method is based upon a linear approximation to the function and it requires only the function values and the gradients to be calculated. The convergence of the procedure can be proved for a fairly general function-class ([2], [3]), and the rate of the convergence turns out to be quadratical. In practice, however the steepest descent method can be very slow.

For getting more rapid convergence, a number of methods, using quadratic approximations to the function, has been developed. Such procedures are those of quasi-Newton methods ([5], [6], [7]), the modified Gauss–Newton method ([8], [9]), and the maximum neighbourhood method ([10], [11], [12]). Each step of these iterative methods usually consists of the following routines:

a) Construct a quadratic approximation to the function.

b) Using the quadratic approximation, determine a curve for the one-dimensional minimization.

c) Search the minimum along this curve.

In this paper the methods for function minimization are discussed from a new point of view based upon the problem *b)*.

In section 2 a class of methods for finding the minimum of a function is defined. These methods are referred to as *asymptotically Newton methods*, the basic properties of which are given below by Theorems 1, 2 and 3.

In section 3 many of the well known methods are discussed as special cases of the asymptotically Newton methods. A new special case which is a generalization of the steepest descent method is also given.

2. The Class of Methods

DEFINITIONS. Let $f(\mathbf{x})$ be a real-valued function defined on the n-dimensional Euclidean space. For fixed $\mathbf{x}_0$ define the set $H\{\mathbf{x}:f(\mathbf{x})\leqq f(\mathbf{x}_0)\}$. The function $f(\mathbf{x})$ satisfies:

CONDITION I. if $f(\mathbf{x})\in C^2$ everywhere on the Euclidean space;

CONDITION II. if there exists a bounded convex set K such that H is a subset of K and the Hessian matrix of $f(\mathbf{x})$ is positive definite on K.

Let $h(x)$ be a positive valued function given in the interval $0 \leqq x < \infty$. Let us suppose that the function $h(x)$ is analytic and satisfies the boundary conditions

$$\lim_{x \to +0} h(x) = c, \quad (c > 0), \tag{2. 1}$$

$$\lim_{x \to +0} xh'(x) = 0, \tag{2. 2}$$

$$\lim_{x \to \infty} xh(x) = 1, \tag{2. 3}$$

$\left(\text{that is the function } h(x) \text{ is asymptotically equal to } \frac{1}{x}\right)$, moreover

$$\varrho'(x) < 0, \quad (0 \leqq x < \infty), \tag{2. 4}$$

and

$$\lim_{x \to \infty} \varrho(x) = -1, \tag{2. 5}$$

where

$$\varrho(x) = \frac{xh'(x)}{h(x)}.$$

Let be a quadratic approximation to the function $f(\mathbf{x})$ given by the formula

$$\Phi(\mathbf{x}'; \mathbf{y}) = f(\mathbf{x}') + \mathbf{q}^*(\mathbf{x}')\mathbf{y} + \tfrac{1}{2}\mathbf{y}^*\mathbf{A}(\mathbf{x}')\mathbf{y}; \quad \mathbf{y} = \mathbf{x} - \mathbf{x}', \ (\mathbf{x}' \in H, \mathbf{x} \in H), \tag{2. 6}$$

where $\mathbf{q}(\mathbf{x})$ is the gradient of $f(\mathbf{x})$, $\left(\mathbf{q}^* = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)\right)$, $\mathbf{A}(\mathbf{x})$ is any symmetrical and positive definite matrix with bounded eigenvalues on H (e.g. the HESSIAN matrix).

The *asymptotically Newton method* for finding the point $\mathbf{x}_M$ where $f(\mathbf{x})$ has a minimum is given by the following steps:

Initial step: Select an estimate $\mathbf{x}_0$ *of* $\mathbf{x}_M$ *and a function* $h(x)$ *satisfying the conditions* (2. 1)—(2. 5).

General routine: Having obtained the estimate $\mathbf{x}_k$ $(k = 0, 1, \ldots)$ *compute* $\mathbf{x}_{k+1}$ *by the formulas*

$$\mathbf{y}_k(\varepsilon) = -\varepsilon h(\varepsilon \mathbf{A}(\mathbf{x}_k))\mathbf{q}(\mathbf{x}_k), \tag{2. 7}$$

$$f(\mathbf{x}_k + \mathbf{y}_k(\varepsilon_k)) \leqq f(\mathbf{x}_k + \mathbf{y}_k(\varepsilon)), \quad (0 \leqq \varepsilon < \infty), \tag{2. 8}$$

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{y}_k(\varepsilon_k). \tag{2. 9}$$

The equations (2. 1)—(2. 5) do not define a unique function $h(x)$ but a class of functions. Hence equations (2. 1)—(2. 9) define a class of methods for finding the least value of a function.

LEMMA 1. *The function* $h(x)$ *is a monotone decreasing function of* x.

The sentence of the Lemma 1 follows from (2. 2) and (2. 4) immediately.

LEMMA 2. *The function $xh(x)$ is a monotone increasing function of x.*

Using (2. 4) and (2. 5) we find that $\varrho(x) > -1$ $(x>0)$ that is

$$xh' + h = (xh)' > 0, \qquad (0 \leqq x < \infty)$$

consequently the function $xh(x)$ is a monotone increasing function of x.

The basic properties of the asymptotically Newton methods are given by the following theorems.

THEOREM 1. *Denote $\|\mathbf{y}\|$ the norm of the vector $\mathbf{y}$ and let α be the angle between $\mathbf{y}(\varepsilon)$ and $\mathbf{q}$. Then $\|\mathbf{y}(\varepsilon)\|$ and α are monotone increasing functions of ε, moreover*

$$\lim_{\varepsilon=0} \|\mathbf{y}(\varepsilon)\| = 0, \tag{2. 10}$$

$$\lim_{\varepsilon=0} \mathbf{y}'(\varepsilon) = -c\mathbf{q} \tag{2. 11}$$

and

$$\lim_{\varepsilon\to\infty} \mathbf{y}(\varepsilon) = -\mathbf{A}^{-1}\mathbf{q}. \tag{2. 12}$$

PROOF. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be the normalized eigenvectors of $\mathbf{A}$ and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues $(\lambda_k \leqq \lambda_j,\ k<j)$. As known, the matrix $\mathbf{A}$ can be written in the form

$$\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^* \tag{2. 13}$$

where the columns of $\mathbf{V}$ are the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and Λ is a diagonal matrix with diagonal elements $\lambda_1, \ldots, \lambda_n$. (The asterisk denotes matrix transposition.) Since the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, we may, accordingly, choose scalars $a_1, a_2, \ldots, a_n$ such that

$$\mathbf{q} = \sum_{k=1}^{n} a_k \mathbf{v}_k = \mathbf{V}\mathbf{a}. \tag{2. 14}$$

Using the formulas (2. 7), (2. 14) and the formula

$$h(\varepsilon\mathbf{A}) = \mathbf{V}h(\varepsilon\Lambda)\mathbf{V}^* \tag{2. 15}$$

valid for any analytic function of the matrix, we have

$$\|\mathbf{y}(\varepsilon)\|^2 = \varepsilon^2 \mathbf{q}^* h^2(\varepsilon A)\mathbf{q} = \varepsilon^2 \mathbf{a}^* V^* V h^2(\varepsilon\Lambda) V^* V\mathbf{a}.$$

Hence

$$\|\mathbf{y}(\varepsilon)\|^2 = \sum_{k=1}^{n} \varepsilon^2 h^2(\varepsilon\lambda_k) a_k^2. \tag{2. 16}$$

Considering Lemma 2, it follows that $\|\mathbf{y}(\varepsilon)\|$ is a monotone increasing function of ε. The relation (2. 10) follows from (2. 1) and (2. 16).

Using the formulas (2. 7) and (2. 15) we have

$$\mathbf{y}(\varepsilon) = -\varepsilon\mathbf{V}h(\varepsilon\Lambda)\mathbf{V}^*\mathbf{q} \tag{2. 17}$$

and

$$\mathbf{y}'(\varepsilon) = -\mathbf{V}\big(h(\varepsilon\mathbf{\Lambda}) + \varepsilon\mathbf{\Lambda}h'(\varepsilon\mathbf{\Lambda})\big)\mathbf{V}^*\mathbf{q}. \tag{2. 18}$$

By (2. 1)—(2. 3) we obtain the relations

$$\lim_{\varepsilon\to\infty} \mathbf{y}(\varepsilon) = -\mathbf{V}\mathbf{\Lambda}^{-1}\mathbf{V}^*\mathbf{q} = -\mathbf{A}^{-1}\mathbf{q}$$

and

$$\lim_{\varepsilon=0} \mathbf{y}'(\varepsilon) = -\mathbf{V}c\mathbf{V}^*\mathbf{q} = -c\mathbf{q}.$$

By definition

$$\cos^2\alpha = \frac{(\mathbf{q}^*\mathbf{y})^2}{\|\mathbf{q}\|^2\,\|\mathbf{y}\|^2}. \tag{2. 19}$$

Inserting (2. 16) and (2. 17) in (2. 19) yields

$$\cos^2\alpha = \frac{\left(\sum_{k=1}^{n} \varepsilon h(\varepsilon\lambda_k) a_k^2\right)^2}{\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} \varepsilon^2 h^2(\varepsilon\lambda_k) a_k^2}.$$

Differentiating we get

$$\frac{d\cos^2\alpha}{d\varepsilon} = \frac{2\sum_{k=1}^{n} h_k a_k^2 \left(\sum_{k=1}^{n-1} \sum_{j=k+1}^{n} (h_k - h_j)(h_k\lambda_j h_j' - h_j\lambda_k h_k') a_k^2 a_j^2\right)}{\sum_{k=1}^{n} a_k^2 \left(\sum_{k=1}^{n} h_k^2 a_k^2\right)^2}, \tag{2. 20}$$

where $h_k = h(\varepsilon\lambda_k)$ and $h_k' = h'(\varepsilon\lambda_k)$. Using (2. 4) we see that

$$\frac{\varepsilon\lambda_j h_j'}{h_j} < \frac{\varepsilon\lambda_k h_k'}{h_k}, \quad (j > k), \tag{2. 21}$$

thus the value of (2. 20) is always negative. Consequently, α is a monotone increasing function of ε, and the theorem is proved.

THEOREM 2. *The function*

$$\varphi(\varepsilon) = \Phi\big(\mathbf{x}; \mathbf{y}(\varepsilon)\big), \quad (\mathbf{x} \in H,\ 0 \leqq \varepsilon < \infty)$$

is a monotone decreasing function of ε.

PROOF. Using the formulas (2. 6) and (2. 7) we find that

$$\varphi'(\varepsilon) = \mathbf{q}^*\big(\varepsilon\mathbf{A}h'(\varepsilon\mathbf{A}) + h(\varepsilon\mathbf{A})\big)\big(\varepsilon\mathbf{A}h(\varepsilon\mathbf{A}) - \mathbf{E}\big)\mathbf{q},$$

where the unit matrix is denoted by $\mathbf{E}$. By (2. 14) and (2. 15) we have

$$\varphi'(\varepsilon) = \sum_{k=1}^{n} \big(\varepsilon\lambda_k h'(\varepsilon\lambda_k) + h(\varepsilon\lambda_k)\big)\big(\varepsilon\lambda_k h(\varepsilon\lambda_k) - 1\big)a_k^2$$

since the function $xh(x)$ is a monotone increasing function of x and $0 \leqq xh(x) < 1$ $(0 \leqq x < \infty)$, the inequality

$$(xh(x))'(xh(x)-1) \leqq 0, \qquad (0 \leqq x < \infty)$$

holds, hence $\varphi'(\varepsilon) \leqq 0$. This proves the theorem.

In view of result (2. 12) it is seen that the name „asymptotically Newton methods" is an appropriate name for the class of methods defined by (2. 1)—(2. 9). Moreover an asymptotically Newton method performs an interpolation between the steepest descent method and the Newton method.

THEOREM 3. *The function $f(\mathbf{x})$ satisfying the Conditions I and II has a unique minimum on the set H and the sequence $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots$ selected by* (2. 7)—(2. 9) *converges to the point $\mathbf{x}_M$ which minimizes $f(\mathbf{x})$ on H.*

PROOF. Without loss of generality we may assume the sequence $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots$ to be infinite. By Conditions I and II a $\varkappa > 0$ can be found such that the inequality

$$f(\mathbf{x}+\mathbf{y}) \leqq f(\mathbf{x}) + \mathbf{q}^*\mathbf{y} + \varkappa \|\mathbf{y}\|^2, \quad (\mathbf{x} \in H, \ \mathbf{x}+\mathbf{y} \in H),$$

holds.

Using (2. 7) and (2. 15) we have

$$(2.22) \quad f(\mathbf{x}_k) - f(\mathbf{x}_k + \mathbf{y}_k(\varepsilon)) \geqq \mathbf{a}^*(\mathbf{x}_k)\mathbf{V}^*\mathbf{V}\varepsilon h(\varepsilon\boldsymbol{\Lambda})(\mathbf{E} - \varepsilon\varkappa h(\varepsilon\boldsymbol{\Lambda}))\mathbf{V}^*\mathbf{V}\mathbf{a}(\mathbf{x}_k) =$$

$$= \sum_{i=1}^{n} \varepsilon h(\varepsilon\lambda_i)(1 - \varepsilon\varkappa h(\varepsilon\lambda_i)) a_i^2(\mathbf{x}_k).$$

Let $\lambda_{\min}$ and $\lambda_{\max}$ be the lower and upper bounds of the eigenvalues of matrix $\mathbf{A}$ on H. Let $\bar{\varepsilon} > 0$ satisfy the inequality

$$\bar{\varepsilon}\lambda_{\max} h(\bar{\varepsilon}\lambda_{\max}) < \frac{\lambda_{\min}}{\varkappa}.$$

Then, for all $\varepsilon \leqq \bar{\varepsilon}$ the right side of (2. 22) is positive. By definition

$$f(\mathbf{x}_k) - f(\mathbf{x}_{k+1}) \geqq f(\mathbf{x}_k) - f(\mathbf{x}_k + \mathbf{y}_k(\varepsilon)), \quad (0 \leqq \varepsilon < \infty).$$

Therefore

$$(2.23) \qquad f(\mathbf{x}_k) - f(\mathbf{x}_{k+1}) \geqq \sum_{i=1}^{n} \varepsilon h(\varepsilon\lambda_i)(1 - \varepsilon\varkappa h(\varepsilon\lambda_i)) a_i^2(\mathbf{x}_k), \quad (\varepsilon \leqq \bar{\varepsilon}).$$

Since $f(\mathbf{x})$ is continuous it achieves a minimum on the bounded and closed set H. Thus, the sequence $f(\mathbf{x}_0), f(\mathbf{x}_1), \ldots, f(\mathbf{x}_k), \ldots$ is bounded from below and has a cluster point

$$\lim_{k \to \infty} f(\mathbf{x}_k) = f_T = f(\mathbf{x}_T), \quad (\mathbf{x}_T \in H),$$

consequently

$$a_i(\mathbf{x}_T) = 0, \qquad i = 1, 2, \ldots, n,$$

that is

$$\mathbf{q}(\mathbf{x}_T) = 0.$$

That this stationary point is actually a minimum, is clear from the fact that $\mathbf{Q}(\mathbf{x}_T)$ is positive definite, where $\mathbf{Q}(\mathbf{x})$ denotes the Hessian matrix of $f(\mathbf{x})$. The unicity is almost trivial. Let the points $\mathbf{x}_T$ and $\mathbf{x}_M$ satisfy the equation

$$\mathbf{q}(\mathbf{x}_T) = \mathbf{q}(\mathbf{x}_M) = 0, \quad (\mathbf{x}_T \neq \mathbf{x}_M).$$

Therefore the function

$$F(t) = \mathbf{q}^*(\mathbf{x}_T + t(\mathbf{x}_M - \mathbf{x}_T))(\mathbf{x}_M - \mathbf{x}_T), \quad (0 \leqq t \leqq 1)$$

vanishes at $t=0$ and $t=1$. Thus by Rolle's theorem there exists a t_b $(0<t_b<1)$ such that.

$$F'(t_b) = (\mathbf{x}_M - \mathbf{x}_T)^* \mathbf{Q}(\mathbf{x}_T + t_b(\mathbf{x}_M - \mathbf{x}_T))(\mathbf{x}_M - \mathbf{x}_T) = 0.$$

Since the matrix $\mathbf{Q}$ is positive definite on a convex set K the contradiction is obvious, and the theorem is proved.

3. Particular methods

The asymptotically Newton method depends on the choice of the function $h(x)$. Various functions can be chosen, each leading to a special method. Now let us consider the most important special cases of the asymptotically Newton methods.

1. *Gradient methods*

The curve for one-dimensional search (2. 7) is not invariant under linear transformations of the **y**-space. Applying the linear transformation

$$\mathbf{w} = \mathbf{W}\mathbf{y}$$

to (2. 6), we find that

$$\Phi(\mathbf{x}_k; \mathbf{y}) = f(\mathbf{x}_k) + \mathbf{q}_k^* \mathbf{W}^{-1}\mathbf{w} + \tfrac{1}{2}\mathbf{w}^* \mathbf{W}^{*-1}\mathbf{A}\mathbf{W}^{-1}\mathbf{w}.$$

Introducing a scaled matrix $\mathbf{A}_w$ and a scaled gradient $\mathbf{q}_w$ of the form

$$\mathbf{A}_w = \mathbf{W}^{*-1}\mathbf{A}\mathbf{W}^{-1},$$

$$\mathbf{q}_w = \mathbf{W}^{*-1}\mathbf{q},$$

the curve for the linear search in the **w**-space is

$$\mathbf{w}(\varepsilon) = -\varepsilon h(\varepsilon \mathbf{W}^{*-1}\mathbf{A}\mathbf{W}^{-1})\mathbf{W}^{*-1}\mathbf{q}.$$

By retransformation we have

$$\mathbf{y}_T(\varepsilon) = -\varepsilon \mathbf{W}^{-1} h(\varepsilon \mathbf{W}^{*-1}\mathbf{A}\mathbf{W}^{-1})\mathbf{W}^{*-1}\mathbf{q}. \tag{3. 1}$$

Clearly, the **y**-space may be scaled in some convenient manner. The choice of scale has, in fact, been widely used as a tool for improving the convergence of the procedure. On the other hand, the choice

$$\mathbf{W} = \mathbf{A}^{\frac{1}{2}}$$

(the square root of a positive definite matrix exists) causes the matrix $\mathbf{A}$ to transform into the unit matrix, and we have

$$\mathbf{y}_T(\varepsilon) = -\varepsilon h(\varepsilon)\mathbf{A}^{-1}\mathbf{q}$$

or

$$\hat{\mathbf{y}}_T(\alpha) = -\alpha\mathbf{A}^{-1}\mathbf{q}, \quad (0 \leqq \alpha \leqq 1). \tag{3.2}$$

The methods defined by (3. 2), (2. 8) and (2. 9) where $\mathbf{A}$ is any positive definite matrix, are called *gradient methods.* A few of them are discussed in [4]. Now we restrict ourselves to refer to the most important cases of the gradient methods.

1. 1 *Steepest descent method:* $\mathbf{A}=\mathbf{E}$. This is the simplest case of the gradient methods.

1. 2 *Modified Newton Method:* $\mathbf{A}=\mathbf{Q}$. ($\mathbf{Q}$ is the HESSian matrix of function $f(\mathbf{x})$.)

The Newton method frequently fails to converge, since the conditions for convergence rely on the initial estimate of the minimum being sufficiently good. This requirement is often impossible to realize in practice. In order to overcome this disadvantage of the Newton method the usual procedure is to obtain α_k such that $f(\mathbf{x}_k+\hat{\mathbf{y}}(\alpha_k))$ is a minimum with respect to α along (3. 2). A further disadvantage of the method is the difficulty of computing the matrix $\mathbf{Q}$. Therefore preferably the quasi-Newton methods are used.

1. 3 *Quasi-Newton methods:* $\mathbf{A}=\mathbf{Q}^{(k)}$, where the matrix $\mathbf{Q}^{(k)}$ is chosen such that it satisfies the relation

$$\mathbf{Q}^{(k)-1}\mathbf{q}(\mathbf{x}_k) \approx \mathbf{Q}^{-1}(\mathbf{x}_k)\mathbf{q}(\mathbf{x}_k). \tag{3.3}$$

A matrix sequence satisfying (3. 3) can be constructed step by step, computing a correction from the values of the first partial derivatives of $f(\mathbf{x})$ to the preceding approximation to $\mathbf{Q}$ [7].

Most of the numerical methods for minimizing a sum of squares of non-linear functions can be regarded as quasi-Newton methods. Let us consider the minimization of the form

$$f(\mathbf{x}) = \sum_{i=1}^{N} f_i^2(\mathbf{x})$$

where the functions $f_i(\mathbf{x})$, (f$=1, \ldots, N$) are non-linear forms of $\mathbf{x}$.

Expanding the functions $f_1(\mathbf{x}), \ldots, f_N(\mathbf{x})$ in a Taylor series through the linear terms we get

$$f(\mathbf{x}_k+\mathbf{y}) \approx f(\mathbf{x}_k)+\mathbf{q}^{(k)*}\mathbf{y}+\tfrac{1}{2}\mathbf{y}^*\mathbf{Q}^{(k)}\mathbf{y}$$

where

$$\begin{aligned} q_i^{(k)} &= \sum_{j=1}^{N} f_j^{(k)} f_{ji}^{(k)}, \qquad (i = 1, \ldots, n), \\ Q_{ij}^{(k)} &= \sum_{l=1}^{N} f_{li}^{(k)} f_{lj}^{(k)}, \qquad (i = 1, \ldots, n;\ j = 1, \ldots, n), \end{aligned} \tag{3.4}$$

and

$$f_j^{(k)} = f_j(\mathbf{x}_k), \quad f_{li}^{(k)} = \left.\frac{\partial f_l(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\mathbf{x}_k}.$$

It is obvious, that $\mathbf{Q}^{(k)}$ defined by (3. 4) satisfies the relation

$$\lim_{\mathbf{x}_k \to \mathbf{x}_M} \mathbf{Q}^{(k)} = \mathbf{Q}(\mathbf{x}_M)$$

if the equation $f(\mathbf{x}_M) = 0$ holds. This type of quasi-Newton method is sometimes referred to as modified Gauss–Newton method [8], or linearization method [9]. It has a simple algorithm, even though it does require the solution of n linear equations at each step, it is perhaps the most powerful procedure for minimizing a sum of squares of non-linear functions. But not infrequently it runs aground because of poor conditioning of the surface $f(\mathbf{x})$.

2. *Generalized steepest descent method*

As it was remarked in section 2, the asymptotically Newton methods perform an interpolation between the steepest descent method and the Newton method. In this method an optimum interpolation is realized by choice $h(x)$ such a way as to guarantee a steepest descent in function $\Phi(\mathbf{x}_k; \mathbf{y})$.

The maximum decrease in function Φ is in the direction of the negative gradient. Thus the curve for steepest descent minimization of function Φ is determined by the differential equation system

$$\mathbf{y}' = -\mathbf{A}\mathbf{y} - \mathbf{q},$$

which has the solution

$$\mathbf{y}(\varepsilon) = -\varepsilon h_s(\varepsilon \mathbf{A})\mathbf{q} \tag{3. 5}$$

where

$$h_s(x) = \frac{1 - e^{-x}}{x}. \tag{3. 6}$$

THEOREM 4. *The algorithm given by the formulas* (3. 5), (2. 8) *and* (2. 9) *is an asimptotically Newton method.*

PROOF. In order to prove this theorem, it is sufficient to verify that relations (2. 1)—(2. 5) hold for $h_s(x)$. This can be done by a simple calculation.

The algorithm given by (3. 6) and (2. 7)—(2. 9) is called *generalized steepest descent method.*

In order to compute the matrix function $h_s(\varepsilon \mathbf{A})$, eigenvectors and eigenvalues of matrix $\mathbf{A}$ must be evaluated. Therefore this method can not be regarded as a good practical algorithm. However, this difficulty can be avoided by using rational approximations of the form

$$h_j(x) = \frac{1 + \frac{x}{2!} + \cdots + \frac{x^{j-1}}{j!}}{1 + \frac{x}{1!} + \cdots + \frac{xj}{j!}}, \qquad (j = 1, 2, \ldots) \tag{3. 7}$$

instead of $h_s(x)$. It can be seen, that the choise $h(x)=h_j(x)$, $(j=1, 2, \ldots)$ also leads to an asymtotically Newton method. The advantage of these approximations is the fact, that the matrix functions $h_j(\varepsilon\mathbf{A})$ $(j=1, 2, \ldots)$ can be calculated by inversion. On the other hand, when $j\to\infty$ the sequence given in (3. 7) rapidly converges to $h_s(x)$. The functions $h_1(x)$, $h_2(x)$ and $h_s(x)$ are illustrated in Figure 1.

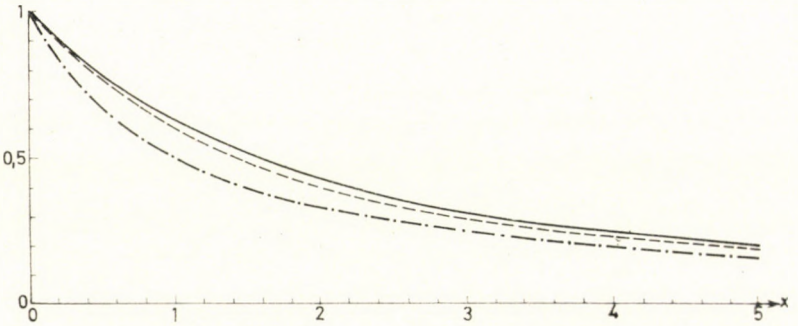


Fig. 1 —— $h_s(x)$; —·— $h_1(x)$; ——— $h_2(x)$

By a modification of $h_2(x)$ we have got the function

$$r(x)=\frac{1+p^2x}{(1+px)^2}, \qquad p=0.375, \tag{3. 8}$$

which is a better approximation to $h_s(x)$ than $h_2(x)$ and has a more simple form. In practice the method defined by (3. 8) and (2. 7)—(2. 9) proved to be a good approximation to the generalized steepest descent method.

A numerical advantage is that the matrix $\mathbf{E}+p\varepsilon\mathbf{A}$ $(\varepsilon>0)$ is always better conditioned than $\mathbf{A}$ itself. Thus the method described above is useful for minimizing a function even though the matrix $\mathbf{A}(\mathbf{x}_k)$ may be ill-conditioned at points x_k, which lie still far from the minimum point.

The first approximation to the generalized steepest descent method can be obtained by using $h_1(x)$. This method is equivalent to the maximum neighbourhood method [11].

3. *The maximum neighbourhood method*

If the function

$$h(x)=\frac{1}{1+x} \tag{3. 9}$$

is chosen, the corresponding asymptotically Newton method defined by (3. 9) and (2. 7)—(2. 9) is called the *maximum neighbourhood method.* This method was first proposed by K. LEVENBERG [10] for minimizing a sum of squares of non-linear functions, and called the *damped least squares method.* But D. W. MARQUARDT in [11] gave it the name "maximum neighbourhood method" because of the following property:

If $\mathbf{y}_0$ *satisfies the equation*

$$(\mathbf{A}(\mathbf{x}_k)+\lambda\mathbf{E})\mathbf{y}_0 = -\mathbf{q}(\mathbf{x}_k),$$

with an arbitrarily chosen $\lambda>0$, *then* $\mathbf{y}_0$ *minimizes* $\Phi(\mathbf{x}_k;\mathbf{y})$ *on the sphere whos-radius* $\|\mathbf{y}\|$ *satisfies* $\|\mathbf{y}\|^2=\|\mathbf{y}_0\|^2$.

Finally it may be noted that an appropriate choice of scale in the **y**-space leads to a fast convergence of the method described. This follows from the fact that generally the contours of the function $f(\mathbf{x}_k+\mathbf{y})-\Phi(\mathbf{x}_k;\mathbf{y})$ are ellipsoids near the point $\mathbf{x}_k$.

REFERENCES

[1] CAUCHY, A. L.: Méthode Générale pour la Résolution des Systèmes d'Equations Simultanées, *C. R. Acad. Sci. Paris* **25** (1847) 536.
[2] CURRY, H. B.: The method of steepest descent for nonlinear minimization problems, *Quart. Appl. Math.* **2** (1944) 258—261.
[3] GOLDSTEIN, A. A.: Cauchy's method of minimization, *Numer. Math.* **4** (1962) 146—150.
[4] SPANG, H. A.: A review of minimization techniques for nonlinear functions, *SIAM Rev.* **4** (1962) 343—360.
[5] DAVIDON, W. C.: Variable matrix method for minimization, *AEC Research and Development Report*, ANL-5990 (1959).
[6] FLETCHER, R. and POWELL, M. J. D.: A rapidly convergent descent method for minimization, *Comput. J.* **6** (1963) 163—168.
[7] BROYDEN, C. G.: Quasi-Newton methods and their application to function minimization, *Math. Comp.* **21** (1967) 368—381.
[8] HARTLEY, H. O.: The modified Gauss-Newton method for the fitting of nonlinear regression functions by least-squares, *Technometrics* **3** (1961) 269—280.
[9] СОКОЛОВ, С. Н. и СИЛИН, И. Н.: Нахожение минимумов функционалов методом линеаризации, *Препринт ОИЯИ. Д-810,* Дубна, 1963.
[10] LEVENBERG, K.: A method for the solution of certain non-linear problems in least squares, *Quart. Appl. Math.* **2** (1944) 164—168.
[11] MARQUARDT, D. W.: An algorithm for least-squares estimation of nonlinear parameters, *J. Soc. Indust. Appl. Math.* **11** (1963) 431—441.
[12] BRAESS, D.: Über Dampfung bei Minimalisierungsverfahren, *Comput. Rev.* **1** (1966) 264—272.

Central Research Institute of Physics, Budapest

(Received May 5, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 301—312.

RANDOM MODELS OF LOGICAL SYSTEMS

by

I. RUZSA

Part I

Models of Valuing Logics

In the present paper I shall deal with *models of some logical systems in the probability calculus*. In these models the logical variables are represented by *random variables*, and the logical operators by *function operators* ranging over random variables (viz. over n-tuples of random variables) and having random variables as their values. Henceforth the models thus generated will be called *random models* (more exactly: *random model systems*).

The results of my investigations might make it possible to generalize or specify several logical systems. One may draw some semantical conclusions from them too. However, the main aim of this paper is to give the mathematical description of the modelling method outlined above, hence the logical and philosophical problems related to it will be at most briefly indicated. A thorough logico-semantical analysis of my mathematical results will be perhaps the subject of another study.

In *Part I* I deal with random models of valuing logics (i.e. logics which are decidable by a finite valuing matrix), including the random model of the two-valued propositional calculus. *Part II* deals with the random models of modal propositional logics of ŁUKASIEWICZ and that of LEWIS. The subject of *Part III* is the random model of quantified logics, including quantified modal logic too. In forthcoming papers I shall deal with a random model of PRIOR's system Q, and that of the intuitionistic logic.

§ 1. Preliminaries. Notation and basic notions

We shall denote the empty set by $\emptyset$, the union, the intersection and the difference of the sets A, B by $A \cup B$, $A \cap B$, and $A \setminus B$ respectively. $A \subseteq B$ denotes that the set B includes the set A.

If A is a set then A^k denotes the Cartesian product $A \times \ldots \times A$, i.e. A^k is the set of ordered k-tuples of the form $(a_1, \ldots, a_k)$ where $a_1 \in A, \ldots, a_k \in A$.

DEFINITION 1. We shall call a quadruple

$$S = (V, O, F, T)$$

an *elementary logical system* (briefly: a *system*), if the following conditions are fulfilled:

V is a denumerably infinite set called *the set of variables*, its elements are the (sentence) variables.

O is a finite union of disjoint finite sets, i.e. $O = O_1 \cup \ldots \cup O_n$, where $n \geqq 1$ and O_n is not empty; the elements of O_k are *the k-placed operators* ($k = 1, 2, \ldots, n$); O is called the set of *operators*; V and O are disjoint.

F is the smallest set satisfying the following conditions: (a) $V \subseteq F$; (b) if $o \in O_k$, $\alpha_1 \in F, \ldots, \alpha_k \in F$, then $o\alpha_1 \ldots \alpha_k \in F$ ($k = 1, 2, \ldots, n$). — F is called *the set of formulas of the system* S; the elements of F are *the formulas of* S.

T is a subset of F ($T \subseteq F$) called *the set of theorems of* S.

The triple

$$S' = (V, O, F)$$

will be called *the language* of the system S.

If we substitute a variable p (in all its occurrence) in a formula α by a formula β, then we denote the resulting formula by "$\alpha(p/\beta)$".

DEFINITION 2. We shall call a triple

$$\Sigma = (I, \mathscr{A}, P)$$

a *probability space*, if

I is a non-empty set;

$\mathscr{A}$ is a σ-algebra of certain subsets of I;

P is a *probability measure* defined on the measurable space $(I, \mathscr{A})$ satisfying the condition $P(I) = 1$.

The elements of $\mathscr{A}$ will be called *events*. The event $I \setminus A$ will be called *the complement* of A ($A \in \mathscr{A}$) and denoted by $\bar{A}$.

A measurable function w defined on I will be called a *random variable on* Σ. — We shall denote the event A_z which contains an element E of I if and only if $w(E) < z$ (where z is a real number) by "$[w < z]$". The meaning of $[w \leqq z]$, $[w = z]$, $[w > z]$ and $[w \geqq z]$ is similar. However we write $P(w < z)$ etc. instead of $P([w < z])$ etc. (where P is the probability measure) provided no misunderstanding can arise.

Let Σ be a probability space, and let w be a random variable on Σ, furthermore let $\langle e_1, \ldots, e_m \rangle$ be a strictly increasing sequence of real numbers ($m \geqq 1$). We say that w is *discrete with respect to the sequence* $\langle e_1, \ldots, e_m \rangle$ if the set $\{e_1, \ldots, e_m\}$ contains all the values of w. If $m = 1$, then we say that w is a *constant* (or *degenerated*) random variable. A constant random variable having the sole value c will be often denoted by its sole value c.

DEFINITION 3. The triple

$$Z = (a, b, \Theta)$$

will be called a *random model structure* (briefly: *rms*) of a language $S' = (V, O, F)$ (see in *Def.* 1.), if the following conditions are fulfilled:

a and b are real numbers, $a < b$. The closed interval $[a, b]$ will be called the *base interval* of the *rms* Z.

Θ is a function defined on O such that if $o \in O_k$ ($k \geqq 1$) then $\Theta(o) = \dot{o}$ is a function operator in k variables, and $\dot{o} = \dot{o}(w_1, \ldots, w_k)$ satisfyes the following condition: if $w_1, \ldots, w_k$ are random variables on a probability space Σ, and $[a, b]$ contains all values of $w_1, \ldots, w_k$, then $\dot{o}(w_1, \ldots, w_k)$ is also a random variable on Σ, and $[a, b]$ contains all values of $\dot{o}(w_1, \ldots, w_k)$ too.

DEFINITION 4. The pair

$$\Gamma=(\Sigma, W)$$

will be called a *random model* belonging to a *rms* Z, if the following conditions are fulfilled:

Σ is a probability space.

W is a function defined on F (i.e. on the set of formulas of the language S') such that the following conditions are fulfilled: (a) If $\alpha\in F$, then $W(\alpha)$ is a random variable on Σ, and the values of $W(\alpha)$ belong to $[a, b]$. (b) If $o\in O_k$ $(k\geqq 1)$, and $\alpha_1, \ldots, \alpha_k$ are elements of F, then $W(o\alpha_1\ldots\alpha_k)=\dot{o}(W(\alpha_1)\ldots, W(\alpha_k))$, where $\dot{o}=\Theta(o)$.

DEFINITION 5. We say that an *rms* $Z=(a, b, \Theta)$ of a language S' is *discrete with respect to the sequence* $\langle a=e_1<e_2<\ldots<e_m=b\rangle$ $(m\geqq 2)$, if Θ has the following property: If $o\in O_k$ $(k\geqq 1)$, $\Theta(o)=\dot{o}$, and $w_1, \ldots, w_k$ are discrete random variables with respect to the sequence $\langle e_1, \ldots, e_m\rangle$, then $\dot{o}(w_1, \ldots, w_k)$ is also discrete to $\langle e_1, \ldots, e_m\rangle$.

Let Z be an *rms* (of a language S') which is discrete with respect to a sequence $\langle e_1, \ldots, e_m\rangle$, and let $\Gamma=(\Sigma, W)$ be a random model belonging to Z. We say that Γ is *discrete with respect to* $\langle e_1, \ldots, e_m\rangle$, if $p\in V$ implies that $W(p)$ is discrete with respect to $\langle e_1, \ldots, e_m\rangle$. (Note that V is the set of variables of S'.) It follows from this stipulation, that if $\alpha\in F$ then $W(\alpha)$ is also discrete with respect to $\langle e_1, \ldots, e_m\rangle$.

We write $[\alpha<z]$ and $P(\alpha<z)$ instead of $[W(\alpha)<z]$ and $P(W(\alpha)<z)$, respectively, since using this notation no misunderstanding can arise.

THEOREM 1. 1. *Let* $S'=(V, O, F)$ *be a language of a system, and let* $Z=(a, b, \Theta)$ *be an rms of* S'. *Assume, that* α *is a formula of* S', *for which there is a real number* c *such that* $a<c\leqq b$, *and in any random model* Γ *belonging to* Z, $[\alpha<c]=\emptyset$ *is fulfilled. Let* p *be a variable* $(p\in V)$, β *be a formula* $(\beta\in F)$, *and denote* $\alpha(p/\beta)$ *by* α_1. *Then in any random model belonging to* Z, $[\alpha_1<c]=\emptyset$ *is also fulfilled.*

PROOF. Let $\Gamma=(\Sigma, W)$ be an arbitrary random model belonging to Z and take $W(\beta)=w$. Consider the random model $\Gamma_1=(\Sigma, W_1)$ which only differs from Γ in $W_1(p)=w$. (It is evident that if Γ exists and belongs to Z, then Γ_1 also exists and belongs to Z.) Now it is clear that $W(\alpha_1)$ (in Γ) and $W_1(\alpha)$ (in Γ_1) are identical random variables on the same probability space Σ. From this and from the assumptions of the theorem it follows that $[\alpha_1<c]=\emptyset$ is fulfilled in Γ.

§ 2. A random model structure of the two-valued propositional calculus

Let us consider the language A_0' of the two-valued propositional calculus: $A_0'=(V, O, F)$; where $O=O_1\cup O_2$, $O_1=\{\neg\}$, $O_2=\{\wedge\}$,

$$O=\{\neg, \wedge\}.$$

"$\neg$" and "$\wedge$" is the negation and conjunction operator, respectively. We shall write "$\alpha\wedge\beta$" instead of "$\wedge\alpha\beta$" $(\alpha\in F, \beta\in F)$, and so we shall also use brackets, in the usual way.

We shall use the operators of disjunction, implication and equivalence as defined operators, denoted by $\vee$, $\rightarrow$ and $\leftrightarrow$, respectively, given by their usual

definitions:

$$(1)\qquad \left.\begin{aligned} \alpha \vee \beta &= \neg(\neg\alpha \wedge \neg\beta), \\ \alpha \to \beta &= \neg(\alpha \wedge \neg\beta), \\ \alpha \leftrightarrow \beta &= (\alpha \to \beta) \wedge (\beta \to \alpha). \end{aligned}\right\} (\alpha \in F, \beta \in F).$$

The *rms* Z_0 of A_0' is given as follows:

$$Z_0 = (0, 1, \Theta),$$

where the function operators $\Theta(\neg) = \dot{\neg}, \Theta(\wedge) = \dot{\wedge}$ are defined by the following stipulations:

$$(2)\qquad \dot{\neg}\, w = 1 - w, \quad \text{and}$$

$$(3)\qquad w_1 \,\dot{\wedge}\, w_2 = \min(w_1, w_2)$$

(we write $\dot{\neg}\, w$ and $w_1 \,\dot{\wedge}\, w_2$ instead of $\dot{\neg}(w)$ and $\dot{\wedge}(w_1, w_2)$, respectively). One can see easily that $\dot{\neg}$ and $\dot{\wedge}$ satisfy the condition prescribed in *Def.* 3.

If we take — in accordance with (1) —

$$w_1 \,\dot{\vee}\, w_2 = \dot{\neg}(\dot{\neg}\, w_1 \,\dot{\wedge}\, \dot{\neg}\, w_2),$$

i.e. if we accept this equality as the definition of $\Theta(\vee) = \dot{\vee}$, then we get

$$w_1 \,\dot{\vee}\, w_2 = 1 - (\min(1 - w_1, 1 - w_2)) = 1 - (1 - \max(w_1, w_2)) = \max(w_1, w_2),$$

i.e.

$$(4)\qquad w_1 \,\dot{\vee}\, w_2 = \max(w_1, w_2).$$

From (2), (3) and (4) we get the following identities easily:

$$(5)\qquad \left\{\begin{array}{ll} w \,\dot{\wedge}\, w = w & w \,\dot{\vee}\, w = w \\ w_1 \,\dot{\wedge}\, w_2 = w_2 \,\dot{\wedge}\, w_1 & w_1 \,\dot{\vee}\, w_2 = w_2 \,\dot{\vee}\, w_1 \\ w_1 \,\dot{\wedge}\, (w_2 \,\dot{\wedge}\, w_3) = (w_1 \,\dot{\wedge}\, w_2) \,\dot{\wedge}\, w_3 & w_1 \,\dot{\vee}\, (w_2 \,\dot{\vee}\, w_3) = (w_1 \,\dot{\vee}\, w_2) \,\dot{\vee}\, w_3 \\ w_1 \,\dot{\wedge}\, (w_2 \,\dot{\vee}\, w_3) = (w_1 \,\dot{\wedge}\, w_2) \,\dot{\vee}\, (w_1 \,\dot{\wedge}\, w_3) & w_1 \,\dot{\vee}\, (w_2 \,\dot{\wedge}\, w_3) = (w_1 \,\dot{\vee}\, w_2) \,\dot{\wedge}\, (w_1 \,\dot{\vee}\, w_3) \\ \dot{\neg}(w_1 \,\dot{\wedge}\, w_2) = \dot{\neg}\, w_1 \,\dot{\vee}\, \dot{\neg}\, w_2 & \dot{\neg}(w_1 \,\dot{\vee}\, w_2) = \dot{\neg}\, w_1 \,\dot{\wedge}\, \dot{\neg}\, w_2 \\ \multicolumn{2}{c}{\dot{\neg}\, \dot{\neg}\, w = w} \end{array}\right.$$

For the constant random variables 0 and 1 we have the following simple laws:

$$(6)\qquad \left\{\begin{array}{ll} \dot{\neg}\, 0 = 1, & \dot{\neg}\, 1 = 0; \\ 0 \,\dot{\wedge}\, 0 = 0 \,\dot{\wedge}\, 1 = 1 \,\dot{\wedge}\, 0 = 0, & \\ 1 \,\dot{\wedge}\, 1 = 1. & \end{array}\right.$$

LEMMA 1. Let w, w_1 and w_2 be arbitrary random variables, and let z be any real number. Then:

(7) $$[\dot{\neg} w < z] = [w > 1 - z]$$

(8) $$[\dot{\neg} w = z] = [w = 1 - z]$$

$$[\dot{\neg} w > z] = [w < 1 - z]$$

(10) $$[w_1 \dot{\wedge} w_2 < z] = [w_1 < z] \cup [w_2 < z]$$

$$[w_1 \dot{\wedge} w_2 \geqq z] = [w_1 \geqq z] \cap [w_2 \geqq z]$$

(11) $$[w_1 \dot{\wedge} w_2 > z] = [w_1 > z] \cap [w_2 > z]$$

(12) $$[w_1 \dot{\vee} w_2 < z] = [w_1 < z] \cap [w_2 < z]$$

$$[w_1 \dot{\vee} w_2 \geqq z] = [w_1 \geqq z] \cup [w_2 \geqq z]$$

$$[w_1 \dot{\vee} w_2 > z] = [w_1 > z] \cup [w_2 > z].$$

PROOF. From (2) we get:

$$[\dot{\neg} w < z] = [1 - w < z] = [w > 1 - z]$$

i.e. (7) is true. We get (8) and (9) similarly.

To get (10), observe that according to (3)

$$[w_1 \dot{\wedge} w_2 < z] = [\min(w_1, w_2) < z],$$

thus we have to prove that

(13) $$E \in [\min(w_1, w_2) < z] \text{ if and only if } E \in [w_1 < z] \cup [w_2 < z].$$

Assume, that $E \in [\min(w_1, w_2) < z]$. Then $\min\big(w_1(E), w_2(E)\big) < z$; i.e. $w_1(E) < z$ *or* $w_2(E) < z$, hence $E \in [w_1 < z] \cup [w_2 < z]$. Conversely, if $E \in [w_1 < z] \cup [w_2 < z]$, then $E \in [w_1 < z]$ *or* $E \in [w_2 < z]$, i.e. $w_1(E) < z$ *or* $W_2(E) < z$, hence $\min\big(w_1(E), w_2(E)\big) < z$, that is $E \in [\min(w_1, w_2) < z]$. Consequently (13) is true, hence (10) is true, too.

The proof of the other identities is very similar, and so is left to the reader.

LEMMA 2. If w is an arbitrary random variable, then $[w \dot{\wedge} \dot{\neg} w > 1/2] = \emptyset$, and $[w \dot{\vee} \dot{\neg} w < 1/2] = \emptyset$.

PROOF. By (11) and (9) we have:

$$[w \dot{\wedge} \neg w > 1/2] = [w > 1/2] \cap [w < 1/2] = \emptyset.$$

Similarly, by (12) and (7) we get:

$$[w \dot{\vee} \dot{\neg} w < \tfrac{1}{2}] = [w < \tfrac{1}{2}] \cap [w > \tfrac{1}{2}] = \emptyset.$$

Let the set T appearing in $A_0 = (V, O, F, T)$ be the set of *valid* formulas (i.e. theorems, or tautologies) of the two-valued propositional calculus. The following theorem presents the first connection between A_0 and Z_0.

THEOREM 2. 1. *If α is a theorem of A_0, then in any random model $\Gamma=(\Sigma, W)$ belonging to Z_0, $[\alpha<1/2]=\emptyset$ is true.*

PROOF. By using the definition (1) and the equivalences (15) below, we can get a special conjunctive normal form of α:

$$\alpha_0=\alpha_1\wedge\ldots\wedge\alpha_m \qquad (m\geqq 1), \tag{14}$$

where α_i is a disjunction of some negated and nonnegated variables. The mentioned equivalences are:

$$\begin{cases} (\beta\wedge\beta)\leftrightarrow\beta & (\beta\vee\beta)\leftrightarrow\beta \\ (\beta_1\wedge\beta_2)\leftrightarrow(\beta_2\wedge\beta_1) & (\beta_1\vee\beta_2)\leftrightarrow(\beta_2\vee\beta_1) \\ (\beta_1\wedge(\beta_2\wedge\beta_3))\leftrightarrow((\beta_1\wedge\beta_2)\wedge\beta_3) & (\beta_1\vee(\beta_2\vee\beta_3))\leftrightarrow((\beta_1\vee\beta_2)\vee\beta_3) \\ (\beta_1\wedge(\beta_2\vee\beta_3))\leftrightarrow((\beta_1\wedge\beta_2)\vee(\beta_1\wedge\beta_3)) & (\beta_1\vee(\beta_2\wedge\beta_3))\leftrightarrow((\beta_1\vee\beta_2)\wedge(\beta_1\vee\beta_3)) \\ \neg(\beta_1\wedge\beta_2)\leftrightarrow(\neg\beta_1\vee\neg\beta_2) & \neg(\beta_1\vee\beta_2)\leftrightarrow(\neg\beta_1\wedge\neg\beta_2) \\ \multicolumn{2}{c}{\neg\neg\beta\leftrightarrow\beta} \end{cases} \tag{15}$$

(Here β, β_1, β_2, β_3, are formulas of A_0.)

The equivalences in (15) are valid formulas of A_0, hence also $\alpha\leftrightarrow\alpha_0$ is valid in A_0. From this and from the assumption that α is valid in A_0 it follows that also α_0 is valid in A_0.

Since the equivalences in (15) are analogous to the identities in (5), it follows that $W(\alpha)=W(\alpha_0)$. Hence (using (14)):

$$[\alpha<1/2]=[\alpha_0<1/2]=[\alpha_1<1/2]\cup\ldots\cup[\alpha_m<1/2]. \tag{16}$$

Since α_0 is a theorem of A_0, it follows that any of the $\alpha_i's$ is a theorem of A_0 too. But this is possible if and only if

$$\alpha_i=p_i\vee\neg p_i\vee\beta_i \qquad (i=1, \ldots, m),$$

where $p_i\in V$ and β_i is an (eventuell empty) disjunction of non-negated and negated variables. Then:

$$[\alpha_i<\tfrac{1}{2}]=[p_i\vee\neg p_i\vee\beta_i<\tfrac{1}{2}]=[p_i\vee\neg p_i<\tfrac{1}{2}]\cap[\beta_i<\tfrac{1}{2}].$$

According to *Lemma* 2, $[p_i\vee\neg p_i<1/2]=\emptyset$, hence $[\alpha_i<1/2]=\emptyset$ $(i=1, \ldots, m)$. From this and from (16) it follows that $[\alpha<1/2]=\emptyset$, qu. e. d.

The following example shows that the result obtained in *Th.* 2. 1. is "best" in the following sense: if we put a number d greater than 1/2 instead of 1/2 into *Th.* 2. 1., then we get a false statement.

The counter-example: Let $\Gamma=(\Sigma, W)$ be a random model belonging to Z_0, where Σ is arbitrary, $W(p)=1/2$ for a variable p $(p\in V)$, and for other variables let W be defined arbitrarily. Then, for $z>1/2$,

$$[p\vee\neg p<z]=[\tfrac{1}{2}\vee(1-\tfrac{1}{2})<z]=[\tfrac{1}{2}<z]\cap[\tfrac{1}{2}<z]=[\tfrac{1}{2}<z]=I.$$

DEFINITION 6. Let $S=(V, O, F, T)$ be a system, let $Z=(a, b, \Theta)$ be an *rms* of S', and let d be a real number for which $a<d\leqq b$ is fulfilled. We say, that Z *is a total d-representation of* S, if $\alpha\in T$ implies that in any random model belonging to Z, $[\alpha<d]=\emptyset$ is true.[1]

Using this definition, we may re-formulate *Th.* 2. 1. as follows: *The rms* Z_0 *is a total* 1/2*-representation of the system* A_0.

DEFINITION 7. Let S, Z and d be as in *Def.* 6. We say that Z *is an adequate total d-representation of* S, if $\alpha\in T$ is true if and only if in any random model belonging to Z, $[\alpha<d]=\emptyset$ is fulfilled. — In the case $d=b$ we say that Z *is a perfect total representation of* S.

THEOREM 2. 2. *The rms* Z_0 *is an adequate total* 1/2*-representation of system* A_0.

PROOF. Having proved *Th.* 2. 1., we only have to prove the following statement: If α is a formula, which is not a theorem of A_0, then there is a random model Γ belonging to Z_0 in which $[\alpha<1/2]\neq\emptyset$. To prove this statement suppose that α is a formula which is not valid in A_0. Then there is a substitution of the variables of α by the truth values "true" and "false", for which — according to the usual reckoning rules of the truth values — α will be "false"; let us denote this substitution by B. Take $\Gamma=(\Sigma, W)$, where Σ is arbitrary, and for $p\in V$

$$W(p)=\begin{cases}1, \text{ if } p \text{ occurs in } \alpha, \text{ and } p \text{ is "true" for } B,\\ 0, \text{ if } p \text{ occurs in } \alpha, \text{ and } p \text{ is "false" for } B,\\ \text{arbitrary, otherwise.}\end{cases}$$

Since the laws in (6) are completely analogous to the reckoning rules of the truth values, it follows immediately that $W(\alpha)=0$, and so $[\alpha<1/2]=[0<1/2]=I$ in Γ.

§ 3. Random model structures of valuing logics

In the present § we shall give a universal method to construct random model structures for logical systems (defined in *Def.* 1.) which have a finite adequate valuing matrix in the sense of the *Definitions* 8-10 given below.[2]

DEFINITION 8. Let $S=(V, O, F, T)$ be a system. We shall call the triple

$$\mathbf{M}=(C, D, \Psi)$$

a *valuing matrix of the language* S' (briefly: a *matrix* of S'), if the following conditions are fulfilled:

C is a set called the set of *values* of $\mathbf{M}$ containing at least two elements.

D is a non-empty proper subset of C called the set of *designated values* of $\mathbf{M}$.

Ψ is a function defined on O such that if $o\in O_k$ $(k\geqq 1)$, then $\Psi(o)=o^*$ is a function ranging over C^k, and the values of o^* are elements of C (i. e. if $(c_1, \ldots, c_k)\in C^k$, then $o^*(c_1, \ldots, c_k)\in C$).

[1] The meaning of the adjective "total" will be cleared up later.

[2] These definitions are generalizations of similar definitions due to TARSKI, see J. ŁUKASIEWICZ—A. TARSKI: Investigations into the sentential calculus. Logic, semantics, metamathematics (by A. TARSKI). Oxford, at the Clarendon Press, 1956; p. 38—59.

MAGYAR TUDOMÁNYOS AKADÉMIA KÖNYVTÁRA

We say that the matrix $\mathbf{M}=(C, D, \Psi)$ is *finite*, if the set C is finite.

DEFINITION 9. Let $S'=(V, O, F)$ be a language, and let $\mathbf{M}=(C, D, \Psi)$ be a matrix of S'. We say that a function f is a *valuing function* of $\mathbf{M}$, if the following conditions are fulfilled:

(a) The domain of f is F, the values of f are elements of C. (b) If $o \in O_k$ $(k \geqq 1)$, $\alpha_1 \in F, \ldots, \alpha_k \in F$, then

$$f(o\alpha_1 \ldots \alpha_k) = o^*(f(\alpha_1), \ldots, f(\alpha_k)),$$

where $o^* = \Psi(o)$.

DEFINITION 10. Let $S=(V, O, F, T)$ be a system, and let $\mathbf{M}=(C, D, \Psi)$ be a matrix of S'. We say that $\mathbf{M}$ is *adequate* to S if the following condition is fulfilled: for an arbitrary formula $\alpha \in F$, $\alpha \in T$ if and only if for an arbitrary valuing function f of $\mathbf{M}$, $f(\alpha) \in D$. (In words: $\mathbf{M}$ is adequate to S, if a formula of S has always designated values if and only if it is a theorem of S.)

DEFINITION 11. Let $\mathbf{M}=(C, D, \Psi)$ be a finite matrix of a language S', and let $[a, b]$ be an interval of real numbers. We say that $\mathbf{M}$ is *appropriate* to $[a, b]$, if the following conditions are fulfilled: (a) $a \in C \setminus D$, and $b \in D$, (b) if $c \in C$, then $a \leqq c \leqq b$; (c) if $c \in C \setminus D$, and $d \in D$ then $c < d$. (In other words: the elements of C constitute a (strictly increasing) sequence of the form

$$a = c_1 < \ldots < c_m = b$$

(where m is the cardinal of C) having the following property: if $c \in C \setminus D$, and $d \in D$, then $c < d$.)

Evidently the restriction that (a finite) $\mathbf{M}$ should be appropriate to a given $[a, b]$ is inessential: if S' has a finite matrix $\mathbf{M}$, then it also has a finite matrix $\mathbf{M}_1$ ippropriate to $[a, b]$.

DEFINITION 12. Let $S'=(V, O, F)$ be a language, $[a, b]$ be an interval of real numbers, and let $\mathbf{M}=(C, D, \Psi)$ be a finite matrix of S' appropriate to $[a, b]$. We say that the *rms*

$$Z=(a, b, \Theta)$$

as *characteristic* to $\mathbf{M}$, if the function Θ is defined as follows:

Let

$$a = c_1 < \ldots < c_m = b \qquad (m \geqq 2) \tag{17}$$

be the strictly increasing sequence of the elements of C. If $o \in O_k$ $(k \geqq 1)$, $\Theta(o) = \dot{o}$, $\Psi(o) = o^*$, $w_1, \ldots, w_k$ are random variables on a probability space $\Sigma = (I, \mathscr{A}, P)$, and $w_1, \ldots, w_k$ are discrete with respect to (17), then

$$\dot{o}(w_1(E), \ldots, w_k(E)) = o^*(w_1(E), \ldots, w_k(E)) \quad \text{for} \quad E \in I. \tag{18}$$

(Note that $w_i(E) \in C$, for $i=1, \ldots, k$, and for any $E \in I$, since w_i is discrete with respect to (17).) In other cases (i.e. if at least one of the random variables $w_1, \ldots, w_k$ is not discrete with respect to (17)) $\dot{o}(w_1, \ldots, w_k)$ can be chosen arbitrarily.

It is clear that Θ, defined as above fulfils the conditions prescribed in *Def.* 3., hence Z is an *rms* of S'. Furthermore, Z is discrete with respect to (17), in the sense of *Def.* 5.

DEFINITION 13. Let S', $[a, b]$ and $\mathbf{M}$ be as in *Def.* 12. We shall define to any formula α ($\alpha \in F$) a function denoted by α^* as follows:

(a) If $\alpha \in F$, and $\langle p_1, \ldots, p_n \rangle$ is a sequence of the distinct variables of α ($n \geqq 1$) arranged, say, according to their first occurrences in α then α^* is a function on C^n, and the values of α^* belong to C.

(b) If $p \in V$ and $c \in C$ then $p^*(c) = c$.

(c) If $o \in O_k$ ($k \geqq 1$), $\alpha_i \in F$, and $\langle p_{i1}, \ldots, p_{ij_i} \rangle$ is the sequence of the variables of α_i for $1 \leqq i \leqq k$, α is $o\alpha_1 \ldots \alpha_k$, $\langle p_1, \ldots, p_n \rangle$ is the sequence of the variables of α, $(c_1, \ldots, c_n) \in C^n$, then

$\alpha^*(c_1, \ldots, c_n) = o^*(\alpha_1^*(c_{11}, \ldots, c_{1j_1}), \ldots, \alpha_k^*(c_{k1}, \ldots, c_{kj_k}))$ where $o^* = \Psi(o)$, and $c_{is} = c_r$ if and only if $p_{is} = p_r$, for $i = 1, \ldots, k$, $s = 1, \ldots, j_i$.

THEOREM 3. 1. Let S', $[a, b]$, and $\mathbf{M}$ be as in *Def.* 12, and let f be a valuing function of $\mathbf{M}$. Then, for any $\alpha \in F$,

$$f(\alpha) = \alpha^*(f(p_1), \ldots, f(p_n)),$$

where $\langle p_1, \ldots, p_n \rangle$ is the sequence of the variables of α ($n \geqq 1$; see *Def. 13*).

PROOF. We shall prove this theorem by induction on the *rank* of α. *The rank* of a formula α will be denoted by $|\alpha|$ and defined as follows: (a) If $\alpha \in V$ (i.e. if α is a variable) then $|\alpha| = 0$. (b) If $|\alpha_1| = r_1, \ldots, |\alpha_k| = r_k$, and $o \in O_k$ ($k \geqq 1$), then $|o\alpha_1 \ldots \alpha_k| = \max(r_1, \ldots, r_k) + 1$.

If $|\alpha| = 0$, i.e. if α is a variable, say p, then our theorem states that

$$f(p) = p^*(f(p)) = f(p)$$

(see (b) in *Def. 13.*), and this is of course true.

Assume that our theorem is valid for any formula β for which $|\beta| < m$ ($m > 0$), and that $|\alpha| = m$. Then $\alpha = o\alpha_1 \ldots \alpha_k$, where $o \in O_k$ ($k \geqq 1$), $|\alpha_1| < m, \ldots, |\alpha_k| < m$. According to (b) in *Def. 9* we have:

$$f(\alpha) = o^*(f(\alpha_1), \ldots, f(\alpha_k)) \tag{19}$$

where $o^* = \Psi(o)$. Since $|\alpha_i| < m$ ($i = 1, \ldots, k$), we may apply our induction assumption to α_i:

$$f(\alpha_i) = \alpha_i^*(f(p_{i1}), \ldots, f(p_{ij_i})) \qquad (i = 1, \ldots, k), \tag{20}$$

where $\langle p_{i1}, \ldots, p_{ij_i} \rangle$ is the sequence of the variables of α_i. From (19) and (20) we get:

$$f(\alpha) = o^*(\alpha_1^*(f(p_{11}), \ldots, f(p_{1j_1})), \ldots, \alpha_k^*(f(p_{k1}), \ldots, f(p_{kj_k}))).$$

According to (c) in *Def. 13* the right side of this equality equals $\alpha^*(f(p_1), \ldots, f(p_n))$. Hence our theorem is valid for α too.

THEOREM 3. 2. *Let S', $[a, b]$ and $\mathbf{M}$ be as in Def. 12, let $Z = (a, b, \Theta)$ be an rms characteristic to $\mathbf{M}$ (in the sense of Def. 12), and let $\Gamma = (\Sigma, W)$ be a random model belonging to Z and discrete with respect to the sequence* (17), *where $\Sigma = (I, \mathcal{A}, P)$. For arbitrary α ($\alpha \in F$) let us denote $W(\alpha)$ by $\dot{\alpha}$. Then: If $\alpha \in F$, $E \in I$, and $\langle p_1, \ldots, p_n \rangle$ is the sequence of the variables of α ($n \geqq 1$), then*

$$\dot{\alpha}(E) = \alpha^*(\dot{p}_1(E), \ldots, \dot{p}_n(E)).$$

PROOF. We shall prove this theorem by induction on the *rank* of α (see the proof of the preceding theorem).

If $|\alpha|=0$, i.e. if α is a variable, say p, then our theorem states that

$$\dot{p}(E) = p^*(\dot{p}(E)) = \dot{p}(E)$$

(see (b) in *Def. 13*), and this holds trivially.

Assume that our theorem is true for any formula β for which $|\beta|<m$ $(m>0)$, and that $|\alpha|=m$. Then $\alpha=o\alpha_1\ldots\alpha_k$, where $o\in O_k$ $(k\geqq 1)$, $|\alpha_1|<m, \ldots, |\alpha_k|<m$. According to (18) in *Def. 12* we have:

$$\dot{\alpha}(E) = \dot{o}(\dot{\alpha}_1(E), \ldots, \dot{\alpha}_k(E)) = o^*(\dot{\alpha}_1(E), \ldots, \dot{\alpha}_k(E)), \tag{21}$$

where $\dot{o}=\Theta(o)$, and $o^*=\Psi(o)$. Since $|\alpha_i|<m$ $(i=1, \ldots, k)$, we may apply our induction assumption to α_i:

$$\dot{\alpha}_i(E) = \alpha_i^*(\dot{p}_{i1}(E), \ldots, \dot{p}_{ij_i}(E)) \qquad (i=1, \ldots, k), \tag{22}$$

where $\langle p_{i1}, \ldots, p_{ij_i}\rangle$ is the sequence of the variables of α_i. From (21) and (22) we get:

$$\dot{\alpha}(E) = o^*(\alpha_1^*(\dot{p}_{11}(E), \ldots, \dot{p}_{1j_1}(E)), \ldots, \alpha_k^*(\dot{p}_{k1}(E), \ldots, \dot{p}_{kj_k}(E))).$$

According to (c) in *Def. 13* the right side of this equality is identical to $\alpha^*(\dot{p}_1(E), \ldots, \dot{p}_n(E))$. Hence our theorem is valid for α too.

COROLLARY. If for some $E\in I$, $\dot{\alpha}(E)=c$ (where $c\in C$), then there is a valuing function f of $\mathbf{M}$, for which $f(\alpha)=c$.

PROOF. According to the assumption and *Th.* 3. 2

$$c = \dot{\alpha}(E) = \alpha^*(\dot{p}_1(E), \ldots, \dot{p}_n(E)). \tag{23}$$

Take: $\dot{p}_1(E)=c_1, \ldots, \dot{p}_n(E)=c_n$. Clearly, $\{c_1, \ldots, c_n\}\subseteqq C$. Choose the function f as follows:

$$f(p) = \begin{cases} c_i, & \text{if} \quad p=p_i, \quad \text{for} \quad i=1, \ldots, n, \\ a & \text{otherwise.} \end{cases}$$

Then, according to *Th. 3. 1*, and using (23), we get that

$$f(\alpha) = \alpha^*(c_1, \ldots, c_n) = \alpha^*(\dot{p}_1(E), \ldots, \dot{p}_n(E)) = \dot{\alpha}(E) = c;$$

qu. e. d.

THEOREM 3. 3. *Let $S=(V, O, F, T)$ be a system, let $[a, b]$ be an interval of real numbers, let $\mathbf{M}=(C, D, \Psi)$ be a finite matrix of S' appropriate to $[a, b]$ and adequate to S, let Z be an rms characteristic to $\mathbf{M}$, let Γ be a random model belonging to Z and discrete with respect to the strictly increasing sequence of the elements of C, and let d be the smallest element of D (i.e. the smallest designated value of $\mathbf{M}$). Then: If $\alpha\in T$, then $[\alpha<d]=\emptyset$.*

PROOF. Since $\alpha\in T$, and $\mathbf{M}$ is adequate to S, it follows that for any valuing function f of $\mathbf{M}$, $f(\alpha)\geqq d$ holds. Henceforth, according to the *Corollary* of *Th. 3. 2*, for any $E\in I$, $\dot{\alpha}(E)\geqq d$ holds too. Consequently, the set $[\alpha<d]$ is empty, indeed.

DEFINITION 14. Let $S=(V, O, F, T)$ be a system, and let $Z=(a, b, \Theta)$ be an *rms* of S' discrete to a sequence $\langle a=e_1<\ldots<e_m=b\rangle$ (in the sense of *Def. 5*), and d be a real number for which $a<d\leqq b$ is fulfilled. We say that Z *is a discrete d-representation of* S *in* $\langle e_1, \ldots, e_m\rangle$, if $\alpha\in T$ implies that in any random model belonging to Z and being discrete to $\langle e_1, \ldots, e_m\rangle$ (in the sense of the second part of *Def. 5.*), $[\alpha<d]=\emptyset$ is true.

Using this definition, we can re-formulate *Th. 3. 3.* as follows: *If the system* S *has a finite adequate matrix, and* $[a, b]$ *is an arbitrary real interval, then there is a finite matrix* $\mathbf{M}$ *of* S' *appropriate to* $[a, b]$ *and adequate to* S *such that any rms* Z *characteristic to* $\mathbf{M}$ *is a discrete d-representation of* S *in* $\langle c_1, \ldots, c_m\rangle$, *where* $\{c_1, \ldots, c_m\}=C$, *and* d *is the smallest element of* D.

DEFINITION 15. Let S, Z, $\langle e_1, \ldots, e_m\rangle$ and d be as in *Def. 14.* We say that Z *is an adequate discrete d-representation of* S *in* $\langle e_1, \ldots, e_m\rangle$, if $\alpha\in T$ is true if and only if in any random model discrete to $\langle e_1, \ldots, e_m\rangle$ and belonging to Z, $[\alpha<d]=\emptyset$ is true. — In the case $d=b$ we say that Z is *a perfect discrete representation* of S in $\langle e_1, \ldots, e_m\rangle$.

THEOREM 3. 4. *Let* S, $[a, b]$ $\mathbf{M}$, Z, *and* d *be as in Th. 3. 3. Let* $\langle c_1, \ldots, c_m\rangle$ *be the strictly increasing sequence of the elements of* C. *Then:* Z *is an adequate discrete d-representation of* S *in* $\langle c_1, \ldots, c_m\rangle$.

PROOF. Having *Th. 3. 3.* we have to prove only the following statement: If $\alpha\in F$ but $\alpha\notin T$, then there is a random model Γ belonging to Z and discrete to $\langle c_1, \ldots, c_m\rangle$ in which $[\alpha<d]\neq\emptyset$.

To construct such a Γ, let f be a valuing function for which $f(\alpha)=c<d$ is true. (There exists such an f, since $\alpha\notin T$, and $\mathbf{M}$ is adequate to S.) Let $\{p_1, \ldots, p_r\}$ be the set of variables occuring in α. We define the probability space $\Sigma=(I, \mathscr{A}, P)$ as follows: Let I be C, let $\mathscr{A}$ contain all subsets of I (i.e. of C), and let $P(\{c_i\})=\frac{1}{m}$ be fullfilled ($i=1, \ldots, m$). (It is clear that Σ is fully determined by these stipulations.) Let the function W be defined as follows (we use the notation $\dot{\alpha}$ for $W(\alpha)$, as above):

$$\dot{p}=\begin{cases} f(p), & \text{if } p\in\{p_1, \ldots, p_r\},\\ b \text{ otherwise (provided that } p\in V).\end{cases}$$

(W is defined for other formulas by *Def. 4.*)

Consider the random model $\Gamma=(\Sigma, W)$ with Σ and W defined above. It is clear that Γ is trivially discrete with respect to $\langle c_1, \ldots, c_m\rangle$. Using *Th. 3. 2*, we have:

$$\dot{\alpha}\equiv\alpha^*(\dot{p}_1, \ldots, \dot{p}_n)\equiv\alpha^*(f(p_1), \ldots, f(p_n)) \tag{24}$$

(using that $\dot{p}_i$ is constant). Furthermore, *Th. 3. 1* gives that

$$c=f(\alpha)=\alpha^*(f(p_1), \ldots, f(p_n)). \tag{25}$$

From (24) and (25) we get that $\dot{\alpha}\equiv c<d$, i.e. $[\alpha=c]=I$, that is $[\alpha<d]=I$ in Γ; qu. e. d.

The next theorem gives the simplest example to *Theorems 3. 3—3. 4.*

THEOREM 3. 5. *The rms* Z_0 *(see in* §. 2*) is a perfect discrete representation of system* A_0 *in* $\langle 0, 1\rangle$.

PROOF. Let C be $\{0, 1\}$, D be $\{1\}$. We define a function Ψ over $\{\neg, \wedge\}$ by the following table:

$\wedge^*$	0	1	$\neg^*$
0	0	0	1
1	0	1	0

The matrix $\mathbf{M}_0 = (C, D, \Psi)$ is appropriate to [0, 1], and — as it is well-known — is adequate to system A_0. It can be seen easily that Z_0 is discreet with respect to $\langle 0, 1\rangle$. Furthermore, Z_0 fulfils the stipulations of *Def. 13.*, i.e. Z_0 is *characteristic* to $\mathbf{M}_0$. In details (see (18) in *Def. 13*);

$$\dot{\neg} w(E) = 1 - w(E) = \neg^*(w(E)),$$

since $\neg^*(c) = 1 - c$. Furthermore:

$$\dot{\wedge}(w_1(E), w_2(E)) = \min(w_1(E), w_2(E)) = \wedge^*(w_1(E), w_2(E)),$$

since $\wedge^*(c_1, c_2) = \min(c_1, c_2)$, as it may be seen immediately from the table given above.

Henceforth, according to *Th. 3. 4.* and *Def. 15.* we get that Z_0 is a perfect discrete representation of A_0 in $\langle 0, 1\rangle$.

In Part II we shall apply our results in the analysis of ŁUKASIEWICZ's modal systems.

Eötvös L. University, Budapest

(Received June 1, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 313–329.

ASYMPTOTICALLY UNIFORM SEQUENCES OF MEASURES

by

D. SZÁSZ

Introduction

Denote by ν^n the n-th convolution power of a probability measure given on a locally compact abelian topological group. Let μ be a — possibly infinite — measure and consider the convolution $\mu * \nu^n$. In connection with the investigation of general branching processes ([6]) the following problem has arisen: under which conditions on ν can we assert for an enough large class of measures μ that $\mu * \nu^n$ tends weakly to the Haar measure? The stated property expresses that the powers ν^n have a smoothing effect on the measure μ. If the group is compact, then results of KLOSS and URBANIK ([4], [7]) state that under fairly general conditions ν^n tends weakly to the normed Haar measure of the group, so we can say that the sequence $\{\nu^n\}$ is asymptotically uniform.

In the locally compact case the sequence $\{\nu^n\}$ tends weakly to the 0-measure, unless ν is degenerate. This is so, because in a locally compact group there exists no finite Haar measure. To express the asymptotic uniformity in this case we have to give a suitable definition of asymptotic uniformity on the basis of a characteristic property of the Haar measure. This may be done in completely different ways: H. HERRMANN and A. RÉNYI gave such definitions, while the smoothing effect described above offers a third possibility. Our aim is to investigate asymptotic uniformity in this last sense.

In § 0 we give some well-known definitions and results of the theory of regularly varying functions and of stable distributions. § 1 deals with the three mentioned definitions of asymptotic uniformity: that of A. RÉNYI, H. HERMANN and that given by the above effect; some useful lemmas and theorems concerning the last definition are proved. Examples show that the three definitions are completely independent of each other. In § 2 we give conditions under which the sequence $\{\nu^n\}$ is asymptotically uniform in our sense. In case of RÉNYI's definition this is done in [5], and in case of HERRMANN's definition in [2] and in a recent paper of J. KERSTAN and K. MATTHES ([8]).

§ 0. Preliminaries

Here we summarize some known definitions and results, which will be used in our proofs.

We say that a positive function $h(t)$ defined for $t>0$ is *slowly varying*, if and only if for arbitrary $x>0$

$$\lim_{t\to\infty}\frac{h(tx)}{h(t)}=1$$

We say that a function $Z(t)$ defined for $t>0$ is *regularly varying* with exponent γ $(-\infty<\gamma<+\infty)$, if and only if it is of the form

$$Z(t)=t^{\gamma}h(t)$$

where $h(t)$ is a slowly varying function.

The definitions above remain reasonable, if the functions are defined or are positive only for large values of x.

THEOREM 0. 1: [1] *If $Z(t)$ is regularly varying with exponent γ and*

$$Z_p(t)=\int_0^t y^pZ(y)\,dy,\quad Z_p^*(t)=\int_t^{\infty} y^pZ(y)\,dy$$

then for $p+\gamma+1\geqq 0$

$$\lim_{t\to\infty}\frac{t^{p+1}Z(t)}{Z_p(t)}=p+\gamma+1$$

and for $p+\gamma+1<0$

$$\lim_{t\to\infty}\frac{t^{p+1}Z(t)}{Z_p^*(t)}=|p+\gamma+1|$$

(The latter equality is also true, if $p+\gamma+1=0$ *and* $Z_{\gamma-1}^*$ *exists.)*

We recall that a nondegenerate probability law $G(x)$ is *stable*, if and only if it is the limit of the distributions of the normed partial sums

$$\frac{\xi_1+\ldots+\xi_n}{B_n}-A_n \tag{1}$$

where $\xi_1,\ldots,\xi_n,\ldots$ are independent, identically distributed random variables with the common distribution $F(x)$ and A_n, B_n are suitable constants. In this case we say that $F(x)$ belongs to the *domain of attraction* of the stable law $G(x)$. The logarithms of the characteristic functions of the stable laws have the form

$$i\gamma t-c|t|^{\alpha}\left(1-i\beta\frac{t}{|t|}w(t,\alpha)\right)$$

where α, β, γ, c are real constants $(c\geqq 0,\ 0<\alpha\leqq 2,\ -1\leqq\beta\leqq 1)$, and

$$w(t,\alpha)=\begin{cases}\operatorname{tg}\dfrac{\pi}{2}\alpha & \alpha\neq 1\\[2ex] \dfrac{2}{\pi}\log|t| & \alpha=1\end{cases}$$

α is called the *exponent* of the stable law. ([3], T. 2. 2. 2).

THEOREM 0. 2 ([3], T. 2. 6. 1) *The distribution function $F(x)$ belongs to the domain of attraction of a stable law with exponent α $(0<\alpha<2)$ if and only if for $|x|\to\infty$*

$$F(x) = \frac{c_1+o(1)}{|x|^\alpha} h(|x|) \qquad x<0$$

(2)

$$1-F(x) = \frac{c_2+o(1)}{x^\alpha} h(x) \qquad x>0$$

where $h(x)$ is an arbitrary slowly varying function, $c_1 \geqq 0$, $c_2 \geqq 0$ and $c_1+c_2>0$.

Let us introduce the notation

$$\chi(x) = F(-x)+1-F(x) \qquad (x>0)$$

It is shown ([3], T. 2. 6. 1) that if $F(x)$ belongs to the domain of attraction of a stable law with exponent α, then in (1) the norming constants B_n can be chosen so that B_n should be the greatest lower bound of those values x, for which

(3) $$\chi(x+0) \leqq \frac{c_1+c_2}{n} \leqq \chi(x-0)$$

LEMMA 0. 3: *If $h(x)$ and B_n are defined by* (2) *resp.* (3), *then for any $\varepsilon>0$*

(4) $$n\frac{h(\varepsilon B_n)}{B_n^\alpha} \sim 1 \qquad (n\to\infty)$$

PROOF. From (2) it follows that

$$\chi(B_n) \sim \frac{c_1+c_2}{B_n^\alpha} h(B_n)$$

but from (3) we have

$$\chi(B_n) \sim \frac{c_1+c_2}{n}$$

thus

$$\frac{nh(B_n)}{B_n^\alpha} \sim 1$$

$h(x)$ is a slowly varying function, so (4) is true.

LEMMA 0. 4: *For any $\varepsilon>0$*

$$\lim_{n\to\infty} nF(-\varepsilon B_n) = c_1\varepsilon^{1-\alpha}$$

$$\lim_{n\to\infty} n(1-F(\varepsilon B_n)) = c_2\varepsilon^{1-\alpha}$$

PROOF. We prove only the first equality:

$$nF(-\varepsilon B_n) = n\frac{c_1+o(1)}{\varepsilon^\alpha B_n^\alpha} h(\varepsilon B_n)$$

If $c_1=0$, then the assertion is trivial, if $c_1>0$, then it is the consequence of Lemma 0.3.

THEOREM 0. 5: ([3], T. 2. 6. 2) *The distribution function $F(x)$ belongs to the domain of attraction of the stable law with exponent 2 (i.e. of the normal law) if and only if one of the following conditions is satisfied*

1^0 *$F(x)$ has finite variance*

2^0 *For $x>0$*

$$\chi(x)=\frac{h(x)}{x^2}$$

where $h(x)$ is an arbitrary slowly varying function.

It is an easy consequence of the convergence theorems that in this case

$$\lim_{n\to\infty} n\chi(B_n x) = 0 \tag{5}$$

for any $x>0$.

In general $(0<\alpha\leqq 2)$, if the $B_n's$ are already given, then the constants A_n can be chosen so that

$$A_n = n\int_{|x|<\varepsilon} x dF(xB_n)+\gamma(\varepsilon) = \frac{n}{B_n}\int_{|x|<\varepsilon B_n} x dF(x)+\gamma(\varepsilon) \tag{6}$$

where ε is a fixed positive number and $\gamma(\varepsilon)$ an arbitrary constant.

Let us denote the distribution function of the random variable (1) by $F_n(x)$, the absolute continuous resp. singular components of $F_n(x)$ by $R_n(x)$ and $S_n(x)$. The density of $R_n(x)$ will be denoted by $p_n(x)$.

THEOREM 0. 6: ([3], T. 4. 4. 1.) *If $g(x)$ denotes the density of a stable law $G(x)$, then*

$$\lim_{n\to\infty}\int_{-\infty}^{+\infty} |p_n(x)-g(x)|\,dx = 0$$

if and only if

(i) *$F(x)$ belongs to the domain of attraction of $G(x)$*

(ii) *There exists a positive integer N, for which $R_N(+\infty)>0$.*

§ 1. Asymptotic uniformity

Let $\{\nu_n\}_{n=1}^{\infty}$ be a sequence of probability measures given on a locally compact abelian topological group G. (Writing R instead of G we shall restrict ourselves to the additive group of real numbers.)

DEFINITION 1. (A. RÉNYI, [5]) We say that the sequence $\{\nu_n\}$ is *asymptotically uniform* in sense 1) on G if and only if for arbitrary bounded open sets U and V

$$\lim_{n\to\infty}\frac{\nu_n(U)}{\nu_n(V)} = \frac{\lambda(U)}{\lambda(V)}$$

where λ is the Haar measure. (It is implicitely required that for enough large values of n $\nu_n(V)>0$.)

DEFINITION 2. (H. HERRMANN, [2]). We say that the sequence $\{\nu_n\}$ is *asymptotically uniform* in sense 2) if and only if for every $x \in G$

$$\lim_{n\to\infty} \mathrm{Var}(\nu_n - \nu_n * \delta_x) = 0$$

where δ_x is the probability measure, concentrated into the point x, $*$ means convolution of measures, and Var ( ) means the total variation of the signed measure in brackets. We remark, that in [2] a weaker form of this definition is also given.

Let us denote

$$m^*(x) = \begin{cases} 1 & x \geqq 0 \\ 0 & x < 0 \end{cases}$$

$$m_*(x) = \begin{cases} 1 & x < 0 \\ 0 & x \geqq 0 \end{cases}$$

DEFINITION 3: Let $\mathfrak{M}$ be a linear class of complex measures μ defined on R. Suppose that $\mathfrak{M}$ contains the measures μ^* and μ_* with

$$\frac{d\mu^*}{d\lambda}(x) = m^*(x) \qquad \frac{d\mu_*}{d\lambda}(x) = m_*(x)$$

We say that the sequence $\{\nu_n\}$ is *asymptotically uniform with respect to the class* $\mathfrak{M}$, if and only if for every $\mu \in \mathfrak{M}$

$$\mu * \nu_n \to \psi(\mu) \cdot \lambda \tag{7}$$

weakly as $n \to \infty$ (i.e. for bounded sets U $\lim_{n\to\infty} (\mu * \nu_n)(U) = \psi(\mu)\lambda(U)$), where $\psi(\mu)$ is a linear functional on $\mathfrak{M}$ such that $\psi(\mu^*)\psi(\mu_*) \neq 0$.

REMARK. The restriction $\psi(\mu^*)\psi(\mu_*) \neq 0$ is advantageous and disadvantageous at the same time. Its disadvantage is that it is difficult to generalize. Its advantage is that it does not allow the sequence ν_n to go out to the infinity and this fact has importance in applications for branching processes.

Obviously the more measures are contained in $\mathfrak{M}$, the more the asymptotic uniformity with respect to $\mathfrak{M}$ requires. $\mathfrak{M}$ would be chosen succesfully if on one side for a large class of measures ν, the sequence $\{\nu^n\}$, (ν^n denotes the n^{th} convolution power of ν) were asymptotically uniform with respect to $\mathfrak{M}$, and on the other side asymptotic uniformity in the sense of Definitions 1 and 2 followed from asymptotic uniformity with respect to $\mathfrak{M}$. The first aim is reached, the second is not.

Let us define $\mathfrak{M}$ as follows: let $\tilde{\mathfrak{B}}$ denote the Banach space of bounded, complex valued functions, defined on the real line, supplied with the uniform norm, and consider the subset $\tilde{\mathfrak{M}}_0$ of $\tilde{\mathfrak{B}}$ consisting of functions of type

$$m(t) = \begin{cases} e^{i\gamma t} & t \geqq a \\ 0 & t < a \end{cases}$$

and

$$m(t) = \begin{cases} e^{i\gamma t} & t < a \\ 0 & t \geqq a \end{cases}$$

where a and γ are arbitrary real constants.

Let $\mathfrak{L}(\tilde{\mathfrak{M}}_0)$ and $[\mathfrak{L}(\tilde{\mathfrak{M}}_0)]=\tilde{\mathfrak{M}}$ denote the linear manifold generated by $\tilde{\mathfrak{M}}_0$ resp. its closure in $\tilde{\mathfrak{B}}$. Let $\mathfrak{M}$ (and $\mathfrak{M}_0$) be the set of complex-measures μ, which are absolutely continuous and the density m of which belongs to $\tilde{\mathfrak{M}}$ ($\tilde{\mathfrak{M}}_0$). $\mathfrak{M}$ can be supplied with a norm in such a way that if m is the density of μ, then the norm of μ is the same as that of m. It is obvious that the fact that here also complex measures are allowed, has no important influence.

In this paper we apply Definition 3 so that $\mathfrak{M}$ means the class defined above and refer to the notion of asymptotic uniformity with respect to $\mathfrak{M}$ as to asymptotic uniformity in sense 3).

We remark that $\tilde{\mathfrak{M}}$ contains every almost periodic function and also the functions of type

$$m(t)=\begin{cases} m_1(t) & t\geqq a \\ 0 & t<a \end{cases}$$

and

$$m(t)=\begin{cases} m_1(t) & t<a \\ 0 & t\geqq a \end{cases}$$

where $m_1(t)$ is arbitrary almost periodic function and a is an arbitrary constant.

First we prove some useful lemmas.

LEMMA 1. 1 *If for a sequence* $\{\nu_n\}$ *of probability measures we know that* (7) *is valid for arbitrary* $\mu\in\mathfrak{M}_0$, *and that* $\psi(\mu^*)\psi(\mu_*)\neq 0$, *then the sequence* $\{\nu_n\}$ *is asymptotically uniform in sense* 3).

PROOF. From the conditions of the lemma it follows that (7) is valid for elements of $\mathfrak{L}(\mathfrak{M}_0)$, and that $\psi(\mu)$ is a linear functional on $\mathfrak{L}(\mathfrak{M}_0)$. What is more $\psi(\mu)$ is also bounded, its norm is 1, so its definition can be extended uniquely to $\mathfrak{M}$ keeping linearity, boundedness and norm.

For every $\mu\in\mathfrak{M}$ we can choose a sequence $\mu_1, \mu_2, \ldots$ such that $\mu_k\in\mathfrak{L}(\mathfrak{M}_0)$ and

$$\mu_k\to\mu$$

in norm. So for fixed U and $\varepsilon>0$ there exists a k_0 such that for $k\geqq k_0$

$$\sup_t |\mu_k(U-t)-\mu(U-t)| \leqq \varepsilon$$

Now for $k\geqq k_0$

$$|(\mu * \nu_n)(U)-(\mu_k * \nu_n)(U)| = \left|\int (\mu_k(U-t)-\mu(U-t))\nu_n(dt)\right| \leqq \varepsilon$$

Further if $k\geqq k_1$, then

$$|\psi(\mu_k)-\psi(\mu)| < \frac{\varepsilon}{\lambda(U)}$$

thus if $k\geqq\max(k_0, k_1)$

$$\begin{aligned} |(\mu * \nu_n)(U)-\psi(\mu)\lambda(U)| &\leqq |(\mu * \nu_n)(U)-(\mu_k * \nu_n)(U)| + \\ &+ |(\mu_k * \nu_n)(U)-\psi(\mu_k)\lambda(U)| + |\lambda(U)(\psi(\mu_k)-\psi(\mu))| \leqq \\ &\leqq 2\varepsilon + |(\mu_k * \nu_n)(U)-\psi(\mu_k)\lambda(U)| \end{aligned}$$

For a fixed k the right side tends to 2ε if $n \to \infty$, but ε is arbitrary, the lemma is proved.

LEMMA 1. 2 *If* $\mu \in \mathfrak{M}$, $\frac{d\mu}{d\lambda}(t) = m(t)$, *and for every* $x \in R$

$$\lim_{n \to \infty} \int m(x-t)\, \nu_n(dt) = c$$

then for every bounded set

$$\lim_{n \to \infty} (\mu * \nu_n)(U) = c\lambda(U)$$

PROOF. $m(t)$ is a bounded function, thus the Lebesgue theorem can be applied.

LEMMA 1. 3. *If the sequence* $\{\nu_n\}$ *is asymptotically uniform in sense* 3), *then for every bounded* A

$$\lim_{n \to \infty} \nu_n(A) = 0 \tag{8}$$

PROOF. We prove (8) for an arbitrary bounded interval $J = (a, b)$. If (8) is not true, then for an $\varepsilon > 0$ and a sequence n_k

$$\nu_{n_k}(J) \geqq \varepsilon \tag{9}$$

Clearly if $I = (\alpha, \beta)$, then

$$(\mu_* * \nu_{n_k})(I) = \int_\alpha^\beta (t-\alpha)\, \nu_{n_k}(dt) + \int_\beta^\infty \lambda(I)\, \nu_{n_k}(dt)$$

and

$$(\mu_* * \nu_{n_k})(I+s) = \int_{\alpha+s}^{\beta+s} (t-\alpha-s)\, \nu_{n_k}(dt) + \int_{\beta+s}^{\infty} \lambda(I)\, \nu_{n_k}(dt)$$

so if the numbers α, β and s satisfy $\alpha < \beta \leqq \alpha + s < \beta + s$, then

$$(\mu_* * \nu_{n_k})(I) - (\mu_* * \nu_{n_k})(I+s) =$$

$$= \int_\alpha^\beta (t-\alpha)\, \nu_{n_k}(dt) + \int_\beta^{\alpha+s} \lambda(I)\, \nu_{n_k}(dt) + \int_{\alpha+s}^{\beta+s} (\lambda(I) - t + \alpha + s)\, \nu_{n_k}(dt) \geqq$$

$$\geqq \int_{\alpha+s}^{\frac{\alpha+\beta}{2}+2} (\lambda(I) - t + \alpha + s)\, \nu_{n_k}(dt) \geqq \frac{\beta-\alpha}{2}\, \nu_{n_k}\left(\alpha+s, \frac{\alpha+\beta}{2}+s\right)$$

If we choose α, β and s so that $\alpha + s = a$, $\frac{\alpha+\beta}{2} + s = b$, then the obtained inequality contradicts to the asymptotic uniformity of ν_n. (The necessary equalities can be reached with $\alpha = 4a - 3b$, $\beta = 2a - b$ and $s = 2(b-a)$.)

LEMMA 1. 4. *If* $\{\nu_n\}$ *is asymptotically uniform in sense* 3), *then if*

$$\frac{d\mu}{d\lambda}(t) = \begin{cases} e^{i\gamma t} & t \geqq a \qquad (t < a) \\ 0 & t < a \qquad (t \geqq a) \end{cases}$$

and $\gamma \neq 0$, *then* $\psi(\mu) = 0$, *where* ψ *is defined by* (7).

PROOF. We can suppose that $a = 0$. If $I = (\alpha, \beta)$, then from the asymptotic uniformity we have

$$\lim_{n\to\infty} (\mu * \nu_n)(I) = \psi(\mu)\lambda(I)$$

and

$$\lim_{n\to\infty} (\mu * \nu_n)\left(I + \frac{\pi}{\gamma}\right) = \psi(\mu)\lambda(I).$$

But

$$(\mu * \nu_n)\left(I + \frac{\pi}{\gamma}\right) = \int_{I+\frac{\pi}{\gamma}} \int_{-\infty}^{x} e^{i\gamma(x-t)} \nu_n(dt)\, dx = \int_I \int_{-\infty}^{x+\frac{\pi}{\gamma}} e^{i\gamma(x-t)} e^{i\pi} \nu_n(dt)\, dx =$$

$$= -\int_I \int_{-\infty}^{x} e^{i\gamma(x-t)} \nu_n(dt)\, dx - \int_I \int_{x}^{x+\frac{\pi}{\gamma}} e^{i\gamma(x-t)} \nu_n(dt)\, dx$$

Obviously

$$\left| \int_I \int_x^{x+\frac{\pi}{\gamma}} e^{i\gamma(x-t)} \nu_n(dt)\, dx \right| \leqq \int_I \int_\alpha^{\beta+\frac{\pi}{\gamma}} \nu_n(dt)\, dx \leqq \lambda(I)\, \nu_n\left([\alpha, \beta + \frac{\pi}{\gamma}]\right) \to 0 \qquad (n \to \infty)$$

so

$$\lim_{n\to\infty} (\mu * \nu_n)\left(I + \frac{\pi}{\gamma}\right) = -\lim_{n\to\infty} (\mu * \nu_n)(I)$$

thus $\psi(\mu) = 0$. Q. e. d.

Simple examples show that Definitions 1, 2 and 3 are completely independent of each other. Let us denote by $\varepsilon(a, b)$ the uniform propability measure on the interval (a, b).

EXAMPLES

1) & 2) $\not\Rightarrow$ 3)

Let

$$\nu_n = \varepsilon(-n^2, n)$$

The asymptotic uniformity of $\{\nu_n\}$ in senses 1) and 2) is obvious. On the other hand one has for $n \to +\infty$

$$\int m_*(x-t)\, \nu_n(dt) = \frac{n-x}{n+n^2} \to 0$$

1) & 3) $\not\Rightarrow$ 2)

Now ν_n is purely atomic, it has atoms of mass $\frac{1}{2n^2}$ in every point of form $k+\frac{i}{n}$, where $-n \leqq k < n$, $0 \leqq i < n$. Asymptotic uniformity in sense 1) is obvious, that in sense 3) can be shown by an easy calculation, and if x is irrational then

$$\operatorname{Var}(\nu_n - \nu_n * \delta_x) = 2$$

2) & 3) $\not\Rightarrow$ 1)

Let

$$\frac{d\nu_n}{d\lambda}(x) = \begin{cases} 2\varepsilon_n x & \text{if} \quad x \in [0,1) \\ \dfrac{1-\varepsilon_n}{2n-1} & \text{if} \quad x \in [-n,0) \cup [1,n) \\ 0 & \text{otherwise} \end{cases}$$

where $\lim_{n\to\infty} \varepsilon_n = 0$. It is easy to see that $\{\nu_n\}$ is asymptotically uniform in senses 2) and 3), but for example

$$\frac{\nu_n((0,\frac{1}{2}))}{\nu_n((0,1))} \to \frac{1}{4} \neq \frac{1}{2}$$

Now we prove two simple theorems which reflect some properties of asymptotic uniformity in sense 3).

THEOREM 1. 5. *If α is an arbitrary probability measure and $\{\nu_n\}$ is asymptotically uniform in sense* 3), *then the sequence $\{\alpha * \nu_n\}$ is also asymptotically uniform in sense* 3).

PROOF. By Lemma 1. 1 it is necessary to prove the validity of (7) only for $\mu \in \mathfrak{M}_0$. Let

$$\frac{d\mu}{d\lambda}(x) = m(x) = \begin{cases} e^{i\lambda x} & x \geqq 0 \\ 0 & x < 0 \end{cases}$$

Then the density of $\alpha * \mu$ has the form

$$n(x) = \int_{-\infty}^{x} e^{i\lambda(x-t)}\,\alpha(dt) = e^{i\lambda x} \int_{-\infty}^{x} e^{-i\lambda t}\,\alpha(dt)$$

Clearly

$$\left| \int_{-\infty}^{x} e^{-i\lambda t}\,\alpha(dt) \right| \leqq \alpha((-\infty, x]) \leqq 1$$

and we can choose $x' < 0$ and $x'' > 0$ so that for a given positive ε

$$\alpha((-\infty, x']) < \varepsilon$$

and

$$\alpha([x'', +\infty)) < \varepsilon$$

Now let

$$n(x) = n_1(x) + n_2(x) + n_3(x) + n_4(x)$$

where

$$n_1(x) = \begin{cases} n(x) & x < x' \\ 0 & \sim \end{cases}$$

$$n_2(x) = \begin{cases} n(x) & x' \leqq x \leqq x'' \\ 0 & \sim \end{cases}$$

$$n_3(x) = \begin{cases} e^{i\lambda x} \int_{-\infty}^{+\infty} e^{-i\lambda t}\alpha(dt) & x > x'' \\ 0 & \sim \end{cases}$$

$$n_4(x) = \begin{cases} n(x) - e^{i\lambda x} \int_{-\infty}^{+\infty} e^{-i\lambda t}\alpha(dt) & x > x'' \\ 0 & \sim \end{cases}$$

Obviously

$$\left| \int n_1(x-t)\nu_n(dt) \right| \leqq \varepsilon$$

and

$$\left| \int n_4(x-t)\nu_n(dt) \right| \leqq \varepsilon$$

further because of Lemma 1.3

$$\left| \int n_2(x-t)\nu_n(dt) \right| \leqq \nu_n([x', x'']) \to 0$$

if $n \to \infty$. If $\lambda \neq 0$, then

$$\int n_3(x-t)\nu_n(dt) = \int_{-\infty}^{+\infty} e^{-i\lambda t}\alpha(dt) \int_{-\infty}^{x-x''} e^{i\lambda(x-u)}\nu_n(du) \to 0 \qquad (n \to \infty)$$

and if $\lambda = 0$, then

$$\int n_3(x-t)\nu_n(dt) = \int_{-\infty}^{x-x''} \nu_n(dt) \to \psi(\mu) \qquad (n \to \infty)$$

So in general for any x

$$\int n(x-t)\nu_n(dt) \to \psi(\mu)$$

if $n \to \infty$, and the theorem is proved.

The proof shows also the fact that the functional $\psi(\mu)$ is the same for the sequence $\{\alpha * \nu_n\}$ as it was for the sequence $\{\nu_n\}$.

THEOREM 1. 6. *If* $\{\nu_n\}$ *is asymptotically uniform in sense 3) and* $h_n = O(1)$, *then* $\{\nu_n * \delta_{h_n}\}$ *is also asymptotically uniform in sense 3).*

PROOF. We assert that if $\mu \in \mathfrak{M}_0$, then for bounded open sets U

$$\int \mu(U - h_n - t)\nu_n(dt) \to \psi(\mu)\lambda(U)$$

where ψ is the functional corresponding to the sequence $\{v_n\}$. If this were not true, then we could choose an $\varepsilon>0$ and a subsequence $\{n_k\}_{n=1}^{\infty}$ such that $h_{n_k}\to h$ $(k\to\infty)$, and

$$\left|\int [\mu(U-h_{n_k}-t)-\mu(U-h-t)]\, v_{n_k}(dt)\right| > \varepsilon \tag{10}$$

Obviously if $k\to\infty$, then uniformly in t

$$\mu(U-h_{n_k}-t)\to\mu(U-h-t)$$

so (10) is impossible.

§ 2. Conditions for the asymptotic uniformity in sense 3) of v^n

Now we turn to give conditions for the asymptotic uniformity of the sequence $\{v^n\}_{n=1}^{\infty}$, where v is a given probability measure. We shall see that in certain cases the asymptotic uniformity of the sequence strongly depends on the behaviour of the constants A_n in (1). This is partly shown by the following theorem.

THEOREM 2. 1. *Suppose that for the distribution function $F(x)$ of the measure v the conditions of Theorem* 0. 6 *are satisfied. Then*

$$A_n = O(1)$$

is a necessary condition for the asymptotic uniformity of $\{v^n\}$, while

$$A_n = o(1)$$

and

$$0 < G(0) < 1$$

are sufficient conditions for that. $G(x)$ is the stable distribution function in Theorem 0. 6.

PROOF. From the inequality

$$R_{n+1}(\infty) \geqq R_n(\infty)(2-R_n(\infty))$$

it follows that

$$R_n(\infty)\to 1 \quad \text{if} \quad n\to\infty \tag{11}$$

If $A_n=O(1)$ is not true, then we can suppose that for a subsequence $\{n_k\}$

$$A_{n_k}\to\infty$$

In this case

$$\int m^*(x-t)\, v^n(dt) = \int_{-\infty}^{x} v^n(dt) = \int_{-\infty}^{x} R_n\left(\frac{dt}{B_n}-A_n\right) + \int_{-\infty}^{x} S_n\left(\frac{dt}{B_n}-A_n\right)$$

Because of (11) the last term tends to 0 if $n\to\infty$, and for the first term we have

$$\int_{-\infty}^{x} R_n\left(\frac{dt}{B_n}-A_n\right) = \int_{-\infty}^{\frac{x}{B_n}-A_n} R_n(ds) = \int_{-\infty}^{\frac{x}{B_n}-A_n} p_n(s)\, ds$$

If $\{v_n\}$ were asymptotically uniform, then for every x

$$\lim_{n\to\infty} \int_{-\infty}^{\frac{x}{B_n}-A_n} p_n(s)\,ds = \psi(\mu^*) \neq 0$$

would be satisfied. But for $x=0$

$$\int_{-\infty}^{-A_n} p_n(s)\,ds \leqq \int_{-\infty}^{-A_n} |p_n(s)-g(s)|\,ds + \int_{-\infty}^{-A_n} g(s)\,ds \leqq$$

$$\leqq \int_{-\infty}^{+\infty} |p_n(s)-g(s)|\,ds + \int_{-\infty}^{-A_n} g(s)\,ds$$

and on the basis of Theorem 0. 6, if $k\to\infty$, then

$$\int_{-\infty}^{-A_{nk}} p_{n_k}(s)\,ds \to 0$$

and this is a contradiction.

Suppose that

$$A_n = o(1)$$

Let

$$m(x) = \begin{cases} e^{i\lambda x} & x \geqq a \\ 0 & x < a \end{cases}$$

Now with the same transformations as above

$$\int m(x-t)\,v^n(dt) = \int_{-\infty}^{\frac{x-a}{B_n}-A_n} e^{i\lambda B_n s} p_n(s)\,ds + o(1)$$

But

$$\left| \int_{-\infty}^{\frac{x-a}{B_n}-A_n} e^{i\lambda B_n s} p_n(s)\,ds - \int_{-\infty}^{\frac{x-a}{B_n}-A_n} e^{i\lambda B_n s} g(s)\,ds \right| \leqq \int_{-\infty}^{+\infty} |p_n(s)-g(s)|\,ds$$

By Theorem 0. 6 the right side tends to 0. Further

$$\frac{x-a}{B_n} - A_n \to 0$$

so if $\lambda=0$, then

$$\int_{-\infty}^{\frac{x-a}{B_n}-A_n} g(s)\,ds \to \int_{-\infty}^{0} g(s)\,ds = G(o)$$

and if $\lambda \neq 0$, then using the Riemann lemma ($B_n \to \infty$),

$$\int_{-\infty}^{\frac{x-a}{B_n}-A_n} e^{i\lambda B_n s} g(s)\, ds \to 0 \qquad (n \to \infty)$$

Q. e. d.

The second part of this theorem gives the basis of the main theorem.

THEOREM 2. 2. *Suppose that for the distribution function $F(x)$ of the measure ν the conditions of Theorem* 0. 6 *are satisfied with some stable law G of exponent α.*

1. *If $\alpha > 1$, then a necessary and sufficient condition to the asymptotic uniformity of $\{\nu^n\}$ is that*

$$\int t\nu(dt) = \int tF(dt) = 0 \tag{12}$$

2. *If $\alpha < 1$, then a necessary and sufficient condition to the asymptotic uniformity of $\{\nu^n\}$ is that*

$$0 < G(0) < 1.$$

3. *If $\alpha = 1$, then a necessary condition to the asymptotic uniformity of $\{\nu^n\}$ is that*

$$F(-x) \sim 1 - F(x) \qquad x \to \infty$$

PROOF. **1.** $\alpha > 1$. (6) shows that a possibility to choose A_n is that

$$A_n = \frac{n}{B_n} \int_{|x| \leqq \varepsilon B_n} x\, dF(x) + \gamma(\varepsilon) \tag{13}$$

where ε is a fixed positive number and $\gamma(\varepsilon)$ is an arbitrary constant. Now

$$A_n = \frac{n}{B_n} \int x\, F(dx) - \frac{n}{B_n} \int_{|x| > \varepsilon B_n} x\, F(dx) + \gamma(\varepsilon) \tag{14}$$

Here

$$\frac{n}{B_n} \int_{|x| > \varepsilon B_n} xF(dx) = \frac{n}{B_n} \left\{ -\varepsilon B_n F(-\varepsilon B_n) - \int_{-\infty}^{-\varepsilon B_n} F(x)\, dx + \right.$$

$$\left. + \varepsilon B_n (1 - F(\varepsilon B_n)) + \int_{\varepsilon B_n}^{\infty} (1 - F(x))\, dx \right\} = \tag{15}$$

$$= -\varepsilon n F(-\varepsilon B_n) + \varepsilon n (1 - F(\varepsilon B_n)) - \frac{n}{B_n} \int_{-\infty}^{-\varepsilon B_n} F(x)\, dx + \frac{n}{B_n} \int_{\varepsilon B_n}^{\infty} (1 - F(x))\, dx$$

If $\alpha \neq 2$, then Theorem 0. 2 says that $1-F(x)$ is regularly varying with exponent $-\alpha$, if $c_2 > 0$, and so by Theorem 0. 1

$$\frac{\int_{\varepsilon B_n}^{\infty} (1-F(x))\,dx}{\varepsilon B_n (1-F(\varepsilon B_n))} \to \frac{1}{\alpha-1} \qquad (n \to \infty)$$

and using Lemma 0. 4

$$\frac{n}{B_n} \int_{\varepsilon B_n}^{\infty} (1-F(x))\,dx \to \frac{1}{\alpha-1} \frac{c_2}{\varepsilon^{\alpha-1}} \tag{16}$$

(If $c_2 = 0$, then $1-F(x)$ can be majorated by a regularly varying function with exponent $-\alpha$, and the right side of (16) will be equal to 0.) Similarly

$$\frac{n}{B_n} \int_{-\infty}^{-\varepsilon B_n} F(x)\,dx \to \frac{1}{\alpha-1} \frac{c_1}{\varepsilon^{\alpha-1}}$$

Now from (15) we have

$$\lim_{n\to\infty} \frac{n}{B_n} \int_{|x|>\varepsilon B_n} xF(dx) = -\frac{c_1}{\varepsilon^{\alpha-1}} + \frac{c_2}{\varepsilon^{\alpha-1}} - \frac{c_1}{(\alpha-1)\varepsilon^{\alpha-1}} + \frac{c_2}{(\alpha-1)\varepsilon^{\alpha-1}} \tag{17}$$

It is known that for $\alpha > 1$

$$\frac{n}{B_n} \to \infty$$

(this follows also from Lemma 0. 3), so if (12) is not true, then $A_n \neq O(1)$, and so by Theorem 2. 1 the sequence may not be asymptotically uniform, the necessity is proved.

If (12) is true, $\gamma(\varepsilon)$ is equal to the right side of (17), then $A_n = o(1)$ and the sufficiency comes from Theorem 2. 1.

Turn to the case $\alpha = 2$. If F has a finite variance, then denoting its expectation and variance by M and D^2, the central limit theorem says, that

$$\frac{\xi_1 + \ldots + \xi_n - n\mathsf{M}}{\sqrt{n}\mathsf{D}} = \frac{\xi_1 + \ldots + \xi_n}{\sqrt{n}\mathsf{D}} - \sqrt{n}\frac{\mathsf{M}}{\mathsf{D}}$$

has a limiting distribution. This means that $A_n = O(1)$ is possible only if $\mathsf{M} = 0$, further in this case every A_n may be chosen 0. Consider the case, when

$$\chi(x) = \frac{h(x)}{x^2}$$

(see Theorem 0. 5). Now from Theorem 0. 1 it follows that

$$\frac{\int_{\varepsilon B_n}^{\infty} \chi(x)\,dx}{\varepsilon B_n \chi(\varepsilon B_n)} \to 1 \qquad (n \to \infty)$$

so from (5)

$$\frac{n}{B_n}\int_{\varepsilon B_n}^{\infty}(1-F(x))\,dx \leqq \frac{n}{B_n}\int_{\varepsilon B_n}^{\infty}\chi(x)\,dx \sim \varepsilon n\chi(\varepsilon B_n) \to 0 \qquad (n\to\infty)$$

Similarly

$$\frac{n}{B_n}\int_{-\infty}^{-\varepsilon B_n}F(x)\,dx \to 0 \qquad (n\to\infty)$$

and the other terms in the right side of (15) also converge to 0, so the proof can be finished, as above.

2. $\alpha<1$.

We calculate again A_n. With integrating by parts

$$A_n = \frac{n}{B_n}\int_{-\varepsilon B_n}^{0} xF(dx)+\frac{n}{B_n}\int_{0}^{\varepsilon B_n} xF(dx)+\gamma(\varepsilon) =$$

(18)

$$= \varepsilon nF(-\varepsilon B_n)-\frac{n}{B_n}\int_{-\varepsilon B_n}^{0}F(x)\,dx+\frac{n}{B_n}\int_{0}^{\varepsilon B_n}(1-F(x))\,dx-\varepsilon n(1-F(\varepsilon B_n))+\gamma(\varepsilon)$$

Using Theorem 0. 1

$$\lim_{n\to\infty}\frac{n}{B_n}\int_{-\varepsilon B_n}^{0}F(x)\,dx = \frac{c_1}{1-\alpha}\varepsilon^{1-\alpha}$$

$$\lim_{n\to\infty}\frac{n}{B_n}\int_{0}^{\varepsilon B_n}(1-F(x))\,dx = \frac{c_2}{1-\alpha}\varepsilon^{1-\alpha}$$

and using Lemma 0. 4

$$\lim_{n\to\infty}A_n = c_1\varepsilon^{1-\alpha}-\frac{c_1}{1-\alpha}\varepsilon^{1-\alpha}+\frac{c_2}{1-\alpha}\varepsilon^{1-\alpha}-c_2\varepsilon^{1-\alpha}+\gamma(\varepsilon) = \frac{(c_2-c_1)\alpha}{1-\alpha}\varepsilon^{1-\alpha}+\gamma(\varepsilon)$$

Choosing

$$\gamma(\varepsilon) = \frac{(c_1-c_2)\alpha}{1-\alpha}\varepsilon^{1-\alpha}$$

we have $A_n=o(1)$, and Theorem 2. 1 may be applied.

3. $\alpha = 1$.

In this case we use (18). Now from Lemma 0. 4

$$\varepsilon n F(-\varepsilon B_n) \to c_1$$
$$\varepsilon n\big(1 - F(\varepsilon B_n)\big) \to c_2$$

so it is sufficient to consider

$$\frac{n}{B_n}\int_0^{\varepsilon B_n} \big(1 - F(x)\big)\,dx - \frac{n}{B_n}\int_{-\varepsilon B_n}^{0} F(x)\,dx = \frac{n}{B_n}\int_0^{\varepsilon B_n} \big(1 - F(x) - F(-x)\big)\,dx \tag{19}$$

From (3)

$$n \sim \frac{c_1 + c_2}{\chi(B_n)} \sim \frac{c_1 + c_2}{\dfrac{c_1 + c_2 + o(1)}{B_n}\, h(B_n)}$$

so

$$\frac{n}{B_n}\int_0^{\varepsilon B_n} \big(1 - F(x) - F(-x)\big)\,dx \sim (c_1 + c_2)\frac{\displaystyle\int_0^{\varepsilon B_n} \big(1 - F(x) - F(-x)\big)\,dx}{B_n\big(1 - F(B_n) - F(-B_n)\big)}$$

If $c_1 \neq c_2$, say $c_1 < c_2$, then for enough large values of x

$$1 - F(x) - F(-x) > 0$$

and $1 - F(x) - F(-x)$ is a regularly varying function of x with exponent -1. Applying Theorem 0. 1 for this function

$$\frac{\displaystyle\int_0^{\varepsilon B_n} \big(1 - F(x) - F(-x)\big)\,dx}{B_n\big(1 - F(B_n) - F(-B_n)\big)} \to \infty$$

and so

$$\frac{\displaystyle\int_0^{\varepsilon B_n} \big(1 - F(x) - F(-x)\big)\,dx}{B_n\big(1 - F(B_n) + F(-B_n)\big)} = \frac{\displaystyle\int_0^{\varepsilon B_n} \big(1 - F(x) - F(-x)\big)\,dx}{B_n\big(1 - F(B_n) - F(-B_n)\big)} \cdot \frac{c_1 - c_2 + o(1)}{c_1 + c_2 + o(1)} \to \infty$$

that is $A_n \neq O(1)$. Thus $c_1 = c_2$ is a necessary condition.

REMARKS 1. In case of $\alpha = 1$ the boundedness of (19) is also a sufficient condition, but it is not a simple one, the boundedness depends not only on the constant c_1 $(= c_2)$ and the function $h(x)$ in (2), but also on the behaviour of the $o(1)$'s (more exactly on the behaviour of their difference).

2. An interesting consequence of the proof is the fact, that if $\xi_1, \ldots, \xi_n, \ldots$ are independent identically distributed random variables, and

$$\frac{\xi_1 + \ldots + \xi_n}{B_n} - A_n$$

has a stable limiting distribution of exponent α, and we suppose that B_n is given in the manner, described in § 0, then if $\alpha > 1$, then the constants A_n can be chosen as

$$A_n = \frac{n}{B_n} \mathsf{M}\xi_1$$

and if $\alpha < 1$, then A_n can be chosen identically 0. This fact is known: see the footnote of translator on p. 175 of the book B. V. GNEDENKO and A. N. KOLMOGOROV: Limit distributions for sums of independent random variables. Translated by. K. L. CHUNG. Addison-Wesley, Inc. Cambridge, Mass. 1954.

REFERENCES

[1] FELLER, W.: *An Introduction to Probability Theory and its Applications*, II, New York—London—Sydney, 1966.
[2] HERRMANN, H.: Glättungseigenschaften der Faltung, *Wiss. Zeitschrift der. F. Schiller Universität Jena* **14** (1965) 221—234.
[3] Ибрагимов, И. А. и Линник, Ю. В.: *Независимые и стационарно связанные случайные величины*, Москва, 1965.
[4] Клосс, Б. М.: О вероятностных распределениях на бикомпактных топологических группах, *Теор. Вероятност. и Применен.* **4** (1959) 255—290.
[5] RÉNYI, A.: *Wahrscheinlichkeitstheorie*, Berlin, 1966.
[6] SZÁSZ, D.: On the general branching process with continuous time parameter, *Studia Sci. Math. Hungar.* **2** (1967) 227—247.
[7] URBANIK, K.: On the limiting probability distribution on a compact topological group, *Fund. Math.* **44** (1957) 243—261.
[8] KERSTAN, J. und MATTHES, K.: Gleichverteilungseigenschaften von Faltungen von Verteilungsgesetzen auf lokalkompakten abelschen Gruppen, I, *Math. Nachr.* **37** (1968) 267—312.

Eötvös L. University, Budapest

(Received June 3, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 331–333.

ON SINE FUNCTIONAL EQUATION

by

PL. KANNAPPAN

Introduction

The functional equation

(B) $$f(x+y)f(x-y) = f(x)^2 - f(y)^2,$$

x, y real where f is a complex-valued function of the real variables, has been treated considerably and is known [5] that either f satisfies

(1) $$f(x+y) = f(x)+f(y),$$

or

(2) $$f(x) = c[g(x)-g(-x)],$$

where c is a complex constant and g satisfies

(3) $$g(x+y) = g(x)g(y).$$

If f is a measurable solution of (B), then f is continuous and either $f(x)=cx$ or $f(x)=c \sin dx$, c, d are complex constants [1], [3], [5]. The aim of this note is to give a functional equation satisfied only by the sine functions. For this, we consider the equation

(C) $$f(x+y+A)f(x-y+A) = f(x)^2 - f(y)^2,$$

where f is a complex-valued, measurable function of the real variables, and $A \neq 0$ is a real constant and prove that $f=0$ or $f(x)=\alpha \sin ax$ or $f(x) = -\alpha \sin bx$, where α is a complex constant and for the constants a and b only a *countable* set of real numbers is admissible. Equation (C) is similar to the equations,

(D) $$f(x-y+A)-f(x+y+A) = 2f(x)f(y)$$

and

(E) $$f(x+y+2A)+f(x-y+2A) = 2f(x)f(y)$$

considered in [2], [4], the former having only the sine function as solutions while the later has only the cosine function as solutions (excluding the trivial solutions).

THEOREM 1. *Let f be a complex-valued function of the reals R, satisfying* (C) *for every x, y in R, where $A \neq 0$ is a real constant. Then the most general solution of* (C) *is $f(x) = h(x-A)$, where h is an arbitrary solution of* (B) *with period $2A$ and only this.*

PROOF. First let us prove that

$$f(x) = h(x-A) \tag{4}$$

is a solution of (C), provided h is a solution of (B) with period $2A$. In fact from (4), (B) and the periodicty of h, we have

$$\begin{aligned} f(x+y+A)f(x-y+A) &= h(x+y)h(x-y) \\ &= h(x+y-2A)h(x-y) \\ &= h(x-A)^2 - h(y-A)^2 \\ &= f(x)^2 - f(y)^2. \end{aligned}$$

Now we show that every solution of (C) is of the form (4). Putting $x=0$, $y=0$ in (C), we obtain

$$f(A)=0. \tag{5}$$

Changing y into $-y$ in (C) and comparing it with (C), we have

$$f(y)^2 = f(-y)^2, \quad \text{for all} \quad y \quad \text{in} \quad R. \tag{6}$$

Letting $y=A$ in (C) and using (5), we get

$$f(x+2A)f(x) = f(x)^2, \quad \text{for all} \quad x \quad \text{in} \quad R. \tag{7}$$

Hence from (7), we conclude that

$$f(x+2A) = f(x), \quad \text{for all} \quad x \quad \text{for which} \quad f(x) \neq 0. \tag{8}$$

Now let $f(x_0)=0$. Replacing x by x_0 and y by x_0+2A in (C) and utilizing (5) and (6), we get $f(x_0+2A) = 0$. Hence (8) holds for all x in R. That is, f is a periodic function with period $2A$.

Here we remark that f is an odd function. For, interchange $x+y$ in (C) and comparing it with (C), we get

$$f(x+y+A)f(x-y+A) = -f(x+y+A)f(y-x+A). \tag{9}$$

Let $f(x_0) \neq 0$. Put $x = x_0 - A$, $y=0$ in (9) and use (8). We obtain

$$f(x_0) = -f(-x_0), \quad \text{for} \quad f(x_0) \neq 0. \tag{10}$$

If $f(x_0)=0$, by (6), $f(x_0) = -f(-x_0)$. Hence f is odd. Changing x by $x+A$ and y by $y+A$ in (C), we obtain

$$f(x+y+3A)f(x-y+A) = f(x+A)^2 - f(y+A)^2,$$

and from (8),

$$f(x+y+A)f(x-y+A) = f(x+A)^2 - f(y+A)^2. \tag{11}$$

Thus for h as defined in (4) for all x in R, it is easy to see from (11) that h satisfies the equation

$$h(x+y)h(x-y) = h(x)^2 - h(y)^2. \tag{B}$$

Also, from (4) and (8), we conclude that h is a periodic function with period $2A$. The proof of the theorem is thus complete.

THEOREM 2. *All the measurable solutions of* (C) *are* $f(x)=\alpha \sin\left(\frac{n\pi x}{A}-n\pi\right)$, $n=0, 1, \ldots, \alpha$, *a complex constant. That is,* $f\equiv 0$, $f(x)=\alpha \sin\frac{2k\pi x}{A}$ *or* $f(x)=-\alpha \sin\frac{(2k+1)\pi x}{A}$ $(k=0, 1, \ldots)$ *are the only measurable solutions of* (C).

PROOF. If f is measurable, then the function h defined by (4) is also measurable. Since h also satisfies (B), h is continuous. Thus f is continuous. The only continuous solutions of (B) are $h(x)=cx$ and $h(x)=\alpha \sin dx$, where c, α and d are complex constants. Since h is of period $2A$, (i) $h(x)=cx$ is possible only when $c=0$, (ii) $h(x)=\alpha \sin dx$ is possible only when $d=\frac{n\pi}{A}$, $(n=0, 1, \ldots)$. Thus we obtain the measurable solutions of (C) as, $f(x)\equiv 0$ and $f(x)=\alpha \sin\left(\frac{n\pi x}{A}-n\pi\right)$, $n=0, 1, \ldots$. When n is even, $n=2k$ $(k=0, 1, \ldots)$, we have $f(x)=\alpha \sin ax$, where $a=\frac{2k\pi}{A}$, $k=0, 1, \ldots$. When n is odd, $n=2k+1$, $k=0, 1, \ldots$ we have $f(x)=-\alpha \sin bx$, where $b=\frac{(2k+1)\pi}{A}$, $k=0, 1, \ldots$. Thus the proof of this theorem is complete.

REFERENCES

[1] ACZÉL, J.: *Lectures on functional equations and their applications*, Academic Press, 1966.
[2] KANNAPPAN, PL.: A functional equation for the cosine, *Canad. Math. Bull.* **2** (1968) 495—498.
[3] KUREPA, S.: Functional equation $F(x+y)F(x-y)=F(x)^2-F(y)^2$ in n-dimensional vector space, *Monatsch. Math.* **64** (1961) 321—329.
[4] VAN VLECK, B.: A functional equation for the sine, *Ann. Math.* **7** (1910) 161—165.
[5] WILSON, W. H.: On certain related functional equations, *Bull. Amer. Math. Soc.* **26** (1919) 300—312.

University of Waterloo, Waterloo, Canada

(Received June 12, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 335–340.

RATIONAL APPROXIMATION IN COMPLEX DOMAINS

by

J. SZABADOS

There is a lot of results which show that the degree of rational approximation is (more or less) better than that of the polynomial, when the considered function has singularities only at finitely many points, or in the neighbourhood of these points, and excluding these points or neighbourhoods, the function will be "better". These results always refer to functions defined on an interval of the real axis.

But there is a possibility for developing such results in complex domains. As far as I know, the first (starting) result in this direction was expounded in [1]. We intend to generalize it in Theorem 1.

First of all, we give the usual definition of the modulus of continuity in the complex unit disk. Let $f(z)$ be analytic in $|z|<1$ and continuous on $|z|\leqq 1$. Then

$$\omega(h) = \sup_{\substack{|z_1-z_2|\leqq h \\ |z_1|,\,|z_2|\leqq 1}} |f(z_1)-f(z_2)| \quad (0 \leqq h \leqq 2)$$

(see e.g. [2], p. 32).

THEOREM 1. *Let $f(z)$ be continuous on $|z|\leqq 1$ and analytic in every simply connected closed subdomain of $C_\varrho=\{z:|z|\leqq\varrho\}(\varrho>1)$ which does not contain s fixed points of $|z|=1$. Then there exist rational functions $r_n(z)$ of degree n such that*

$$|f(z)-r_n(z)| \leqq \frac{64\varrho s^2}{\varrho-1}\,\omega\left(\frac{\log^2 n}{n^2}\right) \quad (|z|\leqq 1) \tag{1}$$

for sufficiently large n's.

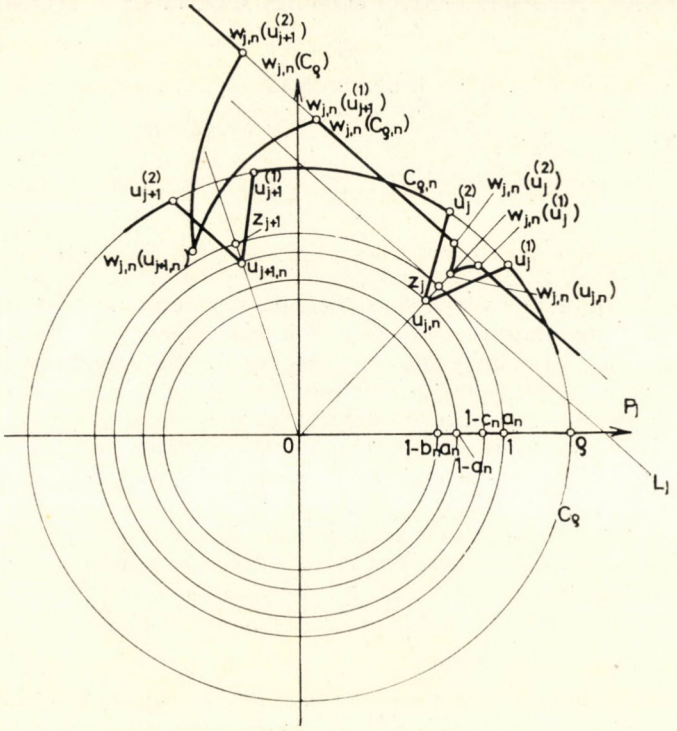
REMARKS. In [1], it was proved essentially the case $s=1$. The order of polynomial approximation is generally not better than $\omega\left(\frac{\log n}{n}\right)$ (see [2], p. 222), and in the case Lip α $(0<\alpha<1)$ can not be $O(n^{-\beta})$ where $\beta>\alpha$ (see [2], pp. 90 and 101). The advantage of the rational approximation is based on the fact that $f(z)$ has singularities on $|z|=1$ *only at finitely many points*, and we are able to construct rational functions whose poles converge to these singularities.

PROOF. The proof is a slight modification of the construction in [1]. Let z_j $(|z_j|=1,\ j=1,\ldots,s)$ be the singularities of $f(z)$,*

$$a_n = 32\frac{\log^2 n}{n^2}, \quad d_n = \left(\frac{1-\sqrt{\tfrac{1}{2}a_n}}{1+\sqrt{\tfrac{1}{2}a_n}}\right)^n = O\left(\frac{1}{n^4}\right), \tag{2}$$

$$g_{j,n}(z) = \left[\frac{z-z_j(1-\sqrt{2a_n})}{z-z_j(1+\sqrt{2a_n})}\right]^n. \tag{3}$$

* In what follows where the index j figures it is to be understood, that $j=1,\ldots,s$.



Denote by L_j the tangent line of $|z|=1$ at the point z_j, and by P_j that closed half-plane bordered by L_j which does not contain $|z|=1$ (see the figure). Then easy to verify that (compare [1])

$$|g_{j,n}(z)| \geqq 1 \quad (z \in P_j) \tag{4}$$

and

$$|g_{j,n}(z)| = d_n \tag{5}$$

for $|z+\frac{1}{2}a_n z_j| = 1-\frac{1}{2}a_n$. Thus, by the maximum-principle,

$$|g_{j,n}(z)| \leqq d_n \quad (|z| \leqq 1-a_n). \tag{6}$$

On the other hand, easy to see by (3) and (5) that

$$|g_{j,n}(z)| \geqq d_n \quad (|z| \geqq 1). \tag{7}$$

Further let

$$w_{j,n}(z) = \frac{1-a_n}{1-b_n a_n} \cdot \frac{\varrho z + (1-b_n a_n)^2 z_j}{z + \varrho z_j} z_j \tag{8}$$

where

$$b_n = \frac{2(\varrho+1)}{\varrho-1+a_n+\sqrt{(\varrho-1+a_n)^2+4a_n(\varrho+1)}} < \frac{\varrho+1}{\varrho-1}. \tag{9}$$

Then

$$w_{j,n}((1-b_na_n)z_j) = (1-a_n)z_j, \quad w_{j,n}((-1+b_na_n)z_j) = (-1+a_n)z_j, \tag{10}$$

$$w_{j,n}(-\varrho z_j) = \infty.$$

By (8) and (10), the $w_{j,n}$ maps the circle $|z| = 1-b_na_n$ into the circle $|z| = 1-a_n$. Thus from (6)

$$|g_{j,n}(w_{j,n}(z))| \leqq d_n \quad (|z| \leqq 1-b_na_n). \tag{11}$$

Easy to verify from (9) that

$$1-b_na_n = \frac{b_n-1}{b_n+1}\varrho, \quad 1-a_n = \frac{(b_n-1)(b_n+1+\varrho)}{b_n(b_n+1)}.$$

Thus, by (8) and (9)

$$|w_{j,n}(z_j)| = \frac{1-a_n}{1-b_na_n}\cdot\frac{\varrho+(1-b_na_n)^2}{\varrho+1} = \frac{b_n+1+\varrho}{b_n}\cdot\frac{(b_n+1)^2+(b_n-1)^2\varrho}{(\varrho+1)(b_n+1)^2} =$$

$$= 1+\frac{[\varrho+1-b_n(\varrho-1)]^2}{b_n(\varrho+1)(b_n+1)^2} > 1,$$

moreover

$$|w_{j,n}(-z_j)| = \frac{1-a_n}{1-b_na_n}\cdot\frac{\varrho-(1-b_na_n)^2}{\varrho-1} > \frac{1-a_n}{1-b_na_n}\cdot\frac{\varrho+(1-b_na_n)^2}{\varrho+1} = |w_{j,n}(z_j)| > 1.$$

So we can choose a c_n such that $1-a_n < c_n < 1$ and for $D_n = \{z: |z| \leqq c_n\}$ we have (with symbolical notation)

$$w_{j,n}(D_n) \supseteq \{z: |z| \leqq 1\}. \tag{12}$$

Now let $u_{j,n} = c_nz_j$ and choose $u_j^{(1)}$ and $u_j^{(2)}$ such that

$$|u_j^{(1)}| = |u_j^{(2)}| = \varrho, \quad \arg z_j = \frac{\arg u_j^{(1)} + \arg u_j^{(2)}}{2}$$

moreover, let $u_j^{(1)}$ and $u_j^{(2)}$ be so near to z_j that there should be no other $u_k^{(l)}$ ($l = 1, 2$, $k \neq j$) point between $u_j^{(1)}$ and $u_j^{(2)}$, and the $w_{j,n}$-images of the line-segments $u_{j,n}u_j^{(1)}$ and $u_{j,n}u_j^{(2)}$ be in P_j (see the figure). The latter is possible because of $\arg u_{j,n} = \arg z_j$, $|w_{j,n}(u_{j,n})| > 1$ and

$$|w_{j,n}(\varrho z_j)| = \frac{1-a_n}{1-b_na_n}\frac{\varrho^2+(1-b_na_n)^2}{2\varrho} > \frac{1-a_n}{1-b_na_n}\cdot\frac{\varrho+(1-b_na_n)^2}{\varrho+1} = |w_{j,n}(z_j)| > 1.$$

Clearly, $w_{j,n}(C_\varrho)$ is a line which touches the circle $|z| = |w_{j,n}(\varrho z_j)|$ at the point $w_{j,n}(\varrho z_j)$. By (12), the $w_{j,n}$-image of each line-segment $u_{k,n}u_k^{(1)}$ and $u_{k,n}u_k^{(2)}$ is external to $|z| \leqq 1$. Denote by $C_{\varrho,n}$ the closed curve put together from the bows $u_{j-1}^{(2)}u_j^{(1)}$,

$u_j^{(2)}u_{j+1}^{(1)}$ and line-segments $u_{j,n}u_j^{(1)}$, $u_{j,n}u_j^{(2)}$. By construction, $f(z)$ is analytic in the simply connected closed domain bounded by $C_{\varrho,n}$.

Now let $R_{ns}(z)$ be the rational function of degree at most ns which interpolates to $f(z)$ in the roots of $g_{j,n}(w_{j,n}(z))$ (with multiplicity n) and in0 , and which has the same poles as $g_{j,n}(w_{j,n}(z))$ (with multiplicity n). Then (see [3], Ch. 8, Theorem 2)

$$(13)\quad f(z)-R_{ns}(z)=\frac{z\cdot\prod_{j=1}^{s}g_{j,n}(w_{j,n}(z))}{2\pi i}\int_{C_{\varrho,n}}\frac{f(\xi)\,d\xi}{\xi(\xi-z)\prod_{j=1}^{s}g_{j,n}(w_{j,n}(\xi))}\qquad(z\in C_{\varrho,n}).$$

Thus, by (11) we have

$$(14)\quad |f(z)-R_{ns}(z)|\leqq c_1\frac{Md_n^s}{a_n}\int_{C_{\varrho,n}}\frac{|d\xi|}{\prod_{j=1}^{s}|g_{j,n}(w_{j,n}(\xi))|}\qquad(|z|\leqq 1-b_na_n)^*$$

where $M=\max_{|z|\leqq C_{\varrho,n}}|f(z)|$. First of all, let $\xi\in C_\varrho\cap C_{\varrho,n}$, then by $w_{j,n}(C_\varrho)\subset P_j$ and (4) we get

$$(15)\quad \prod_{j=1}^{s}|g_{j,n}(w_{j,n}(\xi))|\geqq 1.$$

On the other hand, when ξ is on the broken line $u_{k,n}^{(1)}u_ku_{k,n}^{(2)}$ then by (4)

$$|g_{k,n}(w_{k,n}(\xi))|\geqq 1,$$

and by (7)

$$|g_{j,n}(w_{j,n}(\xi))|\geqq d_n\quad(j\neq k).$$

Thus in each cases we have from (14)

$$(16)\quad |f(u)-R_{ns}(u)|\leqq c_2M\frac{d_n^s}{a_n}\frac{1}{d_n^{s-1}}\leqq c_3M\frac{1}{n^2\cdot\log^2 n}\qquad(|u|\leqq 1-b_na_n).$$

Put here $u=(1-b_na_n)z$, $|z|=1$. Evidently

$$(17)\quad |f(z)-f((1-b_na_n)z)|\leqq\omega(b_na_n)\leqq\frac{32(\varrho+1)}{\varrho-1}\omega\left(\frac{\log^2 n}{n^2}\right)\qquad(|z|\leqq 1).$$

(16) and (17) gives

$$|f(z)-R_{ns}((1-b_na_n)z)|\leqq\frac{64\varrho}{\varrho-1}\omega\left(\frac{\log^2 n}{n^2}\right)\qquad(|z|\leqq 1).$$

From this easy to obtain (1), qu. e. d.

THEOREM 2. *Let $1<r<R$, $f(z)$ continuous on $|z|\leqq r$ and analytic in every simply connected closed subdomain of $|z|\leqq R$ which does not contain s fixed points of $|z|=r$.*

* $c_1, c_2, \ldots$ denote explicite positive numerical constants.

Then there exist rational functions $r_n(z)$ of degree at most n such that

$$|f(z)-r_n(z)| \leqq \frac{M_R[R\pi+(R-r)s](R+1)}{\pi r(r-1)}\left[1-\sqrt{\frac{1}{2}\frac{R-1}{R+1}\left(1-\frac{1}{r}\right)}\right]^{n/s} \quad (|z| \leqq 1) \tag{18}$$

for sufficiently large n's where $M_R = \max_{|z|\leqq R} |f(z)|$.

REMARKS. The order of the polynomial approximation is generally not better than $O\left(\frac{1}{n^{c_4}}\right)$ (see [2], pp. 156 and 166). Our result (18) gives a better estimation if

$$s < \frac{\log \dfrac{1}{1-\sqrt{\dfrac{1}{2}\dfrac{R-1}{R+1}\left(1-\dfrac{1}{r}\right)}}}{\log r} \stackrel{\text{def}}{=} F(r, R). \tag{19}$$

Evidently with fixed R $(>r)$

$$\lim_{r\to 1+0} F(r, R) = \infty$$

i.e. (18) is advantageous when $r>1$ is near to 1 (then s can be large). The estimation (18) is as better as large R is. Roughly speaking, (18) is useful if s is not too large.

PROOF. Consider the function $f(rz)$. It is continuous on $|z|\leqq 1$ and is analytic in every simply connected subdomain of $|z|\leqq \varrho = R/r$ not containing s fixed points of $|z|=1$. Apply the proof of the previous theorem with

$$a_n = \frac{R-1}{R+1}\left(1-\frac{1}{r}\right) \tag{20}$$

instead of (2). Then

$$d_n < \left[1-\sqrt{\frac{1}{2}\frac{R-1}{R+1}\left(1-\frac{1}{r}\right)}\right]^n \tag{21}$$

and by (9)

$$b_n = \frac{2\cdot\dfrac{R+r}{r}}{\dfrac{R-r}{r}+\dfrac{(R-1)(r-1)}{(R+1)r}+\sqrt{\left[\dfrac{R-r}{r}+\dfrac{(R-1)(r-1)}{(R+1)r}\right]^2+\dfrac{4(R-1)(r-1)(R+r)}{(R+1)r^2}}} = \tag{22}$$

$$= \frac{2(R+r)(R+1)}{R^2-2r+1+\sqrt{(R^2-2r+1)^2+4(R^2-1)(r-1)(R+r)}} =$$

$$= \frac{2(R+r)(R+1)}{R^2-2r+1+R^2-1+2Rr-2R} = \frac{R+1}{R-1}.$$

Thus we get by (13), (21), (20) and (22)

$$|f(rz)-R_{ns}(z)| \leqq \frac{1}{r}\frac{d_n}{2\pi}[2\varrho\pi+2s(\varrho-1)]\cdot\frac{M_R}{\frac{1}{2}a_n(b_n-1)}=$$

$$=\frac{2M_R\left[\frac{R}{r}\pi+s\cdot\frac{R-r}{r}\right]}{\pi r\cdot\frac{R-1}{R+1}\left(1-\frac{1}{r}\right)\frac{2}{R-1}}\cdot d_n=$$

$$=\frac{M_R[R\pi+(R-r)s](R+1)}{\pi r(r-1)}\cdot\left[1-\sqrt{\frac{1}{2}\frac{R-1}{R+1}\left(1-\frac{1}{r}\right)}\right]^n$$

for

$$|z| \leqq 1-b_n a_n = 1-\frac{R+1}{R-1}\cdot\frac{R-1}{R+1}\left(1-\frac{1}{r}\right)=\frac{1}{r},$$

where $R_{ns}(z)$ is a rational function of degree ns. Putting z instead of rz and n instead of ns, we have (18), qu. e. d.

REFERENCES

[1] SZABADOS, J.: Rational approximation on the complex unit circle, *Ann. Univ. Sci. Budapest, Sect. Math.* **11** (1968) 105—108.

[2] SEWELL, W. E.: *Degree of approximation by polynomials in the complex domain*, Princeton, 1942.

[3] WALSH, J. L.: *Interpolation and approximation by rational functions in the complex domain*, Amer. Math. Soc., Colloquium Publications, Vol. XX, 1960.

Eötvös Loránd University, Budapest

(Received June 17, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 341—343.

UNE INÉGALITÉ DE TYPE DE HARNACK CONCERNANT LES FONCTIONS BIHARMONIQUES

par

G. ADLER

Soit $v(r, \varphi)$ une fonction harmonique et non-négative dans le cercle $r < R$ (où r et φ désignent des coordonnées polaires). Alors l'inégalité de HARNACK s'écrit de la manière suivante (voir [1]):

$$\frac{R-r}{R+r} v(0,0) \leqq v(r, \varphi) \leqq \frac{R+r}{R-r} v(0,0). \tag{1}$$

Nous nous sommes proposés d'établir une inégalité analogue pour les fonctions biharmoniques et non-négatives dans un cercle, c'est-à-dire de trouver une majoration pour ces fonctions en termes de certaines de leurs données se rapportant au centre du cercle. Le premier problème qui se présente c'est de trouver le nombre minimum de ces données qui suffisent pour une majoration de telle sorte.

Soit $u(x, y)$ une fonction biharmonique et non-négative dans le cercle $n < R$. L'exemple de la fonction ax^2 ($a > 0$ arbitrairement grande) montre que la quantité $u(0, 0)$ même ensemble avec $|\operatorname{grad} u(0, 0)|$ ne suffisent pas pour majorer u. Nous conjecturons que $u(0, 0)$ et $\Delta u(0, 0)\left(\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ sont les données appropriées pour la majoration en question. Dans notre majoration nous nous sommes servis d'une hypothèse supplémentaire, notamment de l'inégalité $\left.\frac{\partial u}{\partial r}\right|_{r=R} \leqq M$ (M constante donnée). Nous en sommes convaincus que l'utilisation de la quantité M dans la majoration en question est superflue et que la nécessité de son application résulte uniquement de l'insuffisance de la technique appliquée. De plus, nous n'avons établi que la majoration analogue à la deuxième inégalité dans (1).

Le résultat que nous avons obtenu s'énonce comme suit.

THÉORÈME. *Soit $u(r, \varphi)$ une fonction biharmonique et non-négative dans le cercle $r < R$, satisfaisant à l'inégalité*

$$\limsup_{r \to R} \frac{\partial u(r, \varphi)}{\partial r} \leqq M. \tag{2}$$

Alors

$$u(r, \varphi) \leqq u(0,0) \frac{2(R+r)^2}{R(R-r)} + \Delta u(0,0) \frac{r(3R^2+r^2)}{2(R-r)} + M \frac{2r^2}{R}.$$

DÉMONSTRATION. On peut supposer, sans restreindre la généralité, que la fonction u soit assez régulière jusqu'à la frontière $r = R$. Dans le cas contraire on peut

effectuer la démonstration pour un cercle $r < R'$ $(R' < R)$ et puis faire le passage à la limite $R' \to R$.

Soient

$$u(R, \varphi) = g(\varphi),$$

$$\left.\frac{\partial u(r, \varphi)}{\partial r}\right|_{r=R} = h(\varphi).$$

La fonction u peut être représentée sous la forme (voir [2]):

$$u(r, \varphi) = -\int_0^{2\pi} H(r, \alpha - \varphi) h(\alpha)\, d\alpha + \int_0^{2\pi} K(r, \alpha - \varphi) g(\alpha)\, d\alpha, \tag{3}$$

où

$$H(r, \alpha) = \frac{(R^2 - r^2)^2}{4\pi R} \frac{1}{R^2 + r^2 - 2Rr\cos\alpha},$$

$$K(r, \alpha) = \frac{(R^2 - r^2)^2}{2\pi R} \frac{R - r\cos\alpha}{(R^2 + r^2 - 2Rr\cos\alpha)^2}.$$

On en tire:

$$u(0, \varphi) = \frac{R}{2\pi}\left[-\frac{1}{2}\int_0^{2\pi} h(\alpha)\, d\alpha + \frac{1}{R}\int_0^{2\pi} g(\alpha)\, d\alpha\right], \tag{4}$$

$$\Delta u(0, \varphi) = \frac{1}{R\pi}\int_0^{2\pi} h(\alpha)\, d\alpha. \tag{5}$$

Étant donné que $g(\alpha) \geqq 0$ (en vertu de $u \geqq 0$) et que

$$K(r, \alpha) \leqq \frac{(R + r)^2}{2\pi R(R - r)},$$

on obtient

$$\int_0^{2\pi} K(r, \alpha - \varphi) g(\alpha)\, d\alpha \leqq \frac{(R + r)^2}{2\pi R(R - r)} \int_0^{2\pi} g(\alpha)\, d\alpha. \tag{6}$$

Écrivons la condition $u \geqq 0$ sous la forme detaillée:

$$-\int_0^{2\pi} H(r, \alpha - \varphi) h(\alpha)\, d\alpha + \int_0^{2\pi} K(r, \alpha - \varphi) g(\alpha)\, d\alpha \geqq 0.$$

Nous affaiblissons cette restriction en considérant seulement le cas où $\varphi = \pi$, et en remplaçant le deuxième terme par sa borne (6):

$$-\int_0^{2\pi} H(r, \alpha - \pi) h(\alpha)\, d\alpha \geqq -\frac{(R + r)^2}{2\pi R(R - r)} \int_0^{2\pi} g(\alpha)\, d\alpha. \tag{7}$$

On a

$$H(r,\alpha)+H(r,\alpha-\pi)=\frac{(R^2-r^2)^2(R^2+r^2)}{2\pi R[(R^2+r^2)^2-4R^2r^2\cos^2\alpha]}\leqq\frac{R^2+r^2}{2\pi R},$$

d'où

$$H(r,\alpha)\leqq -H(R,\alpha-\pi)+\frac{R^2+r^2}{2\pi R}. \tag{8}$$

En vertu de la condition (2)

$$-h(\alpha)+M\geqq 0.$$

Nous obtenons ainsi, à l'aide de (8) et de (7),

$$\int_0^{2\pi}H(r,\alpha)[-h(\alpha)+M]\,d\alpha\leqq\int_0^{2\pi}\left[-H(r,\alpha-\pi)+\frac{R^2+r^2}{2\pi R}\right][-h(\alpha)+M]\,d\alpha\leqq \tag{9}$$

$$\leqq\frac{(R+r)^2}{2\pi R(R-r)}\int_0^{2\pi}g(\alpha)\,d\alpha-\frac{R^2+r^2}{2\pi R}\int_0^{2\pi}h(\alpha)\,d\alpha-M\int_0^{2\pi}H(r,\alpha-\pi)\,d\alpha+M\frac{R^2+r^2}{R}.$$

Il suffit de démontrer le théorème, sans altérer la généralité, pour $\varphi=0$. Il découle de (3), pour $\varphi=0$, en vertu des inégalités (6) et (9):

$$u(r,0)\leqq\frac{(R+r)^2}{\pi R(R-r)}\int_0^{2\pi}g(\alpha)\,d\alpha-\frac{R^2+r^2}{2\pi R}\int_0^{2\pi}h(\alpha)\,d\alpha-$$

$$-M\int_0^{2\pi}[H(r,\alpha)+H(r,\alpha-\pi)]\,d\alpha+M\frac{R^2+r^2}{R}.$$

Il en résulte le théorème à l'aide de (4), (5) et de la formule

$$\int_0^{2\pi}[H(r,\alpha)+H(r,\alpha-\pi)]\,d\alpha=\frac{R^2-r^2}{R}.$$

* * *

Je suis bien reconnaissant à M. E. MAKAI qui m'a bien voulu montrer, au moyen de la construction très élégante d'un contre-exemple, que les seules quantités $u(0,0)$ et $\Delta u(0,0)$ (sans faisant intervenir l'inégalité (2)) ne suffisent pas pour la majoration désirée si l'on substitue la condition $u\geqq 0$ par la condition affaiblie (7).

BIBLIOGRAPHIE

[1] COURANT, R. und HILBERT, D.: *Methoden der Mathematischen Physik*, II, Verlag von J. Springer, Berlin, 1937. (p. 245.)

[2] FRANK, PH. und VON MISES, R.: *Die Differential- und Integralgleichungen der Mechanik und Physik*, I, Dover (New York) — Vieweg (Braunschweig), 1961.

Institut de Mathématique de l'Académie des Sciences de Hongrie, Budapest

(Reçu le 19 juin 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 345—354.

ОБ ОЦЕНКЕ ОСТАТОЧНОГО ЧЛЕНА В ЦЕНТРАЛЬНОЙ ПРЕДЕЛЬНОЙ ТЕОРЕМЕ ДЛЯ ВЫБОРОК, ВЗЯТЫХ ИЗ КОНЕЧНЫХ МНОЖЕСТВ*

A. BIKELIS

Рассмотрим асимптотическое поведение функции распределения

$$F_{n,s_n}(x) = \frac{1}{\binom{n}{s_n}} \sum_{a_{ni_1}+\ldots+a_{ni_{s_n}} < x\sigma_n} 1$$

$$1 \leqq i_1 < i_2 < \ldots < i_{s_n} \leqq n$$

всевозможных $\binom{n}{s_n}$ сумм составленных случайным выбором $s_n \left(\leqq \frac{n}{2}\right)$ различных элементов из некоторой генеральной совокупности $a_{n1},\ldots,a_{nn}$ конечных вещественных чисел. Здесь

$$\sigma_n^2 = \frac{s_n}{n}\left(1-\frac{s_n}{n}\right)\sum_{k=1}^{n} a_{nk}^2 .$$

Как известно [1], не ограничивая общности можно считать, что $\sum_{k=1}^{n} a_{nk} = 0$.

P. Erdős и A. Rényi показали [1], что при выполнении условия типа Линдеберга для всех x

$$\lim_{n\to\infty} F_{ns_n}(x) = \Phi(x),$$

где $\Phi(x)$ — функция распределения нормального закона с параметрами (0, 1). Проведенные ими исследования полностью покривают результаты по этому вопросу С. Н. Бернштейна [5], стр. 255, W. G. Madov и F. N. David (см. [1].) В 1961 году J. Hájek [2] доказал, что условия P. Erdős и A. Rényi являются необходимыми и достаточными, более того, он исследовал условия при которых $F_{n,s_n}(x)$ сходится к безгранично делимым законам.

Здесь мы исследуем скорость сходимости $F_{n,s_n}(x)$ к функции распределения $F(x)$, а именно, докажем следующую теорему.

Теорема. *Существует положительное постоянное C_1 такое, что*

$$\sup_x |F_{n,s_n}(x) - \Phi(x)| \leqq \frac{C_1}{\sqrt{s_n}} \frac{\max\limits_{1\leqq k\leqq n} |a_{nk}|}{\left(\frac{1}{n}\sum\limits_{k=1}^{n} a_{nk}^2\right)^{1/2}}.$$

* Эта работа была выполнена под руководством проф. A. Rényi во время стажировки автора в Будапештском университете им. L. Eötvös.

Далее, через $C_1, C_2, \ldots$ обозначим положительные константы. Приведем несколько примеров из работы [1].

Пример 1. Если $0 < c \leqq |a_{nk}| \leqq C$, то

$$\sup_x |F_{n,s_n}(x) - \Phi(x)| \leqq \frac{C_2}{\sqrt{s_n}}.$$

Пример 2. Если $a_{nk} = \frac{(-1)^k}{\sqrt{k}}$, то

$$\sup_x |F_{n,s_n}(x) - \Phi(x)| \leqq \frac{C_3\sqrt{n}}{\sqrt{s_n \ln n}}.$$

Пример 3. Если $0 < ck^2 \leqq |a_{nk}| \leqq Ck^2$, то

$$\sup_x |F_{n,s_n}(x) - \Phi(x)| \leqq \frac{C_4}{\sqrt{s_n}}.$$

В частности, если $s_n = \frac{n}{4}$, то

$$\sup_x |F_{n,s_n}(x) - \Phi(x)| \leqq \frac{2C_4}{\sqrt{n}}.$$

Утверждение теоремы является очевидным следствием из трех доказываемых ниже лемм и следующей теоремы ESSEEN [3], стр. 24.

Теорема. *Пусть A, T, ε — постоянные, $F(x)$ — неубывающая функция, $G(x)$ — функция ограниченной вариации, $f(t)$ и $g(t)$ — их характеристические функции. Если*

1. $F(-\infty) = G(-\infty)$, $F(+\infty) = G(+\infty)$;

2. *$G'(x)$ существует при всех x и $|G'(x)| < A$;*

3. $$\int_{-T}^{T} \left| \frac{f(t) - g(t)}{t} \right| dt = \varepsilon,$$

то каждому числу $k > 1$ соответствует число $c(k)$, зависящее только от k такое, что

$$|F(x) - G(x)| \leqq k\frac{\varepsilon}{2\pi} + c(k)\frac{A}{T}.$$

При этом $C(2) = \frac{24}{\pi}$.

Перед тем как сформулировать леммы, следуя P. ERDŐS и A. RÉNYI введем в рассмотрение характеристическую функцию, соответствующую функции распределения $F_{n,s_n}(x)$:

$$(1) \qquad \varphi_{n,s_n}\left(\frac{t}{\sigma_n}\right) = \int_{-\infty}^{\infty} e^{itx}\, dF_{n,s_n}(x) = \frac{1}{\binom{n}{s_n}} \sum_{i \leqq i_1 < i_2 < \ldots < i_{s_n} \leqq n} e^{i\frac{t}{\sigma_n}(a_{ni_1} + a_{ni_2} + \ldots + a_{ni_{s_n}})}$$

Ее (см. [1]) можно записать в виде интеграла

$$\varphi_{n,s_n}\left(\frac{t}{\sigma_n}\right)=\frac{1}{2\pi\sqrt{npq}\,P_n(s_n)}\int\limits_{-\pi\sqrt{npq}}^{\pi\sqrt{npq}}\prod_{k=1}^{n}\left\{qe^{-ip\left(\frac{\theta}{\sqrt{npq}}+\frac{ta_{nk}}{\sigma_n}\right)}+pe^{iq\left(\frac{\theta}{\sqrt{npq}}+\frac{ta_{nk}}{\sigma_n}\right)}\right\}d\theta.$$

Здесь $p=\frac{s_n}{n}$, $q=1-p$ и

$$P_n(s_n)=\binom{n}{s_n}p^{s_n}q^{n-s_n}.$$

Далее положим

$$\varrho_k(\theta,t)=qe^{-ip\left(\frac{\theta}{\sqrt{npq}}+\frac{ta_{nk}}{\sigma_n}\right)}+pe^{iq\left(\frac{\theta}{\sqrt{npq}}+\frac{ta_{nk}}{\sigma_n}\right)}.$$

Если примем, что

$$\prod_{l=1}^{0}\varrho_l(\theta,0)\varrho_l(0,t)=\prod_{l=n+1}^{n}\varrho_l(\theta,t)=1,$$

то нетрудно проверить, что имеет место равенство

$$\prod_{k=1}^{n}\varrho_k(\theta,t)-\prod_{k=1}^{n}\varrho_k(\theta,0)\varrho_k(0,t)=$$

$$=\sum_{j=1}^{n}\prod_{l=1}^{j-1}\varrho_l(\theta,0)\varrho_l(0,t)[\varrho_j(\theta,t)-\varrho_j(\theta,0)\varrho_j(0,t)]\prod_{k=j+1}^{n}\varrho_k(\theta,t).$$

Поскольку

$$P_n(s_n)=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}(q+pe^{i\theta})^n e^{-inp\theta}\,d\theta,$$

то

$$\prod_{k=1}^{n}\left(qe^{-\frac{itpa_{nk}}{\sigma_k}}+pe^{\frac{itqa_{nk}}{\sigma_n}}\right)=\frac{1}{2\pi\sqrt{npq}\,P_n(s_n)}\int\limits_{-\pi\sqrt{npq}}^{\pi\sqrt{npq}}\prod_{k=1}^{n}\varrho_k(\theta,0)\varrho_k(0,t)\,d\theta.$$

Следовательно

$$\varphi_{n,s_n}\left(\frac{t}{\sigma_n}\right)-\prod_{k=1}^{n}\left(qe^{-\frac{itpa_{nk}}{\sigma_n}}+pe^{\frac{itqa_{nk}}{\sigma_n}}\right)= \tag{2}$$

$$=\frac{1}{2\pi\sqrt{npq}\,P_n(s_n)}\int\limits_{-\pi\sqrt{npq}}^{\pi\sqrt{npq}}\sum_{j=1}^{n}\prod_{l=1}^{j-1}\varrho_l(\theta,0)\varrho_l(0,t)\,[\varrho_j(\theta,t)-\varrho_j(\theta,0)\varrho_j(0,t)]\times$$

$$\times\prod_{k=j+1}^{n}\varrho_k(\theta,t)\,d\theta.$$

Лемма 1. *Для всех* $|t| \leqq \frac{24-\pi^2}{48\pi b_n}\left(\frac{1}{560}\right)^2$ *имеет место неравенство*

$$\left|\varphi_{n,s_n}\left(\frac{t}{\sigma_n}\right) - \prod_{k=1}^{n}\left(qe^{-ip\frac{ta_{nk}}{\sigma_n}} + pe^{iq\frac{ta_{nk}}{\sigma_n}}\right)\right| \leqq \frac{C_6 \max\limits_{1\leqq k\leqq n}|a_{nk}|(|t|+t^4)}{\sigma_n\sqrt{npq}\,P_n(s_n)}\, e^{-c_7 t^2}.$$

Здесь и в дальнейшем $b_n = \frac{\max\limits_{1\leqq k\leqq n}|a_{nk}|}{\sigma_n}$.

Лемма 2. *При* $|t| \leqq \frac{3}{32\, b_n}$

$$\left|\prod_{k=1}^{n}\left(qe^{-ip\frac{ta_{nk}}{\sigma_n}} + pe^{iq\frac{ta_{nk}}{\sigma_n}}\right) - e^{-\frac{t^2}{2}}\right| \leqq \frac{8b_n|t|^3}{3}\, e^{-\frac{t^2}{4}}.$$

Лемма 3. *Для всех* $|t| \leqq \frac{24-\pi^2}{48\pi\, b_n}\left(\frac{1}{560}\right)^2$ *существуют положительные константы* C_8 *и* C_9 *такие, что*

$$\left|\varphi_{n,s_n}\left(\frac{t}{\sigma_n}\right) - e^{-\frac{t^2}{2}}\right| \leqq C_8 b_n(|t|+t^4)e^{-c_9 t^2}.$$

Доказательство леммы 1. Обозначим

$$g_k(\theta, t) = \exp\left\{-\frac{1}{2}\left(\frac{pq}{npq}\,\theta^2 + 2\,\frac{pqa_{nk}}{\sigma_n\sqrt{npq}}\,\theta t + \frac{pqa_{nk}^2}{\sigma_n^2}\,t^2\right)\right\}.$$

В силу того, что $\sum\limits_{k=1}^{n} a_{nk} = 0$, имеем

$$\sum_{j=1}^{n}\prod_{l=1}^{j-1} g_l(\theta,0)g_l(0,t)\,[g_j(\theta,t) - g_j(\theta,0)g_j(0,t)]\prod_{k=j+1}^{n} g_k(\theta,t) =$$

$$= \prod_{k=1}^{n} g_k(\theta,t) - \prod_{k=1}^{n} g_k(\theta,0)g_k(0,t) \equiv 0.$$

Следовательно,

$$(3)\qquad \sum_{j=1}^{n}\prod_{l=1}^{j-1} \varrho_l(\theta,0)\varrho_l(0,t)\,[\varrho_j(\theta,t) - \varrho_j(\theta,0)\varrho_j(\theta,t)]\prod_{k=j+1}^{n} \varrho_k(\theta,t) = \sum_{j=1}^{n}(Y_j + I_j),$$

где

$$(4)\qquad Y_j = [\varrho_j(\theta,t) - \varrho_j(\theta,0)\varrho_j(0,t)]\left\{\prod_{l=1}^{j-1}\varrho_l(\theta,0)\varrho_l(0,t)\prod_{k=j+1}^{n}\varrho_k(\theta,t) - \right.$$

$$\left. - \prod_{l=1}^{j-1} g_l(\theta,0)g_l(0,t)\prod_{k=j+1}^{n} g_k(\theta,t)\right\}$$

и

$$I_j = \{[\varrho_j(\theta,t) - \varrho_j(\varrho,0)\varrho_j(0,t)] - [g_j(\theta,t) - g_j(\theta,0)g_j(0,t)]\}\times$$

$$\times \prod_{l=1}^{j-1} g_l(\theta,0)g_l(0,t)\prod_{k=j+1}^{n} g_k(\theta,t).$$

Сперва оценим Y_j. По определению $\varrho_j(\theta, t)$

$$\varrho_j(\theta, t)-\varrho_j(\theta, 0)\varrho_j(0, t) = pq\left(1-e^{\frac{ita_{nj}}{\sigma_n}}\right)\left(1-e^{\frac{i\theta}{\sqrt{npq}}}\right)e^{-ip\left(\frac{\theta}{\sqrt{npq}}+\frac{ta_{nj}}{\sigma_n}\right)}.$$

Из этого равенства следует, что

$$|\varrho_j(\theta, t)-\varrho_j(\theta, 0)\varrho_j(0, t)| \leqq \frac{pq|\theta|\cdot|t||a_{nj}|}{\sigma_n\sqrt{npq}}. \tag{5}$$

Очевидно

$$|\varrho_k(\theta, t)|^2 = p^2+q^2+2pq\cos\left(\frac{\theta}{\sqrt{npq}}+\frac{ta_{nk}}{\sigma_n}\right), \tag{6}$$

следовательно, при $|t| \leqq \frac{\pi}{4b_n}$ и $|\theta| \leqq \frac{\pi}{4}\sqrt{npq}$ имеет место оценка

$$|\varrho_k(\theta, t)|^2 \geqq \frac{1}{2}.$$

Этот факт позволяет нам применить многомерную формулу TAYLOR-a:

$$\prod_{l=1}^{j-1}\varrho_l(\theta, 0)\varrho_l(0, t)\prod_{k=j+1}^{n}\varrho_k(\theta, t) = \exp\left\{\sum_{l=1}^{j-1}\ln\varrho_l(\theta, 0)\varrho_l(0, t)+\sum_{k=j+1}^{n}\ln\varrho_k(\theta, t)\right\} =$$

$$= \exp\left\{\sum_{l=1}^{j-1}\left[\left(-\frac{ip}{\sqrt{npq}}+p\frac{iq}{\sqrt{npq}}\right)\theta+\left(-q\frac{ipa_{nl}}{\sigma_n}+p\frac{iqa_{nl}}{\sigma_n}\right)t\right]+\right.$$

$$+\sum_{k=j+1}^{n}\left[\left(-q\frac{ip}{\sqrt{npq}}+p\frac{iq}{\sqrt{npq}}\right)\theta+\left(-q\frac{ipa_{nk}}{\sigma_n}+p\frac{iqa_{nk}}{\sigma_n}\right)t-\right.$$

$$-\frac{1}{2}\sum_{l=1}^{j-1}\left[\left(\frac{p^2q}{npq}+\frac{pq^2}{npq}\right)\theta^2+\left(\frac{p^2qa_{nl}}{\sigma_n^2}+\frac{pq^2a_{nl}}{\sigma_n^2}\right)t^2\right]-$$

$$\left.\left.-\frac{1}{2}\sum_{k=j+1}^{n}\left[\left(\frac{p^2q}{npq}+\frac{pq^2}{npq}\right)\theta^2+2\left(\frac{p^2qa_{nk}}{\sigma_n\sqrt{npq}}+\frac{pq^2a_{nk}}{\sigma_n\sqrt{npq}}\right)+\left(\frac{qp^2a_{nk}^2}{\sigma_n^2}+\frac{pq^2a_{nk}^2}{\sigma_n^2}\right)t^2\right]+r_n(\theta, t)\right\}$$

в вышеуказанных интервалах изменения t и θ. Здесь $r_n(\theta, t)$—остаточный член в формуле TAYLOR-a. Так как он очень грамоздкий, то его в явном виде выписать не будем.

Замечая, что

$$\prod_{l=1}^{j-1}g_l(\theta, 0)g_l(0, t)\prod_{k=j+1}^{n}g_k(\theta, t) = \exp\left\{-\frac{\theta^2+t^2}{2}+\frac{pq\theta^2}{2npq}+\frac{pqt^2a_{nj}^2}{2\sigma_n^2}-\sum_{k=j+1}^{n}\frac{pq\theta ta_{nk}}{\sigma_n\sqrt{npq}}\right\},$$

последнее равенство можем записать в виде

$$\prod_{l=1}^{j-1}\varrho_l(\theta, 0)\varrho_l(0, t)\prod_{k=j+1}^{n}\varrho_k(\theta, t) = \prod_{l=1}^{j-1}g_l(\theta, 0)g_l(0, t)\prod_{k=j+1}^{n}g_k(\theta, t)-$$

$$-(1-e^{r_n(\theta, t)})\exp\left\{-\frac{\theta^2+t^2}{2}+\frac{pq\theta^2}{2npq}+\frac{pqt^2a_{nj}^2}{2\sigma_n^2}-\sum_{k=j+1}^{n}\frac{pq\theta ta_{nk}}{\sigma_n\sqrt{npq}}\right\}.$$

Из этого соотношения вытекает, что

$$(7)\qquad \left|\prod_{l=1}^{j-1}\varrho_l(\theta,0)\varrho_l(0,t)\prod_{k=j+1}^{n}\varrho_k(\theta,t)-\prod_{l=1}^{j-1}g_l(\theta,0)g_l(0,t)\prod_{k=j+1}^{n}g_k(\theta,t)\right|\leqq$$

$$\leqq |r_n(\theta,t)|\exp\left\{-\frac{t^2+\theta^2}{2}+\frac{pq\theta^2}{2npq}+\frac{pqt^2a_{nj}^2}{2\sigma_n^2}-\sum_{k=j+1}^{n}\frac{pq\theta t a_{nk}}{\sigma_n\sqrt{npq}}+|r_n(\theta,t)|\right\}.$$

Поскольку сумма $\sum_{k=1}^{n}a_{nk}$ равна нулю, а $\sigma_n^2=pq\sum_{k=1}^{n}a_{nk}$, то

$$(8)\qquad \left|\sum_{k=j+1}^{n}\frac{pq\theta t a_{nk}}{\sigma_n\sqrt{npq}}\right|\leqq\frac{\theta^2}{2\sqrt{2}}+\frac{t^2}{2\sqrt{2}}.$$

Осталось оценить сверху $r_n(\theta,t)$ при $|t|\leqq\frac{13}{560\,b_n}$ и $|\theta|\leqq\frac{\sqrt{npq}}{560}$. Имеем

$$(9)\qquad \begin{aligned}|r_n(\theta,t)|\leqq{}&\frac{(\sqrt{2})^3}{6}\left(\frac{p^2+q^2}{\sqrt{npq}}|\theta|^3+\frac{pq(p^2+q^2)|t|^3}{\sigma_n^2}\sum_{k=1}^{n}|a_{nk}|^3+\right.\\&\left.+\frac{3pq(p^2+q^2)|t|\theta^2}{\sigma_n npq}\sum_{k=1}^{n}|a_{nk}|+\frac{3pq(p^2+q^2)t^2|\theta|}{\sigma_n^2\sqrt{npq}}\sum_{k=1}^{n}a_{nk}^2\right)+\\&+\frac{(\sqrt{2})^3}{2}\sum_{l=1}^{j-1}\left(\frac{2pq|\theta|}{\sqrt{npq}}+\frac{2pq|t||a_{nl}|}{\sigma_n}\right)\left(\frac{pq\theta^2}{npq}+\frac{2pq|\theta||t||a_{nl}|}{\sigma_n\sqrt{npq}}+\frac{pqt^2a_{nl}^2}{\sigma_n^2}\right)+\\&+\frac{(\sqrt{2})^3}{2}\sum_{k=j+1}^{n}\left(\frac{2pq|\theta|}{\sqrt{npq}}+\frac{2pq|t||a_{nk}|}{\sigma_n}\right)\left(\frac{pq\theta^2}{npq}+\frac{8p^2q^2|\theta||t||a_{nk}|}{\sigma_n\sqrt{npq}}+\frac{pqt^2a_{nk}^2}{\sigma_n^2}\right)+\\&+\frac{(\sqrt{2})^3}{6}\sum_{k=1}^{n}\left(\frac{2pq|\theta|}{\sqrt{npq}}+\frac{2pq|t||a_{nk}|}{\sigma_n}\right)^3\leqq 13\sqrt{2}\frac{|\theta|^3}{\sqrt{npq}}+\\&+\frac{13\sqrt{2}}{6}\frac{|t|^3\max\limits_{1\leqq k\leqq n}|a_{nk}|}{\sigma_n}+\frac{5\sqrt{2}}{2}\frac{|t|\theta^2\max\limits_{1\leqq k\leqq n}|a_{nk}|}{\sigma_n}+\frac{5\sqrt{2}}{2}\cdot\frac{t^2|\theta|}{\sqrt{npq}},\end{aligned}$$

а также

$$(10)\qquad |r_n(\theta,t)|\leqq\frac{t^2}{10}+\frac{\theta^2}{8}.$$

В силу (4, 5, 7—10) существуют константы $C_{10}>0$ и $C_{11}>0$ такие, что

$$(11)\qquad \left|\sum_{j=1}^{n}Y_j\right|\leqq$$

$$\leqq\sum_{j=1}^{n}\frac{pq|t||\theta||a_{nj}|}{\sigma_n\sqrt{npq}}\left(\frac{13\sqrt{2}\,|\theta|^3}{\sqrt{npq}}+\frac{13\sqrt{2}\,|t|^3b_n}{6}+\frac{5\sqrt{2}\,|t|\theta^2b_n}{1}+\frac{5\sqrt{2}}{2}\frac{|\theta|t^2}{\sqrt{npq}}e^{-c_{10}(\theta^2+t^2)}\right)\leqq$$

$$\leqq 13\sqrt{2}\,b_n(\theta^4|t|+t^4|\theta|+t^2|\theta|^3+\theta^2|t|^3)e^{-c_{11}(t^2+\theta^2)}$$

для всех $|t|\leqq\frac{13}{560b_n}$ и $|\theta|\leqq\frac{\sqrt{npq}}{560}$.

Теперь оценим сумму $\sum_{j=1}^{n} I_j$. Очевидно

$$I'_j = [\varrho_j(\theta, t) - \varrho_j(\theta, 0)\varrho_j(0, t)] - [g_j(\theta, t) - g_j(\theta, 0)g_j(0, t)] =$$

$$= pq\left(1 - e^{\frac{ita_{nj}}{\sigma_n}}\right)\left(1 - e^{\frac{i\theta}{\sqrt{npq}}}\right)e^{-ip\left(\frac{\theta}{\sqrt{npq}} + \frac{ta_{nj}}{\sigma_n}\right)} - \left[\exp\left\{-\frac{1}{2}\left(\frac{pq}{npq}\theta^2 + \right.\right.\right.$$

$$\left.\left.\left. + \frac{2pq\theta t a_{nj}}{\sigma_n\sqrt{npq}} + \frac{pqt^2a_{nj}^2}{\sigma_n^2}\right)\right\} - \exp\left\{-\frac{1}{2}\left(\frac{pq}{npq}\theta^2 + \frac{pqt^2a_{nj}^2}{\sigma_n^2}\right)\right\}\right].$$

С помощью формулы TAYLOR-a I'_j можем записать в виде

$$I'_j = pq\left[\frac{ita_{nj}}{\sigma_n} + \frac{1}{2}\left(\frac{ita_{nj}}{\sigma_n}\right)^2 e^{\frac{it\varepsilon_1 a_{nj}}{\sigma_n}}\right]\left[\frac{i\theta}{\sqrt{npq}} + \frac{1}{2}\left(\frac{i\theta}{\sqrt{npq}}\right)^2 e^{\frac{i\theta\varepsilon_2}{\sqrt{npq}}}\right] \cdot$$

$$\cdot\left[1 - ip\left(\frac{\theta}{\sqrt{npq}} + \frac{ta_{nj}}{\sigma_n}\right)\exp\left\{-ip\varepsilon_3\left(\frac{\theta}{\sqrt{npq}} + \frac{ta_{nj}}{\sigma_n}\right)\right\}\right] -$$

$$- \left[-\frac{1}{2}\left(\frac{pq}{npq}\theta^2 + \frac{2pqa_{nj}}{\sigma_n\sqrt{npq}}\theta t + \frac{pqa_{nj}^2}{\sigma_n^2}t^2\right) + \right.$$

$$+ \frac{1}{4}\left(\frac{pq}{npq}\theta^2 + 2\frac{pqa_{nj}}{\sigma_n\sqrt{npq}}\theta t + \frac{pqa_{nj}^2}{\sigma_n^2}t^2\right)^2 \exp\left\{-\frac{\varepsilon_4}{2}\left(\frac{pq}{npq}\theta^2 + 2\frac{pqa_{nj}}{\sigma_n\sqrt{npq}}\theta t + \frac{pqa_{nj}^2}{\sigma_n^2}t^2\right)\right\} +$$

$$\left. + \frac{1}{2}\left(\frac{pq}{npq}\theta^2 + \frac{pqa_{nj}^2}{\sigma_n^2}t^2\right) - \frac{1}{4}\left(\frac{pq}{npq}\theta^2 + \frac{pqa_{nj}^2}{\sigma_n^2}t^2\right)^2 \exp\left\{-\frac{\varepsilon_5}{2}\left(\frac{pq}{npq}\theta^2 + \frac{pqa_{nj}^2}{\sigma_n^2}t^2\right)^2\right\}\right].$$

Здесь $0 < \varepsilon_j < 1$, $j = 1, 2, 3, 4, 5$. Отсюда для $|t| \leqq \frac{13}{560\, b_n}$ и $|\theta| \leqq \frac{\sqrt{npq}}{560}$ получаем

$$|I'_j| \leqq \frac{2pq|t|^2|\theta|a_{nj}^2}{\sigma_n^2\sqrt{npq}} + \frac{2pq\theta^2|t||a_{nj}|}{\sigma_n npq} + \frac{2p^2q^2t^2\theta^2a_{nj}^2}{\sigma_n^2 npq} + \frac{p^2q|\theta|^3|t||a_{nj}|}{\sigma_n(\sqrt{npq})^3} +$$

$$+ \frac{p^2q^2|\theta||t|^3|a_{nj}|^3}{2\sigma_n^3\sqrt{npq}} + \frac{p^2qt^2|\theta|^3a_{nj}^2}{2\sigma_n^2(\sqrt{npq})^3} + \frac{p^2q\theta^2|t|^3|a_{nj}|^3}{4\sigma_n^3 npq} + \frac{p^3q^3\theta^2t^4a_{nj}^4}{2\sigma_n^4 npq}.$$

Так как для тех же t и θ

$$\prod_{l=1}^{j-1} g_l(\theta, 0)g_l(0, t) \prod_{k=j+1}^{n} g_k(\theta, t) \leqq 2e^{-\frac{\theta^2+t^2}{10}},$$

то

$$\left|\sum_{j=1}^{n} I_j\right| \leqq 2b_n(t^2|\theta| + \theta^2|t| + t^2\theta^2 + |t||\theta|^3 + |\theta||t|^3 + t^2|\theta|^3 + \theta^2|t|^3 + \theta^2t^4)e^{-\frac{t^2+\theta^2}{10}} \tag{12}$$

Из (3), (11) и (12) вытекает, что

$$(13)\qquad \left|\sum_{j=1}^{n}\prod_{l=1}^{j-1}\varrho_l(\theta,0)\varrho_l(0,t)[\varrho_j(\theta,t)-\varrho_j(\theta,0)\varrho_j(0,t)]\prod_{k=j+1}^{n}\varrho_k(\theta,t)\right| \leqq$$

$$\leqq 13\sqrt{2}\,b_n(|t|\theta^4+t^4|\theta|+t^2|\theta|^3+\theta^2|t|^3+|t|\theta^2+|\theta|t^2+t^2\theta^2+|t||\theta|^3+$$

$$+|\theta||t|^3+\theta^2t^4)e^{-c_{12}(t^2+\theta^2)}$$

при $|t|\leqq\frac{13}{560\,b_n}$ и $|\theta|\leqq\frac{\sqrt{npq}}{560}$.

Теперь оценим подинтегральное выражение

$$\Delta_j=\prod_{l=1}^{j-1}\varrho_l(\theta,0)\varrho_l(0,t)[\varrho_j(\theta,t)-\varrho_j(\theta,0)\varrho_j(0,t)]\prod_{k=j+1}^{n}\varrho_k(\theta,t)$$

для $|t|\leqq\frac{24-\pi^2}{48\pi\,b_n}\left(\frac{1}{560}\right)^2$ и $\frac{\sqrt{npq}}{560}\leqq|\theta|\leqq\pi\sqrt{npq}$. В силу (5) и (6)

$$|\Delta_j|\leqq\frac{pq|t\theta a_{nj}|}{\sigma_n\sqrt{npq}}\prod_{l=1}^{j-1}\left(p^2+q^2+2pq\cos\frac{\theta}{\sqrt{npq}}\right)\left(p^2+q^2+2pq\cos\frac{ta_{nl}}{\sigma_n}\right)\cdot$$

$$\cdot\prod_{k=j+1}^{n}\left(p^2+q^2+2pq\cos\left(\frac{\theta}{\sqrt{npq}}+\frac{ta_{nk}}{\sigma_n}\right)\right)\leqq\frac{pq|t\theta a_{nj}|}{\sigma_n\sqrt{npq}}\cdot$$

$$\exp\left\{-2pq\left(\sum_{l=1}^{j-1}\left(\sin^2\frac{\theta}{2\sqrt{npq}}+\sin^2\frac{ta_{nl}}{2\sigma_n}\right)+\sum_{k=j+1}^{n}\sin^2\left(\frac{\theta}{2\sqrt{npq}}+\frac{ta_{nk}}{2\sigma_n}\right)\right)\right\}\leqq$$

$$\leqq\frac{pq|t\theta a_{nj}|}{\sigma_n\sqrt{npq}}\exp\left\{-\frac{t^2+\theta^2}{2}+\frac{\theta^2}{2n}+\frac{t^2a_{nj}^2}{2\sigma_n^2}-\frac{\theta^4}{24n^2pq}-\frac{pqt^4a_{nj}^4}{24\sigma_n^4}+\frac{\theta^4}{24npq}+\right.$$

$$\left.+\sum_{j=1}^{n}\frac{pqt^4a_{nj}^4}{24\sigma_n^4}-\sum_{k=j+1}^{n}\frac{pq\theta ta_{nk}}{\sigma_n\sqrt{npq}}+pq\sum_{k=j+1}^{n}\left(\frac{|t||\theta|^3|a_{nk}|}{4\sigma_n(\sqrt{npq})^3}+\frac{3\theta^2t^2a_{nk}^2}{8npq\sigma_n^2}+\frac{|\theta||t|^3|a_{nk}|^3}{4\sigma_n^3\sqrt{npq}}\right)\right\}.$$

Далее, при вышеуказанном изменении t и θ, получаем

$$\left|\sum_{k=j+1}^{n}\frac{pq\theta ta_{nk}}{\sigma_n\sqrt{npq}}\right|\leqq\frac{24-\pi^2}{48\pi b_n}\left(\frac{1}{560}\right)^2|\theta|,$$

при этом

$$-\frac{\theta^2}{2}+\frac{\theta^4}{24npq}+\left|\sum_{k=j+1}^{n}\frac{pq\theta ta_{nk}}{\sigma_n\sqrt{npq}}\right|\leqq-\frac{24-\pi^2}{48}\theta^2.$$

Из трех последних неравенств вытекает

$$|\Delta_j|\leqq\frac{C_{14}pq|t\theta a_{nj}|}{\sigma_n\sqrt{npq}}e^{-c_{13}(t^2+\theta^2)}.$$

Очевидно

$$\sum_{j=1}^{n} |\Delta_j| \leqq C_{14}|\theta t|\sqrt{n}\, b_n e^{-c_{15}(t^2+\theta^2)} \leqq C_{16}|t\theta|\, b_n e^{-c_{17}(t^2+\theta^2)} \tag{14}$$

при $|t| \leqq \frac{24-\pi^2}{48\pi b_n}\left(\frac{1}{560}\right)^2$ и $\frac{\sqrt{npq}}{560} < |\theta| \leqq \pi\sqrt{npq}$.

Из (2), (13) и (14) вытекает, что

$$\left|\varphi_{n,s_n}\left(\frac{t}{\sigma_n}\right) - \prod_{k=1}^{n}\left(qe^{-\frac{itpa_{nk}}{\sigma_n}} + pe^{\frac{ita_{nk}}{\sigma_n}}\right)\right| \leqq \frac{1}{2\pi\sqrt{npq}\,P_n(s_n)} \cdot$$

$$\cdot \int_{-\frac{\sqrt{npq}}{560}}^{\frac{\sqrt{npq}}{560}} 13\sqrt{2}\, b_n(\theta^4|t| + |\theta|\,t^4 + t^2|\theta|^3 + \theta^2|t|^3 + |t|\,\theta^2 + |\theta|\,t^2 + t^2\theta^2 + |t|\,|\theta|^3 +$$

$$+ |t|^3|\theta| + \theta^2 t^4) e^{-c_{18}(t^2+\theta^2)}\, d\theta + \frac{1}{2\pi\sqrt{npq}\,P_n(s_n)} \int_{\frac{\sqrt{npq}}{560} \leqq |\theta| \leqq \pi\sqrt{npq}} \frac{C_{19}pq|t\theta a_{nj}|}{\sigma_n\sqrt{npq}} \cdot$$

$$\cdot e^{-c_{20}(t^2+\theta^2)}\, d\theta \leqq \frac{C_{21}\, b_n(|t|+t^4)}{\sqrt{npq}\,P_n(s_n)} e^{-c_{22}t^2}$$

при $|t| \leqq \frac{24-\pi^2}{48\pi b_n}\left(\frac{1}{560}\right)^2$. Лемма 1 доказана.

Доказательство леммы 2. Надо заметить, что для $1 \leqq k \leqq n$ и $|t| \leqq \frac{\pi}{4b_n}$ имеет место неравенство

$$\left|qe^{-ip\frac{ta_{nk}}{\sigma_n}} + pe^{iq\frac{ta_{nk}}{\sigma_n}}\right|^2 \leqq \frac{1}{2}.$$

Поэтому можно применить формулу TAYLOR-а для указанных t:

$$\prod_{k=1}^{n}\left(qe^{-ip\frac{ta_{nk}}{\sigma_n}} + pe^{iq\frac{ta_{nk}}{\sigma_n}}\right) = e^{-\frac{t^2}{2}R_n(t)},$$

где

$$R_n(t) = \frac{(it)^3}{6\sigma_n^3}\sum_{k=1}^{n}\left[\ln\left(qe^{-ip\frac{t_0a_{nk}}{\sigma_n}} + pe^{iq\frac{t_0a_{nk}}{\sigma_n}}\right)\right]'''_{|t_0=\varepsilon t}, \qquad 0 < \varepsilon < 1.$$

В силу леммы 3 работы [4]

$$\sum_{k=1}^{n}\left|\left[\ln\left(qe^{-ip\frac{t_0a_{nk}}{\sigma_n}} + pe^{iq\frac{t_0a_{nk}}{\sigma_n}}\right)\right]'''\right| \leqq 16pq\sum_{k=1}^{n}|a_{nk}|^3$$

при $|t| \leqq \frac{3}{32\, b_n}$. Кроме того,

$$\frac{16pq|t|^3}{6\sigma_n^3}\sum_{k=1}^{n}|a_{nk}|^3 \leqq \frac{t^2}{4}.$$

Из двух последних неравенств и того, что

$$\left|\prod_{k=1}^{n}\left(qe^{-ip\frac{ta_{nk}}{\sigma_n}}+pe^{iq\frac{ta_{nk}}{\sigma_n}}\right)-e^{-\frac{t^2}{2}}\right| \leqq$$

$$\leqq \frac{16pq|t|^3}{6\sigma_n^3}\sum_{k=1}^{n}|a_{nk}|^3 \exp\left\{-\frac{t^2}{2}+\frac{16pq|t|^3}{6\sigma_n^3}\sum_{k=1}^{n}|a_{nk}|^3\right\}$$

следует утверждение леммы 2.

Доказательство леммы 3. Для всех значений s_n и $n \geqq n_0(\varepsilon)$ имеет место неравенство

$$\sqrt{s_n}\,P_n(s_n)=\binom{n}{s_n}\left(\frac{s_n}{n}\right)^{s_n}\left(1-\frac{s_n}{n}\right)^{n-s_n}\sqrt{s_n} \geqq (1-\varepsilon)\frac{2\sqrt{2}}{e^2}.$$

Действительно,

$$\left(1+\frac{1}{s}\right)^{s+\frac{1}{2}} \geqq e, \quad \text{если} \quad s \geqq 2,$$

и

$$\left(1+\frac{1}{n}\right)^{n-1} \leqq e, \quad \text{если} \quad n \geqq 1.$$

Поэтому последовательность

$$a(n,s)=\binom{n}{s}\left(\frac{s}{n}\right)^{s}\left(1-\frac{s}{n}\right)^{n-s}\sqrt{s} \quad (s=2,3,\ldots,n)$$

возрастает:

$$a(n,s) \geqq a(n,2) \to \frac{2\sqrt{2}}{e^2}, \quad n \to \infty.$$

Кроме того

$$a(n,1) \to \frac{1}{e}, \quad n \to \infty.$$

Теперь утверждение леммы 3 немедленно вытекает из леммы 1 и 2.

В заключение хочу сердечно поблагодарить проф. A. Rényi за постановку задачи и ценные советы при ее решении.

ЛИТЕРАТУРА

[1] Erdős, P. and Rényi, A.: On a central limit theorem for samples from a finite population, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **4** (1959) 49—61.

[2] Hajek, J.: Limiting distributions in simple random sampling for a finite population, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **5** (1960) 361—374.

[3] Ибрагимов, И. А. Линник, Ю. В.: *Независимые и стационарно связанные величины,* Изд. Наука, Москва, 1965.

[4] Викялис, А.: Асимптотические разложения для характеристических функций, *Лит. мат. сб.* **8** (2) (1968).

[5] Бернштейн, С. Н. *Теория вероятностей,* Москва, Гостехиздат, 1949.

Вильнюсский Гос. Университет им. В. Капсукаса, Вильнюс

(Поступила 27-ого июня 1968 г.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 355—365.

ON THE ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF $y'' + \sum_{i=0}^{n} a_i(x) y^{2i+1} = 0$

by

L. HATVANI

§ 1. Introduction

In this paper we shall study the asymptotic properties of the solutions of the equation

$$y'' + \sum_{i=0}^{n} a_i(x) y^{2i+1} = 0 \quad (0 \leqq x < \infty) \tag{1.1}$$

where the functions $a_i(x)$ $(i = 0, 1, 2, \ldots, n)$, defined on the interval $[0, \infty)$, are nonnegative and continuously differentiable. We shall seek for conditions which assure that all the solutions of (1. 1) are bounded, or satisfy even

$$\lim_{x \to \infty} y(x) = 0 \tag{1.2}$$

and at the same time we shall consider the behaviour of the function $y'(x)$ for large values of x.

A. MEIR, D. WILLETT and J. S. W. WONG in their papers for the linear and nonlinear equations

$$y'' + a(x) y = 0 \quad (0 \leqq x < \infty) \tag{1.3}$$

$$y'' + a(x) y^{2k+1} = 0 \quad (0 \leqq x < \infty;\ k = 0, 1, 2, \ldots) \tag{1.4}$$

respectively, proved the following theorems:

THEOREM 1. (A. Meir, D. Willett and J. S. W. Wong [1])

If $a(x) \in C^3[0, \infty)$ is a monotone nondecreasing, nonnegative function for which $\lim_{x \to \infty} a(x) = \infty$ and

$$\int_0^x \left| (a^{-\alpha}(t))''' \right| dt = o(a^{1-\alpha}(x)) \quad (x \to \infty)$$

for some real number α $(0 < \alpha < 1)$, then all the solutions of the equation (1. 3) *satisfy the relation* (1. 2).

THEOREM 2. (D. Willett, J. S. W. Wong [3])

Suppose that $a(x) \in C^3[0, \infty)$, $a(x) \geqq \nu > 0$ *and introduce the following notations:*

$$\psi(x) = a^{-\lambda}(x) \quad \textit{where} \quad \lambda = \frac{1}{k+2},$$

$$\Psi_k(x) = \psi^{k+1}(x) \int_0^x [\psi'''(t)]_- \, dt \,^1$$

and

$$\Phi_k(x) = \psi^{k+1}(x) \int_0^x |\psi'''(t)| \, dt.$$

A. If $k = 0$ *and* $\Psi_0(x) \leqq 2$ *for all* $x \geqq x_0$, *or* $k \geqq 1$ *and* $\Psi_k(x)$ *is bounded for* $x \geqq x_0$, *then all the solutions of* (1. 4) are *bounded.*

B. Suppose that $\lim_{x \to \infty} a(x) = \infty$. *If* $k = 0$ *and* $\limsup_{x \to \infty} \Phi_0(x) < 2$, or $k \geqq 1$ *and* $\limsup_{x \to \infty} \Phi_k(x) = 0$, *then all the solutions of* (1. 4) *satisfy the relation* (1. 2).

In § 2 of our paper we shall extend Theorems 1 and 2 to the equation (1. 1 in sharpened forms.

In § 3 we find a sufficient condition assuring (1. 2).

In the conditions of the theorems we introduce a function $q(x)$ with some properties T, T_β defined as follows:

DEFINITION 1. Let the equation (1. 1) be given. A positive function $q(x)$ defined on an appropriate interval $[x_0, \infty)$ $(x_0 \geqq 0)$ is said to have the *property* T if there exists at least one j $(0 \leqq j \leqq n)$ such that

$$\liminf_{x \to \infty} \frac{a_j(x)}{q(x)} > 0.$$

DEFINITION 2. Let the equation (1. 1) be given. A positive function $q(x)$ on an appropriate interval $[x_0, \infty)$ $(x_0 \geqq 0)$ will be called having the *property* T_β (where β is a positive number) if $q(x)$ is continuously differentiable, has property T, and moreover the inequalities

$$a_i'(x) \leqq \beta a_i(x) \frac{q'(x)}{q(x)}$$

hold for $i = 0, 1, 2, \ldots, n$, on $[x_0, \infty)$.

If the system $\{a_i(x)\}_{i=0}^n$ consists of positive functions, and there exist a number j $(0 \leqq j \leqq n)$ and a function $f(x)$, nonnegative and measurable on an appropriate interval $[x_0, \infty)$ such that

$$\max_{0 \leqq i \leqq n} \left\{ \frac{a_i'(x)}{a_i(x)} \right\} = \frac{a_j'(x)}{a_j(x)}$$

[1] For any real number a, we denote by $[a]_+$ the *positive* and by $[a]_-$ the *negative* part of a, i.e. $[a]_+ = \max\{0, a\}$ and $[a]_- = \max\{0, -a\}$.

on $[x_0, \infty)$ and

$$a_j^{\frac{1}{\beta}-1}(x)\exp\left(\frac{1}{\beta}\int_{x_0}^{x} f(s)\,ds\right) = O(1) \quad (x\to\infty)$$

then by differentiation it can be immediately seen that the function

$$q(x) = ca_j^{\frac{1}{\beta}}(x)\exp\left(\frac{1}{\beta}\int_{x_0}^{x} f(s)\,ds\right) \quad (c>0,\ \beta>0)$$

has the property T_β.

In the proofs of the theorems of this paper we shall refer to the following properties of the function

$$V(x, y, y'; q) = \frac{1}{q(x)}\left(y'^2 + \sum_{i=0}^{n}\frac{1}{i+1}a_i(x)y^{2(i+1)}\right) \tag{1.5}$$

where the function $q(x)$ has the property T.

Supposing $y=y(x)$ to be a solution of (1. 1) on $[x_0, \infty)$ and introducing the notation

$$v=v(x)=V(x, y(x), y'(x);\ q(x)) \tag{1.6}$$

we have:

(a) $v(x)$ is a nonnegative continuous function on the interval $[x_0, \infty)$.

(b) If $v(x)$ is bounded then the solution $y(x)$ is bounded too and $y'(x)=$ $=O(q^{1/2}(x))$ $(x\to\infty)$.

(c) If $\lim_{x\to\infty} v(x)=0$ then $\lim_{x\to\infty} y(x)=0$ and $y'(x)=o(q^{1/2}(x))$ $(x\to\infty)$.

(d) If the function $q(x)$ is nondecreasing and has the property T_β $(0<\beta\leqq 1)$ then the function $v(x)$ is nonincreasing on $[x_0, \infty)$.

The property (a) is obvious, (b) and (c) can be obtained immediately combining (1. 6) and the definition of the property T.

ad(d): The function $q(x)$ has the property T_β with $\beta\leqq 1$, hence

$$\left(\frac{a_i(x)}{q(x)}\right)' \leqq 0 \qquad (i=0, 1, 2, \ldots, n;\ x\geqq x_0). \tag{1.7}$$

Since the function $y=y(x)$ is a solution of (1. 1), we have

$$v' = -\frac{y'^2q'}{q^2} + \sum_{i=0}^{n}\frac{1}{i+1}\left(\frac{a_i}{q}\right)' y^{2(i+1)}.$$

From $q'(x)\geqq 0$ and (1. 6) we deduce that $v'(x)\leqq 0$ for $x\geqq x_0$ showing that (d) is true. In what follows, according to these properties, the function $V(x, y, y'; q)$ will be called the *Liapunov function for the equation* (1. 1).

§ 2.

The first lemma of this § is a generalization of a result of UTZ [4].

LEMMA 2. 1. *If* $a_i'(x) \leqq 0$ $(i=0, 1, \ldots, n)$ *on an appropriate interval* (b, ∞) *and*

$$\lim_{x\to\infty} a_l(x) > 0 \tag{2. 1}$$

for at least one l $(0 \leqq l \leqq n)$, *then every solution of* (1. 1) *satisfies* $|y(x)| < C_1$, $|y'(x)| < C_2$.

PROOF. Multiplying the equation (1. 1) by $y'(x)$ and integrating from b to x $(x > b)$, we obtain

$$c = y'^2(x) + \sum_{i=0}^{n} \frac{1}{i+1} a_i(x) y^{2(i+1)}(x) - \int_b^x \left(\sum_{i=0}^{n} \frac{1}{i+1} a_i'(t) y^{2(i+1)}(t) \right) dt \geqq$$

$$\geqq y'^2(x) + \frac{1}{l+1} a_l(x) y^{2(l+1)}(x).$$

In view of (2. 1) this proves the lemma.

LEMMA 2. 2. *If* $\int_0^\infty a_l(x)dx = \infty$ *for at least one* l $(0 \leqq l \leqq n)$, *then all the solutions of* (1. 1) *are oscillatory.*

PROOF. Suppose that (1. 1) has a nonoscillatory solution, that is there exists a solution $y(x)$ and a positive constant b such that $y(x) \neq 0$ when $x \geqq b$. Since, with $y(x)$, $-y(x)$ also is a solution of (1. 1), we may assume without loss of generality that $y(x) > 0$, if $x \geqq b$. Then from (1. 1) it follows that $y''(x) < 0$ on $[b, \infty)$, i.e. $y'(x)$ is a strictly decreasing function, therefore $\lim_{x\to\infty} y'(x) = \alpha$ exists and evidently $\alpha \geqq 0$. Thus $y(x)$ is an increasing function on $[b, \infty)$. Integrating the equation (1. 1) from b to x $(x > b)$, we obtain the relation

$$y'(b) - y'(x) = \int_b^x \left(\sum_{i=0}^{n} a_i(t) y^{2i+1}(t) \right) dt \geqq \int_b^x a_l(t) y^{2l+1}(t)\, dt \geqq y^{2l+1}(b) \int_b^x a_l(t)\, dt$$

which shows that $y'(x)$ is unbounded, since $\int_b^\infty a_l(t)dt = \infty$. This contradicts the existence of $\lim_{x\to\infty} y'(x)$, consequently all solutions of (1. 1) are oscillatory.

THEOREM 2. 1. *Suppose that* $\int_0^\infty a_l(x)dx = \infty$ *for at least one* l $(0 \leqq l \leqq n)$.

If there exists a function $q = q(x) \in C^3[x_0, \infty)$ *having the property T such that*

$$a_i'(x) \leqq \alpha \frac{a_i(x)}{q(x)} ([q'(x)]_+ - (i+2)[q'(x)]_-) \qquad (i = 0, 1, 2, \ldots, n;\ x \geqq x_0) \tag{2. 2}$$

furthermore

$$(2.3)_1 \qquad q^{\alpha}(x)\int_{x_0}^{x}[(q^{-\alpha})'''(t)]_-\,dt \leqq 2a_0(x) \quad \text{if} \quad j=0,$$

$$(2.3)_2 \qquad \frac{1}{q^{1-\alpha}(x)}\int_{x_0}^{x}[(q^{-\alpha})'''(t)]_-\,dt < K \quad \text{if} \quad j\neq 0$$

(see j in Definition 1) *on the interval* $[x_0, \infty)$ *for some real numbers* α $(0<\alpha<1)$ *and* K, *then all the solutions of the equation* (1. 1) *are bounded.*

PROOF. Let $y=y(x)$ be a solution of (1. 1) on $[x_0, \infty)$ and introduce the notation

$$(2.4) \qquad D = D(x) = q^{1-\alpha}(x)v(x)-(q^{-\alpha})'(x)y(x)y'(x)+\tfrac{1}{2}(q^{-\alpha})''(x)y^2(x).$$

Using (1. 1), we obtain by derivation the identity

$$(2.5) \qquad D' = \frac{1}{2}(q^{-\alpha})'''y^2+q^{-1-\alpha}\sum_{i=0}^{n}\frac{1}{i+1}(qa_i'-\alpha(i+2)q'a_i)y^{2(i+1)}.$$

Taking into account that $q' = [q']_+ - [q']_-$, from condition (2. 2) obviously

$$(2.6) \qquad q(x)a_i(x) \leqq \alpha(i+2)q'(x)a_i(x) \qquad (i=0, 1, 2, \ldots, n;\ x\geqq x_0).$$

Suppose that $y(x)$ is not bounded. Since the assumption of the Lemma 2. 2 holds, $y(x)$ is oscillatory. Therefore there exists a sequence $\{x_k\}_{k=1}^{\infty}$, such that $x_0 < x_1 < x_2 < x_3 < \ldots$; $y'(x_k)=0$ $(k=1, 2, 3, \ldots)$, $\lim_{k\to\infty} x_k = \infty$,

$$(2.7) \qquad \begin{cases} y(x_k) \geqq y(x) & (x \leqq x_k) \\ \lim_{k\to\infty} y(x_k) = \infty. \end{cases}$$

Integrating (2. 5) from ξ to x_k $(\xi \geqq x_0;\ x_k > \xi)$ and using the inequalities (2. 6) and

$$v(x_k) \geqq \frac{1}{j+1}\frac{a_j(x_k)}{q(x_k)}y^{2(j+1)}(x_k)$$

we get

$$(2.8) \qquad D(\xi) \geqq D(x_k)-\frac{1}{2}\int_{\xi}^{x_k}(q^{-\alpha})'''(t)\,y^2(t)\,dt \geqq$$

$$\geqq q^{1-\alpha}(x_k)v(x_k)+\frac{1}{2}y^2(x_k)(q^{-\alpha})''(\xi)-\frac{1}{2}y^2(x_k)\int_{\xi}^{x_k}(q^{-\alpha})'''(t)[y^2(t)y^{-2}(x_k)-1]\,dt.$$

Hence by (2. 7)

$$(2.9) \qquad D(\xi) \geqq \tfrac{1}{2}y^2(x_k)(q^{-\alpha})''(\xi)+$$

$$+q^{1-\alpha}(x_k)y^2(x_k)\left[\frac{1}{j+1}\frac{a_j(x_k)}{q(x_k)}y^{2j}(x_k)-\frac{1}{2}\frac{1}{q^{1-\alpha}(x_k)}\int_{\xi}^{x_k}[(q^{-\alpha})'''(t)]_-\,dt.\right.$$

Suppose that there exists a ξ ($\xi \geqq x_0$) such that $(q^{-\alpha})''(\xi) > 0$. Then by $(2.3)_1$, $(2.3)_2$, (2.7) and the definition of the property T it is clear that the right member of (2.9) becomes unbounded as $k \to \infty$, yielding a contradiction, whence it follows that $y(x)$ is bounded. If $(q^{-\alpha})''(x) \leqq 0$ on $[x_0, \infty)$ then

$$q''(x) \geqq \frac{(1+\alpha)q'^2(x)}{q(x)} \geqq 0,$$

i.e. the function $q'(x)$ has the same sign for all large values of x. If $q'(x) \leqq 0$, then from (2.2), it follows that $a_i'(x) \leqq 0$ $(i = 0, 1, 2, \ldots, n)$, thus in view of Lemma 2.1 our statement is true. If $q'(x) \geqq 0$, then from (2.2) we get the inequality

$$a_i'(x)q(x) \leqq \alpha a_i(x)q'(x) \leqq a_i(x)q'(x) \qquad (i = 0, 1, 2, \ldots, n;\ x \geqq x_0)$$

i.e.

$$\left(\frac{a_i(x)}{q(x)}\right)' \leqq 0 \qquad (i = 0, 1, 2, \ldots, n;\ x \geqq x_0), \tag{2.10}$$

but then by properties (b) and (d) of the Liapunov function $v(x)$, we obtain that all the solutions of (1.1) are bounded. This completes the proof.

THEOREM 2.2. *If there exists a function $q = q(x) \in C^3[x_0, \infty)$ having the property T, $\lim_{x\to\infty} q(x) = \infty$, and*

$$a_i'(x) \leqq \alpha \frac{a_i(x)}{q(x)}([q'(x)]_+ - (i+2)[q'(x)]_-) \qquad (i = 0, 1, 2, \ldots, n;\ x \geqq x_0) \tag{2.2}$$

furthermore

$$\lim_{x\to\infty} \frac{1}{q^{1-\alpha}(x)} \int_{x_0}^{x} |(q^{-\alpha})'''(t)|\, dt = 0 \tag{2.11}$$

for some number α $(0 < \alpha < 1)$, then for all solutions of the equation (1.1) $\lim_{x\to\infty} y(x) = 0$ is valid.

PROOF. From the condition $\lim_{x\to\infty} q(x) = \infty$ and the definition of the property T it follows that $a_j(x) \geqq \nu > 0$ on some interval $[b, \infty)$, thus the assumptions of the Theorem 2.1 hold, consequently all solutions of the (1.1) are bounded.

Let $y = y(x)$ be a solution of (1.1) on $[x_0, \infty)$ and $y^2(x) < M$. Suppose the contrary i.e. that $y(x)$ does not tend to 0 as $x \to \infty$. Since $y(x)$ is oscillatory (see the proof of the Theorem 2.1), there exists a sequence $\{x_k\}_{k=1}^{\infty}$ such that $x_0 < x_1 < x_2 < x_3 < \ldots$; $\lim_{k\to\infty} x_k = \infty$, $y'(x_k) = 0$ $(k = 1, 2, 3, \ldots)$ and $\lim_{k\to\infty} y^2(x_k) > 0$. Since in the proof of the Theorem 2.1 we have obtained formula (2.8) without the use of the properties (2.7) of the sequence $\{x_k\}_{k=1}^{\infty}$, the formula is true also for this sequence $\{x_k\}_{k=1}^{\infty}$. Similarly as in the proof of the Theorem 2.1, from (2.8) we obtain the estimates:

$$D(\xi) \geqq \tfrac{1}{2} y^2(x_k)(q^{-\alpha})''(\xi) + \tag{2.12}$$

$$+ q^{1-\alpha}(x_k)\left[\frac{1}{j+1}\frac{a_j(x_k)}{q(x_k)} y^{2(j+1)}(x_k) - M\frac{1}{q^{1-\alpha}(x_k)} \int_{\xi}^{x_k} |(q^{-\alpha})'''(t)|\, dt\right].$$

We see from (2. 11) and the definition of the sequence $\{x_k\}_{k=1}^{\infty}$ that the right member of (2. 12) becomes unbounded as $k\to\infty$, which is a contradiction, consequently $\lim_{x\to\infty} y(x)=0$ holds for all solutions of (1. 1).

The last theorem of this § contains a result concerning the derivate of the solution too.

THEOREM 2. 3. *If there exists a nondecreasing function* $q=q(x)\in C^3[x_0,\infty)$ *with property* T_β $(0<\beta\leqq 1)$ *and such that*

$$\lim_{x\to\infty} q(x)=\infty \tag{2. 13}$$

and

$$\lim_{x\to\infty}\frac{1}{q^{1-\alpha}(x)}\int_{x_0}^{x}|(q^{-\alpha})'''(t)|\,dt=0 \tag{2. 14}$$

for some real number α $(0<\alpha<1)$, *then for all solutions* $y(x)$ *of the equation* (1. 1)

$$\lim_{x\to\infty} y(x)=0 \quad \textit{and} \quad y'(x)=o(q^{1/2}(x)) \quad (x\to\infty)$$

are valid.

PROOF. Let $y=y(x)$ be a solution of (1. 1). In view of the properties (c) and (d) of the Liapunov function $v(x)$ the theorem will be proved, if we show that $\lim_{x\to\infty} v(x)=0$ holds. Suppose that $\lim_{x\to\infty} v(x)=s>0$. If ε is an arbitrary positive number, there exists such a $\xi=\xi(\varepsilon)$ that $\xi\geqq x_0$ and

$$s\leqq v(x)\leqq(1+\varepsilon)s \qquad (x\geqq\xi).$$

Combining the condition (2. 13) with the definition of the property T_β, we see that $a_j(x)\geqq\nu>0$ (see j in Definition 1) for some positive number ν and for sufficiently large values of x, thus the assumption of the Lemma 2. 2 holds, i.e. $y(x)$ oscillates. Hence there exists a sequence $\{x_k\}_{k=1}^{\infty}$ such that $x_k>\xi$, $y(x_k)=0$ $(k=1,2,3,\ldots)$ and $\lim_{k\to\infty} x_k=\infty$.

The identity (2. 5) can be written in the form

$$D'=\frac{1}{2}(q^{-\alpha})'''y^2+q^{1-\alpha}\sum_{i=0}^{n}\frac{1}{i+1}\left(\frac{a_i}{q}\right)'y^{2(i+1)}+q^{-\alpha}q'\sum_{i=0}^{n}\frac{1-\alpha(i+2)}{i+1}\frac{a_i}{q}y^{2(i+1)}$$

too. Integrating from ξ to x_k and taking into account the definition of the sequence $\{x_k\}_{k=1}^{\infty}$ and the property T_β of $q(x)$ it follows the inequality

$$q^{1-\alpha}(x_k)v(x_k)\leqq\frac{1}{2}\int_{\xi}^{x_k}|(q^{-\alpha})'''(t)|y^2(t)\,dt+$$

$$+(1-2\alpha)\int_{\xi}^{x_k}q^{-\alpha}(t)q'(t)\left(\sum_{i=0}^{n}\frac{1}{i+1}\frac{a_i(t)}{q(t)}y^{2(i+1)}(t)\right)dt+O(1)\quad(k\to\infty),$$

whence by the reason of the definition of ξ and of the boundedness of $y(x)$ it follows that

$$q^{1-\alpha}(x_k)s \leqq \frac{1}{2}K\int_{\xi}^{x_k}|(q^{-\alpha})'''(t)|\,dt+[1-2\alpha]_+\frac{1}{1-\alpha}q^{1-\alpha}(x_k)(1+\varepsilon)s+O(1) \quad (k\to\infty).$$

Multiplying both sides of this inequality by $q^{-1+\alpha}(x_k)s^{-1}$, by (2. 13) we obtain as $k\to\infty$ the relation

$$1 \leqq \frac{1}{1-\alpha}(1+\varepsilon)[1-2\alpha]_+ .$$

This yields the desired contradiction with any $\varepsilon>0$ if $\alpha\geqq\frac{1}{2}$, and with $\varepsilon<\frac{\alpha}{1-2\alpha}$ if $\alpha<\frac{1}{2}$. Therefore $\lim_{x\to\infty} v(x)=0$, which completes the proof.

REMARK 2. 1. If the equation (1. 1) is linear ($n=0$) and the function $a(x)=a_0(x)$ satisfies the conditions of the Theorem 1 mentioned in the introduction, then under the choices $q(x)=a(x)$ and $\beta=1$ the assumptions of the Theorem 2. 3 are also satisfied. Therefore the Theorem 2. 3 contains and sharpens the result of MEIR, WILLETT and WONG [1].

REMARK 2. 2. If instead of the condition (2. 2) of the Theorems 2. 1 and 2. 2 we require the inequalities

$$(2.2)' \qquad a_i'(x) \leqq \alpha(i+2)\frac{a_i(x)}{q(x)}q'(x) \qquad (i=0,1,2,\ldots,n;\ x\geqq x_0)$$

$$(2.15) \qquad 0<\alpha\leqq\frac{1}{n+2}$$

then the conclusions of the Theorems 2. 1 and 2. 2 remain true. Indeed, in the course of the proofs the relation (2. 2) plays a role only by the deduction of the inequalities (2. 6) and (2. 10), but we may obtain these inequalities obviously also from the relations (2. 2)′ and (2. 15).

These versions of the Theorems 2. 1 and 2. 2 contain the result of WILLETT and WONG 3 (Theorem 2) if $q(x)=a(x)$ and $\alpha=\frac{1}{k+2}$.

§ 3.

THEOREM 3. 1. *Suppose that there exists a positive number v such that*

$$(3.1) \qquad a_0(x)\geqq v$$

on some interval $[\xi,\infty)$, and a function $q=q(x)\in C^2[x_0,\infty)$ having the property T_β ($\beta\leqq 1$), nondecreasing on $[x_0,\infty)$ ($x_0\geqq\xi$) and such that

$$(3.2) \qquad \lim_{x\to\infty} q(x)=\infty$$

furthermore

$$\lambda_q = \lambda = \limsup_{x\to\infty} \frac{1}{q^\alpha(x)} \int_{x_0}^{x} q^{\alpha-1}(t)|q''(t)|\,dt < \frac{2k}{\left(2+\frac{1}{1-\alpha}\right)\alpha} \tag{3.3}$$

for some number α $(0<\alpha<1)$, *where* $0<k\leqq 1$ *and* $a_0(x)\geqq k$ $(x\geqq x_0)$.

Then for all solutions $y(x)$ *of the equations* (1.1)

$$\lim_{x\to\infty} y(x) = 0 \quad \textit{and} \quad y'(x) = o\left(q^{1/2}(x)\right) \quad (x\to\infty) \tag{3.4}$$

are valid.

PROOF. Let $y=y(x)$ be a solution of (1.1). According to the properties of the Liapunov function $v(x)$ it is sufficient to prove, that $\lim_{x\to\infty} v(x)=s=0$.

Suppose that $s>0$. If ε is a positive number, then there exists an $\eta=\eta(\varepsilon)$ such that

$$0 < s \leqq v(x) < s+\varepsilon \qquad (x\geqq\eta). \tag{3.5}$$

Let γ $(0<\gamma<1)$ be an arbitrary number. By the form of the derivative of the function $v(x)$ we get

$$(q^\alpha v)' = (1-\gamma)(q^\alpha)'v+\alpha\gamma q^{\alpha-2}q'\left(y'^2+\sum_{i=0}^{n}\frac{1}{i+1}a_i y^{2(i+1)}\right)- \tag{3.6}$$

$$-q^{\alpha-2}q'y'^2+q^\alpha\sum_{i=0}^{n}\frac{1}{i+1}\left(\frac{a_i}{q}\right)'y^{2(i+1)}.$$

Since the function $q(x)$ is nondecreasing and has the property T_β $(\beta\leqq 1)$ we obtain, using the inequality

$$y'^2+\sum_{i=0}^{n}\frac{1}{i+1}a_i(x)y^{2(i+1)} \leqq -(yy')'+2y'^2$$

from (3.6) the estimate

$$(q^\alpha v)' \leqq (1-\gamma)(q^\alpha)'v-\alpha\gamma q^{\alpha-2}q'(yy')'+[2\gamma\alpha-1]_+ q^{\alpha-1}q'v.$$

Integrating this inequality from η to x $(x>\eta)$ and applying (3.5) it follows

$$sq^\alpha \leqq \left[1-\gamma+\frac{[2\gamma\alpha-1]_+}{\alpha}\right](s+\varepsilon)q^\alpha+\alpha\gamma\left|\int_\eta^x q^{\alpha-2}q'(yy')'\,dt\right|+O(1) \quad (x\to\infty). \tag{3.7}$$

Integrating by parts, we can immediately get the identity

$$\int_\eta^x q^{\alpha-2}q'(yy')'\,dt = \tag{3.8}$$

$$= [q^{\alpha-2}q'yy']_\eta^x - \int_\eta^x q^{\alpha-2}q''yy'\,dt+(2-\alpha)\int_\eta^x q^{\alpha-3}q'^2yy'\,dt.$$

In view of (3. 1) there exists a number k such that $0<k\leqq 1$ and $a_0(x)\geqq k$ on $[\eta, \infty)$, but then

$$|yy'| \leqq \frac{1}{2}\left(y'^2+\frac{a_0}{k}y^2\right) \leqq \frac{1}{2k}vq < \frac{1}{2k}(s+\varepsilon)q. \tag{3. 9}$$

By repeated using of (3. 9) from (3. 8) it can be obtained the estimates

$$\left|\int_\eta^x q^{\alpha-2}q'(yy')'\,dt\right| \leqq \tag{3. 10}$$

$$\leqq \frac{1}{2k}(s+\varepsilon)\left\{\left(1+\frac{2-\alpha}{\alpha-1}\right)q^{\alpha-1}(x)q'(x)+\left(\frac{2-\alpha}{1-\alpha}+1\right)\int_\eta^x q^{\alpha-1}|q''|\,dt\right\}+O(1) \leqq$$

$$\leqq \frac{1}{2k}(s+\varepsilon)\left(2+\frac{1}{1-\alpha}\right)\int_\eta^x q^{\alpha-1}|q''|\,dt+O(1) \quad (x\to\infty).$$

Substitute this in (3. 7) and multiply both sides by $q^{-\alpha}$:

$$s \leqq \left[1-\gamma+\frac{[2\gamma\alpha-1]_+}{\alpha}\right](s+\varepsilon)+ \tag{3. 11}$$

$$+h\gamma(s+\varepsilon)\frac{1}{q^\alpha}\int_\eta^x q^{\alpha-1}|q''|\,dt+\frac{1}{q^\alpha}O(1) \quad (x\to\infty)$$

where $h=\dfrac{\alpha}{2k}\left(2+\dfrac{1}{1-\alpha}\right)$.

According to (3. 3) there exists a sequence $\{x_k\}_{k=1}^\infty$ such that $x_k>\eta$ $(k=1,2,3,\ldots)$, $\lim\limits_{k\to\infty} x_k=\infty$ and

$$\lim_{k\to\infty}\frac{1}{q^\alpha(x_k)}\int_\eta^{x_k} q^{\alpha-1}(t)|q''(t)|\,dt=\lambda.$$

Consider (3. 11) with $x=x_k$ $(k=1, 2, 3, \ldots)$. Letting $k\to\infty$, (3. 2) and (3. 3) imply

$$s \leqq \left[1+\gamma(h\lambda-1)+\frac{[2\gamma\alpha-1]_+}{\alpha}\right](s+\varepsilon). \tag{3. 12}$$

Put

$$0<\gamma<\min\left\{1, \frac{1}{2\alpha}, (1-h\lambda)^{-1}\right\}.$$

Then by (3. 12)

$$0<\frac{s\gamma(1-h\lambda)}{1-\gamma(1-h\lambda)} \leqq \varepsilon$$

which yields a contradiction, i.e. $s=0$ thus the proof is complete.

REMARK. If instead of the condition (3. 3) we require the inequality

$$(3.3)' \qquad \mu_q = \mu = \limsup_{x\to\infty} \frac{1}{\ln q(x)} \int_{x_0}^{x} q^{-1}(t)|q''(t)|\,dt < \frac{2}{3}k$$

then the conclusions of the theorem remain true. This version of our theorem can be proved similarly taking $\ln q$ instead of the function q^{α} in the identity (3. 6).

REFERENCES

[1] MEIR, A., WILLETT, D. and WONG, J. S. W.: A stability condition for $y''+p(x)y = 0$, *Michigan Math. J.* **13** (1966) 169—170.
[2] MEIR, A., WILLETT, D. and WONG, J. S. W.: On the asymptotic behavior of the solutions of $x''+a(t)x = 0$, *Michigan Math. J.* **14** (1967) 47—51.
[3] WILLETT, D. and WONG, J. S. W.: Some Boundedness and Unboundedness Results for the Equation $x''+a(t)x^{2n-1} = 0$, *Monatsh. Math.* **71** (1967) 461—465.
[4] UTZ, W. R.: Properties of solutions of $u''+g(t)u^{2n-1} = 0$, *Monatsh. Math.* **69** (1965), 353—361.
[5] CESARI, L.: *Asymptotic behavior and stability problems in ordinary differential equations*, Springer-Verlag, 1959, 80—86.

József A. University, Institut Bolyai, Szeged

(Received June 27, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 367—370.

FACTORIZATION OF GENERALIZED POISSON DISTRIBUTIONS

by

R. CUPPENS and E. LUKÁCS

1. Introduction. Let $f(t)$ be the characteristic function of an infinitely divisible distribution which has the canonical representation

$$\log f(t) = \int_0^\infty (e^{itu} - 1)\, dN(u) \tag{1.1}$$

where $N(u)$ is a non-decreasing function of bounded variation. The corresponding distribution is called a generalized Poisson distribution and was studied by F. E. SATTERTHWAITE [1] and other authors. A. RÉNYI raised in a conversation the problem of determining the factors of these distributions.

For the formulation of the result it is necessary to introduce the following terminology. Let $\mu(x)$ be a function of bounded variation, the point a is said to be a point of constancy of $\mu(x)$ if $\mu(a+\varepsilon) - \mu(a-\varepsilon) = 0$ for ε sufficiently small. The complement of the set of all points of constancy is called the spectrum of μ and is denoted by $S(\mu)$. Let A be a set of real numbers, we define the set $(n)A$ by induction writing $(1)A = A$ and $(n)A = (n-1)A(+)A$ where $(+)$ indicates the vectorial sum of two sets. We denote by $\infty A = \bigcup_{n=1}^{\infty} (n)A$.

THEOREM. *Let $f(t)$ be a characteristic function which admits the representation* (1.1). *Then every factor $f_1(t)$ of $f(t)$ can be written in the form*

$$\log f_1(t) = ita + \int_0^\infty (e^{itu} - 1)\, dN_1(u) \tag{1.2}$$

where a is real and $N_1(u)$ is a function of bounded variation (not necessarily non-decreasing) which satisfies the condition $S(N_1) \subset \overline{\infty S(N)}$. (The horizontal bar denotes the closure).

It is known that there exist characteristic functions which are not infinitely divisible but admit a representation similar to the canonical representation of infinitely divisible characteristic functions but with a „spectral function" which is not monotone but of bounded variation. Examples of such characteristic functions were given by D. A. RAIKOV [7] and by H. CRAMÉR [8]. CRAMÉR's example has also the form (1.2).

In section 2 we present some properties of boundary characteristic functions, in section 3 we prove the theorem and extend it to the multivariate case in section 4.

2. Boundary characteristic function. A characteristic function $f(t)$ is called a boundary characteristic function if there exists a complex valued function $A(z)$ of the complex variable $z = t+iy$ which is regular in the rectangle $\{|t|<\Delta, 0<y<b\}$ (respectively in $\{|t|<\Delta, -a<y<0\}$) and has the property that $\lim_{y\to 0} A(t+iy) = f(t)$ for $|t|<\Delta$ and $y>0$ (respectively $|t|<\Delta$ and $y<0$). These functions were introduced by J. MARCINKIEWICZ [2] and studied recently by C. G. ESSEEN [3] and B. RAMACHANDRAN [4]. We state now some properties of boundary characteristic functions which we shall use.

LEMMA 1. *A boundary characteristic function $f(t)$ is analytic in a horizontal strip $0<\operatorname{Im} z<b$ and can be represented in this strip by the Fourier-Stieltjes integral $f(z)=\int_{-\infty}^{\infty} e^{izx}dF(x)$. Here $F(x)$ is the distribution function corresponding to $f(t)$.*

LEMMA 2. *Let $F(x)$ be a distribution function and $f(t)$ its characteristic function. $F(x)$ is bounded to the left if, and only if, $f(t)$ is regular in the upper half plane and if*

$$|f(z)| \leqq e^{c|z|}$$

for some $c>0$ and $\operatorname{Im} z>0$.

For the PROOF of the lemmas we refer to the papers quoted above.

3. Proof of the theorem. It follows easily from the assumption (1. 1) that the characteristic function $f(t)$ can be written as

$$f(t) = c\left\{1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_0^{\infty} e^{itu}\,dN^{*n}(u)\right\}$$

where $c = \exp\left\{-\int_0^{\infty} dN(u)\right\}$ and where $N^{*n}(u)$ denotes the n-fold convolution of $N(u)$. The corresponding distribution function $F(x)$ is therefore

$$F(x) = c\left\{\varepsilon(x)+\sum_{n=1}^{\infty}\frac{N^{*n}(x)}{n!}\right\}+k$$

where k is a constant and $\varepsilon(x)$ is the degenerate distribution. The point $x=0$ is a discontinuity point of $F(x)$ and we conclude that

$$S(F) = \overline{\{0\}\cup\infty S(N)} \subset [0,\infty]. \tag{3. 1}$$

Let

$$f(t)=f_1(t)f_2(t) \tag{3. 2}$$

be a decomposition of $f(t)$ and let $F_1(x)$ be the distribution function of $f_1(t)$. One concludes easily from (3. 1) and (3. 2) that $F_1(x)$ is bounded to the left and can assume without loss of generality that lext $[F_1]=0$. The point $x=0$ is then a discontinuity point of $F_1(x)$. We can therefore write

$$F_1(x) = c_1\varepsilon(x)+G(x)$$

where $G(x)$ is a non-decreasing function of bounded variation such that $G(x)$ is continuous at $x=0$ and that

$$S(G) \subset \overline{\infty S(N)}. \tag{3.3}$$

It follows from lemma 2 that $f_1(t)$ is regular in the upper half plane and we see from lemma 1 that

$$f_1(t) = c_1 + \int_0^\infty e^{itx}\, dG(x) \quad (\operatorname{Im} t \geqq 0).$$

It is therefore possible to select $r>0$ such that

$$\int_0^\infty e^{-rx}\, dG(x) < c_1. \tag{3.4}$$

We introduce the function

$$H(x) = \int_0^x e^{-ru}\, dG(u) \tag{3.5}$$

and have then

$$f_1(t+ir) = c_1 + \int_0^\infty e^{itx}\, dH(x) \quad (\operatorname{Im} t \geqq -r).$$

Therefore

$$\log f_1(t+ir) = \log c_1 + \sum_{n=1}^\infty \frac{(-1)^{n-1}}{nc_1^n} \left[\int_0^\infty e^{itx}\, dH(x)\right]^n$$

where (3.4) assures the convergence of the series. Let

$$H_1(x) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{nc_1^n} H^{*n}(x) \tag{3.6}$$

then

$$\log f_1(t+ir) = \log c_1 + \int_0^\infty e^{itx}\, dH_1(x) \quad (\operatorname{Im} t \geqq -r).$$

We write

$$N_1(x) = \int_0^x e^{ru}\, dH_1(u) \tag{3.7}$$

and see that

$$\log f_1(t) = \log c_1 + \int_0^\infty e^{itx}\, dN_1(x) \quad (\operatorname{Im} t \geqq 0).$$

This is the representation (1.2) where c_1 is determined by the condition that $\log f_1(0)=0$. We still have to show that $S(N_1) \subset \overline{\infty S(N)}$. This follows immediately from (3.3), (3.5), (3.6) and (3.7).

4. The multivariate case. We denote by $f(\mathbf{t}) = f(t_1, t_2, \ldots, t_n) = \mathscr{E}\{\exp(i\mathbf{tX})\}$ a multivariate characteristic function. Here $\mathbf{tX} = t_1X_1 + \ldots + t_nX_n$ is the inner (scalar) product of the vectors $\mathbf{t} = (t_1, \ldots, t_n)$ and $\mathbf{X} = (X_1, \ldots, X_n)$. We consider a function

$$f(\mathbf{t}) = \exp\left\{\int_D (e^{i\mathbf{tu}} - 1)\, dN(\mathbf{u})\right\}. \tag{4. 1}$$

Here D is the positive octant, $D = \{\mathbf{x}: x_1 \geqq 0, \ldots, x_n \geqq 0\}$ while $N(\mathbf{u})$ is a positive and finite measure on D. It is well known ([5] 214—221) that the canonical representation of infinitely divisible laws is also valid in the multivariate case, hence (4. 1) is the characteristic function of an infinitely divisible law.

THEOREM. *Let $f(\mathbf{t})$ be a characteristic function which admits the representation* (4. 1). *Then every factor $f_1(\mathbf{t})$ of $f(\mathbf{t})$ can be written in the form*

$$f_1(\mathbf{t}) = \exp\left\{i\mathbf{ta} + \int_D (e^{i\mathbf{ta}} - 1)\, dN_1(\mathbf{u})\right\}$$

where $\mathbf{a}$ *is a real vector and $N_1(\mathbf{u})$ is a finite signed measure on D.*

The proof is completely analogous to the one given in section 3. It is naturally necessary to replace the lemmas of section 2 by the corresponding multivariate results. These can be found in [6], chapter 3.

REFERENCES

[1] SATTERTHWAITE, F. E.: Generalized Poisson distribution, *Ann. Math. Statist.* **13** (1942) 410—417.
[2] MARCINKIEWICZ, J.: Sur les fonctions indépendentes, III, *Fund. Math.* **31** (1938), 86—102.
[3] ESSEEN, C. G.: On infinitely divisible one-sided distributions, *Math. Scand.* **17** (1965), 65—76.
[4] RAMACHANDRAN, B.: One one-sided distribution functions, *Sankhya Ser. A.* **28** (1966) 315—318.
[5] LÉVY, P.: *Théorie de l'addition des variable aléatoires*, Gauthier-Villars, Paris, 1937.
[6] CUPPENS, R.: Décomposition des fonctions caractéristiques des vecteurs aléatoires, *Publ. Inst. Statist. Univ. Paris*, **16** (1968), 61—153.
[7] RAIKOV, D. A.: On the decomposition of Gauss and Poisson laws, *Izv. Akad. Nauk. SSSR Ser. Mat.* **2** (1938) 91—124
[8] CRAMÉR, H.: On the factorization of certain probability distributions, *Ark. Mat.* **1** (1949) 61—65.

The Catholic University of America

(Received July 9, 1968.)
(Revised September 25, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 371–377.

ÜBER „LÜCKENHAFTE" ORTHOGONALPOLYNOMSYSTEME

von

M. SALLAY

1. Es sei $0=\lambda_0<\lambda_1<\ldots<\lambda_n<\ldots$ $(\lambda_i-\lambda_{i-1}>0)$ eine Folge ganzer Zahlen, und betrachten wir in dem Intervall [0, 1] das System der x-Potenzen $\{x^{\lambda_i}\}$ $(i=0, 1, \ldots, n)$. D. NEWMAN hat in seiner Arbeit [6] den folgenden Satz bewiesen:

Es sei $f(x)$ im [0, 1] L_2-integrierbare Funktion; dann existieren die Koeffizienten c_{λ_i} $(i=0, \ldots, n)$, für welche

$$\left\| f(x) - \sum_{i=0}^{n} c_{\lambda_i} x^{\lambda_i} \right\|_{L_2} \leqq 3\omega^{(2)}(f; \varepsilon_\Lambda)$$

besteht, mit

$$\omega^{(2)}(f; \delta) = \sup_{t \leqq \delta} \left[\int_0^1 (f(x+t)-f(x))^2 \, dx \right]^{1/2}$$

und

$$\varepsilon_\Lambda = \prod_{i=1}^{n} \frac{\lambda_i - \frac{1}{2}}{\lambda_i + \frac{3}{2}}.$$

Da in dem Fall $\lambda_i = 2i$ $\varepsilon_\Lambda = \dfrac{3}{4n+3}$ ist, enthielt der NEWMANsche Satz den wohlbekannten Satz von D. JACKSON.

Im weiteren setzen wir voraus, dass $\lambda_n \to \infty$ strebt und die Reihe

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty \tag{1}$$

divergiert. Dann bildet — nach dem Satz von H. MÜNZ [8] und O. SZÁSZ [9] die Folge von x-Potenzen $\{x^{\lambda_i}\}$ ein vollständiges System.

P. LESKY in der Arbeit [5] beschäftigte sich mit der Orthogonalisierung der x-Potenzen mit ganzzahligen Exponenten $\{1, x, x^{\alpha-1}, x^{\alpha+s}, x^{\alpha+s+1}, \ldots\}$, $(s=1,2,\ldots)$, unter denen aufeinander folgende fehlen.

Es sei nun $w(x)$ in dem Intervall [0, 1] gegebene Gewichtsfunktion. Setzen wir voraus, dass die Folge $\{\lambda_i\}$ die Bedingung (1) befriedigt. Von der Idee von P. Lesky ausgehend, orthogonalisieren wir das System $\{x^{\lambda_i}\}$ bezüglich der Gewichtsfunktion $w(x)$. Auf diese Art ergibt sich nach (1) ein vollständiges und normiertes Orthogonalsystem $\{\tilde{p}_{\lambda_i}(x)\}$, wo $\tilde{p}_{\lambda_i}(x)$ ein Polynom von x höchstens λ_i-ten Grades ist, und in dem Polynom treten nur die x-Potenzen mit den Exponenten λ_j $(j=0, 1, \ldots, i)$ auf. Wir werden das System $\{\tilde{p}_{\lambda_i}(x)\}$ „lückenhaftes" Orthogonalpolynomsystem nennen.

Es sei $f(x)\in L_2$, $x\in[0, 1]$ und

$$E_{\lambda_n}^{(2)}(f; w; \tilde{p}_{\lambda_n}(x)) \overset{\text{def}}{=} \operatorname*{Min}_i \left[\int_0^1 \left(f(x) - \sum_{i=0}^{n} a_{\lambda_i}\tilde{p}_{\lambda_i}(x)\right)^2 w(x)\,dx\right]^{1/2}.$$

Aus der Vollständigkeit des Systems $\{\tilde{p}_{\lambda_i}\}$ folgt, dass

$$a_{\lambda_i} = \int_0^1 f(x)\tilde{p}_{\lambda_i}(x)w(x)\,dx$$

ist, woraus die Relation

$$\left(\sum_{i=n+1}^{\infty} |a_{\lambda_i}^2|\right)^{1/2} = E_{\lambda_n}^{(2)}(f; w; \tilde{p}_{\lambda_n}(x))$$

folgt.

In der vorliegenden Arbeit untersuchen wir die folgende Frage.

Man betrachte die Orthogonalpolynomentwicklung der Funktion $f(x)\in L_2$ in dem Intervall [0, 1]. Neben welchen Bedingungen bezüglich $w(x)$ und $f(x)$ wird die Reihe $\sum_{i=0}^{\infty} a_{\lambda_i}\tilde{p}_{\lambda_i}(x)$ absolut konvergent sein?

2. Das Problem der absoluten Konvergenz der Orthogonalentwicklung haben in seinen Arbeiten [7], [4], [1] und [2] B. STECKIN, G. ALEXITS und G. FREUD untersucht. Wir zeigen, dass man die Resultate von G. Freud auch für lückenhaftes Orthogonalpolynomsystem übertragen kann.

SATZ I. *Es befriedigen die Zahlen* λ_i $(i=1, \ldots, n)$ *die Bedingung* (1), *und es sei* $w(x)$ *im* [0, 1] *Gewichtsfunktion, für welche*

$$0 < m \leqq w(x)$$

besteht. Setzen wir ferner voraus, dass für $f\in L_2$

$$\sum_{n=1}^{\infty} \frac{\sqrt{\lambda_n}}{n} E_{\lambda_n}^{(2)}(f; w; \tilde{p}_{\lambda_n}(x)) < \infty \tag{2}$$

gilt. Dann ist die Reihe $\sum_{i=0}^{\infty} a_{\lambda_i}\tilde{p}_{\lambda_i}(x)$ *in jedem inneren Punkt des Intervalls* [0, 1] *absolut konvergent und in jedem inneren Teilintervall von* (0, 1) *gleichmässig absolut konvergent.*

Wenn wir für $w(x)$ auch die Bedingung

$$0 < m \leqq w(x) < M \tag{3}$$

voraussetzen, folgt aus dem Newmanschen Satz die Ungleichung

$$E_{\lambda_n}^{(2)}(f; w; \tilde{p}_{\lambda_n}(x)) \leqq 3M\omega^{(2)}(f; \varepsilon_A). \tag{4}$$

Nach dem Satz I und aus der Ungleichung (4) ergibt sich der folgende Satz:

SATZ II. *Es seien die Bedingungen* (1) *und* (3) *befriedigt, ferner sei*

$$\sum_{n=1}^{\infty} \frac{\sqrt{\lambda_n}}{n} \omega^{(2)}(f; \varepsilon_{\Lambda_n}) < \infty.$$

Dann ist die Reihe $\sum_{i=0}^{\infty} a_{\lambda_i} \tilde{p}_{\lambda_i}(x)$ in jedem inneren Punkt des Intervalls [0, 1] absolut konvergent und in jedem inneren Teilintervall von (0, 1) gleichmässig absolut konvergent.

Vor dem Beweis des Satzes I zeigen wir folgendes:

LEMMA 1. Setzen wir voraus, dass $w(x)$ (1) befriedigt, dann ist in jedem inneren Punkt von [0, 1]

$$\sum_{i=0}^{n} [\tilde{p}_{\lambda_i}(x)]^2 \leqq O(\lambda_n), \tag{5}$$

und diese Abschätzung ist in jedem inneren Teilintervall gleichmässig.

Es sei $\{p_m(x)\}$ ein beliebiges Orthogonalpolynomsystem bezüglich $w(x)$. $\Pi_m(x)$ durchlaufe alle Polynome höchstens m-ten Grades, für welche

$$\int_0^1 \Pi_m^2(x) w(x)\, dx \leqq 1$$

ist. Wir suchen das Maximum von $\Pi_m^2(x)$. Aus der Orthogonaleigenschaft des Systems folgt, dass das Maximum von $\Pi_m^2(x)$ gleich $\sum_{k=0}^{m} p_m^2(x)$ ist. (S. z. B. [2]). Es bezeichne $\tilde{\Pi}_{\lambda_n}(x)$ die Polynome höchstens λ_n-ten Grades, für welche

$$\int_0^1 \tilde{\Pi}_{\lambda_n}^2(x) w(x)\, dx \leqq 1$$

ist und in welchem die x Potenzen nur mit den Exponenten λ_i $(i = 0, 1, \ldots, n)$ auftreten.

Da

$$\sum_{i=0}^{n} \tilde{p}_{\lambda_i}^2(x) = \operatorname*{Max}_{\tilde{\Pi}_{\lambda_n}} |\tilde{\Pi}_{\lambda_n}^2(x)| \leqq \operatorname*{Max}_{\tilde{\Pi}_{\lambda_n}} |\Pi_{\lambda_n}^2(x)| \leqq \sum_{i=0}^{\lambda_n} p_i^2(x)$$

ist, und nach (1)

$$\sum_{i=0}^{\lambda_n} p_i^2(x) = O(\lambda_n)$$

besteht (s. [3]), ergibt sich die Behauptung.

BEWEIS DES SATZES I. Es sei

$$s_\nu(x) = \sum_{i=2^\nu+1}^{2^{\nu+1}} |a_{\lambda_i} \tilde{p}_{\lambda_i}(x)|.$$

Da $E_{\lambda_n}^{(2)}(f;\ w;\ \tilde{p}_{\lambda_n}(x))$ eine monoton abnehmende Funktion von n ist, ist aus (5)

$$s_\nu(x) \leqq \left[\sum_{2^\nu+1}^{2^{\nu+1}} a_{\lambda_i}^2 \sum_{2^\nu+1}^{2^{\nu+1}} \tilde{p}_{\lambda_i}^2(x)\right]^{1/2} \leqq E_{\lambda_{2^\nu}}^{(2)} \cdot \left[\sum_{i=1}^{2^{\nu+1}} \tilde{p}_{\lambda_i}^2(x)\right]^{1/2} =$$

$$= E_{\lambda_{2^\nu}}^{(2)} \cdot O(\sqrt{\lambda_{2^{\nu+1}}}) = O(\sqrt{\lambda_{2^{\nu+1}}})\, 2^{-(\nu+1)} \sum_{m=2^{\nu-1}+1}^{2^\nu} E_{\lambda_m}^{(2)} = O\left\{\sum_{m=2^{\nu-1}+1}^{2^\nu} \frac{\sqrt{\lambda_m}}{m} E_{\lambda_m}^{(2)}\right\}.$$

Man addiere diese Ungleichung $m=1$ bis $m=\infty$, so ergibt sich

$$\sum_{i=0}^{\infty} |a_{\lambda_i} \tilde{p}_{\lambda_i}(x)| \leqq \sum_{n=1}^{\infty} \frac{\sqrt{\lambda_n}}{n} E_{\lambda_n}^{(2)}(f; w; \tilde{p}_{\lambda_n}),$$

w. z. b. w.

3. Im weiteren setzen wir voraus, dass die Funktion $f(x)$ im $[0, 1]$ r-mal stetig differenzierbar ist.

Wir können die folgende Erweiterung des NEWMANschen Satzes beweisen:

SATZ III. *Es sei* $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_n < \ldots (\lambda_i - \lambda_{i-1} > 0)$, *($\lambda_i$ ganz) und* $\{x^{\lambda_i}\}$ *System der x-Potenzen mit den Exponenten* λ_i $(i=0, 1, \ldots, n)$. *Existieren die Koeffizienten* $\tilde{c}_{\lambda_i}$ $(i=0, 1, \ldots, n)$ *für welche*

$$\left\| f(x) - \sum_{i=0}^{n} \tilde{c}_{\lambda_i} x^{\lambda_i} \right\|_{L_2} \leqq 3^{r+1} \left(\prod_{k=0}^{r-1} \varepsilon_{\Lambda(k)}\right) \omega^{(2)}(f^{(r)}; \varepsilon_{\Lambda(r)})$$

besteht mit

$$\varepsilon_{\Lambda(0)} = \varepsilon_\Lambda, \quad \varepsilon_{\Lambda(k)} = \prod_{\lambda_i > k_i \leqq n} \frac{(\lambda_i - k) - \frac{1}{2}}{(\lambda_i - k) + \frac{3}{2}}.$$

Wir beweisen den Satz III durch vollständige Induktion.

Es sei $g(x)$ im $[0, 1]$ differenzierbare Funktion. Bezeichnen wir den Stetigkeitsmodul von $f(x)$ in dem Raum L_2 mit $\omega^{(2)}(f; \delta)$. Es ist leicht zu zeigen (S. z. B. [6], Lemma 1), dass die Ungleichung

$$\omega^{(2)}(g; \delta) \leqq \delta \|g'(x)\|_{L_2} \tag{6}$$

besteht.

Wir beweisen den Satz erstens für $r=1$. Definieren wir die Folge der ganzen Zahlen $\{\mu_i\}$ folgenderweise:

$$\mu_0 = 0, \quad \mu_1 = \lambda_1 - 1, \quad \mu_2 = \lambda_2 - 1, \ldots, \mu_n = \lambda_n - 1 \qquad (\text{für } \lambda_1 \neq 1)$$

$$\mu_0 = 0, \quad \mu_1 = \lambda_2 - 1, \quad \mu_2 = \lambda_3 - 1, \ldots, \mu_{n-1} = \lambda_n - 1 \quad (\text{für } \lambda_1 = 1),$$

und betrachten wir das System der x-Potenzen $\{x^{\mu_i}\}$. Da $f'(x) \in L_2$, aus dem NEWMANschen Satz folgt, dass die Koeffizienten α_{μ_i} existieren, für welche

$$\left\| f'(x) - \sum_{i=0}^{n} \alpha_{\mu_i} x^{\mu_i} \right\|_{L_2} \leqq 3\omega^{(2)}(f'; \varepsilon_{\Lambda(1)}) \tag{7}$$

ist mit

$$\varepsilon_{\Lambda^{(1)}} = \begin{cases} \prod_{i=1}^{n} \frac{\mu_i - \frac{1}{2}}{\mu_i + \frac{3}{2}} = \prod_{i=1}^{n} \frac{(\lambda_i - 1) - \frac{1}{2}}{(\lambda_i - 1) + \frac{3}{2}} & (\text{für } \lambda_1 \neq 1) \\ \prod_{i=1}^{n-1} \frac{\mu_i - \frac{1}{2}}{\mu_i + \frac{3}{2}} = \prod_{i=2}^{n} \frac{(\lambda_i - 1) - \frac{1}{2}}{(\lambda_i - 1) + \frac{3}{2}} & (\text{für } \lambda_1 = 1) \end{cases} = \prod_{\lambda_i > 1,\, i \leqq n} \frac{(\lambda_i - 1) - \frac{1}{2}}{(\lambda_i - 1) + \frac{3}{2}}.$$

Es sei die Funktion

$$\Phi(x) = f(x) - \sum_{i=0}^{n} \frac{\alpha_{\mu_i}}{\mu_i + 1} x^{\mu_i + 1}.$$

Aus (7) folgt, dass

$$\|\Phi'(x)\|_{L_2} = \left\| f'(x) - \sum_{i=0}^{n} \alpha_{\mu_i} x^{\mu_i} \right\|_{L_2} \leqq 3\omega^{(2)}(f'; \varepsilon_{\Lambda^{(1)}})$$

ist. Wieder aus dem „Newmanschen Satz" folgt, dass die Koeffizienten β_{λ_i} mit

$$\left\| \Phi(x) - \sum_{i=0}^{n} \beta_{\lambda_i} x^{\lambda_i} \right\|_{L_2} \leqq 3\omega^{(2)}(\Phi; \varepsilon_{\Lambda^{(0)}})$$

existieren.

Da die Funktion $\Phi(x)$ differenzierbar ist, folgt aus (6) die Ungleichung

$$\left\| \Phi(x) - \sum_{i=0}^{n} \beta_{\lambda_i} x^{\lambda_i} \right\|_{L_2} \leqq 3^2 \varepsilon_{\Lambda^{(0)}} \omega^{(2)}(f'; \varepsilon_{\Lambda^{(1)}}).$$

Da das zweite Glied von $\Phi(x)$ Polynom der x-Potenzen mit den Exponenten $\mu_i + 1 = \lambda_i$ ist, ergibt sich mit der Wahl der Koeffizienten

$$\tilde{c}_0 = \beta_0, \quad \tilde{c}_{\lambda_i} = \beta_{\lambda_i} + \frac{\alpha_{\lambda_i}}{\lambda_i}$$

der Satz III.

Es sei nun $f(x)$ im $[0, 1]$ r-mal stetig differenzierbar. Setzen wir voraus, dass die Behauptung für $r-1$ besteht.

Es seien die ganzen Zahlen ν_i gleich

$$\nu_0 = 0, \quad \nu_1 = \lambda_j - r \qquad (j = \min i \;\; \lambda_i > r)$$

$$\nu_i = \lambda_{j+i} \lambda_{j+i} - r \qquad (i = 1, 2, \ldots, m = n - j + 1).$$

Betrachten wir das System $\{x^{\nu_i}\}$. Bezeichnen wir mit γ_{ν_i} die Koeffizienten, für welche

$$\left\| f^{(r)} - \sum_{i=0}^{m} \gamma_{\nu_i} x^{\nu_i} \right\|_{L_2} \leqq 3\omega^{(2)}(f^{(r)}; \varepsilon_{\Lambda^{(r)}}) \tag{8}$$

besteht mit

$$\varepsilon_{\Lambda^{(r)}} = \prod_{i=1}^{m} \frac{\nu_i - \frac{1}{2}}{\nu_i + \frac{3}{2}} = \prod_{\lambda_i > r,\, i \leqq n} \frac{(\lambda_i - r) - \frac{1}{2}}{(\lambda_i - r) + \frac{3}{2}}.$$

Es sei

$$\Phi_r(x) = f(x) - \sum_{i=0}^{n-j+1} \frac{\gamma_{\nu_i}}{(\nu_i + 1) \ldots (\nu_i + r)} x_i^{\nu_i + r}. \tag{9}$$

Da $\Phi(x)$ r-mal stetig differenzierbar ist und wegen (8) ist

$$\|\Phi_r^{(r)}(x)\|_{L_2} \leqq 3\omega^{(2)}(f^{(r)}; \varepsilon_{\Lambda^{(r)}}). \tag{10}$$

Nach der Induktionsbedingung existieren doch die Koeffizienten δ_{λ_i} mit

$$\left\| \Phi_r(x) - \sum_{i=0}^{n} \delta_{\lambda_i} x^{\lambda_i} \right\|_{L_2} \leqq 3^r \left(\prod_{i=0}^{r-2} \varepsilon_{\Lambda^{(i)}} \right) \omega^{(2)}(\Phi^{(r-1)}; \varepsilon_{\Lambda^{(r-1)}}) \leqq 3^r \left(\prod_{i=0}^{r-1} \varepsilon_{\Lambda^{(i)}} \right) \|\Phi^{(r)}\|_{L_2},$$

woraus wegen (9) und (10) die Behauptung folgt.

Aus dem Satz III folgt, dass für die r-mal stetig differenzierbaren Funktionen $f(x)$

$$E_{\lambda_n}^{(2)}(f; w; \tilde{p}_{\lambda_i}(x)) \leqq 3^{r+1} \left(\prod_{k=0}^{r-1} \varepsilon_{\Lambda^{(k)}} \right) \omega^{(2)}(f^{(r)}; \varepsilon_{\Lambda^{(r)}})$$

besteht.

4. Betrachten wir nun die Reihe

$$\sum_{i=0}^{\infty} a_{\lambda_i} \tilde{p}_{\lambda_i}^{(r)}(x),$$

die durch gliedweise Differenzieren aus der Reihe $\sum_{i=0}^{\infty} a_{\lambda_i} \tilde{p}_{\lambda_i}(x)$ entsteht.

G. FREUD in seiner Arbeit [3] versuchte das Problem der gliedweisen Differenzierung einer orthogonalen Polynomreihe. Wir können sein Resultat auch für lückenhaftes Orthogonalpolynomsystem übertragen.

SATZ IV. *Es sei* $w(x)$ *Gewichtsfunktion in dem Intervall* [0, 1] *mit*

$$0 < m \leqq w(x) < M,$$

und es sei $f(x)$ *im* [0, 1] r*-mal stetig differenzierbare Funktion, für welche die Reihe*

$$\sum_{n=0}^{\infty} \left(\prod_{k=1}^{r} \varepsilon_{\Lambda^{(k)}} \right) \frac{\lambda_n^{r+\frac{1}{2}}}{n} \omega^{(2)}\left(f^{(r)}; \frac{1}{n} \right) < \infty$$

konvergiert. Dann ist die Reihe $\sum_{i=0}^{\infty} a_{\lambda_i} \tilde{p}_{\lambda_i}^{(r)}(x)$ *in jedem inneren Teilintervall von* (0, 1) *absolut konvergent und stellt da die Funktion* $f^{(r)}(x)$ *dar.*

Der Beweisgang geht dem Beweis des Satzes I von G. FREUD in [3] und dem Beweis des Satzes I ähnlicherweise.*

* Aus dem Hilfssatz I von [3] ist es leicht zu zeigen, dass

$$\sum_{i=0}^{n} [\tilde{p}_{\lambda_i}^{(r)}(x)]^2 \leqq O(\lambda_n^{2r+1})$$

ist.

LITERATURVERZEICHNIS

[1] FREUD, G.: Über die absolute Konvergenz von orthogonalen Polynomreihen, *Acta Math. Acad. Sci. Hungar.* (1953) 127—155.
[2] FREUD, G.: Über die starke (C, 1) Summierbarkeit von orthogonalen Polynomreihen, *Acta Math. Acad. Sci. Hungar.* **3** (1952) 83—88.
[3] FREUD, G.: Über das gliedweise Differenzieren einer orthogonalen Polynomreihe, *Acta Math. Acad. Sci. Hungar.* **4** (1955) 221—226.
[4] ALEXITS, G.: Sur la convergence et la sommabilité presque partout des séries de polynomes orthogonaux, *Acta Sci. Math. (Szeged)* **12** (1950) 223—250.
[5] LESKY, P.: Die Orthogonalisierung der x-Potenzen. *Sitzungsber. der Österr. Akad. der Wiss. Abt.* II. 170, Bd. 5 bis 7 *Heft.* 1962, *Wien*, 1962.
[6] NEWMAN, D.: On the Münz-Jackson theorem, *Amer. J. Math.* **87** (1965) 940—944.
[7] СТЕЧКИН, Ш. Б.: Об абсолутной сходимости ортогональных рядов. *Мат. Сборник* **21** (71) (1951) 225—232.
[8] MÜNZ, Ch.: Über den Approximationssatz von Weierstrass, *Schwarz Festschrift*, 1914.
[9] SZÁSZ, O.: Über die Approximation stetiger Funktionen durch lineare Aggregate von Potenzen. *Math. Ann.* **77** (1916) 482—496.

Mathematisches Institut der Ungarischen Akademie der Wissenschaften, Budapest

(Eingegangen: 20. Juli, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 379–384.

EIN BEITRAG ZUR THEORIE DES LAGRANGESCHEN INTERPOLATIONSVERFAHRENS

von

G. FREUD

Einleitung

Der erste Paragraph vorliegender Arbeit enthält einen Satz über die Konvergenz bzw. Konvergenzgeschwindigkeit des Lagrangeschen Interpolationsverfahrens in einem Endpunkte der Interpolationsinvervalles $[-1, +1]$, falls im n-ten Schritte als Knotenpunkte der Interpolation die Nullstellen des Orthogonalpolynoms $p_n(d\alpha; x)$ bezüglich einer recht allgemeinen Belegung $d\alpha$ gewählt werden. Der Träger dieser Belegung soll in $[-1, +1]$ liegen, so dass die Knotenpunkte in $[-1, +1]$ fallen (vgl. G. FREUD [1], Satz I. 2. 5.) Für die Nullstellen der Tschebischeffschen Polynome oder der LEGENDREschen Polynome als Knotenpunkte sind diese Abschätzungen der Grössenordnung nach bestmöglich.

In den zweiten Paragraphen wird der erhaltene Satz auf die gleichmässige Abschätzung des Fehlers des Interpolationsverfahrens in $[-1, +1]$ angewandt. Das Ergebnis ergibt z. B. für die Nullstellen der LEGENDREschen Polynome als Knotenpunkte die richtige Grössenordnung. Es sei hier in einigen Zeilen ein allgemeines Prinzip erörtert, welches sowohl hier, wie in meiner (auch hier benützten) früheren Arbeit G. FREUD [2] angewandt wurde.

LEMMA. *Für ein Polynom $\omega(x)$ mit lauter verschiedenen Nullstellen bezeichnen wir mit $L(\omega; f; x)$ das Lagrangesche Interpolationspolynom in der Veränderlichen x, welches die Funktion $f(x)$ in den Nullstellen von $\omega(x)$ interpoliert.**

Sind nun $\omega_1(x)$ und $\omega_2(x)$ zwei Polynome mit lauter verschiedenen Nullstellen und ohne gemeinsamen Nullstellen, dann gilt

$$(*) \qquad L(\omega_1\omega_2; f; x) = L(\omega_1; f; x) + \omega_1(x) L\{\omega_2; [f - L(\omega_1; f)]\omega_1^{-1}; x\}.$$

Der Beweis dieses Lemmas erfolgt unmittelbar, und sei dem Leser überlassen.

In meiner Arbeit G. FREUD [2] habe ich $(*)$ mit $\omega_1(x) = 1 - x^2$, $\omega_2(x) = p_n(d\alpha; x)$ angewandt. In § 2 vorliegender Arbeit kommt $(*)$ mit $\omega_1(x) = p_n(d\alpha; x)$ und $\omega_2(x) = 1 - x^2$ zur Anwendung.

§ 1. Über das Verhalten Lagrangescher Interpolationspolynome in einem Endpunkt des Interpolationsintervalls

Es werden die gleichen Bezeichnungen verwendet, wie in meinem Buche [1].

SATZ 1. *Es sei* $Tr(d\alpha) \subseteq [-1, +1]$,

$$(1) \qquad \int_{-1}^{+1} (1-x)^{\varrho-1}\, d\alpha(x) = I_\varrho(d\alpha) < \infty$$

* Der Grad von $L(\omega; f; x)$ ist um Eins kleiner, als der Grad von $\omega(x)$.

und $f \in \mathrm{Lip}\, \varrho$ $(0 < \varrho \leqq 1)$, *dann gilt mit einer absoluten Konstanten* $A > 0$

$$|f(1) - L_n(d\alpha; f; 1)| \leqq A[I_\varrho(d\alpha) p_{n-1}(d\alpha; 1) p_n(d\alpha; 1)]^{1/2} n^{-\varrho}. \tag{2}$$

Gilt sogar gleichmässig bezüglich $x, x+h \in [-1, +1]$

$$f(x+h) - f(x) = o(|h|^\varrho) \tag{3}$$

für $h \to 0$, *dann ist*

$$f(1) - L_n(d\alpha; f; 1) = [p_{n-1}(d\alpha; 1) p_n(d\alpha; 1)]^{1/2} o(n^{-\varrho}). \tag{4}$$

Für die Interpolation über die Nullstellen der Tschebyscheffschen Polynome ergibt sich an der rechten Seite von (2) bzw. (4) $O(n^{-\varrho})$, bzw. $o(n^{-\varrho})$. Das wurde schon früher als Ergebniss einer vom Verfasser vorgeschlagenen Untersuchung von O. Kis [5] gezeigt. Satz 1 zeigt, dass das immer der Fall ist, wenn die Folge $\{p_n(d\alpha; 1)\}$ beschränkt ist.

Es gelten aus Symmetriegründen analoge Abschätzungen für die Stelle $x = -1$:

Satz 1*. *Es sei* $Tr(d\alpha) \subseteq [-1, +1]$,

$$\int_{-1}^{+1} (1+x)^{-1+\varrho}\, d\alpha(x) = I_\varrho^*(d\alpha) < \infty \tag{1*}$$

und $f \in \mathrm{Lip}\, \varrho$ $(0 < \varrho \leqq 1)$, *dann gilt*

$$|f(-1) - L_n(d\alpha; f; -1)| \leqq A\left[I_\varrho^*(d\alpha) |p_{n-1}(d\alpha; -1) p_n(d\alpha; -1)|\right]^{1/2} n^{-\varrho}, \tag{2*}$$

und aus der Bedingung (3) *folgt*

$$f(-1) - L_n(d\alpha; f; -1) = |p_{n-1}(d\alpha; -1) p_n(d\alpha; -1)|^{1/2} o(n^{-\varrho}). \tag{4*}$$

Beweis des Satzes 1. Aus $f \in \mathrm{Lip}\, \varrho$ bzw. (3) folgt die Existenz einer Folge $\{\psi_n(x)\}$, so dass ψ_n eine Polynom höchstens $n-1$-ten Grades ist und

$$|f(x) - \psi_n(x)| \leqq \delta_n (1-x^2)^{\varrho/2} n^{-\varrho} \tag{5}$$

gültig ist. Dabei ist δ_n eine beschränkte Zahlenfolge, bzw. — falls (3) erfüllt ist — eine Nullfolge. (Vgl. I. G. Gopengaus [4], S. A. Teliakowski [6]). Aus (5) folgt $\psi_n(1) = f(1)$, so dass

$$|f(1) - L_n(d\alpha; f; 1)| = |\psi_n(1) - L_n(d\alpha; f; 1)| =$$

$$= |L_n(d\alpha; \psi_n - f; 1)| \leqq \sum_{k=1}^{n} |l_n(d\alpha; x_{kn})|\, |\psi_n(x_{kn}) - f(x_{kn})|$$

ist. Aus (5) erhalten wir weiter

$$|f(1) - L_n(d\alpha; f; 1)| \leqq \delta_n n^{-\varrho} \sum_{k=1}^{n} |l_n(d\alpha; 1; x_{kn})| (1-x_{kn}^2)^{\varrho/2} \leqq \tag{6}$$

$$\leqq \delta_n n^{-\varrho} \left\{ \sum_{k=1}^{n} |l_n(d\alpha; 1; x_{kn}) p_{n-1}(d\alpha; x_{kn})| \right\}^{1/2} \cdot$$

$$\cdot \left\{ \sum_{k=1}^{n} |l_n(d\alpha; 1; x_{kn}) p_{n-1}^{-1}(d\alpha; x_{kn})| (1-x_{kn}^2)^{\varrho} \right\}^{1/2}.$$

Nach Formel III. (6. 3) aus [1] gilt

$$l_n(d\alpha; 1; x_{kn}) = \lambda_n(d\alpha; x_{kn}) \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \frac{p_{n-1}(d\alpha; x_{kn})}{1 - x_{kn}} p_n(d\alpha; 1). \tag{7}$$

Es ist $p_n(d\alpha; 1) > 0$, da der Leitkoeffizient $\gamma_n(d\alpha)$ von $p_n(d\alpha; x)$ positiv ist und alle Nullstellen von $p_n(d\alpha; x)$ in $(-1, +1)$ liegen.

Aus (7) folgt

$$\operatorname{sign} l_n(d\alpha; 1; x_{kn}) = \operatorname{sign} p_{n-1}(d\alpha; x_{kn}) \tag{8}$$

und

$$\sum_{k=1}^{n} |l_n(d\alpha; 1; x_{kn}) p_{n-1}(d\alpha; x_{kn})| = \sum_{k=1}^{n} l_n(d\alpha; 1; x_{kn}) p_{n-1}(d\alpha; x_{kn}) = \tag{9}$$

$$= p_{n-1}(d\alpha; 1).$$

Andererseits ist wegen (7)

$$\sum_{k=1}^{n} |l_n(d\alpha; 1; x_{kn}) p_{n-1}^{-1}(d\alpha; x_{kn})| (1 - x_{kn}^2)^{\varrho} \leqq$$

$$\leqq 2 \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} p_n(d\alpha; 1) \sum_{k=1}^{n} \lambda_n(d\alpha; x_{kn})(1 - x_{kn})^{\varrho - 1},$$

so dass infolge Hilfssatz III. 1. 5 und I. 7. 2. von [1]

$$\sum_{k=1}^{n} |l_n(d\alpha; 1; x_{kn}) p_{n-1}^{-1}(d\alpha; x_{kn})| (1 - x_{kn}^2)^{\varrho} \leqq 4 p_n(d\alpha; 1) I_{\varrho}(d\alpha) \tag{10}$$

gültig ist. Aus (6), (9) und (10) ergibt sich

$$|f(1) - L_n(d\alpha; f; 1)| \leqq 2[I_{\varrho}(d\alpha) p_{n-1}(d\alpha; 1) p_n(d\alpha; 1)]^{1/2} \delta_n n^{-\varrho} \tag{11}$$

bzw.

$$|f(-1) - L_n(d\alpha; f; -1)| \leqq 2\left[I_{\varrho}^*(d\alpha) |p_{n-1}(d\alpha; -1) p_n(d\alpha; -1)|\right]^{1/2} \delta_n n^{-\varrho} \tag{11*}$$

w. z. b. w.

§ 2. Über die gleichmässige Konvergenz des Interpolationsverfahrens

Es bezeichne wie in [2] $L_n^*(d\alpha; f; x)$ das Lagrangesche Interpolationspolynom höchstens $n+1$-ten Grades, welches an den Stellen -1, $+1$ und x_{kn} $(k = 1, 2, \ldots, n)$ die gleichen Werte wie $f(x)$ annimmt.

SATZ 2. *Es sei* $m > 0$, *ferner sei für fast alle* $x \in [-1, +1]$

$$\alpha'(x) \leqq m(1 - x^2)^{1/2} \tag{12}$$

und

$$J_{\varrho}(d\alpha) = \int_{-1}^{+1} (1 - x^2)^{\varrho - 1} \, d\alpha(x) < \infty; \tag{13}$$

dann folgt aus $f \in \mathrm{Lip}\, \varrho$ *bzw. aus* (3)

$$|f(x)-L_n(d\alpha;f;x)| \leqq 4[m^{-1}J_\varrho(d\alpha)]^{1/2}\delta_n n^{1/2-\varrho} \tag{14}$$

wo δ_n *gemäss* (5) *eine beschränkte Folge bzw. — unter der Voraussetzung* (3) *— eine Nullfolge ist.*

Der Beweis dieses Satzes ergibt sich genau so, wie der Beweis des Hauptsatzes der Arbeit [2], welcher der Spezialfall $\varrho = 1/2$ dieses Satzes ist. Die Einzelheiten seien dem Leser überlassen.

HILFSSATZ 3. Es gilt die Identität

$$L_n^*(d\alpha;f;x) = L_n(d\alpha;f;x)+\frac{p_n(d\alpha;x)}{p_n(d\alpha;1)}\frac{1+x}{2}[f(1)-L_n(d\alpha;f;1)]+ \tag{15}$$

$$+\frac{p_n(d\alpha;x)}{p_n(d\alpha;-1)}\frac{1-x}{2}[f(-1)-L_n(d\alpha;f;-1)].$$

BEWEIS. Es ist klar, dass die rechte an den Stellen $+1$, -1 und x_{kn} $(k=1,2,\dots n)$ den gleichen Wert wie $f(x)$ annimmt. (Spezialfall des Lemmas aus der Einleitung.)

HILFSSATZ 4. *Aus* $Tr(d\alpha) \subseteq [-1, +1]$ *folgt*

$$\frac{p_{n-1}(d\alpha;1)}{p_n(d\alpha;1)} \leqq 2\frac{\gamma_n(d\alpha)}{\gamma_{n-1}(d\alpha)}. \tag{16}$$

BEWEIS. Wir setzen $x=1$ in die Rekursionsformel (vgl. I. (2. 4) in [1])

$$xp_n(d\alpha;x) = \frac{\gamma_n(d\alpha)}{\gamma_{n+1}(d\alpha)}p_{n+1}(d\alpha;x)+a_np_n(d\alpha;x)+\frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}p_{n-1}(d\alpha;x)$$

und beachten, dass für jedes ν $p_\nu(d\alpha;1)>0$ und $\gamma_\nu(d\alpha)>0$ ist, ferner dass

$$|a_n| = \left|\int_{-1}^{+1} xp_n^2(d\alpha;x)\,d\alpha(x)\right| \leqq \int_{-1}^{+1} p_n^2(d\alpha;x)\,d\alpha(x) = 1$$

gilt. Wir erhalten

$$\frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\frac{p_{n-1}(d\alpha;1)}{p_n(d\alpha;1)} \leqq 1-a_n \leqq 2$$

w. z. b. w.

SATZ 5. *Unter den Voraussetzungen des Satzes* 2 *gilt*

$$|f(x)-L_n(d\alpha;f;x)| \leqq 4[m^{-1}J_\varrho(d\alpha)]^{1/2}\delta_n n^{1/2-\varrho}+ \tag{17}$$

$$+K(d\alpha)|p_n(d\alpha;x)|\,\delta_n n^{-\varrho} \qquad (-1 \leqq x \leqq +1)$$

wo $K(d\alpha)$ *nur von* $d\alpha$ *abhängt.*

BEWEIS. Es folgt aus (11), (11*), Satz 2 und Hilfssatz 3 unter Beachtung von $I_\varrho(d\alpha) \leqq 2J_\varrho(d\alpha)$ und $I_\varrho^*(d\alpha) \leqq 2J_\varrho(d\alpha)$

$$|f(x) - L_n(d\alpha; f; x)| \leqq$$

$$(18) \quad \leqq \left\{4[m^{-1}J_\varrho(d\alpha)]^{1/2} + \left[2\frac{p_{n-1}(d\alpha;1)}{p_n(d\alpha;1)}J_\varrho(d\alpha)\right]^{1/2}\frac{(1+x)}{2}|p_n(d\alpha;x)|n^{-1/2} + \right.$$

$$\left. + \left[2\left|\frac{p_{n-1}(d\alpha;-1)}{p_n(d\alpha;-1)}\right|J_\varrho(d\alpha)\right]^{1/2}\frac{(1-x)}{2}|p_n(d\alpha;x)|n^{-1/2}\right\}\delta_n n^{1/2-\varrho}.$$

Infolge (12) ist $(1-x^2)^{-1/2}\log\alpha'(x) \in L$, so dass nach bekannten Satze (vgl. [1], Tabelle V. A)

$$(19) \qquad \lim_{n\to\infty}\frac{\gamma_n(d\alpha)}{\gamma_{n-1}(d\alpha)} = 2$$

ist. Es ergibt sich (17) aus (18) und (19), w. z. b. w.

Zum Schluss wenden wir Satz 5 auf die Interpolation über die Nullstellen der Jacobischen Polynome $P_n^{(\alpha,\beta)}(x)$ an. Nach G. SZEGŐ [3] Satz 7. 32. 2 und Formeln 4. 34, 4. 33 gilt für die entsprechenden normierten Orthogonalpolynome mit der Gewichtsfunktion $\alpha'(x) = (1-x)^\alpha(1+x)^\beta$

$$(20) \qquad |p_n^{(\alpha,\beta)}(x)| \leqq \begin{cases} c\min\{(1-x)^{-\alpha/2-1/4},\ n^{1/2+\alpha}\} & \text{für} \quad x\in[0,1] \\ c\min\{(1+x)^{-\beta/2-1/4},\ n^{1/2+\beta}\} & \text{für} \quad x\in[-1,0]. \end{cases}$$

Die Abschätzungen, welche sich aus (17) und (20) ergeben, haben wir in nachstehender Tafel zusammengestellt.

$$|f(x) - L_n(d\alpha; f; x)| \leqq \delta_n\varepsilon_n(x)$$

$-\varrho < \alpha \leqq 0$	$\varepsilon_n(x) = O(n^{1/2-\varrho})$	$(0 \leqq x \leqq 1)$
$\alpha > 0$	$\varepsilon_n(x) = O(n^{-\varrho})\{n^{1/2} + \min[(1-x)^{-\alpha/2-1/4}, n^{1/2+\alpha}]\}$	$(0 \leqq x \leqq 1)$
$-\varrho < \beta \leqq 0$	$\varepsilon_n(x) = O(n^{1/2-\varrho})$	$(-1 \leqq 0 \leqq 0)$
$\beta > 0$	$\varepsilon_n(x) = O(n^{-\varrho})\{n^{1/2} + \min[(1+x)^{-\beta/2-1/4}, n^{1/2+\beta}]\}$	$(-1 \leqq x \leqq 0)$

LITERATURVERZEICHNIS

[1] FREUD, G.: *Orthogonale Polynome*, Birkhäuser Verl., Basel, 1968.
[2] FREUD, G.: Über eine Klasse Lagrangescher Interpolationsverfahren, *Studia Sci. Math. Hungar.* **3** (1968) 249—255.
[3] SZEGŐ, G.: *Orthogonal Polynomials*, 2 edition, Amer. Math. Soc., New York, 1959.
[4] Гопенгауз, И. Г.: К теореме А. Ф. Тимана о приближении функций многочленами на конечном отрезке, *Математические Записки* **1** (1967) 163—172.
[5] Киш, О.: Замечания о порядке сходимости Лагранжева интерполирования, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*
[6] Теляковский С. А.: Две теоремы о приближении функций алгебраическими многочленами, *Мат. Сборник*, **70** (1966) 252—265.

Mathematisches Institut der Ungarischen Akademie der Wissenschaften, Budapest

(Eingegangen: 9. August, 1968.)

ERRATUM to my paper „On rational approximation of absolutely continuous functions", *Studia Sci. Mat. Hung.* **3** (1968) 383 — 386.

Formula (13) should read

$$\lambda_{9^{2N}} > C_4^{-1} r^{1/2} 9^{-N} \left[\frac{1}{9^{3(N+1)}} - C_3 r^{-3/2} \frac{1}{9^{3N}} \right] > C_4^{-1} r^{1/2} 9^{-N} \cdot \frac{1}{2} \frac{1}{9^{3(N+1)}} > \frac{C_5}{9^{4N}} .$$

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 385—387.

NOTIZ ZUR EULERSCHEN CHARAKTERISTIK OFFENER UND ABGESCHLOSSENER EUKLIDISCHER POLYEDER

von

H. HADWIGER

Wie bekannt, lässt sich die Eulersche Charakteristik der Polyeder des euklidischen Raumes auf rein elementargeometrische Weise begründen. Existenz, Eindeutigkeit und Haupteigenschaften können aufgrund eines nach der Raumdimension weitergreifenden rekursiven Verfahrens sichergestellt werden, wie dies vom Verfasser (vgl. [1], S. 103 ff) etwas allgemeiner für Körper des Konvexringes dargelegt worden ist. Speziell für Polyeder ist die erwähnte Methode auch von H. LENZ (vgl. [2], S. 291 ff) ausführlicher erklärt worden. — In der vorliegenden Note soll darauf hingewiesen werden, dass sich nach dem nämlichen Rekursionsprozess in leicht differenzierter Art eine Begründung der Eulerschen Charakteristik für abgeschlossene — aber andererseits auch für offene Polyeder darstellen lässt, wobei sich im gleichen elementaren Rahmen aus der Wechselwirkung beider Begründungen weitere mitteilenswerte Relationen ergeben. An sich sind diese Beziehungen zwar nicht neu; wie dem Verfasser von H. E. DEBRUNNER in freundlicher Weise mitgeteilt worden ist, lassen sich die fraglichen Relationen innerhalb der Topologie mühelos aus dem ALEXANDERschen Dualitätssatz folgern. Jedoch scheinen die Formeln in der hier vorliegenden einfachen konkreten Gestalt im zuständigen Fachschrifttum kaum berücksichtigt worden zu sein.

Es sei $n \geqq 1$, E^n der n-dimensionale euklidische Raum, $\mathfrak{X}^n$ die Klasse der beschränkten abgeschlossenen Polyeder $A \subset E^n$. Weiter bezeichne $\mathfrak{U}^n \subset \mathfrak{X}^n$ die Teilklasse der konvexen Polyeder. Es gibt dann eine eindeutig bestimmte Abbildung $\chi: \mathfrak{X}^n \to Z$ von $\mathfrak{X}^n$ in die additive Gruppe der ganzen Zahlen derart, dass die Postulate

$$\chi\emptyset = 0; \tag{1}$$

$$\chi P = 1, \quad P \in \mathfrak{U}^n, \quad P \neq \emptyset; \tag{2}$$

$$\chi(A \cup B) + \chi(A \cap B) = \chi A + \chi B, \quad A, B \in \mathfrak{X}^n \tag{3}$$

erfüllt sind, wobei $\emptyset$ das leere Polyeder bedeutet. Die Funktion χ ist die Charakteristik von Euler-Poincaré. Weiterhin gilt die Relation

$$\chi A = \Sigma_t[\chi(A \cap H_t) - \chi(A \cap H_{t+0})]. \tag{4}$$

Hierbei ist $H_t = H + tu$, wo H einen durch den Ursprung des E^n hindurchgehenden $(n-1)$-dinemsionalen Unterraum von E^n, u einen auf H orthogonal stehenden Einheitsvektor, t einen reellen Parameter und schliesslich $\chi(A \cap H_{t+0})$ den Grenzqert $\lim \chi(A \cap H_{t+d})$, $d > 0$, $d \to 0$ anzeigen. Die in (4) vorgeschriebene Summation erstreckt sich über alle t, $-\infty < t < \infty$, wobei vermerkt werden muss, dass der Summand für alle t mit endlich vielen Ausnahmen verschwindet. — Da $A \cap H_t \in \mathfrak{X}^{n-1}$

MAGYAR TUDOMÁNYOS AKADÉMIA KÖNYVTÁRA

gilt, ist wie oben angedeutet eine auf (4) basierende rekursive Begründung von χ über $\mathfrak{X}^n$ möglich.

Es sei nun andererseits $\mathfrak{P}^n$ die Klasse der beschränkten offenen Polyeder $B \subset E^n$, $\mathfrak{B}^n \subset \mathfrak{P}^n$ die Teilklasse der konvexen Polyeder. In analoger Weise lässt sich zeigen:

Es gibt eine eindeutige Abbildung $\chi\colon \mathfrak{P}^n \to Z$ von $\mathfrak{P}^n$ in die additive Gruppe der ganzen Zahlen so, dass die Postulate

$$(1^\circ) \qquad \chi\emptyset = 0;$$

$$(2^\circ) \qquad \chi Q = 1, \quad Q \in \mathfrak{B}^n, \quad Q \neq \emptyset;$$

$$(3^\circ) \qquad \chi(A \cup B) + \chi(A \cap B) = \chi A + \chi B, \quad A, B \in \mathfrak{P}^n$$

erfüllt werden. Wieder ergibt sich die Charakteristik χ. Ferner gilt erneut eine rekursive Beziehung, nämlich

$$(4^\circ) \qquad \chi B = -\Sigma_t[\chi(B \cap H_t) - \chi(B \cap H_{t+0})]$$

mit den analogen Festsetzungen wie bei (4); man beachte jedoch den Vorzeichenwechsel! — Da $B \cap H_t \in \mathfrak{P}^{n-1}$ gilt, lässt sich basierend auf (4°) in analoger Weise eine rekursive Begründung von χ über $\mathfrak{P}^n$ durchführen.

Steht nun die Charakteristik χ für beide Klassen $\mathfrak{X}^n$ und $\mathfrak{P}^n$ zur Verfügung, so lassen sich die nachfolgend angeführten Beziehungen nachweisen:

$$(5) \qquad A \in \mathfrak{X}^n, \ B \in \mathfrak{P}^n, \ B \subset A \Rightarrow \chi A = \chi(A-B) + (-1)^n \chi B;$$

$$(6) \qquad A \in \mathfrak{X}^n, \ B \in \mathfrak{P}^n, \ A \subset B \Rightarrow \chi B = \chi(B-A) + (-1)^n \chi A,$$

Hierbei ist in (5) $A - B = A \cap B^* \in \mathfrak{X}^n$ und in (6) $B - A = B \cap A^* \in \mathfrak{P}^n$; Der Stern zeigt den Übergang zur komplementären Menge an.

Die Relationen (5) und (6) sind für $n = 1$ trivial richtig. Für $n > 1$ lassen sie sich mit Heranziehung beider Rekursionen (4) und (4°) in naheliegender Weise nach dem Induktionsverfahren sicherstellen; man hat lediglich zu beachten, dass in beiden Fällen $A \cap H_t \in \mathfrak{X}^{n-1}$, $B \cap H_t \in \mathfrak{P}^{n-1}$, in (5) $B \cap H_t \subset A \cap H_t$ und in (6) $A \cap H_t \subset B \cap H_t$ für alle t gilt.

Als Korollar lässt sich noch die Formel

$$(7) \qquad A \in \mathfrak{X}^n \Rightarrow \chi A = \chi \hat{A} + (-1)^n \chi A^\circ$$

gewinnen, wo $\hat{A}$ den Rand und A° den offenen Kern von A bezeichnen. — In der Tat genügt es, in (5) $B = A^\circ$ zu setzen und $\hat{A} = A - A^\circ$ zu beachten.

Endlich wollen wir „ausreichend normale" Polyeder in Betracht ziehen, deren Charakteristik mit derjenigen ihres offenen Kernes übereinstimmt. Es resultiert aus (7) die Relation

$$(8) \qquad A \in \mathfrak{X}^n, \ \chi A = \chi A^\circ \Rightarrow \chi \hat{A} = [1 - (-1)^n] \chi A,$$

welche die Charakteristik des Randpolyeders mit derjenigen des Polyeders selbst in Beziehung bringt.

Natürgemäss ist es jetzt wünschbar, die für die Gültigkeit von (8) verbindliche „Normalität" eines Polyeders rein geometrisch zu kennzeichnen. Diese Frage soll an anderer Stelle untersucht werden.

Abschliessend haben wir noch darauf hinzuweisen, dass sich die Existenz der Eulerschen Charakteristik simultan für offene und abgeschlossene Polyeder auch im Rahmen einer von V. KLEE [3] entwickelten rein verbands kombinatorischen Theorie ergibt.

LITERATURVERZEICHNIS

[1] HADWIGER, H.: Eulers Charakteristik und kombinatorische Geometrie, *J. reine angew. Math.* **194** (1955) 101—110.
[2] LENZ, H.: *Grundlagen der Elementarmathematik*, Berlin, 1961.
[3] KLEE, V.: The Euler characteristic in Combinatorial geometry, Amer. Math. Monthly **70** (1963) 119—127.

Universität Bern Mathematisches Institut, Bern

(Eingegangen: 22. August, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 389–399.

ENTROPY OF POINT PROCESSES

by

J. FRITZ

This paper deals with the problem to define entropy for point processes. The first results in this direction has been published by M. RUDEMO [1]. Here we apply the results of I. CSISZÁR [2] on generalized entropy, and for point processes in Euclidean spaces we give another interpretation of this definition.

I should like to express my thanks to IMRE CSISZÁR for his valuable advices and for the correction of the manuscript.

1. Generalized entropy

In this section we summarize briefly some results on generalized entropy which are needed in our investigations. (See I. CSISZÁR [2] § 2—3.) Let $(\Omega, \mathscr{A}, \Lambda)$ be a σ-finite measure space, and let P be an arbitrary probability measure on $\mathscr{A}$.

DEFINITION 1. The generalized entropy (Λ-entropy) of P on the σ-algebra $\mathscr{C} \subset \mathscr{A}$ is defined by

$$H_\Lambda(P:\mathscr{C}) = -\infty \quad \text{if} \quad P \not\ll \Lambda \text{ on } \mathscr{C} \tag{1.1}$$

and

$$H_\Lambda(P:\mathscr{C}) = -\int_\Omega f_{\mathscr{C}}(\omega) \log f_{\mathscr{C}}(\omega)\, \Lambda(d\omega) \quad \text{if} \quad P \ll \Lambda \tag{1.2}$$

on $\mathscr{C}$ and $f_{\mathscr{C}}(\omega)$ is its Radon-Nikodym derivative. If $P \ll \Lambda$ but the integral (1.2) does not exist, then $H_\Lambda(P:\mathscr{C})$ is not defined.

In the case, when there exists a finite or countable family $\{A_i\}_{i\in I}$ of atoms of the σ-algebra $\mathscr{C} \subset \mathscr{A}$, such that $\sum_{i\in I} P(A_i) = 1$, then

$$H_\Lambda(P:\mathscr{C}) = \sum_{i\in I} P(A_i) \log \frac{\Lambda(A_i)}{P(A_i)} \tag{1.3}$$

provided that the right hand side exists.

An important consequence of the convexity of the function $x \log x$ is that $H_\Lambda(P:\mathscr{C})$ exists, if $H_\Lambda(P:\mathscr{C}) < +\infty$ for some σ-algebra $\mathscr{C} \subset \mathscr{C}$, and then $H_\Lambda(P:\mathscr{C}) \leqq H_\Lambda(P:\mathscr{C})$. The following statement is a bit more difficult, it is Theorem 3. of CSISZÁR [2].

THEOREM 1. *Let Γ be a system of σ-algebras $\mathscr{C} \subset \mathscr{A}$ with the following properties:*

(i) *For each $\mathscr{C}_1, \mathscr{C}_2 \in \Gamma$ there exists a $\mathscr{C} \in \Gamma$ such that $\mathscr{C}_1, \mathscr{C}_2 \subset \mathscr{C}$.*

(ii) $\mathscr{A} = \sigma(\bigcup \{\mathscr{C};\ \mathscr{C} \in \Gamma\})$.

(iii) $H_\Lambda(P:\mathscr{C}) < +\infty$ *for some* $\mathscr{C} \in \Gamma$.

Then $H_\Lambda(P:\mathscr{A})$ exists and

$$H_\Lambda(P:\mathscr{A}) = \inf\{H_\Lambda(P:\mathscr{C});\ \mathscr{C} \in \Gamma,\ H_\Lambda(P:\mathscr{C}) \text{ exists}\}. \tag{1.4}$$

Now we define the conditional generalized entropy. Let $\mathscr{A}_1 \subset \mathscr{A}$ be a σ-algebra and let Λ_1 be a σ-finite measure on $\mathscr{A}_1$, such that $\Lambda \ll \Lambda_1$ on $\mathscr{A}_1$. For a fixed $A \in \mathscr{A}$, $\Lambda_A(C) = \Lambda(AC)$ is a σ-finite measure, and $\Lambda_A \ll \Lambda_1$ on $\mathscr{A}_1$. Set $\varphi_A(\omega) = \dfrac{\Lambda_A(d\omega)}{\Lambda_1(d\omega)}$. If $P \ll \Lambda_1$ on $\mathscr{A}_1$, then

$$\Lambda'(A) = \int_\Omega \varphi_A(\omega) P(d\omega) \tag{1.5}$$

is defined and it is a σ-finite measure on $\mathscr{A}$.

It is easy to see, that $P \ll \Lambda$ on $\mathscr{A}$, if and only if, $P \ll \Lambda_1$ on $\mathscr{A}_1$ and $P \ll \Lambda'$ on $\mathscr{A}$, further

$$H_\Lambda(P:\mathscr{A}) = H_{\Lambda_1}(P:\mathscr{A}_1) + H_{\Lambda'}(P:\mathscr{A}) \tag{1.6}$$

provided that its terms exists, and the sum on the right hand side is defined. This relation suggests to define the conditional $\Lambda|\Lambda_1$-entropy of P on $\mathscr{A}$ given the σ-algebra $\mathscr{A}_1$ as $H_{\Lambda'}(P:\mathscr{A})$, provided that $P \ll \Lambda_1$ on $\mathscr{A}_1$ and $H_{\Lambda'}(P:\mathscr{A})$ exists. (See CSISZÁR [2] § 3.)

In this paper we need this definition only in the following special case. Let $\mathscr{A}_2 \subset \mathscr{A}$ be another σ-algebra such that $\mathscr{A}_1$ and $\mathscr{A}_2$ are qualitatively independent — i.e. $A_1 A_2 = \emptyset$ for an $A_1 \in \mathscr{A}_1$, $A_2 \in \mathscr{A}_2$ only if $A_1 = \emptyset$ or $A_2 = \emptyset$ — and $\mathscr{A} = \sigma(\mathscr{A}_1 \cup \mathscr{A}_2)$. Further, let Λ_2 be such a σ-finite measure on $\mathscr{A}_2$ that

$$\Lambda(A_1 A_2) = \Lambda_1(A_1)\Lambda_2(A_2) \quad \text{if} \quad A_1 \in \mathscr{A}_1,\ A_2 \in \mathscr{A}_2. \tag{1.7}$$

Because of the qualitative independence of $\mathscr{A}_1$ and $\mathscr{A}_2$, Λ_1 and Λ_2 determine the measure Λ on $\mathscr{A} = \sigma(\mathscr{A}_1 \cup \mathscr{A}_2)$ by (1.7) uniquely, thus we may write $\mathscr{A} = \mathscr{A}_1 \times \mathscr{A}_2$ and $\Lambda = \Lambda_1 \times \Lambda_2$.

In this case (1.5) implies that $\Lambda' = P_1 \times \Lambda_2$ on $\mathscr{A} = \mathscr{A}_1 \times \mathscr{A}_2$, where P_1 denotes the restriction of P to $\mathscr{A}_1$, consequently $H_{\Lambda'}(P:\mathscr{A})$ does not depend on the choice of Λ_1.

DEFINITION 2. In the above case we define the conditional Λ_2-entropy of P on $\mathscr{A}$ given the σ-algebra $\mathscr{A}_1$ as

$$H_{\Lambda_2}(P|\mathscr{A}_1) = H_{\Lambda'}(P:\mathscr{A}) \tag{1.8}$$

provided that $H_{\Lambda'}(P:\mathscr{A})$ exists, where $\Lambda' = P_1 \times \Lambda_2$ on $\mathscr{A} = \mathscr{A}_1 \times \mathscr{A}_2$.

With this notation (1. 6) becomes

$$H_A(P: \mathscr{A}) = H_{A_1}(P: \mathscr{A}_1) + H_{A_2}(P|\mathscr{A}_1). \tag{1.9}$$

Further

$$H_{A_2}(P|\mathscr{A}_1) \leqq H_{A_2}(P: \mathscr{A}_2) \tag{1.10}$$

and equality holds if, and only if, $\mathscr{A}_1$ and $\mathscr{A}_2$ are P-independent.

We see that the well known properties of the conditional entropy are valid for $H_{A_2}(P|\mathscr{A}_1)$, too.

2. Point processes

Let us consider an arbitrary separable measurable space $(R, \mathscr{R})$; i.e. let $\mathscr{R}$ be a σ-algebra of subsets of R generated by a countable family of sets $\{G_i;\ i \in I\}$. Then the atoms of the σ-algebra $\mathscr{R}$ are the nonempty sets of type $F = \cap \{E_i;\ i \in I\}$, where $E_i = G_i$ or $E_i = R - G_i$ for each $i \in I$. Of course, the set of atoms of $\mathscr{R}$ does not depend on the choice of the generating family $\{G_i;\ i \in I\}$. Further, in a separable measurable space $(R, \mathscr{R})$ every measurable set is a (not necessarily countable) union of atoms. Thus, without any loss of generality, we may assume that the atoms of $\mathscr{R}$ are exactly the elements of R.

A σ-finite integral valued measure ω on $\mathscr{R}$ will be called a counting measure. Because of the separability of $(R, \mathscr{R})$, each counting measure ω on $\mathscr{R}$ has the form

$$\omega(F) = \sum_{i \in I} \delta(F|x_i) \quad \text{if} \quad F \in \mathscr{R}, \tag{2.1}$$

where $\delta(F|x)$ denotes the indicator of the set F. In this representation $\{x_i;\ i \in I\}$ is a finite or countable sequence of not necessarily different elements of R. The elements of this sequence are characterized by the property $\omega(\{x\}) > 0$, and an $x \in R$ occurs in it exactly $\omega(\{x\})$-times, thus the representation (2. 1) is unique. As ω is a σ-finite measure, $\omega(\{x\}) < +\infty$ for each $x \in R$.

In order to define a point process in $(R, \mathscr{R})$, we construct the probability space $(\Omega, \mathscr{A}, P)$ as follows. The realizations ω of the process are counting measures on $(R, \mathscr{R})$; the set of all these counting measures is Ω. For an $\omega \in \Omega$ the elements x of R with $\omega(\{x\}) > 0$ are called the points of the process if the elementary event ω has occured, or briefly, the points of the realization ω. If $\omega(\{x\}) = 1$, then x is a single point, and x is a multiple point with multiplicity k if $\omega(\{x\}) = k > 1$. We introduce also the notation $N(F, \omega) = \omega(F)$, where $F \in \mathscr{R}$, $\omega \in \Omega \cdot N(F, \omega)$ is called the number of points of the process in F (with their multiplicity) if the elementary event ω has occured.

The algebra of events $\mathscr{A}$ will be the σ-algebra of subsets of Ω generated by the functions $N = N(F, \omega)$; that is the minimal σ-algebra with respect to which all the random variables $N(F) = N(F, \omega)$ if $F \in \mathscr{R}$, are measurable. We shall say that a point process in $(R, \mathscr{R})$ is given if we are given a probability measure P on $\mathscr{A}$.

The average number of points (with their multiplicity) in the set $F \in \mathscr{R}$ is

$$v(F) = E(N(F)) = \int_{\Omega} N(F, \omega)\, P(d\omega). \tag{2.2}$$

The set function v is a measure on $\mathscr{R}$, we assume that it is a σ-finite one. Then the counting measures $\omega \in \Omega$ are uniformly σ-finite with probability one: if $\bigcup_{i=1}^{\infty} G_i = R$, $G_i \in \mathscr{R}$ and $v(G_i) < +\infty$ for each $i = 1, 2, 3, \ldots$, then $\omega(G_i) = N(G_i, \omega) < +\infty$ for each i with probability one.

In general, the set $\underline{D}(G)$ of realizations ω having at least one multiple point in the measurable set G need not belong to $\mathscr{A}$, but the the σ-finiteness of v and the separability of the measurable space $(R, \mathscr{R})$ imply the existence of such an event $C \in \mathscr{A}$ that $P(C) = 1$ and $D(G) = \underline{D}(G)C$ belongs to $\mathscr{A}$ for each $G \in \mathscr{R}$. Namely, let $\{G_i\}_{i=1}^{\infty}$ be a sequence of sets of $\mathscr{R}$ such that $\bigcup_{i=1}^{\infty} G_i = R$ and $v(G_i) < +\infty$ for each $i = 1, 2, \ldots$, and let C be the set of those realizations ω, for which $N(G_i, \omega) < +\infty$ for each i. Then $C \in \mathscr{A}$ and because of the σ-finiteness of v, $P(C) = 1$. On the other hand, from the separability of $(R, \mathscr{R})$ easily follows that $\underline{D}(G)C = D(G) \in \mathscr{A}$, which proves our statement. We see that the choice of the event C depends only on the measure v. In the most familiar cases there are no multiple points of the process with probability one, i.e. $P(D(G)) = 0$ for each $G \in \mathscr{R}$.

We shall denote by $A(F_2, F_2, \ldots, F_j; k_1, k_2, \ldots, k_j)$ the event that $N(F_i) = k_i$ for $i = 1, 2, \ldots, j$; where $F_1, F_2, \ldots, F_j \in \mathscr{R}$ and $k_1, k_2, \ldots, k_j$ are nonnegative integers. These events generate $\mathscr{A}$ even if $F_1, \ldots, F_j$ range over only an algebra $\mathscr{R}_0$ generating $\mathscr{R}$, thus $\mathscr{A}$ is a separable σ-algebra, too.

In many cases we are interested in the behaviour of our point process only in a measurable subset G of R. Such a problem is formulated in terms of the events of the σ-algebra $\mathscr{A}_G = \sigma\{A(F, k);\ F \in \mathscr{R},\ F \subseteq G,\ k \geqq 0 \text{ integer}\}$. This is exactly the σ-algebra generated by the restriction $N_G(F, \omega) = N(F \cap G, \omega) = N_G$ of N to G. Obviously, the mapping $A \to A\Omega_G$ is an isomorphism of $\mathscr{A}_G$ to the σ-algebra $\mathscr{A}_G^0 = \{A\Omega_G;\ A \in \mathscr{A}_G\}$, where Ω_G denotes the set of counting measures ω vanishing outside G. Further, it is easy to see, if $G_1 \cap G_2 = \emptyset$, then $\mathscr{A}_{G_1}$ and $\mathscr{A}_{G_2}$ are qualitatively independent, and $\mathscr{A}_{G_1 \cup G_2} = \mathscr{A}_{G_1} \times \mathscr{A}_{G_2}$.

Another approach to the above problem is to consider such a probability measure P_G on $\mathscr{A}$ instead of P that

$$P_G(A) = P(A) \quad \text{if} \quad A \in \mathscr{A}_G \tag{2.3}$$

and

$$P_G(\Omega_G) = 1. \tag{2.4}$$

Then the mapping $A \to A\Omega_G$ preserves P_G, and $N = N_G$ with P_G-probability one. This P_G is constructed as follows: For each $A \in \mathscr{A}$ there exists an $A_G \in \mathscr{A}_G$ such that $A\Omega_G = A_G\Omega_G$, because the events $A \in \mathscr{A}$ with this property form a σ-algebra, and for the events $A(F, k)$ generating $\mathscr{A}$, we have $A(F, k)\Omega_G = A(F \cap G, k)\Omega_G$. As $B\Omega_G = \emptyset$ for a $B \in \mathscr{A}_G$ only if $B = \emptyset$, A_G is uniquely determined for each $A \in \mathscr{A}$. The mapping $A \to A_G$ is a homomorphism of $\mathscr{A}$ to $\mathscr{A}_G$, where $A = A_G$ if and only if $A \in \mathscr{A}_G$; thus

$$P_G(A) = P_G(A_G) = P(A_G) \tag{2.5}$$

defines a probability measure P_G on $\mathscr{A}$, and it is the only one with the above properties. We remark that the image of the event

$$(2.6) \quad A^{\circ}(F_1, \dots, F_j; k_1, \dots, k_j) = A(F_1, \dots, F_j; k_1, \dots, k_j) A\left(R - \bigcup_{i=1}^{j} F_i; 0\right)$$

by the mapping $A \to A_G$ will be the event

$$(2.7) \quad A_G^0(F_1, \dots, F_j; k_1, \dots, k_j) = A\left(F_1 \cap G, \dots, F_j \cap G, G - \bigcup_{i=1}^{j} F_i; k_1, \dots, k_j, 0\right),$$

thus their P_G-probabilities are equal.

3. The generalized entropy of a point process

In order to define the generalized entropy of our point process, we have to give a measure Λ on $\mathscr{A}$. Then $H_\Lambda(P_G : \mathscr{A})$ will be the Λ-entropy of the process in the measurable set G. Of course, the most convenient choice of Λ depends on the concrete problem. Here we investigate a simple class of measures having very advantageous properties. For each σ-finite measure $\lambda, \vartheta, \dots$ on $\mathscr{R}$ we define a σ-finite measure $\Lambda, \Theta, \dots$ on $\mathscr{A}$ by the following construction.

Let λ be an arbitrary σ-finite measure on $\mathscr{R}$. For each natural integer s set $(R^s, \mathscr{R}^s, \lambda^s) = \bigtimes_{k=1}^{s} (R, \mathscr{R}, \lambda) \cdot R^0$ is defined as a set having the only element $\mathbf{x}^0 = \emptyset$, $\mathscr{R}^0 = \{\emptyset, R^0\}$ and $\lambda^0(R^0) = 1$; i.e. $(R^0, \mathscr{R}^0, \lambda^0)$ is also a measure space. The union of the disjoint sets $R^0, R^1, R^2, \dots$ will be denoted by R^+. Similarily, $F^s = \bigtimes_{k=1}^{s} F$, $F^0 = \{\mathbf{x}^0\}$ and $F^+ = \bigcup_{s=0}^{\infty} F^s$ for every $F \in \mathscr{R}$. The elements of R^+ have the form $\mathbf{x} = (x_1, x_2, \dots, x_s)$ if $\mathbf{x} \in R^s$ and $\mathbf{x} = \mathbf{x}^0$ if $\mathbf{x} \in R^0$. Observe, that to every $\mathbf{x} \in R^+$ there corresponds a finite counting measure $\omega_{\mathbf{x}}$ in a very natural way, namely $\omega_{\mathbf{x}}(F) = \sum_{i=1}^{s} \delta(F | x_i)$ if $\mathbf{x} = (x_1, x_2, \dots, x_s)$ and $\omega_{\mathbf{x}^0} \equiv 0$.

Now $\mathscr{R}^+ = \left\{ \bigcup_{s=0}^{\infty} F_s;\ F_s \in \mathscr{R}^s \right\}$ is a σ-algebra of subsets of R^+, and

$$(3.1) \qquad \lambda^+(\mathbf{F}) = \sum_{s=0}^{\infty} \frac{1}{s!} \lambda^s(\mathbf{F} \cap R^s) \quad \text{if} \quad \mathbf{F} \in \mathscr{R}^+$$

defines a σ-finite measure λ^+ on $\mathscr{R}^+$. Of course, every F^s and F^+ belongs to $\mathscr{R}^+$ if $F \in \mathscr{R}$, further $\lambda^+(F^s) = \frac{1}{s!} \lambda(F)^s$ for $s = 0, 1, 2, \dots$ and $\lambda^+(F^+) = e^{\lambda(F)}$.

The measure Λ will be defined by the following mapping T of $\mathscr{A}$ into the subsets of R^+:

$$(3.2) \qquad TA = \{\mathbf{x};\ \mathbf{x} \in R^+, \omega_{\mathbf{x}} \in A\} \quad \text{if} \quad A \in \mathscr{A}.$$

A relevant property of T is that $TA = \emptyset$ if, and only if, $N(R, \omega) = +\infty$ for each

$\omega \in A$. It is easy to see that T preserves the set-theoretic operations, and $TA(F, k) \in \mathscr{R}^+$ if $F \in \mathscr{R}$, consequently T is a homomorphism of $\mathscr{A}$ into $\mathscr{R}^+$. Thus

$$\Lambda(A) = \lambda^+(TA) \tag{3. 3}$$

defines a σ-finite measure Λ on $\mathscr{A}$ such that $\Lambda(A) = 0$ if $N(R, \omega) = +\infty$ on A. Taking into account that

$$TA^\circ(F_1, \ldots, F_j; k_1, \ldots, k_j) = \bigcup \left\{ \bigtimes_{n=1}^{j} E_n; E_n = F_i \, k_i\text{-times}, \, i = 1, \ldots, j \right\} \tag{3. 4}$$

where $F_1, \ldots, F_j$ are disjoint sets of $\mathscr{R}$ and $s = \sum_{i=1}^{j} k_i$, we have

$$\Lambda\left(A^\circ(F_1, \ldots, F_j; k_1, \ldots, k_j)\right) = \prod_{i=1}^{j} \frac{1}{k_i!} \lambda(F_i)^{k_i}. \tag{3. 5}$$

DEFINITION 3. The entropy of our point process in the measurable set G is defined as the Λ-entropy $H_\Lambda(P_G : \mathscr{A})$ of P_G on $\mathscr{A}$, provided that $H_\Lambda(P_G : \mathscr{A})$ exists. We shall denote this entropy by $H_\lambda(P : G)$ expressing the fundamental role of the measure λ in its definition.

In the case, when $P_G \not\ll \Lambda$, then $H_\lambda(P : G) = -\infty$. We remark that $P_G \not\ll \Lambda$ if the process has multiple points in G with positive probability and the measure λ is nonatomic. Now let us assume that $P_G \ll \Lambda$ on $\mathscr{A}$ and $f_G(\omega)$ is its Radon—Nikodym derivative. $f_G(\omega)$ vanishes outside Ω_G and

$$H_\lambda(P : G) = -\int_{\Omega} f_G(\omega) \log f_G(\omega) \Lambda(d\omega). \tag{3. 6}$$

Here $f_G(\omega) = 0$ even if $N(R, \omega) = +\infty$ and

$$f_G(\omega) = P\big(A(G, s)\big) f_s(x_1, x_2, \ldots, x_s) \tag{3. 7}$$

if $N(R, \omega) = N(G, \omega) = s$ and $\omega = \omega_{\mathbf{x}}$, $\mathbf{x} = (x_1, x_2, \ldots, x_s)$; where f_s is a nonnegative, measurable symmetric function on R^s vanishing outside G^s and

$$\int_{G^s} f_s(\mathbf{x}) \frac{1}{s!} \lambda^s(d\mathbf{x}) = 1. \tag{3. 8}$$

The function $\frac{1}{s!} f_s$ is called the s-dimensional symmetrized density function of the process in G with respect to λ. From (3. 7) and (3. 6) we have

$$H_\lambda(P : G) = -\sum_{s=0}^{\infty} w_s \log w_s - \sum_{s=1}^{\infty} \frac{1}{s!} w_s \int_{G^s} f_s \log f_s \lambda^s(d\mathbf{x}), \tag{3. 9}$$

where $w_s = P\big(A(G, s)\big)$. This is a well known formula for one dimensional point processes, when λ is the Lebesque-measure. (See RUDEMO [1] and MCFADDEN [4]).

Now we derive a useful formula for the transformation of entropy, when λ is replaced by another measure. Let $\vartheta \ll \lambda$ be an arbitrary σ-finite measure on $\mathscr{R}$,

set $\frac{\vartheta(dx)}{\lambda(dx)} = \varphi(x) \cdot \vartheta$ induces by (3. 1), (3. 2) and (3. 3) a σ-finite measure Θ on $\mathscr{A}$, $\Theta \ll \Lambda$ and

$$\frac{\Theta(d\omega)}{\Lambda(d\omega)} = \prod_{i=1}^{j} \varphi(x_i) \quad \text{if} \quad \omega = \omega_{\mathbf{x}} \quad \text{and} \quad \mathbf{x} = (x_1, \ldots, x_s) \in R^+. \tag{3. 10}$$

Thus in the case when $P_G \ll \Theta$ and the terms of (3. 11) are defined, from (3. 9) we obtain the following relation:

$$H_\theta(P: G) = H_\lambda(P: G) + \int_G \varrho(x) \log \varphi(x) \lambda(dx), \tag{3. 11}$$

where

$$\varrho(x) = \frac{\nu(dx)}{\lambda(dx)} = \sum_{s=1}^{\infty}{}' s w_s \int_G \ldots \int_G f_s(x, x_2, \ldots, x_s) \lambda(dx_2) \ldots \lambda(dx_s) \tag{3. 12}$$

is the density function of the point process with respect to λ.

Finally we define the conditional entropy $H_\lambda(P: G_1 | G_2)$ of our point process in G_1 given the points of the process in G_2 ($G_1 \cap G_2 = \emptyset$), in the sense of Definition 2. For this purpose we have to define a measure Λ_G on $\mathscr{A}_G$ such that

$$H_\lambda(P: G) = H_{\Lambda_G}(P_G: \mathscr{A}_G) \quad \text{for every} \quad G \in \mathscr{R} \tag{3. 13}$$

and

$$\Lambda_{G_1 \cup G_2}(A_1 A_2) = \Lambda_{G_1}(A_1) \Lambda_{G_2}(A_2) \tag{3. 14}$$

if $G_1 \cap G_2 = \emptyset$ and $A_1 \in \mathscr{A}_{G_1}$, $A_2 \in \mathscr{A}_{G_2}$; that is $\Lambda_{G_1 \cup G_2} = \Lambda_{G_1} \times \Lambda_{G_2}$. This Λ_G will be defined by

$$\Lambda_G(A) = \Lambda(A\Omega_G). \tag{3. 15}$$

Then

$$\Lambda_G(A) = 0 \quad \text{if} \quad N(R - G) > 0 \quad \text{on} \quad A, \tag{3. 16}$$

further

$$\Lambda_G(A^\circ(F_1, \ldots, F_j; k_1, \ldots, k_j)) = \prod_{i=1}^{j} \frac{1}{k_i!} \lambda(F_i \cap G)^{k_i} = \tag{3. 17}$$

$$= \Lambda_G(A_G^0(F_1, \ldots, F_j; k_1, \ldots, k_j))$$

if $F_1, F_2, \ldots, F_j$ are disjoint sets of $\mathscr{R}$, and

$$\Lambda_G(A) = 0 \quad \text{if} \quad N(G, \omega) = +\infty \quad \text{on} \quad A. \tag{3. 18}$$

Thus (3. 14) holds for events of $\mathscr{A}_{G_1}$ respectiely $\mathscr{A}_{G_2}$ of type as in (3. 17) or (3. 18), but these events generate $\mathscr{A}_{G_1}$ respectively $\mathscr{A}_{G_2}$, consequently (3. 14) is true for each $A_1 \in \mathscr{A}_{G_1}$, $A_2 \in \mathscr{A}_{G_2}$.

On the other hand, as the mapping $A \to A\Omega_G$ preserves P_G and $A\Omega_G = A_G\Omega_G$ for each $A \in \mathscr{A}$, we see that

$$H_\Lambda(P_G: \mathscr{A}) = H_{\Lambda_G}(P_G: \mathscr{A}_G^0) = H_{\Lambda_G}(P: \mathscr{A}_G) \tag{3. 19}$$

Thus the conditional entropy $H_\lambda(P:G_1|G_2)$ can be defined in the sense of Definition 2. as

$$H_\lambda(P: G_1|G_2) = H_{\mathcal{A}_{G_1}}(P_{G_1 \cup G_2}|\mathcal{A}_{G_2}), \text{ where } G_1 \cap G_2 = \emptyset. \tag{3. 20}$$

From (1. 9) we obtain

$$H_\lambda(P:G_1 \cup G_2) = H_\lambda(P:G_2) + H_\lambda(P:G_1|G_2). \tag{3. 21}$$

Further, (1. 10) becomes

$$H_\lambda(P:G_1|G_2) \leqq H_\lambda(P:G_1), \tag{3. 22}$$

where equality holds if, and only if, $\mathcal{A}_{G_1}$ and $\mathcal{A}_{G_2}$ are P-independent, that is

$$H_\lambda(P:G_1 \cup G_2) \leqq H_\lambda(P:G_1) + H_\lambda(P:G_2) \tag{3. 23}$$

with equality if, and only if $\mathcal{A}_{G_1}$ and $\mathcal{A}_{G_2}$ are P-independent.

4. The dimensional entropy

In this section we give another definition for the entropy of an r-dimensional point process, which is essentially equivalent to the above one, if λ is the Lebesque-measure.

An approach to the definition of the entropy of an abstract random variable is the method of RÉNYI and BALATONI [3]. (See CSISZÁR [2] § 4, too.) The crucial step of their definition is the construction of an approximating sequence of discrete random variables; the asymptotic behaviour of the Shannon-entropies of these random variables suggests the way of the definition of the entropy. Taking into account the structure of an r-dimensional point process, this method can be applied in our case, too. We define the entropy of our point process in every Borel set G, where the average number of points is finite.

Let us consider an r-dimensional point process. This means that R is the r-dimensional Euclidean space and $\mathcal{R}$ denotes the σ-algebra of its Borel subsets. An r-dimensional interval of type

$$I_n = \left\{(x_1, \ldots, x_r): \frac{k_i}{n} \leqq x_i < \frac{k_i+1}{n},\ i = 1, 2, \ldots, r\right\}, \tag{4. 1}$$

where $k_1, k_2, \ldots, k_r$, n are integers and $n>0$, will be called a cell of the σ-algebra $\mathcal{R}^{(n)}$, generated by these intervals. The σ-algebra $\mathcal{A}_G^{(n)}$ of subsets of Ω, generated by the restriction $N_G^{(n)}$ of N_G to $\mathcal{R}^{(n)}$ is an atomic σ-algebra, and in case of $N(G) < +\infty$ with probability one, the atoms of $\mathcal{A}_G^{(n)}$ having positive probability are events of type

$$A_G^0(F_1^{(n)}, \ldots, F_j^{(n)}; k_1, \ldots, k_j) = \tag{4. 2}$$

$$= A\left(F_1^{(n)} \cap G, \ldots, F_j^{(n)} \cap G, G - \bigcup_{i=1}^{j} F_i^{(n)}; k_1, \ldots, k_j, 0\right)$$

where $F_1^{(n)}, \ldots, F_j^{(n)}$ are cells of $\mathscr{R}^{(n)}$. This means, if $N(G) < +\infty$ with probability one, then $N_G^{(n)}$ is a discrete random variable taking values in the set of finite integral valued measures on $\mathscr{R}^{(n)}$ vanishing outside G. The sequence $N_G^{(n)}$ approximates N_G in the sense that $\mathscr{A}_G = \sigma\left\{\bigcup_{n=1}^{\infty} \mathscr{A}_G^{(n)}\right\}$. The following theorem characterizes the behaviour of the Shannon-entropies $H(N_G^{(n)})$.

THEOREM 2. *If* $v(G) < +\infty$, *then*

$$H(N_G^{(n)} | N_G^{(m)}) \leqq r v(G) \log\left(\frac{n}{m}+2\right) \tag{4.3}$$

where $H(N_G^{(n)}|N_G^{(m)})$ *denotes the conditional entropy of* $N_G^{(n)}$ *given* $N_G^{(m)}$.

PROOF. The assumption $v(G) < +\infty$ implies that $N_G^{(n)}$ is a discrete random variable for each n. Now, let us be given any atom $A_G^{(m)} = A_G^0(F_1^{(m)}, \ldots, F_j^{(m)}; k_1, \ldots, k_j)$ of $\mathscr{A}_G^{(m)}$. Under the condition that $A_G^{(m)}$, obtains, $N_G^{(n)}$ can take on at most

$$\prod_{i=1}^{j}\left(\frac{n}{m}+2\right)^{rk_i} = \left(\frac{n}{m}+2\right)^{rN(G)} \tag{4.4}$$

different values, namely the number of those cells of $\mathscr{R}^{(n)}$ having nonempty intersection in any fixed cell $F_i^{(m)}$ of $\mathscr{R}^{(m)}$ is at most $\left(\frac{n}{m}+2\right)^r$, and the k_i points of the process in $F_i^{(m)}$ can be distributed in these cells of $\mathscr{R}^{(n)}$ at most in $\left(\frac{n}{m}+2\right)^{rk_i}$ different ways. Consequently

$$H(N_G^{(n)} | A_G^{(m)}) \leqq rN(G) \cdot \log\left(\frac{n}{m}+2\right). \tag{4.5}$$

Taking the expectation of (4. 5) we obtain the required statement.

As $H(N_G^{(1)}, N_G^{(n)}) = H(N_G^{(n)})$, from (4. 3) follows that

$$H(N_G^{(1)}) \leqq H(N_G^{(n)}) \leqq H(N_G^{(1)}) + r \cdot v(G) \cdot \log(n+2). \tag{4.6}$$

Thus either each $H(N_G^{(n)})$ is finite or all are infinite. It is easy to see that $H(N_G^{(1)})$ is always finite if G is bounded. Relation (4. 6) suggests the following definition. (See BALATONI and RÉNYI [3]).

DEFINITION 4. In the case, when $H(N_G^{(1)}) < +\infty$ and $d = rv(G) < +\infty$, we define the d-dimensional entropy of the process in the Borel set G by

$$H_d(N_G) = \lim_n \left(H(N_G^{(n)}) - d \log n\right), \tag{4.7}$$

provided that this limit exists. $H_d(N_G)$ is called the d-dimensional entropy of N_G, too. If $H(N_G^{(1)}) = +\infty$, then we don't define $H_d(N_G)$.

THEOREM 3. *If the average number of points in the Borel set* G *is finite,* $d = r \cdot v(G)$ *and* $H(N_G^{(1)}) < +\infty$, *then the d-dimensional entropy of* N_G *exists, and*

$$H_d(N_G) = \inf_n \left(H(N_G^{(n)}) - d \log n\right). \tag{4.8}$$

PROOF. Let h be the greatest lower bound of the sequence $H_d^{(n)} = H(N_G^{(n)}) - d \log n$; we know that $h < +\infty$. We have to show that, if $c > h$, then $H_d^{(n)} < c$ for every n large enough. According to the definition of h, there exists an m such that $H_d^{(m)} < c$. For every n there holds the relation

$$H_d^{(n)} \leqq H(N_G^{(m)}) + H(N_G^{(n)} | N_G^{(m)}) - d \cdot \log n, \tag{4.9}$$

whence by Theorem 2. we have

$$H_d^{(n)} \leqq H_d^{(m)} + d \cdot \log \left(1 + 2\frac{m}{n}\right) \tag{4.10}$$

which proves the theorem.

Finally, we treate the connection of Definition 3. and Definition 4. We shall see that they are essentially equivalent if the process has no multiple points with probability one.

THEOREM 4. *Let us denote by λ the r-dimensional Lebesque measure, and let Λ be the measure induced by λ on $\mathscr{A}$. If $H_d(N_G)$ is defined (i.e. $H(N_G^{(1)}) < +\infty$, and $v(G) < +\infty$), further*

$$H_\Lambda(P_G : \mathscr{A}_R^{(1)}) > -\infty \tag{i}$$

$$P(D(G)) = 0, \tag{ii}$$

then $H_\lambda(P:G)$ also exists and they are equal.

PROOF. In view of Theorem 1. it is sufficient to show that the sequence

$$\Delta^{(n)}(G) = H(N_G^{(n)}) - d \log n - H_\Lambda(P_G : \mathscr{A}_R^{(n)}) \tag{4.11}$$

converges to 0. The atoms with positive probability of $\mathscr{A}_G^{(n)}$ and of $\mathscr{A}_R^{(n)}$ respectively, are the events $A_G^0(F_1^{(n)}, \ldots, F_j^{(n)}; k_1, \ldots, k_j)$ respectively $A^0(F_1^{(n)}, \ldots, F_j^{(n)}; k_1, \ldots, k_j)$, where $F_1^{(n)}, \ldots, F_j^{(n)}$ are cells of $\mathscr{R}^{(n)}$, and the P_G-probabilities of the corresponding atoms are equal. As $N(G)$ is measurable on $\mathscr{A}_G^{(n)}$, and $d \cdot \log n = E(\log n^{-rN(G)})$, from

$$\Lambda\left(A^\circ(F_1^{(n)}, \ldots, F_j^{(n)}; k_1, \ldots, k_j)\right) = n^{-rN(G)} \prod_{i=1}^{j} \frac{1}{k_i!} \tag{4.12}$$

we see that

$$\Delta^{(n)}(G) = E\left(\eta^{(n)}(G)\right) \tag{4.13}$$

where $\eta^{(n)}(G) = \sum_{K=2}^{\infty} \xi_K^{(n)}(G) \cdot \log k!$, and $\xi_K^{(n)}(F) = \xi_K^{(n)}(F, \omega)$ denotes the number of those cells $F^{(n)}$ of $\mathscr{R}^{(n)}$ for which $N(G \cap F \cap F^{(n)}, \omega) = k$. $\xi_K^{(n)}(\cdot, \omega)$, thus $\eta^{(n)}(\cdot, \omega)$, too, is a measure on $\mathscr{R}^{(1)}$, and

$$\eta^{(n)}(F^{(1)}) = \log N_G(F^{(1)})! = \eta^{(1)}(F^{(1)}) \tag{4.14}$$

if $F^{(1)}$ is any cell of $\mathscr{R}^{(1)}$; consequently

$$\eta^{(n)}(G) \leqq \eta^{(1)}(G). \tag{4.15}$$

Further, if $N(G, \omega) < +\infty$, and ω has no multiple points in G, then for an n which is large enough, all the points of ω in G will be in distinct cells of $\mathscr{R}^{(n)}$, thus from $\nu(G) < +\infty$ and assumption (ii) we obtain that

$$\lim_n \eta^{(n)}(G) = 0 \tag{4. 16}$$

with probability one.

On the other hand, assumptions (i) and $H(N_G^{(1)}) < +\infty$ imply

$$E(\eta^{(1)}(G)) < +\infty, \tag{4. 17}$$

whence the statement of the theorem follows by the dominated convergence theorem.

Observe that (4. 17) follows also from

$$E(N(G) \log N(G)) < +\infty \tag{4. 18}$$

and if G is bounded, then they are equivalent. But, assumption (i) is a very natural one, namely if it does not hold, then $H_A(P_G : \mathscr{B}) = -\infty$ for each σ-algebra $\mathscr{B} \supset \mathscr{A}_G^{(1)}$.

REFERENCES

[1] RUDEMO, M.: Dimension and entropy for a class of stochastic processes, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **9** (1964) 73—87.
[2] CSISZÁR, I.: On generalized entropy, *Studia Sci. Math. Hungar.* **4** (1969).
[3] BALATONI, J. and RÉNYI, A.: Remarks on entropy (in Hungarian), *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **1** (1956) 9—40.
[4] MCFADDEN, J.: The entropy of a point process, *J. Industrial Math.* (1965).
[5] FRITZ, J.: Information theory and thermodynamics of gas systems, *Proc. Coll. Inf. Theory*, Debrecen, 1967.
[6] MOYAL, J.: The general theory of stochastic population processes, *Acta Math.* **108** (1962) 1—31.
[7] LOÈVE, M.: *Probability theory*, Van Nostrand Toronto, New York, London, 1955.

Mathematical Institute of the Hungarian Academy of Sciences, Budapest

(Received September 23, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 401–419.

ON GENERALIZED ENTROPY

by

I. CSISZÁR

§ 1. Introduction

In information theory, the following quantities as measures of the „amount of information" are very important:

(i) *Entropy:* Let $\mathscr{P} = \{p_i\}_{i \in I}$ be a discrete probability distribution (i.e., $p_i \geqq 0$, $\sum_{i \in I} p_i = 1$) where I is a finite or countably infinite set of indices; the entropy of $\mathscr{P}$ as defined by SHANNON [13] is

$$H(\mathscr{P}) = -\sum_{i \in I} p_i \log_2 p_i. \tag{1.1}$$

(ii) *Differential entropy:* let $\mathscr{P}$ be a continuous probability distribution in the r-dimensional Euclidean space, given by a density function $f(\mathbf{x})$ $\left(f(\mathbf{x}) \geqq 0, \int f(\mathbf{x})\, d\mathbf{x} = 1\right)$ where $\mathbf{x} = (x_1, \ldots, x_r)$, $d\mathbf{x} = dx_1 \ldots dx_r$. The differential entropy of $\mathscr{P}$ as defined by Shannon [13] or r-dimensional entropy of $\mathscr{P}$ as defined by RÉNYI [8] is

$$H_r(\mathscr{P}) = -\int f(\mathbf{x}) \log_2 f(\mathbf{x})\, d\mathbf{x}. \tag{1.2}$$

(iii) *I-divergence:* let μ_1 and μ_2 be two probability measures on an arbitrary measurable space $(X, \mathscr{X})$. The I-divergence of μ_1 and μ_2 as defined by KULLBACK [5] is

$$I(\mu_1 : \mu_2) = \begin{cases} \int \log_2 \dfrac{\mu_1(dx)}{\mu_2(dx)}\, \mu_1(dx) & \text{if} \quad \mu_1 \ll \mu_2 \\ +\infty & \text{if} \quad \mu_1 \not\ll \mu_2. \end{cases} \tag{1.3}$$

A particularly important special case of I-divergence is

(iii a) The *mutual information* of two arbitrary random variables ξ_1 and ξ_2. Let ξ_1 and ξ_2 take their values in the (arbitrary) measurable spaces $(X_1, \mathscr{X}_1)$ and $(X_2, \mathscr{X}_2)$; let μ_{ξ_1}, μ_{ξ_2} and $\mu_{\xi_1 \xi_2}$ denote the distribution of ξ_1, the distribution of ξ_2 and the joint distribution of ξ_1 and ξ_2, respectively (μ_{ξ_1}, μ_{ξ_2} and $\mu_{\xi_1 \xi_2}$ are probability measures on $(X_1, \mathscr{X}_1)$, $(X_2, \mathscr{X}_2)$ and $(X_1 \times X_2, \mathscr{X}_1 \times \mathscr{X}_2)$, respectively). The mutual information of ξ_1 and ξ_2 is defined as

$$I(\xi_1, \xi_2) = I(\mu_{\xi_1 \xi_2} : \mu_{\xi_1} \times \mu_{\xi_2}). \tag{1.4}$$

The information quantities (i)—(iii) are of similar type and as a common generalization of them one may consider the entropy of a probability measure μ

on an arbitrary measurable space $(X, \mathcal{X})$ with respect to a σ-finite measure λ given on the same measurable space:

$$H_\lambda(\mu) = \begin{cases} -\int f(x) \log_2 f(x) \lambda(dx) & \text{if } \mu \ll \lambda \\ -\infty & \text{if } \mu \not\ll \lambda \end{cases} \tag{1. 5}$$

where $f(x)$ denotes the Radon—Nikodym derivative of μ with respect to λ (if $\mu \ll \lambda$).

If $X = \{x_i\}_{i \in I}$ is a finite or countably infinite set, $\mathcal{X}$ consists of all subsets of X and the probability measure $\mu = \mathcal{P}$ is defined by the measures of the one-point-sets: $\mu(\{x_i\}) = p_i$ $(i \in I)$ and if we choose for λ the counting measure, (1. 5) reduces to (1. 1). In a similar way, if X is the r-dimensional Euclidean space, $\mathcal{X}$ consisting of the Borel sets, and if the probability measure $\mu = \mathcal{P}$ is absolutely continuous with respect to the r-dimensional Lebesgue measure, playing the role of λ, (1. 5) reduces to (1. 2). At last, according to (1. 3) and (1. 5) we clearly have

$$I(\mu_1 : \mu_2) = -H_{\mu_2}(\mu_1). \tag{1. 6}$$

The term „generalized entropy" is due to PEREZ [6]. He has considered, however, only the case that λ is a probability measure. The quantity (1. 5), with an arbitrary σ-finite λ has been treated in several papers of ROSENBLATT—ROTH, see e.g. [10], [11]. [See also Ghurye, S. G. Information and sufficient sub-fields, Annals Math. Statist. **38** (1968), 2056—2066; added in proof.]

The aim of this paper is to give an account of some basic properties of generalized entropy (mostly well-known in the case that λ is a probability measure, see [3], [5], [6]) and to call attention to connections of generalized entropy with some other information-theoretic concepts, in particular, with the concept of dimensional entropy (see [8]). The main motivation has been to provide a background for investigations concerning the entropy of point processes, see the forthcoming paper of J. FRITZ [4].

Section 2 is centered around the generalization of a theorem of DOBRUŠIN [3].

Section 3 is devoted to the definition and basic properties of conditional generalized entropy.

In Section 4, the connections of the theories of generalized entropy and of dimensional entropy are pointed out. Some of the ideas of this section are due to J. FRITZ.

§ 2. Basic properties of generalized entropy

Let $(X, \mathcal{X})$ be a measurable space, let μ be a probability measure and λ a σ-finite measure on $\mathcal{X}$.

DEFINITION 1. If $\mathcal{Y}$ is a sub-σ-algebra of $\mathcal{X}$, the generalized entropy of μ with respect to λ, or briefly, the λ-entropy of μ on $\mathcal{Y}$ is defined as

$$H_\lambda(\mu, \mathcal{Y}) = -\int f_{\mathcal{Y}}(x) \log_2 f_{\mathcal{Y}}(x) \lambda(dx) \tag{2. 1}$$

if $\mu \ll \lambda$ on $\mathcal{Y}$, where $f_{\mathcal{Y}}(x)$ denotes the Radon—Nikodym derivative of μ with respect to λ with underlying σ-algebra $\mathcal{Y}$ (if the integral does not exist, $H_\lambda(\mu, \mathcal{Y})$ is not defined) and we set $H_\lambda(\mu, \mathcal{Y}) = -\infty$ if $\mu \not\ll \lambda$ on $\mathcal{Y}$.

If $\mathscr{Y}=\mathscr{X}$, we write $H_\lambda(\mu)$ and $f(x)$ instead of $H_\lambda(\mu, \mathscr{X})$ and $f_{\mathscr{X}}(x)$; for this case, definition 1 clearly reduces to (1. 5).

DEFINITION 2. If ξ is a random variable with values in $(X, \mathscr{X})$ (i.e. a measurable mapping of a probability space $(\Omega, \mathscr{F}, P)$ into $(X, \mathscr{X})$) and λ is a σ-finite measure on $\mathscr{X}$, the λ-entropy of ξ is defined as $H_\lambda(\xi)=H_\lambda(\mu_\xi)$ where μ_ξ denotes the distribution of ξ, i.e. the probability measure on $\mathscr{X}$ defined by

$$\mu_\xi(A)=P(\xi\in A)=P(\{\omega\colon \xi(\omega)\in A\}) \quad \text{if} \quad A\in\mathscr{X}. \tag{2. 2}$$

A finite or countably infinite class $\mathfrak{A}=\{A_i\}_{i\in I}$ of subsets of X is called a *partition* of $(X, \mathscr{X})$ if $A_i\in\mathscr{X}\,(i\in I)$, $A_i\cap A_j=\emptyset$ $(i\neq j)$ and $\bigcup_{i\in I} A_i=X$.

If μ is a probability measure and λ a σ-finite measure on $\mathscr{X}$, let us denote by $R=R_{\lambda,\mu}$ the class of all such partitions of $(X, \mathscr{X})$ for which $\lambda(A_i)<\infty$ $(i\in I)$ and the sum of the positive terms on the right hand side of

$$H_\lambda(\mu, \mathfrak{A})=\sum_{i\in I}\mu(A_i)\log_2\frac{\lambda(A_i)}{\mu(A_i)} \tag{2. 3}$$

is finite. Observe that if we denote by $\mathscr{A}$ the σ-algebra with atoms A_i $(i\in I)$ then

$$H_\lambda(\mu, \mathfrak{A})=H_\lambda(u, \mathscr{A}) \quad \text{if} \quad \mathfrak{A}\in R. \tag{2. 4}$$

We agree to understand

$$0\cdot\log_2\frac{a}{0}=0; \quad b\cdot\log_2\frac{0}{b}=-\infty \quad \text{if} \quad b>0. \tag{2. 5}$$

Then $H_\lambda(\mu, \mathfrak{A})$ is well defined by (2.3) for every $\mathfrak{A}\in R$ and we have $-\infty\leqq$ $\leqq H_\lambda(\mu, \mathfrak{A})<+\infty$.

If $\mathfrak{A}=\{A_i\}_{i\in I}$ and $\mathfrak{B}=\{B_j\}_{j\in J}$ are two partitions of $(X, \mathscr{X})$ such that to each $j\in J$ there exists $i\in I$ with $B_j\subset A_i$, we shall write $\mathfrak{A}\prec\mathfrak{B}$ or $\mathfrak{B}\succ\mathfrak{A}$.

LEMMA 1. *$\mathfrak{B}\succ\mathfrak{A}\in R$ implies $\mathfrak{B}\in R$ and*

$$H_\lambda(\mu, \mathfrak{B})\leqq H_\lambda(\mu, \mathfrak{A})<+\infty. \tag{2. 6}$$

PROOF. Substituting in the inequality

$$\log_2 u\leqq\log_2 u_0+b(u-u_0) \quad \left(b=\frac{\log_2 e}{u_0}\right) \tag{2. 7}$$

$$u=\frac{\lambda(B_j)}{\mu(B_j)}, \quad u_0=\frac{\lambda(A_i)}{\mu(A_i)},$$

multiplying both sides by $\mu(B_j)$[1] and summing for the indices $j\in J$ with $B_j\subset A_i$ we obtain, since $\sum_{j\colon B_j\subset A_i}\mu(B_j)=\mu(A_i)$, $\sum_{j\colon B_j\subset A_i}\lambda(A_i)=\lambda(A_i)$,

$$\sum_{j\colon B_j\subset A_i}\mu(B_j)\log_2\frac{\lambda(B_j)}{\mu(B_j)}\leqq\mu(A_i)\log_2\frac{\lambda(A_i)}{\mu(A_i)}. \tag{2. 8}$$

[1] Observe that — due to the conventions (2. 5) — the inequality obtained in this way is valid even if $\mu(B_j)$ or $\lambda(B_j)$ or both vanish.

Summing for all $i \in I$, the inequality (2. 8) obviously implies (2. 6).

LEMMA 2. *If $\mu \ll \lambda$ and $\mathfrak{A} \in R$, we have*

$$-\int f(x) \log_2 f(x) \lambda(dx) \leqq H_\lambda(\mu, \mathfrak{A}) \tag{2. 9}$$

where $f(x)$ denotes the Radon—Nikodym-derivative $\dfrac{\mu(dx)}{\lambda(dx)}$.

PROOF. Substituting in (2. 7) $u = \dfrac{1}{f(x)}$, $u_0 = \dfrac{\lambda(A_i)}{\mu(A_i)}$, multiplying by $f(x)$ and integrating on A_i we obtain, similarly as in the proof of lemma 1,

$$\int_{A_i} f(x) \log_2 \frac{1}{f(x)} \lambda(dx) \leqq \mu(A_i) \log_2 \frac{\lambda(A_i)}{\mu(A_i)}. \tag{2. 10}$$

Summing for all $i \in I$, the inequality (2. 10) gives rise to (2. 9).

Observe, that the class R is a directed set with respect to the relation $\prec$. Lemma 1 implies that the limit $\lim\limits_{R} H_\lambda(\mu, \mathfrak{A})$ exists and equals $\inf\limits_{\mathfrak{A} \in R} H_\lambda(\mu, \mathfrak{A})$.

THEOREM 1. *If $R = R_{\lambda,\mu}$ is non-void, we have*

$$\inf_{\mathfrak{A} \in R} H_\lambda(\mu, \mathfrak{A}) = H_\lambda(\mu). \tag{2. 11}$$

Similarly, if $\mathscr{Y}$ is a sub-σ-algebra of $\mathscr{X}$ and the subclass $R_{\mathscr{Y}}$ of R — consisting of those partitions $\mathfrak{A} = \{A_i\}_{i \in I}$ for which $A_i \in \mathscr{Y}$, $i \in I$ — is non-void, we have

$$\inf_{\mathfrak{A} \in R_{\mathscr{Y}}} H_\lambda(u, \mathfrak{A}) = H_\lambda(\mu, \mathscr{Y}). \tag{2. 12}$$

PROOF. Obviously, it is enough to prove the first statement. If μ is not absolutely continuous with respect to λ and $\mathfrak{A} = \{A_i\}_{i \in I}$ is an arbitrary partition belonging to R, for some $i \in I$ there exists $A \in \mathscr{X}$, $A \subset A_i$ with $\mu(A) > \lambda(A) = 0$. Replace in $\mathfrak{A}$ the set A_i by A and $A_i \setminus A$; for the partition $\mathfrak{B} \succ \mathfrak{A}$ obtained in this way we have $H_\lambda(\mu, \mathfrak{B}) = -\infty$. Thus we have to consider the case $\mu \ll \lambda$ only.

Lemma 2 implies

$$\inf_{\mathfrak{A} \in R} H_\lambda(\mu, \mathfrak{A}) \geqq -\int f(x) \log_2 f(x) \lambda(dx) = H_\lambda(\mu); \tag{2. 13}$$

in particular, the integral is well-defined and it is $< +\infty$. Furthermore, if $\varepsilon > 0$, let h_i, $i = 0, \pm 1, \pm 2, \ldots$ be positive numbers such that $h_i < h_{i+1}$, $\lim\limits_{i \to -\infty} h_i = 0$, $\lim\limits_{i \to +\infty} h_i = +\infty$ and

$$\log_2 h_{i+1} - \log_2 h_i < \varepsilon, \qquad i = 0, \pm 1, \pm 2, \ldots; \tag{2. 14}$$

consider the partition $\mathfrak{A} = \{A_i\}_{i \in I}$ where the index set I consists of the integers and of the further index α and

$$A_i = \{x : h_i \leqq f(x) < h_{i+1}\} \quad \text{if} \quad i = 0, \pm 1, \pm 2, \ldots \tag{2. 15}$$

$$A_\alpha = \{x : f(x) = 0 \quad \text{or} \quad f(x) = +\infty\}. \tag{2. 16}$$

Then, by the definition of A_i and of $f(x)$, we have $h_i\lambda(A_i) \leqq \mu(A_i) \leqq h_{i+1}\lambda(A_i)$ if $i \in I$, $i \neq \alpha$; thus, if $x \in A_i$, both $f(x)$ and $\frac{\mu(A_i)}{\lambda(A_i)}$ are in the interval $[h_i, h_{i+1})$ and therefore, using (2. 14)

$$\int_{A_i} f(x) \log_2 f(x) \lambda(dx) \leqq \int_{A_i} f(x) \left(\log_2 \frac{\mu(A_i)}{\lambda(A_i)} + \varepsilon \right) \lambda(dx) = \tag{2. 17}$$

$$= -\mu(A_i) \log_2 \frac{\lambda(A_i)}{\mu(A_i)} + \varepsilon \mu(A_i).$$

Of course, for $i = \alpha$

$$\int_{A_\alpha} f(x) \log_2 f(x) \lambda(dx) = -\mu(A_\alpha) \log_2 \frac{\lambda(A_\alpha)}{\mu(A_\alpha)} = 0. \tag{2. 18}$$

(2. 17) and (2. 18) give rise to

$$\int f(x) \log_2 f(x) \lambda(dx) \leqq -H_\lambda(\mu, \mathfrak{A}) + \varepsilon. \tag{2. 19}$$

As $\varepsilon > 0$ has been arbitrary, (2. 19) implies that the right hand side of (2. 13) is greater than or equal to the left hand side. The proof of Theorem 1 is complete.

REMARK. The above proof shows, too, that if $\mu \ll \lambda$ and $\int f(x) \log_2 f(x) \lambda(dx) > -\infty$ then R is non-void; thus, in this case, the condition of Theorem 1 that R is non-void may be omitted.

Let now $\mathscr{X}_0$ be an algebra of subsets of X generating the σ-algebra $\mathscr{X}$. Let $\mathfrak{A}_0 = \{A_i^0\}_{i \in I} \in R$ be a fixed partition with $A_i^0 \in \mathscr{X}_0$, $i \in I_0$. Let us denote by R_0 the class of partitions $\mathfrak{B} = \{B_j\}_{j \in J}$ of $(X, \mathscr{X})$ with the following properties:

(i) $\mathfrak{B} > \mathfrak{A}_0$; $B_j \in \mathscr{X}_0$ for each $j \in J$.

(ii) There is only a finite number of B_j's not equal to some A_i^0 $(i \in I_0)$.

THEOREM 2. *Let R^* be a class of partitions of X such that to every $\mathfrak{B} \in R_0$ there exist $\mathfrak{A} \in R^*$ with $\mathfrak{A} > \mathfrak{B}$. Then*

$$H_\lambda(\mu) = \inf_{\mathfrak{A} \in R^*} H_\lambda(\mu, \mathfrak{A}). \tag{2. 20}$$

COROLLARY. If $\mathfrak{A}_n$, $n = 1, 2, \ldots$ is an increasing sequence of partitions of $(X, \mathscr{X})$ (i.e., $\mathfrak{A}_{n+1} > \mathfrak{A}_n$, $n = 1, 2, \ldots$) which generates the σ-algebra $\mathscr{X}$ (i.e., the smallest σ-algebra containing all sets belonging to some $\mathfrak{A}_n$ is $\mathscr{X}$) and $\mathfrak{A}_1 \in R$ then

$$H_\lambda(\mu) = \lim_{n \to \infty} H_\lambda(\mu, \mathfrak{A}_n). \tag{2. 21}$$

PROOF. In view of Theorem 1 and lemma 1 it is enough to show that to every partition $\mathfrak{A}' > \mathfrak{A}_0$ there exists $\mathfrak{A} \in R^*$ with $H_\lambda(\mu, \mathfrak{A})$ arbitrarily close to $H_\lambda(\mu, \mathfrak{A}')$.

For the sake of definiteness assume that $H_\lambda(\mu, \mathfrak{A}')$ is finite; if $H_\lambda(\mu, \mathfrak{A}') = -\infty$, in the following proof only obvious modifications are needed.

Let $\mathfrak{A}' = \{A_{ik}\}_{i \in I, k \in I}$ where $\bigcup_{k \in I} A_{ik} = A_i^0$ $(i \in I_0)$.

Let ε be an arbitrary positive number and let I_0' be a finite subset of I_0 such that

$$\sum_{i \notin I_0'} \mu(A_i^0) \log_2 \frac{\lambda(A_i^0)}{\mu(A_i^0)} < \varepsilon \tag{2.22}$$

and

$$\sum_{i \in I_0'} \sum_{k \in I} \mu(A_{ik}) \log_2 \frac{\lambda(A_{ik})}{\mu(A_{ik})} < H_\lambda(\mu, \mathfrak{A}') + \varepsilon \tag{2.23}$$

(the existence of such I_0' follows from $\mathfrak{A}_0 \in R$ and the assumption $H_\lambda(\mu, \mathfrak{A}') > -\infty$).

As the algebra $\mathscr{X}_0$ generates the σ-algebra $\mathscr{X}$, for arbitrary $\varepsilon' > 0$, there exist $B_{ik} \in \mathscr{X}_0$, $B_{ik} \subset A_i^0$ such that[2]

$$\mu(A_{ik} \circ B_{ik}) < \varepsilon', \quad \lambda(A_{ik} \circ B_{ik}) < \varepsilon' \qquad (i \in I_0,\ k \in I); \tag{2.24}$$

here we made use of the condition $\lambda(A_{ik}) \leqq \lambda(A_i^0) < \infty$.

Let I' be a finite subset of I to be specified later; without any loss of generality we may and do assume $I' = \{1, \ldots, M-1\}$. Consider the sets

$$C_{ik} = B_{ik} \setminus \bigcup_{j=1}^{k-1} B_{ij} \quad (1 \leqq j \leqq M-1), \quad C_{iM} = A_i^0 \setminus \bigcup_{j=1}^{M-1} B_{ij}. \tag{2.25}$$

Then

$$A_{ik} \circ C_{ik} = (A_{ik} \setminus C_{ik}) \cup (C_{ik} \setminus A_{ik}) \subset (A_{ik} \setminus B_{ik}) \cup$$

$$\cup \left(\bigcup_{j=1}^{k-1} (A_{ik} \cap B_{ij}) \right) \cup (B_{ik} \setminus A_{ik}) \subset \bigcup_{j=1}^{k} (A_{ij} \circ B_{ij});$$

thus, using (2.24)

$$\mu(A_{ik} \circ C_{ik}) < M\varepsilon', \quad \lambda(A_{ik} \circ C_{ik}) < M\varepsilon' \qquad (k = 1, \ldots, M-1), \tag{2.26}$$

furthermore

$$\mu(C_{iM}) < \mu\left(A_i^0 \setminus \bigcup_{j=1}^{M-1} A_{ij} \right) + M^2\varepsilon', \quad \lambda(C_{iM}) < \lambda\left(A_i^0 \setminus \bigcup_{j=1}^{M-1} A_{ij} \right) + M^2\varepsilon'. \tag{2.27}$$

Let now $I' \subset I$ be chosen in such a way that for every $i \in I_0'$

$$\sum_{k \in I'} \mu(A_{ik}) \log_2 \frac{\lambda(A_{ik})}{\mu(A_{ik})} \geqq \sum_{k \in I} \mu(A_{ik}) \log_2 \frac{\lambda(A_{ik})}{\mu(A_{ik})} + \varepsilon'' \tag{2.28}$$

and

$$\mu\left(A_i^0 \setminus \bigcup_{j=1}^{M-1} A_{ij} \right) < \varepsilon'', \quad \lambda\left(A_i^0 \setminus \bigcup_{j}^{M-1} A_{ij} \right) < \varepsilon'' \tag{2.29}$$

where ε'' is a positive number to be specified later. According to (2.26), (2.27) and (2.28), if ε' and ε'' have been chosen small enough, we have

$$\sum_{k=1}^{M} \mu(C_{ik}) \log_2 \frac{\lambda(C_{ik})}{\mu(C_{ik})} \geqq \sum_{k \in I'} \mu(A_{ik}) \log_2 \frac{\lambda(A_{ik})}{\mu(A_{ik})} + \frac{\varepsilon}{N}, \tag{2.30}$$

[2] The symbol $\circ$ denotes symmetric difference: $A \circ B = (A \setminus B) \cup (B \setminus A)$.

where N denotes the size of I_0'; in particular, we may assume $\varepsilon'' < \frac{\varepsilon}{N}$. Then, according to (2. 22), (2. 23), (2. 28) and (2. 30), for the partition $\mathfrak{B} \in R_0$ consisting of the sets C_{ik} $(i \in I_0', k = 1, \ldots, M)$ and A_i^0 $(i \in I_0 \setminus I_0')$ there holds

$$H_\lambda(\mu, \mathfrak{B}) < H_\lambda(\mu, \mathfrak{A}') + 4\varepsilon \tag{2. 31}$$

(we made use of the fact that, in view of (2. 8), the condition (2. 22) implies $\sum_{i \in I_0'} \sum_{k \in I} \mu(A_{ik}) \log_2 \frac{\lambda(A_{ik})}{\mu(A_{ik})} < \varepsilon$, as well).

The proof of theorem 2 is complete.

To prove the corollary, we have to take as $\mathscr{X}_0$ the algebra of sets which are finite unions of sets belonging to some $\mathfrak{A}_n$ $(n = 1, 2, \ldots)$ to let $R^* = \{\mathfrak{A}_n\}$, and to refer to lemma 1.

Also the following generalization of the corollary of theorem 2 is valid:

THEOREM 3. *Let $\{\mathscr{Y}_\alpha\}_{\alpha \in A}$ be an increasing net of sub-σ-algebras of $\mathscr{X}$,* i.e. *let A be a directed set and let $\mathscr{Y}_{\alpha_1} \supset \mathscr{Y}_{\alpha_2}$ if $\alpha_1 \succ \alpha_2$. Let $\mathscr{Y}$ denote the smallest σ-algebra containing all $\mathscr{Y}_\alpha$'s. Then*

$$\lim_A H_\lambda(\mu, \mathscr{Y}_\alpha) = H_\lambda(\mu, \mathscr{Y}) \tag{2. 32}$$

provided that $R_{\mathscr{Y}_\alpha}$ is non-void for some $\alpha \in A$.

PROOF. Without any loss of generality, we may assume $\mathscr{Y} = \mathscr{X}$. Theorem 1 implies that the limit on the left hand side of (2. 32) exists and equals

$$\inf_{\alpha \in A} H_\lambda(\mu, \mathscr{Y}_\alpha) \geqq H_\lambda(\mu). \tag{2. 33}$$

Furthermore, $\mathscr{X}_0 = \bigcup_{\alpha \in A} \mathscr{Y}_\alpha$ is an algebra generating — by assumption — the σ-algebra $\mathscr{X}$. Let $\mathfrak{A}_0$ be a partition belonging to $R_{\mathscr{Y}_{\alpha_0}}$ for some $\alpha_0 \in A$ and let R^* be the class of all partitions $\mathfrak{A} \succ \mathfrak{A}_0$ which belong to $R_{\mathscr{Y}_\alpha}$ for some $\alpha \in A$. Then, by theorem 2,

$$H_\lambda(\mu) = \inf_{\mathfrak{A} \in R^*} H_\lambda(\mu, \mathfrak{A}) \tag{2. 34}$$

and as $H_\lambda(\mu, \mathscr{Y}_\alpha) \leqq H_\lambda(\mu, \mathfrak{A})$ if $\mathfrak{A} \in R_{\mathscr{Y}_\alpha}$, (2. 34) gives rise to

$$H_\lambda(\mu) \geqq \inf_{R_\alpha \neq \mathbb{O}} H_\lambda(\mu, \mathscr{Y}_\alpha) = \inf_{\alpha \in A} H_\lambda(\mu, \mathscr{Y}_\alpha). \tag{2. 35}$$

(2. 34) and (2. 35) imply $H_\lambda(\mu) = \inf_{\alpha \in A} H_\lambda(\mu, \mathscr{Y}_\alpha)$ completing the proof.

In view of (2. 4), theorem 3 is, indeed, a generalization of the corollary of Theorem 2. Let us emphasize that the condition that $R_{\mathscr{Y}_\alpha}$ is non-void for some $\alpha \in A$, as well as the corresponding condition in the corollary of theorem 2 that $\mathfrak{A}_1 \in R$ (or, at least, $\mathfrak{A}_n \in R$ for some n) cannot be omitted, in general. E.g., let X be the real line, $\mathscr{X}$ the σ-algebra of Borel-sets, and let $\mathfrak{A}_n$ be the partition of X to the diadic intervals $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$. Let λ be the Lebesgue measure and let μ be a discrete probabi-

lity distribution concentrated on the integers $k \geqq 2$, defined by $\mu(\{k\}) = \frac{c}{k \log^2 k}$, where c is a norming constant. Then $H_\lambda(\mu, \mathfrak{A}_n) = +\infty$ for each n, while $H_\lambda(\mu) = -\infty$. A similar counterexample may be constructed even with $\mu \ll \lambda$. In fact, let the probability distribution μ be defined by the density function $f(x) = \frac{c}{\log^2 k}$ if $k \leqq x \leqq k + \frac{1}{k}$ $(k = 2, 3, \ldots)$ and $f(x) = 0$ elsewhere, where c is the same norming constant as above. Then $H_\lambda(\mu, \mathfrak{A}_n) = +\infty$ for each n, while $H_\lambda(\mu) = -\int f(x) \log_2 f(x) dx$ is finite.

If λ is a probability measure (or, more generally, a finite measure), the trivial partition consisting of X alone always belongs to R. Thus, in this case, the condition of the corollary of theorem 2 as well as that of theorem 3 is automatically satisfied. Actually, for the case that λ is a probability measure, when $H_\lambda(\mu)$ reduces to the I-dinvergence $I(\mu : \lambda)$ (apart from a — sign) the statements of theorems 1—3 are well known. In this respect, the contribution of this paper consists only of the generalization to the σ-finite case. The proofs of theorems 1 and 2 presented here are adaptations of DOBRUŠIN's proofs [3], given originally for the particular case (iii a) (see § 1). Theorem 3, for the case that λ is a probability measure and $A = \{1, 2, \ldots\}$, appears in [6], with a proof based on martingale theorems. Of course, theorem 2 can be deduced from theorem 3.

We conclude this section by a property of λ-entropy which is well known and basic in the particular cases (i) and (iii) referred to in § 1 (cf. [13], [5]) and which provides an intuitive meaning of λ-entropy in the general case, as well.

THEOREM 4. *Let $(X^n, \mathcal{X}^n)$ be the Cartesian product of n exemplars of $(X, \mathcal{X})$ and let μ^n and λ^n be the corresponding product measures obtained from μ and λ, respectively. Let us denote by $\beta_n^* = \beta_n^*(\alpha)$ the infimum of $\lambda^n(A)$ for the sets $A \in \mathcal{X}^n$ with $\mu(A^n) \geqq \alpha$. Then, for every $0 < \alpha < 1$, we have*

$$\lim_{n \to \infty} \frac{1}{n} \log_2 \beta_n^* = H_\lambda(\mu) \tag{2.36}$$

provided that $H_\lambda(\mu)$ exists.

PROOF. If $\mu \not\ll \lambda$, there exists a set $B \in \mathcal{X}$ with $\lambda(B) = 0$, $\mu(B) = \varepsilon > 0$. Then, if n is so large that $(1-\varepsilon)^n < 1 - \alpha$, the set $A = X^n \setminus (X \setminus B)^n$ having λ^n-measure 0 satisfies $\mu^n(A) > \alpha$, hence $\beta_n^* = 0$; i.e., in this case, (2. 36) is trivially valid. If $\mu \ll \lambda$, $f(x) = \frac{\mu(dx)}{\lambda(dx)}$, we have $\mu^n \ll \lambda^n$ with Radon—Nikodym derivative

$$\frac{\mu^n(dx_1 \ldots dx_n)}{\lambda^n(dx_1 \ldots dx_n)} = \prod_{i=1}^n f(x_i). \tag{2.37}$$

Let

$$\eta_n = \eta_n(x_1, \ldots, x_n) = -\frac{1}{n} \sum_{i=1}^n \log_2 f(x_i) \tag{2.38}$$

and denote by $F_n(\varepsilon)$ the subset of X^n where $|\eta_n - H_\lambda(\mu)| < \varepsilon$ if $H_\lambda(\mu)$ is finite or

$\eta_n > \frac{1}{\varepsilon}$ $\left(\text{or } \eta_n < -\frac{1}{\varepsilon}\right)$ if $H_\lambda(\mu) = +\infty$ (or $H_\lambda(\mu) = -\infty$). Then the weak law of large numbers implies

$$\lim_{n\to\infty} \mu^n(F_n(\varepsilon)) = 1. \tag{2. 39}$$

On the other hand, on account of (2. 37) and (2. 38), we have

$$2^{n(H_\lambda(\mu)-\varepsilon)}\mu^n(F_n(\varepsilon)) \leqq \lambda^n(F_n(\varepsilon)) \leqq 2^{n(H_\lambda(\mu)+\varepsilon)}\mu^n(F_n(\varepsilon)) \tag{2. 40}$$

if $H_\lambda(\mu)$ is finite and

$$\lambda^n(F_n(\varepsilon)) \begin{cases} \geqq 2^{n\frac{1}{\varepsilon}} & \text{if} \quad H_\lambda(\mu) = +\infty \\ \leqq 2^{n-\frac{1}{\varepsilon}} & \text{if} \quad H_\lambda(\mu) = -\infty. \end{cases} \tag{2. 41}$$

From (2. 39), (2. 40) and (2. 41) — as $\varepsilon > 0$ has been arbitrary — (2. 36) readily follows.

§ 3. Conditional entropy

Our next aim is to give a proper definition of conditional generalized entropy, with respect to a σ-algebra $\mathscr{Y} \subset \mathscr{X}$.

Let $(X, \mathscr{X})$ be a measurable space, let $\mathscr{Y}$ be a sub-σ-algebra of $\mathscr{X}$, let μ be a probability measure and λ a σ-finite measure on $\mathscr{X}$, and let the restriction of μ and λ to $\mathscr{Y}$ be denoted by $\bar{\mu}$ and $\bar{\lambda}$, respectively. Observe, that $\bar{\lambda}$ may or may not be σ-finite on $\mathscr{Y}$. Let λ_1 be a σ-finite measure on $\mathscr{Y}$ such that

$$\bar{\lambda} \ll \lambda_1 \quad (\text{on } \mathscr{Y}). \tag{3. 1}$$

Let $\bar{\lambda}_A$ $(A \in \mathscr{X})$ denote the measure on $\mathscr{Y}$ defined by

$$\bar{\lambda}_A(B) = \lambda(A \cap B) \quad (B \in \mathscr{Y}). \tag{3. 2}$$

Then, in view of the assumption (3. 1), $\bar{\lambda}_A \ll \lambda_1$, thus one may speak of the ($\mathscr{Y}$-measurable) Radon—Nikodym derivative

$$g_A(x) = \frac{\bar{\lambda}_A(dx)}{\lambda_1(dx)}. \tag{3. 3}$$

If $\bar{\mu} \ll \lambda_1$, let us define

$$\lambda'(A) = \int g_A(x)\mu(dx) = \int g_A(x)\bar{\mu}(dx) \quad (A \in \mathscr{X}). \tag{3. 4}$$

As different versions of the Radon—Nikodym derivative $g_A(x)$ may differ only on a λ_1-null-set, under the condition $\bar{\mu} \ll \lambda_1$ the integral on the right hand side of (3. 4) is uniquely defined, and it is easy to see that λ' is a σ-finite measure on $\mathscr{X}$.

We shall write

$$f(x) = \frac{\mu(dx)}{\lambda(dx)} \quad \text{if} \quad \mu \ll \lambda \tag{3. 5}$$

$$\bar{f}(x) = \frac{\bar{\mu}(dx)}{\lambda_1(dx)} \quad \text{if} \quad \bar{\mu} \ll \lambda_1 \tag{3. 6}$$

(of course, the Radon—Nikodym derivative $\frac{\bar{\mu}(dx)}{\lambda_1(dx)}$ is understood with respect to $\mathscr{Y}$).

THEOREM 5. *The following assertions are valid:*
(i) $\bar{\mu} \ll \lambda_1$ *implies* $\lambda' \ll \lambda$; *in this case*[3]

$$\frac{\lambda'(dx)}{\lambda(dx)} = \bar{f}(x) \quad [\lambda]. \tag{3.7}$$

(ii) $\mu \ll \lambda$ *if and only if* $\bar{\mu} \ll \lambda_1$ *and* $\mu \ll \lambda'$; *in this case*

$$\frac{\mu(dx)}{\lambda'(dx)} = \frac{f(x)}{\bar{f}(x)} \quad [\lambda']. \tag{3.8}$$

(iii) *we have*

$$H_\lambda(\mu) = H_{\lambda_1}(\bar{\mu}) + H_{\lambda'}(\mu) \tag{3.9}$$

provided that the right hand side is defined. Here, of course, $H_{\lambda_1}(\bar{\mu})$ *means* $H_{\lambda_1}(\bar{\mu}, \mathscr{Y}) = H_{\lambda_1}(\mu, \mathscr{Y})$.

PROOF. (i) If $\bar{\mu} \ll \lambda_1$, we have, according to (3. 4), (3. 6) and (3. 3)

$$\lambda'(A) = \int g_A(x)\bar{\mu}(dx) = \int g_A(x)\bar{f}(x)\lambda_1(dx) = \int \bar{f}(x)\bar{\lambda}_A(dx) \tag{3.10}$$

for every $A \in \mathscr{X}$. In view of (3. 2), the last integral in (3. 10) is equal to $\int_A \bar{f}(x)\lambda(dx)$. Thus assertion (i) is proved.

(ii) $\mu \ll \lambda$ implies $\bar{\mu} \ll \bar{\lambda}$, thus, in view of (3. 1) $\bar{\mu} \ll \lambda_1$; moreover, in this case λ' is defined and, according to (i), $\lambda'(A) = 0$ $(A \in \mathscr{X})$ implies $\bar{f}(x) = 0$ $[\lambda]$ on A, i.e. $A \subset N_1 \cup N_2$ where $N_1 = \{x : \bar{f}(x) = 0\} \in \mathscr{Y}$ and $\lambda(N_2) = 0$. (3. 6) implies $\mu(N_1) = \bar{\mu}(N_1) = 0$ and $\mu \ll \lambda$ implies $\mu(N_2) = 0$. Thus, if $\lambda'(A) = 0$, we have $\mu(A) = 0$ as well, i.e., $\mu \ll \lambda'$.

The converse implication is trivial, as, by (i), $\bar{\mu} \ll \lambda_1$ implies $\lambda' \ll \lambda$.

(3. 8) follows from the "chain rule" $\frac{\mu(dx)}{\lambda(dx)} = \frac{\mu(dx)}{\lambda'(dx)} \cdot \frac{\lambda'(dx)}{\lambda(dx)}$ using (3. 5) and (3. 7).

(iii) The second term on the right hand side of (3. 9) is defined only if $\bar{\mu} \ll \lambda_1$. In this case, if $\mu \not\ll \lambda'$, we have $H_{\lambda'}(\mu) = -\infty$ and, by (ii), $\mu \not\ll \lambda$, thus $H_\lambda(\mu) = -\infty$ as well.

In the case $\bar{\mu} \ll \lambda_1$, $\mu \ll \lambda'$ we have

$$H_{\lambda_1}(\bar{\mu}) = -\int \bar{f}(x) \log_2 \bar{f}(x)\lambda_1(dx) = -\int \log_2 \bar{f}(x)\bar{\mu}(dx) = \tag{3.11}$$

$$= -\int \log_2 \bar{f}(x)\mu(dx) = -\int f(x) \log_2 \bar{f}(x)\lambda(dx)$$

and, in view of (3. 8) and (3. 7)

$$H_{\lambda'}(\mu) = -\int \frac{f(x)}{\bar{f}(x)} \log_2 \frac{f(x)}{\bar{f}(x)} \lambda'(dx) = -\int f(x) \log_2 \frac{f(x)}{\bar{f}(x)} \lambda(dx) \tag{3.12}$$

[3] The symbol $[\lambda]$ means as usual: almost everywhere with respect to λ.

provided that the integrals defining $H_{\lambda_1}(\mu)$ and $H_{\lambda'}(\mu)$ do exist. Thus, if the sum $H_{\lambda_1}(\mu)+H_{\lambda'}(\mu)$ is not of the form $+\infty-\infty$, (3. 11) and (3. 12) gives rise to

$$H_{\lambda_1}(\bar{\mu})+H_{\lambda'}(\mu) = -\int f(x)\log_2 f(x)\lambda(dx) = H_\lambda(\mu). \tag{3. 13}$$

The proof of Theorem 5 is complete.

Equation (3. 9) suggests to interprete $H_{\lambda'}(\mu)$ as conditional entropy with respect to the σ-algebra $\mathscr{Y}\subset\mathscr{X}$.

DEFINITION 3. If λ is a σ-finite measure on $\mathscr{X}$ and λ_1 is a σ-finite measure on $\mathscr{Y}\subset\mathscr{X}$ with $\bar{\lambda}\ll\lambda_1$ (where the bar denotes restriction to $\mathscr{Y}$), the conditional $\lambda|\lambda_1$-entropy of a probability distribution μ on $\mathscr{X}$ with respect to $\mathscr{Y}$ is defined as

$$H_{\lambda|\lambda_1}(\mu|\mathscr{Y}) = H_{\lambda'}(\mu) \tag{3. 14}$$

provided that $\bar{\mu}\ll\lambda_1$ and $H_{\lambda'}(\mu)$ exists. Here the σ-finite measure λ' on $\mathscr{X}$ is defined by (3. 4) and (3. 3).

According to (i) and (ii) of Theorem 5, we have

$$H_{\lambda|\lambda_1}(\mu|\mathscr{Y}) = -\int f(x)\log_2\frac{f(x)}{\bar{f}(x)}\lambda(dx) \quad \text{if} \quad \mu\ll\lambda \tag{3. 15}$$

(if the integral does not exist, $H_{\lambda|\lambda_1}(\mu|\mathscr{Y})$ is not defined) and

$$H_{\lambda|\lambda_1}(\mu|\mathscr{Y}) = -\infty \quad \text{if} \quad \bar{\mu}\ll\lambda_1,\ \mu\not\ll\lambda \tag{3. 16}$$

(if $\bar{\mu}\not\ll\lambda_1$, $H_{\lambda|\lambda_1}(\mu|\mathscr{Y})$ is not defined).

With the notation (3. 14) the identity (3. 9) becomes

$$H_\lambda(\mu) = H_{\lambda_1}(\mu,\mathscr{Y})+H_{\lambda|\lambda_1}(\mu|\mathscr{Y}). \tag{3. 17}$$

REMARK. The measure λ' defined by (3. 4) plays the same role as the measure μ_{12} in [1] and [2]. Analogously to the definition of μ_{12} in [2], one could define λ' for the case $\bar{\mu}\not\ll\lambda_1$ as well, thus one could speak of $H_{\lambda|\lambda_1}(\mu|\mathscr{Y})$ even if $\bar{\mu}\not\ll\lambda_1$; from the point of view of identity (3. 17) the way how $H_{\lambda|\lambda_1}(\mu|\mathscr{Y})$ is defined for $\bar{\mu}\not\ll\lambda_1$ is irrelevant as in this case $H_\lambda(\mu) = H_{\lambda_1}(\mu) = -\infty$.

An important special case of Theorem 5 arises if $\mathscr{Y}_1$ and $\mathscr{Y}_2$ are qualitatively independent[4] sub-σ-algebras of $\mathscr{X}$, such that the smallest σ-algebra containing both $\mathscr{Y}_1$ and $\mathscr{Y}_2$ equals $\mathscr{X}$, and the measure λ has the property

$$\lambda(B_1\cap B_2)=\lambda_1(B_1)\lambda_2(B_2) \qquad (B_1\in\mathscr{Y}_1,\ B_2\in\mathscr{Y}_2) \tag{3. 18}$$

where λ_1 and λ_2 are (σ-finite) measures on $\mathscr{Y}_1$ and $\mathscr{Y}_2$ respectively. As the class of finite disjoint unions of sets $B_1\cap B_2$ $(B_1\in\mathscr{Y}_1, B_2\in\mathscr{Y}_2)$ is an algebra generating the σ-algebra $\mathscr{X}$, λ is uniquely determined by λ_1 and λ_2. The point worth emphasizing is that, in this case, the conditional $\lambda|\lambda_1$-entropy $H_{\lambda|\lambda_1}(\mu|\mathscr{Y}_1)=H_\lambda(\mu)$ depends only on λ_2 and not on λ_1. In fact, if $A = B_1\cap B_2$, $B_1\in\mathscr{Y}_1$, $B_2\in\mathscr{Y}_2$, we have, in view of (3. 18), $g_A(x)=\lambda_2(B_2)$ if $x\in B_1$ and $g_A(x)=0$ elsewhere, thus, by (3. 4),

$$\lambda'(A)=\mu(B_1)\lambda_2(B_2). \tag{3. 19}$$

[4] I.e., $B_1\in\mathscr{Y}_1$, $B_2\in\mathscr{Y}_2$, $B_1\neq\emptyset$, $B_2\neq\emptyset$ implies $B_1\cap B_2\neq\emptyset$.

Using the qualitative independence of $\mathcal{Y}_1$ and $\mathcal{Y}_2$, cf. Footnote[4], it is easy to check that (3. 19) gives rise to a uniquely defined σ-finite measure on the algebra of finite disjoint unions of sets $B_1 \cap B_2$ $(B_1 \in \mathcal{Y}_1, B_2 \in \mathcal{Y}_2)$ thus the measure λ' is uniquely defined on $\mathcal{X}$, as well, without any reference to λ_1 or λ.

DEFINITION 4. Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be qualitatively independent sub-σ-algebras of $\mathcal{X}$ such that the smallest σ-algebra containing both $\mathcal{Y}_1$ and $\mathcal{Y}_2$ equals $\mathcal{X}$. Let μ be a probability measure on $\mathcal{X}$ and λ_2 a σ-finite measure on $\mathcal{Y}_2$. Then

$$H_{\lambda_2}(\mu|\mathcal{Y}_1) = H_{\lambda'}(\mu) \tag{3. 20}$$

will be called the conditional λ_2-entropy of μ, with respect to the σ-algebra $\mathcal{Y}_1$.

With this notation, the identity (3. 9) becomes

$$H_\lambda(\mu) = H_{\lambda_1}(\mu, \mathcal{Y}_1) + H_{\lambda_2}(\mu|\mathcal{Y}_1). \tag{3. 21}$$

Let now ξ_1 and ξ_2 be two random variables with values in the measurable spaces $(X_1, \mathcal{X}_1)$ and $(X_2, \mathcal{X}_2)$, respectively. Let $(X, \mathcal{X})$ be the product space $(X, \mathcal{X}) = (X_1, \mathcal{X}_1) \times (X_2, \mathcal{X}_2)$ and let the joint distribution of ξ_1 and ξ_2 be defined, in the analogy of (2. 2), as the probability measure $\mu_{\xi_1\xi_2}$ on $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ defined by

$$\mu_{\xi_1\xi_2}(A) = P((\xi_1, \xi_2) \in A) \quad \text{if} \quad A \in \mathcal{X}_1 \times \mathcal{X}_2. \tag{3. 22}$$

DEFINITION 5. Let λ_2 be a σ-finite measure on $\mathcal{X}_2$. The conditional λ_2-entropy of ξ_2 given ξ_1 is defined as

$$H_{\lambda_2}(\xi_2|\xi_1) = H_{\mu_{\xi_1} \times \lambda_2}(\mu_{\xi_1\xi_2}) \tag{3. 23}$$

provided that the right hand side exists.

Observe that the conditional λ_2-entropy defined above is the same as the conditional λ_2-entropy of the probability measure $\mu_{\xi_1\xi_2}$ with respect to the σ-algebra $\mathcal{Y}_1 = \{B_1 : B_1 = A_1 \times X_2, A_1 \in \mathcal{X}_1\}$ in the sense of definition 4 (if we interprete λ_2 as a measure on $\mathcal{Y}_2 = \{B_2 : B_2 = X_1 \times A_2, A_2 \in \mathcal{X}_2\}$ rather than on $\mathcal{X}_2$). In fact, if $B_1 = A_1 \times X_2 \in \mathcal{Y}_1$, we have $\mu_{\xi_1\xi_2}(B_1) = \mu_{\xi_1}(A_1)$, thus (3. 19) means $\lambda'(A) = \mu_{\xi_1}(A_1)\lambda_2(A_2)$ if $A = A_1 \times A_2 \in \mathcal{X}$ and this amounts to $\lambda' = \mu_{\xi_1} \times \lambda_2$.

If $\mu_{\xi_1\xi_2} \ll \mu_{\xi_1} \times \lambda_2$, let us denote by $f(x_2|x_1)$ the Radon—Nikodym derivative of $\mu_{\xi_1\xi_2}$ with respect to $\mu_{\xi_1} \times \lambda_2$. Then $f(x_2|x_1)$ is the conditional density function of ξ_2 with respect to λ_2, in the sense that for every $A_2 \in \mathcal{X}_2$ we have

$$\int_{A_2} f(x_2|x_1)\lambda_2(dx_2) = P(\xi_2 \in A_2|\xi_1 = x_1) \quad [\mu_{\xi_1}]. \tag{3. 24}$$

In fact, the left hand side of (3. 24) is $\mathcal{X}_1$-measurable, and its integral over a set $A_1 \in \mathcal{X}_1$ is equal to the integral of $f(x_2|x_1)$ over $A_1 \times A_2$ with respect to $\mu_{\xi_1} \times \lambda_2$, i.e. — by the definition of $f(x_2|x_1)$ — to $\mu_{\xi_1\xi_2}(A_1 \times A_2) = P(\xi_1 \in A_1, \xi_2 \in A_2)$.

In the case $\mu_{\xi_1\xi_2} \ll \mu_{\xi_1} \times \lambda_2$, definition 5 reduces to

$$H_{\lambda_2}(\xi_2|\xi_1) = -\int\int f(x_2|x_1) \log_2 f(x_2|x_1) \lambda_2(dx_2)\mu_{\xi_1}(dx_1), \tag{3. 25}$$

providing an intuitive interpretation of conditional λ_2-entropy; in fact, the quantity

$$H_{\lambda_2}(\xi_2|\xi_1 = x_1) = -\int f(x_2|x_1) \log_2 f(x_2|x_1) \lambda_2(dx_2) \tag{3. 26}$$

may be interpreted as the λ_2-entropy of ξ_2 under the condition $\xi_1 = x_1$ and then (3. 25) becomes

$$H_{\lambda_2}(\xi_2|\xi_1) = \int H_{\lambda_2}(\xi_2|\xi_1 = x_1)\mu_{\xi_1}(dx_1). \tag{3. 27}$$

In particular, if $X_2 = \{x_i\}_{i\in I}$ is a finite or countable set, $\mathscr{X}_2$ is the σ-algebra of all subsets of X_2 and λ_2 is the counting measure, (3. 26) reduces to

$$H(\xi_2|\xi_1 = x_1) = -\sum_{i\in I} P(\xi_2 = x_i|\xi_1 = x_1)\log_2 P(\xi_2 = x_i|\xi_1 = x_1) \tag{3. 28}$$

and (3. 27) reduces to the familiar definition of $H(\xi_2|\xi_1)$.

The concept of conditional λ-entropy of a random variable with respect to another one is of course by no means new; for the case that the conditional distribution of ξ_2 given $\xi_1 = x_1$ is absolutely continuous with respect to λ_2, with density $f(x_2|x_1)$, the formulas (3. 26) and (3. 27) are familiar, see e. g. [10], [11]. In this respect, the contribution of this paper consists in deriving these formulas from a more general definition. The following theorem shows that conditional λ-entropy can play, in many respects, the same role that usual conditional entropy does in the discrete case. Without any claim of novelty, for the sake of completeness, we shall give the simple proof, as well.

THEOREM 6. *If λ_1 and λ_2 are arbitrary σ-finite measures on $(X_1, \mathscr{X}_1)$ and $(X_2, \mathscr{X}_2)$, respectively, we have*

$$H_{\lambda_1\times\lambda_2}(\xi_1, \xi_2) = H_{\lambda_1}(\xi_1) + H_{\lambda_2}(\xi_2|\xi_1) \tag{3. 29}$$

$$I(\xi_1, \xi_2) = H_{\lambda_2}(\xi_2) - H_{\lambda_2}(\xi_2|\xi_1) \tag{3. 30}$$

$$I(\xi_1, \xi_2) = H_{\lambda_1}(\xi_1) + H_{\lambda_2}(\xi_2) - H_{\lambda_1\times\lambda_2}(\xi_1, \xi_2) \tag{3. 31}$$

provided, in each case, that the right hand side exists. There hold, too, the inequalities

$$H_{\lambda_2}(\xi_2|\xi_1) \leqq H_{\lambda_2}(\xi_2) \tag{3. 32}$$

$$H_{\lambda_1\times\lambda_2}(\xi_1, \xi_2) \leqq H_{\lambda_1}(\xi_1) + H_{\lambda_2}(\xi_2) \tag{3. 33}$$

provided that both sides exist, with equalities if and only if ξ_1 and ξ_2 are independent or if both sides are infinite.

PROOF. Equation (3. 29) is a particular case of (3. 21), namely that of $\mu = \mu_{\xi_1\xi_2}$, $\lambda = \lambda_1\times\lambda_2$, $\mathscr{Y}_1 = \{B_1 : B_1 = A_1\times X_2,\ A_1\in\mathscr{X}_1\}$ (cf. definitions 2 and 5).

To verify (3. 30) observe that if $\mu_{\xi_1\xi_2} \not\ll \mu_{\xi_1}\times\lambda_2$ then in the case $\mu_{\xi_2} \not\ll \lambda_2$ the right hand side of (3. 30) is of the form $+\infty - \infty$, while in the case $\mu_{\xi_2} \ll \lambda_2$ we have $\mu_{\xi_1\xi_2} \not\ll \mu_{\xi_1}\times\mu_{\xi_2}$ as well, thus $I(\xi_1, \xi_2) = +\infty$, and (3. 30) holds if the right hand side is defined. Therefore one may restrict attention to the case $\mu_{\xi_1\xi_2} \ll \mu_{\xi_1}\times\lambda_2$. In this case, of course, $\mu_{\xi_2} \ll \lambda_2$, too. Denoting the corresponding Radon—Nikodym derivatives by $f(x_2|x_1)$ and $f_2(x_2)$, respectively, observe that $f_2(x_2)$ is the Radon—Nikodym derivative of $\mu_{\xi_1}\times\mu_{\xi_2}$ with respect to $\mu_{\xi_1}\times\lambda_2$, as well. Hence, as by the definition of $f_2(x_2)$ we have

$$\mu_{\xi_1\xi_2}(\{(x_1, x_2) : f_2(x_2) = 0\}) = \mu_{\xi_2}(\{x_2 : f_2(x_2) = 0\}) = 0, \tag{3. 34}$$

thus $f(x_2|x_1)>0$ implies $f_2(x_2)>0$ $[\mu_{\xi_1}\times\lambda_2]$, one concludes that $\mu_{\xi_1\xi_2}$ is absolutely continuous with respect to $\mu_{\xi_1}\times\mu_{\xi_2}$ with Radon—Nikodym derivative $a(x_1, x_2)=\frac{f(x_2|x_1)}{f_2(x_2)}$. In view of (1. 4), definition 2 and (3. 25), we arrive at

$$I(\xi_1, \xi_2) = \iint \log_2 a(x_1, x_2)\mu_{\xi_1\xi_2}(dx_1, dx_2) = \tag{3. 35}$$

$$= \iint \frac{f(x_2|x_1)}{f_2(x_2)} \log_2 f(x_2|x_1)\mu_{\xi_1}(dx_1)\mu_{\xi_2}(dx_2) - \iint \log_2 f_2(x_2)\mu_{\xi_1\xi_2}(dx_1, dx_2) =$$

$$= \iint f(x_2|x_1)\log_2 f(x_2|x_1)\mu_{\xi_1}(dx_1)\lambda_2(dx_2) - \int \log_2 f_2(x_2)\mu_{\xi_2}(dx_2) =$$

$$= -H_{\lambda_2}(\xi_1|\xi_2) + H_{\lambda_2}(\xi_2)$$

provided that the right hand side exists. The identity (3. 31) follows from (3. 29) and (3. 30). The inequalities (3. 32) and (3. 33) are immediate consequences of (3. 30) and (3. 31), respectively.

The conditional λ-entropy of a random variable with respect to another has been a particular case of conditional $\lambda|\lambda_1$-entropy in the sense of definition 3. Another important particular case of definition 3 arises if λ is a probability measure and one takes $\lambda_1=\bar{\lambda}$ (the bar denoting restriction to the σ-algebra $\mathscr{Y}\subset\mathscr{X}$).

In this case $H_{\lambda_1}(\bar{\mu})$ reduces to $H_\lambda(\mu, \mathscr{Y}) = -I(\mu:\lambda, \mathscr{Y})$ and g_A will be the conditional probability of $A\in\mathscr{X}$ with respect to $\mathscr{Y}$ (with underlying probability measure λ). Substituting in (3. 4) this g_A, we can define the conditional I-divergence (with respect to $\mathscr{Y}$) of μ and λ as

$$I(\mu:\lambda|\mathscr{Y}) = I(\mu:\lambda') = -H_{\lambda'}(\mu), \tag{3. 36}$$

provided that λ' is defined, cf. (3. 4), i. e. that $\bar{\mu}\ll\bar{\lambda}$ (as to the possibility of defining $I(\mu:\lambda|\mathscr{Y})$ also for $\bar{\mu}\not\ll\bar{\lambda}$ cf. the remark to theorem 5.)

According to (3. 14), (3. 15) and (3. 16), the conditional I-divergence (3. 36) equals

$$I(\mu:\lambda|\mathscr{Y}) = \begin{cases} \int f(x)\log_2\frac{f(x)}{\bar{f}(x)}\lambda(dx) & \text{if} \quad \mu\ll\lambda, \\ +\infty & \text{if} \quad \mu\not\ll\lambda,\ \bar{\mu}\ll\bar{\lambda}. \end{cases} \tag{3. 37}$$

On account of (3. 9) and (3. 36) we have

$$I(\mu:\lambda) = I(\mu:\lambda, \mathscr{Y}) + I(\mu:\lambda|\mathscr{Y}). \tag{3. 38}$$

The identity $I(\mu:\lambda) = I(\mu:\lambda, \mathscr{Y}) + I(\mu:\lambda')$ is, of course, well known; it has been used, e.g., to the study of statistical data reduction problems, see [7]. The interpretation of the second term on the right hand side as conditional I-divergence, however, seems to not have been recognised.

A particular case of conditional I-divergence is, however, well known as conditional mutual information. To arrive at this concept, consider three random variables ξ_1, ξ_2, ξ_3 with values in the measurable spaces $(X_1, \mathscr{X}_1)$, $(X_2, \mathscr{X}_2)$ and $(X_3, \mathscr{X}_3)$ respectively. Let $(X, \mathscr{X})$ be the product space $\bigtimes_{i=1}^{3}(X_i, \mathscr{X}_i)$; let $\mu_{\xi_i}, \mu_{\xi_i\xi_j}$

$(i \neq j)$ and $\mu_{\xi_1\xi_2\xi_3}$ denote the distribution of ξ_i, the joint distribution of ξ_i and ξ_j and the joint distribution of ξ_1, ξ_2 and ξ_3, respectively. Then the mutual information of (ξ_1, ξ_2) and ξ_3 is defined, according to (1. 4), as

$$I((\xi_1, \xi_2), \xi_3) = I(\mu_{\xi_1\xi_2\xi_3} : \mu_{\xi_1\xi_2} \times \mu_{\xi_3}). \tag{3. 39}$$

Let us choose as $\mathscr{Y}$ the copy of $\mathscr{X}_1 \times \mathscr{X}_3$ embedded into $\mathscr{X}$, i.e. $\mathscr{Y} = \{A_1 \times X_2 \times A_3 = A_1 \in \mathscr{X}_1,\ A_3 \in \mathscr{X}_3\}$ and let $\mu_{\xi_1\xi_2\xi_3}$ and $\mu_{\xi_1\xi_2} \times \mu_{\xi_3}$ play the role of μ and λ, respectively. Then the auxiliary measure λ' is defined — in accordance with (3. 4) — by

$$\lambda'(A) = \int g_A(x_1, x_3)\mu_{\xi_1\xi_3}(dx_1, dx_3) \tag{3. 40}$$

where $g_A(x_1, x_3)$ is the conditional probability of $A \in \mathscr{X}$ with respect to $\mathscr{Y}$, with underlying probability measure $\mu_{\xi_1\xi_2} \times \mu_{\xi_3}$. It is easy to check that for $A = A_1 \times \times A_2 \times A_3$ $(A_i \in \mathscr{X}_i,\ i = 1, 2, 3)$ we have

$$g_A(x_1, x_3) = P(\xi_2 \in A_2 | \xi_1 = x_1)\chi_{A_1}(x_1)\chi_{A_3}(x_3) \tag{3. 41}$$

where χ_{A_i} denotes the indicator function of the set A_i. Thus

$$\lambda'(A_1 \times A_2 \times A_3) = \int\limits_{A_1 \times A_3} P(\xi_2 \in A_2 | \xi_1 = x_1)\mu_{\xi_1\xi_3}(dx_1, dx_3) = \tag{3. 42}$$

$$= \int\limits_{A_1} P(\xi_2 \in A_2 | \xi_1 = x_1)P(\xi_3 \in A_3 | \xi_1 = x_1)\mu_{\xi_1}(dx_1).$$

The average conditional information of ξ_2 and ξ_3 given ξ_1 is defined as

$$I(\xi_2, \xi_3 | \xi_1) = I(\mu_{\xi_1\xi_2\xi_3} : \mu_{\xi_1\xi_2} \times \mu_{\xi_3} | \mathscr{Y}) = I(\mu_{\xi_1\xi_2\xi_3} : \lambda') \tag{3. 43}$$

where the probability measure λ' is uniequely determined by (3. 42).

As the restrictions of $\mu_{\xi_1\xi_2\xi_3}$ and $\mu_{\xi_1\xi_2} \times \mu_{\xi_3}$ to $\mathscr{Y}$ are $\mu_{\xi_1\xi_3}$ and $\mu_{\xi_1} \times \mu_{\xi_3}$, respectively, (3. 38) reduces to KOLMOGOROV's identity

$$I((\xi_1, \xi_2), \xi_3) = I(\xi_1, \xi_3) + I(\xi_2, \xi_3 | \xi_1) \tag{3. 44}$$

(cf. [3]).

§ 4. Generalized entropy and dimensional entropy

The concept of information-theoretic dimension and dimensional entropy has been introduced by A. RÉNYI [8]. If ξ is a random variable taking values in the r-dimensional Euclidean space E^r, let us approximate the distribution of ξ by a sequence of discrete distributions $\mathscr{P}_n = \{p_{nk_1,\ldots,k_r}\}$ $(n = 1, 2, \ldots;\ k_i = 0, \pm 1, \ldots, i = 1, \ldots, r)$ where

$$p_{nk_1\ldots k_r} = P\left(\frac{k_i}{n} \leqq \xi^i < \frac{k_i + 1}{n},\ i = 1, \ldots, r\right); \tag{4. 1}$$

here ξ^i stands for the i'th coordinate of ξ.

The information-theoretic dimension of ξ is defined as

$$\lim_{n\to\infty} \frac{H(\mathscr{P}_n)}{\log_2 n} = d \tag{4.2}$$

(provided that $H(\mathscr{P}_1) < +\infty$ and the limit (4. 2) exists) where

$$H(\mathscr{P}_n) = -\sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_r=-\infty}^{\infty} p_{nk_1\ldots k_r} \log_2 p_{nk_1\ldots k_r}; \tag{4.3}$$

if also the limit

$$\lim_{n\to\infty} H(\mathscr{P}_n) - d\log_2 n = H_d(\xi) \tag{4.4}$$

exists, it is called the d-dimensional entropy of ξ.

RÉNYI has shown that the information-theoretic dimension of a random variable taking values in E^r and having an absolutely continuous distribution equals the geometric dimension r of the space (provided that $H(\mathscr{P}_1) < +\infty$) and the r-dimensional entropy of ξ is

$$H_r(\xi) = -\int f(\mathbf{x}) \log_2 f(\mathbf{x})\, d\mathbf{x} \tag{4.5}$$

where $f(\mathbf{x})$ is the density function of ξ (here $\mathbf{x}$ and $d\mathbf{x}$ stand for $(x_1, \ldots, x_r)$ and $dx_1 \ldots dx_r$, respectively). This means that $H_r(\xi)$ is equal to the λ-entropy of ξ, in the sense of definition 2, if we choose λ to be the r-dimensional Lebesgue measure.

Also the dimension and dimensional entropy of certain stochastic processes have been defined, see [12], [4]. In this section we attempt to indicate how the theory of dimensional entropy can be incorporated into the theory of generalized entropy.

Let $(X, \mathscr{X})$ be a measurable space and λ a σ-finite measure on $(X, \mathscr{X})$. If μ is a probability measure on $(X, \mathscr{X})$ and $\mathfrak{A} = \{A_i\}_{i\in I}$ is a partition of $(X, \mathscr{X})$, let $H(\mu, \mathfrak{A})$ denote the entropy of μ on $\mathfrak{A}$, i.e.

$$H(\mu, \mathfrak{A}) = -\sum_{i\in I} \mu(A_i) \log_2 \mu(A_i). \tag{4.6}$$

Furthermore, let us write

$$d(\mu, \mathfrak{A}) = \sum_{i\in I} \mu(A_i) \log_2 \lambda(A_i) \tag{4.7}$$

provided that the right hand side exists.

Then, if $H(\mu, \mathfrak{A}) < +\infty$ and $d(\mu, \mathfrak{A})$ exists, we have, cf. (2. 3)

$$H_\lambda(\mu, \mathfrak{A}) = H(\mu, \mathfrak{A}) + d(\mu, \mathfrak{A}). \tag{4.8}$$

DEFINITION 6. Let $\mathfrak{A}_n$, $n = 1, 2, \ldots$ be a sequence of partitions $\mathfrak{A}_n = \{A_i^n\}$ of $(X, \mathscr{X})$ and let $\mathscr{M}$ be a class of probability distributions on $(X, \mathscr{X})$. A real-valued function $d(n)$ will be called a dimension function with respect to the sequence of partitions $\mathfrak{A}_n$ and to the class of distributions $\mathscr{M}$ if

$$d(\mu, \mathfrak{A}_n) - d(n) \to 0 \quad \text{as} \quad n\to\infty \quad \text{for every} \quad \mu\in\mathscr{M}. \tag{4.9}$$

THEOREM 7. *Let $\mathfrak{A}_n$, $n=1, 2, \ldots$ be an increasing sequence of partitions of $(X, \mathscr{X})$ and let $d(n)$ be a dimension function with respect to the sequence $\{\mathfrak{A}_n\}$ and to a class $\mathscr{M}$ of probability distributions on $(X, \mathscr{X})$. Let $\mathscr{Y}$ be the smallest σ-algebra containing all the sets A_i^n $(n=1, 2, \ldots;\ i \in I)$. Then*

$$\lim_{n\to\infty} \left(H(\mu, \mathfrak{A}_n) + d(n)\right) = H_\lambda(\mu, \mathscr{Y}) \tag{4. 10}$$

provided that $H(\mu, \mathfrak{A}_n) < +\infty$ for some n for which $d(\mu, \mathfrak{A}_n)$ exists, and is $< +\infty$.

PROOF. The statement of Theorem 7 is an obvious consequence of (4. 8), (4. 9) and the corollary of theorem 2 (with $\mathscr{Y}$ playing the role of $\mathscr{X}$).

An important particular case is when the partitions $\mathfrak{A}_n = \{A_i^n\}_{i\in I}$ consist of sets of equal λ-measure

$$\lambda(A_i^n) = c(n) \quad (i \in I). \tag{4. 11}$$

In this case

$$d(\mu, \mathfrak{A}_n) = \log_2 c(n) \qquad n=1, 2, \ldots \tag{4. 12}$$

thus $d(n) = \log_2 c(n)$ is a dimension function with respect to the class of all probability distributions on $(X, \mathscr{X})$. E.g., let $X = E^r$, $\mathscr{X}$ the σ-algebra of Borel sets, λ the Lebesgue measure and let $\mathfrak{A}_n$ consist of the sets

$$A^n_{k_1,\ldots,k_r} = \left\{(x_1, \ldots, x_r):\ \frac{k_i}{2^n} \leqq x_i < \frac{k_i+1}{2^n},\ i = 1, \ldots, r\right\};$$

then $c(n) = 2^{-rn}$, $d(n) = -rn$ and (4. 10) gives rise to

$$\lim_{n\to\infty} \left(H(\mu, \mathfrak{A}_n) - rn\right) = H_\lambda(\mu) \tag{4. 13}$$

provided that $H(\mu, \mathfrak{A}_1) < +\infty$.

In particular, if ξ is a random variable with values in E^r and $\mu = \mu_\xi$ is the distribution of ξ, with the notation (4. 1) we arrive at

$$\lim_{n\to\infty} \left(H(\mathscr{P}_{2^n}) - r \log_2 2^n\right) = H_\lambda(\xi). \tag{4. 14}$$

Observe that (4. 14) is a bit weaker result than RÉNYI's theorem (4. 5), as (4. 14) asserts the limit-relation (4. 4) only for a subsequence of the positive integers, namely for those of form 2^n. To obtain (4. 5), a simple additional argument is needed.

LEMMA 3. *If n is a positive integer and $\varepsilon > 0$, there exists $m_0 = m_0(n, \varepsilon)$ with the property that for every positive integer $m \geqq m_0$*

$$H(\mathscr{P}_m) - r \log_2 m \leqq H(\mathscr{P}_n) - r \log_2 n + \varepsilon. \tag{4. 15}$$

PROOF. Let the random variables ξ_n be defined by

$$\xi_n = (k_1, \ldots, k_n) \quad \text{if} \quad \frac{k_i}{n} \leqq \xi^i < \frac{k_i+1}{n}, \qquad i = 1, \ldots, r \tag{4. 16}$$

where ξ^i stands for the i'th coordinate of ξ. Then, according to (4. 1),

$$H(\mathscr{P}_n) = H(\xi_n). \tag{4. 17}$$

Under the condition that ξ_n takes on some given value, ξ_m can take on no more than $\left(\frac{m}{n}+2\right)^r$ different values. Hence follows

$$H(\xi_m) \leqq H(\xi_n, \xi_m) = H(\xi_n) + H(\xi_m|\xi_n) \leqq H(\xi_n) + r \log_2 \left(\frac{m}{n}+2\right) \tag{4. 18}$$

and thus

$$H(\xi_m) - r \log_2 m \leqq H(\xi_n) - r \log_2 n + r \log_2 \left(1+2\frac{n}{m}\right). \tag{4. 19}$$

In view of (4. 17), lemma 3 is proved.

Now lemma 2 clearly implies

$$H(\mathscr{P}_m) - r \log_2 m \geqq H_\lambda(\xi) \tag{4. 20}$$

for every m. This means that (4. 14) and lemma 3 gives rise to (4. 15).

In order to find an abstract generalization of (4. 5), one could look for general conditions ensuring the validity of the conclusion of theorem 7 for certain non-increasing sequences of partitions, as well. In this paper, however, we do not enter this problem. Concerning a more complex example of dimensional entropy we refer to the forthcoming paper of J. FRITZ [4].

REMARK. As it has been pointed out in [1] and [2], the I-divergence of two probability distributions can be considered as a particular member of a wide class of measures of divergence of distributions, called f-divergences, sharing many important properties of the I-divergence. This fact suggests to define the generalized f-entropy of a probability distribution μ with respect to a σ-finite measure λ, as well, in such a way that the relation of f-divergence and generalized f-entropy be the same as that of f-divergence and generalized (Shannon) entropy. The results of section 2 (except for Theorem 4) carry over with very slight changes[5] for generalized f-entropies, as well; the case of f-divergences has been considered in [1] in detail.

The results of section 3 do not seem capable of a sensible generalization in this direction. As to section 4, while nothing can be done for generalized f-entropies in general, to consider partitions $\mathfrak{A}_n$ with the property (4. 11) is a proper approach to the theory of dimensional entropy of order α, in the sense of RÉNYI [9].

[5] Perhaps the only essential difference is that the f-entropy may conceivably be finite even if $\mu \not\ll \lambda$.

REFERENCES

[1] CSISZÁR, I.: Information-type measures of divergence of probability distributions, I—II, (In Hungarian, with English summary) *Magyar Tud. Akad. Mat. Fiz. Oszt. Közl.* **17** (1967) 123—149 and 267—291.

[2] CSISZÁR, I.: Information-type measures of difference of probability distributions and indirect observations, *Studia Sci. Math. Hungar.* **2** (1967) 299—318.

[3] Добрушин, Р. Л.: Общая формулировка основной теоремы Щеннона в теории информации, *Успехи Мат. Наук* **14** (1959) 3—104.

[4] FRITZ, J.: Entropy of point processes, *Studia Sci. Math. Hungar.* **4** (1969).

[5] KULLBACK, S.: *Information theory and statistics,* Wiley, New York, 1959.

[6] PEREZ, A.: Notions généralisées d'incertitude, d'entropie et d'information du point de vue de la théorie de martingales, *Transactions of the First Prague Conference on Information Theory, Statistical Decision Functions and Random Processes*, Prague, 1957, 183—208.

[7] PEREZ, A.: Information, ε-sufficiency and data reduction problems, *Kybernetika* **1** (1965) 297—322.

[8] RÉNYI, A.: On the dimension and entropy of probability distributions, *Acta Math. Acad. Sci. Hungar.* **10** (1959) 193—215.

[9] RÉNYI, A.: On measures of entropy and information, *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, Vol. 1, Berkeley, 1961, 541—561.

[10] Розенблат—Рот, М.: Нормированная ε-энтропия множеств и теория передачи информации, *Transactions of the Second Prague Conference on Information Theory, Statistical Decision Functions, Random Processes*, Prague, 1960, 569—577.

[11] Розенблат—Рот, М.: Понятие энтропии в теории вероятностей и его применения в теории передачи информации по каналам связи. Теор. Вероятност. и Применен. **9** (1964) 238—261.

[12] RUDEMO, M.: Dimension and entropy for a class of stochastic processes, *A Magyar Tudományos Akadémia Matematikai Kutató Intézetének Közleményei* **9** (1964) 73—87.

[13] SHANNON, C. E. and WEAVER, W.: *The Mathematical Theory of Communication*, University of Illinois Press, Urbana, 1949.

Mathematical Institute of the Hungarian Academy of Sciences, Budapest

(Received September 26, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 421–440.

TAUBERIAN THEOREMS FOR UNIVALENT FUNCTIONS

by

G. HALÁSZ

1. Throughout this paper we assume, without constantly repeating it, that

$$w(z) = a_1 z + a_2 z^2 + \dots \qquad (|z| < 1)$$

represents a univalent function in the unit disc $|z| < 1$. This simple topological property has, as is well-known, far reaching function-theoretic consequences. Can it serve also as a TAUBERian condition?

FEJÉR's elementary theorem "from Abel summability of Σa_n follows convergence if $\Sigma n|a_n|^2 < +\infty$" admits such an interpretation, the square sum (multiplied by π) being the area of the image under a conformal mapping $w(z)$. We look, however, for cases where the fact that $w(z)$ is univalent plays an essential role. An example is the problem raised by G. PIRANIAN and CH. POMMERENKE (see [1], Problem 2) whether Abel summability implies convergence already under the minimal condition $a_n \to 0$. We believe that it is not true but prove here some positive results.

Common premise in all of these: $w(z)$ is univalent and its series is Abel summable at $z = 1$,

$$w(r) \to A \quad \text{as} \quad r \to 1 - 0.$$

We begin with the simple

THEOREM 1. *Σa_n is summable even by Cesaro means*

$$\sigma_N^{(\alpha)} = \frac{1}{\binom{\alpha+N}{N}} \sum_{n=1}^{N} a_n \binom{\alpha+N-n}{N-n}$$

of every order $\alpha > 2$ (to the same sum A).

For the coefficients holds Littlewood's sharp estimate $a_n = O(n)$. Therefore, it is still possible that Theorem 1 remains true for $(C, 1+\varepsilon)$ summability ($(C, \alpha) =$ = Cesaro ... of order α) and $(C, 1)$ boundedness. This is of course the most that can be expected: $(C, 1)$ summability has already the necessary condition $a_n = o(n)$. We can prove only

THEOREM 2. *For the first Cesaro means*

$$\sigma_N^{(1)} = O(\log N).$$

It is our feeling that the log factor cannot be dropped altogether. Therefore it is of interest to find subclasses of univalent functions for which the best possible result holds.

THEOREM 3. $\sigma_N^{(1)} = O(1)$ *in case* $w(z)$ *is star-like, i.e. maps* $|z|<1$ *onto a domain starlike with respect to the origin.*

We mention only that $(C, 1+\varepsilon)$ summability follows already from $(C, 1)$ boundedness (Theorem 3) and (C, α) summability (Theorem 1) by an elementary Tauberian theorem (see [2]).

For this class of functions even the above question of PIRANIAN and POMMERENKE can be answered in the affirmative.

THEOREM 4. *If* $w(z)$ *is star-like,* $a_n \to 0$, *then* $\sum_{n=1}^{\infty} a_n$ *converges (to its Abel sum A).*

There is a little flaw in these two theorems, namely that star-likeness can be characterized very simply by function-theoretic means:

$$\operatorname{Re} z\frac{w'}{w}(z) > 0 \quad (|z|<1)$$

is both necessary and sufficient (see [3]), so that they are purely analytic theorems. Therefore, we give a wider class of functions with an exclusively geometric definition. It is not our intention, however, to give as general a class as would be possible.

Let us regard the image of $|z|<1$ under the mapping $w(z)$ on the w plane. Every circle $|w|=R$ intersects it (if it does) in one or more open arcs and every arc divides it into two domains. We call a "tongue" corresponding to such an arc that one of the two domains which does not contain the origin. First for the sake of simplicity only, we assume that this lies entirely in $|w|>R$. Let us now choose different arcs, possibly on different circles $|w|=R_i$, but in such a way that the tongues corresponding to them be disjoint. Let Δ_i denote the angular measure of an arc and suppose secondly that

$$\sum \Delta_i^{\alpha} \leqq C$$

with constants α, $C>0$ for any described choice of arcs.

If a domain possesses these two properties, we call it, as well as the function mapping $|z|<1$ onto it, admissible.

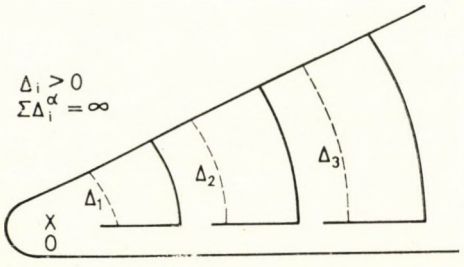


Fig. 1

Star-like domains are really all admissible: the first condition is trivially fulfilled, while tongues are disjoint if and only if the beams of rays which lead to the arcs are disjoint in which case we have $\Sigma\Delta_i \leqq 2\pi$, i.e. the second condition with $\alpha=1$, $C=2\pi$. In Fig. 1 we show an inadmissible domain.

THEOREM 3′. *Theorem* 3 *holds for admissible functions as well.*

THEOREM 4′. *Let $w(z)$ be admissible and suppose also that the image contains no circle with centrepoint on $|w|=R$ and radius $(1/2-\delta)R^*$ with a fixed $\delta>0$ for large enough R. In this case $\sum_{n=1}^{\infty} a_n$ converges.*

At the end of the paper we shall discuss some other results, e.g. that in the above interpretation of FEJÉR's theorem the requirement of finite area can be weakened to finite logarithmic area. It will be clear then, how to obtain them.

2. The proof of Theorem 1 is quite simple, but for orders less than 2 more precise methods are needed. The common part in the proof of these theorems (the function-theoretic part) was found by us in [4] and [5] in investigating the mean values of general multiplicative numbertheoretic functions. As we learned afterwards, this part somewhat resembles to HAYMAN's paper [6], but also it essentially differs from it. As a main tool a differential inequality is used here. We also learned that another type of differential inequality had been used by CLUNIE and POMMERENKE [7].

3. First step in the proofs. If we are to prove convergence or summability for $\sum_{u=1}^{\infty} a_n$, then $w(z)=\sum_{n=1}^{\infty} a_n z^n$ must satisfy certain conditions. E.g. if $s_n = a_1+\ldots+a_n$ tends to A, then

$$\frac{w(z)}{1-z} = \sum_{n=0}^{\infty} s_n z^n = A\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} o(1)z^n = \frac{A}{1-z} + o\left(\frac{1}{1-|z|}\right),$$

$$w(z) = A + o\left(\frac{|1-z|}{1-|z|}\right)$$

uniformly as $|z| \to 1-0$. (C, 1) summability would imply

$$w(z) = A + o\left(\left[\frac{|1-z|}{1-|z|}\right]^2\right).$$

In this "first step" we prove at least these necessary conditions.

Let us begin with the case of summability (Theorem 1, 2, 3, 3′). Here there will be no need for special properties such as star-likeness etc.

We know that

$$|w(r)-A| < \varepsilon$$

for r sufficiently close to 1. Of course the Abel sum A does not belong to the image so that $w(z)$ omits values from the circle of radius ε round $w(r)$ on the w plane. But then KOEBE's theorem (see [3]) can be applied e.g. in the circle $|z-r| < 1-r$. To transform it into the standard form, let

$$f(u) = \frac{w(r+u[1-r])-w(r)}{w'(r)(1-r)} = u+\ldots \quad (|u|<1).$$

* The 1/2 could be enlarged.

This function will then omit values from the circle round 0 of radius $\frac{\varepsilon}{|w'(r)|(1-r)}$. By the "$^1/_4$ theorem" of Koebe

$$\frac{1}{4} < \frac{\varepsilon}{|w'(r)|(1-r)}, \quad |w'(r)|(1-r) < 4\varepsilon.$$

Koebe's inequality for $|f(u)|$ gives then the uniform bound in $|u| \leqq {}^1/_2$

$$|f(u)| \leqq 2, \quad |w(r+u[1-r]) - w(r)| \leqq 2|w'(r)|(1-r) < 8\varepsilon \qquad (|u| \leqq {}^1/_2)$$

and in particular,

$$|w(re^{i\vartheta}) - w(r)| < 8\varepsilon, \quad |w(re^{i\vartheta}) - A| < 9\varepsilon \quad \left(|\vartheta| \leqq \frac{1-r}{2}\right).$$

Next we can apply the same argument to the two circles $\left|z - re^{\pm i\frac{1-r}{2}}\right| < 1-r$, ε to be replaced by 9ε and increasing the range of ϑ in this way by $\frac{1-r}{2}$ at a time, after the mth step we get

$$|w(re^{i\vartheta}) - A| < 9^m\varepsilon \quad \left(|\vartheta| \leqq m\frac{1-r}{2}\right).$$

In other words, $w(z)$ tends to A as $z = re^{i\vartheta}$ tends to 1 in $|\vartheta| \leqq K(1-r)$ with any temporarily fixed K ($\geqq 2\pi$). We call it a Stolz angle.

Let now $z = re^{i\vartheta}$ ($|\vartheta| \leqq \pi$) be outside this angle. The ray leading through it intersects the boundary of the angle at $z_1 = r_1 e^{i\vartheta}$ where $|\vartheta| = K(1-r_1)$, i.e. $r_1 = 1 - \frac{|\vartheta|}{K}$ ($\geqq {}^1/_2$). The difference

$$\log|w(z)| - \log|w(z_1)| \leqq \int_{r_1}^{r} \left|\frac{w'}{w}(xe^{i\vartheta})\right| dx$$

can be estimated by the Koebe inequality concerning $\frac{f'}{f}$ applied to $f(z) = w(z)$ in the whole circle $|z| < 1$:

$$\left|\frac{w'}{w}(xe^{i\vartheta})\right| \leqq \frac{1}{x} + \frac{2}{1-x} = \frac{2}{1-x} + O(1) \quad \left(\frac{1}{2} \leqq x < 1\right):$$

$$\log|w(z)| - \log|w(z_1)| \leqq \int_{r_1}^{r} \frac{2}{1-x}\,dx + O(1) = \log\left(\frac{1-r_1}{1-r}\right)^2 + O(1),$$

$$|w(z)| \leqq c_1 |w(z_1)| \left(\frac{1-r_1}{1-r}\right)^2$$

or by $1 - r_1 = \frac{|\vartheta|}{K}$

$$|w(z)| \leqq c_1 |w(z_1)| \left(\frac{|\vartheta|}{K(1-r)}\right)^2 \leqq c_2 |w(z_1)| \left(\frac{|1-z|}{K(1-r)}\right)^2.$$

Now, in Theorem 2, 3 and 3′ where O-estimations suffice, we put e.g. $K=2\pi$. $w(z_1)=O(1)$ as we know already, z_1 being in the Stolz angle and our last inequality becomes

$$|w(z)| \leqq c_3 \left(\frac{|1-z|}{1-|z|}\right)^2 \tag{1}$$

for z outside the Stolz angle. Inside it, the inequality means just $w(z)=O(1)$, so that it holds everywhere.

For Theorem 1 $o(\ )$ remainder term is needed. Therefore, we have to choose K large and in order to make again $w(z_1)$ bounded, uniformly in K, we have to ensure that z_1 is sufficiently close to 1. This will happen if $|1-z| < \delta$ with a $\delta > 0$ depending on K and we may write

$$|w(z)| \leqq c_2 |w(z_1)| \left(\frac{|1-z|}{K(1-|z|)}\right)^2 \leqq c_4 \left(\frac{|1-z|}{K(1-|z|)}\right)^2 \leqq c_4 \left(\frac{|1-z|}{K(1-|z|)}\right)^{2+\eta}.$$

In $|1-z| \geqq \delta$ we make use of (1),

$$|w(z)| \leqq c_3 \left(\frac{|1-z|}{1-|z|}\right)^2 \leqq \left(\frac{|1-z|}{K(1-|z|)}\right)^{2+\eta}$$

provided $|z|$ is sufficiently near 1. Summing up,

$$w(z) = A + o\left(\left[\frac{|1-z|}{1-|z|}\right]^{2+\eta}\right) \quad (\eta > 0). \tag{2}$$

In fact, in the Stolz angle it simply means $w(z) \to A$ and outside it the main term A, by $\frac{|1-z|}{K(1-|z|)} \geqq c_5$, as well as $w(z)$, according to our estimations, will be small compared to $\left(\frac{|1-z|}{1-|z|}\right)^{2+\eta}$, choosing first K large and then $1-|z|$ small.

Concerning ordinary convergence,

$$w(z) = A + o\left(\frac{|1-z|}{1-|z|}\right) \tag{3}$$

is to be proved. The square in the previous cases resulted from the 2 in our estimation

$$\left|\frac{w'}{w}(z)\right| \leqq \frac{2}{1-|z|} + O(1).$$

If we can replace it by 1 or even by a fixed $d<1$, then a repetition of the argument for Theorem 1 gives even

$$w(z) = A + o\left(\left[\frac{|1-z|}{1-|z|}\right]^d\right)$$

with a little larger $d(<1)$.

For this purpose, in Theorem 4, $a_n \to 0$ and the property of star-likeness are available. The former will be exploited through its immediate consequence

$$w(z) = \sum_{n=1}^{\infty} a_n z^n = \sum_{n=1}^{\infty} o(1) z^n = o\left(\frac{1}{1-|z|}\right)$$

and the latter, as we have already remarked, through

$$u(z) = \operatorname{Re} z\frac{w'}{w}(z) > 0, \quad \left[z\frac{w'}{w}(z)\right]_{z=0} = 1 \quad (|z| < 1).$$

It is well-known* (see [8]) that in this case there exists a positive measure μ on the periphery $|\xi| = 1$ with total measure $\mu(|\xi| = 1) = 1$ and

$$z\frac{w'}{w}(z) = \int_{|\xi|=1} \frac{\xi + z}{\xi - z}\, d\mu_\xi = 1 + 2z \int_{|\xi|=1} \frac{d\mu_\xi}{\xi - z}.$$

To get an expression for $|w(z)|$, let us divide by z and integrate from $1/2$ to z_1. Taking real parts

$$\log |w(z_1)| = 2 \int_{|\xi|=1} \log \frac{1}{|\xi - z_1|}\, d\mu_\xi + O(1).$$

Owing to

$$w(z_1) = o\left(\frac{1}{1-|z_1|}\right),$$

$$2 \int_{|\xi|=1} \log \frac{1}{|\xi - z_1|}\, d\mu_\xi \leqq \log \frac{1}{1-|z_1|} + O(1) + \log o(1).$$

On the subarc $|\xi - z_1| > 1$ the integrand is negative but not less than $-\log 2$ and the contribution of this arc can be incorporated into the $O(1)$ term. By narrowing further down the range of integration to $|\xi - z_1| < \delta$ $(> 1 - |z_1|$ lest it should be void), no additional error term emerges in the inequality, the integrand being already positive. Since $\log o(1)$ tends to $-\infty$, uniformly as $|z_1| \to 1 - 0$, let us fix $|z_1| = r_1$ $(\geqq 1/2)$ in such a way that the right hand side should be less than, say, $\log \frac{1}{1-r_1} - 1$. If we choose afterwards e. g. $\delta = 2(1 - r_1)$, then we have for the measure of any arc, though small but of fixed length,

$$2 \log \frac{1}{\delta}\, \mu(|\xi - z_1| < \delta) \leqq 2 \int_{|\xi - z_1| < \delta} \log \frac{1}{|\xi - z_1|}\, d\mu_\xi \leqq \log \frac{1}{1-r_1} - 1,$$

$$2\mu(|\xi - z_1| < \delta) = \frac{\log \frac{1}{1-r_1} - 1}{\log \frac{1}{1-r_1} - \log 2} \overset{\text{def}}{=} d\ (< 1).$$

* We do not need this theorem. The Poisson integral for $|\xi| = r < 1$ would do.

Let us use this in the integral representation of $z\frac{w'}{w}(z)$. If z and z_1 have the same argument, then for $|\xi - z_1| \geqq \delta$ $\dfrac{1}{\xi - z}$ is bounded and so is its integral. On the other hand,

$$\left| \int\limits_{|\xi - z_1| < \delta} \frac{d\mu_\xi}{\xi - z} \right| \leqq \frac{1}{1-|z|}\, \mu(|\xi - z_1| < \delta) \leqq \frac{1}{1-|z|} \cdot \frac{d}{2}$$

and the whole expression,

$$\left| z\frac{w'}{w}(z) \right| = \left| 1 + 2z \int\limits_{|\xi|=1} \frac{d\mu_\xi}{\xi - z} \right| \leqq O(1) + 2|z| \frac{d}{2(1-|z|)}, \quad \left| \frac{w'}{w}(z) \right| \leqq \frac{d}{1-|z|} + O(1)$$

$$(\tfrac{1}{2} \leqq |z| < 1)$$

as we stated.*

In Theorem 4′ the non-existence of certain circles is assumed to guarantee this. Suppose that $\left|\dfrac{w'}{w}(z)\right| \geqq \dfrac{d}{1-|z|}$ and let $|w(z)| = R$. In the usual application of Koebe's $^1/_4$ theorem, put

$$f(u) = \frac{w\left(\dfrac{u+z}{1+u\bar{z}}\right) - w(z)}{w'(z)(1-|z|^2)} = u + \ldots \quad (|u| < 1).$$

The image of this function contains a circle of radius $^1/_4$ round 0, hence that of w contains one of radius

$$\frac{|w'(z)|(1-|z|^2)}{4} = \left|\frac{w'}{w}(z)\right| R \frac{1-|z|^2}{4} \geqq \frac{d}{1-|z|} R \frac{1-|z|^2}{4} = \frac{d}{4} R(1+|z|)$$

round the point $w(z)$ on $|w| = R$. By assumption it cannot exceed $(^1/_2 - \delta)R$:

$$\frac{d}{4}(1+|z|) \leqq \frac{1}{2} - \delta, \quad d < 1 - \delta$$

for $|z|$ sufficiently near 1.

It is true that the assumption concerned $R > R_0$, but since $w(z)$ tends uniformly to the boundary (in the spherical metric) as $|z| \to 1-0$, the image cannot contain such large circles round $w(z)$ even if $w(z)$ falls in $|w| \leqq R_0$, at least for $|z|$ close enough to 1.

4. Proof of Theorem 1. From the first step, (2)

$$w(z) = A + o\left(\left[\frac{|1-z|}{1-|z|}\right]^{2+\eta}\right).$$

* This, we have learned, was essentially done by **Pommerenke** "On star-like and convex functions", Journal of London Math. Soc. **37** (1962), 209—224.

This is, by the way, a necessary condition for $(C, 1+\eta)$ summability. We can deduce from it only (C, α) summability, $\alpha > 2+\eta$. The definition of the latter is

$$\sum_{n=1}^{N} a_n \binom{\alpha+N-n}{N-n} = A\binom{\alpha+N}{N} + o\left(\binom{\alpha+N}{N}\right).$$

$\binom{\alpha+n}{n}$ is the coefficient of z^n in $\frac{1}{(1-z)^{\alpha+1}}$, hence the sum is the Nth coefficient of $\frac{w(z)}{(1-z)^{\alpha+1}}$:

$$\frac{1}{2\pi i}\int_{|z|=r} \frac{w(z)}{z^{N+1}(1-z)^{\alpha+1}}\,dz.$$

Replacing $w(z)$ by its asymptotic relation,

$$\frac{1}{2\pi i}\int_{|z|=r} \frac{A}{z^{N+1}(1-z)^{\alpha+1}}\,dz + o(1)\int_{|z|=r} \frac{\left(\frac{|1-z|}{1-r}\right)^{2+\eta}}{r^{N+1}|1-z|^{\alpha+1}}\,|dz| =$$

$$= A\binom{\alpha+N}{N} + o\left(\frac{1}{r^{N+1}(1-r)^{2+\eta}}\right)\int_{|z|=r}\frac{|dz|}{|1-z|^{\alpha-1-\eta}}.$$

The main term is the one required. On account of $\binom{\alpha+N}{N} \sim \text{const}\cdot N^{\alpha}$, it remains to prove that the second one is $o(N^{\alpha})$.

Generally ($z = re^{i\vartheta}$)

$$(4)\qquad \int_{|z|=r}\frac{|dz|}{|1-z|^{\beta}} = \int_{|\vartheta|\leqq 1-r} + \int_{|\vartheta|\geqq 1-r} \leqq \frac{2(1-r)}{(1-r)^{\beta}} + 2^{\beta}\int_{1-r\leqq|\vartheta|<\infty}\frac{d\vartheta}{|\vartheta|^{\beta}} = \frac{c_6}{(1-r)^{\beta-1}},$$

provided $\beta > 1$. In our case this means $\alpha - 1 - \eta > 1$, $\alpha > 2+\eta$ and if so, the remainder term becomes

$$o\left(\frac{1}{r^{N+1}(1-r)^{2+\eta}(1-r)^{\alpha-2-\eta}}\right) = o\left(\frac{1}{r^{N+1}(1-r)^{\alpha}}\right).$$

At the optimal choice $r = 1-\frac{1}{N}$ $r^{N+1} = \left(1-\frac{1}{N}\right)^{N+1} \to e^{-1}$ and our quantity has the right magnitude.

Q. e. d.

PROOF OF THEOREM 2. The first Cesaro means are given by

$$\frac{1}{N+1}\sum_{n=1}^{N} a_n(N+1-n)$$

so that we have to prove $\left(\text{if again } r = 1-\frac{1}{N}\right)$

$$\left|\sum_{n=1}^{N} a_n(N+1-n)\right| = \left|\frac{1}{2\pi i}\int_{|z|=r} \frac{w(z)}{z^{N+1}(1-z)^2}\,dz\right| \leqq c_7 \int_{-\pi}^{\pi} \frac{|w(re^{i\vartheta})|}{|1-re^{i\vartheta}|^2}\,d\vartheta = O(N\log N).$$

The estimation of the first step, (1),

$$\frac{|w(re^{i\vartheta})|}{|1-re^{i\vartheta}|^2} = O\left(\frac{1}{(1-r)^2}\right)$$

provides a uniform bound for the integrand (which trivially would give only $O(N^2)$). The following device is able to yield sharp bounds for integrals on using the maximum of the integrand only.

Let

$$U(r) = \int_{-\pi}^{\pi} |F(re^{i\vartheta})|\,d\vartheta.$$

In subsequent cases $F(z)$ will be a power of a regular and non-zero function in $0<|z|<1$ and therefore we allow that $F(z)$ can be multi-valued but $|F(z)|$, $\frac{F'}{F}(z)$ and so $|F'(z)|$ are all single-valued.

Let us differentiate $U(r)$. On the basis of the elementary fact $|f(x)|' \leqq |f'(x)|$,

$$U'(r) = \int_{-\pi}^{\pi} \frac{\partial}{\partial r}|F(re^{i\vartheta})|\,d\vartheta \leqq \int_{-\pi}^{\pi} |F'(re^{i\vartheta})|\,d\vartheta.$$

We shall write $|F'|$ in the form $\left|\frac{F'}{F}\right||F|$ and factor out the logarithmic derivative. But on the set $\Omega = \Omega(r) = \Omega(r,\delta)$ where $\left|\frac{F'}{F}(re^{i\vartheta})\right| \geqq \frac{\delta}{1-r}$ (δ fixed) is large, we estimate trivially:

$$\int_{\Omega} |F'(re^{i\vartheta})|\,d\vartheta \leqq |\Omega(r)| \max_{\vartheta\in[-\pi,\pi]} |F'(re^{i\vartheta})|.$$

($|\Omega|$ denotes the measure). In order to express it in terms of $|F|$, by Cauchy's inequality for the derivative in the circle $|z-re^{i\vartheta}| \leqq \frac{1-r}{2}$

$$|F'(re^{i\vartheta})| \leqq \frac{1}{\frac{1-r}{2}} \max_{|z|\leqq\frac{1+r}{2}} |F(z)|$$

and

$$\int_{\Omega} |F'(re^{i\vartheta})|\,d\vartheta \leqq 2\frac{|\Omega(r)|}{1-r} \max_{|z|\leqq\frac{1+r}{2}} |F(z)|.$$

The complement $\bar{\Omega}$ of Ω is characterized by $\left|\frac{F'}{F}(re^{i\vartheta})\right| < \frac{\delta}{1-r}$ and here we proceed as was indicated:

$$\int_{\bar{\Omega}} |F'(re^{i\vartheta})|\, d\vartheta = \int_{\bar{\Omega}} \left|\frac{F'}{F}(re^{i\vartheta})\right| |F(re^{i\vartheta})|\, d\vartheta \leqq \frac{\delta}{1-r} \int_{-\pi}^{\pi} |F(re^{i\vartheta})|\, d\vartheta = \frac{\delta}{1-r} U(r).$$

Summing up, we have to solve the differential inequality

$$U'(r) \leqq \frac{\delta}{1-r} U(r) + 2\frac{|\Omega(r)|}{1-r} \max_{|z| \leqq \frac{1+r}{2}} |F(z)|.$$

If we multiply by $(1-r)^{\delta}$, the terms with $U'(r)$ and $U(r)$ form a derivative:

$$[U(r)(1-r)^{\delta}]' \leqq 2\frac{|\Omega(r)|}{(1-r)^{1-\delta}} \max_{|z| \leqq \frac{1+r}{2}} |F(z)|$$

or integrating from $^1/_2$ to r

$$U(r) \leqq \frac{2}{(1-r)^{\delta}} \int_{\frac{1}{2}}^{r} \frac{|\Omega(x)|}{(1-x)^{1-\delta}} \max_{|z| \leqq \frac{1+x}{2}} |F(z)|\, dx + O\left(\frac{1}{(1-r)^{\delta}}\right) \tag{5}$$

and all depends on the estimation of the measure $|\Omega|$ and $\max |F|$.

In the present case $F(z) = \frac{w(z)}{(1-z)^2}$ and according to the first step, (1)

$$\max_{|z| \leqq \frac{1+r}{2}} |F(z)| \leqq \frac{c_3}{\left(1 - \frac{1+r}{2}\right)^2} = \frac{4c_3}{(1-r)^2}.$$

As to the logarithmic derivative,

$$\frac{F'}{F}(z) = \frac{w'}{w}(z) + \frac{2}{1-z}.$$

For the sake of completeness we shall reproduce the proof of the result

$$\int_{-\pi}^{\pi} \left|\frac{w'}{w}(re^{i\vartheta})\right|^2 d\vartheta = O\left(\frac{1}{1-r} \log \frac{1}{1-r}\right)$$

due to BIERNACKI [9]. The same holds true for $\frac{1}{1-z}$ even without the log factor

(see (4) in the proof of Theorem 1) and so (by $|a+b|^2 \leqq 2(|a|^2+|b|^2)$) also for $\frac{F'}{F}(z)$. Then $\left|\frac{F'}{F}\right|$ cannot be large on a large set:

$$|\Omega(r)|\left(\frac{\delta}{1-r}\right)^2 \leqq \int_{-\pi}^{\pi}\left|\frac{F'}{F}(re^{i\vartheta})\right|^2 d\vartheta \leqq c_8\frac{1}{1-r}\log\frac{1}{1-r},$$

$$|\Omega(r)| \leqq c_9(\delta)(1-r)\log\frac{1}{1-r}.$$

Substituting these in our inequality (5)

$$U(r) \leqq \frac{2}{(1-r)^\delta}\int_{\frac{1}{2}}^{r}\frac{c_9(\delta)(1-x)\log\frac{1}{1-x}}{(1-x)^{1-\delta}}\cdot\frac{4c_3}{(1-x)^2}dx + O\left(\frac{1}{(1-r)^\delta}\right) \leqq$$

$$\leqq c_{10}(\delta)\frac{\log\frac{1}{1-r}}{(1-r)^\delta}\int_{\frac{1}{2}}^{r}\frac{dx}{(1-x)^{2-\delta}} + O\left(\frac{1}{(1-r)^\delta}\right) \leqq$$

$$\leqq c_{11}(\delta)\frac{\log\frac{1}{1-r}}{(1-r)^\delta}\cdot\frac{1}{(1-r)^{1-\delta}} + O\left(\frac{1}{(1-r)^\delta}\right) \leqq c_{12}(\delta)\frac{\log\frac{1}{1-r}}{1-r} = c_{12}(\delta)N\log N,$$

provided $2-\delta > 1$, e.g. $\delta = 1/2$.

The proof of Biernacki's result consists in an application of the area principle. The logarithmic area of the image of $r \leqq |z| \leqq \frac{1+r}{2}$ is given, on the one hand, by

$$\int_{r}^{\frac{1+r}{2}}\int_{-\pi}^{\pi}\left|\frac{w'}{w}(xe^{i\vartheta})\right|^2 d\vartheta\, x\, dx \geqq \frac{1-r}{2}r^2\int_{-\pi}^{\pi}\left|\frac{w'}{w}(re^{i\vartheta})\right|^2 d\vartheta,$$

the square integral of the regular function $z\frac{w'}{w}(z)$ being increasing. On the other hand, it can be transformed into an integral

$$\iint\frac{1}{R^2}d\Theta R\, dR$$

over the image on the w plane. Now, this image is bounded away from 0 (if $r \geqq 1/2$) and is contained in $|w| \leqq \frac{c_{13}}{(1-r)^2}$ by KOEBE's theorem, hence the integral is majorized by

$$2\pi\int_{c_{14}}^{\frac{c_{13}}{(1-r)^2}}\frac{1}{R}dR \leqq c_{15}\log\frac{1}{1-r}$$

and comparing the two inequalities, the promised result

$$\int_{-\pi}^{\pi} \left| \frac{w'}{w}(re^{i\vartheta}) \right|^2 d\vartheta = O\left(\frac{1}{1-r} \log \frac{1}{1-r} \right)$$

follows.

PROOF OF THEOREM 3. It is clear from the previous case that if

$$\int_{-\pi}^{\pi} \left| \frac{w'}{w}(re^{i\vartheta}) \right|^2 d\vartheta = O\left(\frac{1}{1-r} \right),$$

then the log factor vanishes and the theorem is proved.

In fact, it is a direct consequence of star-likeness. Since the function

$$z \frac{w'}{w}(z) = \sum_{n=0}^{\infty} c_n z^n$$

has positive real part, its coefficients, as is well-known, are bounded* and by PARSEVAL's equality

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} r^2 \left| \frac{w'}{w}(re^{i\vartheta}) \right|^2 d\vartheta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} = O\left(\sum_{n=0}^{\infty} r^{2n} \right) = O\left(\frac{1}{1-r} \right)$$

as stated.

PROOF OF THEOREM 4. In the case of convergence, proceeding just as before, an undesirable log factor would appear because of the exponent 1 (from, so to speak, $\int \frac{|dz|}{|1-z|}$). The difficulty can be circumvented by appealing to the elementary Tauberian theorem called Tauber's second theorem (see [2]): It is enough to prove

$$\sum_{n=1}^{N} n a_n = o(N),$$

since this and Abel summability together imply convergence.

The quantity is a partial sum of the derivative,

$$\left| \sum_{n=1}^{N} n a_n \right| = \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{w'(z)}{z^N (1-z)} dz \right| \leqq c_{16} \int_{-\pi}^{\pi} \left| \frac{w'(re^{i\vartheta})}{1-re^{i\vartheta}} \right| d\vartheta \quad \left(r = 1 - \frac{1}{N} \right).$$

We dispose of the range $|\vartheta| \leqq K(1-r)$ first. In the course of the first step we actually proved that $w'(z) = o\left(\frac{1}{1-|z|} \right)$ in this range, but it follows from $w(z) - A = o(1)$

* E.g. because $r^n |c_n| = \left| \frac{1}{\pi} \int_{-\pi}^{\pi} u(re^{i\vartheta}) e^{-in\vartheta} d\vartheta \right| \leqq \frac{1}{\pi} \int_{-\pi}^{\pi} u(re^{i\vartheta}) d\vartheta = 2u(0), \quad r \to 1,$

(in a Stolz angle), anyway, by the much used Cauchy estimation. Hence

$$\int\limits_{|\vartheta|\leqq K(1-r)} \left|\frac{w'(re^{i\vartheta})}{1-re^{i\vartheta}}\right| d\vartheta \leqq o\left(\frac{1}{1-r}\right)\frac{1}{1-r}\,2K(1-r) = o\left(\frac{1}{1-r}\right) = o(N).$$

On the remaining part we write w' in the form $\frac{w'}{w}\,w$ and apply HÖLDER's inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$

$$(6)\qquad \int\limits_{|\vartheta|\geqq K(1-r)} \left|\frac{w'(re^{i\vartheta})}{1-re^{i\vartheta}}\right| d\vartheta \leqq \left(\int\limits_{-\pi}^{\pi}\left|\frac{w'}{w}(re^{i\vartheta})\right|^p d\vartheta\right)^{1/p}\left(\int\limits_{|\vartheta|\geqq K(1-r)}\left|\frac{w(re^{i\vartheta})}{1-re^{i\vartheta}}\right|^q d\vartheta\right)^{1/q}.$$

Here we could take the Schwarz case $p=q=2$. As we remarked in the proof of Theorem 3, $w(z)$ being star-like, the first integral is $O\left(\frac{1}{(1-r)^{p-1}}\right)\left[=O\left(\frac{1}{1-r}\right)\right]$. From the second we factor out a small power, taking care that an exponent greater than 1 should remain:

$$\int\limits_{|\vartheta|\geqq K(1-r)}\left|\frac{w(re^{i\vartheta})}{1-re^{i\vartheta}}\right|^q d\vartheta \leqq \max_{|\vartheta|\geqq K(1-r)}\left|\frac{w(re^{i\vartheta})}{1-re^{i\vartheta}}\right|^{\frac{q-1}{2}}\int\limits_{-\pi}^{\pi}\left|\frac{w(re^{i\vartheta})}{1-re^{i\vartheta}}\right|^{\frac{q+1}{2}} d\vartheta.$$

The result (3) of the first step concerning convergence reads

$$w(z) = A + o\left(\frac{|1-z|}{1-|z|}\right),\quad \left|\frac{w(z)}{1-z}\right| \leqq \frac{|A|}{|1-z|} + o\left(\frac{1}{1-r}\right).$$

For the above max it means for small $1-r$, z being far from 1,

$$\max_{|\vartheta|\geqq K(1-r)}\left|\frac{w(re^{i\vartheta})}{1-re^{i\vartheta}}\right|^{\frac{q-1}{2}} \leqq \frac{c_{17}}{[K(1-r)]^{\frac{q-1}{2}}}$$

while on the whole circle uniformly

$$\left|\frac{w(re^{i\vartheta})}{1-re^{i\vartheta}}\right|^{\frac{q+1}{2}} \leqq \frac{c_{18}}{(1-r)^{\frac{q+1}{2}}}.$$

This latter is needed if we are to apply to the remaining integral our inequality (5),

$$\int\limits_{-\pi}^{\pi}|F(re^{i\vartheta})|\,d\vartheta \leqq \frac{2}{(1-r)^{\delta}}\int\limits_{\frac{1}{2}}^{r}\frac{|\Omega(x)|}{(1-x)^{1-\delta}}\max_{|z|\leqq\frac{1+x}{2}}|F(z)|\,dx + O\left(\frac{1}{(1-r)^{\delta}}\right)$$

with $F(z) = \left(\frac{w(z)}{1-z}\right)^{\frac{q+1}{2}}$. Here, as in Theorem 3, $|\Omega(r)| \leqq c_{19}(\delta)(1-r)$ and we get further

$$\leqq \frac{c_{20}(\delta)}{(1-r)^{\delta}} \int_{\frac{1}{2}}^{r} \frac{1-x}{(1-x)^{1-\delta}(1-x)^{\frac{q+1}{2}}}\, dx + O\left(\frac{1}{(1-r)^{\delta}}\right) \leqq$$

$$\leqq \frac{c_{20}(\delta)}{(1-r)^{\delta}} \cdot \frac{c_{21}(\delta)}{(1-r)^{\frac{q+1}{2}-\delta-1}} + O\left(\frac{1}{(1-r)^{\delta}}\right) \leqq \frac{c_{22}(\delta)}{(1-r)^{\frac{q-1}{2}}},$$

provided $\frac{q+1}{2} - \delta > 1$, e.g. $\delta = \frac{q-1}{4}$. Collecting our estimations to the Hölder inequality (6),

$$\int_{|\vartheta| \leqq K(1-r)} \left|\frac{w'(re^{i\vartheta})}{1-re^{i\vartheta}}\right| d\vartheta \leqq \left(\frac{c_{23}}{(1-r)^{p-1}}\right)^{1/p} \left(\frac{c_{17}}{[K(1-r)]^{\frac{q-1}{2}}} \frac{c_{22}(\delta)}{(1-r)^{\frac{q-1}{2}}}\right)^{1/q} =$$

$$= \frac{c_{24}}{K^{\frac{q-1}{2q}}} \frac{1}{(1-r)^{1-\frac{1}{p}+1-\frac{1}{q}}} = \frac{c_{24}}{K^{\frac{q-1}{2q}}} \frac{1}{1-r} = \frac{c_{24}}{K^{\frac{q-1}{2q}}} N.$$

Choosing K large, we see that this contribution, too, is small compared to N and the proof is completed.

PROOF OF THEOREM 3′ AND 4′. As to Theorem 3′, if one repeats the method of Theorem 2, one finds, as in Theorem 3, that the only estimation missing is

1. $|\Omega(r)| \left(= \text{measure of the set } \left|\frac{w'}{w}(re^{i\vartheta})\right| \geqq \frac{\delta}{1-r}\right) \leqq c_{25}(\delta)(1-r)$.*

In Theorem 4 we used Hölder's inequality with general p and q in order to point out that in Theorem 4′, in addition to 1., only

2. $$\int_{-\pi}^{\pi} \left|\frac{w'}{w}(re^{i\vartheta})\right|^{p} d\vartheta = O\left(\frac{1}{(1-r)^{p-1}}\right)$$

for a p however large, is needed. We derive these from the fact that $w(z)$ is admissible. Although 1. follows from 2., we begin with 1., giving a $c_2(\delta)$ explicitly and then deduce 2. from this sharpened form.

We could operate on the w plane but it is more convenient to take $s = \sigma + it = \log w$. Let us join on the w plane 0 and e.g. a boundary point of the image of $w(z)$ nearest 0 with a straight segment. In the domain split this way, $s = \log w$ is

* Formerly Ω was defined by $\left|\frac{w'}{w} + \frac{C}{1-z}\right| \geqq \frac{\delta}{1-r}$. But, since the inequality in 1. is obviously satisfied by $\frac{1}{1-z}$, the two Ω's are equivalent from the point of view of this inequality.

single-valued and maps it onto a strip-like simply connected domain D on the s plane (see Fig. 2). If the above nearest point is on $|w|=\varrho$, then let us first restrict ourselves to the part $\sigma = \operatorname{Re} s \geqq \log \varrho + 2$ of D so the slit do not cause any inconvenience. Every vertical line (corresponding to circles on the w plane) intersects D in open intervals (corresponding to arcs) and every interval cuts D into two domains. From now on, that part which does not contain $-\infty$ as a boundary point will be called a tongue (and not its counterpart on the w plane), corresponding to the vertical interval or even to a point on the interval. The first assumption in the definition of admissibility says that it lies always to the right of the interval.

Let $z_0 = re^{i\vartheta}$ $(r \geqq {}^3/_4)$ be a point of Ω: $\left|\frac{w'}{w}(z_0)\right| \geqq \frac{\delta}{1-r}$ $(0 < \delta \leqq 2)$ and also $\log |w(z_0)| \geqq \log \varrho + 2$. Let us regard $s = s(z) = \log w(z)$ in the circle $|z - z_0| \leqq \frac{1-r}{2}$. For the derivative at the centrepoint $|s'(z_0)| = \left|\frac{w'}{w}(z_0)\right| \geqq \frac{\delta}{1-r}$ and by the ${}^1/_4$ theorem the image contains a circle of radius $\delta/8$ or a square of side length $\delta/8$ round $s_0 = \sigma_0 + it_0 = s(z_0)$. It is natural to expect that if the s image of $|z| = r$ has points far to the right of s_0, then, as passing through s_0, it must be near the boundary of D so that D cannot contain such a large square. This is roughly what we prove first.

Let us regard the tongue corresponding to $s_0 + \delta/16$ and suppose that it contains a point $s^* = \sigma^* + it^*$ with $\sigma^* \geqq \sigma_0 + L$ $(L \geqq 3)$ the inverse image of which, $z^* = re^{i\vartheta^*}$ is on $|z| = r$ as well. For ${}^1/_2 \leqq r_1 < r$ the image curve of the segment $z = xe^{i\vartheta^*}$ $r_1 \leqq x \leqq r$ is not longer than

$$\int_{r_1}^{r} |s'(xe^{i\vartheta^*})|\, dx = \int_{r_1}^{r} \left|\frac{w'}{w}(xe^{i\vartheta^*})\right| dx < \int_{r_1}^{r} \frac{3}{1-x}\, dx = 3 \log \frac{1-r_1}{1-r}.$$

If this does not exceed $L-1$, then the curve cannot get out of the tongue and even its endpoint $s_1 = s(z_1) = s(r_1 e^{i\vartheta^*})$ will remain inside the tongue. We define, therefore, r_1 by

$$3 \log \frac{1-r_1}{1-r} = L-1, \quad 1-r_1 = (1-r)e^{\frac{L-1}{3}}.$$

($r_1 < r$ is trivial, while $r_1 \geqq {}^1/_2$ means

$$(1-r)e^{\frac{L-1}{3}} \leqq \frac{1}{2}, \quad L \leqq 3 \log \frac{1}{1-r} + 1 - 3 \log 2.$$

But this is fulfilled automatically for $1-r$ small enough, since the image of $|z| = r$ runs entirely in $\sigma \leqq 2 \log \frac{1}{1-r} + O(1)$ by Koebe's theorem.)

Next we intersect the image of $r_1 < |z| < 1$ with the vertical strip $|\sigma - \sigma_0| < \delta/16$ and select the component containing our square. (We might have remarked that owing to $L \geqq 3$ $1 - r_1 > 2(1-r)$ so that the square, being assumed in $|z - z_0| < \frac{1-r}{2}$, really belongs to the intersection.) This component can be bounded by the vertical

lines $\sigma = \sigma_0 \pm \delta/16$, by the image of $|z| = r_1$ and by the boundary of D. Now, either upwards or downwards from the square, the last possibility cannot occur. Otherwise we could join s_0 both upwards and downwards (within the vertical strip) with boundary points of D by a curve each, without intersecting the image of $|z| = r_1$. The two curves together split D to pieces. We recall that tongues are located to the right of the interval or the point defining them. Therefore, the tongue corresponding to $s_0 + \delta/16$ lies entirely in one piece, the complement of the tongue (with respect to D) corresponding to $s_0 - \delta/16$ lies in another. They belong indeed to different pieces, since the two points $s_0 \pm \delta/16$ are on different sides of the splitting curves.

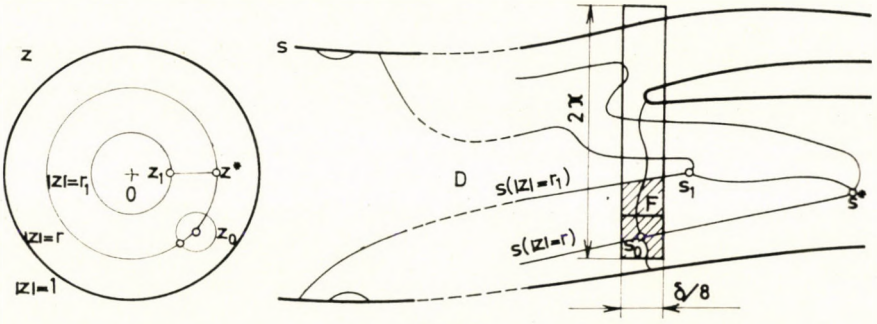


Fig. 2

The image of $|z| = r_1$ has points in the first named tongue, namely s_1 and also it has points in the complement of the second, e.g. it always reaches the two parallel lines corresponding to the Schlitz on the w plane. This means that different pieces can be joined (by the image of $|z| = r_1$) avoiding the splitting curves. It is impossible. In Fig. 2 the upper part of the above component is such that it is not bounded by the boundary of D. Let then F denote the domain (hachured in the figure) consisting of this upper part together with the square.

Let us regard in F the inverse function $z = z(s)$ or rather the harmonic function

$$g(s) = \log \frac{1}{|z(s)|}.$$

It is everywhere positive but on the non-linear part of the boundary of F which, as we know, consists solely of the image of $|z| = r_1$, it is as large as

$$\log \frac{1}{r_1} \geqq 1 - r_1.$$

To minorate $g(s)$, let us enlarge F to a rectangle of height 2π (see the figure). The lower horizontal side is fully contained in D, hence the upper can contain no point from D, different points of D being incongruent mod $2\pi i$. Therefore, F is really a subdomain to the rectangle. Let $h(s)$ be a harmonic function in the rectangle with boundary values 0 except on the upper horizontal side where it should not exceed $1 - r_1$. Such a $h(s)$ minorates $g(s)$ in F, for on the linear parts of its boundary the

latter is positive while the former is zero, and on the remaining parts the latter is at least $1-r_1$ but the former cannot exceed $1-r_1$ anywhere. As a consequence,

$$h(s_0) \leqq g(s_0) = \log \frac{1}{r} \leqq 2(1-r).$$

The function ($s = \sigma + it$)

$$\left(e^{\frac{8\pi t}{\delta}} - e^{-\frac{8\pi t}{\delta}}\right) \cos \frac{8\pi\sigma}{\delta}$$

is harmonic $\left(\text{being imaginary part to } 2 \sin \frac{8\pi s}{\delta}\right)$ and vanishes on $\sigma = \pm\delta/16$ and $t=0$ while for $t=2\pi$ it is at most

$$e^{\frac{16\pi^2}{\delta}} - e^{-\frac{16\pi^2}{\delta}} < e^{\frac{16\pi^2}{\delta}}.$$

Multiplying by $(1-r_1)e^{-\frac{16\pi^2}{\delta}}$, it can serve, after a shift of the rectangles onto each other, for a $h(s)$. To s_0 corresponds the point $\sigma=0$, $t=\delta/16$ so that

$$(1-r_1)e^{-\frac{16\pi^2}{\delta}}\left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right) = h(s_0) \leqq 2(1-r).$$

By the definition $1-r_1 = (1-r)e^{\frac{L-1}{3}}$ of r_1

$$(1-r)e^{\frac{L-1}{3}} e^{-\frac{16\pi^2}{\delta}}\left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right) \leqq 2(1-r), \quad L \leqq \frac{500}{\delta}.$$

Recalling the meaning of L, our result says: If s_0 is the image of a $z_0 = re^{i\vartheta}$ from $\Omega\left(\text{i. e. } \left|\frac{w'}{w}(z_0)\right| \geqq \frac{\delta}{1-r}\right)$ with $\operatorname{Re} s_0 \geqq \log \varrho + 2$, then the tongue corresponding to $s_0 + \delta/16$ contains points from the image of $|z|=r$, if any, only in $\sigma \leqq \operatorname{Re} s_0 + \frac{500}{\delta}$.

We have to prove that there are „few" such points z_0.

Let us chose points $z_k = re^{i\vartheta_k}$ ($k=1, \ldots, M$) from Ω with $|\vartheta_i - \vartheta_j| \geqq 4(1-r) \bmod 2\pi$ ($r \geqq 1/2$, $i \neq j$). The restriction is imposed to make the circles $|z-z_k| < \frac{1-r}{2}$ and so the above squares round the images $s_k = s(z_k)$, disjoint. Suppose first that all s_k lie in $\sigma \leqq \log \varrho + 2$.

We have M tongues corresponding to the points $s_k + \delta/16$. Since two tongues are either disjoint or one contains the other, each of them is a subset of a tongue which is already no (proper) part of any other. Let $s_i + \delta/16$ define such a maximal tongue. How many subtongues can it have? If $s_k + \delta/16$ defines one of them, then $\operatorname{Re} s_i \leqq \operatorname{Re} s_k$ and also $\operatorname{Re} s_k \leqq \operatorname{Re} s_i + \frac{500}{\delta}$ by what we proved before. The square round s_k belongs therefore to

$$\operatorname{Re} s_i - \frac{\delta}{16} \leqq \sigma \leqq \operatorname{Re} s_i + \frac{500}{\delta} + \frac{\delta}{16}.$$

This part of D is at most of area

$$2\pi\left(\frac{500}{\delta}+2\frac{\delta}{16}\right)\leqq\frac{3500}{\delta}$$

while the area of a square is $(\delta/8)^2$ so that a maximal tongue can contain not more than

$$\frac{\frac{3500}{\delta}}{\left(\frac{\delta}{8}\right)^2}\leqq\frac{10^6}{\delta^3}$$

subtongues (including itself).

To estimate now the number of maximal tongues, we remark that they are (identical ones counted now only once) disjoint. If the Δ_i denote the length of the open intervals defining them, then according to the assumption

$$\sum\Delta_i^\alpha\leqq C.$$

But $\Delta_i\geqq\delta/8$ so that there are at most

$$\frac{C}{(\delta/8)^\alpha}=\frac{c_{26}}{\delta^\alpha}$$

maximal tongues. Hence the total number M of tongues (of points z_k) cannot exceed

$$\frac{c_{26}}{\delta^\alpha}\cdot\frac{10^6}{\delta^3}=\frac{c_{27}}{\delta^{\alpha+3}}.$$

If now we allow points with $\operatorname{Re}s_k<\log\varrho+2$, then, possibly, their squares consist of separate parts, this fact, however, does not effect the area. Since certainly $\operatorname{Re}s_k\geqq a$ $(=\text{const.})$ for $r\geqq{}^1/_2$, these points are not more than

$$\frac{2\pi\left(\log\varrho+2-a+2\frac{\delta}{16}\right)}{(\delta/8)^2}\leqq\frac{c_{28}}{\delta^2}$$

in number and adding it to the above estimate,

$$M\leqq\frac{c_{29}}{\delta^{\alpha+3}}$$

in any case.

Let $\vartheta_1\in[-\pi,\pi]$ denote the first point ϑ of the closed set Ω with $\vartheta\geqq-\pi+4(1-r)$, $\vartheta_2\in[-\pi,\pi]$ the first $\vartheta\in\Omega$ with $\vartheta\geqq\vartheta_1+4(1-r)$, $\vartheta_3\in[-\pi,\pi]$ the first $\vartheta\in\Omega$ with $\vartheta\geqq\vartheta_2+4(1-r)$ and so on. In this way we get a finite number of points from Ω the mutual distance of which, $|\vartheta_i-\vartheta_j|\geqq4(1-r)\ (\mathrm{mod}\ 2\pi)$. Their number M cannot, therefore, exceed $\frac{c_{29}}{\delta^{\alpha+3}}$. The M intervals $[\vartheta_k-4(1-r),\ \vartheta_k+4(1-r)]$ and $[-4(1-r),4(1-r)]$ cover the whole of Ω, so that for its measure

$$|\Omega|\leqq(M+1)8(1-r)\leqq\frac{c_{30}}{\delta^{\alpha+3}}(1-r).$$

For δ fixed this is nothing else than 1. in an explicit form.

To estimate the integral of $\left|\frac{w'}{w}(re^{i\vartheta})\right|^p$ with its aid, let $\delta=\delta_m=2^{-m}$ and let us divide $[-\pi, \pi]$ into sets with

$$\frac{\delta_m}{1-r} \leqq \left|\frac{w'}{w}(re^{i\vartheta})\right| < \frac{\delta_{m-1}}{1-r}.$$

Since $\left|\frac{w'}{w}(re^{i\vartheta})\right| \leqq \frac{3}{1-r}$ $(r \geqq 1/2)$, it is sufficient to deal with $m=-1, 0, 1, \ldots$. This set is contained in our $\Omega=\Omega(r, \delta)$ with $\delta=\delta_m$ so that its measure is at most

$$\frac{c_{30}}{\delta_m^{\alpha+3}}(1-r) = c_{30}2^{(\alpha+3)m}(1-r).$$

(For very large m this is worse than the trivial 2π but for our purpose it is indifferent.) On the other hand, the integrand is bounded by

$$\left(\frac{\delta_{m-1}}{1-r}\right)^p = \frac{2^p}{(1-r)^p}2^{-pm}$$

and for the contribution of this set

$$\int_{\frac{\delta_m}{1-r} \leqq \left|\frac{w'}{w}\right| < \frac{\delta_{m-1}}{1-r}} \left|\frac{w'}{w}(re^{i\vartheta})\right|^p d\vartheta \leqq c_{30}2^{(\alpha+3)m}(1-r)\frac{2^p}{(1-r)^p}2^{-pm} =$$

$$= \frac{c_{31}(p)}{(1-r)^{p-1}}2^{-(p-\alpha-3)m}.$$

Summing over m, the series is convergent if only $p > \alpha+3$ and yields the necessary estimation.

Q. e. d.

5. Corollaries

Let

$$f(z) = a_1z+a_2z^2+\ldots \qquad (|z|<1)$$

be regular for $|z|<1$ and convergent at $z=1$. If $z=T(v)$ stands for any conformal mapping of the unit circle onto itself with $T(1)=1$ but not the identity, then Turán's theorem [10] states that

$$f(T(v)) = b_0+b_1v+\ldots \qquad (|v|<1)$$

can diverge at $v=1$. The proof of our Theorem 4 or 4′ shows that it cannot happen for star-like or admissible functions (the first step being even superfluous, since the necessary condition is not spoiled by the transformation $z=T(v)$).

In Theorem 4 (using the full strenght of the result of the first step with $d<1$) we could prove even $(C, -\delta)$ summability with a positive δ depending on the function or: $a_n \to 0$ implies $a_n=o(n^{-\delta})$ for star-like $w(z)$.*

* This was also proved previously by Pommerenke, see the above footnote.

It is clear how to replace angular measure in the definition of admissibility (for Theorem 4′) by certain area, showing that even for a wide class of domains with infinite logarithmic area Abel summability implies convergence. But in the case of finite logarithmic area, i.e.

$$\int_{\frac{1}{2}}^{1}\int_{-\pi}^{\pi}\left|\frac{w'}{w}(re^{i\vartheta})\right|^2 d\vartheta r\,dr < +\infty$$

there is no need for the geometric considerations of Theorem 4′. $\frac{w'}{w}(z) = o\left(\frac{1}{1-|z|}\right)$ and $\int_{-\pi}^{\pi}\left|\frac{w'}{w}(re^{i\vartheta})\right|^2 d\vartheta = o\left(\frac{1}{1-r}\right)$ follow trivially and much less would be already enough for the method.

Added in the proof. In a subsequent paper Proof W. K. Hayman will make great improvements. Just to mention two: Contrary to our conjecture expressed above, Theorem 3 (and 3′) hold for any univalent function. Univalency and $a_n \to 0$ do not, however, imply convergence. His example diverges everywhere on the periphery.

REFERENCE

[1] Lecture Notes of the Summer Institute on Entire Functions and Related Parts of Analysis Univ. of California, San Diego, La Jolla, California, 1966.

[2] HARDY, G. H.: *Divergent Series*, Oxford, Clarendon Press, 1949.

[3] HAYMAN, W. K.: *Multivalent Functions*, Cambridge, University Press, 1958.

[4] HALÁSZ, G.: Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen, *Acta Math. Acad. Sci. Hungar.* **19** (1968) 365—403.

[5] HALÁSZ, G.: Über die Konvergenz multiplikativer zahlentheoretischer Funktionen, *Stud. Sci. Math. Hungar.* **4** (1969).

[6] HAYMAN, W. K.: The Asymptotic Behaviour of p-valent Functions, *Proc. London Math. Soc.* **5** (1955) 257—284.

[7] CLUNIE, J. and POMMERENKE, CH.: On the coefficients of univalent functions, *Michigan Math. J.* **14** (1967) 71—78.

[8] NEVANLINNA, R.: *Eindeutige analytische Funktionen*, Berlin—Göttingen—Heidelberg, Springer Verlag, 1953.

[9] BIERNACKI, M.: Sur les fonctions en moyenne multivalentes, *Bull. Sci. Math.* **70** (1946) 51.

[10] TURÁN, P.: A remark concerning the behaviour of a power series on the periphery of its convergence circle. *Publ. de l'Inst. Math. de l'Acad. Serbe des Sci.* **12** (1956) 19—26.

Mathematical Institute of the Hungarian Academy of Sciences, Budapest

(Received September 26, 1968)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 441—445.

REMARKS ON A THEOREM OF R. M. ROBINSON

by

L. FEJES TÓTH

How should n points be arranged on the unit sphere so as to maximize the least distance between pairs of points? This problem, which is connected also with biology and information theory [1, 2, 3, 4] attracted the interest of many mathematicians. Special attention is due to the problem of finding the values of n such that $a_{n-1} = a_n$, where a_n denotes the maximum of the minimum distance. The first such number is known to be $n = 6$. Are there other numbers n with this property, and if so which ones? The following theorem [5] may be considered as a first approach to this difficult problem, which is unsolved so far.

THEOREM OF ROBINSON. *If n points are placed on a sphere in such a way that each point is as near to five others as two nearest points are to each other, then* $n = 12$, 24, 48, 60 *or* 120. *For any n, the configuration is unique up to rotation and reflection.*

For $n = 12$, 24 and 60 the points form the vertices of an icosahedron $\{3, 5\} = (3, 3, 3, 3, 3)$ and the Archimedean solids $(4, 3, 3, 3, 3)$ and $(5, 3, 3, 3, 3)$, respectively. The configurations corresponding to the remaining two cases $n = 48$ and 120, discovered by ROBINSON, are described in [5].

ROBINSON also pointed out that no analogous theorem exists if we replace the number five by four or three or by a number greater than five. For, in the first two cases there are infinitely many configurations, while in the third case there is none. Nevertheless ROBINSON's theorem can be completed in a natural way as follows.

We define the *Newton number* of a convex body b as the maximal number of non-overlapping congruent replicas of b which can be brought into contact with b. This name refers to the controversy between NEWTON and D. GREGORY about the Newton number of a ball, which has been decided 180 years later in favour of Newton. Again, we define a *maximal packing* of congruent convex bodies as a packing in which each body is touched by as many others as its Newton number.

Now we can state the following

THEOREM. *If on a sphere n congruent circles form a maximal packing then* $n = 1$, 2, 3, 4, 6, 8, 9, 12, 24, 48, 60 *or* 120. *For any* $n > 3$ *the configuration is unique up to rotation and reflection.*

If the radius r of the circles equals the face-inradius r_p of the tessellation $\{p, 3\}$, we obviously have the face-incircles of $\{p, 3\}$, i.e. 3, 4, 6 or 12 circles according as $p = 2$, 3, 4 or 5, The cases when $r > r_3$ are also trivial: for $r > \pi/2$ we have a single circle, for $\pi/2 \geqq r > \pi/3 = r_2$ a pair of touching circles and for $r_2 \geqq r > r_3$ three circles

mutually touching each other. If $r_3 > r > r_4$, the Newton number is equal to three. Thus we must have at least four circles. But since four circles with $r < r_3$ cannot form a maximal packing, and more than four circles with $r > r_4$ cannot be packed on the sphere at all, there is no maximal packing for $r_3 > r > r_4$. On the other hand, we will prove that for $r_4 > r > r_5$ there are two maximal packings, one consisting of 8 circles centered at the vertices of (4, 3, 3, 3), the other consisting of 9 circles arranged in a configuration shown in Fig. 4. For $r \leqq r_5$ the enumeration of maximal circle-packings is equivalent with the problem solved by ROBINSON's theorem.

In the following investigation of the case when $r_5 < r < r_4$ we do not make use of the method developed in [5]. Now each circle is touched by four others. We claim that

a) to each pair of touching circles there is a third touching both.

Let p be a pair of touching circles a and b, and suppose, contrary to *a)*, that no circle touches both a and b. Since apart from each other, both a and b have three neighbours, p is surrounded by six circles c, d, e, f, g, h, in this cyclic order, the first three touching a and the rest touching b. We claim that the angles $\sphericalangle HBA$ and $\sphericalangle BAC$ add up to more than 3α, where the capital letters denote the centers of the corresponding circles and α is an angle of an equilateral triangle with side-length $2r$. To see this, we obviously may suppose that c and h abut (Fig. 1). Then $ABHC$ is a rhombus with sides equal to $2r$ and diagonals longer than $2r$. But since the area of a rhombus with a constant side-length decreases if its shorter diagonal decreases, the sum of the angles of $ABHC$ is greater than in the limiting cases when $AH = 2r$ or $BC = 2r$, i.e. greater than 6α. Thus we have $\sphericalangle HBA + \sphericalangle BAC > 3\alpha$ and, similarly, $\sphericalangle EAB + \sphericalangle ABF > 3\alpha$. Consequently, the sum of the angles $\sphericalangle CAD$, $\sphericalangle DAE$, $\sphericalangle EAB$, $\sphericalangle ABF$, $\sphericalangle FBG$, $\sphericalangle GBH$, $\sphericalangle HBA$ and $\sphericalangle BAC$ is, on the one hand greater than $(6+4)\alpha > 10 \cdot \frac{2\pi}{5}$, on the other hand, equal to $2 \cdot 2\pi$. This contradiction proves the proposition *a)*.

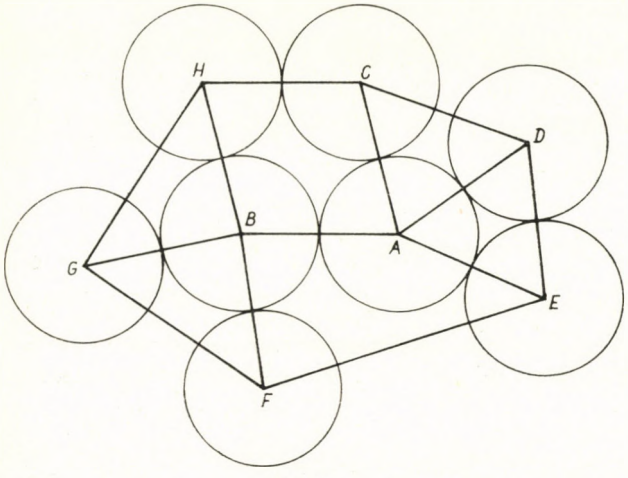


Fig. 1

In a similar way one can show that

b) to each triple of mutually abutting circles there is a fourth touching two circles of the triple.

Indeed, in the opposite case, the angles around the centers of the three circles (Fig. 2) would add up, on the one hand, to more than $(9+6)\alpha > 15 \cdot \frac{2\pi}{5}$, on the other hand, to $3 \cdot 2\pi$.

We use the symbols (ab) and (abc) to express that the circles in parentheses mutually abut. According to propositions *a)* and *b)* we can start with four circles a, b, c and d such that (abc) and (bcd). There is exactly one circle e other than a, b and d such that (ce). By *a)* we then have a circle x such that (cex). This circle must be identical either with a or d, since otherwise c would have five neighbours: a, b, d, e and x. Suppose that $x \equiv d$, so that (ced). Similarly, there is a circle f such that either (def) or (bdf).

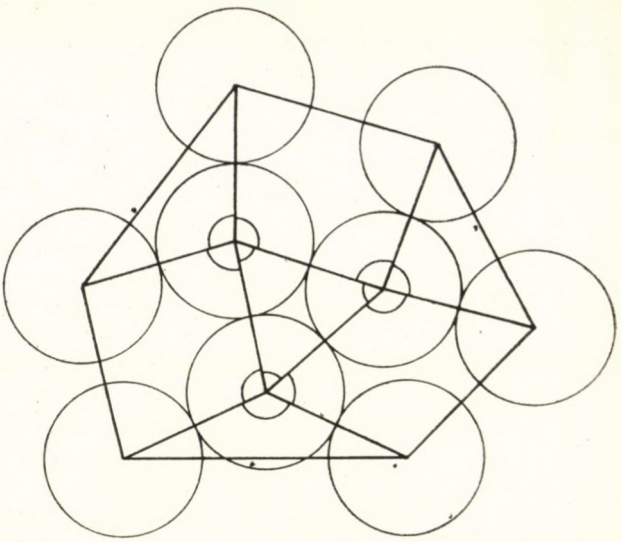


Fig. 2

Case (def). Since c has already four neighbours, there must be a circle g such that (efg). Again, since d has already four neighbours, there is a circle h such that (fhg), etc. (Fig. 3). The circles are centered at the vertices of an Archimedean antiprism $(k, 3, 3, 3)$, and the assumption that $r_5 < r < r_4$ implies that $k = 4$.

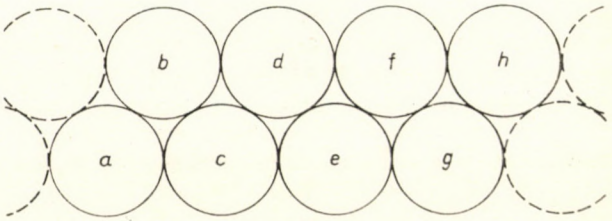


Fig. 3

Case (bdf). If any two of the circles a, e and f touch, then (aef) and we have the face-incircles of $\{4, 3\}$. This contradicts the assumption that $r < r_4$. Therefore each of the circles a, e and f has two further neighbours (Fig. 4). These neighbours cannot be all different from one another because then we would have twelve circles, which, because of $r > r_5$, is impossible. Consequently there is a circle g other than d such that, say, (eg) and (fg). This implies the existence of two new circles h and i such that (egh) and

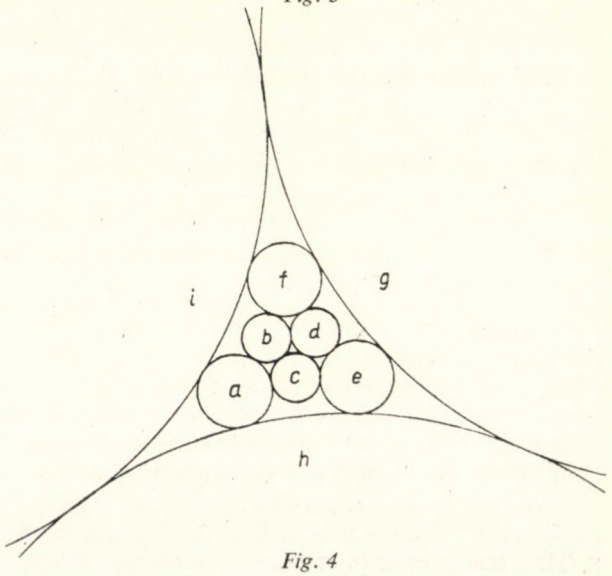


Fig. 4

(*fgi*). These circles are different from d because otherwise d would have g as a a fifth neighbour. On the other hand, since there is no circle of radius r other than d and g touching both e and f, the circles h and i are also different from one another. Again, there are two circles one touching two of the triple (*egh*) and the other touching two of the triple (*fgi*). Since each circle except a, h and i already has four neighbours, and a does not touch e, f or g, these circles are identical with i and h. Thus we have (*ghi*), which, by symmetry implies (*ahi*).

This completes the enumeration of the maximal packings of circles with $r > r_5$.

In the maximal packing of 4, 6, 8, 9, 12 and 24 circles the centers have been proved to form an extremal arrangement regarding the problem mentioned first [4, 6, 7]. As to the remaining cases, Robinson calls the configuration of 48 and 120 points „excellent" and the configuration of 60 points „good".

In Euclidean plane the only maximal circle-packing consists of the face-incircles of {6, 3}. What about hyperbolic plane? Is the number of maximal packings of circles with radius lying in the interval $(0, r_7)$ finite? The same question may be raised for the intervals (r_7, r_8), (r_8, r_9), etc.

It may be conjectured that in Euclidean 3-space all maximal sphere-packings are built up of hexagonal layers, so that the Dirichlet-cell of each sphere is either a rhombic or a trapezo-rhombic dodecahedron [4, p. 295]. This conjecture may be supported by the following considerations.

Let u be a unit ball, and $u_1, \ldots, u_{12}$ twelve unit balls touching u. Since a further unit ball u_{13} cannot touch u, the centers of u and u_{13} have a distance greater than 2. The minimum, D, of this distance for all possible arrangements of $u_1, \ldots, u_{13}$ is conjectured to be $14/\sqrt{27} = 2{\cdot}69 \ldots$ [4, p. 297].

Both in the rhombic and trapezo-rhombic arrangement each of the balls $u_1, \ldots, u_{12}$ is fixed by the others. Now the least distance between the centers of non-abutting balls equals $\sqrt{8}$, which, of course, is greater than D. But it seems very likely that in any other position some of the balls $u_1, \ldots, u_{12}$ have a little play, so that there will be two non-abutting balls the centers of which have a distance less than D, thus hindering each other to be touched by twelve unit balls.

To recapitulate, we formulate the following problem: In a certain arrangement of $u_1, \ldots, u_{12}$ other than the rhombic or trapezo-rhombic arrangement, let d be the least distance between the centers of non-abutting balls. Prove (or disprove) that $d < D$.

After I called the attention of Professor Robinson to this problem, he soon proved that $d < 2.54$, so that the desired result concerning maximal sphere-packings would be obtained by proving that $D > 2.54$. Robinson made the conjecture that the maximum of d is equal to $\sqrt{(42 - 2\sqrt{57})/5} \approx 2.32$. This bound is attained when the points of tangency of u and $u_1, \ldots, u_{12}$ consist of the vertices of an equilateral triangle of side-length 60° centered at the north pole, of the images of the vertices reflected in the sides and the images of these six points obtained by a glide-reflection in and along the equator which carries the three southernmost points of the northern hemisphere 60° from their original positions.

We still mention a result of another type concerning the notion of a maximal packing. It is known that in Euclidean plane the Newton number of a square equals 8. Thus the faces of {4, 4} form a maximal packing. The same is true for all spherical

and Euclidean tessellations $\{p, q\}$, $\frac{1}{p}+\frac{1}{q} \geqq \frac{1}{2}$. On the other hand, it may be conjectured that the number of the regular hyperbolic tessellations $\{p, q\}$, $\frac{1}{p}+\frac{1}{q} < \frac{1}{2}$, whose faces form a maximal packing is finite.

An upper bound for the Newton number of a convex Euclidean disc in terms of its diameter and width is given in [8].

REFERENCES

[1] VAN DER WAERDEN, B. L.: Pollenkörner, Punktverteilungen auf der Kugel und Informationstheorie, *Die Naturwissenschaften* **48** (1961) 189—92.
[2] ŞERBAN, M. and STROILĂ, C.: Un modele mathématique du noyau celulaire, *Rev. Roumaine Math. Pur. Appl.* **11** (1966) 287—316.
[3] GOLDBERG, M.: Viruses and a mathematical problem, *J. Mol. Biol.* **24** (1967) 337—8.
[4] FEJES TÓTH, L.: *Regular Figures*, Pergamon Press, Oxford—London—New York—Paris 1964.
[5] ROBINSON, R. M.: Finite sets of points on a sphere with each nearest to five others, *Math. Ann.* **179** (1969) 296—318.
[6] SCHÜTTE, K. and VAN DER WAERDEN, B. L.: Auf welcher Kugel haben 5, 6, 7, 8 oder 9 Punkte mit Mindestabstand Eins Platz? *Math. Ann.* **123** (1951) 96—124.
[7] ROBINSON, R. M.: Arrangement of 24 points on a sphere, *Math. Ann.* **144** (1961) 17—48.
[8] FEJES TÓTH, L.: On the number of equal discs that can touch another of the same kind, *Studia Sci. Math. Hungar.* **2** (1967) 363—67.

Mathematical Institute of the Hungarian Academy of Sciences, Budapest

(Received October 7, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 447—463.

AUTOMATISCHE PROGRAMMIERUNG ZUR BERECHNUNG DER PARTIELL-REKURSIVEN FUNKTIONEN

von

R. PÉTER

J. C. SHEPHERDSON mit H. E. STURGIS[1] und Verfasser[2] haben — unabhängig von einander — gezeigt, dass falls eine partiell-rekursive Funktion und eine Stelle gegeben ist, dann für eine entsprechende Maschine von einfachem Anweisungsystem ein Programm derartig angegeben werden kann, dass im Laufe dessen Durchführung durch die Maschine der Wert der gegebenen partiell-rekursiven Funktion berechnet wird, wenn dies existiert; im entgegengesetzten Fall aber die Maschine ewig weiterläuft ohne etwas zu berechnen.

Es erhebt sich die Frage, ob die Maschine auch selber ein solches Programm verfertigen kann, wenn in ihrem Speicher ein fixiertes sog. programmierendes Programm gesetzt wird, ferner, als Ausgangsinformation, ein definierendes Gleichungssystem der betreffenden partiell-rekursiven Funktion, das Zeichen dieser Funktion, und die betreffende Stelle.

Die Existenz eines solchen programmierenden Programms folgt unmittelbar aus dem genannten Ergebnis der zitierten Arbeiten und aus der Tatsache, dass die bekannte KLEENEsche explizite Form so gestaltet werden kann (indem eine aus den Argumenten $n_1, n_2, \ldots, n_r$ bestehende Stelle einer r-stelligen Funktion etwa durch die Zahl

$$p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$$

repräsentiert wird, wo p_i für $i = 1, 2, \ldots$ die wachsende Folge der Primzahlen ist), dass eine universelle partiell-rekursive Funktion ψ angegeben wird, welche den Wert einer beliebigen partiell-rekursiven Funktion φ an einer gegebenen Stelle annimmt, wenn für ihre Argumente die betreffende Stelle und die Gödel-Nummer eines definierenden Gleichungssystems von φ gesetzt wird; und daher würde es genügen für die Berechnung der einzigen partiell-rekursiven Funktion ψ ein Programm zu konstruieren. Die Konstruktion eines programmierenden Programms auf Grund dieses Grundgedankens wäre jedoch nicht so einfach.

Vielleicht ist es nicht ohne Interesse einen direkten konstruktiven Beweis für die Existenz des genannten programmierenden Programs zu geben, ohne den Umweg über die KLEENEsche explizite Form, und ohne die Gödelisierung des definierenden Gleichungssystems. Dies wird in den Folgenden durchgeführt werden.

I enthält die Festlegung des Bezeichnungsystems für die Gleichungssysteme, durch welche partiell-rekursive Funktionen definiert werden. In **II** gebe ich die

[1] J. C. SHEPHERDSON and H. E. STURGIS, Computability of Recursive Functions, *Journal of the ACM*, 10 (1963) S. 217—255.

[2] R. PÉTER, Programmierung und partiell-rekursive Funktionen, *Acta Math. Acad. Sci. Hungaricae*, **14** (1963) S. 173—401.

einfachen Grundanweisungen unserer Maschine an, und weitere (Makro-) Anweisungen, welche auf diese zurückgeführt werden können. In **III** werden die zu unserem Zweck notwendigen Subrutinen konstruiert. Endlich wird in **IV** das gesuchte programmierende Programm angegeben.

I

Nehmen wir an, dass in die Speicherzellen der Maschine (neben endlich viele weitere, zur Aufzeichnung des Programms notwendige Zeichenketten) „Worte einer Wortemenge über das endliche Alphabet"

$$\mathfrak{A} = \{a_1, a_2, \ldots, a_s\}$$

eingetragen werden können, worunter endliche Ketten (Folgen ohne Trennungszeichen zwischen ihren Gliedern) von Elementen aus $\mathfrak{A}$ (kurz: von „Buchstaben") verstanden werden. Die Elemente von $\mathfrak{A}$ werden die in den definierenden Gleichungssystemen auftretenden Zeichen sein. Damit $\mathfrak{A}$ endlich ausfallen soll, werde ich zur Bezeichnung der Variablen nur die beiden Grundzeichen

$$x \quad \text{und} \quad |$$

benutzen; aus diesen kann eine unendliche

$$x|,\ x||,\ x|||, \ldots$$

Variablenzeichenfolge aufgebaut werden. Die Zeichen für natürliche Zahlen werden aus dem Grundzeichen 0 ausgehend durch das übliche obere Strichzeichen der Nachfolgerfunktion gebildet:

$$0,\ 0',\ 0'',\ 0''', \ldots .$$

Doch für Zellenadressen werden die Zeichen

$$\Lambda,\ |,\ ||,\ |||, \ldots$$

günstiger sein, wo Λ das Symbol für „leer" ist. Auf eine durch k Striche bezeichnete Adresse werde ich mich im Text mit der Bezeichnung „k" berufen.

Wir brauchen auch unendliche Folgen von Funktionszeichen bei jeder Variablenanzahl; deise können z. B. durch Verwendung von Paaren je anders bezeichneter Indizes neben f konstruiert werden; z. B. wird — nach Aufnahme etwa von $+$ als neues Grundzeichen —

$$fxxx++$$

das 2-te unter den Zeichen für 3-stellige Funktionen sein. Sei unter diesen auch ein „nulltes", durch

$$fxxx$$

bezeichnet; und ähnlich für jede Variablenanzahl. Machen wir die Vereinbarung, dass die zu berechnende Funktion immer durch ein nulltes Zeichen bezeichnet werden soll.

Ausser den bisher aufgezählten Grundzeichen gehören zu unserem Aphabet $\mathfrak{A}$ noch das Komma, der Strichpunkt, die Anfangsklammer und die Endklammer; insgesamt enthält es also $s=10$ Elemente.

Unter den Worten über das Alphabet $\mathfrak{A}$ werden 0 und die Variablenzeichen

$$x|,\ x||,\ x|||,\ \ldots$$

Terme genannt; ferner, falls

$$t \quad \text{bzw.} \quad t_1, t_2, \ldots, t_r$$

bereits Terme sind, so gilt auch t' bzw. bei beliebigem i auch

$$f\underbrace{xx\ldots x}_{r\text{-mal}}\underbrace{+ + \cdots +}_{i}(t_1, t_2, \ldots, t_r)$$

als Term. Wenn t_1 und t_2 Terme sind, so heisst

$$t_1 = t_2$$

eine Gleichung. Ein definierendes Gleichungssystem für eine durch

$$f\underbrace{xx\ldots x}_{r\text{-mal}}$$

bezeichnete partiell-rekursive Funktion ist eine endliche Folge von Gleichungen, wenn daraus, sooft $n_1, \ldots, n_r$ Zahlzeichen, und zwar Zeichen für solche Argumente sind, für welche diese Funktion definiert ist, eine „Ableitung" zur Gleichung

$$f\underbrace{xx\ldots x}_{r}(n_1, \ldots, n_r) = n$$

führt, wo n das Zeichen des Wertes der betreffenden Funktion an der betrachteten Stelle ist; und keine Ableitung daraus zu einer Gleichung

$$f\underbrace{xx\ldots x}_{r}(n_1, \ldots, n_r) = m$$

führen kann, wo m ein von n abweichendes Zahlzeichen ist. Dabei wird unter Ableitung eine endliche Folge von Gleichungen verstanden, deren jedes Glied entweder eine der Definitionsgleichungen ist, oder durch einen der folgenden drei Schritte entsteht:

1. Substitution von 0 für jedes Vorkommen eines Variablenzeichens in einem vorherigen Glied der Folge,

2. Substitution von

$$x\underbrace{||\ldots|}_{i\text{-mal}}'$$

für jedes Vorkommen von

$$x\underbrace{||\ldots|}_{i\text{-mal}}$$

in einem vorherigen Glied der Folge,

3. Ersetzen eines Teils t_1 durch t_2 in einem vorherigen Glied der Folge, falls unter den vorherigen Gliedern der Folge auch $t_1 = t_2$ oder $t_2 = t_1$ vorkommt.

In einem Gleichungssystem kommen natürlich nur endlich viele Variablenzeichen vor (es könnte ohne Einschränkung der Allgemeinheit auch verlangt werden, dass diese immer von $x|$ an mit lückenlos zunehmende Indizes bezeichnet werden sollen).

II

Als Grundanweisungen übernehme ich aus der unter[1] zitierten Arbeit (mit einer kleinen Abweichung in den Bezeichnungen), wo m, n Speicheradressen bezeichnen,

$$\langle n \rangle$$

immer den Inhalt der Adresse n bezeichnet (für welche keine Schranke vorgeschrieben wird), und a_i ein beliebiges Element des benutzten Alphabets ist (also bei uns $i = 1, 2, \ldots, 10$ sein kann):

$P^{(a_i)}(n)$: Füge a_i an das Ende von $\langle n \rangle$.

$D(n)$: Streiche den ersten Buchstaben von $\langle n \rangle$ (falls $\langle n \rangle = \Lambda$, tue nichts).

$J^{(a_i)}(n)[c]$: Wenn $\langle n \rangle$ mit a_i beginnt, springe zur durch die Zahl c bezeichneten Anweisung (kurz: zur Anweisung c).

In der genannten Arbeit wurden auf diese auch die folgenden weiteren Anweisungen zurückgeführt:

$C(m, n)$ (das ich auch durch $C_{\mathscr{S}_1}(m, n)$ bezeichne, um herforzuheben, dass seine Zurückführung auf die Grundanweisungen auch ein Hilfsregister $\mathscr{S}_1$ bedarf, welches zum Schluss entleert wird; ähnliche Bezeichnungen werde ich auch später gebrauchen): Kopiere $\langle m \rangle$ — auch in m beibehaltend — in n (d.h. an Stelle von $\langle n \rangle$).

$\bar{J}(n)[c]$: Falls n nicht leer ist, springe zur Anweisung c.

$J[c]$: Springe (unbedingt) zur Anweisung c.

$J(n)[c]$: Falls n leer ist, springe zur Anweisung c.

$\Lambda(n)$: Entleere n (d. h. bringe Λ hinein).

$CP(m: n)$: Füge $\langle m \rangle$ an das Ende von $\langle n \rangle$, bei Entleerung von m.

Falls $F^{(a_i)}$ für $i = 1, 2, \ldots, 10$ bereits eingeführte Subrutinen sind, bezeichne $\sum_{i=1}^{10} (n) F^{(a_i)}$: beginnt $\langle n \rangle$ mit a_i, so durchführe $F^{(a_i)}$.

Bei den Zurückführungen auf die Grundanweisungen kommt meistens auch ein Hinweis auf eine nicht leere Adresse vor; eine solche Adresse wird aber in unserem Programm gewiss immer vorhanden sein (z. B. zu Beginn „1" und nach dem ersten Schritt immer „2").

Ich werde noch die folgenden Anweisungen gebrauchen, für welche ich gleich auch die sie auf Grundanweisungen zurückführende Subrutinen angebe:

$\bar{J}^{(a_i)}(n)[c]$: Beginnt $\langle n \rangle$ nicht mit a_i, so springe zur Adresse c.

Subrutine:

$J(n)[c]$

$J^{(a_1)}(n)[c]$

.....

$J^{(a_{i-1})}(n)[c]$

$J^{(a_{i+1})}(n)[c]$

.....

$J^{(a_{10})}(n)[c]$

$J(m, n)[c]$: Ist keines von m, n leer, und beginnen $\langle m \rangle$ und $\langle n \rangle$ mit demselben Buchstaben, so springe zur Anweisung c.

Subrutine:

$$\sum_{i=1}^{10} (m) J^{(a_i)}(n)[c]$$

$\bar{J}(m, n)[c]$: Beginnen $\langle m \rangle$ und $\langle n \rangle$ nicht mit demselben Buchstaben (auch wenn genau eines der Adressen m, n leer ist), so springe zur Anweisung c.

Subrutine:

$\bar{J}(m)[1]$

$\bar{J}(n)[c]$

1. $\sum_{i=1}^{10} (m) \bar{J}^{(a_i)}(n)[c]$

$J\{m, n\}[c]$: Wenn $\langle m \rangle$ und $\langle n \rangle$ übereinstimmen, so springe zur Anweisung c.

Subrutine:

$C_{S_1}(m, S_2)$

$C_{S_1}(n, S_3)$

1. $\bar{J}(S_2, S_3)[3]$

$\bar{J}(S_2)[2]$

$J[c]$

2. $D(S_2)$

$D(S_3)$

$J[1]$

3. $A(S_2)$

$A(S_3)$

Will man die dabei benutzten Hilfsregister hervorheben, so verwendet man die Bezeichnung:

$$J_{S_1,S_2,S_3}\{m, n\}[c].$$

$\bar{J}\{m, n\}[c](\bar{J}_{S_1,S_2,S_3}\{m, n\}[c])$: Stimmen $\langle m \rangle$ und $\langle n \rangle$ nicht überein, so springe zur Anweisung c.

Subrutine:

$C_{S_1}(m, S_2)$

$C_{S_1}(n, S_3)$

1. $\bar{J}(S_2, S_3)[2]$

$J(S_2)[N]$

$D(S_2)$

$D(S_3)$

$J[1]$

2. $\Lambda(S_2)$

$\Lambda(S_3)$

$J[c]$

Das hier — und auch in den Späteren — verwendete N bezeichnet immer die nächste (auf die Subrutine folgende) Anweisung des Programms.

Die folgende Anweisung wird nur in solchen Fällen angewandt, wobei die Adresse n nicht leer, und sowohl $\langle m \rangle$ als auch $\langle n \rangle$ eine Kette lauter Striche ist. (Die ersten Buchstaben zweier derartigen Ketten stimmen dann und nur dann nicht überein, wenn genau eine der beiden Ketten leer ist.)

$\mathrm{Max}(m, n)$ $(=\mathrm{Max}_{S_1,S_2,S_3}(m, n))$: Bringe das Nicht-Kürzere von $\langle m \rangle$ und $\langle n \rangle$ in m, bei Auslöschung des Kürzeren.

Subrutine:

$C_{S_1}(m, S_2)$

$C_{S_1}(n, S_3)$

1. $\bar{J}(S_2, S_3)[2]$

$J(S_2)[3]$

$D(S_2)$

$D(S_3)$

$J[1]$

2. $J(S_3)[3]$

$C_{S_1}(n, m)$

$\Lambda(S_3)$

3. $\Lambda(n)$

In den Folgenden gebe ich Subrutinen bezüglich der partiell-rekursiven Funktionen definierenden Gleichungen und der Ableitungen aus diesen.

III

Mit Hilfe der zuletzt beschriebenen Anweisung Max(m, n) ergibt sich

Maxind(m, n) ($=\mathrm{Maxind}_{S_1,S_2,S_3,S_4,S_5}(m, n)$): Kopiere den maximalen in $\langle m \rangle$ enthaltenen Variablenindex in n.

Subrutine: $C_{S_1}(m, S_4)$

1. $J(S_4)[N]$

   $J^{(\mathrm{I})}(S_4)[3]$

2. $D(S_4)$

   $J[1]$

3. $P^{(\mathrm{I})}(S_5)$

   $D(S_4)$

   $J^{(\mathrm{I})}(S_4)[3]$

   $\mathrm{Max}_{S_1,S_2,S_3}(n, S_5)$

   $J(S_4)[N]$

   $J[2]$

Es wird auch die „Länge" eines (Gleichungbezeichnenden) Wortes eine Rolle spielen, das durch soviele Striche angegeben werden kann, wieviele Buchstaben das betreffende Wort enthält (mehrfach auftretende Buchstaben mehrfach gerechnet):

Long(m, n) ($=\mathrm{Long}_{S_1,S_2}(m, n)$): Setze nach $\langle n \rangle$ (das in den Anwendungen leer sein wird) soviele Striche, wieviele Buchstaben $\langle m \rangle$ enthält.

Subrutine: $C_{S_1}(m, S_2)$

1. $J(S_2)[N]$

   $D(S_2)$

   $P^{(\mathrm{I})}(n)$

   $J[1]$

Auch die Vertauschung der Seiten einer Gleichung wird notwendig sein. Die folgende Subrutine wird nur dann benutzt, wenn $\langle m \rangle$ eine Gleichung ist.

Vert(m, n) ($=\mathrm{Vert}_{S_1}(m, n)$): Setze $\langle m \rangle$ (in m beibehaltend) mit vertauschten Seiten in n.

Subrutine: $C_{S_1}(m, n)$

1. $J^{(=)}(n)[2]$

   $\sum_{i=1}^{10}(n)P^{(a_i)}(S_1)$

   $D(n)$

   $J[1]$

2. $D(n)$

   $P^{(=)}(n)$

   $CP(S_1 : n)$

Ferner werden wir erkennen müssen, ob die rechte Seite einer Gleichung $\langle m \rangle$ ein Zahlzeichen ist:

$\mathrm{Num}(m)[c](=\mathrm{Num}_{S_1,S_2}(m)[c])$: Ist die rechte Seite von $\langle m \rangle$ ein Zahlzeichen, so springe zur Anweisung c.

Subrutine: $C_{S_1}(m, S_2)$

1. $J^{(=)}(S_2)[2]$

   $D(S_2)$

   $J[1]$

2. $D(S_2)$

   $J^{(0)}(S_2)[3]$

   $\Lambda(S_2)$

   $J[N]$

3. $D(S_2)$

4. $J(S_2)[c]$

   $J^{(')}(S_2)[5]$

   $\Lambda(S_2)$

   $J[N]$

5. $D(S_2)$

   $J[4]$

Nun soll die Subrutine für den ersten Ableitungsschritt verfertigt werden. (In den Anwendungen wird dabei $\langle k \rangle$ immer der Indexteil $||...|$ eines Variablenzeichens $x||...|$ sein.)

$\mathrm{Subst}^{0}(m, n, k)$ $(=\mathrm{Subst}^{0}_{S_1,S_2,S_3,S_4,S_5}(m, n, k))$: Es soll $\langle m\rangle$ derart modifiziert in n übertragen (aber auch in m beibehalten) werden, dass für jedes Vorkommen von $x\langle k\rangle$ worauf kein Strich folgt in $\langle m\rangle$ das Zeichen 0 substituiert wird.

Subrutine: $C_{S_1}(m, S_4)$

1. $J^{(x)}(S_4)[2]$
   $J(S_4)[N]$
   $\sum_{i=1}^{10}(S_4)P^{(a_i)}(n)$
   $D(S_4)$
   $J[1]$
2. $D(S_4)$
3. $J^{(|)}(S_4)[5]$
   $J_{S_1,S_2,S_3}\{S_5, k\}[4]$
   $P^{(x)}(n)$
   $CP(S_5: n)$
   $J[1]$
4. $P^{(0)}(n)$
   $A(S_5)$
   $J[1]$
5. $D(S_4)$
   $P^{(|)}(S_5)$
   $J[3]$

Die Subrutine für den zweiten Ableitungsschritt wird folgenderweise bezeichnet:

$\mathrm{Subst}^{x'}(m, n, k)$ $(=\mathrm{Subst}^{x'}_{S_1,S_2,S_3,S_4,S_5}(m, n, k))$: Es soll $\langle m\rangle$ derart modifiziert in n übertragen (aber auch in m beibehalten) werden, dass für jedes Vorkommen von $x\langle k\rangle$ worauf kein Strich folgt in $\langle m\rangle$ das Wort $x\langle k\rangle'$ substituiert wird.

Dies ergibt sich genau so, wie die Subrutine für $\mathrm{Subst}^0(m, n, k)$, mit dem einzigen Unterschied, dass statt der Anweisungen

4. $P^{(0)}(n)$

$A(S_5)$

die folgenden Anweisungen aufgenommen werden:

$P^{(')}(S_5)$

$CP(S_5 : n)$

Nun folgt die Subrutine für den dritten Ableitungsschritt. (In den Anwendungen wird dabei $\langle k \rangle$ immer eine Gleichung und $\langle l \rangle$ immer eine Kette von Strichen sein.)

$\mathrm{Ers}(m, n, k, l)[c] (= \mathrm{Ers}_{S_1,S_2,S_3,S_4}(m, n, k, l)[c])$: Es soll — falls möglich — $\langle$m$\rangle$ derart modifiziert in n übertragen (aber auch in m beibehalten) werden, dass falls ein — mit seinem i-ten Buchstaben beginnender — Teil von $\langle m \rangle$ mit der linken Seite von $\langle k \rangle$ übereinstimmt, dann dieser Teil durch die rechte Seite von $\langle k \rangle$ ersetzt wird, wobei i um 1 mehr als die Länge von $\langle l \rangle$ ist; und dann springe zur Anweisung c.

Subrutine:

$C_{S_1}(l, S_2)$

$C_{S_1}(m, S_3)$

$C_{S_1}(k, S_4)$

1. $J(S_2)[2]$

$J(S_3)[5]$

$\sum_{i=1}^{10} (S_3) P^{(a_i)}(n)$

$D(S_3)$

$D(S_2)$

$J[1]$

2. $\bar{J}(S_3, S_4)[6]$

3. $D(S_3)$
   $D(S_4)$
   $J^{(=)}(S_4)[4]$
   $\bar{J}(S_3, S_4)[6]$
   $J[3]$
4. $D(S_4)$
   $CP(S_4 : n)$
   $CP(S_3 : n)$
   $J[c]$
5. $A(S_2)$
   $J[7]$
6. $A(S_3)$
7. $A(S_4)$
   $A(n)$

Endlich gebe ich eine Subrutine an, das im „Finish" gebraucht werden kann, wo nach Entstehung je eines neuen Gleichunges untersucht wird, ob sie eine Gleichung der Form

$$fxx\ldots x(n_1, \ldots, n_r) = n^* \quad \text{oder} \quad n^* = fxx\ldots x(n_1, \ldots, n_r)$$

ist, mit einem Zahlzeichen n^*; diese bestimmt dann sicher den gesuchten Funktionswert. (In den Anwendungen wird S_6 leer sein, $\langle k \rangle$ wird mit $fxx\ldots x(n_1, \ldots, n_r)$ übereinstimmen — es wird auch benutzt, dass dies keine Variablen enthält — und l wird die für das Ergebnis vorbehaltene Adresse sein.)

$\mathrm{Fin}(m, k, l)\,(= \mathrm{Fin}_{S_1,S_2,S_3,S_4,S_5,S_6,S_7}(m, k, l))$: Ist $\langle m \rangle$ ein Wort der Form $\langle k \rangle = n^*$ oder $n^* = \langle k \rangle$, wobei n^* ein Zahlzeichen $0''\cdots'$ ist, so bringe n^* in l und dann stop!

Subrutine:

$\mathrm{Num}_{S_1,S_2}(m)[4]$
$\mathrm{Vert}_{S_1}(m, S_5)$
$\mathrm{Num}_{S_1,S_2}(S_5)[3]$

1. $A(S_5)$
   $J[N]$
2. $C_{S_1}(S_6, l)$
   stop!
3. $\mathrm{Ers}_{S_1,S_2,S_3,S_4}(k, S_6, S_5, S_7)[2]$
   $J[1]$
4. $\mathrm{Ers}_{S_1,S_2,S_3,S_4}(k, S_6, m, S_7)[2]$

Dabei wird benutzt, dass ein *echter* Anfangsteil von

$$fxx...x(n_1, ..., n_r)$$

mit der linken Seite einer Gleichung nie übereinstimmen kann.

IV

Nun sind die Mitteln beisammen um das in Frage stehende programmierende Programm anzugeben.

In Adresse „2" soll immer das zu berechnende

$$\underbrace{fxx...x}_{r\text{-mal}}(n_1, ..., n_r)$$

gesetzt werden (wobei die n_i Abkürzungen der Zahlzeichen

$$\overbrace{0''...'}^{n_i\text{-mal}}$$

sind); Adresse „1" wird nach gewissen vorbereitenden Schritten für das Ergebnis vorbehalten; in Adresse „5" wird immer der grösste Variablenindex $||...|$ gesetzt, das im definierenden Gleichungssystem vorkommt.

In die Adressen „$3k$" für $k = 1, 2, ...$ werden Gleichungen gesetzt: zu Beginn die Definitionsgleichungen, dann die durch Ableitungsschritte allmählich zustande kommenden Gleichungen. Diese sollen als in besondere Klassen eingeteilt betrachtet werden: die Definitionsgleichungen bilden die 0-te Klasse; auf diese folgt die 1-te Klasse der aus ihnen durch einen Ableitungsschritt gebildeten Gleichungen; auf diese folgt die 2-te Klasse, die aus jene Gleichungen besteht, welche aus den Definitionsgleichungen durch zwei nacheinander ausgeführte Ableitungsschritte entstehen, usw.; Gleichungen höherer Klassen kommen immer in Adressen grösserer Nummer.

In die Adresse „$3k+1$" kommt für $k = 1, 2, ...$ die grösste Adresse der $k-1$-ten Gleichungsklasse; zu Beginn wird also in „4" die grösste der Definitionsgleichungen enthaltenden Adressen gesetzt, und die weiteren Adressen „$3k+1$" werden nach das Unterbringen des letzten Gleichunges je einer Klasse ausgefüllt.

Unter den Adressen der Form „$3k+2$" werden jene, für welche $2 \leqq k \leqq 8$ gilt, die zu den in **II** und **III** angegebenen Subrutinen notwendigen, durch

$$S_1, S_2, S_3, S_4, S_5, S_6, S_7$$

bezeichneten Register-Adressen sein; auf diese folgen für $9 \leqq k \leqq 22$ die Adressen der zu den Folgenden notwendigen weiteren Register, die ich mit R bezeichnen werde, mit verschiedenen (ihre Rollen einigermassen andeutenden) Indizes versehen. (Die Adressen „$3k+2$" für $k \geqq 23$ stehen zum Unterbringen des programmierenden Programms zur Verfügung.)

Genauer: Sei die zu den Subrutinen „Subst" notwendige, immer ein Variablenindex enthaltende Adresse durch R_v bezeichnet, und die zur Subrutine „Ers" not-

wendige, immer eine Kette von Strichen enthaltende Adresse durch R_i. Die Länge des Inhaltes der letzteren wird immer mit der Länge der eben betrachteten Gleichung verglichen; zu diesem Zweck soll deren Länge immer in eine durch R_L bezeichnete Adresse gesetzt werden. Die Adresse der eben entstehenden Gleichung wird immer in eine durch R_e bezeichnete Adresse gesetzt.

Wird die Bildung einer neuen Gleichungsklasse begonnen, so müssen wir immer die erste und die letzte bereits fertige Klasse vor Augen halten (bei Bildung der 1-ten Klasse fallen diese beiden zusammen). Die Adressen der zur ersten gehörigen Gleichungen sind jene, die grösser als der Inhalt der Adresse | und höchstens so gross wie der Inhalt der Adresse |||| sind; diese beiden Adressen | und |||| sollen in durch R_I^* bzw. $R_{I'}^*$ bezeichneten Registern gespeichert werden. Die Adressen der zur letztgebildeten Klasse gehörigen Gleichungen sind jene, die grösser als der Inhalt der vorletzten, und höchstens so gross, wie der Inhalt der letzten nicht leeren Adresse „$3k+1$" sind; diese beiden (im Laufe der Berechnung veränderlichen) Adressen sollen in durch R_X^* bzw. $R_{X'}^*$ bezeichneten Registern gespeichert werden. Das Zeichen $*$ will eine Fixierung der Inhälte der Register während der Bildung einer Klasse andeuten; für Änderungen dieser Inhälte sollen Arbeitszellen R_I; $R_{I'}$; R_X; $R_{X'}$ zur Verfügung stehen. Endlich werden noch die Inhälte von R_I und $R_{X'}$ (die Gleichungsadressen sind) in durch $R_{\langle I\rangle}$ bzw. $R_{\langle X\rangle}$ bezeichnete Adressen gesetzt.

Das in Adressen der Form „$3k+2$" für $k \geqq 23$ eingetragene Programm wird dadurch in Gang gesetzt, dass in „1" als Ausgangsinformation das „Wort"

$$fxx\ldots x(n_1, \ldots, n_r);\ G_1;\ G_2;\ \ldots;\ G_g$$

gesetzt wird, wobei f das Zeichen der zu berechnenden Funktion ist, worauf eine r-gliedrige Kette von x-Zeichen folgt; die Zeichen n_i für $i=1, 2, \ldots, r$ die Zahlzeichen

$$\overbrace{0'\ldots'}^{n_i\text{-mal}}$$

der betrachteten Argumente, und die Zeichen G_i für $i=1, 2, \ldots, g$ die gegebenen Definitionsgleichungen bedeuten.

Das Programm beginnt mit vorbereitenden Schritten, durch welche (1) der Anfang

$$fxx\ldots x(n_1, \ldots, n_r)$$

von $\langle|\rangle$ an seine endgültige Stelle, in „2" übertragen wird, (2) der in $\langle|\rangle$ enthaltene grösste Variablenindex in „5" kopiert wird, (3) die Gleichungen $G_1, G_2, \ldots, G_g$ bei Entleerung von „1" an ihre Stellen „3", „6", ..., „$3g$" übertragen werden, (4) in R_I^* und in R_X^* die Adresse „1", in $R_{I'}^*$ und in $R_{X'}^*$ die Adresse „4", in $\langle R_{X'}^*\rangle$ (das jetzt „4" ist) die Adresse „$3g$" gebracht wird.

So lautet der Beginn des Programms (mit der Abkürzung

$$\{A\}^n$$

falls eine Anweisung A n-mal nacheinander auszuführen ist):

1. $J^{(;)}(|)[2]$

$\sum_{i=1}^{10}(|)P^{(a_i)}(||)$

$D(|)$

$J[1]$

2. $D(|)$

Maxind $(|, |||||)$

3. $\{P^{(|)}(R_e)\}^3$

4. $J^{(;)}(|)[5]$

$\sum_{i=1}^{10}(|)P^{(a_i)}(\langle R_e\rangle)$

$D(|)$

$J(|)[6]$

$J[4]$

5. $D(|)$

$\mathrm{Fin}(\langle R_e\rangle, ||, |)$

$J[3]$

6. $P^{(|)}(R_I^*)$

$P^{(|)}(R_X^*)$

$\{P^{(|)}(R_{I'}^*)\}^4$

$\{P^{(|)}(R_{X'}^*)\}^4$

$C(R_e, \langle R_{X'}^*\rangle)$

Nun folgen die vorbereitenden Schritte jeder neuen Klassenbildung:

7. $C(R_I^*, R_I)$

$C(R_{I'}^*, R_{I'})$

$C(R_X^*, R_X)$

$C(R_{X'}^*, R_{X'})$

$C(\langle R_I\rangle, R_{\langle I\rangle})$

$C(\langle R_X\rangle, R_{\langle X\rangle})$

Falls die Definitionsgleichungen überhaupt Variablen enthalten (also die Adresse „5" nicht leer ist), beginnt die Bildung einer neuen Gleichungsklasse mit Ableitungsschritte „Subst". Nur durch Anwendungen auf Gleichungen der zuletzt gebildeten Klasse können solche Schritte neue Gleichungen liefern. Die Adresse der ersten Gleichung dieser Klasse ist um 3 länger als der Inhalt von $R_{\langle X\rangle}$; und die Adresse, worin die erste neue Gleichung aufzubauen ist (die erste leere Adresse der Form „$3k$") ist um 3 länger, als der Inhalt von R_e. Bei der Bildung je einer Gleichung wird der Inhalt von R_e um je 3 verlängert, was aber, falls der angewandte Schritt zu keiner neuen Gleichung führt, wieder rückgängig zu machen ist.

So lautet die Fortsetzung unseres Programms:

8. $J(|||||)[14]$
9. $\{P^{(|)}(R_{\langle X\rangle})\}^3$
10. $\{P^{(|)}(R_e)\}^3$

    $P^{(|)}(R_v)$

    $\mathrm{Subst}^0(\langle R_{\langle X\rangle}\rangle, \langle R_e\rangle, R_v)$

    $\bar{J}\{\langle R_{\langle X\rangle}\rangle, \langle R_e\rangle\}[11]$

    $\Lambda(\langle R_e\rangle)$

    $\{D(R_e)\}^3$

    $J[12]$
11. $\mathrm{Fin}(\langle R_e\rangle, ||, |)$

    $\{P^{(|)}(R_e)\}^3$

    $\mathrm{Subst}^{x'}(\langle R_{\langle X\rangle}\rangle, \langle R_e\rangle, R_v)$
12. $J\{R_v, |||||\}[13]$

    $J[10]$
13. $\Lambda(R_v)$

    $\bar{J}\{R_{\langle X\rangle}, \langle R^*_{X'}\rangle\}[9]$

    $C(\langle R^*_{\langle X\rangle}\rangle, R_{\langle X\rangle})$
14. ...

Hier beginnt die Bildung neuer Gleichungen durch Ableitungsschritte „Ers". Bei diesen spielen immer zwei Gleichungen eine Rolle; gehören diese zur i-ten bezw. zur j-ten Klasse, so gehört eine aus ihnen durch „Ers" gebildete Gleichung zur $i+j+1$-ten Klasse. So erhalten wir sämtliche durch „Ers" gebildete Gleichungen einer entstehenden neuen Klasse, indem wir die Ableitungsschritte „Ers" erst auf Gleichungspaare anwenden, deren erstes Glied zur letzten, zweites Glied zur ersten bereits fertigen Klasse gehört, dann auf Gleichungspaare, deren erstes Glied zur

vorletzten, zweites Glied zur zweiten fertigen Klasse gehört, usw. bis zu den Gleichungspaaren, deren erstes Glied zur ersten, zweites Glied zur letzten fertigen Klasse gehört. Haben wir inzwischen die Behandlung eines Klassenpaars beendet, so gehen wir dadurch auf das nächste Klassenpaar über, dass wir die Inhälte von R_I und von $R_{I'}$ um 3 verlängern, und die Inhälte von R_X und von $R_{X'}$ um 3 verkürzen; und dies tun wir bis zur Entleerung der Adresse $\langle R_X \rangle$.

Die Länge des ersten Gliedes des eben betrachteten Gleichungspaars wird immer in R_L gesetzt. R_i ist zu Beginn leer.

Die Klassenbildung wird (neben Entleerung der Adresse R_L) damit abgeschlossen, dass die Inhälte von R_X^* und von $R_{X'}^*$ um 3 verlängert, und der Inhalt von R_e in die Adresse $\langle R_{X'}^* \rangle$ kopiert wird.

So lautet der restliche Teil unseres Programms:

14. $\{P^{(|)}(R_{\langle X \rangle})\}^3$
    $\Lambda(R_L)$
    $\mathrm{Long}(\langle R_{\langle X \rangle} \rangle, R_L)$
    $C(\langle R_I \rangle, R_{\langle I \rangle})$
15. $\{P^{(|)}(R_{\langle I \rangle})\}^3$
16. $\{P^{(|)}(R_e)\}^3$
    $\mathrm{Ers}(\langle R_{\langle X \rangle} \rangle, \langle R_e \rangle, \langle R_{\langle I \rangle} \rangle, R_i)[17]$
    $\{D(R_e)\}^3$
    $J[18]$
17. $\mathrm{Fin}(\langle R_e \rangle, ||, |)$
18. $J\{R_i, R_L\}[19]$
    $P^{(|)}(R_i)$
    $J[16]$
19. $\Lambda(R_i)$
    $\bar{J}\{R_{\langle I \rangle}, \langle R_{I'} \rangle\}[15]$
    $\bar{J}\{R_{\langle X \rangle}, \langle R_{X'} \rangle\}[14]$
    $J(\langle R_X \rangle)[20]$
    $\{P^{(|)}(R_I)\}^3$
    $\{P^{(|)}(R_{I'})\}^3$
    $\{D(R_X)\}^3$
    $\{D(R_{X'})\}^3$
    $C(\langle R_X \rangle, R_{\langle X \rangle})$
    $J[14]$

20. $\Lambda(R_L)$
$\{P^{(1)}(R_X^*)\}^3$
$\{P^{(1)}(R_{X'}^*)\}^3$
$C(R_e, \langle R_{X'}^* \rangle)$
$J[7]$

Gelangt die Maschine im Laufe der Durchführung dieses Programms nach endlich vielen Schritten zum „stop", dann befindet sich der gesuchte Funktionswert in Adresse „1". Gelangt die Maschine nie zum „stop", so ist die betrachtete Funktion an der betreffenden Stelle nicht definiert.

Eötvös L. Universität, Budapest

(Eingegangen: 7. Oktober, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 465—469.

ZUR KONVERGENZ VON FOURIER-REIHEN QUADRATISCH INTEGRIERBARER FUNKTIONEN

von

H. KLEINDIENST

Mit C_n bzw. $C_n^{(m)}$ seien die von E. MAKAI in [1] eingeführten Konstanten bezeichnet. Man zeigt leicht, dass folgendes gilt

$$C_n = \sup_{N\in\mathcal{N}} \sup_{\|f\|=1} \left| \int_0^{2\pi} s_{N(x)}(x;f)\,dx \right| = \tag{1}$$

$$= \sup_{N\in\mathcal{N}} \left\{ \frac{1}{\pi} \int_0^{2\pi}\int_0^{2\pi} D_{\min(N(x),N(y))}(x-y)\,dx\,dy \right\}^{\frac{1}{2}}$$

$$C_n^{(m)} = \sqrt{\pi} \sup_{\|p_n\|=1} \max_{I_n^{(m)}} \frac{1}{m} \left| \sum_{r=1}^{m}{}' s_{n_r}\left(\frac{2\pi}{m} r; p_n\right) \right| = \tag{2}$$

$$= \frac{1}{m} \max_{I_n^{(m)}} \left(\sum_{p,q=1}^{m}{}' D_{\min(n_p,n_q)}(x_p - x_q) \right)^{\frac{1}{2}}.$$

Zum Beweis siehe [1] und [2]. Dabei ist $f\in L^2$, $s_k(x;f)$ die k-te Teilsumme der Fourier-Reihe $S[f]$ an der Stelle x, $\mathcal{N}$ die Menge derjenigen Funktionen N, die nur ganzzahlige Werte aus $[0, n]$ haben und so beschaffen sind, dass $s_{N(x)}(x;f)$ messbar ist. Ferner ist p_n ein trigonometrisches Polynom n-ter Ordnung und $I_n^{(m)} = (n_1, \ldots, n_m)$ ein m-Tupel ganzer Zahlen n_r mit $0 \leqq n_r \leqq n$ $(r = 1, \ldots, m)$.

Die Konstanten C_n bzw. $C_n^{(m)}$ spielen eine entscheidende Rolle in der Theorie der Konvergenz f. ü. von Fourier—Reihen quadratisch integrierbarer Funktionen, wie die folgenden Sätze zeigen.

SATZ 1. *Das trigonometrische System ist genau dann ein Konvergenzsystem bzgl. L^2, wenn die Zahlenfolge (C_n) beschränkt ist.*

SATZ 2. *Das trigonometrische System ist genau dann ein Konvergenzsystem bzgl. L^2, wenn es zu jedem n eine natürliche Zahl m_n gibt, so dass die Folge $(C_n^{(m_n)})$ beschränkt ist.*

Es soll hier nur Satz 2 bewiesen werden, für den Beweis von Satz 1 siehe [1], [2], [3]. Der Beweis wird dadurch erbracht, dass die Gültigkeit folgender Ungleichungskette nachgewiesen wird.

$$2\sqrt{\pi}\,C_n^{(m)} - 2\pi\sqrt{2\pi}\,\frac{n(n+1)}{m} \leqq C_n \leqq 2\sqrt{\pi}\,C_n^{(m)}. \tag{3}$$

Die Richtigkeit von $C_n \leqq 2\sqrt{\pi}\, C_n^{(m)}$ wurde schon von E. MAKAI bewiesen [1]. Die linksstehende Ungleichung erhält man folgendermassen. Nach Definition der Konstanten C_n gilt für jedes $f \in L^2$ mit $\|f\| = 1$ und jede Funktion $N \in \mathcal{N}$

$$C_n \geqq \left| \int_0^{2\pi} s_{N(x)}(x; f)\, dx \right|$$

$\alpha)$ Konstruktion der Funktion $f \in L^2$.
Es sei m eine natürliche Zahl. Dann sei $I_n^{(m)} = (n_1, \ldots, n_m)$ eines derjenigen m-Tupel ganzer Zahlen n_r mit $0 \leqq n_r \leqq n$ $(r = 1, \ldots, m)$ für das gilt

$$C_n^{(m)} = \frac{1}{m} \left(\sum_{p,q=1}^{m} D_{\min(n_p, n_q)} \left(\frac{2\pi}{m} (p-q) \right) \right)^{\frac{1}{2}}.$$

Ferner sei $g(x) = 2 \cdot \sum_{q=1}^{m} D_{n_q}(x - x_q)$ mit $x_q = \frac{2\pi}{m} q$. Dann wird

$$f = g \cdot \|g\|^{-1}$$

gesetzt. Mit $\varepsilon_{n_r, \nu} = 1$ für $|\nu| \leqq n_r$ und $\varepsilon_{n_r, \nu} = 0$ für $|\nu| > n_r$ ergibt sich für g die folgende Darstellung.

$$g(x) = 2 \cdot \sum_{q=1}^{m} D_{n_q}(x - x_q) = \sum_{q=1}^{m} \sum_{\nu=-n}^{n} \varepsilon_{n_q, \nu} e^{i\nu(x - x_q)} =$$

$$= \sum_{\nu=-n}^{n} \left(\sum_{q=1}^{m} \varepsilon_{n_q, \nu} e^{-i\nu x_q} \right) e^{i\nu x} = \sum_{\nu=-n}^{n} c_\nu e^{i\nu x} \quad \text{mit} \quad c_\nu = \sum_{q=1}^{m} \varepsilon_{n_q, \nu} e^{-i\nu x_q}.$$

Aus der Parsevalschen Gleichung

$$\sum_{\nu=-n}^{n} |c_\nu|^2 = \frac{1}{2\pi} \|g\|^2$$

ergibt sich für die Norm von g

$$\|g\|^2 = 2\pi \sum_{\nu=-n}^{n} \left| \sum_{q=1}^{m} \varepsilon_{n_q, \nu} e^{-i\nu x_q} \right|^2 = 2\pi \sum_{\nu=-n}^{n} \left(\sum_{q=1}^{m} \varepsilon_{n_q, \nu} e^{-i\nu x_q} \right) \left(\sum_{p=1}^{m} \varepsilon_{n_p, \nu} e^{i\nu x_p} \right) =$$

$$= 2\pi \sum_{\nu=-n}^{n} \sum_{p,q=1}^{m} \varepsilon_{\min(n_p, n_q), \nu} e^{i\nu(x_p - x_q)} = 4\pi \sum_{p,q=1}^{m} D_{\min(n_p, n_q)}(x_p - x_q) = 4\pi m^2 (C_n^{(m)})^2,$$

d.h. $\|g\| = 2\sqrt{\pi} m C_n^{(m)}$.

Weiter wird nun $s_{N(x)}(x; f)$ bei zunächst beliebigen N betrachtet. Mit $f = g \cdot \|g\|^{-1}$ gilt

$$s_{N(x)}(x; f) = \frac{1}{\|g\|} \sum_{\nu=-N(x)}^{N(x)} c_\nu e^{i\nu x} = \frac{2}{\|g\|} \sum_{q=1}^{m} D_{\min(N(x), n_q)}(x - x_q)$$

$\beta)$ Wahl von $N(x)$.

Wird das Intervall $[0, 2\pi]$ in m-Teilintervalle $I_p=(x_{p-1}, x_p)$ $(p=1, \ldots, m)$ unterteilt und $N(x)=n_p$ gesetzt falls $x\in I_p$ ist, so gilt — da es auf eine Menge vom Masse Null nicht ankommt — bei Verwendung der soeben konstruierten Funktionen f und N die folgende Abschätzung

$$\frac{\|g\|}{2}\left|\int_0^{2\pi} s_{N(x)}(x;f)\,dx\right| = \left|\int_0^{2\pi}\sum_{q=1}^{m} D_{\min(N(x),n_q)}(x-x_q)\,dx\right| =$$

$$= \left|\sum_{q=1}^{m}\sum_{p=1}^{m}\int_{x_{p-1}}^{x_p} D_{\min(N(x),n_q)}(x-x_q)\,dx\right| \geqq \left|\sum_{p,q=1}^{m}\int_{x_{p-1}}^{x_p} D_{\min(n_p,n_q)}(x_p-x_q)\,dx\right| -$$

$$- \sum_{p,q=1}^{m}\int_{x_{p-1}}^{x_p} |D_{\min(n_p,n_q)}(x-x_q)-D_{\min(n_p,n_q)}(x_p-x_q)|\,dx =$$

$$= \frac{2\pi}{m}\sum_{p,q=1}^{m} D_{\min(n_p,n_q)}(x_p-x_q) -$$

$$- \sum_{p,q=1}^{m}\int_{x_{p-1}}^{x_p}\left|\sum_{\mu=1}^{\min(n_p,n_q)}\left(\cos\mu(x-x_q)-\cos\mu(x_p-x_q)\right)\right| dx \geqq$$

$$\geqq 2\pi m(C_n^{(m)})^2 - \sum_{p,q=1}^{m}\int_{x_{p-1}}^{x_p}\left|\sum_{\mu=1}^{\min(n_p,n_q)} 2\sin\mu\frac{x+x_p-2x_q}{2}\sin\mu\frac{x-x_p}{2}\right| dx.$$

Da $|x-x_p|<|x_{p-1}-x_p|=\dfrac{2\pi}{m}$ für $x\in I_p$ ist, gilt $2\left|\sin\mu\dfrac{x-x_p}{2}\right|<\mu\dfrac{2\pi}{m}$ und man erhält weiterhin

$$\left|\int_0^{2\pi} s_{N(x)}(x;f)\,dx\right| \geqq 2\sqrt{\pi}C_n^{(m)} - \frac{2}{\|g\|}\sum_{q=1}^{m}\sum_{p=1}^{m}\int_{x_{p-1}}^{x_p}\left(\frac{2\pi}{m}\sum_{\mu=1}^{n}\mu\right)dx =$$

$$= 2\sqrt{\pi}C_n^{(m)} - \frac{8\pi^2}{\|g\|}\cdot\frac{n(n+1)}{2} = 2\sqrt{\pi}C_n^{(m)} - 2\pi\sqrt{\pi}\frac{n(n+1)}{mC_n^{(m)}}.$$

Um das Ergebnis auf der linken Seite der Ungleichung (3) zu erhalten, wird noch eine untere Schranke für die $C_n^{(m)}$ benötigt. Es gilt

(4) $$C_n^{(m)} \geqq \sqrt{\frac{1}{2}+\left[\frac{n}{m}\right]}, \quad ([x] \text{ größte ganze Zahl} \leqq x).$$

BEWEIS. Es sei $n_r=n$ für $r=1, \ldots, m$ gewählt und

$$\sigma_n^{(m)}(n) = \sum_{p,q=1}^{m} D_n\left(\frac{2\pi}{m}(p-q)\right).$$

Dann gilt

$$\sigma_n^{(m)}(n) = m \cdot \sum_{q=0}^{m-1} D_n\left(\frac{2\pi}{m} q\right) = m\left(D_n(0) + \sum_{q=1}^{m-1}\left(\frac{1}{2} + \sum_{v=1}^{n} \cos\left(\frac{2\pi}{m} q\right) v\right)\right) =$$

$$= m\left(n + \frac{m}{2} + \sum_{v=1}^{n}\sum_{q=1}^{m-1} \cos\left(\frac{2\pi}{m} v\right) q\right) = m\left(\frac{m}{2} + n + \sum_{v=1}^{n}\left(-\frac{1}{2} + D_{m-1}\left(\frac{2\pi}{m} v\right)\right)\right).$$

Für $v \neq km$ $(k = 1, 2, \ldots)$ ist $D_{m-1}\left(\frac{2\pi}{m} v\right) = -\frac{1}{2}$, für $v = km$ ist $D_{m-1}\left(\frac{2\pi}{m} v\right) =$ $= D_{m-1}(0) = m - \frac{1}{2}$. Daher ist

$$\sum_{v=1}^{n} D_{m-1}\left(\frac{2\pi}{m} v\right) = -\frac{n}{2} + m\left[\frac{n}{m}\right],$$

denn wegen $v \leqq n$ treten in der Summe $\left[\frac{n}{m}\right]$ Glieder, auf bei denen v ein Vielfaches von m ist. Somit erhält man

$$\sigma_n^{(m)}(n) = \frac{m^2}{2} + m^2\left[\frac{n}{m}\right].$$

Nach Definition der $C_n^{(m)}$ folgt damit die Richtigkeit von (4). Aus $C_n^{(m)} \geqq \sqrt{\frac{1}{2}}$ — was man übrigens auch sofort erhält, wenn man in (2) $I_n^{(m)} = (0, \ldots, 0)$ wählt — folgt die in (3) behauptete Abschätzung, womit auch Satz 2 bewiesen ist. Weiterhin ergibt sich aus (3) die Grenzbeziehung

$$C_n = \lim_{m \to \infty} 2\sqrt{\pi} C_n^{(m)}.$$

Mit Hilfe des Resultates von L. CARLESON, wonach das trigonometrische System ein Konvergenzsystem bzgl. L^2 ist, lässt sich aus den Sätzen 1 und 2 sowie den Ungleichungen in (3) sofort das nachstehende Ergebnis folgern.

SATZ 3. *Die Zahlenfolgen* (C_n) *und* $(C_n^{(m_n)})$ *mit* $m_n \geqq n^2$ *sind beschränkt.*

Letztere Aussage enthält insbesondere den Beweis für die Richtigkeit einer Vermutung von E. MAKAI. Wie weiterhin (4) zeigt, ist für die Beschränktheit der Folge $(C_n^{m_n})$ notwendig, dass $\frac{n}{m_n} = O(1)$ ist.*

* In der nachfolgenden Arbeit von E. MAKAI [4] wird ergänzend gezeigt, dass diese Bedingung nicht nur notwendig, sondern auch hinreichend ist.

LITERATURVERZEICHNIS

[1] MAKAI, E.: Sur la convergence des séries de Fourier des fonctions de carré sommable, *C. R. Acad. Sci. Paris* **257** (1963) 1893—1895.
[2] KLEINDIENST, H.: Dissertation, Hannover, 1968.
[3] ZYGMUND, A.: *Trigonometric Series*, II, Cambridge, 1959, 165—66.
[4] MAKAI, E.: A remark to the preceding paper of Kleindienst, *Studia Sci. Math. Hungar.* **4** (1969)

Technische Universität Hannover, Institut B für Mathematik

(Eingegangen: 5. November, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 471–472.

A REMARK TO THE PRECEDING PAPER OF H. KLEINDIENST

by

E. MAKAI

In this note the same notations will be used as in the preceding paper of H. KLEINDIENST [2]. The theorem of L. CARLESON [1] is equivalent to the statement that the sequence $\{C_n\}$, $n=1, 2, \ldots$ is bounded. The inequality

$$C_n \leqq 2\sqrt{\pi}\, C_n^{(m)} \tag{1}$$

was shown in [3]. A lower bound for $C_n^{(m)}$ was found by KLEINDIENST [2], namely

$$C_n^{(m)} \geqq \sqrt{\frac{1}{2}+\left[\frac{n}{m}\right]} \quad ([x]=\text{greatest integer contained in } x). \tag{2}$$

Here we shall prove

$$2\sqrt{\pi}\, C_n^{(m)} \leqq \left(1+2\pi\frac{n}{m}\right) C_n \tag{3}$$

which together with inequality (2) yields a statement comprising KLEINDIENST's Theorem 3:

A sequence $C_n^{(m_n)}$ (m_n *natural numbers*, $n=1, 2, \ldots$) *is bounded if and only if* $n/m_n = O(1)$.

From Lemma 2 of [3] and its proof follow that (*a*) in formula (2) of [2] the sign "sup" can be exchanged into "max", (*b*) there exist a real trigonometric polynomial $p^* = \Sigma c_\nu e^{i\nu x}$ ($c_\nu = \bar{c}_{-\nu}$) of order n and of unit norm and a sequence of integers n_1^*, n_2^*, ..., n_m^* satisfying $0 \leqq n_r^* \leqq n$ such that if $x_r = 2\pi r/m$, then

$$C_n^{(m)} = \sqrt{\pi} \cdot \frac{1}{m} \sum_{r=1}^{m} s_{n_r^*}(x_r; p^*). \tag{4}$$

We remark here the elementary fact that

$$\|p^{*\prime}\| = \sqrt{2\pi}\left[\sum_{\nu=-n}^{n} \nu^2 c_\nu^2\right]^{\frac{1}{2}} \leqq \sqrt{2\pi n}\left[\sum_{\nu=-n}^{n} c_\nu^2\right]^{\frac{1}{2}} = n. \tag{5}$$

By the definition of C_n we have

$$\begin{aligned} C_n &= \sup_{N\in\mathcal{N}} \sup_{\|f\|=1} \left| \int_0^{2\pi} s_{N(x)}(x; f)\, dx \right| \\ &\geqq \sup_{N\in\mathcal{N}} \int_0^{2\pi} s_{N(x)}(x; p^*)\, dx \geqq \sum_{r=1}^{m} \int_{x_{r-1}}^{x_r} s_{n_r^*}(x; p^*)\, dx. \end{aligned} \tag{6}$$

Let now $0 \leqq \eta_r \leqq 2\pi/m$ be a quantity for which

$$s_{n_r}^*(x_r - \eta_r; p^*) = \min_{x_{r-1} \leqq x \leqq x_r} s_{n_r}^*(x; p^*).$$

Then

$$\frac{2\pi}{m} \sum_{r=1}^{m} s_{n_r}^*(x_r - \eta_r; p^*) \leqq \sum_{r=1}^{m} \int_{x_{r-1}}^{x_r} s_{n_r}^*(x; p^*)\, dx \leqq C_n. \tag{7}$$

On the other hand by (1) we have from (4) that

$$\frac{2\pi}{m} \sum_{r=1}^{m} s_{n_r}^*(x_r; p^*) = 2\sqrt{\pi} C_n^{(m)} \geqq C_n. \tag{8}$$

Introducing the continuous function

$$g(\vartheta) = \frac{2\pi}{m} \sum_{r=1}^{m} s_{n_r}^*(x_r - \vartheta\eta_r; p^*)$$

of ϑ we infer from (7) and (8) that $g(1) \leqq C_n$ and $g(0) \geqq C_n$. Hence there exists in the closed interval [0, 1] a quantity ϑ_0 such that $g(\vartheta_0) = C_n$. Using again (8) we have

$$2\sqrt{\pi} C_n^{(m)} - C_n = \frac{2\pi}{m} \sum_r \{s_{n_r}^*(x_r; p^*) - s_{n_r}^*(x_r - \vartheta_0\eta_r; p^*)\} = \frac{2\pi}{m} \sum_r \int_{x_r - \vartheta_0\eta_r}^{x_r} s_{n_r}^*(x; p^{*\prime})\, dx =$$

$$= \frac{2\pi}{m} \sum_r \left\{ \int_{x_{r-1}}^{x_r - \vartheta_0\eta_r} s_0(x; p^{*\prime})\, dx + \int_{x_r - \vartheta_0\eta_r}^{x_r} s_{n_r}^*(x; p^{*\prime})\, dx \right\} \leqq \frac{2\pi}{m} \int_0^{2\pi} \max_{k=0,1,\ldots,n} s_k(x; p^{*\prime})\, dx.$$

We distinguish now two cases: (i) $\|p^{*\prime}\| = 0$, (ii) $\|p^{*\prime}\| \neq 0$. Incidentally, case (i) occurs only if $n = 0$. Supposing it, however, for any n, the last integral would vanish and we would be led to an inequality consistent with (3). In case (ii) we have by (5) and by the definition of C_n

$$2\sqrt{\pi} C_n^{(m)} - C_n \leqq \frac{2\pi}{m} \|p^{*\prime}\| \left| \int_0^{2\pi} \max_k s_k\left(x; \frac{p^{*\prime}}{\|p^{*\prime}\|}\right) dx \right| \leqq \frac{2\pi}{m} n C_n,$$

an inequality equivalent to (3).

REFERENCES

[1] CARLESON, L.: On convergence and growth of partial sums of Fourier series, *Acta Math.* **116** (1966) 135—157.

[2] KLEINDIENST, H.: Zur Konvergenz von Fourier-Reihen quadratisch integrierbarer Funktionen, *Stud. Sci. Math. Hung.* **4** (1969) 465—469.

[3] MAKAI, E.: On the summability of the Fourier series of L^2 integrable functions, I., *Publ. Math.* **11** (1964) 101—118.

Mathematical Institute of the Hungarian Academy of Sciences, Budapest

(Received November 8, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 473—475.

PROPERTIES OF GRAPHS WITH CONSTRAINTS ON DEGREES

by

J. A. BONDY

1. Introduction

A graph G is *Hamiltonian* if there is a simple circuit which passes through each vertex of G. Such a circuit is called a *Hamiltonian circuit* of G. G is *λ-connected* if the removal of fewer than λ vertices neither disconnects G nor reduces it to the trivial graph consisting of a single vertex. The *degree* $d(v)$ of a vertex v of G is the number of edges of G incident with v. Let D be an n-tuple $(d_1, d_2, \ldots, d_n)$ of non-negative integers. We shall say that a graph G of order n is a *realization* of D if the vertices $\{v_i\}_1^n$ of G can be arranged so that $d(v_i) = d_i$ for all i. PÓSA [4] and CHARTRAND, KAPOOR and KRONK [1] have given conditions on D which suffice to ensure that any realization of D is, respectively, Hamiltonian and λ-connected. In this paper we show that these conditions can be somewhat weakened. We also give conditions on D which ensure that any realization of D has at most c components. All graphs considered will be finite, undirected, loopless, and without multiple edges. D will be assumed ordered so that $d_1 \leqq d_2 \leqq \ldots \leqq d_n$. We write $\mathcal{G}(D)$ for the set of graphs which are realizations of D, and $|S|$ for the cardinal number of the set S.

2. Hamiltonian circuits

THEOREM 1. *Let D satisfy the following condition:*

$$\text{(A)} \qquad d_k \leqq k,\ d_l \leqq l\ (k \neq l) \Rightarrow d_k + d_l \geqq n,$$

where $n \geqq 3$. Then if $G \in \mathcal{G}(D)$, G is Hamiltonian.

PROOF. The argument is similar to that used by NASH—WILLIAMS in his proof of PÓSA's theorem [3]. Let $G \in \mathcal{G}(D)$. Suppose that G has two components K, L, of orders k, l, respectively. Then each of the k vertices of K can be adjacent only to other vertices of K. Hence $d_k \leqq k-1$. Similarly $d_l \leqq l-1$, and therefore $d_k + d_l \leqq k+l-2 < n$. But this contradicts (A). Hence G has just one component and is connected.

Now let $P = v_0, v_1, \ldots, v_p$, be a simple path of maximum length (p) in G (we represent P only by its vertices in sequential order), chosen so that $d(v_0) + d(v_p)$ is as large as possible. By the definition of p, v_0 is adjacent only to vertices of P. Let S be the set of all vertices v_i such that v_0 is adjacent to v_{i+1}. If $v_i \in S$, $v_i, v_{i-1}, \ldots, v_0, v_{i+1}, v_{i+2}, \ldots, v_p$, is a path of length p in G, and therefore, by the definition of P, $d(v_i) \leqq d(v_0) = k$, say. Also $|S| = k$. Hence there are k vertices of degree at most k and so $d_k \leqq k$. Similarly if $d(v_p) = l$ then $d_l \leqq l$. If $k \neq l$ this implies, by (A), that $k + l \geqq n$. If $k = l$ then there are $k+1$ vertices of degree at most k (the vertices in S

together with v_p). Hence $d_k \leqq k$, $d_{k+1} \leqq k$ and, again by (A), we have $2k \geqq n$. Thus it is always the case that $d(v_0)+d(v_p) \geqq n$. Therefore v_p is adjacent to some vertex $v_j \in S$ and so $v_j, v_{j-1}, \ldots, v_0, v_{j+1}, v_{j+2}, \ldots, v_p, v_j$, is a circuit C of length $p+1$ in G. If $p+1$ were less than n then, since G is connected, there would be a vertex v of G not in P, with v adjacent to some vertex of P. But then v together with the vertices of C would be the vertices of a path of length $p+1$. Hence $p+1 = n$ and C is a Hamiltonian circuit of G.

COROLLARY 1. 1. (PÓSA [4]) Let D satisfy the following conditions:

(i) $$d_k \leqq k \Rightarrow d_k \geqq \frac{n-1}{2},$$

(ii) $$\text{If } n \text{ odd, } d_{\frac{n+1}{2}} \geqq \frac{n+1}{2},$$

where $n \geqq 3$. Then if $G \in \mathscr{G}(D)$, G is Hamiltonian.

Pósa's argument can also be used to prove Theorem 1, and yields the following result when applied to bipartite graphs.

THEOREM 1A. Suppose G is bipartite, with the same number of vertices in each part. Let D satisfy the following condition:

$$d_k \leqq k,\ d_l \leqq l\ (k \neq l) \Rightarrow d_k + d_l \geqq \frac{n}{2},$$

where $n \geqq 4$. Then if $G \in \mathscr{G}(D)$, G is Hamiltonian.

When in addition we know that G has degrees $\{d_i\}_1^{n/2}$ in one part, and degrees $\{d'_i\}_1^{n/2}$ in the other part, we obtain a generalization of a result of Moon and Moser [2]:

THEOREM 1B. Suppose G is bipartite with vertices $\{v_i\}_1^{n/2}$ in one part, and vertices $\{v'_i\}_1^{n/2}$ in the other part. Let D satisfy the following condition:

$$d_k \leqq k,\ d'_l \leqq l \Rightarrow d_k + d'_l \geqq \frac{n}{2}.$$

where $n \geqq 4$ and d_i, d'_i respectively are the degrees of vertices v_i, v'_i with $d_1 \leqq d_2 \leqq \ldots \leqq d_{n/2}$, $d'_1 \leqq d'_2 \leqq \ldots \leqq d'_{n/2}$. Then if $G \in \mathscr{G}(D)$, G is Hamiltonian.

3. λ-connectedness

THEOREM 2. *Let D satisfy the following condition:*

(B) $$d_k \geqq k+\lambda-1, \quad (1 \leqq k \leqq n-d_{n-\lambda+1}-1).$$

Then if $G \in \mathscr{G}(D)$, G is λ-connected.

PROOF. Let $G \in \mathscr{G}(D)$ and let S be any cut-set of vertices of G, with $|S| = s$. Suppose that K is a component of $G-S$ of minimum order, say k. Then each vertex of K can have degree at most $k-1$ in $G-S$, and hence at most $k+s-1$ in G. There-

fore $d_k \leqq k+s-1$, and, by (B), this implies that either $s \geqq \lambda$ or $k > n-d_{n-\lambda+1}-1$. Now there are $n-k-s$ vertices of G not in S or K and therefore, by a similar argument to the above, each of these vertices has degree at most $n-k-1$ in G. Hence there are at most s vertices (those in S) with degree greater than $n-k-1$ in G. Therefore $d_{n-s} \leqq n-k-1$. Now if $k > n-d_{n-\lambda+1}-1$ then $d_{n-s} < d_{n-\lambda+1}$ and so $s \geqq \lambda$. Hence we always have $s \geqq \lambda$ and this implies that G is λ-connected.

COROLLARY 2. 1. (Chartrand, Kapoor, Kronk [1]). Let D satisfy the following conditions:

(i) $$d_k \geqq k+\lambda-1, \quad \left(1 \leqq k \leqq \frac{n-\lambda}{2}\right)$$

(ii) $$d_{n-\lambda+1} \geqq \frac{n+\lambda}{2}-1.$$

Then if $G \in \mathscr{G}(D)$, G is λ-connected.

4. Number of components

THEOREM 3. *Let D satisfy the following condition:*

(C) $$d_k \geqq k, \quad (1 \leqq k \leqq m).$$

Then if $G \in \mathscr{G}(D)$, G has at most $\left[\frac{n+m-d_n}{m+1}\right]$ components.

PROOF. Let $G \in \mathscr{G}(D)$. Suppose that G has c components, the smallest having order k, and the largest having order l. Then there are at least k vertices of degree at most $k-1$. Hence $d_k \leqq k-1$ and, by (C), we have $k \geqq m+1$. Also, every vertex of G has degree at most $l-1$. Therefore $d_n \leqq l-1$, i.e. $l \geqq d_n+1$. Now $(c-1)k+l \leqq n$ and hence $(c-1)(m+1) \leqq n-d_n-1$. Thus $c \leqq \frac{n-d_n-1}{m+1}+1$, and so $c \leqq \left[\frac{n+m-d_n}{m+1}\right]$.

I am indebted to Professor C. St. J. A. NASH—WILLIAMS for prompting this investigation.

REFERENCES

[1] CHARTRAND, G., KAPOOR, S. F. and KRONK, H. V.: A sufficient condition for n-connectedness of graphs, *Mathematika* **15** (1968) 51—52.

[2] MOON, J. W. and MOSER, L.: On Hamiltonian Bipartite Graphs, *Israel J. Math* **1** (1963) 163—165.

[3] NASH—WILLIAMS, C. ST. J. A.: On Hamiltonian circuits in finite graphs, *Proc. Amer. Math. Soc.* **17** (1966), 466—467.

[4] PÓSA, L.: A theorem concerning Hamiltonian lines, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **7** (1962) 225—226.

University of Waterloo, Waterloo, Ontario, Canada

(Received November 24, 1968.)

Studia Scientiarum Mathematicarum Hungarica 4 (1969) 477—479.

CONVERGENCE ABSOLUE ET CHANGEMENT DE VARIABLE

par

L. ALPÁR

La note présente n'est qu'une simple remarque attirant l'attention sur le fait que deux propositions obtenues respectivement par G. HALÁSZ ([3], p. 64) et par nous même ([1], théorème 1, p 61, et théorème 2, p 64) contiennent des résultats plus généraux que ceux qui ont été formulés dans les travaux cités. Ces théorèmes améliorés s'énoncent comme suit:

THÉORÈME 1. — *Étant données la fonction* $h(z) = (z-\zeta)/(1-\bar{\zeta}z)$ $(0<|\zeta|<1)$ *et la suite* $\{\alpha_n\}_0^\infty$, *avec* $0<\alpha_n<2$, $\alpha_n \to 0$, $\alpha_n \log n \to \infty$ *pour* $n \to \infty$, *il existe une fonction*

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$$

analytique pour $|z|<1$, *ainsi qu'une suite* $\{\varepsilon_\nu\}_0^\infty$, *avec* $0<\varepsilon_\nu<1$, $\varepsilon_\nu \to 0$ *pour* $\nu \to \infty$, *telle que*

$$\sum_{\nu=0}^{\infty} |a_\nu|^{\varepsilon_\nu} < \infty, \tag{1}$$

et si l'on pose

$$f(h(z)) = g(z) = \sum_{n=0}^{\infty} b_n z^n \tag{2}$$

où $g(z)$ *est aussi analytique pour* $|z|<1$, *on a*

$$\sum_{n=1}^{\infty} |b_n|^{2-\alpha_n} = \infty. \tag{3}$$

En tenant compte d'un autre résultat ([2], théorème 2, p. 288) nous pouvons y ajouter que la série (2) est en même temps uniformément convergente sur la circonférence $|z|=1$.

La seconde proposition, dans laquelle t désigne une variable réelle, concerne les séries de Fourier.

THÉORÈME 2. — *Si, sur un segment I arbitrairement petit, f est deux fois dérivable, avec* $f''(t)>\varkappa>0$ *(ou* $f''(t)<-\varkappa<0$*), et si la suite* $\{\alpha_n\}_{-\infty}^{\infty}$ *remplit les conditions:* $0<\alpha_n<2$, $\alpha_n \to 0$, $\alpha_n \log |n| \to \infty$ *pour* $|n| \to \infty$, *il existe une fonction*

$$F(t) = \sum_{\nu=-\infty}^{\infty} A_\nu e^{i\nu t},$$

ainsi qu'une suite $\{\varepsilon_\nu\}_{-\infty}^{\infty}$, *avec* $0<\varepsilon_\nu<1$, $\varepsilon_\nu\to 0$ *pour* $|\nu|\to\infty$, *telle que*

$$\sum_{\nu=-\infty}^{\infty}|A_\nu|^{\varepsilon_\nu}<\infty, \tag{4}$$

et si l'on pose

$$F(f(t))=G(t)=\sum_{n=-\infty}^{\infty}B_n e^{int}, \tag{5}$$

on a

$$\sum_{n=-\infty}^{\infty}|B_n|^{2-\alpha_n}=\infty.$$

D'après les théorèmes 2 et 3 de la note [1], si $f(t)\equiv f(t+2\pi)\pmod{2\pi}$, $G(t)$ est une fonction continue ou la série (5) est même uniformément convergente selon que $f(t)$ est seulement continue ou analytique sur l'axe des t.

Dans les notes [3] et [1] ont été établis de théorèmes analogues avec $\varepsilon_\nu=1$ pour tout ν; c'est-à-dire ces théorèmes concernent les cas où les séries (1) et (4) sont absolument convergentes au sens ordinaire. HALÁSZ a obtenu son résultat en construisant une fonction particulière $f_0(z)=\sum_{\nu=0}^{\infty}a_\nu^{(0)}z^\nu$ telle que $\sum_{\nu=0}^{\infty}|a_\nu^{(0)}|<\infty$ et dont la transformée (2) satisfait à la relation (3). Or, il est simple de voir que la série formée des mêmes $a_\nu^{(0)}$ vérifie aussi la relation (1) avec une suite $\{\varepsilon_\nu\}_0^\infty$ convenablement choisie, où $0<\varepsilon_\nu<1$, $\varepsilon_\nu\to 0$ pour $\nu\to\infty$.

En effet, il construit, selon une certaine règle, deux suites: l'une d'entiers positifs $\{\nu_k\}_0^\infty$ et l'autre de nombres positifs $\{u_k\}_0^\infty$, d'où nous retenons seulement le fait que

$$0<u_k<\frac{c}{2^k}\qquad(k=0,1,2,\ldots)$$

où $c>0$ est une constante. On met ensuite $|a_{\nu_k}^{(0)}|=u_k$ et $a_\nu^{(0)}=0$ si $\nu\neq\nu_k$. Les ε_ν peuvent donc être arbitraire quand $\nu\neq\nu_k$ et les ε_{ν_k} doivent remplir la condition

$$\sum_{k=0}^{\infty}2^{-k\varepsilon_{\nu_k}}<\infty. \tag{6}$$

On a alors

$$\sum_{\nu=0}^{\infty}|a_\nu^{(0)}|^{\varepsilon_\nu}=\sum_{k=0}^{\infty}u_k^{\varepsilon_{\nu_k}}<c_1\sum_{k=0}^{\infty}2^{-k\varepsilon_{\nu_k}}.$$

Il est évident qu'il existe une infinité de suites $\{\varepsilon_{\nu_k}\}$ réalisant (6). On peut poser p.e

$$\varepsilon_{\nu_k}\geqq\frac{(1+\delta)\log k}{k\log 2}\qquad(0<\delta<1).$$

On obtient ainsi

$$\sum_{\nu=0}^{\infty}|a_\nu^{(0)}|^{\varepsilon_\nu}<c_2\sum_{k=1}^{\infty}k^{-1-\delta}<\infty.$$

Pour établir le théorème 2 on peut procéder de la même façon, avec la différence simple que cette fois-ci $k=0,\ \pm1,\ \pm2,\ldots$.

MTA Könyvtára
Periodika 4456/1969

RÉFÉRENCES

[1] ALPÁR, L.: Sur certaines théorèmes de Carleman et de Hardy, *Studia Sci. Math. Hungar.* **4** (1969) 59—67.

[2] ALPÁR, L.: Sur certaines transformées des séries de puissances absolument convergentes sur la frontière de leur cercle de convergence, *Magyar. Tud. Akad. Mat. Kutató Int. Közl.* **7** (1962) 297—316.

[3] HALÁSZ, G.: On Taylor series absolutely convergent on the circumference of the circle of convergence I, *Publ. Math. Debrecen* **14** (1967) 63—68.

Institut de Mathématique de l'Académie des Sciences de Hongrie, Budapest

(Reçu le 29 novembre 1968.)

ERRATUM TO THE PAPER OF B. D. MALVIYA, „ON THE ABSOLUTE RIESZ SUMMABILITY OF A SERIES RELATED TO A FOURIER SERIES" 3/1968/5—11.

Page 6: Delete the first line from the bottom.

Page 6: In the statement of Theorem A add the following: *is summable* $|R, e^{(\log n)^{\Delta}}, 1|$, *where* $0<\alpha<1$ *and* $\Delta = 1+\frac{1}{\alpha}$.

INDEX

Bajraktarević, M.: Über die Vergleichbarkeit der mit Gewichtsfunktionen gebildeten Mittelwerte 3
Burgess, D. A.: A problem on semi-groups 9
Hoàng Tuy et Nguên quang Thái: Sur deux problèmes d'affectation 13
Athreya, K. B.: On a characteristic property of Polya's urn 31
Forgó, F.: Relationship between mixed zero-one integer linear programming and certain quadratic programming problems 37
Chakrabarti, A.: Derivation of certain finite Kontorovitch type of integral transforms..... 45
Tomescu, I.: Une évaluation du nombre chromatique d'un graphe fini 55
Alpár, L.: Sur certains théorèmes de Carleman et de Hardy..... 59
Fényes, T.: On the operational solution of certain non-linear singular integral equations..... 69
Fejes Tóth, L.: Über die Nachbarschaft eines Kreises in einer Kreispackung 93
Handa B. R., and Mohanty, S. G.: On some distributions concerning a restricted random walk 99
Kosik, P.: Die Lösung eines Wärmeleitungsproblems für Systeme, bestehend aus Stäben und Wärmebehältern 109
Elbert, Á.: Remark on a paper of I. Bihari and T. Fényes 127
Schmidt, W. M.: Disproof of some conjectures on Diophantine approximation 137
Balogh, A.: On some problems of estimation after selection in the reliability theory..... 145
Ambartzumian, R. V.: Correlation properties of the intervals in the superpositions of independent stationary recurrent point processes 161
Halász, G.: Über die Konvergenz multiplikativer zahlentheoretischer Funktionen..... 171
Freud, G.: Lagrangesche Interpolation über die Nullstellen der Hermiteschen Orthogonalpolynome 179
Tasche, M.: Über Funktionen, die Δ^2-stetig bzw. stetig von 2. Ordnung sind 191
Rhemtulla, A. R.: On a problem of L. Fuchs 195
Жук, В. В.: О приближении периодических функций линейными процессами 201
Majthay, A.: On basis optimality in linear programming 207
Majthay, A.: On complementary pivot theory..... 213
Fejes Tóth, G.: Kreisüberdeckungen der Sphäre 225
Balázs, M. and Goldner, G.: On an iterative method with difference quotients of the second order 249
Elbert, Á.: On the solutions of the differential equation $y''+q(x)y = 0$, where $[q(x)]^v$ is concave, II. 257
Révész, Gy.: Syntactic analysis and unilateral context-sensitive grammars 267
Kis, O.: Замечания об оценке погрешности интерполирования 279
Varga, L.: A class of methods for function minimization 291
Ruzsa, I.: Random models of logical systems 301
Szász, D.: Asymptotically uniform sequences of measures 313
Kannappan, Pl.: On sine functional equation..... 331
Szabados, J.: Rational approximation in complex domains 335
Adler, G.: Une inégalité de type de Harnack concernant les fonctions biharmoniques 341
Bikelis, A.: Об оценке остаточного члена в центральной предельной теореме для выборок взятых из конечных множеств 345

Hatvani, L.: On the asymptotic behaviour of the solutions of $y''+\sum_{i=0}^{n} a_i(x)y^{2i+1} = 0$ 355

Cuppens, R. and Lukács, E.: Factorization of generalized Poisson distributions 367

Sallay, M.: Über „lückenhafte" Orthogonalpolynomsysteme 371
Freud, G.: Ein Beitrag zur Theorie des Lagrangeschen Interpolationsverfahrens 379
Hadwiger, H.: Notiz zur Eulerschen Charakteristik offener und abgeschlossener euklidischen Polyeder 385
Fritz, J.: Entropy of point processes 389
Csiszár, I.: On generalized entropy 401
Halász, G.: Tauberian theorems for univalent functions 421
Fejes Tóth, L.: Remarks on a theorem of R. M. Robinson 441
Péter, R.: Automatische Programmierung zur Berechnung der partiel-rekursiven Funktionen .. 447
Kleindienst, H.: Zur Konvergenz von Fourier-Reihen quadratisch integrierbarer Funktionen .. 465
Makai, E.: A remark to the preceding paper of H Kleindienst 471
Bondy, J. A.: Properties of graph with constraints on degrees 473
Alpár, L.: Convergence absolue et changement de variable 477
Erratum to the paper of B. D. Malviya "On absolute Riesz summability" 481

MAGYAR
TUDOMÁNYOS AKADÉMIA
KÖNYVTÁRA

Printed in Hungary

A kiadásért felel az Akadémiai Kiadó igazgatója — Műszaki szerkesztő: Farkas Sándor
A kézirat nyomdába érkezett: 1969. I. 28. — Terjedelem; 42,50 (A/5) ív, 38 ábra

69-7131 — Szegedi Nyomda

Die *Studia Scientiarum Mathematicarum Hungarica* ist eine Halbjahrsschrift der Ungarischen Akademie der Wissenschaften. Sie veröffentlicht Originalbeiträge aus dem Bereiche der Mathematik in deutscher, englischer, französischer oder russischer Sprache. Es erscheint jährlich ein Band.

Adresse der Redaktion: Budapest V., Reáltanoda u. 13—15, Ungarn.
Technischer Redaktor: Gy. Petruska.

Abonnementspreis pro Band (pro Jahr): 165.— Ft. Bestellbar bei Buch- und Zeitungs-Aussenhandelsunternehmen *Kultúra* (Budapest 62, P.O.B. 149), oder bei den Vertretungen im Ausland.
Austauschabmachungen können mit der Bibliothek des Mathematischen Instituts (Budapest V., Reáltanoda u. 13—15) getroffen werden.
Die zur Veröffentlichung bestimmten Manuskripte sind in zwei Exemplaren an die Redaktion zu schicken.

Studia Scientiarum Mathematicarum Hungarica est une revue biannuelle de l'Académie Hongroise des Sciences publiant des essais originaux, en francais, anglais, allemand ou russe, du domaine des mathématiques.

Rédaction: Budapest V. Reáltanoda u. 13—15, Hongrie.
Rédacteur téchnique: Gy. Pertuska

Le prix de l'abonnement: 165 Forints par an (volume). On s'abonne chez *Kultúra*, Société pour le Commerce de Livres et Journaux (Budapest 62, P.O.B. 149) ou chez ses représentants à l'étranger.
Pour établir des relations d'échange on est prié de s'adresser à la Bibliotheque de l'Institut de Mathématique (Budapest V., Reáltanoda u. 13—15).
Onfest prié d'envoyer les articles destinés à la publication en deux exemplaires à l'adresse de la rédaction.

Studia Scientiarum Mathematicarum Hungarica — выходит два раза в год в издании Академии Наук Венгрии. Журнал публикует оригинальные исследования в области математики и немецком, английском, французском и русском языках. Отдельные вышуски составляют ежегодно один том.

Адрес редакции: Budapest V., Reáltanoda u. 13—15, Венгрия.
Технический редактор: Gy. Petruska.

Подписная цена на год (за один том): 165 форинтов. Подписка на журнал принимается Внешнеторговым предприятием „Культура" (Budapest 62, P. O. B. 149) или его представительствами за границей.
По поводу отношения обмена просим обращаться к Библиотеке Института Математики (Budapest V., Reáltanoda u. 13—15).
Работы, предназначенные для опубликования в журнале следует направлять по адресу редакции в двух экземплярах.

All the reviews of the Hungarian Academy of Sciences may be obtained among others from the following bookshops:

ALBANIA

Ndermarja Shtetnore e Botimeve
Tirana

AUSTRALIA

A. Keesing
Box 4886, GPO
Sidney

AUSTRIA

Globus Buchvertrieb
Salzgries 16
Wien I.

BELGIUM

Qffice International de Libirairie
30, Avenue Marnix
Bruxelles 5
Du Monde Entier
5, Place St. Jean
Bruxelles

BULGARIA

Raznoiznos
1 Tzar Assen
Sofia

CANADA

Pannonia Books
2 Spadina Road
Toronto 4, Ont.

CHINA

Waiwen Shudian
Peking
P.O.B. Nr. 88.

CHECHOSLOVAKIA

Artia A. G.
Ve Smeckách 30
Praha II.
Postova Novinova Sluzba
Dovoz tisku
Vinohradska 46
Praha 2
Postova Novinova Sluzba
Dovoz tlace
Leningradska 14
Bratislava

DENMARK

Ejnar Munksgaard
Nörregade 6
Kopenhagen

FINLAND

Akateeminen Kirjakauppa
Keskuskatu 2
Helsinki

FRANCE

Office International de Documentation et Libraire
48, rue Gay Lussac
Paris 5

GERMAN DEMOCRATIC REPUBLIC

Deutscher Buch-export und Import
Leninstraße 16.
Leipzig C. I.
Zeitungsvertriebsamt
Clara Zetkin Straße 62.
Berlin N. W.

GERMAN FEDERAL REPUBLIC

Kunst und Wissen
Erich Bieber
Postfach 46.
7 Stuttgart S.

GREAT BRITAIN

Collet's' Subscription Dept.
44—45 Museum Streetf
London W.C.I.
Robert Maxwell and Co. Ltd.
Waynflete Bldg. The Plain
Oxford

HOLLAND

Swetz and Zeitlinger
Keizersgracht 471—487
Amsterdam C.
Martinus Nijhof
Lange Voorhout 9
The Hague

INDIA

Current Technical Literature
Co. Private Ltd.
Head Pffice:
India House OPP.
GPO Post Box 1374f
Bombay I.

ITALY

Santo Vanasia
71 Via M. Macchi
Milano
Libreria Commissionaria Sansoni
Via La Marmora 45
Firenze

JAPAN

Nauka Ltd.
2 Kanada-Zimbocho 2-chome
Chiyoda-ku
Tokyo
Maruzen and Co. Ltd.
P.O. Box 605
Tokyo
Far Eastern Booksellers
Kanada P.O. Box 72
Tokyo

KOREA

Chulpanmul
Korejskoje Obschestvo po Exportui Importu Proizvedenij Pechati
Phenjan

NORWAY

Johan Grundt Tanum
Karl Johansgatan 43
Oslo

POLAND

RUCH
ul. Wilcza 46.
Warszawa

ROUMANIA

Cartimex
Str. Aristide Briand 14—18.
Bucuresti

SOVIET UNION

Mezhdunarodnaja Kniga
Moscow
G—200

SWEDEN

Almquist and Wiksell
Gamla Brogatan 26
Stockholm

USA

Stechert Hafner Inc.
31 East 10th Street
New York 3 N. Y.
Walter J. Johnson
111 Fifth Avenue
New York 3 N. Y.

VIETNAM

Xunhasaba
Service d'Export et d'Import des Livres et Périodiques
19. Tran Quoc Toan
Hanoi

YUGOSLAVIA

Forum
Vojvode Misiva broj 1.
Novi Sad
Jugoslovenska Kniga
Terazije 27.
Beograd

X. 15. 1969. Index: 26 748